Quantum Hackenbush

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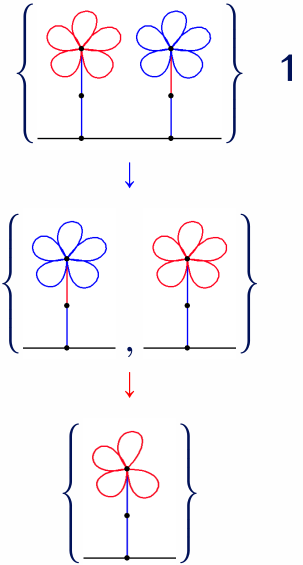
Universiteit Leiden
The Netherlands

Quantum Hackenbush

- "Hackenbush?"
- Quantum(-inspired) Hackenbush
- Results
- Future research



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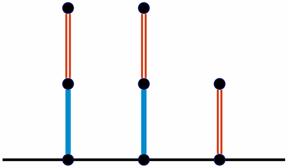
  - Short (finite and loopfree)
- Played on graph with blue and red edges.



Combinatorial games

Definition

A *short game* is an ordered pair

$$G = \left\{ G_1^L, \ldots, G_m^L \;\middle|\; G_1^R, \ldots, G_n^R \right\}$$

of Left options G^L and Right options G^R.

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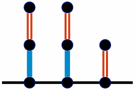
$G =$ 

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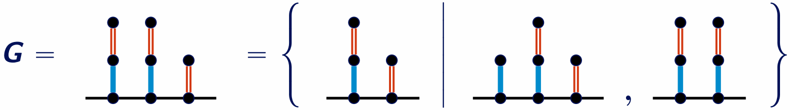

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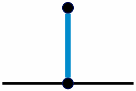
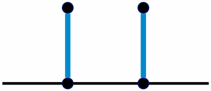
- **Advantage:** by *how much* does a player win?

Values of combinatorial games

$G_1 =$  , $G_2 =$ 

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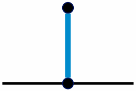
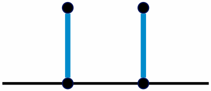
$G_1 =$  , $G_2 =$ 

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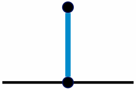
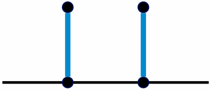
$G_1 =$  , $G_2 =$ 

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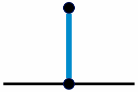

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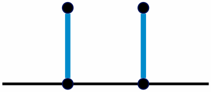

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Alternating turns: $G \to G^L \to G^{LR} \to G^{LRL} \to \cdots$

Why give Left two turns in a row?

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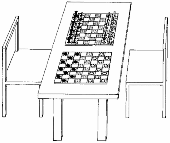

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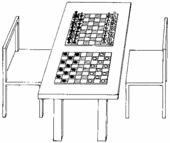
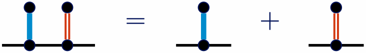


$$\boldsymbol{G} + \boldsymbol{H} \to \boldsymbol{G}^{\boldsymbol{L}} + \boldsymbol{H} \to \boldsymbol{G}^{\boldsymbol{L}} + \boldsymbol{H}^{\boldsymbol{R}} \to \boldsymbol{G}^{\boldsymbol{LL}} + \boldsymbol{H}^{\boldsymbol{R}}$$

Disjunctive sums in HACKENBUSH



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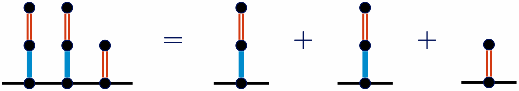

$$= \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}} + (-\mathbf{1}) = \mathbf{0}$$

Simplest number theorem

Theorem

A short game G, where $G^L < G^R$ for all G^L, G^R, is:

- Integer $G^L < n < G^R$ smallest in absolute value, if it exists.
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Examples:

- $\{ \mid \} = 0$
- $\{0 \mid \} = 1$
- $\{-999, -21 \mid 42, 181\} = 0$
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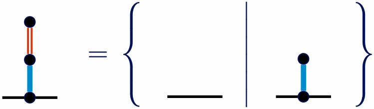
Not all games are numbers! Take $\{0 \mid 0\} = *$.

“Half” a move



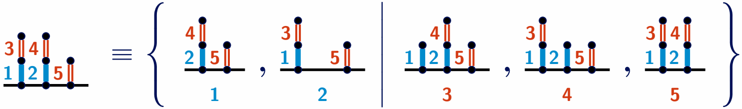
$$= \{\mathbf{0} \mid \mathbf{1}\} = \frac{\mathbf{1}}{\mathbf{2}}$$

Quantum-inspired combinatorial games

Altered variant of a ruleset, in which *superposed moves* are allowed, leading to *superposed game states*.

Preliminaries

A *labelled short game* $\widehat{\boldsymbol{G}}$:

$\widehat{\boldsymbol{G}} \equiv$ 

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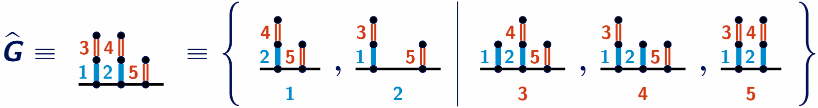
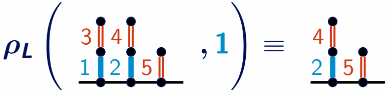


Definition

A *ruleset* is a pair of move functions $\mathcal{R} = (\rho_L, \rho_R)$.



How do we label moves?

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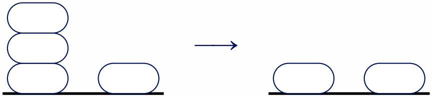
- HACKENBUSH: ***move*** $\iff$ ***edge_id***.
- Not so easy for other games, such as NIM:

Definition

NIM is played on a number of heaps, each containing at least one token. On their turn, the player selects one heap to remove a (nonzero) number of tokens from.



What constitutes a Nim move?

Definition 1

Nim-Subtract: the player selects a heap and removes a (nonzero) number of tokens off the top.

Definition 2

Nim-Decrease: the player selects a heap and cuts it at some height, leaving a smaller heap.

NIM-SUBTRACT labelling

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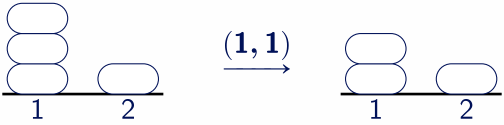

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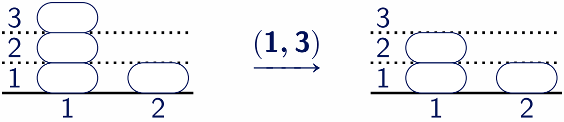
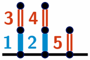


Main idea

$\Big\{$  $\Big\}$

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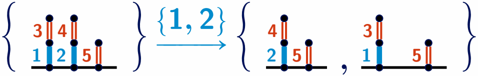

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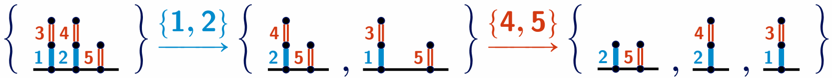
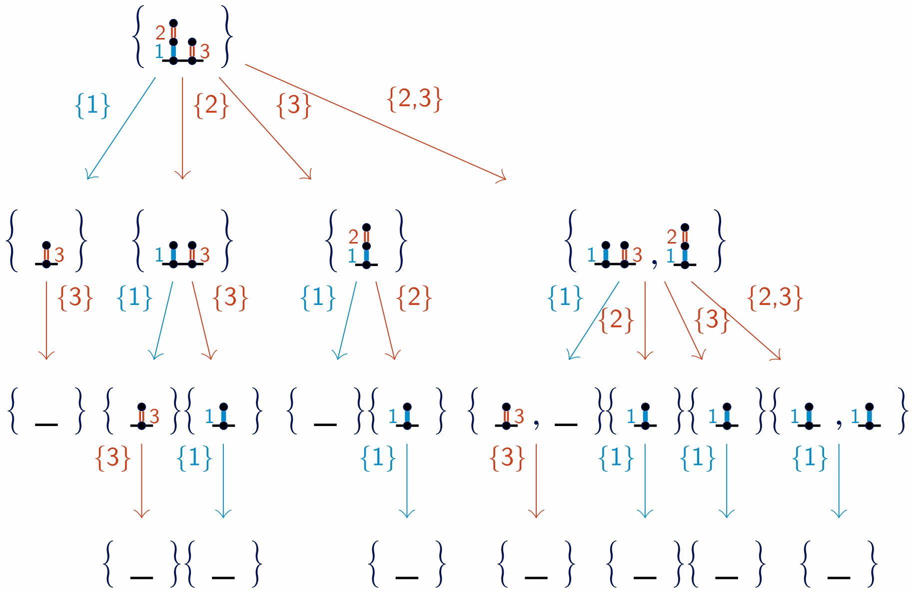


Quantum Hackenbush



The flavours

When is an unsuperposed move allowed?

- $\mathcal{A}$: Never.
- $\mathcal{B}$: Never, except if the only option.
- $\mathcal{C}$: Only if legal in all realisations.
- $\mathcal{C}'$: Only if legal in all realisations in which the player still has at least one legal move.
- $\mathcal{D}$: Always.

Some properties of Hackenbush

- **Dead-ending**: moves cannot "come back".

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- **Dead-ending**: moves cannot "come back".
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- **Persistent**: interpretation of move does not change.
- **Consistent**: any move always has the same effect.

Useful theorem

Theorem

Under flavour $\boldsymbol{f} \in \{\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{D}}\}$, if a realisation is *weakly covered*, then it can be left out of the superposed game state without changing its legal superposed move options.

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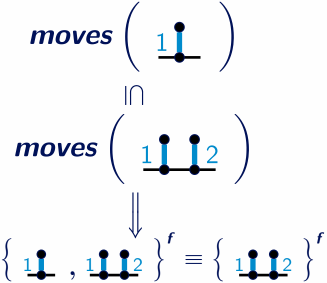

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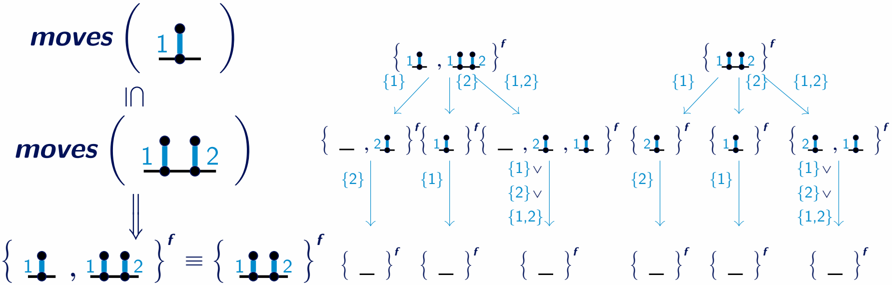
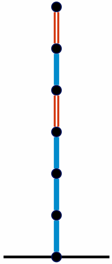


What is the value of a single stalk?



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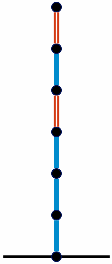

Theorem by Thea van Roode

Starting from the ground, count number of edges until first colour change, sign determined by colour of grounded edge. Then, divide by **2** for every next edge, add for blue and subtract for red.

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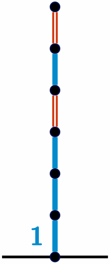

$= \mathbf{1}$

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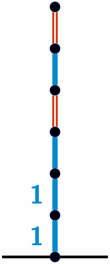

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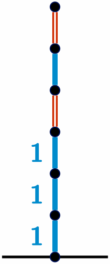

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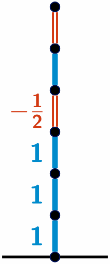

$$= 1 + 1 + 1 - \frac{1}{2}$$

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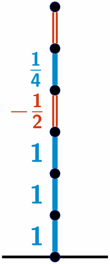

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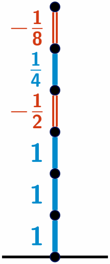

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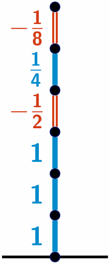

$$= 1 + 1 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = 2\frac{5}{8}$$

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And a Quantum Hackenbush stalk? (1/3)

Theorem

Under flavour $\boldsymbol{f} \in \{\mathcal{B}, \mathcal{D}\}$, any Hackenbush stalk has the same value as in the classical case.

Proof

The realisation containing the tallest stalk weakly covers all other realisations.

And a QUANTUM HACKENBUSH stalk? (2/3)

Theorem

Under flavour $\mathcal{A}$, any HACKENBUSH stalk has the same value as in the classical case, ignoring the first edge of either colour.

$\Bigg\{$  $\Bigg\}^{\mathcal{A}}$

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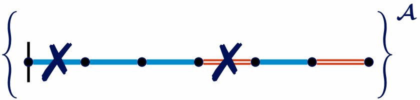

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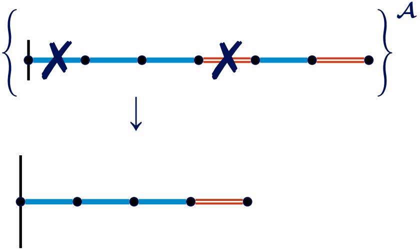

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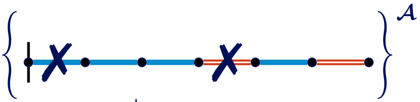
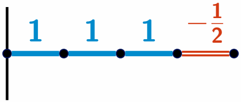


$\downarrow$



$$= \mathbf{1} + \mathbf{1} + \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{2}\frac{\mathbf{1}}{\mathbf{2}}$$

And a Quantum Hackenbush stalk? (3/3)

	4	5	6
000100	$\{1 \mid 1\}$	$\{2 \mid 2\}$	$\{3 \mid 3\}$
001000		$\{2 \mid 1\}$	$\{3 \mid 2\}$
001001		$\{1 \mid 1, \{1 \mid 1\}\}$	$\{2 \mid 2, \{2 \mid 2\}\}$
001011		$\{\frac{1}{2} \mid \frac{1}{2}\}$	$\{1\frac{1}{2} \mid 1\frac{1}{2}\}$
001100		$\{\frac{1}{2} \mid \frac{1}{2}\}$	$\{1\frac{1}{2} \mid 1\frac{1}{2}\}$
010000			$\{3 \mid 1\}$
010001			$\{2 \mid 1, \{2 \mid 1\}\}$
010010			$\{1\frac{1}{2} \mid 1, \{1 \mid 1\}\}$
010011			$\{1 \mid 1, \{1 \mid 1, \{1 \mid 1\}\}\}$
010100			$\{\frac{3}{4} \mid \frac{3}{4}\}$
010110			$\{\frac{1}{2}, \{\frac{1}{2} \mid \frac{1}{2}\} \mid \frac{1}{2}\}$
010111			$\{\frac{1}{2} \mid \frac{1}{4}\}$
011000			$\{\frac{3}{4} \mid \frac{1}{2}\}$
011001			$\{\frac{1}{2} \mid \frac{1}{2}, \{\frac{1}{2} \mid \frac{1}{2}\}\}$
011011			$\{\frac{1}{4} \mid \frac{1}{4}\}$
011100			$\{\frac{1}{4} \mid \frac{1}{4}\}$

Values of binary-encoded stalks under flavour $\mathcal{C}$.

Outcome classes

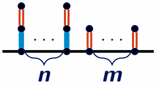
$\mathcal{A}$	0	1	2	3	4	5
0	$\mathscr{P}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
1	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
2	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
3	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
4	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
5	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
6	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
7	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$			
8						

$\mathcal{B}$	0	1	2	3	4	5
0	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
1	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
2	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
3	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
4	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
5	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
6	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
7	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$			
8						

$\mathcal{C}$	0	1	2	3	4	5
0	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
1	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
2	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
3	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
4	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
5	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
6	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
7	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$			
8						

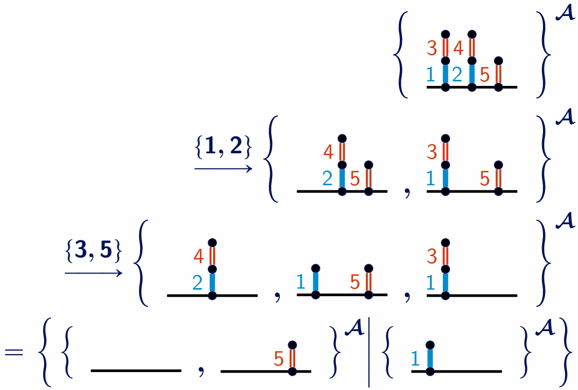
$\mathcal{C}'$	0	1	2	3	4	5
0	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
1	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
2	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
3	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
4	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
5	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
6	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
7	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$			
8						

$\mathcal{D}$	0	1	2	3	4	5
0	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
1	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
2	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
3	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
4	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
5	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{P}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
6	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
7	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$
8	$\mathscr{L}$	$\mathscr{L}$	$\mathscr{L}$			



Outcome classes for different ***n*** *(vertical) and* ***m*** *(horizontal). Max. move width* **2**.

$\mathscr{N}$-positions



$$= \{\mathbf{0} \mid \mathbf{0}\} = *$$

Future research

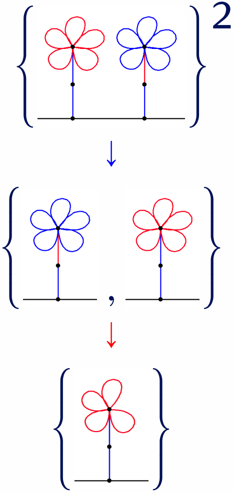
- Proving conjectures.
- Different classes of positions.
- Efficient representation of superposed game states.
- Quantum circuit to play Quantum Hackenbush.

Finishing up

Summary and conclusion

- Values.
- Defining quantum-inspired combinatorial games.
- QUANTUM HACKENBUSH stalks.
- Outcome classes, $\mathscr{N}$-positions.

Think (and taste) carefully. . .



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https://designer.microsoft.com