# Control Systems

G V V Sharma*

Contents



*Abstract*—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. 

## 1 Mason's Gain Formula

## 2 Bode Plot

### 2.1 Introduction

### 2.2 Example

## 3 Second order System

### 3.1 Damping

### 3.2 Example

## 4 Routh Hurwitz Criterion

### 4.1 Routh Array

### 4.2 Marginal Stability

### 4.3 Stability

4.1. Unit Step response of a linear time invariant (LTI) system is given by

$$y(t) = (1 - e^{-2t})u(t) \tag{4.1.1}$$

Assuming zero initial condition, the transfer function of the system is
**Solution:** We can convert this step response into s-domain using the Laplace Transform

$$Y(s) = \mathcal{L}(y(t)) \tag{4.1.2}$$

$$where \quad \mathcal{L}(y(t)) = \int_{-\infty}^{\infty} y(t)e^{-st}dt \tag{4.1.3}$$

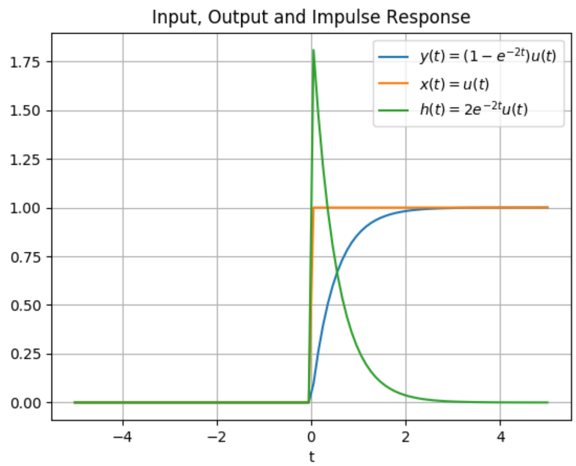
$$Y(s) = \mathcal{L}(y(t)) \tag{4.1.4}$$

$$\implies Y(s) = \mathcal{L}(u(t) - e^{-2t}u(t)) \tag{4.1.5}$$

$$\implies Y(s) = \mathcal{L}(u(t)) - \mathcal{L}(e^{-2t}u(t)) \tag{4.1.6}$$

$$\implies Y(s) = \frac{1}{s} - \frac{1}{s+2} \tag{4.1.7}$$

$$\implies Y(s) = \frac{2}{s(s+2)} \tag{4.1.8}$$




$$X(s) = \frac{1}{s} \tag{4.1.9}$$

The Transfer Function $H(s)$ is given by

$$H(s) = \frac{Y(s)}{X(s)} \tag{4.1.10}$$

$$H(s) = \frac{2}{s+2} \tag{4.1.11}$$

Hence the Transfer Function $H(s)$ is $\frac{2}{s+2}$

4.2. Plotting the input, output and impulse response using Python **Solution:**

```
codes/EE18BTECH11021.py
```

4.3. Comment on the stabilty of the system from the obtained transfer function
**Solution:** The Transfer Function is

$$H(s) = \frac{2}{s+2} \tag{4.3.1}$$

which has only 1 pole at s = -2, which is on the left half of s-plane therefore the system is stable
We can see that the output function attains a steady state of y = 1 as t tends to infinity, this is the verification that the system is stable

# 5 State-Space Model

## 5.1 Second Order System

# 6 Nyquist Plot

# 7 Phase Margin

# 8 Gain Margin

# 9 Compensators

## 9.1 Phase Lead

9.1. What is the purpose of a phase lead compensator?
**Solution:**
- Phase lead compensators are used to produce an output having a phase lead when an input is applied
- The major application is to help improve the Phase Margin (P.M) of the system.

9.2. Through an example show how the compensator in Problem 7.1 can be used in a control system
Consider a control system with the Transfer Function

$$G(s) = \frac{1}{s(3s+1)} \tag{9.2.1}$$

$$G(\jmath\omega) = \frac{1}{(j\omega)(3j\omega+1)} \tag{9.2.2}$$

$$\implies \left|G(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} \tag{9.2.3}$$

$$\angle G(\jmath\omega) = -\tan^{-1}(3\omega) - 90^\circ \tag{9.2.4}$$

At Gain Crossover,

$$\left|G(\jmath\omega)\right| = 1 \tag{9.2.5}$$

$$\implies \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} = 1 \tag{9.2.6}$$

$$\implies \omega_{gc} = 0.531 \tag{9.2.7}$$

$$\angle G(\jmath\omega) = -147.88^\circ \implies PM = 32.12^\circ \tag{9.2.8}$$

Phase Margin is only 32.12°which is below the acceptable range of Phase Margin i.e 45°.
Now we will apply Lead Compensator

$$D(s) = \frac{3(s + \frac{1}{3T})}{(s + \frac{1}{T})} \tag{9.2.9}$$

Choose T = 1 for this compensator
We have,

$$D(s) = \frac{3(s + \frac{1}{3})}{(s + 1)} \tag{9.2.10}$$

By cascading the Compensator and the Open Loop Transfer Function, we get

$$D(s)G(s) = \frac{1}{s(3s + 1)} \frac{3(s + \frac{1}{3})}{(s + 1)} \tag{9.2.11}$$

$$Let G_1(s) = D(s)G(s) \tag{9.2.12}$$

$$\implies G_1(s) = \frac{1}{s(s + 1)} \tag{9.2.13}$$

Calculating Phase Margin for cascaded Transfer Function

$$G_1(s) = \frac{1}{s(s + 1)} \tag{9.2.14}$$

$$G_1(\jmath\omega) = \frac{1}{(j\omega)(j\omega + 1)} \tag{9.2.15}$$

$$\implies \left|G_1(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{\omega^2 + 1}\right)} \tag{9.2.16}$$

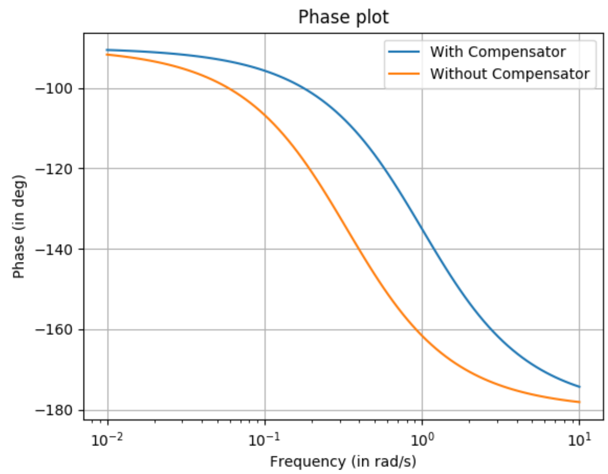
$$\angle G_1(\jmath\omega) = -\tan^{-1}(\omega) - 90° \tag{9.2.17}$$

At Gain Crossover,

$$\left|G_1(\jmath\omega)\right| = 1 \tag{9.2.18}$$

$$\implies \frac{1}{\omega\left(\sqrt{\omega^2 + 1}\right)} = 1 \tag{9.2.19}$$

$$\implies \omega_{gc} = 0.786 \tag{9.2.20}$$




$$\angle G(\jmath\omega) = -128.167° \implies PM = 51.83° \tag{9.2.21}$$

As we can see, the Phase Margin has gone above 45°which is the acceptable range of Phase Margin.
Therefore, through this example we have shown that a Lead Compensator is helpful in improving the Phase Margin of the System.

9.3. Verify the above improvement in Phase Margin with the help of a Python Code **Solution:**

```
codes/EE18BTECH11021_2.py
```

## 10 Oscillator

## 11 Controllers

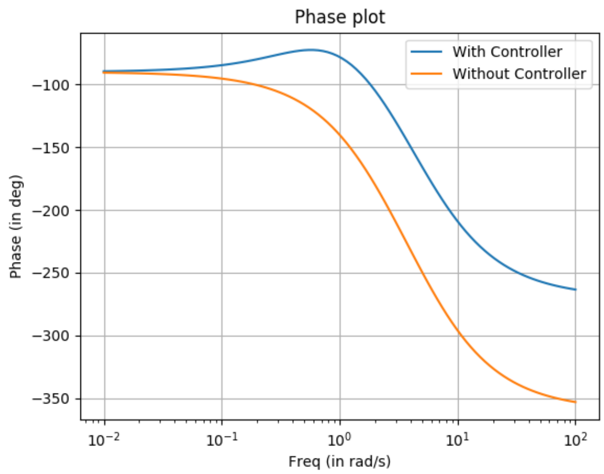
11.1. For a unity Feedback system

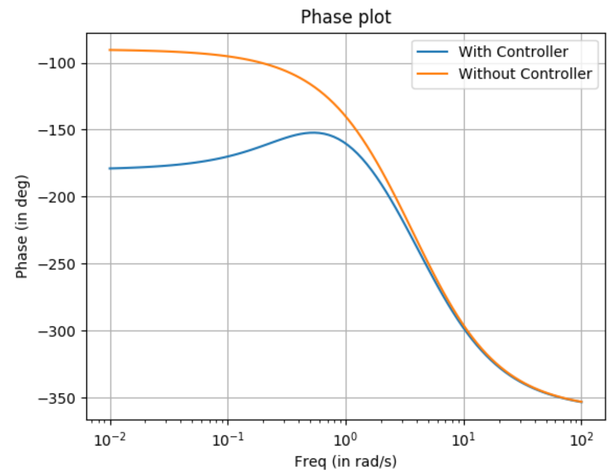
$$G(s) = \frac{K}{s(s + 2)(s + 4)(s + 6)} \tag{11.3.1}$$

Design a PD Controller with $K_v = 2$ and Phase Margin 30°
**Solution:** PD Controller is cascaded with the given G(s). The Transfer Function of the PD Controller is $K_p(1 + T_d s)$

$$G_c(s) = \frac{K_p(1 + T_d s)K}{s(s + 2)(s + 4)(s + 6)} \tag{11.3.2}$$

$$K_v = \lim_{s \to 0} sG_1(s) = 2 \tag{11.3.3}$$




If we choose $K_p = 1$

$$\implies K = 96 \tag{11.3.4}$$

For Phase Margin 30°, at Gain Crossover Frequency w

$$\tan^{-1}(T_d\omega) - \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{\omega}{4})\tan^{-1}(\frac{\omega}{6}) = -60 \tag{11.3.5}$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_d^2 w^2 + 1}}{w\sqrt{(w^2+4)(w^2+16)(w^2+36)}} = 1 \tag{11.3.6}$$

Solving these equations is really complicated, By Hit and Trial, one of the best combinations is

$$w = 4 \tag{11.3.7}$$

$$T_d = 1.884 \tag{11.3.8}$$

We get a Phase Margin of 30.31°

11.2. Verify using a Python Plot
**Solution:**

| codes/EE18BTECH11021_3.py |
|---|

11.3. Design a PI Controller with $K_v = \infty$ and Phase Margin 30°
**Solution:** PI Controller is cascaded with the given G(s). The Transfer Function of the PD Controller is $K_p(1 + \frac{1}{T_i s})$

$$G_1(s) = \frac{K_p(1 + \frac{1}{T_i s})K}{s(s+2)(s+4)(s+6)} \tag{11.3.9}$$

Choose $K_pK = 96$ This can be written as

$$G_1(s) = \frac{96(T_i s + 1)}{T_i s^2(s+2)(s+4)(s+6)} \tag{11.3.10}$$

For Phase Margin 30°, at Gain Crossover Frequency w

$$\tan^{-1}(T_i\omega) - \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{\omega}{4})\tan^{-1}(\frac{\omega}{6}) = 30 \tag{11.3.11}$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_i^2 w^2 + 1}}{T_i^2 w^2\sqrt{(w^2+4)(w^2+16)(w^2+36)}} = 1 \tag{11.3.12}$$

By Hit and Trial, one of the best combinations is

$$w = 0.75 \tag{11.3.13}$$

$$T_i = 2.713 \tag{11.3.14}$$

We get a Phase Margin of 25.53°

11.4. Verify using a Python Plot
**Solution:**

| codes/EE18BTECH11021_4.py |
|---|

11.5. Design a PID Controller with $K_v = \infty$ and Phase Margin 30°
**Solution:** PID Controller is cascaded with

the given G(s). The Transfer Function of the PD Controller is $K_p + K_d s + \frac{K_i}{s}$

$$G_1(s) = \frac{(K_p + K_d s + +\frac{K_i}{s})K}{s(s+2)(s+4)(s+6)} \quad (11.3.15)$$

Choose $K_p K = 96$ This can be written as

$$G_1(s) = \frac{96(T_i T_d s^2 + T_i s + 1)}{T_i s^2(s+2)(s+4)(s+6)} \quad (11.3.16)$$

For Phase Margin 30°, at Gain Crossover Frequency w

$$\tan^{-1}(\frac{T_i\omega}{1 - TiT_d w^2}) - \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{\omega}{4})\tan^{-1}(\frac{\omega}{6}) = 30 \quad (11.3.17)$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{(1 - TiT_d w^2)^2 + T_i^2}}{T_i^2 w^2\sqrt{(w^2+4)(w^2+16)(w^2+36)}} \quad (11.3.18)$$

By Hit and Trial, one of the best combinations is

$$w = 1 \quad (11.3.19)$$

$$T_i = 1.738 \quad (11.3.20)$$

$$T_d = 0.4 \quad (11.3.21)$$

We get a Phase Margin of 30°

11.6. Verify using a Python Plot

**Solution:**

codes/EE18BTECH11021_5.py