ASTRONOMICAL ALGORITHMS

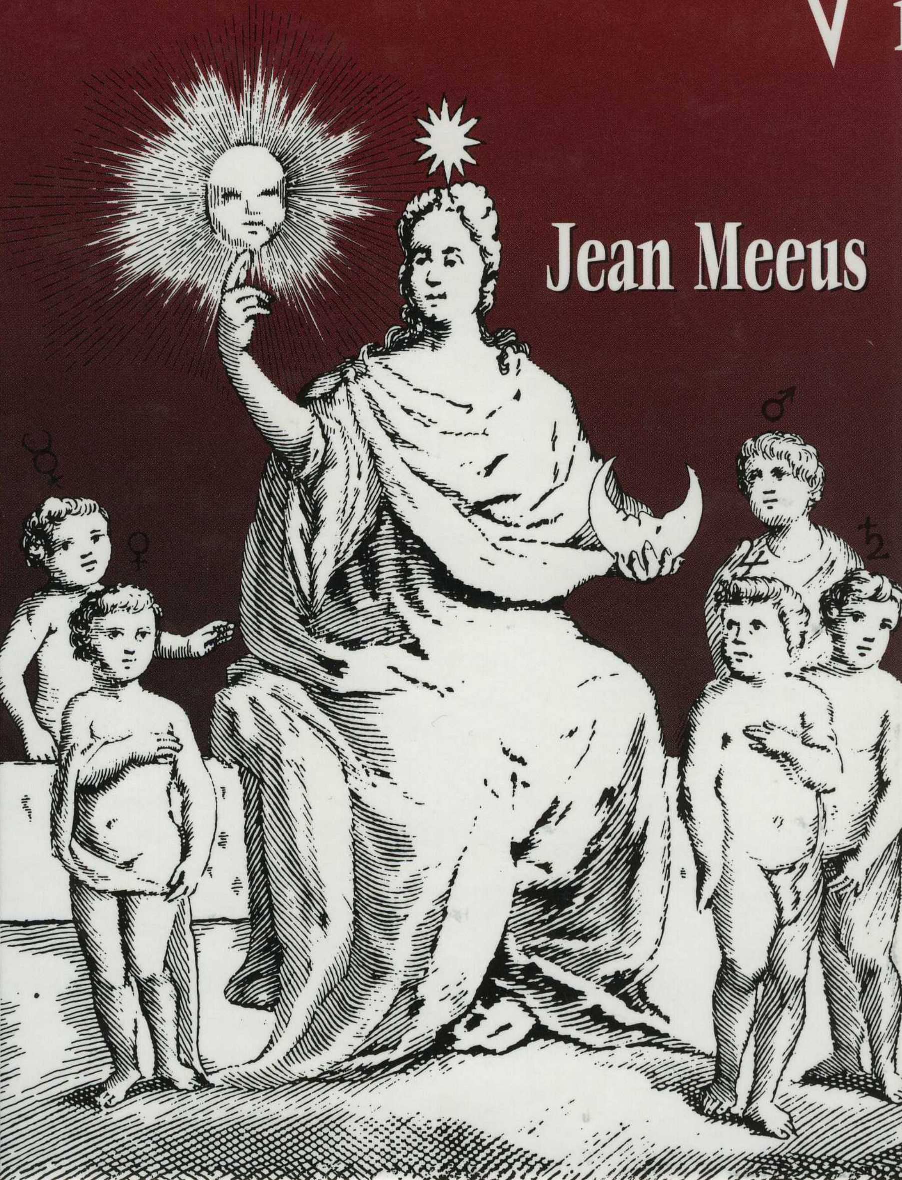
SECOND EDITION

$$v = 42.1219\sqrt{\frac{1}{r}-\frac{1}{2a}}$$

$$v = 42.1219\sqrt{\frac{1}{r}-\frac{1}{2a}}$$

$$v = 42.1219\sqrt{\ }$$

Jean Meeus



ASTRONOMICAL

SECOND EDITION

ALGORITHMS

Jean Meeus

Willmann-Bell, Inc.

P.O. Box 35025 • Richmond, Virginia 23235 • ☎ (804) 320-7016 • FAX (804) 272-5920
www.willbell.com

Published by Willmann-Bell, Inc.
P.O. Box 35025, Richmond, Virginia 23235

First English Edition 1991
Second English Edition 1998

Printed in the United States of America

Library of Congress Cataloging-in-Publication Data.

Meeus, Jean.
Astronomical algorithms / Jean Meeus. -- 2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-943396-61-1
1. Astronomy - Data processing. 2.Astronomy - Problems, exercises, etc. 3. Algorithms. I. Title.
QB51.3.E43M42 1998 98-55091
520'.285--dc21 CIP

98 99 00 01 02 03 04 05 06 9 8 7 6 5 4 3 2 1

Foreword (to the first edition)

People who write their own computer programs often wonder why the machine gives inaccurate planet positions, an unreal eclipse track, or a faulty Moon phase. Sometimes they insist, bewildered, "and I used double precision, too." Even commercial software is sometimes afflicted with gremlins, which comes as quite a shock to anyone caught up in the mystique and presumed infallibility of computers. Good techniques can help us avoid erroneous results from a flawed program or a simplistic procedure — and that's what this book is all about.

In the field of celestial calculations, Jean Meeus has enjoyed wide acclaim and respect since long before microcomputers and pocket calculators appeared on the market. When he brought out his *Astronomical Formulae for Calculators* in 1979, it was practically the only book of its genre. It quickly became the "source among sources", even for other writers in the field. Many of them have warmly acknowledged their debt (or should have), citing the unparalleled clarity of his instructions and the rigor of his methods.

And now this Belgian astronomer has outdone himself yet again! Virtually every previous handbook on celestial calculations (including his own earlier work) was forced to rely on formulae for the Sun, Moon, and planets that were developed in the last century — or at least before 1920. The past 10 years, however, have seen a stunning revolution in how the world's major observatories produce their almanacs. The Jet Propulsion Laboratory in California and the U.S. Naval Observatory in Washington, D.C., have perfected powerful new machine methods for modeling the motions and interactions of bodies within the solar system. At the same time in Paris, the Bureau des Longitudes has been a beehive of activity aimed at describing these motions analytically, in the form of explicit equations.

Yet until now the fruits of this exciting work have remained mostly out of reach of ordinary people. The details have existed mainly on reels of magnetic tape in a form comprehensible only to the largest brains, human or electronic. But *Astronomical Algorithms* changes all that. With his special knack for computations of all sorts, the author has made the essentials of these modern techniques available to us all.

We also stand at a confusing crossroads for astronomy. In just the last few years the International Astronomical Union has introduced subtle changes in the reference frame used for the coordinates of celestial objects, both within and far beyond our solar system. So sweeping are these revisions that a highly respected work for professional astronomers, the *Explanatory Supplement to the Astronomical Ephemeris*, published in 1961, is now seriously out of date. While the technical journals have seen a flurry of scientific papers on these issues, the book you're holding now is the first to offer succinct and practical methods for coping with the changeover. It will be many years before astronomical data bases and catalogues are fully converted to the new system, and anyone who needs a detailed understanding of what's going on will appreciate this book's many comments about the FK4 and FK5 reference frames, "equinox error", and the distinction between "J" and "B" when placed before an epoch like 2000.0.

Scarcely any formula is presented without a fully worked numerical example — so crucial to the debugging process. The emphasis throughout is on testing, on the proper arrangement of formulae, and on not pushing them beyond the time span over which they are valid. Chapter 2 contains much wisdom of this sort, growing out of the author's long experience with various computers and their languages. He alerts us to other pitfalls throughout the text. Anyone who tries to chart the path of a comet, for instance, soon encounters Kepler's equation. It has so vexed astronomers over the years that literally hundreds of solutions have been proposed; the striking graphs in Chapter 30 give a good idea why.

Whenever I read about interpolation techniques, as in Chapter 3, I'm reminded poignantly of Comet Kohoutek. News of its discovery caused a great stir in the spring of 1973, and then it let observers down with a lackluster performance. But this comet also taught me an important mathematical lesson. After preparing a chart of its motion from a list of ephemeris points, I noticed that it was going to pass very near the Sun and tried several interpolation schemes in hopes of finding out what the exact time and minimum distance would be. Much to my surprise, they all failed to give an answer matching what was perfectly obvious from my chart! Readers of this book can save themselves a similar frustration by paying close attention to the remarks on page 111.

When he's not busy writing or conducting seminars on computing techniques, Meeus likes to seize hold of an astronomical problem with great zeal, especially if he senses it is a calculation that has never been done before. Once I asked him about the dates in the past and future when the Moon reaches its most extreme near and far distances from the Earth. Within weeks he had created a table much like that given in Table 50.C of this book. He later confided that this calculation had taken 470 hours on his HP-85 computer, consuming 12 kilowatt-hours of electricity.

On another occasion I heard about a program that was much too large for the mainframe computer he was using at the time. So he devised a scheme to avoid storing the vast number of coefficients in the computer's limited memory; his Fortran program simply read and rewound the same magnetic tape 915 times in the course of generating the hour-by-hour lunar ephemeris he sought. No problem, except that the computer-room operators began to take notice, getting mildly perturbed!

Astronomical calculations have a variety of uses, some scarcely foreseen by the person making them. As long ago as 1962, for example, Meeus published an article in the British Astronomical Association *Journal* about a rare and remarkable forthcoming event. If any observers happened to be on Mars on 1984 May 11, he explained, they should be able to see the silhouette of Earth pass directly across the face of the Sun. Among his readers was the science-fiction writer Arthur C. Clarke, who later incorporated the calculations in a short story, *Transit of Earth*. The piece tells of an astronaut, stranded on the red planet, who barely manages to witness this event before his oxygen supply runs out.

Many of the topics in this book are targeted at serious observers of the sky. Thus, Chapter 53 can help in predicting the illumination at a specific spot on the Moon, for any date and time. Observers often want to know the exact moments when sunlight will just glance across a particular crater, sinuous rille, or gently sloping lunar dome, because oblique lighting is ideal for telescopic scrutiny, making subtle reliefs stand out better than in most of NASA's closeup spacecraft photographs. This chapter can also help us find when the Moon will undergo extreme librations, turning craters near the limb our way.

Chapter 44 holds a special treat for students of Jupiter. First there is a simple method for locating the four famous satellites, quite adequate for identifying them in your own telescope or on historical drawings back to the time of Galileo. Then comes a second set of formulae of the utmost accuracy. Here the computer hobbyist can have a field day, creating observing schedules not only for ordinary satellite eclipses and transits but also for the mutual events between one satellite and another. Astronomy journals have been lax in forecasting these dramatic events, so that many of them have gone unobserved except by accident. For handling the Jovian moons, the routines presented in this book rival or exceed in accuracy those used by the great national almanac offices.

Other unusual topics are offered, like the method in Chapter 52 for computing the dates when the Moon's declination becomes extreme. This is no frivolous calculation, for the very issue came up in recent findings about a century-old murder trial involving the Illinois lawyer and soon-to-be U.S. President Abraham Lincoln. Historians had long tried to reconcile conflicting testimony about the Moon and its role in allowing a witness to see the details of the murder. Some suggested that Lincoln, as lawyer for the defense, may have tampered with an almanac. Not until 1990 was this curious situation explained, and Lincoln's integrity upheld, when Donald W. Olson and Russell Doescher noticed something quite unusual about the Moon on the night in question: 1857 August 29. As any user of this book can confirm, the Moon had a far southerly declination that night, nearly the most extreme value possible in its 18.6-year cycle, and this circumstance made the time of moonset appear quite at odds with its phase. Here is a beautiful instance of astronomers stepping in, bringing their special knowledge and calculations to bear on a longstanding puzzle for historians.

We now live in a thrilling time for practitioners of the number-crunching art. The four-function pocket calculators that were so costly 20 years ago are now incorporated as a gimmick on certain wristwatches. The memory capacity of the 1K RAM board in the pioneering MITS Altair microcomputer is exceeded 500-fold by a single chip in some of today's laptop and notebook computers. Who knows what other marvels lie just ahead? By presenting these astronomical algorithms in standard mathematical notation, rather than in the form of program listings, the author has made them accessible to users of a wide variety of machines and computer languages — including those not yet invented.

Roger W. Sinnott
Sky & Telescope magazine

Introduction

When, in 1978, I wrote the first (Belgian) edition of my *Astronomical Formulae for Calculators*, the industry of microcomputers was just starting its worldwide expansion. Because these "personal computers" were not yet within reach of everybody, the aforesaid book was written mainly for the users of *pocket* calculating machines and therefore calculation methods requiring a large amount of computer memory, or many steps in a program, were avoided as far as possible, or kept to a minimum.

The present work is a greatly revised version of the former one. It is, in fact, a completely new book. The subjects have been expanded and the content has been improved. Changes were needed to take into account new resolutions of the International Astronomical Union, particularly the adoption of the new standard epoch J2000.0, while moreover I profited by the new planetary and lunar theories constructed at the Bureau des Longitudes, Paris.

As Gerard Bodifée wrote in the Preface of my previous work:

> Anyone who endeavours to make astronomical calculations has to be very familiar with the essential astronomical conceptions and rules and he must have sufficient knowledge of elementary mathematical techniques. As a matter of fact he must have a perfect command of his calculating machine, knowing all possibilities it offers the competent user. However, all these necessities don't suffice. Creating useful, successful and beautiful programs requires much practice. Experience is the mother of all science. This general truth is certainly valid for the art of programming. Only by experience and practice can one learn the innumerable tricks and dodges that are so useful and often essential in a good program.

Astronomical Algorithms intends to be a guide for the (professional or amateur) astronomer who wants to do calculations. An algorithm (from the Arabic mathematician *Al-Khārezmi*) is a set of rules for getting something done; for us it is a mathematical procedure, a sequence of reasonings and operations which provides the solution to a given problem.

This book is not a general textbook on astronomy. The reader will find no theoretical derivations. Some definitions are kept to a minimum. Nor is this a textbook on mathematics or a manual for microcomputers. The reader is assumed to be able to use his machine properly.

Except in a few rare cases, no programs are given in this book. The reasons are clear. A program is useful only for one computer language. Even if we consider BASIC only, there are so many versions of this language that a given program cannot be used as such by everybody without making the necessary changes. Every calculator thus must learn to create his own programs. There is the added circumstance that the precise contents of a program usually depend on the specific goals of the computation, that are impossible to anticipate by anybody else.

The few programs we give are in standard BASIC. They can easily be converted into FORTRAN or any other programming language.

Of course, in the formulae we still use the classical mathematical symbols and notations, not the symbolism used in program languages. For example, we write $\sqrt{a}$ instead of SQR(A), or $a(1-e)$ instead of A * (1 − E), or $\cos^2 x$ instead of COS(X)^2 or cos(X) * * 2.

The writing of a program to solve some astronomical problem will require a study of more than one chapter of this book. For instance, in order to create a program for the calculation of the altitude of the Sun for a given time on a given date at a given place, one must first convert the date and time to Julian Day (Chapter 7), then calculate the Sun's longitude for that instant (Chapter 25), its right ascension and declination (Chapter 13), the sidereal time (Chapter 12) and finally the required altitude of the Sun (Chapter 13).

This book is restricted to the "classical", mathematical astronomy, although a few astronomy oriented mathematical techniques are dealt with, such as interpolation, fitting curves, and sorting data. But astrophysics is not considered at all. Moreover, it is clear that not all topics of mathematical astronomy could have been covered in this book. So nothing is said about orbit determination, occultations of stars by the Moon, meteor astronomy, or eclipsing binaries. For solar eclipses, the interested reader will find Besselian elements and many useful formulae in *Elements of Solar Eclipses 1951 to 2200* by the undersigned (1989). Elements and formulae about transits of Mercury and Venus across the Sun's disk are provided in my *Transits* (1989). These two books are published by Willmann-Bell, Inc.

The author wishes to express his gratitude to Dr. S. De Meis (Milan, Italy), to A. Dill (Germany), and to E. Goffin and C. Steyaert (Belgium), for their valuable advice and assistance.

Jean Meeus

Note to the second edition

In this second edition several misprints and errors have been corrected. The principal change in the new edition is the addition of some material, such as expressions for the times of the stations of the planets (Chapter 36), a list of constants (Appendix I), expressions for the heliocentric coordinates of the giant planets from 1998 to 2025 (Appendix IV), and new chapters about the Jewish and Moslem Calendars, and the satellites of Saturn.

J.M.

Contents

Some Symbols and Abbreviations

a	Semimajor axis (of an orbit)
e	Eccentricity (of an orbit)
h	Altitude above the horizon
i	Orbital inclination
n	Mean daily motion
q	Perihelion distance, in AU
r	Radius vector, or distance of a body to the Sun, in AU
v	True anomaly
A	Azimuth
H	Hour angle
M	Mean anomaly
R	Distance from Earth to Sun, in AU
T	Time in Julian centuries (36525 days) from J2000.0
α	Right ascension
δ	Declination
ε	Obliquity of the ecliptic (ε_0 is used for the mean obliquity)
θ	Sidereal time (θ_0 is the sidereal time at Greenwich)
π	Parallax
π	Longitude of perihelion
τ	Time in Julian millennia (365250 days) from J2000.0
φ	Geographical latitude
φ'	Geocentric latitude
Δ	Distance to the Earth, in AU
Δ	is used to indicate a correction or a difference, for instance $\Delta\alpha$
ΔT	Difference TD − UT
$\Delta\varepsilon$	Nutation in obliquity
$\Delta\psi$	Nutation in longitude
AU	Astronomical Unit
INT	Integer part of a number
JD	Julian Day
JDE	Julian Ephemeris Day
TD	Dynamical Time
UT	Universal Time

Following an old, general astronomical practice, small superior symbols are placed immediately above the decimal point, not after the last decimal. For instance, $28\overset{\circ}{.}5793$ means 28.5793 degrees. See, for instance, the *Circulars* of the International Astronomical Union, or the great astronomical almanacs.

Moreover, note carefully the difference between hours with decimals, and hours-minutes-seconds. For example, $1\overset{\mathrm{h}}{.}30$ is *not* 1 hour and 30 minutes, but 1.30 hours, that is 1 hour and 30 *hundreths* of an hour, or 1 hour and 18 minutes.

Do not use the symbols $'$ and $''$ for minutes and seconds of time: they are used for minutes and seconds of a *degree* (or arcminutes and arcseconds, respectively). Minutes and seconds of time have the symbols m and s. For example,

$$\text{the angle } 23°26'44'', \quad \text{but the instant } 15^{\mathrm{h}}22^{\mathrm{m}}07^{\mathrm{s}}.$$

Indeed, we have

$$\begin{aligned} 1' &= \text{one minute of } \textit{arc} &&= 1/60\text{th of a } \textit{degree} \\ 1^{\mathrm{m}} &= \text{one minute of } \textit{time} &&= 1/60\text{th of an } \textit{hour} \end{aligned}$$

Do not use the symbol $\pm$ for "approximately". That symbol means "plus *or* minus" (or "plus *and* minus"). For instance, the square root of 25 is ± 5, which means $+5$ or -5. Writing $\pi = \pm 3$ is incorrect, because π is equal to neither $+3$ nor -3. The correct symbol to be used here is $\approx$. For example, $1002 \approx 1000$.

In general, we shall use the "scientific" form for calendar dates, which reads from the largest to the smallest unit of time, for example 1993 November 6. It contrasts with the common "American" form (November 6, 1993) and with the "European" form (6 November 1993). Anyway, it is recommended to spell out the month, because one person's "11/6/93" is another's "6/11/93".

It is recommended to write the year number out in full, not trimmed to the last two digits. For example, the solar eclipse of February 1998, not February 98 nor February '98.

Chapter 1

Hints and Tips

To explain how to calculate or to program on a computer is out of the scope of this book. The reader should, instead, study carefully his instructions manual. However, even writing good programs cannot be learned in the lapse of time of one day. It is an art which can be acquired only progressively. Only by practice can one learn to write better and shorter programs. In this first Chapter, we will give some practical hints and tips, which may be of general interest.

Trigonometric functions of large angles

Large angles frequently appear in astronomical calculations. In Example 25.a we find that on 1992 October 13.0 the mean longitude of the Sun is −2318.19280 degrees. Even larger angles are found for rapidly moving objects such as the Moon and the bright satellites of Jupiter, or the rotations of the planets (see, for instance, the angle W in step 9 of Example 42.a).

It may be necessary to reduce the angles to the interval 0–360 degrees, because some pocket calculators or some program languages give incorrect values for the trigonometric functions of large angles. Try, for instance, to calculate the sine of 36 000 030 degrees. The result must be 0.5 exactly.

Angle modes

The majority of calculating machines do not calculate directly the trigonometric functions of an angle which is given in degrees, minutes and seconds. Before performing the trigonometric functions, the angle should be converted to degrees and *decimals*. Thus, to calculate the cosine of 23°26′49″, first convert this angle to 23.446 944 44 degrees, and *then* use the COS function.

There is the added complication that most programming languages can calculate only in radians, not in degrees. It is an infernal nuisance having to convert degrees to radians all the time, but in most computer languages this has to be done before calculating a trigonometric function of an angle given in degrees. To convert an angle from degrees to radians, multiply it by $\pi/180 = 0.017\,453\,292\,519\,942...$

Right ascensions

Right ascensions are generally expressed in hours, minutes, and seconds of time. To calculate the trigonometric function of a right ascension, it is necessary to convert that value to degrees (and then in radians, if needed). Remember that one hour corresponds to 15 degrees.

Example 1.a — Calculate $\tan \alpha$, where $\alpha = 9^h14^m55^s.8$.

We first convert α to hours and decimals:

$$9^h14^m55^s.8 = 9 + 14/60 + 55.8/3600 = 9.248\,833\,333 \text{ hours.}$$

Then, multiplying by 15, we obtain $\alpha = 138°.73250$.

Multiplying this value by $\pi/180 = 0.017\,453\,292\,5...$ gives α in radians. We then find

$$\tan \alpha = -0.877\,517.$$

The correct quadrant

When the sine, the cosine or the tangent of an angle is known, the angle itself can be obtained by using the "inverse" function arcsine (ASN or ASIN), arccosine (ACS or ACOS), or arctangent (ATN or ATAN). Note that, unfortunately, the functions arcsine and arccosine are absent in many programming languages.

The inverse trigonometric functions (arcsine, arccosine, arctangent) are not single valued. For instance, if $\sin \alpha = 0.5$, then $\alpha = 30°$, $150°$, $390°$, etc. For this reason, the programming languages return inverse trigonometric functions correctly over only half the range of 0 to 360 degrees: arcsine and arctangent give an angle lying between -90 and $+90$ degrees (that is, between $-\pi/2$ and $+\pi/2$ radians), while arccosine gives a value between 0 and $+180$ degrees (between 0 and π radians).

For example, try $\cos 147°$. The answer is -0.8387, which reverts to $147°$ when you take the inverse function. But now try $\cos 213°$. The answer is again -0.8387 which, when you take its arccosine, gives $147°$.

Hence, whenever the inverse function of SIN, COS, or TAN is taken, an ambiguity arises which has to be cleared up by one or other means *when it is necessary*. Each problem must be examined separately.

For instance, formulae (13.4) and (25.7) give the sine of the declination of a celestial body. The function arcsine then will always give this declination in the correct quadrant, because all declinations lie between -90 and $+90$ degrees. So, no special test should be performed here.

This is also the case for the angular separation whose cosine is given by formula (17.1). Indeed, any angular separation is in the range of $0°$ to $+180°$, which matches the range of the inverse cosine function.

But consider the conversion from right ascension (α) and declination (δ) to celestial longitude (λ) and latitude (β) by means of the following formulae

$$\cos\beta\sin\lambda = \sin\delta\sin\varepsilon + \cos\delta\cos\varepsilon\sin\alpha$$
$$\cos\beta\cos\lambda = \cos\delta\cos\alpha$$

Call A and B the second members. Then, dividing the first equation by the second one, we obtain $\tan\lambda = A/B$. Applying the function arctangent to the quotient A/B will yield the angle λ between $-90°$ and $+90°$, with an ambiguity of $\pm 180°$. This ambiguity can be removed with the following test: if $B < 0$, add 180° to the result. However, some computer languages contain the useful "second" arctangent function, ATN2 or ATAN2, which uses the *two* arguments A and B separately and returns the angle in the proper quadrant. For instance, suppose that $A = -0.5712$, $B = -0.9139$; then ATN(A/B) will give the angle 32°, while ATN2(A,B) will yield the correct value $-148°$, or $+212°$.

The input of negative angles

Angles expressed in degrees, minutes, and seconds can be input as three different numbers (in BASIC: INPUT D, M, S). For instance, the angle 21°44′07″ can be entered as the three numbers 21, 44, and 7. Then, in the program the angle H in degrees is calculated by means of the instruction H = D + M/60 + S/3600.

In such a case, care must be taken for negative angles. If the angle is, for example, −13°47′22″, then this means −13° and −47′ and −22″. In this case, the three numbers are D = −13, M = −47, and S = −22. All three numbers have the same sign!

Mislead by the notation −13°47′22″, one can have the tendency to input −13, +47, and +22 instead, and in that case the angle entered would actually be −12°12′38″. It is possible to write the program in such a way that similar errors are corrected automatically:

```
200   INPUT D, M, S
210   IF D < 0 THEN M = -ABS(M) : S = -ABS(S)
220   H = D + M/60 + S/3600
```

In line 210, the minutes and seconds are made negative when the degrees are negative. The two ABS functions make sure that no error is made when M and S are actually entered as negative numbers.

This procedure does not work, however, when the angle is between 0° and −1°. If the angle is, for instance, equal to −0°32′41″, then we have D = −0, which a computer automatically converts to 0, which is not negative, so the machine will conclude that the angle is +0°32′41″ instead. One solution (in BASIC) is to enter the degrees as a "string" instead of a numeric variable, hence by means of INPUT D$ instead of INPUT D. Then one can use the VAL function and test on the first character of the string D$.

Powers of time

Some quantities are calculated by means of a formula containing powers of the time (T, T^2, T^3, ...). It is important to note that such polynomial expressions are valid only for values of T that are not too large. For instance, the formula

$$e = 0.04638122 - 0.000027293T + 0.0000000789T^2 \tag{1.1}$$

gives the eccentricity e of the orbit of Uranus; T is the time measured in Julian centuries (36525 days) from the beginning of the year 2000. It is evident that this formula is valid for only a limited number of centuries before and after A.D. 2000, for instance for T lying between -30 and $+30$. For $|T|$ much larger than 30, the above expression is no longer valid. For $T = -3307.9$ the formula would give $e = 1$, and an incompetent person, thinking that "the computer cannot make errors", would deduce that in the year -328790 the orbit of Uranus was parabolic and hence that this planet originates from outside our solar system — bringing us in the realm of pseudoscience.

In fact, the eccentricity e of a planet's orbit varies rather irregularly in the course of time, though it cannot exceed a well-defined upper limit. But for a time interval of a few millennia the eccentricity can be accurately represented by a polynomial of the second degree such as (1.1).

One should further carefully note the difference between periodic terms (terms in sine and/or cosine), which remain small throughout the centuries, and secular terms (terms in T, T^2, T^3, ...) which increase more and more rapidly with time. A term in T^2, which is very small when T is small, becomes increasingly important for larger values of $|T|$. Thus, for large values of $|T|$ it is meaningless to take into account small periodic terms if terms in T^2, etc., are neglected in the calculation.

Avoiding powers

Suppose that one wants to calculate the value of the polynomial

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4$$

with A, B, C, D, and E constants, and x a variable. Now, one may write the program to calculate this polynomial directly term after term and adding all terms, so that for each given x the machine obtains the value of the polynomial. However, instead of calculating all the powers of x, it appears to be wiser to write the polynomial as follows:

$$y = A + x(B + x(C + x(D + xE)))$$

In this expression all power functions have disappeared and only additions and multiplications are to be performed. This way of expressing a polynomial is called

Horner's method, an approach especially well suited for automatic calculation because powers are avoided.

Also, it may be wise to calculate the square of a number A by means of A * A instead of using the power function. We calculated the squares of the first 200 positive integers on the HP-85 microcomputer. Using the procedure

```
FOR I = 1 TO 200
K = I^2
NEXT I
```

The complete calculation took 10.75 seconds. But when the second line was replaced by K = I * I, then the calculation time was only 0.96 second!

To shorten a program

To make a program as short as possible is not always an art for art's sake, but sometimes a necessity as long as the memory capacities of the calculating machine have their limits.

There exist many tricks to make a program shorter, even for simple calculations. Suppose that one wants to calculate the sum S of many terms:

$$\begin{aligned} S = {} & 0.0003233 \sin (2.6782 + 15.54204T) \\ & + 0.0000984 \sin (2.6351 + 79.62980T) \\ & + 0.0000721 \sin (1.5905 + 77.55226T) \\ & + 0.0000198 \sin (3.2588 + 21.32993T) \\ & + \dots\dots \end{aligned}$$

First, because the coefficients of all sines are small numbers, one can avoid typing in all those decimals by taking as unit the last decimal (10^{-7} in this case). So, instead of 0.0003233, etc., we use 3233, etc. Then, *after* the sum of the terms has been calculated, we divide the result by 10^7.

Secondly, it would be unwise to write all those terms explicitly in the program. Instead, we could make use of a so-called *loop*. Each of the above terms is of the form $A \sin (B + CT)$, so we put all values A, B, C as DATA in the program. Suppose there are 50 terms. Then the program will look like this:

```
100  S = 0
110  RESTORE 170
120  FOR J = 1 TO 50
130  READ A, B, C
140  S = S + A * SIN(B + C * T)
150  NEXT J
160  S = S/10000000
170  DATA 3233, 2.6782, 15.54204, 984, etc....
```

Safety tests

Include a safety test in case an "impossible" situation might occur, for example in order to stop the calculation when, after a specified number of iterations, the required accuracy has not been reached.

Or consider the case of the occultation of a star by the Moon. In a program for local circumstances, the times of disappearance and of reappearance of the star are calculated. It may happen, however, that the star is not occulted as seen from the given place; in such a case, the times of ingress and egress do not exist, and trying to calculate them would correspond to calculating the square root of a negative number. To avoid this problem, the program should be written in such a way that first of all the value of the star's least distance to the center of the lunar disk (as seen from the given place) is calculated; if, and only if, this distance is smaller than the radius of the Moon's disk, can the times of ingress and egress be calculated.

Debugging

After a program has been written, it must be checked for errors, which are called *bugs*. The process of locating the bugs and correcting them is known as *debugging*. Several types of errors can occur when programming in any language:

a. *syntax errors* violate the rules of the language, such as spelling, a forgotten parenthesis, or other conventions specific to each language. For instance, in BASIC,

A = SIM(B)	should be	A = SIN(B)
P = SQR(ABS(A + B)	should be	P = SQR(ABS(A + B))

b. *semantic errors*, such as a forgotten line. For instance, GOTO 800 when no line labelled 800 exists in the program.

c. *run-time errors*, which occur during the execution of a program. For example:

A = SQR(B). The variable B is calculated during execution of the program, but its value happens to be negative;

ON X GOTO 1000, 2000, 3000, but X is larger than 3.

d. other programmer's errors. The following ones happen frequently:

- Typing the letter O ("oh") instead of the digit zero (0 or Ø), or vice versa, or typing the digit 1 instead of the letter I.
- The name of a variable is used twice in the program (with different meanings).
- A variable has not been defined, and therefore the program assumes its value is zero.
- Error in copying down a numerical constant (such as 127.3 instead of 127.03), or 15 instead of .15), typing an * instead of a +, etc.

- Incorrect units are used. For instance, an angle is expressed in degrees instead of radians, or a right ascension expressed in hours has not been converted to degrees or radians.
- The angle is in the wrong quadrant. See "The correct quadrant" on page 8.
- The natural logarithm of a number has been used instead of its logarithm to the base 10 — see Chapter 56.
- Rounding errors. For example, the cosine of an angle d has been calculated, from which one wants to deduce that angle. This does not work well when the angle is very small. Indeed, if d is very small, its cosine is almost equal to 1 and varies quite slowly as a function of d. In that case, the value of d is ill-defined and cannot be calculated accurately.

  For instance, $\cos 15'' = 0.999\,999\,997$ but $\cos 0''$ is 1 exactly. If one expects that the angle d can be very small, then its value should be calculated by means of another method. See, for instance, Chapter 17.
- Single precision is used instead of double precision. In QuickBASIC, even if the variable G has been declared to be of the double-precision type, the statement G = .1 gives a result of lower accuracy, namely 0.100 000 014 901 16. One should write G = .1# here.
- An iteration procedure which does not guarantee convergence in some cases. See Chapters 5 (Iteration) and 30 (Equation of Kepler).
- An incorrect method of calculation has been used. For example, to interchange two numbers X and Y, an extra variable A is needed (*):

Incorrect procedure	*Correct procedure*
Y = X	A = Y
X = Y	Y = X
	X = A

In QuickBASIC, GWBASIC, and some other BASIC versions, there exists the SWAP function: SWAP(X, Y) interchanges the numbers X and Y.

(*) This is not quite exact. Theoretically, it *is* possible to interchange two numbers without using a third, auxiliary variable, as follows:

X = X + Y
Y = X − Y
X = X − Y

But, of course, this is rather a curiosity than a useful method, because the execution of these operations requires extra computer time, and because rounding errors can occur.

Checking the results

Of course, a program should not only be "grammatically" correct: it must give correct results. *Test* your program using a known solution. If, for instance, you wrote a program for the calculation of planetary positions or for the times of lunar phases, compare your results with the values given in an astronomical almanac.

Test your program for some "special" cases. For instance, are the results still correct for a negative value of the declination? Or for a declination lying between $0°$ and $-1°$? Or if the observer's latitude is exactly zero? Or for negative values of the time T?

Chapter 2

About Accuracy

The following topics will be considered in this Chapter: the accuracy needed for a particular problem, the accuracy with which a given programming language works, and finally the accuracy of the published results.

The accuracy needed for a given problem

The accuracy needed in a calculation depends on its aim. For example, if one wants to calculate the position of a planet with the goal of obtaining the times of rising and setting for a given place, an accuracy of 0.001 or even 0.01 degree will be sufficient. The reason is evident: the apparent diurnal motion of the celestial sphere corresponds to a rotation over one degree during a time interval of four minutes, and so an error of 0.01 degree in the object's position will result in an error of only 0.04 minute (approximately) in its time of rising or setting. Taking hundreds of periodic terms into account in order to obtain the planet's position to an accuracy of $0''.01$ would just be a waste of effort and of computer time for *this* problem.

But if the position of the planet is needed to calculate the occultation of a star by that planet, then an accuracy of better than $1''$ will be necessary by reason of the small size of the planet's disk.

A program written for one aim may not be suitable for another application. Suppose that, for the calculation of the position of a star, a program uses the low-accuracy method for the precession (see Chapter 21). While the results will be good enough for the observer who wants to find celestial objects with a telescope on a parallactic mounting, that program will be completely worthless when *accurate* results are required, for instance in occultation work, or for the calculation of close conjunctions.

If a given accuracy is required, one has to use an algorithm that really provides this precision. John Mosley [1] mentions a commercially available program which calculates planetary positions; but because perturbations are not taken into account, the positions of Saturn, Uranus, and Neptune can be up to 1 degree off, even though displayed to the nearest arcsecond!

To obtain a better accuracy it is often necessary to use another method of calculation, not just to keep more decimals in the result of an approximate calculation. For example, if one has to know the position of Mars with an accuracy of 0.1 degree, it suffices to use an unperturbed elliptical orbit (Keplerian motion). But if the position of Mars is to be known with a precision of 10″ or better, perturbations due to the other planets have to be calculated and the program will be a much longer one.

The programmer, who knows his formulae and the desired accuracy in a given problem, must himself consider which terms, if any, may be omitted in order to keep the program handsome and as short as possible. For instance, the mean geometric longitude of the Sun, referred to the mean equinox of the date, is given by

$$L = 280°27'59''.245 + 129\,602\,771''.380\,T + 1''.0915\,T^2$$

where T is the time in Julian centuries of 36525 ephemeris days from the epoch 2000 January 1.5 TD. In this expression, the last term (secular acceleration of the Sun) is smaller than 1″ if $|T| < 0.95$, that is, between the years 1905 and 2095. If an accuracy of 1″ is sufficient, the term in T^2 may thus be dropped for any instant in that period. But for the year +100 we have $T = -19$, so that the last term becomes 394″, which is larger than 0.1 degree.

The computer's accuracy

This is a much more complex problem. The program language should work with a sufficient number of significant digits. Note that this is not the same as the number of decimals! For instance, the number 0.0000183 has seven decimals, but only three significant digits. The significant digits of a number are those digits which are left over when the leading and trailing zeros are suppressed.

On a machine rounding operations to 6 significant figures, the result of 1 000 000 + 2 will just be 1 000 000.

There can be dangerous situations, for instance when the difference is made of two *nearly-equal* numbers. Suppose that the following subtraction is performed:

$$6.92736 - 6.92735 = 0.00001.$$

Each number is given to six figures, but subtracting them gives a number with just *one* significant figure! Moreover, the two given numbers perhaps have already been rounded. If such is the case, then the situation can even be worse. Suppose that the two numbers are actually 6.927 3649 and 6.927 3451. Then the correct result of the subtraction is 0.000 0198, which is almost twice the previous result!

Six or eight significant digits, as was the general rule for the early microcomputers, or is nowadays often the case in "single precision", are generally not sufficient for mathematical astronomy.

For many applications, it is necessary that the machine calculates with a larger number of significant digits than it is required in the final result. Let us consider, for example, the following formula giving the mean longitude L' of the Moon for any given instant, in degrees (Chapter 47):

$$L' = 218.3164477 + 481\,267.881\,234\,21\,T - 0.001\,5786\,T^2 + 0.000\,0019\,T^3$$

where T is the time measured in Julian centuries of 36525 days elapsed since the standard epoch 2000 January 1.5 TD (JDE 2451545.0). Suppose now that we wish to obtain the Moon's mean longitude to an accuracy of 0.001 degree. Because longitudes are restricted to the interval 0–360 degrees, one might think that a language calculating with only six significant digits internally will be just sufficient for our purpose (3 digits before, and 3 digits after the decimal point). This is not the case in the present problem, however, because L' can reach large values before it is reduced to less than 360 degrees.

For instance, let us calculate L' for $T = 0.4$ which corresponds to 2040 January 1 at 12^h TD. We find $L' = 192\,725^\circ.469$, which reduced to $125^\circ.469$, the correct answer. But if the machine works with only six significant digits, it will not find $L' = 192\,725^\circ.469$, but rather $192\,725^\circ$ (six digits!), which will reduce to 125°, so in this case the final result is only to the nearest degree, and the error is 0.469 degree or 28′; and this happens for only 40 years after the starting epoch. Under such circumstances it is just impossible to calculate eclipses or occultations.

To find out with which internal accuracy a programming language works, the following short program (in BASIC) can be used.

```
10  X = 1
20  J = 0
30  X = X * 2
40  IF X + 1 < > X THEN 60
50  GOTO 80
60  J = J + 1
70  GOTO 30
80  PRINT J, J * 0.30103
90  END
```

Here, J is the number of significant bits in the mantissa of a floating number, while $0.30103J$ is the number of significant digits in a *decimal* number. The constant 0.30103 is $\log_{10} 2$. For instance, the HP-85 computer gives $J = 39$, whence 11.7 digits. With the HP-UX Technical Basic 5.0, working on the HP-Integral microcomputer, we find $J = 52$, whence 15.6 internal digits. The QuickBASIC 4.5 gives $J = 63$, whence 19.0 digits.

However, this accuracy refers only to simple arithmetics, not to the trigonometric functions. Although the GWBASIC has $J = 55$, that is 16.6 internal digits, it gives the sines with only 7 correct decimals; the last nine figures are all wrong!

One rapid way to check the accuracy of trigonometric functions is PRINT 4 * ATN(1). If the computer works in radians, this must give the famous number $\pi = 3.14\,15\,92\,65\,35\,89\,79\ldots$ Or one may calculate the sine of an angle whose value is accurately known, for instance SIN(0.61 rad) = 0.572 867 460 100 48...

Rounding is inevitable in a computer. Consider for instance the value 1/3 = 0.33333333... Because the machine cannot handle an infinite number of decimals, such a number must necessarily be truncated somewhere.

Rounding errors can *accumulate* from one calculation to the next. In most cases this is of no important because the errors almost cancel each other, but in some arithmetical applications the accumulated error can increase beyond any limit. Although this topic is outside of the scope of this book, we shall mention two cases.

Consider the following program.

```
10  X = 1/3
20  FOR J = 1 TO 30
30  X = (9 * X + 1) * X - 1
40  PRINT J, X
50  NEXT J
60  END
```

The operation on line 30 actually replaces X by itself. Yet on most computers the results diverge. The above-mentioned HP-UX Technical Basic yields

0.333 333 333 333 308	after 4 steps
0.333 326 162 117 054	after 14 steps
0.215 899 338 763 055	after 19 steps
286.423...	after 24 steps

and a value of the order of 10^{217} after 30 steps!

The difference in accuracy between microcomputers or even hand-held calculators can be demonstrated by a simple test [2]: repeatedly squaring the number 1.0000001. After 27 times, the result to ten significant figures must be 674 530.4707. The results for some machines or programming languages are as follows:

674 494.06	on the HP-67 calculator
674 514.87	on the HP-85 and on the HP-48s calculator
674 520.61	on the TI-58 calculator
674 530.4755	on the HP-Integral (HP-UX Technical Basic)
674 530.4755	in QuickBASIC 4.5

But that is still not the end of the story. There are two basically different ways for the internal representation of numerical information into a computer. Some machines, such as the older HP-85, use the BCD (Binary Coded Decimal) scheme for representing numbers internally, but in most other cases the binary representation is used.

BCD is a scheme where the actual value of *each digit* of a number is stored individually. This allows numbers to be represented exactly, to the specified digits of precision of the given machine or programming language. Binary, on the other hand, represents all numbers as some combination of powers of 2. In binary, fractions are also represented as being powers of 2, so it is impossible to represent numbers which are not exact combinations of negative powers of 2 in a binary system. For instance, 1/10 is not rationally expressed as combinations of negative powers of 2, because $1/10 = 1/16 + 1/32 + 1/128\ldots$.

Binary arithmetic functions are usually faster in their execution than BCD counterparts, but the inconvenience is that some numbers, even with a small number of decimals, are not represented exactly.

As a consequence, the result of an arithmetic operation may be incorrect, even when numbers with only a few decimals are involved. Suppose that $X = 4.34$. Then the correct result of the operation H = INT(100 * (X − INT(X))) is 34. However, many computer languages give H = 33 here. The reason is that in this case the value of X is represented internally as 4.3399999998, or something like that.

Another surprising example is

$$2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2 - 3$$

On many computers, the result is *not* zero! On the HP-Integral, using the HP-UX Technical Basic 5.0, the result is 8.88×10^{-16}. But on the same machine

$$0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 2 - 3$$

does give zero, so the order in which the operations are performed can be of importance here!

Surprisingly, $2 + (5 * 0.2) - 3$ gives exactly zero on the HP-Integral, and so does the following:

```
A = 0.2 + 0.2 + 0.2 + 0.2 + 0.2
B = 2 + A
C = B - 3
PRINT C
```

Consider the following program:

```
10  FOR I = 0 TO 100 STEP 0.1
20  U = I
30  NEXT I
40  PRINT U
50  END
```

Here, I and U take the successive values from 0 to 100 with steps of 0.1, and the last value of U must be exactly 100. The HP-85 does give 100 indeed, but QuickBASIC 4.5 gives 99.999 999 999 9986, which can have a disastrous consequence in some applications. The error is due to the fact that the step value of 0.1 is translated into binary as 0.0999999.... The difference with 0.1 is very small, but because there are 1000 steps, the final error is 1000 times as large as that small difference. In this case, one remedy may consist in taking an integer value for the step:

```
10  FOR J = 0 TO 1000
20  I = J/10
30  U = I
40  NEXT J
50  PRINT U
60  END
```

We may find other surprises with A = 3 * (1/3), PRINT INT(A), whose result is correctly 1 in some programming languages, but zero in others. Or try, for instance, A = 0.1, PRINT INT(1000 * A).

Another interesting test is

```
INPUT A
B = A/10
C = 10 * B
PRINT A - C
```

The result must be zero. But for some numbers A the answer can be different.

One easy way to find out if a computer language works in BCD or not, consists of looking at the largest possible integer value, that is, a number defined as an INTEGER. If this is a "nice, round" number, this indicates that the machine works in BCD. For example, on the HP-85 that largest integer is 99 999 (or $10^5 - 1$). But if the largest possible integer is a "strange" number (in fact, a power of 2 minus one), this means that the computer does not work in BCD. On the old TRS-80, that largest integer is 32767 (or $2^{15} - 1$), while for QuickBASIC 4.5 it is 2 147 483 647 (or $2^{31} - 1$).

Rounding by inexact arithmetics can yield other surprising results. In most programming languages, the result of SQR(25) − 5 is *not* zero! This can be a problem when testing on the result. Is 25 a perfect square? One might think the answer is no, since the computer tells us that SQR(25) − INT(SQR(25)) is not zero!

Important! If you are comparing INTEGER numbers, no special precautions are necessary. However, if you are comparing so-called REAL values, especially those which are the results of calculations and functions, it is possible to run into problems. The equality test may fail due to rounding or other errors caused by the inherent limitations of machines. A repeating decimal or irrational number cannot be represented exactly in any finite machine.

Rounding the final result

Results should be rounded correctly and meaningfully, where it is necessary. Rounding should be made to the *nearest* value. For instance, 15.88 is to be rounded to 15.9, or to 16, not to 15. However, calendar dates and years are exceptions. For example, March 15.88 denotes an instant belonging to March 15: it means 0.88 day after March 15, 0^h. So, if we read that an event occurs on March 15.88, it takes place on March 15, not on March 16. Similarly, 1987.69 denotes an instant belonging to the year 1987, not 1988; it is 0.69 year after the start of A.D. 1987.

Only meaningful digits should be retained. For example, Müller's formula for calculating the visual magnitude of Jupiter is

$$m = -8.93 + 5 \log r\Delta$$

where r is Jupiter's distance to the Sun, Δ its distance to the Earth (both in astronomical units), and the logarithm is to the base 10. Now, on 1992 May 14, at 0^h TD, we have

$$r = 5.417149$$
$$\Delta = 5.125382$$

whence $m = -1.712514898$. But giving all these decimals, under the pretext that they were given like this by the computer, would be ridiculous and would give the reader a false impression of high accuracy. Since the constant -8.93 in Müller's formula is given to 0.01 magnitude, no higher accuracy can be expected in the result. And, in any case, the meteorological phenomena in the atmosphere of Jupiter are such that the magnitude of that giant planet cannot be predicted with an accuracy better than 0.01 or even 0.1.

As another example, John Mosley [3] mentions a commercially available program giving rising and setting times of heavenly bodies to the nearest 0.1 second, which is impossibly precise.

Some "feeling" and sufficient astronomical knowledge are necessary here. For instance, it would be completely irrelevant to give the illuminated fraction of the Moon's disk accurate to 0.000000001.

The rounding should be performed *after* the whole calculation has been made, not before the start or before the input of the data into the computer.

Example: Calculate 1.4 + 1.4 to the nearest integer. If we first round the given numbers, we obtain 1 + 1 = 2. In fact, 1.4 + 1.4 = 2.8, which rounds to 3.

Here is another example. At its opposition date, 1996 July 18, the declination of Neptune was $\delta = -20°24'$. What was the planet's altitude h_m at the transit through the southern meridian at Sonneberg Observatory, Germany, to the nearest degree? The Observatory's latitude is $\varphi = +50°23'$. The formula to be used is

$$h_m = 90° - \varphi + \delta$$

The answer is $h_m = 90° - 50°23' - 20°24' = 19°13'$, whence 19°. Rounding

φ and δ to the nearest degree *before* the calculation would yield the incorrect result $90° - 50° - 20° = 20°$.

A similar error occurs when distances, already rounded to the nearest mile, are converted to kilometers. In this case the value of 17 km, for instance, will never be reached, because

10 miles will give 16.09 km, which is rounded to 16 km,
11 miles will give 17.70 km, which is rounded to 18 km.

Right ascensions and declinations. — Since 24 hours correspond to 360 degrees, one hour corresponds to 15°, one minute of *time* corresponds to 15 minutes of *arc*, and one second of time to 15 seconds of arc: during a time interval of one second the Earth rotates over an arc of 15″.

For this reason, if the declination of a celestial body is given, for instance, to 1″, then its right ascension should be given to the nearest *tenth* of a second of time, since otherwise the declination would be given with a much greater accuracy than the right ascension. The following table gives the approximate correspondence between the accuracies in right ascension (α) and in declination (δ). For example, if δ is given with an accuracy of 1′, then α must be given to the nearest 0.1 minute of time. As examples, the position of Nova Cygni 1975 with different accuracies is given.

in α	in δ	Example (Nova Cygni 1975)	
1^m	$0\overset{\circ}{.}1$	$\alpha = 21^h10^m$	$\delta = +47\overset{\circ}{.}9$
$0\overset{m}{.}1$	$1'$	$21^h09\overset{m}{.}9$	$+47°57'$
1^s	$0'.1$	$21^h09^m53^s$	$+47°56'.7$
$0\overset{s}{.}1$	$1''$	$21^h09^m52\overset{s}{.}8$	$+47°56'41''$
$0\overset{s}{.}01$	$0''.1$	$21^h09^m52\overset{s}{.}83$	$+47°56'41''.2$

As a final remark, let us mention that trailing zeros can be important. For instance, 18.0 is not the same as 18. The former value means that the actual number lies between 17.95 and 18.05, while the second value has been rounded to the nearest integer and can actually be equal to any number between 17.5 and 18.5. For this reason, trailing zeros *must* be given in the result to indicate the accuracy: a star of magnitude 7 is not the same as a star of magnitude 7.00.

REFERENCES

1. John Mosley, *Sky and Telescope*, Vol. 78, p. 300 (September 1989).
2. F. Gruenberger, "Computer Recreation", *Scientific American*, Vol. 250, p. 10 (April 1984).
3. John Mosley, *Sky and Telescope*, Vol. 81, p. 201 (February 1991).

Chapter 3

Interpolation

The astronomical almanacs or other publications contain numerical tables giving some quantities y for *equidistant* values of an argument x. For example, y is the right ascension of the Sun, and the values x are the different days of the year at 0^h.

Interpolation is the process of finding values for instants, quantities, etc., intermediate to those given in a table.

Of course, the "table" should not necessarily be taken from a book, but may have been calculated in a computer program. Suppose that the position of the Sun is to be calculated for many (> 3) instants of the *same* day. Then one may calculate the Sun's position for 0^h, 12^h, and 24^h of that day, and then use these values to perform the interpolation for every given instant. This will require less computer time than calculating the position of the Sun directly for every instant.

In this Chapter we will consider two cases: interpolation from three or from five tabular values. In both cases we will also show how an extremum or a zero of the function can be found. The case of only two tabular values will not be considered here, for in that case the interpolation can but be linear, and this will give no difficulty at all.

Three tabular values

Three tabular values y_1, y_2, y_3 of the function y are given, corresponding to the values x_1, x_2, x_3 of the argument x. Let us form the table of differences

$$\begin{array}{cccc} x_1 & y_1 & & \\ & & a & \\ x_2 & y_2 & & c \\ & & b & \\ x_3 & y_3 & & \end{array} \tag{3.1}$$

where $a = y_2 - y_1$ and $b = y_3 - y_2$ are called the *first differences*. The *second* difference c is equal to $b - a$, that is

$$c = y_1 + y_3 - 2y_2$$

Generally, the differences of the successive orders are gradually smaller in absolute value. Interpolation from three tabular values is permitted when the second differences are almost constant in that part of the table, that is, when the third differences are almost zero. Some good sense and experience are needed here. For example, the Moon's position can be interpolated accurately from three positions given at hourly interval, but not when the interval is one day.

Let us consider, for instance, the distance of Mars to the Earth from 5 to 9 November 1992, at 0^h TD. The values are given in astronomical units, and the differences are in units of the sixth decimal:

1992 November 5	0.898013			
		−6904		
6	0.891109		+21	
		−6883		+2
7	0.884226		+23	
		−6860		+2
8	0.877366		+25	
		−6835		
9	0.870531			

Since the third differences are almost zero, we may interpolate from only three tabular values.

The central value x_2 must be chosen in such a way that it is that value of x that is closest to the value of x for which we want to perform the interpolation. For example, if from the table above we must deduce the value of the function for November 7 at 22^h14^m, then y_2 is the value for November 8.00. In that case we should consider the tabular values for November 7, 8, and 9, namely the table

$$\begin{array}{lll} \text{November } 7 & y_1 = 0.884226 & \\ \phantom{\text{November }} 8 & y_2 = 0.877366 & \\ \phantom{\text{November }} 9 & y_3 = 0.870531 & \end{array} \qquad (3.2)$$

and the differences are

$$a = -0.006860 \qquad c = +0.000025$$
$$b = -0.006835$$

Let n be the *interpolating factor*. That is, if the value y of the function is required for the value x of the argument, we have $n = x - x_2$ in units of the tabular interval. The value n is positive if $x > x_2$, that is for a value "later" than x_2, or from x_2 towards the bottom of the table. If x precedes x_2, then $n < 0$.

If y_2 has been correctly chosen, then n will be between -0.5 and $+0.5$, although the following formulae will also give correct results for all values of n between -1 and $+1$.

The interpolation formula is

$$y = y_2 + \frac{n}{2}(a + b + nc) \qquad (3.3)$$

Example 3.a — From the table (3.2), calculate the distance of Mars to the Earth on 1992 November 8, at 4^h21^m TD.

We have $4^h21^m = 4.35$ hours and, since the tabular interval is 1 day or 24 hours, $n = 4.35/24 = +0.18125$.

Formula (3.3) then gives $y = 0.876\,125$, the required value.

If the tabulated function reaches an *extremum* (that is, a maximum or a minimum value), this extremum can be found as follows. Let us again form the difference table (3.1) for the appropriate part of the ephemeris. The extreme value of the function is

$$y_m = y_2 - \frac{(a+b)^2}{8c} \tag{3.4}$$

and the corresponding value of the argument x is given by

$$n_m = -\frac{a+b}{2c} \tag{3.5}$$

in units of the tabular interval, and again measured from the central value x_2.

Example 3.b — Calculate the time of passage of Mars through the perihelion in May 1992, and the value of its radius vector at that instant.

The following values for the distance Sun–Mars have been calculated at intervals of four days:

1992 May	12.0 TD	1.381 4294 AU
	16.0	1.381 2213
	20.0	1.381 2453

The differences are

$$a = -0.000\,2081 \qquad b = +0.000\,0240 \qquad c = +0.000\,2321$$

from which we deduce

$$y_m = 1.381\,2030 \quad \text{and} \quad n_m = +0.39\,660$$

Hence, the least distance from Mars to the Sun is 1.381 2030 astronomical units. The corresponding time is found by multiplying 4 days (the tabular interval) by +0.39 660. This gives 1.58 640 days, or 1 day and 14 hours later than the central time, that is 1992 May 17, at 14^h TD.

[Of course, if n_m were negative, the extremum would take place *earlier* than the central time.]

The value of the argument x for which the function y becomes zero can be found by again forming the difference table (3.1) for the appropriate part of the ephemeris. The interpolating factor corresponding to a zero of the function is then given by

$$n_0 = \frac{-2y_2}{a + b + cn_0} \tag{3.6}$$

This equation can be solved by first putting $n_0 = 0$ in the second member. Now the formula gives an approximate value for n_0. This value is then used to calculate the right hand side again, which gives a still better value for n_0. This process, called *iteration* (Latin: *iterare* = to repeat), can be continued until the value found for n_0 no longer varies, to the precision of the computer.

Example 3.c — Given the following values for the declination of Mercury,

1973 February	26.0 TD	−0° 28′ 13″.4
	27.0	+0 06 46.3
	28.0	+0 38 23.2

calculate when the planet's declination was zero.

Firstly, we convert the tabulated values into seconds of a degree and then form the differences:

$$\begin{array}{lll} y_1 = -1693.4 & & \\ & a = +2099.7 & \\ y_2 = +\ 406.3 & & c = -202.8 \\ & b = +1896.9 & \\ y_3 = +2303.2 & & \end{array}$$

Formula (3.6) then becomes

$$n_0 = \frac{-812.6}{+3996.6 - 202.8\,n_0}$$

Putting $n_0 = 0$ in the second member, we find $n_0 = -0.20332$. Repeating the calculation, we find successively -0.20125 and -0.20127. Hence, $n_0 = -0.20127$. The tabular interval being one day, Mercury crossed the celestial equator on

1973 February 27.0 − 0.20127 = February 26.79873
= February 26, at 19^h10^m TD.

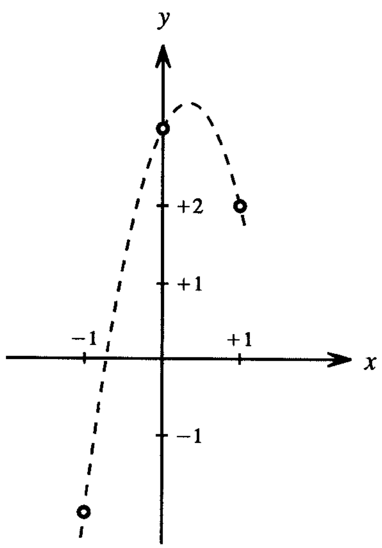
For the calculation of the value of the interpolating factor n_0 for which the function is zero, formula (3.6) is excellent when, as in Example 3.*c*, the function is "almost a straight line" in the interval considered. If, however, the curvature of the function is important, use of the formula may require a large number of iterations; moreover, it can lead to divergence even when starting from an almost correct value for n_0. In this case, a better method for calculating n_0 is as follows: the *correction* to the assumed value of n_0 is

$$\Delta n_0 = -\frac{2y_2 + n_0(a + b + cn_0)}{a + b + 2cn_0} \tag{3.7}$$

The calculation should be repeated, using the new value of n_0, until n_0 no longer varies.

Example 3.d — Consider the following values of a function:

$$\begin{aligned} x_1 &= -1 & y_1 &= -2 \\ x_2 &= 0 & y_2 &= +3 \\ x_3 &= +1 & y_3 &= +2 \end{aligned}$$



These three points actually define the parabola $y = 3 + 2x - 3x^2$, which has a strong curvature between $x = -1$ and $x = +1$ (see the Figure at left).

Starting with $n_0 = 0$, formula (3.6) gives successively

−1.5
−0.461 538...
−0.886 363...
−0.643 902...
−0.763 027...
−0.699 450...

and so on. The correct value of the *sixth* decimal is obtained after not less than 24 iterations. But if we use formula (3.7), again starting with n_0 = zero, we find successively

−1.5
−0.886 363 636 364
−0.732 001 693 959
−0.720 818 540 935
−0.720 759 221 726
−0.720 759 220 056
−0.720 759 220 056

so the 12th decimal is correctly obtained with only six iterations in this case.

Five tabular values

When the third differences may not be neglected, more than three tabular values must be used. Taking five consecutive tabular values, y_1 to y_5, we form, as before, the table of differences

$$
\begin{array}{ccccc}
 & y_1 & & & & \\
 & & A & & & \\
 & y_2 & & E & & \\
 & & B & & H & \\
n \downarrow & y_3 & & F & & K \\
 & & C & & J & \\
 & y_4 & & G & & \\
 & & D & & & \\
 & y_5 & & & &
\end{array}
$$

where $A = y_2 - y_1$, $H = F - E$, etc. If n is the interpolating factor, measured from the central value y_3 in units of the tabular interval, positively towards y_4, the interpolating formula is

$$y = y_3 + \frac{n}{2}(B + C) + \frac{n^2}{2}F + \frac{n(n^2-1)}{12}(H + J) + \frac{n^2(n^2-1)}{24}K$$

which may also be written (3.8)

$$y = y_3 + n\left(\frac{B+C}{2} - \frac{H+J}{12}\right) + n^2\left(\frac{F}{2} - \frac{K}{24}\right) + n^3\left(\frac{H+J}{12}\right) + n^4\left(\frac{K}{24}\right)$$

Example 3.e — Consider the following values of the equatorial horizontal parallax of the Moon:

1992 February 27.0 TD	54′36″.125
27.5	54 24.606
28.0	54 15.486
28.5	54 08.694
29.0	54 04.133

The differences in arcseconds are

$$
\begin{array}{llll}
A = -11.519 & & & \\
 & E = +2.399 & & \\
B = -9.120 & & H = -0.071 & \\
 & F = +2.328 & & K = -0.026 \\
C = -6.792 & & J = -0.097 & \\
 & G = +2.231 & & \\
D = -4.561 & & &
\end{array}
$$

We see that the third differences (H and J) may not be neglected, unless an accuracy of 0".1 is sufficient.

Let us now calculate the Moon's parallax on February 28 at 3^h20^m TD. The tabular interval being 12 hours, we have

$$n = \frac{3^h20^m}{12^h} = \frac{3.333\,333}{12} = +0.277\,7778$$

Formula (3.8) then gives

$$y = 54'15''.486 - 2''.117 = 54'13''.369$$

The interpolating factor n_m corresponding to an extremum of the function can be obtained by solving the equation

$$n_m = \frac{6B + 6C - H - J + 3n_m^2(H + J) + 2n_m^3 K}{K - 12F} \tag{3.9}$$

As before, this may be performed by iteration, firstly putting $n_m = 0$ in the second member. Once n_m is found, the corresponding value of the function can be calculated by means of formula (3.8).

The interpolating factor n_0 corresponding to a zero of the function may be found from

$$n_0 = \frac{-24y_3 + n_0^2(K - 12F) - 2n_0^3(H + J) - n_0^4 K}{2\,(6B + 6C - H - J)} \tag{3.10}$$

where, again, n_0 can be found by iteration, starting by putting $n_0 = 0$ in the second member.

The remark made on pages 26–27 about formula (3.6) holds here too. If the curvature of the function in the considered interval is important, a better method for calculating n_0 is as follows. Calculate

$$M = \frac{K}{24} \qquad N = \frac{H + J}{12} \qquad P = \frac{F}{2} - M \qquad Q = \frac{B + C}{2} - N$$

Then the correction to the assumed value of n_0 is

$$\Delta n_0 = -\frac{Mn_0^4 + Nn_0^3 + Pn_0^2 + Qn_0 + y_3}{4Mn_0^3 + 3Nn_0^2 + 2Pn_0 + Q} \tag{3.11}$$

and, again, the calculation should be repeated with the new value of n_0 until n_0 no longer varies.

Exercise. — From the following values of the heliocentric latitude of Mercury, find the instant when the latitude was zero, by using formula (3.10).

1988 January 25.0 TD	$-1°11'21''.23$
26.0	$-0\ 28\ 12.31$
27.0	$+0\ 16\ 07.02$
28.0	$+1\ 01\ 00.13$
29.0	$+1\ 45\ 46.33$

Answer: Mercury reached the ascending node of its orbit for $n_0 = -0.361413$, that is on 1988 January 26.638587, or January 26 at 15^h20^m TD.

Using only the three central values and formula (3.6), one would find $n_0 = -0.362166$, a difference of 0.000753 day, or 1.1 minute, with respect to the previous result.

Important remarks

1. Interpolation cannot be performed on complex (*) quantities directly. These quantities should be converted, in advance, into a single, suitable unit. For instance, angles expressed in degrees, minutes, and seconds should be converted either to degrees and decimals, or to arcseconds, before they can be used for interpolation.

2. *Interpolating times and right ascensions.* — We draw attention to the fact that times and right ascensions jump to zero when the value of 24 hours is reached. This should be taken into account when interpolation is performed on tabular values. Suppose, for example, that we wish to calculate the right ascension of Mercury for the instant 1992 April 6.2743 TD, using the three following values:

1992 April 5.0 TD	$\alpha = 23^h51^m56^s.04$
6.0	23 56 28.49
7.0	0 01 00.71

Not only is it necessary to convert these values to hours and decimals, but the last value should be written as $24^h01^m00^s.71$, otherwise the machine will consider that, from April 6.0 to 7.0, the value of α *decreases* from 23^h56^m.... to 0^h01^m....

We find a similar situation in some other cases. For instance, here is the longitude of the central meridian of the Sun for a few dates:

(*) By definition, a *complex* number is a number composed of different units, having among them a ratio different from a power of 10. Examples of "complex" quantities are $10^h29^m55^s$; $23°26'44''$; £, shillings, pence; yd, ft, inch; $a + bi$.

1992 June 14.0 UT	37°.96
15.0	24.72
16.0	11.48
17.0	358.25

It is evident that the variation is approximately -13.24 degrees per day. Hence, one should *not* interpolate directly between 11.48 and 358.25. Either the first value should be written as 371°.48, or the second value should be considered as being equal to -1.75 degrees.

3. As much as possible, avoid making an interpolation for $|n| > 0.5$. In any case, the interpolating factor n should be restricted between the limits -1 and $+1$. This same rule applies to the calculation of an extremum (n_m) or a zero (n_0) of the function. Choose the central value of y in such a way that this is the tabular value which is closest to the extremum or to the zero. Of course, the exact value of n_m or n_0 is not known in advance, but an approximate value can be calculated first, after which the choice of the central value (y_3 or y_2) of the function can be changed accordingly.

If the chosen value is too far from the zero or from the extremum, the formulae given in this Chapter for calculating these points will give incorrect or even absurd results. Let us give an example. We know that $\sin x$ reaches a maximum for $x = 90°$. But consider the following sines, with ten decimals:

sin 29°	0.484 809 6202
sin 30°	0.500 000 0000
sin 31°	0.515 038 0749
sin 32°	0.529 919 2642
sin 33°	0.544 639 0350

Using the *three* central values, formula (3.4) gives $y_m = 1.22827$ instead of 1 exactly, and (3.5) yields $n_m = +95.35$, indicating that the maximum occurs for 31° + 95°.35 = 126°.35, instead of 90°.

Using all *five* values, formula (3.9) gives $n_m = +57.30$, whence the maximum taking place at 88°.30, from which the value of 0.99348 is found for that maximum. Although these results are much better than those obtained with only three points, they are still unsatisfactory!

Interpolation to halves

If the values y_1, y_2, y_3, y_4 of the function are given for four equally-spaced abscissae x_1, x_2, x_3, and x_4, then the value of the function for the point exactly half-way between x_2 and x_3 is easily calculated by means of the following formula, which is valid when the fourth differences of the tabulated values are negligible:

$$y = \frac{9\,(y_2 + y_3) - y_1 - y_4}{16} \tag{3.12}$$

Example 3.f — Given the following values for the apparent right ascension of the Moon, calculate the right ascension for 11^h00^m TD.

1994 March 25	8^h TD	$\alpha = 10^h18^m48^s.732$
	10	10 23 22.835
	12	10 27 57.247
	14	10 32 31.983

Converting the minutes and seconds, after 10^h, into seconds, we change the four given data into

$$\begin{aligned} y_1 &= 1128.732 \text{ seconds} \\ y_2 &= 1402.835 \\ y_3 &= 1677.247 \\ y_4 &= 1951.983 \end{aligned}$$

Formula (3.12) then gives $y = 1540.001$ seconds $= 25^m40^s.001$, so that the required right ascension is $\alpha = 10^h25^m40^s.001$.

Interpolation with unequally-spaced abscissae : Lagrange's interpolation formula

When the abscissae (the values of the independent x coordinate) of the given points are not equally spaced, the interpolation formula of Lagrange may be used. (Of course, this formula may also be used when the points are evenly spaced).

This simple formula, developed by the French mathematician J.L. Lagrange (1736–1813), determines a polynomial of degree $n - 1$ matching n given points exactly. If the given points are x_i, y_i ($i = 1$ to n), the formula is, for a given x,

$$y = y_1 L_1 + y_2 L_2 + \ldots\ldots + y_n L_n \tag{3.13}$$

where

$$L_i = \prod \frac{x - x_j}{x_i - x_j} \qquad (j = 1 \text{ to } n,\ j \neq i)$$

The Π means that the product of the fractions should be calculated for all values $j = 1$ to n, except for $j = i$. That is,

$$L_i = \frac{(x - x_1)\,(x - x_2)\,\ldots.\,(x - x_{i-1})\,(x - x_{i+1})\,\ldots.\,(x - x_n)}{(x_i - x_1)\,(x_i - x_2)\,\ldots.\,(x_i - x_{i-1})\,(x_i - x_{i+1})\,\ldots.\,(x_i - x_n)}$$

Important : The values x_i of the given points must all be different!

The following program in BASIC can be used.

```
10  DIM X(50), Y(50)
20  PRINT "NUMBER OF GIVEN POINTS = ";
30  INPUT N
40  IF N < 2 OR N > 50 THEN 20
50  PRINT
60  FOR I = 1 TO N
70  PRINT "X, Y FOR POINT No."; I
80  INPUT X(I), Y(I)
90  IF I = 1 THEN 130
100 FOR J = 1 TO I - 1
110 IF X(I) = X(J) THEN PRINT "THIS VALUE OF X HAS ALREADY
      BEEN USED!" : GOTO 70
120 NEXT J
130 NEXT I
140 PRINT : PRINT "POINT X FOR INTERPOLATION = ";
150 INPUT Z
160 V = 0
170 FOR I = 1 TO N
180 C = 1
190 FOR J = 1 TO N
200 IF J = I THEN 220
210 C = C * (Z - X(J))/(X(I) - X(J))
220 NEXT J
230 V = V + C * Y(I)
240 NEXT I
250 PRINT : PRINT "INTERPOLATED VALUE = "; V
260 PRINT : PRINT "STOP (0) OR INTERPOLATION AGAIN (1) ";
270 INPUT A
280 IF A = 0 THEN END
290 IF A = 1 THEN 140
300 GOTO 260
```

The program first asks how many known values you are going to enter from a table and allows you to input these one at a time. Then it asks you repeatedly for intermediate values of interest, returning the interpolated value for each.

A remarkable feature of Lagrange interpolation is that the values entered initially do not have to be in order, or evenly spaced. Accuracy is usually better with uniform spacing, however.

As an exercise, try the program on the following six given points.

x = *angle in degrees*	y = *sine*
29.43	0.491 359 8528
30.97	0.514 589 1926
27.69	0.464 687 5083
28.11	0.471 165 8342
31.58	0.523 688 5653
33.05	0.545 370 7057

Asking for the sine of 30°, you should obtain 0.5 exactly. It is remarkable that, even for the remote values $x = 0°$ and $x = 90°$, the Lagrange interpolation formula performed with these six data points yields the still rather good values +0.0000482 and +1.00007, respectively, the correct values being 0 and +1 exactly.

The expression (3.13) is a polynomial of degree $n - 1$, and it is the *unique* polynomial of that degree which takes the values $y_1, y_2, \ldots, y_n$ for $x = x_1, x_2, \ldots, x_n$. But Lagrange's formula has the disadvantage that in itself it gives no indication of the number of points required to secure a desired degree of accuracy. However, when we wish to express the interpolating polynomial explicitly as a function of the variable x rather than making an actual interpolation, the use of Lagrange's formula is advantageous.

Example 3.g — Construct the (unique) 3rd-order polynomial passing through the following values:

x :	1	3	4	6
y :	−6	6	9	15

By substituting the given values of x and y into (3.13), we obtain

$$y = (-6)\frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} + (6)\frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}$$
$$+ (9)\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} + (15)\frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}$$

which upon simplification reduces to

$$y = \frac{1}{5}(x^3 - 13x^2 + 69x - 87)$$

Chapter 4

Curve Fitting

In many cases, the result of a large number of observations is a series of points in a graph, each point being defined by an x-value and an y-value. It may be necessary to draw, through the points, the "best" fitting curve.

Several curves can be fitted through a series of points: a straight line, an exponential, a polynomial, a logarithmic curve, and so on.

To avoid individual judgment, it is necessary to agree on a definition of a "best fitting" curve. Consider Figure 1 in which the N data points are given by (X_1, Y_1), (X_2, Y_2), $\ldots$, (X_N, Y_N). The values of X are supposed to be rigorously exact, while the Y-values are measured quantities, hence subject to an error.

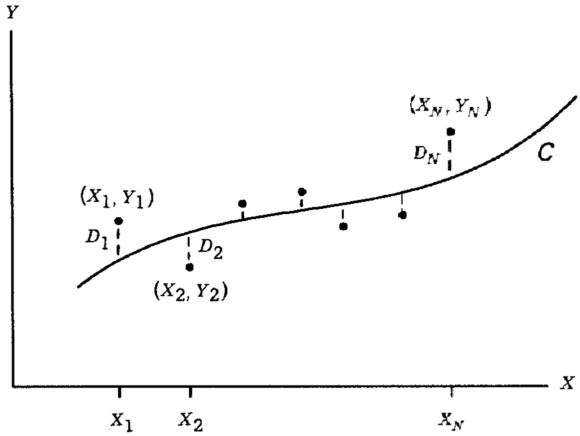


Figure 1

For a given value of X, say X_1, there will be a difference between the value Y_1 and the corresponding value as determined from the curve C. As indicated in the figure, we denote this difference by D_1, which is sometimes referred to as *deviation*, *error*, or *residual* and may be positive, negative, or zero. Similarly, corresponding to the values $X_2, \ldots, X_N$ we obtain the deviations $D_2, \ldots, D_N$.

A measure of the "goodness of fit" of the curve C to the given data is provided by the quantity $D_1^2 + D_2^2 + \ldots. + D_N^2$. If this is small the fit is good; if it is large the fit is bad. We therefore make the following definition: of all curves approximating a given set of data points, the curve having the property that ΣD_i^2 is a minimum, is called a best fitting curve. The Σ means "sum of".

A curve having this property is said to fit the data in the *least square sense* and is called a *least square curve*.

As has been said above, all values of the independent variable X are supposed to be exact. Of course, it is possible to define another least square curve by considering perpendicular distances from each of the data points to the curve instead of vertical distances; however, this is not used too often.

In this Chapter we will consider principally the case where the best fitting curve is a straight line, a problem called *linear regression*.

The name "regression" may seem strange, because in the calculation of the best curve nothing "regresses"! Alt [1] writes:

> Die Benennung Regression wurde von *Galton* (1822–1911) eingeführt, der die Körperlängen von Eltern und Kindern verglich und dabei beobachtete, daß zwar im allgemeinen große Väter große Söhne haben, daß diese Beziehung jedoch nicht immer stimmt, da die Körpergröße der Söhne im Mittel etwas kleiner ist, als die der Väter, umgekehrt aber kleine Eltern im Mittel etwas größere Kinder haben. Diesen 'Rückschlag' in Richtung auf die Durchschnittgröße der Bevölkerung bezeichnete er als Regression.

A better term is *curve fitting*, and in the case of a straight line it is a linear curve fitting.

Linear curve fitting (linear regression)

We wish to calculate the coefficients of the linear equation

$$y = ax + b \tag{4.1}$$

using the least-squares method. The slope a and the y-intercept b can be calculated by means of the formulae

$$\left.\begin{aligned} a &= \frac{N\Sigma xy - \Sigma x\,\Sigma y}{N\Sigma x^2 - (\Sigma x)^2} \\[2ex] b &= \frac{\Sigma y\,\Sigma x^2 - \Sigma x\,\Sigma xy}{N\Sigma x^2 - (\Sigma x)^2} \end{aligned}\right\} \tag{4.2}$$

where N is the number of points. Note that both fractions have the same denominator. The sign Σ indicates the summation. Thus, Σx is the sum of all the x-values, Σy the sum of all y-values, Σx^2 the sum of the squares of all x-values,

Σxy the sum of the products xy of all the couples of values, etc. Note that Σxy is not the same as $\Sigma x \times \Sigma y$ (the sum of the products is not the same as the product of the sums), and that $(\Sigma x)^2$ is not the same as Σx^2 (the square of the sum is not the same as the sum of the squares)!

An interesting astronomical application is to find the relation between the intrinsic brightness of a comet and its distance to the Sun. The apparent magnitude m of a comet can generally be represented by a formula of the form

$$m = g + 5 \log \Delta + \kappa \log r$$

Here, Δ and r are the distances in astronomical units of the comet to the Earth and to the Sun, respectively. The logarithms are to the base 10. The absolute magnitude g and the coefficient κ must be deduced from the observations. This can be performed when the magnitude m has been measured during a sufficiently long period. More precisely, the range of r should be sufficiently large. For each value of m, the values of Δ and r must be deduced from an ephemeris or calculated from orbital elements.

In this case, the unknowns are g and κ. The formula above can be written

$$m - 5 \log \Delta = \kappa \log r + g$$

which is of the form (4.1), when we write $y = m - 5 \log \Delta$, and $x = \log r$. The quantity y may be called the "heliocentric" magnitude, because the effect of the variable distance to the Earth has been removed.

Example 4.a — Table 4.A contains visual magnitude estimates m of the periodic comet Wild 2 (1978*b*), made by John Bortle. The corresponding values of r and Δ have been calculated from orbital elements [2]. The quantities x and y are used to calculate the sums Σx, Σy, Σx^2, and Σxy. We find

$$N = 19 \qquad \Sigma x = 4.2805 \qquad \Sigma x^2 = 1.0031$$
$$\Sigma y = 192.0400 \qquad \Sigma xy = 43.7943$$

whence, by formulae (4.2),

$$a = 13.67 \qquad b = 7.03$$

Consequently, the "best" straight line fitting the observations is

$$y = 13.67\,x + 7.03$$

or $$m - 5 \log \Delta = 13.67 \log r + 7.03$$

Hence, for the periodic comet Wild 2 in 1978, we have

$$m = 7.03 + 5 \log \Delta + 13.67 \log r$$

TABLE 4.A

1978, UT		m	r	Δ	$x = \log r$	$y = m - 5\log\Delta$
Febr.	4.01	11.4	1.987	1.249	0.2982	10.92
	5.00	11.5	1.981	1.252	0.2969	11.01
	9.02	11.5	1.958	1.266	0.2918	10.99
	10.02	11.3	1.952	1.270	0.2905	10.78
	25.03	11.5	1.865	1.335	0.2707	10.87
March	7.07	11.5	1.809	1.382	0.2574	10.80
	14.03	11.5	1.772	1.415	0.2485	10.75
	30.05	11.0	1.693	1.487	0.2287	10.14
April	3.05	11.1	1.674	1.504	0.2238	10.21
	10.06	10.9	1.643	1.532	0.2156	9.97
	26.07	10.7	1.582	1.592	0.1992	9.69
May	1.08	10.6	1.566	1.610	0.1948	9.57
	3.07	10.7	1.560	1.617	0.1931	9.66
	8.07	10.7	1.545	1.634	0.1889	9.63
	26.09	10.8	1.507	1.696	0.1781	9.65
	28.09	10.6	1.504	1.703	0.1772	9.44
	29.09	10.6	1.503	1.707	0.1770	9.44
June	2.10	10.5	1.498	1.721	0.1755	9.32
	6.09	10.4	1.495	1.736	0.1746	9.20

Coefficient of Correlation

A correlation coefficient is a statistical measure of the degree to which two variables are related to each other. In the case of a linear equation, the coefficient of correlation is

$$r = \frac{N\Sigma xy - \Sigma x\,\Sigma y}{\sqrt{N\Sigma x^2 - (\Sigma x)^2}\;\sqrt{N\Sigma y^2 - (\Sigma y)^2}} \tag{4.3}$$

This coefficient is always between $+1$ and -1. A value of $+1$ or -1 would indicate that the two variables are totally correlated; it would denote a perfect linear relationship, all the points representing paired values of x and y falling exactly on the straight line representing this relationship. If $r = +1$, an increase of x corresponds to an increase of y (Figure 2). If $r = -1$, there is again a perfect linear relationship, but y decreases when x increases (see Figure 3).

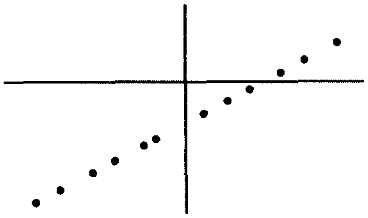


Figure 2

Perfect linear relationship; positive correlation

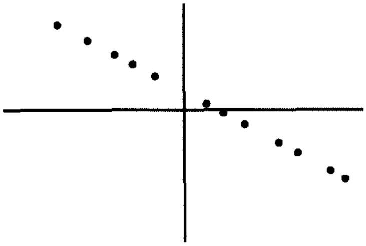


Figure 3

Perfect linear relationship; negative correlation

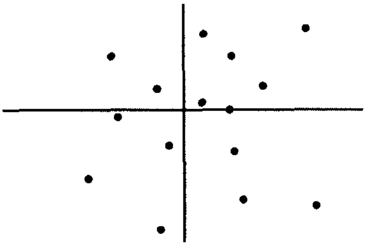


Figure 4

No correlation

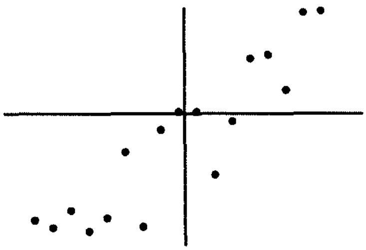


Figure 5

Some correlation

When r is zero, there is no relationship between x and y (Figure 4). In practice, however, when there is no relationship, one may find that r is not exactly zero, due to fortuitous coincidences that generally occur except for an infinite number of points.

When $|r|$ is between 0 and 1, there is a trend between x and y, although there is no strict relationship (Fig. 5). Here, again, *if* there is actually a strict relationship between the two variables, the calculation may give a value of r that is not exactly equal to $+1$ or to -1, by reason of inaccuracies inherent to all measures.

Note that r is a dimensionless quantity: it does not depend on the units employed. The sign of r only tells us whether y is increasing or decreasing when x increases. The important fact is not the sign, but the magnitude of r, because it is this magnitude which indicates how well the linear approximation is.

It must be emphasized that the computed value of r in any case measures the degree of relationship relative to the assumed type of function, namely the linear equation. Thus, if the value of r appears to be nearly zero, it means that there is almost no *linear* correlation between the variables. However, it does not necessarily mean that there is no correlation at all, since there may actually be a high *non-linear* correlation between the variables. As an example, consider the seven points

x	-4	-3	-2	-1	0	$+1$	$+2$
y	-6	-1	$+2$	$+3$	$+2$	-1	-6

Formula (4.3) yields r = zero, although the points lie *exactly* on the parabola $y = 2 - 2x - x^2$ (Fig. 6).

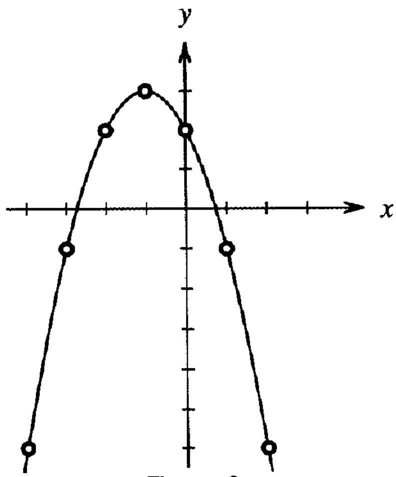


Figure 6

We must be careful not to improperly deduce causation from correlation. A high correlation coefficient (that is, near $+1$ or -1) does not necessarily indicate a direct, physical dependence of the variables. Thus, if we consider a sufficiently large number of administrative territories, one can find a high correlation between the number of beds in the psychiatric hospitals and the number of television receivers of each territory. A high *mathematical* correlation, indeed, but a physical nonsense.

Example 4.b — Table 4.B gives, for each of the twenty-two sunspot maxima which have occurred from 1761 to 1989, the time interval x, in months, since the previous sunspot minimum, and the height y of the maximum (highest smoothed monthly mean). We find

$\Sigma x = 1120$; $\Sigma y = 2578.9$; $\Sigma x^2 = 60608$; $\Sigma y^2 = 340225.91$;

$\Sigma xy = 122337.1$; $N = 22$; and then, by formulae (4.2) and (4.1),

$$y = 244.18 - 2.49x \tag{4.4}$$

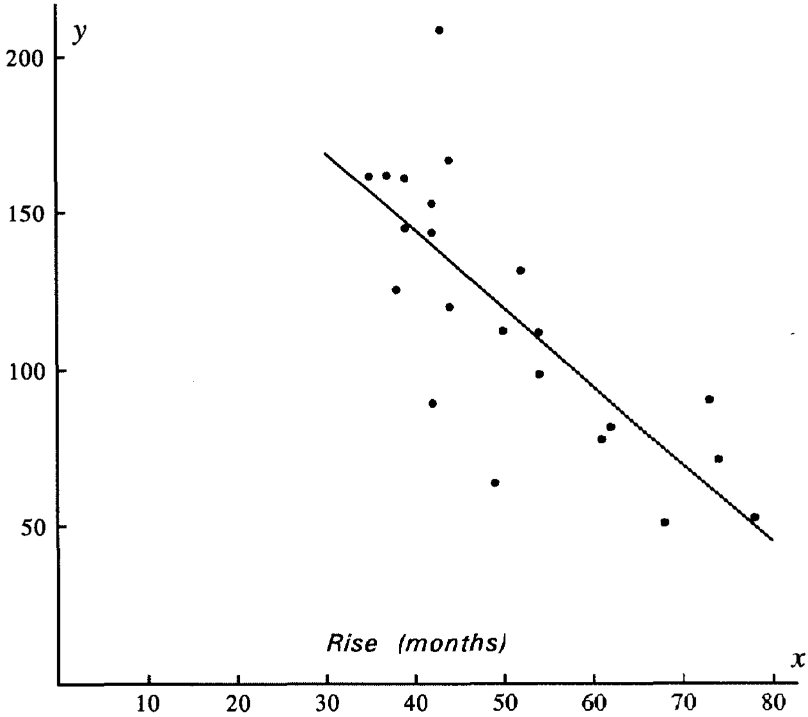


Figure 7

TABLE 4.B

Epoch of maximum	x	y	*Epoch of maximum*	x	y
1761 June	73	90.4	1884 Jan.	61	78.1
1769 Oct.	38	125.3	1893 Aug.	42	89.5
1778 May	35	161.8	1905 Oct.	49	63.9
1787 Nov.	42	143.4	1917 Aug.	50	112.1
1804 Dec.	78	52.5	1928 June	62	82.0
1816 March	68	50.8	1937 May	44	119.8
1829 June	74	71.5	1947 July	39	161.2
1837 Feb.	42	152.8	1957 Nov.	43	208.4
1847 Nov.	52	131.3	1969 Feb.	54	111.6
1860 July	54	98.5	1979 Nov.	44	167.1
1870 July	39	144.8	1989 Oct.	37	162.1

Equation (4.4) represents the best straight line fitting the given 22 points. These points and the line are shown in Figure 7.

From formula (4.3) we find $r = -0.767$. This shows that there exists an evident trend to connection, and the negative sign of r indicates that the correlation between x and y is negative: the *longer* the duration of the rise from a minimum to the next maximum of the sunspot activity, the *lower* this maximum generally is.

Note that here, as in all statistic studies, the sample must be sufficiently large in order to give a meaningful result. A correlation coefficient close to $+1$ or to -1 has no physical meaning if it is based on too small a number of cases. With too few cases the correlation coefficient can accidentally be quite large.

TABLE 4.C

year	*x*	*y*	*year*	*x*	*y*	*year*	*x*	*y*	*year*	*x*	*y*
1901	2.7	700	1925	44.3	1075	1949	134.7	521	1973	38.0	690
1902	5.0	762	1926	63.9	896	1950	83.9	951	1974	34.5	1039
1903	24.4	854	1927	69.0	837	1951	69.4	878	1975	15.5	734
1904	42.0	663	1928	77.8	882	1952	31.5	926	1976	12.6	541
1905	63.5	912	1929	64.9	688	1953	13.9	557	1977	27.5	855
1906	53.8	821	1930	35.7	953	1954	4.4	741	1978	92.5	767
1907	62.0	622	1931	21.2	858	1955	38.0	616	1979	155.4	839
1908	48.5	678	1932	11.1	858	1956	141.7	795	1980	154.6	913
1909	43.9	842	1933	5.7	738	1957	190.2	801	1981	140.5	1016
1910	18.6	990	1934	8.7	707	1958	184.8	834	1982	115.9	800
1911	5.7	741	1935	36.1	916	1959	159.0	560	1983	66.6	689
1912	3.6	941	1936	79.7	763	1960	112.3	962	1984	45.9	931
1913	1.4	801	1937	114.4	900	1961	53.9	903	1985	17.9	758
1914	9.6	877	1938	109.6	711	1962	37.5	862	1986	13.4	946
1915	47.4	910	1939	88.8	928	1963	27.9	713	1987	29.2	908
1916	57.1	1054	1940	67.8	837	1964	10.2	785	1988	100.2	1005
1917	103.9	851	1941	47.5	744	1965	15.1	1073	1989	157.6	639
1918	80.6	848	1942	30.6	841	1966	47.0	1054	1990	142.6	759
1919	63.6	980	1943	16.3	738	1967	93.8	707	1991	145.7	794
1920	37.6	760	1944	9.6	766	1968	105.9	776	1992	94.3	916
1921	26.1	417	1945	33.2	745	1969	105.5	776	1993	54.6	857
1922	14.2	938	1946	92.6	861	1970	104.5	727	1994	29.9	894
1923	5.8	917	1947	151.6	640	1971	66.6	691	1995	17.5	763
1924	16.7	849	1948	136.3	792	1972	68.9	710	1996	8.6	745

As an exercise, show that there is no correlation between the rainfall at the Uccle Observatory, Belgium, and the sunspot activity, using the data of Table 4.C, where

x = yearly mean of the definitive Zürich sunspot numbers,
y = total annual rainfall at Uccle, in millimeters.

Answer: The correlation coefficient is $r = -0.054$, which shows that there is no significant correlation between x and y.

Quadratic curve fitting

Suppose that we wish to draw, through a set of N given points (x, y), the best quadratic function

$$y = ax^2 + bx + c$$

This is a parabola with vertical axis.

Let

$$\begin{aligned} P &= \Sigma x \\ Q &= \Sigma x^2 \\ R &= \Sigma x^3 \\ S &= \Sigma x^4 \\ T &= \Sigma y \\ U &= \Sigma xy \\ V &= \Sigma x^2 y \end{aligned}$$

$$D = NQS + 2PQR - Q^3 - P^2S - NR^2 \tag{4.5}$$

Then we have

$$\left.\begin{aligned} a &= \frac{NQV + PRT + PQU - Q^2T - P^2V - NRU}{D} \\ b &= \frac{NSU + PQV + QRT - Q^2U - PST - NRV}{D} \\ c &= \frac{QST + QRU + PRV - Q^2V - PSU - R^2T}{D} \end{aligned}\right\} \tag{4.6}$$

General curve fitting (multiple linear regression)

The principle of the best fitting straight line can be extended to other functions and with more than two unknown linear coefficients.

Let us consider the case of a linear combination of *three* functions. Suppose that we know that

$$y = af_0(x) + bf_1(x) + cf_2(x)$$

where f_0, f_1, and f_2 are three known functions of x, but that the coefficients a, b, and c are not known. Suppose, moreover, that the value of y is known for at least three values of x. Then the coefficients a, b, c can be found as follows.

Calculate the sums

$$\begin{aligned} M &= \Sigma f_0^2 & U &= \Sigma y f_0 \\ P &= \Sigma f_0 f_1 & V &= \Sigma y f_1 \\ Q &= \Sigma f_0 f_2 & W &= \Sigma y f_2 \\ R &= \Sigma f_1^2 \\ S &= \Sigma f_1 f_2 \\ T &= \Sigma f_2^2 \end{aligned}$$

Then

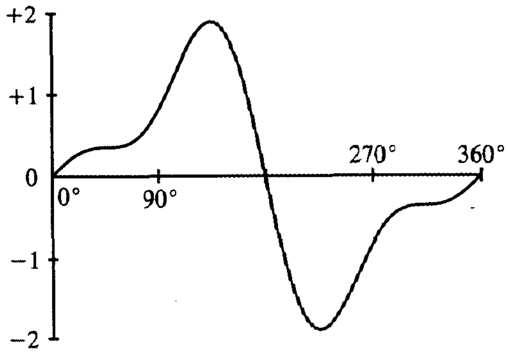
$$\left.\begin{aligned} D &= MRT + 2PQS - MS^2 - RQ^2 - TP^2 \\ a &= \frac{U(RT - S^2) + V(QS - PT) + W(PS - QR)}{D} \\ b &= \frac{U(SQ - PT) + V(MT - Q^2) + W(PQ - MS)}{D} \\ c &= \frac{U(PS - RQ) + V(PQ - MS) + W(MR - P^2)}{D} \end{aligned}\right\} \qquad (4.7)$$

Example 4.c — We know that y is of the form

$$y = a \sin x + b \sin 2x + c \sin 3x$$

and that y takes the following values:

x (degrees)	y
3	0.0433
20	0.2532
34	0.3386
50	0.3560
75	0.4983
88	0.7577
111	1.4585
129	1.8628
143	1.8264
160	1.2431
183	−0.2043
200	−1.2431
218	−1.8422
230	−1.8726
248	−1.4889
269	−0.8372
290	−0.4377
303	−0.3640
320	−0.3508
344	−0.2126



Find the values of the coefficients a, b, c.

We leave it as an exercise to the reader. The function is

$$y = 1.2 \sin x - 0.77 \sin 2x + 0.39 \sin 3x$$

and is illustrated in the Figure above. The reader will *not* find 1.2, −0.77, and +0.39 *exactly*, because in the table the values of y are given with only four decimals.

Let us consider the special case $y = ax^2 + bx + c$. Here we have

$$f_0 = x^2$$
$$f_1 = x$$
$$f_2 = 1$$

resulting in $T = N$ (the number of given points) and $Q = R$. The formulae (4.7) then reduce to (4.5) and (4.6), with other notations.

As another special case, consider $y = af(x)$ with only one unknown coefficient. The latter is easily found from

$$a = \frac{\Sigma y.f}{\Sigma f^2} \tag{4.8}$$

Example 4.d — $y = a\sqrt{x} \quad (x >= 0)$

Find a for the best fitting curve through the following points:

x :	0	1	2	3	4	5
y :	0	1.2	1.4	1.7	2.1	2.2

Here, $f(x) = \sqrt{x}$, so Σf^2 is simply the sum of the x-values. Formula (4.8) gives

$$a = \frac{15.2437}{15}$$

so the required function is

$$y = 1.016\sqrt{x}$$

REFERENCES

1. Helmut Alt, *Angewandte Mathematik, Finanz-Mathematik, Statistik, Informatik für UPN-Rechner*, p. 125 (Vieweg, Braunschweig, 1979).
2. International Astronomical Union *Circular* No. 3177 (1978 February 24).

Chapter 5

Iteration

Iteration (from the Latin *iterare* = to repeat) is a method consisting of repeating a calculation several times, until the value of an unknown quantity is obtained. Generally, after each repetition of the calculation, one obtains a result that is closer to the exact solution. We have already seen the use of iteration in Chapter 3, for solving equations (3.6), (3.7), (3.9), (3.10), and (3.11).

Iteration is used, for instance, when there is no method for calculating the unknown quantity directly in an easy way. Examples are:

— solving the equation of the fifth degree $x^5 + 17x - 8 = 0$;

— the calculation of the times of beginning and end of a solar eclipse, or of an occultation of a star by the Moon, for a given place at the Earth's surface;

— the equation of Kepler $E = M + e \sin E$ (see Chapter 30), where E is the unknown quantity.

To perform an iteration, one must start with an *approximate* value for the unknown quantity, and use must be made of a formula, or of a set of formulae, in order to obtain a *better* value for the unknown. This process is then *repeated* (iteration) until the required accuracy is reached.

A classical example is the calculation of the square root of a number. Of course, this method has nowadays lost its interest (except in special cases), because all pocket calculators and all program languages already possess the function $\sqrt{\ }$ or SQR. The calculation proceeds as follows.

Let N be the number whose square root is requested. Start with an approximate value n for this root; if none is known, the value 1 can be used. Divide N by n, and take the arithmetic mean of the quotient and n. The result is a better value for the square root. In other words, a better value is given by $(n + N/n)/2$. Then the calculation must be repeated.

Example 5.a — Calculate $\sqrt{159}$ to eight decimals.

We know that $12 \times 12 = 144$, so that 12 is an approximate value of the square root of 159. We divide 159 by 12, and find the quotient 13.25. The arithmetic mean of 12 and 13.25 is 12.625, which is a better value for the required square root.

We now divide 159 by 12.625; the quotient is 12.59406. The mean of 12.625 (the previous result) and 12.59406 is 12.60953, which is a still better value for the square root.

In that way, we find successively

12 = starting value
12.625 000 00
12.609 529 70
12.609 520 21
12.609 520 21

As you see, 12.60952021 yields 12.60952021 again, so this is the required square root of 159.

Example 5.b — Calculate the (only) real root of the equation

$$x^5 + 17x - 8 = 0 \tag{5.1}$$

Because there is no method or formula for the direct calculation of the roots of an equation of the fifth degree, we will have recourse to the iteration procedure. In equation (5.1) we put x^5 in the second member and solve for x; this gives

$$x = \frac{8 - x^5}{17} \tag{5.2}$$

The unknown quantity is now present in the right-hand member too, but that does not matter, as we shall see. We start by letting $x = 0$ in the right-hand member. Formula (5.2) then yields

$$x = 8/17 = 0.470\,588\,235$$

which is already a better value than $x = 0$. We now put the value $x = 0.470\,588\,235$ in the right-hand member, and now the formula gives $x = 0.469\,230\,684$. After four more iterations, we obtain the definitive value, namely $x = 0.469\,249\,878$.

The iteration process is not always without problems, however, as it is shown in the following example.

Example 5.c — Consider the equation $x^5 + 3x - 8 = 0$.

As in the preceding example, we put x^5 in the right-hand member, and we obtain

$$x = \frac{8 - x^5}{3}$$

If we start, here again, with $x = 0$, we obtain successively

$$\begin{array}{l} 0.0000 \quad \text{(starting value)} \\ 2.6667 \\ -42.2826 \\ 45\,049\,099 \\ -6.18 \times 10^{37} \\ \text{etc....} \end{array}$$

and so the method does not work in this case! The successive results diverge; in absolute value they grow bigger and bigger. They go "in the wrong direction".

Why did the method work in Example 5.b, but not in Example 5.c? When x lies between 0 and 1, then x^5 too is between 0 and 1. Moreover, x^5 is then smaller than x. This is the reason why in Example 5.b the results of the successive iterations *converge* to a well-defined value, the root of the equation. This root lies between 0 and 1.

But, as we shall see, the root of the equation in Example 5.c is larger than 1. When $x > 1$, then $x^5 > x > 1$, and a small increase of x gives rise to a much larger increase of x^5. For $x = 2$, we have already $x^5 = 32$.

Consequently, the iteration procedure, performed in the same way as in Example 5.b, cannot converge to the required result: the successive values diverge. However, it *is* possible to get the answer, on the condition that we write the iteration formula in another form.

Example 5.d — Let us again consider the equation $x^5 + 3x - 8 = 0$, but now we take into account the fact that the root is larger than 1, and hence that $x^5 > x$. For this reason, we do *not* put x^5 in the right-hand member here. Instead, we keep x^5 in the first member, so the equation becomes

$$x^5 = 8 - 3x \qquad \text{or} \qquad x = \sqrt[5]{8 - 3x}$$

Starting again with $x = 0$, we obtain the required root after 14 iterations, namely, $x = 1.321\,785\,627$.

In example 5.b, we searched for the root of the equation

$$x^5 + 17x - 8 = 0$$

However, we can write this equation as

$$x(x^4 + 17) = 8, \qquad \text{whence} \qquad x = \frac{8}{x^4 + 17}$$

We now can use this latter formula instead of (5.2). As an exercise, solve this equation by iteration; you should obtain the same result as in Example 5.b.

If we wish to work similarly for the equation of Example 5.c, we obtain the iteration formula

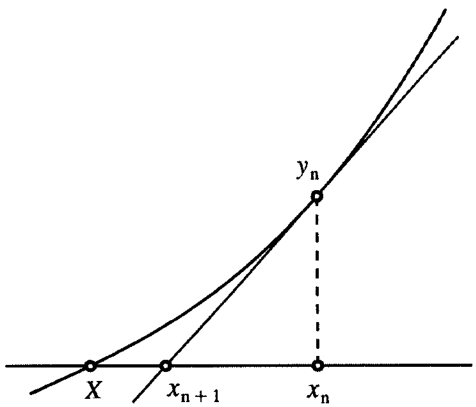
$$x = \frac{8}{x^4 + 3}$$

If we again start by putting the value $x = 0$ in the right-hand member, we obtain $x = 8/3 = 2.666\ldots$. But then comes the surprise: after a few iterations, the successive results jump unceasingly from 2.666 223 459 to 0.149 436 927, and back. As you see, the iteration method does not succeed in all cases; much depends on the form of the iteration formula.

As another example, consider the equation $\sin\varphi = 3\cos\varphi$. Putting $\varphi = 0°$ in the right member yields $\sin\varphi = 3$, an impossibility. Putting, instead, $\varphi = 90°$ in the second member gives $\sin\varphi = 0$, whence $\varphi = 0°$, which brings us back to the first case.

But if we write the equation as $\cos\varphi = (\sin\varphi)/3$ then, starting with $\varphi = 0°$, we reach the solution $\varphi = 71^\circ\!.565051$ after a few iterations.

Or consider the equation $\sin\varphi = \cos 2\varphi$. Evidently, the solution is $\varphi = 30°$, because $\sin 30° = \cos 60°$. If we start by putting $\varphi = 29°$ in the second member of that equation, the results of the successive iterations diverge. If, however, we write the equation the other way, namely, $\cos 2\varphi = \sin\varphi$, then the successive results converge!



As a further illustration of the iteration procedure, let us consider Newton's method for searching the solution of an equation with one unknown by successive approximations.

Let $f(x)$ be a function of x, and we want to find for what value of x that function is zero. Let $f'(x)$ be the derivative function of $f(x)$. If x_n is an assumed value for the root X, then calculate the value y_n of the function $f(x)$, and the value y'_n of the derivative $f'(x)$, for that value of x. The value y'_n is the slope of the tangent to the curve at the point x_n, y_n — see the Figure on the preceding page. Then, a better value for the unknown quantity is given by

$$x_{n+1} = x_n - \frac{y_n}{y'_n}$$

The calculation is then repeated using this new value of x, until the final value X is reached.

In this procedure, the choice of a good starting value for x can be a problem. For example, for the equation

$$x^5 - 3x - 8 = 0$$

the derivative function is $5x^4 - 3$ and, if we start with $x = 0$, we obtain oscillating values, as shown in the box at right.

The reason is that the function reaches a maximum value for $x = -0.88$, so that the tangents on both sides of that point have slopes in opposite directions.

But if we start with $x = 1$, then the correct value (to 9 decimal places) is reached after 11 iterations, as shown in the second box.

0.000 000 000
−2.666 666 667
−2.126 929 222
−1.672 392 941
−1.227 532 073
−0.376 965 299
−2.749 036 974
−2.194 266 642
−1.731 201 846
−1.293 218 529
−0.588 844 800
−3.216 865 068
−2.572 967 056
−2.049 930 312
−1.603 831 481
−1.145 086 796

+1.000 000 000
+6.000 000 000
+4.803 458 391
+3.850 111 311
+3.095 824 107
+2.510 476 381
+2.080 081 724
+1.807 461 730
+1.690 945 284
+1.671 102 262
+1.670 579 511
+1.670 579 156
+1.670 579 156

Test on "smaller than"

When an iteration procedure is used, one should — as has been mentioned above — repeat the calculation until the result no longer varies. In other words, as long as the last result differs from the previous one, a new iteration must be performed. But here we are faced with a small problem, due to the fact that the computer does not calculate "exactly".

Consider the following equation of the third degree

$$s^3 + 3s - W = 0$$

which appears in the calculation of the motion in a parabolic orbit (see Chapter 34). W is a given constant, while s is the unknown quantity. This equation can very easily be solved by iteration. Start from *any* value; a good choice is $s = 0$. Then a better value for s is

$$\frac{2s^3 + W}{3(s^2 + 1)}$$

After some iterations the correct value of s is obtained. Take, for instance, the case $W = 0.9$. The calculation performed on the HP-85 microcomputer gives the following successive results:

0.000 000 000 000
0.300 000 000 000
0.291 743 119 266
0.291 724 443 641
0.291 724 443 546
0.291 724 443 548
0.291 724 443 548

and hence the exact value (with twelve significant digits) is 0.291 724 443 548. But if we repeat the calculation on the same machine for $W = 1.5$, we have a surprise: the machine does not stop and finds successively:

0.000 000 000 000
0.500 000 000 000
0.466 666 666 667
0.466 220 600 162
0.466 220 523 909
0.466 220 523 911
0.466 220 523 910
0.466 220 523 908
0.466 220 523 911
0.466 220 523 910
0.466 220 523 908

and forever again ...911, ...910, ...908. However, we tried this calculation (for $W = 1.5$) with two other programming languages, and the iteration procedure *did* converge; but then it did not converge for *other* values of the constant W.

A remedy for this trouble consists of testing on "smaller than" instead of on "equal to". In other words, let the iteration process stop when the difference between the new value of s and the previous one is, *in absolute value*, less than a given quantity, for instance 10^{-10}.

The binary search

There is a procedure which is absolutely foolproof, because it can neither stall nor diverge, and always converges in a fixed amount of time to the most exact value of the root the programming language is capable. The method does not try to find successively better values of the root. Instead, it just uses a *binary search* to locate the correct value of the root.

Let us explain the procedure by reconsidering the equation of Example 5.b, namely $x^5 + 17x - 8 = 0$.

For $x = 0$ and $x = 1$, the first member of this equation takes the values -8 and $+10$, respectively. So we know that the root lies between 0 and 1 (*).

Let us now try $x = 0.5$, which is the arithmetical mean of 0 and 1. For $x = 0.5$, the function takes the value $+0.53125$, which has the opposite sign of the function's value for $x = 0$. So we now know that the root is between 0 and 0.5.

We now try $x = 0.25$, which is the arithmetical mean of 0 and 0.5. And so on.

After each step, the interval in which the root necessarily must be, is halved. After 32 steps the value of the root is known with nine exact decimals. (In Example 5.b, the same accuracy was obtained after only six steps. But, as we already pointed out, the binary search is a method which is absolutely safe, and it can be used when the "ordinary" iteration procedure is likely to fail).

With the binary search, one knows in advance the accuracy after n steps: it is the initial interval divided by 2^n.

For the example given above, the program in BASIC can be written as follows. Line 60 is not actually needed; it has been included to show the successively better values of x.

```
10  DEF FNA(X) = X*(X^4 + 17) - 8
20  X1 = 0 : Y1 = FNA(X1)
30  X2 = 1 : Y2 = FNA(X2)
40  FOR J = 1 TO 33
50  X = (X1 + X2)/2
60  PRINT J, X
70  Y = FNA(X)
80  IF Y = 0 THEN PRINT J, X : END
90  IF Y*Y1 > 0 THEN 120
100 X2 = X : Y2 = Y
110 GOTO 130
120 X1 = X : Y1 = Y
130 NEXT J
140 END
```

(*) This is true only if the function is continuous in the interval considered. From the fact that $\tan 86° > 0$ and $\tan 93° < 0$, we may *not* conclude that $\tan x$ becomes zero for a value of x between 86° and 93°.

Chapter 6

Sorting Numbers

Computers are more than calculating machines. They can store and handle data. One example of handling is to rearrange or sort data. Sorting is a function with almost universal application for all users of computers. In astronomy, examples are: sorting stars by right ascension, or by declination; sorting times chronologically; sorting minor planets by increasing semimajor axis, or sorting their names alphabetically. Different algorithms are available to perform sorting. In this Chapter we shall give three methods, provide the BASIC programs, and compare the calculation times.

One of the simplest sorting algorithms is given in Table 6.A under the name "SIMPLE SORT". We start from N numbers $X(1)$, $X(2)$, ..., $X(N)$. The values of these elements are arbitrary, and the same value may occur more than once.

After the execution of the routine the numbers $X(I)$ are sorted in increasing order. If one wants them in decreasing order, one should, on line 120, replace $>=$ by $<=$; or, alternatively, one may replace $X(I)$ by $-X(I)$.

At each step, two elements are permuted. Successively, the smallest element is placed in front (for $I = 1$), then the second, and so on, up to $N - 1$. Note that on line 100 the index I should go till $N - 1$, not till N.

This method is also called "straight insertion". The time needed to sort N numbers depends, of course, on the type of the computer and on the program language, but in any case the sorting time will approximately be proportional to N^2. This means that the method is unsuitable for large N.

The method we called "BETTER" is somewhat faster, but again the sorting time is approximately proportional to N^2. Its principle is simple: find the smallest element, and place it in front by permuting two elements.

When the set of data to be sorted is large, a much better method is "QUICKSORT", which was invented by C.A.R. Hoare. The program itself is longer, but the computer time is considerably shorter. Moreover, when N is sufficiently large, the computer time is approximately proportional to N, not to N^2. (In fact, it is nearly proportional to $N \log N$).

TABLE 6.A

Three sorting programs in BASIC

SIMPLE SORT

```
100 FOR I = 1 TO N-1
110 FOR J = I+1 TO N
120 IF X(J) > = X(I) THEN 160
130 A = X(I)
140 X(I) = X(J)
150 X(J) = A
160 NEXT J
170 NEXT I
```

BETTER

```
100 FOR I = 1 TO N-1
110 M = X(I)
120 K = I
130 FOR J = I+1 TO N
140 IF X(J) < M THEN
      M = X(J) : K = J
150 NEXT J
160 A = X(I) : X(I) = M : X(K) = A
170 NEXT I
```

QUICKSORT

```
100 DIM L(30), R(30)
110 S = 1 : L(1) = 1 :  R(1) = N
120 L = L(S) : R = R(S)
130 S = S-1
140 I = L : J = R
150 V = X(INT((L+R)/2))
160 IF X(I) > = V THEN 190
170 I = I+1
180 GOTO 160
190 IF V > = X(J) THEN 220
200 J = J-1
210 GOTO 190
220 IF I>J THEN 250
230 W = X(I) : X(I) = X(J) :
      X(J) = W
240 I = I+1 : J = J-1
250 IF I< =J THEN 160
260 IF J-L < R-I THEN 320
270 IF L > = J THEN 300
280 S = S+1
290 L(S) = L : R(S) = J
300 L = I
310 GOTO 360
320 IF I > = R THEN 350
330 S = S+1
340 L(S) = I : R(S) = R
350 R = J
360 IF L < R THEN 140
370 IF S < > 0 THEN 120
```

The QUICKSORT sorting technique needs two small auxiliary one-dimensional arrays: L(M) and R(M). M is at least the smallest integer larger than $\log_2 N$. A value of M = 30 is certainly sufficient for all practical purposes.

In Table 6.B we mention the calculation times for some values of N on the HP-85 microcomputer for the three programs mentioned in Table 6.A. As we already said, the times will be different on other computers, but in any case we find that these times increase rapidly for larger values of N, except for the QUICKSORT algorithm.

TABLE 6.B

Calculation times (in seconds) of the three sorting algorithms on the HP-85 microcomputer

N	SIMPLE SORT	BETTER	QUICKSORT
10	0.73	0.51	0.70
20	3.92	2.11	1.84
40	15.4	7.81	4.43
60	38.0	17.0	8.63
80	63.8	29.1	11.3
100	104.3	44.6	14.6
150	254	98.6	24.1
200	453	174	32.9
300	1002	387	56.7
500			97.7
1000			218
1500			342
2000			472

To gain some idea of the calculation speeds for larger values of N, we did appeal to a faster computer; the programs were written in FORTRAN and were compiled. The results are given in Table 6.C. The superiority of QUICKSORT is conspicuous here. For $N = 300$, the calculation time is still 15% of that with BETTER (Table 6.B); but for 15000 numbers it is only one third of 1 per cent!

TABLE 6.C

Calculation times (in seconds) of the three sorting algorithms on a "big" computer

N	SIMPLE SORT	BETTER	QUICKSORT
1 000	13	10	< 1
2 000	51	40	1
3 000	114	90	1
4 000	206	159	2
5 000	321	249	2
10 000	1272	994	5
15 000		2236	7
20 000			10
25 000			12
30 000			15

In some cases there is even no need to write a program. For instance, the old TRS-80 Model I contained a built-in function which sorted 1000 numbers in 9 seconds, and 8000 numbers in 83 seconds. It appears that the sorting time is approximately proportional to N here, not to N^2, so probably the QUICKSORT method was used.

To conclude, we can recommend the "straight insertion" (SIMPLE SORT) if the set of data to be sorted is not too large, for instance for $N < 200$. For larger sets it is well worth while to use QUICKSORT.

Besides numerical data, often strings (names) are to be sorted, such as X$(1) = "Ceres", X$(2) = "Pallas", etc. Each character has its own value. The complete list with all signs constitutes the so-called ASCII table, a part of which is given in Table 6.D. [ASCII = "American Standard Code for Information Interchange".]

TABLE 6.D

Visible ASCII Characters

After each character its decimal code is given

space	32	8	56	P	80	h	104
!	33	9	57	Q	81	i	105
"	34	:	58	R	82	j	106
#	35	;	59	S	83	k	107
$	36	<	60	T	84	l	108
%	37	=	61	U	85	m	109
&	38	>	62	V	86	n	110
'	39	?	63	W	87	o	111
(	40	@	64	X	88	p	112
)	41	A	65	Y	89	q	113
*	42	B	66	Z	90	r	114
+	43	C	67	[	91	s	115
,	44	D	68	\	92	t	116
-	45	E	69	]	93	u	117
.	46	F	70	^	94	v	118
/	47	G	71	_	95	w	119
0	48	H	72	`	96	x	120
1	49	I	73	a	97	y	121
2	50	J	74	b	98	z	122
3	51	K	75	c	99	{	123
4	52	L	76	d	100	|	124
5	53	M	77	e	101	}	125
6	54	N	78	f	102	~	126
7	55	O	79	g	103		

Chapter 7

Julian Day

In this Chapter we give a method for converting a date, given in the Julian or in the Gregorian calendar, into the corresponding Julian Day number (JD), or vice versa.

General remarks

The Julian Day number or, more simply, the *Julian Day* (*) (JD) is a continuous count of days and fractions thereof from the beginning of the year -4712. By tradition, the Julian Day begins at Greenwich mean *noon*, that is, at 12^h Universal Time. If the JD corresponds to an instant measured in the uniform scale of Dynamical Time, the expression *Julian Ephemeris Day* (JDE) (**) is often used. For example,

$$\begin{aligned} 1977 \text{ April } 26.4 \text{ UT} &= \text{JD } 2443\,259.9 \\ 1977 \text{ April } 26.4 \text{ TD} &= \text{JDE } 2443\,259.9 \end{aligned}$$

In the methods described below, the Gregorian calendar reform is taken into account. Thus, the day following 1582 October 4 (Julian calendar) is 1582 October 15 (Gregorian calendar).

(*) In many books we read "Julian *Date*" instead of "Julian Day". A *date* consists of a year number, a month, and a day of the month, in any calendar. For me, a Julian date is a date in the Julian calendar, just as a Gregorian date refers to the Gregorian calendar. The JD has nothing to do with the Julian calendar.

(**) Not JED as it is sometimes written. The "E" is a sort of index appended to "JD": JDE = $(\text{Julian Day})_{\text{Ephemeris}}$. The name *Ephemeris* comes from "Ephemeris Time", the old name for the uniform Dynamical Time. The abbreviation JDE has been used in the *Minor Planet Circulars* until 1991 inclusively, when it was changed to JDT. Here the "T" means Terrestrial Dynamical Time (see Chapter 10). But what if we want to refer to the Barycentric Dynamical Time, or in cases where the very small difference between TDT and TDB does not matter? For this reason, I prefer to continue to use the abbreviation JDE.

The Gregorian calendar was not at once officially adopted by all countries. This should be kept in mind when making historical research. In Great Britain, for instance, the change was made as late as in 1752, and in Turkey not before 1927.

The Julian calendar was established in the Roman Empire by Julius Caesar in the year -45 and reached its final form about the year $+8$. Nevertheless, we shall follow the astronomers' practice consisting of extrapolating the Julian calendar indefinitely to the past. In this system we can speak, for instance, of the solar eclipse of August 28 of the year -1203, although at that remote time the Roman Empire was not yet founded and the month of August was still to be conceived!

There is a disagreement between astronomers and historians about how to count the years preceding the year 1. In this book, the "B.C." years are counted astronomically. Thus, the year before the year $+1$ is the year zero, and the year preceding the latter is the year -1. The year which the historians call 585 B.C. is actually the year -584. (Do *not* use the mention "B.C." when using negative years! "-584 B.C.", for instance, is incorrect.)

The astronomical counting of the negative years is the only one suitable for arithmetical purposes. For example, in the historical practice of counting, the rule of divisibility by 4 revealing the Julian leap years no longer exists; these years are, indeed, 1, 5, 9, 13, ... B.C. In the astronomical sequence, however, these leap years are called 0, -4, -8, -12 ..., and the rule of divisibility by 4 subsists.

We will indicate by INT(x) the greatest integer less than or equal to x. For example:

INT(7/4) = 1	INT(5.02) = 5
INT(8/4) = 2	INT(5.9999) = 5

There may be a problem with negative numbers. In most programming languages, INT(x) has the definition given above. In that case we have, for instance, INT(-7.83) = -8, because -7 is indeed larger than -7.83.

But in other languages, such as FORTRAN 77, INT is the integer part of the *written* number, that is, the part of the number that precedes the decimal point. In that case, INT(-7.83) is -7. This is called *truncation*, and some programming languages have both functions: INT(x) having the first of the above-mentioned meanings, and TRUNC(x) or FIX(x).

Hence, take care when using the INT function for negative numbers. (For positive numbers, both meanings yield the same result). In the formulae given in this Chapter, however, the argument of the INT function is always positive.

Calculation of the JD

The following method is valid for positive as well as for negative years, but not for negative JD.

Let Y be the year, M the month number (1 for January, 2 for February, etc., to 12 for December), and D the day of the month (with decimals, if any) of the given calendar date.

- If $M > 2$, leave Y and M unchanged.
  If $M = 1$ or 2, replace Y by $Y - 1$, and M by $M + 12$.
  In other words, if the date is in January or February, it is considered to be in the 13th or 14th month of the preceding year.

- In the *Gregorian* calendar, calculate

$$A = \mathrm{INT}\left(\frac{Y}{100}\right) \qquad\qquad B = 2 - A + \mathrm{INT}\left(\frac{A}{4}\right)$$

  In the *Julian* calendar, take $B = 0$.

- The required Julian Day is then

$$\begin{aligned} \mathrm{JD} = \mathrm{INT}\,(365.25\,(Y + 4716)) + \mathrm{INT}\,(30.6001\,(M + 1)) \\ + D + B - 1524.5 \end{aligned} \tag{7.1}$$

The number 30.6 (instead of 30.6001) will give the correct result, but 30.6001 is used so that the proper integer will always be obtained. [In fact, instead of 30.6001, one may use 30.601, or even 30.61.] For instance, 5 times 30.6 gives 153 exactly. However, most computer languages would not represent 30.6 exactly — see in Chapter 2 what we said about BCD — and hence might give a result of 152.9999998 instead, whose integer part is 152. The calculated JD would then be incorrect.

In formula (7.1), the constant 4716 has been added to the argument of the first INT function, in order to avoid trouble for negative years.

Example 7.a — Calculate the JD corresponding to 1957 October 4.81, the time of launch of Sputnik 1.

Here we have $Y = 1957$, $M = 10$, $D = 4.81$.
Because $M > 2$, we leave Y and M unchanged.
The date is in the Gregorian calendar, so we calculate

$A = \mathrm{INT}(1957/100) = \mathrm{INT}(19.57) = 19$

$B = 2 - 19 + \mathrm{INT}(19/4) = 2 - 19 + 4 = -13$

$\mathrm{JD} = \mathrm{INT}(365.25 \times 6673) + \mathrm{INT}(30.6001 \times 11) + 4.81 - 13 - 1524.5$

$\mathrm{JD} = 2436\,116.31$

Example 7.b — Calculate the JD corresponding to January 27 at 12^h of the year 333.

Because $M = 1$, we have $Y = 333 - 1 = 332$ and $M = 1 + 12 = 13$.

Because the date is in the Julian calendar, we have $B = 0$.

$\mathrm{JD} = \mathrm{INT}(365.25 \times 5048) + \mathrm{INT}(30.6001 \times 14) + 27.5 + 0 - 1524.5$

$\mathrm{JD} = 1842\,713.0$

The following list gives the JD corresponding to some calendar dates. These data may be useful for testing a program.

2000	Jan.	1.5	2451 545.0	1600	Dec.	31.0	2305 812.5
1999	Jan.	1.0	2451 179.5	837	Apr.	10.3	2026 871.8
1987	Jan.	27.0	2446 822.5	−123	Dec.	31.0	1676 496.5
1987	June	19.5	2446 966.0	−122	Jan.	1.0	1676 497.5
1988	Jan.	27.0	2447 187.5	−1000	July	12.5	1356 001.0
1988	June	19.5	2447 332.0	−1000	Feb.	29.0	1355 866.5
1900	Jan.	1.0	2415 020.5	−1001	Aug.	17.9	1355 671.4
1600	Jan.	1.0	2305 447.5	−4712	Jan.	1.5	0.0

If one is interested only in dates between 1900 March 1 and 2100 February 28, then in formula (7.1) we have $B = -13$.

In some applications it is needed to know the Julian Day JD_0 corresponding to January 0.0 of a given year. This is the same as December 31.0 of the preceding year. For a year in the *Gregorian* calendar, this can be calculated as follows.

$$Y = \text{year} - 1 \qquad\qquad A = \text{INT}\left(\frac{Y}{100}\right)$$

$$JD_0 = \text{INT}(365.25\,Y) - A + \text{INT}\left(\frac{A}{4}\right) + 1721\,424.5$$

For the years 1901 to 2099 inclusively, this reduces to

$$JD_0 = 1721\,409.5 + \text{INT}(365.25 \times (\text{year} - 1))$$

When is a given year a leap year ?

In the ***Julian calendar***, a year is a leap (or bissextile) year of 366 days if its numerical designation is divisible by 4.
All other years are common years (365 days).
For instance, the years 900 and 1236 were bissextile years, while 750 and 1429 were common years.

The same rule holds in the ***Gregorian calendar***, with the following exception: the centurial years that are *not* divisible by 400, such as 1700, 1800, 1900, 2100, are common years. The other century years, which *are* divisible by 400, are leap years, for instance 1600, 2000, and 2400.

The *Modified Julian Day* (MJD) sometimes appears in modern work, for instance when mentioning orbital elements of artificial satellites. Contrary to the JD, the Modified Julian Day begins at Greenwich mean *midnight*. It is equal to

$$\text{MJD} = \text{JD} - 2400\,000.5$$

and therefore MJD = 0.0 corresponds to 1858 November 17 at 0^h UT.

Calculation of the Calendar Date from the JD

The following method is valid for positive as well as for negative years, but not for negative Julian Day numbers.

Add 0.5 to the JD, and let Z be the integer part, and F the fractional (decimal) part of the result.

If $Z < 2299\,161$, take $A = Z$.

If Z is equal to or larger than 2291 161, calculate

$$\alpha = \text{INT}\left(\frac{Z - 1867\,216.25}{36524.25}\right)$$

$$A = Z + 1 + \alpha - \text{INT}\left(\frac{\alpha}{4}\right)$$

Then calculate

$$B = A + 1524$$

$$C = \text{INT}\left(\frac{B - 122.1}{365.25}\right)$$

$$D = \text{INT}(365.25\,C)$$

$$E = \text{INT}\left(\frac{B - D}{30.6001}\right)$$

The day of the month (with decimals, if any) is then

$$B - D - \text{INT}(30.6001\,E) + F$$

The month number m is $\quad E - 1 \quad$ if $E < 14$
$\quad E - 13 \quad$ if $E = 14$ or 15

The year is $\quad C - 4716 \quad$ if $m > 2$
$\quad C - 4715 \quad$ if $m = 1$ or 2

Contrary to what has been said about formula (7.1), in the formula for E the number 30.6001 may *not* be replaced by 30.6, even if the computer calculates exactly. Otherwise, one would obtain February 0 instead of January 31, or April 0 instead of March 31.

Example 7.c — Calculate the calendar date corresponding to JD 2436 116.31.

2436 116.31 + 0.5 = 2436 116.81

Z = 2436 116 and F = 0.81

Because Z > 2299 161, we have

$$\alpha = \text{INT}\left(\frac{2436\,116 - 1867\,216.25}{36524.25}\right) = 15$$

$$A = 2436\,116 + 1 + 15 - \text{INT}\left(\frac{15}{4}\right) = 2436\,129$$

Then we find

B = 2437 653 C = 6673 D = 2437 313 E = 11

day of month = 4.81

month $m = E - 1 = 10$ (because $E < 14$)

year = $C - 4716 = 1957$ (because $m > 2$)

Hence, the required date is 1957 October 4.81.

Exercise : Calculate the calendar dates corresponding to

JD = 1842 713.0 and JD = 1507 900.13.

Answers: 333 January 27.5 and −584 May 28.63.

Time interval in days

The number of days between two calendar dates can be found by calculating the difference between their corresponding Julian Days.

Example 7.d — The periodic comet Halley passed through the perihelion of its orbit on 1910 April 20 and on 1986 February 9. What is the time interval between these two passages?

1910 April 20.0 corresponds to JD 2418 781.5
1986 Febr. 9.0 corresponds to JD 2446 470.5

The difference is 27 689 days.

Exercise : Find the date exactly 10 000 days after 1991 July 11.

Answer: 2018 November 26.

Day of the week

The day of the week corresponding to a given date can be obtained as follows. Compute the JD for that date at 0^h UT, add 1.5, and divide the result by 7. The remainder of this division will indicate the weekday, as follows: if the remainder is 0, it is a Sunday, 1 a Monday, 2 a Tuesday, 3 a Wednesday, 4 a Thursday, 5 a Friday, and 6 a Saturday.

The week was not modified in any way by the Gregorian reform of the Julian calendar. Thus, in 1582, Thursday October 4 was followed by Friday October 15.

Example 7.e — Find the weekday of 1954 June 30.

1954 June 30.0 corresponds to JD 2434 923.5

2434 923.5 + 1.5 = 2434 925

The remainder of the division of 2434 925 by 7 is 3. Hence it was a Wednesday.

Day of the Year

The number N of a day in the year can be computed by means of the following formula [1].

$$N = \mathrm{INT}\left(\frac{275\,M}{9}\right) - K \times \mathrm{INT}\left(\frac{M+9}{12}\right) + D - 30$$

where M is the month number, D the day of the month, and

$K = 1$ for a leap (bissextile) year,
$K = 2$ for a common year.

N takes integer values, from 1 on January 1, to 365 (or 366 in leap years) on December 31.

Example 7.f — 1978 November 14.

Common year, $M = 11$, $D = 14$, $K = 2$.

One finds $N = 318$.

Example 7.g — 1988 April 22.

Leap year, $M = 4$, $D = 22$, $K = 1$.

One finds $N = 113$.

Let us now consider the reverse problem: the day number N in the year is known, and the corresponding date is required, namely the month number M and the day D of that month. The following algorithm was found by A. Pouplier, of the Société Astronomique de Liège, Belgium [2].

As above, take

$K = 1$ in the case of a leap year,
$K = 2$ in the case of a common year.

$$M = \mathrm{INT}\left(\frac{9\,(K+N)}{275} + 0.98\right)$$

If $N < 32$, then $M = 1$

$$D = N - \mathrm{INT}\left(\frac{275\,M}{9}\right) + K \times \mathrm{INT}\left(\frac{M+9}{12}\right) + 30$$

REFERENCES

1. Nautical Almanac Office, U.S. Naval Observatory, Washington, D.C., *Almanac for Computers for the Year 1978*, page B2.
2. A. Pouplier, letter to Jean Meeus, 1987 April 10.

Chapter 8

Date of Easter

In this Chapter we give a method for calculating the date of the Christian Easter Sunday of a given year. For the Jewish Pesach, see next Chapter.

Gregorian Easter

The following method has been given by Spencer Jones in his book *General Astronomy* (pages 73–74 of the edition of 1922). It has been published again in the *Journal of the British Astronomical Association*, Vol. 88, page 91 (December 1977) where it is said that it was devised in 1876 and appeared in Butcher's *Ecclesiastical Calendar*.

Unlike the formula given by Gauss, this method has no exception and is valid for all years in the Gregorian calendar, hence from the year 1583 on. The procedure for finding the date of Easter is as follows:

Divide	by	Quotient	Remainder
the year x	19	—	a
the year x	100	b	c
b	4	d	e
$b + 8$	25	f	—
$b - f + 1$	3	g	—
$19a + b - d - g + 15$	30	—	h
c	4	i	k
$32 + 2e + 2i - h - k$	7	—	l
$a + 11h + 22l$	451	m	—
$h + l - 7m + 114$	31	n	p

Then $\quad n \;=\;$ number of the month (3 = March, 4 = April),

$p+1 \;=\;$ day of that month upon which Easter Sunday falls.

If the programming language has no "modulo" function or no "remainder" function, the calculation of the remainder of a division must be programmed carefully. Suppose that the remainder of the division of 34 by 30 should be found. On the HP-48s calculator, for instance, we find

$$34/30 = 1.133\,333\,333\,33$$

the fractional part of which is 0.133 333 333 33. When multiplied by 30, this gives 3.999 999 9999. This result differs from 4, the correct value, and may give a wrong date for Easter at the end of the calculation.

Try your program on the following years:

1991 → March 31	1954 → April 18
1992 → April 19	2000 → April 23
1993 → April 11	1818 → March 22

The extreme dates of Easter are March 22 (as in 1818 and 2285) and April 25 (as in 1886, 1943, 2038).

The rule for finding the date of Easter Sunday is well known: Easter is the first Sunday *after* the Full Moon that happens on or next after the March equinox. Actually, the rules for finding the Easter date were fixed long ago by the Christian clergy. For the purposes of these rules, the Full Moon is reckoned according to an ecclesiastical computation and is not the real, astronomical Full Moon. Likewise, the equinox is always assumed to fall on March 21; actually, it can occur a day or two sooner.

In 1967, for instance, the equinox was on March 21, and the Full Moon on March 26 (UT dates). The first Sunday after March 26 was April 2. Nevertheless, Easter Sunday was on March 26.

During the period 1900–2100, the purely astronomical rule yields another date for Easter Sunday than the ecclesiastical rule for the following years: 1900, 1903, 1923, 1924, 1927, 1943, 1954, 1962, 1967, 1974, 1981, 2038, 2049, 2069, 2076, 2089, 2095, and 2096. See also Chapter 60 of my *Mathematical Astronomy Morsels* (Willmann-Bell, ed.; 1997).

A period of 5 700 000 years is required for the cyclical recurrence of the Gregorian Easter dates. It has been found that, in the long run, the most frequent Gregorian Easter date is April 19.

Julian Easter

In the Julian calendar, the date of Easter can be found as follows.

Divide	by	Quotient	Remainder
the year x	4	—	a
the year x	7	—	b
the year x	19	—	c
$19c + 15$	30	—	d
$2a + 4b - d + 34$	7	—	e
$d + e + 114$	31	f	g

Then f = number of the month (3 = March, 4 = April),
$g + 1$ = day of that month upon which Easter Sunday falls.

The date of the *Julian* Easter has a periodicity of 532 years. For instance, we find April 12 for the years 179, 711, and 1243.

Chapter 9

Jewish and Moslem Calendars

It is not the aim of this Chapter to describe the principles of the Jewish and Moslem calendars. We shall just give some calculation methods which are easily programmable on a computer or on a pocket calculator. The algorithms given here were published by Denis Savoie in 1990 and 1991 in *Observations et Travaux*, a publication of the Société Astronomique de France.

In what follows we will denote by $[a]_b$ the remainder of the division of a by b, a and b being integers. For instance, $[16]_7 = 2$ and $[21]_7 = 0$.

INT(x) will mean the integer part of x. It is, in fact, the greatest integer which is not greater than x. For instance, INT(19) and INT(19.95) are both equal to 19. Great care should be taken when the value in negative. Some programming languages have both the INT and the FIX functions. For positive numbers these functions give the same results. But, for instance, INT$(-2.4) = -3$, the correct answer, while FIX$(-2.4) = -2$.

Jewish Calendar

The Jewish (or Hebrew) calendar is luni-solar, being ruled by both the lunation (the synodic lunar month) and the tropical year. The Jewish month has 29 or 30 days, and the year has 12 or 13 months. Moreover, both types of years can vary in three ways, so a Jewish *common* year may contain 353, 354, or 355 days, and an *embolismic* or leap year 383, 384, or 385 days. The names of the months and their lengths are given in Table 9.A.

The Jewish Easter, or *Pesach*, always falls on 15 Nisan.

Let A be the year number in the Jewish calendar, and X the year in the Julian or Gregorian calendar. Then the date in year X on which 15 Nisan occurs can be found by the following formulae due to Gauss.

$$C = \text{INT}\left(\frac{X}{100}\right) \qquad\qquad S = \text{INT}\left(\frac{3C - 5}{4}\right)$$

$$A = X + 3760 \qquad\qquad a = [12X + 12]_{19} \qquad\qquad b = [X]_4$$

TABLE 9.A

Classification of Years in the Jewish Calendar

Month	Common Year			Embolismic (Leap) Year		
	Deficient	*Regular*	*Complete*	*Deficient*	*Regular*	*Complete*
Tishri	30	30	30	30	30	30
Heshvan	29	29	30	29	29	30
Kislev	29	30	30	29	30	30
Tevet	29	29	29	29	29	29
Shevat	30	30	30	30	30	30
Adar	29	29	29	30	30	30
Veadar				29	29	29
Nisan	30	30	30	30	30	30
Iyar	29	29	29	29	29	29
Sivan	30	30	30	30	30	30
Tammuz	29	29	29	29	29	29
Av	30	30	30	30	30	30
Elul	29	29	29	29	29	29
Sum	353	354	355	383	384	385

$$Q = -1.904412361576 + 1.554241796621\,a + 0.25\,b - 0.003177794022\,X + S$$

$$j = [\mathrm{INT}(Q) + 3X + 5b + 2 - S]_7$$

$$r = Q - \mathrm{INT}(Q)$$

If $X < 1583$, or in order to obtain Q in the Julian calendar, take $S = 0$.

One distinguishes the following four cases:

1. if $j = 2$, 4, or 6, then $D = \mathrm{INT}(Q) + 23$;
2. if $j = 1$, $a > 6$, and $r >= 0.632870370$, then $D = \mathrm{INT}(Q) + 24$;
3. if $j = 0$, $a > 11$, and $r >= 0.897723765$, then $D = \mathrm{INT}(Q) + 23$;
4. in all other cases, $D = \mathrm{INT}(Q) + 22$.

The Pesach then falls on D March or, if $D > 31$, on $(D - 31)$ April.

Once the date of the Pesach is obtained, just add 163 days to obtain the date of the beginning (1 Tishri) of the *next* Jewish year. The Jewish year A always begins in September or October of the Julian or Gregorian year $X = A - 3761$.

If A is the Jewish year number, then take the remainder $[A]_{19}$. If this remainder is 0, 3, 6, 8, 11, 14, or 17, then that year has 13 months; otherwise it is a common year of 12 months.

Example 9.a — Calculate the date of 15 Nisan in the Gregorian year $X = 1990$.

We find successively $C = 19$; $S = 13$; $a = 9$; $b = 2$; $Q = 19.2599537042$; $\mathrm{INT}(Q) = 19$; $j = 3$; $r = 0.2599537042$.

We are in the fourth case, so $D = 19 + 22$. Hence, the date is $19 + 22 - 31 =$ 10 April. The Jewish year is $A = 1990 + 3760 = 5750$.

Adding 163 days, we find 1990 September 20. This is the Gregorian date corresponding to 1 Tishri 5751. Because $[5751]_{19} = 13$, the Jewish year 5751 is a common year.

To find the number of days (whether 353, 354, or 355) in that year, the simplest way is to search the Gregorian date corresponding to the beginning of the *next* Jewish year, and to make the difference. We find that 1 Tishri 5752 corresponds to 1991 September 9, so the year $A = 5751$ has 354 days.

Moslem Calendar

The Moslem (or Islamic) calendar is purely lunar, as it follows the lunar phase cycle without regard for the tropical year.

The year contains twelve months. The months have alternately 30 and 29 days, except the last month which can have 29 or 30 days — see Table 9.B. Consequently, the Moslem year has 354 or 355 days; it is shorter than the Gregorian year by about 11 days. As a result, the cycle of twelve lunar months regresses through the seasons over a period of about 33 Gregorian years.

TABLE 9.B

Months of the Moslem Calendar

1.	Muharram	30 days	7.	Rajab	30 days
2.	Safar	29	8.	Sha'ban	29
3.	Rabi'al-Awwal	30	9.	Ramadan	30
4.	Rabi'ath-Thani	29	10.	Shawwal	29
5.	Jumada l-Ula	30	11.	Dhu l-Qa'da	30
6.	Jumada t-Tania	29	12.	Dhu l-Hijja	29 or 30

The algorithms given below, due to M. Francœur (1841) and modified by Denis Savoie and the present author, will give meaningless results for dates earlier than 622 July 16 of the Julian calendar, corresponding to the beginning of the Islamic era, 1 Muharram A.H. 1 (A.H. = *Anno Hegirae*).

Conversion of a Moslem date to a Gregorian (or Julian) date

Let H, M, and D be the year, the month number, and the day of the month in the Moslem calendar. Then calculate

$$
\begin{aligned}
N &= D + \mathrm{INT}(29.5001(M-1) + 0.99) \\
Q &= \mathrm{INT}(H/30) \\
R &= [H]_{30} \\
A &= \mathrm{INT}((11R + 3)/30) \\
W &= 404Q + 354R + 208 + A \\
Q1 &= \mathrm{INT}(W/1461) \\
Q2 &= [W]_{1461} \\
G &= 621 + 4 \times \mathrm{INT}(7Q + Q1) \\
K &= \mathrm{INT}(Q2/365.2422) \\
E &= \mathrm{INT}(365.2422K) \\
J &= Q2 - E + N - 1 \\
X &= G + K
\end{aligned}
$$

If $J > 366$ and $[X]_4 = 0$, then subtract 366 from J, and add 1 to X.
If $J > 365$ and $[X]_4 > 0$, then subtract 365 from J, and add 1 to X.

Then J is the number of the day in the *Julian* year X. To convert to the Gregorian calendar (if the date is later than 1582 October 4), and to find the month and the day of the month, one can proceed as follows.

$$JD = \mathrm{INT}(365.25(X-1)) + 1721423 + J$$

$$\alpha = \mathrm{INT}\left(\frac{JD - 1867216.25}{36524.25}\right) \qquad \beta = JD + 1 + \alpha - \mathrm{INT}\left(\frac{\alpha}{4}\right)$$

However, if $JD < 2299161$, then take $\beta = JD$.

$$b = \beta + 1524 \qquad c = \mathrm{INT}\left(\frac{b - 122.1}{365.25}\right)$$

$$d = \mathrm{INT}(365.25c) \qquad e = \mathrm{INT}\left(\frac{b-d}{30.6001}\right)$$

Then the day of the month is $b - d - \mathrm{INT}(30.6001e)$
and the month number m is $e - 1$ if $e < 14$
$e - 13$ if $e > 13$

The year is $c - 4716$ if $m > 2$, or $c - 4715$ if $m < 3$.

If $[11R + 3]_{30} > 18$, then H is a leap year of 355 days, otherwise it is a common year of 354 days.

Example 9.b — Find the Julian or Gregorian date corresponding to the first day of the Moslem year 1421.

Here we have $H = 1421$, $M = 1$, $D = 1$, and we find successively:

$N = 1$; $Q = 47$; $R = 11$; $A = 4$; $W = 23094$; $Q1 = 15$; $Q2 = 1179$; $G = 1997$; $K = 3$; $E = 1095$; $J = 84$; $X = 2000$.

This gives the 84th day of the year 2000 in the *Julian* calendar. Continuing, we obtain

$JD = 2451641$; $\alpha = 16$; $\beta = 2451654$; $b = 2453178$; $c = 6716$; $d = 2453019$; $e = 5$; day $= 6$; month $= 4$; year $= 2000$.

Hence, 1 Muharram 1421 corresponds to 6 April of the Gregorian year 2000.

Because $[11R + 3]_{30} = 4$, which is not larger than 18, the Moslem year 1421 is a common year of 354 days.

Conversion of a Gregorian (or Julian) date to a Moslem date

If the date is given in the Gregorian calendar, we first have to convert it to the corresponding date in the Julian calendar. This can be done as follows.

Let X, M, D be the given year, month number, and day of the month in the Gregorian calendar.

If $M < 3$, subtract 1 from X, and add 12 to M.

Calculate

$$\alpha = \mathrm{INT}\left(\frac{X}{100}\right) \qquad \beta = 2 - \alpha + \mathrm{INT}\left(\frac{\alpha}{4}\right)$$

$$b = \mathrm{INT}(365.25X) + \mathrm{INT}(30.6001\,(M+1)) + D + 1722519 + \beta$$

With this value of b, calculate c, d, e, and the new values (Julian calendar) for the day D, the month number M, and the year X as before (page 74).

The date being now Julian, proceed as follows.

If $[X]_4 = 0$, then $W = 1$, otherwise $W = 2$.

$$N = \mathrm{INT}\left(\frac{275M}{9}\right) - W \times \mathrm{INT}\left(\frac{M+9}{12}\right) + D - 30$$

$$A = X - 623$$

$$B = \mathrm{INT}\left(\frac{A}{4}\right)$$

$$C = [A]_4 \qquad C1 = 365.2501\,C \qquad C2 = \mathrm{INT}(C1)$$

If $C1 - C2 > 0.5$, add 1 to $C2$.

$$D' = 1461B + 170 + C2$$

$$Q = \mathrm{INT}\left(\frac{D'}{10631}\right)$$

$$R = [D']_{10631}$$

$$J = \mathrm{INT}\left(\frac{R}{354}\right)$$

$$K = [R]_{354}$$

$$O = \mathrm{INT}\left(\frac{11J + 14}{30}\right)$$

$$H = 30Q + J + 1 \qquad JJ = K - O + N - 1$$

JJ is the number of the day in the Moslem year H. If $JJ > 354$, we have to look if H is a common year or a leap year, in order to know whether we should subtract 354 or 355 days. This can be done as follows.

$$CL = [H]_{30} \qquad DL = [11\,CL + 3]_{30}$$

If $DL < 19$, subtract 354 from JJ, and add 1 to H.

If $DL > 18$, subtract 355 from JJ, and add 1 to H.

Finally, if $JJ = 0$, then put $JJ = 355$, and subtract 1 from H.

Now, the day number JJ should be converted to the month number m and the day d of the month:

$$S = \mathrm{INT}\left(\frac{JJ - 1}{29.5}\right) \qquad m = 1 + S \qquad d = \mathrm{INT}(JJ - 29.5S)$$

If $JJ = 355$, then $m = 12$ and $d = 30$.

Example 9.c — Find the Moslem date corresponding to the Gregorian date 1991 August 13.

Here we have $X = 1991$, $M = 8$, $D = 13$. We find successively:

$\alpha = 19$; $\beta = -13$; $b = 2450006$; $c = 6707$; $d = 2449731$; $e = 8$;
$D = 31$; $M = 7$; $X = 1991$.

So the date in the *Julian* calendar is 1991 July 31.

$W = 2$; $N = 212$; $A = 1368$; $B = 342$; $C = C1 = C2 = 0$; $D' = 499832$;
$Q = 47$; $R = 175$; $J = 0$; $K = 175$; $O = 0$; $H = 1411$; $JJ = 386$.

Because $JJ > 354$, we calculate $CL = 1$ and $DL = 14$. Because DL is smaller than 19, we subtract 354 from JJ and add 1 to H, obtaining $JJ = 32$, $H = 1412$.

Then $m = 2$, $d = 2$. So the date is 2 Safar of A.H. 1412.

Chapter 10

Dynamical Time and Universal Time

The Universal Time (UT), or Greenwich Civil Time, is based on the rotation of the Earth. The UT is necessary for civil life and for the astronomical calculations where local hour angles are involved. (Universal Time is erroneously called "Greenwich Mean Time" in Great Britain and by most navigators. In astronomy, "mean" time has a precise meaning. By definition, mean time is measured from the superior transit of the mean Sun, hence from mean *noon*. It is the *civil* time which begins at midnight, so GMT and UT differ by twelve hours.)

However, the Earth's rotation is generally slowing down. Moreover, this occurs with unpredictable irregularities. For this reason, the UT is not a uniform time.

But the astronomers need a uniform time scale for their accurate calculations (celestial mechanics, orbits, ephemerides). From 1960 to 1983, in the great astronomical almanacs such as the *Astronomical Ephemeris*, use was made of a uniform time scale called the *Ephemeris Time* (ET) and defined by the laws of dynamics: it was based on the planetary motions. In 1984, the ET was replaced by the *Dynamical Time*, which is defined by atomic clocks. The Dynamical Time is, in fact, a prolongation of the Ephemeris Time.

One distinguishes a Barycentric Dynamical Time (TDB) and a Terrestrial Dynamical Time (TDT). These times differ by at most 0.0017 second, the difference being related to the motion of the Earth on its elliptical orbit around the Sun (relativistic effect). Because this very small difference can be neglected for most practical purposes, we will make no distinction between TDB and TDT, and we will name both simply "Dynamical Time", or TD by dropping the last letter from both TDB and TDT. Hence, our abbreviation TD does *not* come from the French "Temps Dynamique", but should be considered as meaning $\text{Time}_{\text{Dynamical}}$.

TDT was later shortened to simply TT ("Terrestrial Time"), an odd name because the mean solar time at Moscow, or the sidereal time at New York, are "terrestrial" times too!

The exact value of the difference $\Delta T = \text{TD} - \text{UT}$ can be deduced only from observations. Table 10.A gives the value of ΔT for the *beginning* of some years. For the years earlier than 1988, they are taken from the *Astronomical Almanac* for 1988 [1]. However, the values earlier than 1955 have been slightly corrected by using Chapront's new value $n' = -25.7376$ "/century^2 for the tidal acceleration of the Moon [2].

For epochs in the *near* future, one may extrapolate the values of Table 10.A. For instance, we can use the provisional values

$$\begin{aligned} \Delta T &= +65 \text{ seconds in } 2000 \\ \Delta T &= +69 \text{ seconds in } 2005 \\ \Delta T &= +80 \text{ seconds in } 2015 \end{aligned}$$

For other epochs outside the time interval of Table 10.A, an *approximate* value of ΔT (in *seconds*) can be calculated by means of the following expressions due to Chapront and Francou [2]:

Let t be the time measured in centuries from the epoch 2000.0 ($t < 0$ before 2000), that is,

$$t = \frac{\text{year} - 2000}{100}$$

Then, before the year +948,

$$\Delta T = 2177 + 497\,t + 44.1\,t^2 \tag{10.1}$$

From +948 to +1600, and after the year +2000,

$$\Delta T = 102 + 102\,t + 25.3\,t^2 \tag{10.2}$$

However, to avoid a discontinuity at A.D. 2000, it is advised to add the correction $+0.37 \times (\text{year} - 2100)$ for the years 2000 to 2100.

With these expressions, the uncertainty of UT can reach as much as two hours back to 4000 B.C. Future improvements of the formulae will benefit the user when converting from TD to UT, but will not change the algorithms, programs, ephemerides, or tables given with the uniform time scale of TD.

The quantity ΔT was slightly negative from A.D. 1871 to 1901. Note that ΔT is positive *both* for the remote past and for the distant future.

Except for the years 1871–1901, an instant given in UT is *later* than the instant in TD having the same numerical value. For example, 1990 January 27, 0^h UT is 57 seconds later than 1990 January 27, 0^h TD. We have UT = TD − ΔT.

Example 10.a — New Moon took place on 1977 February 18 at $3^h37^m40^s$ Dynamical Time (see Example 49.a).

At that instant, ΔT was equal to +48 seconds. Consequently, the corresponding Universal Time of that lunar phase was

$$3^h37^m40^s - 48^s = 3^h36^m52^s.$$

TABLE 10.A

ΔT = TD − UT (in seconds) for the beginning of some years

year	ΔT	year	ΔT	year	ΔT	year	ΔT	year	ΔT
1620	+121	1700	+ 7	1780	+16	1860	+ 7.7	1940	+24.3
1622	112	1702	7	1782	16	1862	7.3	1942	25.3
1624	103	1704	8	1784	16	1864	6.2	1944	26.2
1626	95	1706	8	1786	16	1866	5.2	1946	27.3
1628	88	1708	9	1788	16	1868	2.7	1948	28.2
1630	+82	1710	+ 9	1790	+16	1870	+ 1.4	1950	+29.1
1632	77	1712	9	1792	15	1872	− 1.2	1952	30.0
1634	72	1714	9	1794	15	1874	− 2.8	1954	30.7
1636	68	1716	9	1796	14	1876	− 3.8	1956	31.4
1638	63	1718	10	1798	13	1878	− 4.8	1958	32.2
1640	+60	1720	+10	1800	+13.1	1880	− 5.5	1960	+33.1
1642	56	1722	10	1802	12.5	1882	− 5.3	1962	34.0
1644	53	1724	10	1804	12.2	1884	− 5.6	1964	35.0
1646	51	1726	10	1806	12.0	1886	− 5.7	1966	36.5
1648	48	1728	10	1808	12.0	1888	− 5.9	1968	38.3
1650	+46	1730	+10	1810	+12.0	1890	− 6.0	1970	+40.2
1652	44	1732	10	1812	12.0	1892	− 6.3	1972	42.2
1654	42	1734	11	1814	12.0	1894	− 6.5	1974	44.5
1656	40	1736	11	1816	12.0	1896	− 6.2	1976	46.5
1658	38	1738	11	1818	11.9	1898	− 4.7	1978	48.5
1660	+35	1740	+11	1820	+11.6	1900	− 2.8	1980	+50.5
1662	33	1742	11	1822	11.0	1902	− 0.1	1982	52.2
1664	31	1744	12	1824	10.2	1904	+ 2.6	1984	53.8
1666	29	1746	12	1826	9.2	1906	5.3	1986	54.9
1668	26	1748	12	1828	8.2	1908	7.7	1988	55.8
1670	+24	1750	+12	1830	+ 7.1	1910	+10.4	1990	+56.9
1672	22	1752	13	1832	6.2	1912	13.3	1992	58.3
1674	20	1754	13	1834	5.6	1914	16.0	1994	60.0
1676	18	1756	13	1836	5.4	1916	18.2	1996	61.6
1678	16	1758	14	1838	5.3	1918	20.2	1998	63.0
1680	+14	1760	+14	1840	+ 5.4	1920	+21.1		
1682	12	1762	14	1842	5.6	1922	22.4		
1684	11	1764	14	1844	5.9	1924	23.5		
1686	10	1766	15	1846	6.2	1926	23.8		
1688	9	1768	15	1848	6.5	1928	24.3		
1690	+ 8	1770	+15	1850	+ 6.8	1930	+24.0		
1692	7	1772	15	1852	7.1	1932	23.9		
1694	7	1774	15	1854	7.3	1934	23.9		
1696	7	1776	16	1856	7.5	1936	23.7		
1698	7	1778	16	1858	7.6	1938	24.0		

Example 10.b — Suppose that the position of Mercury should be calculated for February 6 at 6^h Universal Time of the year +333.

Here we have $T = (333.1 - 2000)/100 = -16.669$, for which formula (10.1) gives the value $\Delta T = +6146$ seconds, or 102 minutes. Hence, TD $= 6^h + 102$ minutes $= 7^h42^m$, and the calculation of the position of Mercury must be performed for 333 February 6 at 7^h42^m TD.

The following approximation for ΔT, valid for the entire time span 1800–1997, represents the values given in Table 10.A with a maximum error of 2.3 seconds.

$$\begin{aligned}\Delta T = {} & -1.02 + 91.02\,\theta + 265.90\,\theta^2 - 839.16\,\theta^3 - 1545.20\,\theta^4 \\ & + 3603.62\,\theta^5 + 4385.98\,\theta^6 - 6993.23\,\theta^7 - 6090.04\,\theta^8 \\ & + 6298.12\,\theta^9 + 4102.86\,\theta^{10} - 2137.64\,\theta^{11} - 1081.51\,\theta^{12}\end{aligned}$$

In this formula, ΔT is expressed in seconds, and θ is the time elapsed since 1900.0 and expressed in Julian centuries (hence $\theta < 0$ before 1900).

The following formula gives ΔT for the shorter time span 1800–1899 with a maximum error of 0.9 second:

$$\begin{aligned}\Delta T = {} & -2.50 + 228.95\,\theta + 5218.61\,\theta^2 + 56\,282.84\,\theta^3 + 324\,011.78\,\theta^4 \\ & + 1\,061\,660.75\,\theta^5 + 2\,087\,298.89\,\theta^6 + 2\,513\,807.78\,\theta^7 \\ & + 1\,818\,961.41\,\theta^8 + 727\,058.63\,\theta^9 + 123\,563.95\,\theta^{10}\end{aligned}$$

For the years 1900 to 1997, the following expression gives ΔT with a maximum error of 0.9 second:

$$\begin{aligned}\Delta T = {} & -2.44 + 87.24\,\theta + 815.20\,\theta^2 - 2637.80\,\theta^3 - 18\,756.33\,\theta^4 \\ & + 124\,906.15\,\theta^5 - 303\,191.19\,\theta^6 + 372\,919.88\,\theta^7 \\ & - 232\,424.66\,\theta^8 + 58\,353.42\,\theta^9\end{aligned}$$

where θ has the same meaning as for the first formula.

Note that these three expressions are empirical formulae, and that *their use is prohibited outside of their defined validity range!* For instance, the second expression would give a value of 70000 seconds for the year 1945!

REFERENCES

1. *Astronomical Almanac for 1988* (Washington, D.C.), pages K8 and K9.
2. J. Chapront, M. Chapront-Touzé, and G. Francou, *Note S055* issued by the Bureau des Longitudes, Paris, in December 1997.

Chapter 11

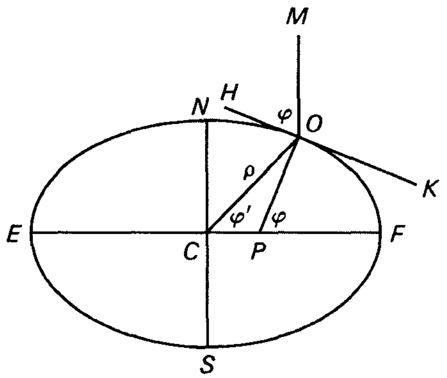
The Earth's Globe

The actual figure of the Earth's surface, including all the inequalities of mountains and valleys, is incapable of geometric definition. Therefore, the ideal figure used in geodesy is that of the mean sea level, extended through the continents. This is the *geoid*, whose surface at every point is perpendicular to the local plumb line.

However, the heterogeneity of the Earth's interior and the attraction of mountains are such that the surface of the geoid is not rigorously represented by any definable solid. An approximation sufficient for most geographical and astronomical purposes is obtained by considering it to be an ellipsoid of revolution.

Geocentric rectangular coordinates of an observer

The Figure represents a meridian cross section of the Earth. C is the Earth's center, N its north pole, S its south pole, EF the equator, HK the horizontal plane of the observer O, and OP the perpendicular to HK. The direction OM, parallel to SN, makes with OH an angle φ which is the *geographical latitude* of O. The angle OPF too is equal to φ. The latitude is positive in the northern hemisphere, negative in the southern hemisphere.



The radius vector OC, joining the observer to the center of the Earth, makes with the equator CF an angle φ' which is the *geocentric latitude* of O. We have $\varphi = \varphi'$ at the poles and at the equator; for all other latitudes

$$|\varphi'| < |\varphi|$$

Let f be the Earth's flattening, and b/a the ratio NC/CF of the polar radius $NC = b$ to the equatorial radius $CF = a$. In 1976 the International Astronomical Union adopted the values

$$a = 6378.14 \text{ km}, \qquad f = \frac{1}{298.257}$$

from which we have

$$b = a(1-f) = 6356.755 \text{ km}$$

$$\frac{b}{a} = 1-f = 0.996\,647\,19$$

The eccentricity e of the Earth's meridian is

$$e = \sqrt{2f-f^2} = 0.081\,819\,22$$

We have the relations

$$f = \frac{a-b}{a} \qquad 1-e^2 = (1-f)^2$$

For a place at sea level,

$$\tan \varphi' = \frac{b^2}{a^2} \tan \varphi$$

If H is the observer's height above sea level in *meters*, the quantities $\rho \sin \varphi'$ and $\rho \cos \varphi'$, which are needed in the calculation of diurnal parallaxes, eclipses and occultations, may be calculated as follows:

$$\tan u = \frac{b}{a} \tan \varphi$$

$$\begin{cases} \rho \sin \varphi' = \dfrac{b}{a} \sin u + \dfrac{H}{6\,378\,140} \sin \varphi \\[2ex] \rho \cos \varphi' = \cos u + \dfrac{H}{6\,378\,140} \cos \varphi \end{cases}$$

The quantity $\rho \sin \varphi'$ is positive in the northern hemisphere and negative in the southern one, while $\rho \cos \varphi'$ is always positive.

The quantity ρ denotes the observer's distance to the center of the Earth (OC in the Figure), the Earth's equatorial radius being taken as unity.

Example 11.a — Calculate $\rho \sin \varphi'$ and $\rho \cos \varphi'$ for the Palomar Observatory, for which

$$\varphi = +33°21'22'', \qquad H = 1706 \text{ meters}.$$

We obtain

$$\begin{aligned} \varphi &= 33^\circ\!.356\,111 \\ u &= 33^\circ\!.267\,796 \\ \rho \sin \varphi' &= +0.546\,861 \\ \rho \cos \varphi' &= +0.836\,339 \end{aligned}$$

Other formulae concerning the Earth's ellipsoid

For a given point on the ellipsoid, the difference between the geographic latitude and the geocentric latitude can be found from

$$\varphi - \varphi' = 692''\!.73 \sin 2\varphi - 1''\!.16 \sin 4\varphi$$

The difference $\varphi - \varphi'$ reaches a maximum value for $u = 45°$. If φ_0 and φ_0' are the corresponding geographic and geocentric latitudes, we have

$$\tan \varphi_0 = \frac{a}{b} \qquad \tan \varphi_0' = \frac{b}{a} \qquad \varphi_0 + \varphi_0' = 90°$$

whence, for the IAU 1976 ellipsoid,

$$\varphi_0 = 45°05'46''\!.36 \qquad \varphi_0' = 44°54'13''\!.64$$

$$\varphi_0 - \varphi_0' = 11'32''\!.73$$

The quantity ρ (for sea level) can be found from

$$\rho = 0.998\,3271 + 0.001\,6764 \cos 2\varphi - 0.000\,0035 \cos 4\varphi$$

The parallel of latitude φ is a circle whose radius is

$$R_p = \frac{a \cos \varphi}{\sqrt{1 - e^2 \sin^2\varphi}}$$

where, as above, e is the eccentricity of the meridian ellipse.

Hence, one degree of longitude, at latitude φ, corresponds to a length of

$$\frac{\pi}{180} R_p$$

The rotational angular velocity of the Earth (with respect to the stars, *not* with respect to the moving vernal equinox) is

$$\omega = 7.292\,114\,992 \times 10^{-5} \text{ radian/second.}$$

Strictly speaking, this is the value at the epoch 1996.5 [1]. It decreases slowly with time because the rotation of the Earth is slowing down — see Chapter 10.

The linear velocity of a point at latitude φ, due to the rotation of the Earth, is ωR_p per second.

The radius of curvature of the Earth's meridian, at latitude φ, is

$$R_m = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2\varphi)^{3/2}}$$

and one degree of latitude corresponds to a length of $\frac{\pi}{180} R_m$.

R_m reaches a minimum value at the equator, $a\,(1 - e^2) = 6335.44$ km, and a maximum value at the poles, $a / \sqrt{1 - e^2} = 6399.60$ kilometers.

Example 11.b — For $\varphi = +42°$, the latitude of Chicago, we find

$R_p = 4747.001$ km

$1°$ of longitude $= 82.8508$ km

linear velocity $= \omega R_p = 0.34616$ km/second

$R_m = 6364.033$ km

$1°$ of latitude $= 111.0733$ km

Distance between two points on the Earth's surface

If the geographic coordinates of two points on the surface of the Earth are known, the shortest distance s between these points, measured along the Earth's surface, can be calculated. Let the first point having longitude and latitude L_1 and φ_1, respectively. Let L_2 and φ_2 be the coordinates of the second point. We will suppose that these points are at sea level.

If no great accuracy is needed, we may consider the Earth as being spherical with a mean radius of 6371 kilometers. Find the angular distance d between the two points by means of the formula

$$\cos d = \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos(L_1 - L_2) \qquad (11.1)$$

which is similar to formula (17.1) for the angular separation between two celestial bodies. Formula (11.1) does not work well when d is very small — see Chapter 17. Then the required linear distance is

$$s = \frac{6371\,\pi d}{180} \text{ kilometers} \qquad (11.2)$$

where d is expressed in degrees.

Higher accuracy is obtained by the following method, due to H. Andoyer [2]; the relative error of the result is of the order of the square of the Earth's flattening.

As before, let a be the equatorial radius of the Earth, and f the flattening. Then calculate

$$F = \frac{\varphi_1 + \varphi_2}{2} \qquad G = \frac{\varphi_1 - \varphi_2}{2} \qquad \lambda = \frac{L_1 - L_2}{2}$$

$$S = \sin^2 G \cos^2 \lambda + \cos^2 F \sin^2 \lambda$$
$$C = \cos^2 G \cos^2 \lambda + \sin^2 F \sin^2 \lambda$$

$$\tan \omega = \sqrt{\frac{S}{C}}$$

$$R = \frac{\sqrt{SC}}{\omega} \qquad \text{where } \omega \text{ is expressed in radians}$$

$$D = 2\omega a \qquad H_1 = \frac{3R - 1}{2C} \qquad H_2 = \frac{3R + 1}{2S}$$

and the required distance will be

$$s = D\,(1 + fH_1 \sin^2 F \cos^2 G - fH_2 \cos^2 F \sin^2 G)$$

Example 11.c — Calculate the geodesic distance between the Observatoire de Paris (France) and the U.S. Naval Observatory at Washington, D.C., adopting the following coordinates:

Paris:	L_1	= 2°20′14″ East	=	−2°20′14″
	φ_1	= 48°50′11″ North	=	+48°50′11″
Washington:	L_2	= 77°03′56″ West	=	+77°03′56″
	φ_2	= 38°55′17″ North	=	+38°55′17″

We find successively

F	+43°.878 8889
G	+ 4°.957 5000
λ	−39°.701 3889
S	0.216 426 96
C	0.783 573 04
ω	27°.724 274 = 0.483 879 87 radian
R	0.851 0555
D	6172.507 km

and finally $s = 6181.63$ kilometers, with a possible error of the order of 50 meters.

If we use the approximate expressions (11.1) and (11.2), we obtain

$$\begin{aligned} \cos d &= 0.567\,146 \\ d &= 55^\circ\!.448\,55 \\ s &= 6166 \text{ km} \end{aligned}$$

REFERENCES

1. International Earth Rotation Service, *Annual Report* for 1996 (Observatoire de Paris, 1997).
2. *Annuaire du Bureau des Longitudes* pour 1950 (Paris), page 145.

Chapter 12

Sidereal Time at Greenwich

We shall denote by Θ_0 the sidereal time at Greenwich at 0^h UT of a given date, and by θ_0 the sidereal time at Greenwich for any given instant UT.

The sidereal time at the meridian of Greenwich, at 0^h Universal Time of a given date, can be obtained as follows.

Calculate the JD corresponding to that date at 0^h UT (Chapter 7). Thus, this is a number ending on .5. Then find T by

$$T = \frac{\mathrm{JD} - 2451\,545.0}{36525} \tag{12.1}$$

The *mean* sidereal time at Greenwich at 0^h UT is then given by the following expression which was adopted in 1982 by the International Astronomical Union:

$$\begin{aligned} \Theta_0 = {} & 6^h41^m50\overset{s}{.}54841 + 8640\,184\overset{s}{.}812\,866\,T \\ & + 0\overset{s}{.}093\,104\,T^2 - 0\overset{s}{.}000\,0062\,T^3 \end{aligned} \tag{12.2}$$

Expressed in *degrees* and decimals, this formula can be written

$$\begin{aligned} \Theta_0 = {} & 100.460\,618\,37 + 36\,000.770\,053\,608\,T \\ & + 0.000\,387\,933\,T^2 - T^3/38\,710\,000 \end{aligned} \tag{12.3}$$

Important: the formulae (12.2) and (12.3) are valid only for those values of T which correspond to 0^h UT of a date. All other values would give incorrect results.

To obtain the sidereal time θ_0 at Greenwich for any instant UT of a given date, multiply that instant by 1.002 737 909 35 and add the result to the sidereal time Θ_0 at 0^h UT.

The mean sidereal time at Greenwich, expressed in *degrees*, can also be found directly for any instant as follows. If JD is the Julian Day corresponding to that instant in UT (not necessarily 0^h), find T by formula (12.1), and then

$$\theta_0 = 280.460\,618\,37 + 360.985\,647\,366\,29\,(\text{JD} - 2451\,545.0) + 0.000\,387\,933\,T^2 - T^3/38\,710\,000 \tag{12.4}$$

If high accuracy is needed, this formula requires the use of a computer language working with a sufficient number of significant digits.

The sidereal time obtained by formulae (12.2), (12.3), or (12.4) is the *mean* sidereal time, that is, the Greenwich hour angle of the mean vernal point (the intersection of the ecliptic of the date with the mean equator of the date).

The *apparent* sidereal time, or the Greenwich hour angle of the true vernal equinox, is obtained by adding the correction $\Delta\psi \cos \varepsilon$, where $\Delta\psi$ is the nutation in longitude, and ε the true obliquity of the ecliptic (see Chapter 22). This correction for nutation is called the *nutation in right ascension* or *equation of the equinoxes*. Because $\Delta\psi$ is a small quantity, the value of ε may be taken to the nearest 10″ here.

If $\Delta\psi$ is expressed in arcseconds (seconds of a degree), the correction in seconds of time is

$$\frac{\Delta\psi \cos \varepsilon}{15}$$

Example 12.a — Find the mean and the apparent sidereal time at Greenwich on 1987 April 10 at 0^h UT.

This date corresponds to JD 2446 895.5, and formula (12.1) gives

$$T = -0.127\,296\,372\,348$$

We then find by means of formula (12.2)

$$\Theta_0 = 6^h41^m50^s.54841 - 1099\,864.18158 \text{ seconds}$$

or, by adding a convenient multiple of 86400 seconds (the number of seconds in one day),

$$\begin{aligned}\Theta_0 &= 6^h41^m50^s.54841 + 23335^s.81842\\ &= 6^h41^m50^s.54841 + 6^h28^m55^s.81842\\ &= 13^h10^m46^s.3668\end{aligned}$$

which is the required mean sidereal time.

From Example 22.a we have, for the same instant, $\Delta\psi = -3''.788$ and $\varepsilon = 23°26'36''.85$. [In fact, these values are for 0^h TD, not for 0^h UT, but here we will neglect the very small variation of $\Delta\psi$ during the time interval $\Delta T = \text{TD} - \text{UT}$.]

Hence the nutation in right ascension is $\dfrac{-3.788}{15} \cos 23°.44357 = -0^s.2317$, and the required apparent sidereal time is

$$13^h10^m46''.3668 - 0^s.2317 = 13^h10^m46^s.1351$$

Example 12.b — Find the mean sidereal time at Greenwich on 1987 April 10 at $19^h21^m00^s$ Universal Time.

First, we calculate the mean sidereal time for that date at 0^h UT. We find $13^h10^m46\overset{s}{.}3668$ (see the previous Example). Then

$$\begin{aligned} & 1.002\,737\,909\,35 \times 19^h21^m00^s \\ = \; & 1.002\,737\,909\,35 \times 69\,660 \text{ seconds} \\ = \; & 69\,850.7228 \text{ seconds} \\ = \; & 19^h24^m10\overset{s}{.}7228 \end{aligned}$$

and the required sidereal time is

$$\begin{aligned} 13^h10^m46\overset{s}{.}3668 + 19^h24^m10\overset{s}{.}7228 &= 32^h34^m57\overset{s}{.}0896 \\ &= 8^h34^m57\overset{s}{.}0896 \end{aligned}$$

Alternatively, we may use formula (12.4). The Julian Day corresponding to 1987 April 10 at $19^h21^m00^s$ UT is

$$\text{JD} = 2446\,896.30625$$

and, by (12.1), the corresponding value of T is $-0.127\,274\,30$. Formula (12.4) then gives

$$\theta_0 = -1677\,831.262\,1266 \text{ degrees}$$

or, by adding a convenient multiple of 360°,

$$\theta_0 = 128\overset{\circ}{.}737\,8734$$

This is the required mean sidereal time in degrees. We obtain it in hours by dividing it by 15 (since one hour corresponds to 15 degrees):

$$\theta_0 = 8\overset{h}{.}582\,524\,89 = 8^h34^m57\overset{s}{.}0896,$$

the same result as above.

Chapter 13

Transformation of Coordinates

We will use the following symbols:

α = right ascension. This quantity is generally expressed in hours, minutes, and seconds of time, and hence should first be converted into degrees (and decimals) and then, if necessary, into radians, before it is used in a formula. Conversely, if α has been obtained by means of a formula and a programming language, it is expressed in radians or in degrees; it may be converted to hours by division of the degrees by 15, and then, if necessary, be converted into hours, minutes, and seconds;

δ = declination, positive if north of the celestial equator, negative if south;

α_{1950} = right ascension referred to the standard equinox of B1950.0;

δ_{1950} = declination referred to the standard equinox of B1950.0;

α_{2000} = right ascension referred to the standard equinox of J2000.0;

δ_{2000} = declination referred to the standard equinox of J2000.0;

λ = ecliptical (or celestial) longitude, measured from the vernal equinox along the ecliptic;

β = ecliptical (or celestial) latitude, positive if north of the ecliptic, negative if south;

l = galactic longitude;

b = galactic latitude;

h = altitude, positive above the horizon, negative below;

A = azimuth, measured westward from the *South*. Note that navigators and meteorologists count the compass direction, or azimuth, from the North (0°), through the East (90°), South (180°), and West (270°). But astronomers disagree (see the box on next page) and we shall measure the azimuth from the South, because the hour angles too are measured from the South, at least for observers in the northern hemisphere. Hence, a celestial body which is exactly on the southern meridian has $A = H = 0°$;

The azimuth: from the North or from the South?

William Chauvenet, on page 20 of his *Manual of Spherical and Practical Astronomy* (5^{th} edition, 1891), Vol. I, wrote: "The origin from which azimuths are reckoned is arbitrary; so also is the direction in which they are reckoned; but astronomers usually take the *south* point of the horizon as the origin,... Navigators, however, usually reckon the azimuth from the north or south points, according as they are in north or south latitude."

S. Newcomb, on p. 95 of his *Compendium of Spherical Astronomy*: "in practice it is measured either from the north or the south point, and in either direction, east or west." — so this great American astronomer had no specific preference.

A. Danjon, on p. 39 of his excellent *Astronomie Générale* (Paris, 1959): "Le point S, origine des azimuts, (...) est l'intersection du méridien et de l'horizon, au sud."

ε = obliquity of the ecliptic; this is the angle between the ecliptic and the celestial equator. The mean obliquity of the ecliptic is given by formula (22.2). If, however, the *apparent* right ascension and declination are used (that is, affected by the aberration and the nutation), the true obliquity $\varepsilon + \Delta\varepsilon$ should be used (see Chapter 22). If α and δ are referred to the standard equinox of J2000.0, then the value of ε for that epoch should be used, namely $\varepsilon_{2000} = 23°26'21''.448 = 23°.439\,2911$. For the standard equinox of B1950.0, we have $\varepsilon_{1950} = 23°.445\,7889$;

φ = the observer's latitude, positive if in the northern hemisphere, negative in the southern one;

H = the local hour angle, measured westwards from the South.

If θ is the local sidereal time, θ_0 the sidereal time at Greenwich, and L the observer's longitude (positive west, negative east from Greenwich), then the local hour angle can be calculated from

$$H = \theta - \alpha \qquad \text{or} \qquad H = \theta_0 - L - \alpha$$

If α is affected by the nutation, then the sidereal time too must be affected by it (see Chapter 12).

For the transformation from equatorial into ecliptical coordinates, the following formulae can be used:

Note on the geographic longitudes

In this work, the geographic longitudes are measured ***positively westwards*** from the meridian of Greenwich, and negatively to the east. This convention has been followed by most astronomers during more than one century — see for instance References 1 to 6. For example, the longitude of Washington, D.C., is +77°04′; that of Vienna, Austria, is −16°23′.

We cannot understand why the International Astronomical Union, having first decided to measure all planetographic longitudes in the direction opposite to that of rotation, then alters the system for the Earth (1982). We shall *not* follow this IAU resolution, and we shall continue to consider *west* longitudes as positive. This is in conformity with the longitude systems on the other planets. On Mars and Jupiter, for instance, the longitudes *are* measured positively to the west, and this is why the longitude of their central meridian, as seen from the Earth, is *increasing* with time.

$$\tan \lambda = \frac{\sin \alpha \cos \varepsilon + \tan \delta \sin \varepsilon}{\cos \alpha} \tag{13.1}$$

$$\sin \beta = \sin \delta \cos \varepsilon - \cos \delta \sin \varepsilon \sin \alpha \tag{13.2}$$

Transformation from ecliptical into equatorial coordinates:

$$\tan \alpha = \frac{\sin \lambda \cos \varepsilon - \tan \beta \sin \varepsilon}{\cos \lambda} \tag{13.3}$$

$$\sin \delta = \sin \beta \cos \varepsilon + \cos \beta \sin \varepsilon \sin \lambda \tag{13.4}$$

Calculation of the local horizontal coordinates:

$$\tan A = \frac{\sin H}{\cos H \sin \varphi - \tan \delta \cos \varphi} \tag{13.5}$$

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos H \tag{13.6}$$

If one wishes to reckon the azimuth from the North instead of the South, add 180° to the value of A given by formula (13.5).

Transformation from horizontal into equatorial coordinates:

$$\tan H = \frac{\sin A}{\cos A \sin \varphi + \tan h \cos \varphi}$$

$$\sin \delta = \sin \varphi \sin h - \cos \varphi \cos h \cos A$$

The current galactic system of coordinates has been defined by the International Astronomical Union in 1959. In the standard equatorial system of B1950.0, the galactic (Milky Way) North Pole has the coordinates

$$\alpha_{1950} = 12^{h}49^{m} = 192^{\circ}\!.25, \qquad \delta_{1950} = +27^{\circ}\!.4$$

and the origin of the galactic longitudes is the point (in western Sagittarius) of the galactic equator which is 33° distant from the ascending node (in western Aquila) of the galactic equator with the equator of B1950.0.

These values have been fixed *conventionally* and therefore must be considered as *exact* for the mentioned equinox of B1950.0.

Transformation from equatorial coordinates, referred to the standard equinox of B1950.0, into galactic coordinates:

$$\tan x = \frac{\sin (192^{\circ}\!.25 - \alpha)}{\cos (192^{\circ}\!.25 - \alpha) \sin 27^{\circ}\!.4 - \tan \delta \cos 27^{\circ}\!.4} \tag{13.7}$$

$$l = 303^{\circ} - x$$

$$\sin b = \sin \delta \sin 27^{\circ}\!.4 + \cos \delta \cos 27^{\circ}\!.4 \cos (192^{\circ}\!.25 - \alpha) \tag{13.8}$$

Transformation from galactic coordinates into equatorial coordinates referred to the standard equinox of B1950.0:

$$\tan y = \frac{\sin (l - 123^{\circ})}{\cos (l - 123^{\circ}) \sin 27^{\circ}\!.4 - \tan b \cos 27^{\circ}\!.4}$$

$$\alpha = y + 12^{\circ}\!.25$$

$$\sin \delta = \sin b \sin 27^{\circ}\!.4 + \cos b \cos 27^{\circ}\!.4 \cos (l - 123^{\circ})$$

If the 2000.0 mean place of the star is given instead of the 1950.0 mean place, then, before using formulae (13.7) and (13.8), convert α_{2000} and δ_{2000} to α_{1950} and δ_{1950}. See Chapter 21.

The formulae (13.1), (13.3), etc., give $\tan \lambda$, $\tan \alpha$, etc., and then λ, α, etc., by the function arctangent. However, the exact quadrant in which the angle is situated is then unknown. To remove the ambiguity of 180°, apply the ATN2 function to the numerator and the denominator of the function (instead of performing the actual division), or use another trick. See "The correct quadrant" in Chapter 1.

Example 13.a — Calculate the ecliptical coordinates of the star Pollux (β Gem), whose equatorial coordinates are

$$\alpha_{2000} = 7^h45^m18^s.946, \qquad \delta_{2000} = +28°01'34''.26.$$

Using the values $\alpha = 116°.328\,942$, $\delta = +28°.026\,183$, and $\varepsilon = 23°.439\,2911$, formulae (13.1) and (13.2) give

$$\tan\lambda = \frac{+1.034\,039\,86}{-0.443\,523\,98} \qquad \text{whence } \lambda = 113°.215\,630;$$

$$\beta = +6°.684\,170.$$

Because α and δ are referred to the standard equinox of 2000.0, λ and β too are referred to that equinox.

Exercise. — Using the values of λ and β found above, find α and δ again by means of formulae (13.3) and (13.4).

Example 13.b — Find the azimuth and the altitude of Venus on 1987 April 10 at $19^h21^m00^s$ UT at the U.S. Naval Observatory at Washington, D.C. (longitude = $+77°03'56'' = +5^h08^m15^s.7$, latitude = $+38°55'17''$).

The planet's apparent equatorial coordinates, interpolated from an ephemeris, are

$$\alpha = 23^h09^m16^s.641, \qquad \delta = -6°43'11''.61$$

These are the *apparent* right ascension and declination of the planet. So we need the *apparent* sidereal time for the given instant.

We first calculate the *mean* sidereal time at Greenwich on 1987 April 10 at $19^h21^m00^s$ UT, and find $8^h34^m57^s.0896$ (see Example 12.b).

By means of the method described in Chapter 22, we find for the same instant:

nutation in longitude: $\Delta\psi = -3''.868$
true obliquity of the ecliptic: $\varepsilon = 23°26'36''.87$

The apparent sidereal time at Greenwich is

$$\theta_0 = 8^h34^m57^s.0896 + \left(\frac{-3.868}{15}\cos\varepsilon\right) \text{seconds} = 8^h34^m56^s.853$$

Hour angle of Venus at Washington:

$$\begin{aligned} H &= \theta_0 - L - \alpha \\ &= 8^h34^m56^s.853 - 5^h08^m15^s.7 - 23^h09^m16^s.641 \\ &= -19^h42^m35^s.488 = -19^h.709\,8578 = -295°.647\,867 \\ &= +64°.352\,133 \end{aligned}$$

Formulae (13.5) and (13.6) then give

$$\tan A = \frac{+0.901\,4712}{+0.363\,6015} \qquad \text{whence} \quad A = +68^\circ\!.0337$$

$$h = +15^\circ\!.1249$$

so the planet is 15 degrees above the horizon between the southwest and the west.

Note that formula (13.6) does not take into account the effect of the atmospheric refraction, nor that of the planet's parallax, nor the dip of the horizon. For the atmospheric refraction, see Chapter 16. The correction for parallax is dealt with in Chapter 40.

As an exercise, find the galactic coordinates of Nova Serpentis 1978, whose equatorial coordinates are

$$\alpha_{1950} = 17^h48^m59^s\!.74, \qquad \delta_{1950} = -14^\circ43'08''\!.2$$

Answer: $l = 12^\circ\!.9593$, $b = +6^\circ\!.0463$.

REFERENCES

1. *The Nautical Almanac and Astronomical Ephemeris for the year 1835*, p. 508 (London, 1833).
2. *The American Ephemeris and Nautical Almanac for the Year 1857*, p. 491 (Washington, 1854).
3. *The Astronomical Ephemeris for the Year 1960*, pp. 434 & fol. (London, 1958).
4. W. Chauvenet, *A Manual of Spherical and Practical Astronomy*, Vol. I, pp. 317 & fol. (Philadelphia, 1891).
5. A. Danjon, *Astronomie Générale*, p. 46 (Paris, 1959).
6. S. Newcomb, *A Compendium of Spherical Astronomy*, p. 119 (New York, 1906).

Chapter 14

The Parallactic Angle, and three other Topics

Suppose that on a bright morning we are looking at the Sun through a piece of dark glass, and that we see a large sunspot near the western ("right") limb of the Sun (Figure 1, A). At noon, the Sun being near the southern meridian, we notice that the spot is lower (Figure 1, B). And in the afternoon, we see that the spot has moved still farther along the Sun's limb (Figure 1, C).

The spot did not actually move that much over the solar disk. It is the whole image of the Sun which rotated clockwise. This can be seen easier with the Moon (Figure 2).

This apparent rotation is easily understood when we consider the diurnal motion of the celestial sphere. Each celestial body describes a parallel circle, a diurnal arc (Figure 3). Only when the Sun (or the Moon) is exactly on the southern meridian, will the celestial north be up, in the direction of the zenith.

The constellations show a similar effect. For an observer in the northern hemisphere of the Earth, the constellation of Orion is inclined to the "left" in the southeast, is upright in the south, and is inclined to the "right" in the southwest.

In Figure 4, the circle represents the disk of the Sun (or that of the Moon). The arc AB is a part of its diurnal arc on the celestial sphere. C is the center of the disk. The direction of the zenith and that of the celestial North are indicated. The latter direction is perpendicular to the arc AB. Z is the *zenith point* of the disk; it is the uppermost point of the disk at the sky as seen by the observer at the given instant.

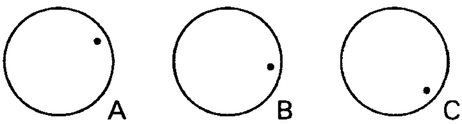


Fig. 1 : The apparent displacement of a sunspot in the course of the day: in the morning (A), near noon (B), and in the afternoon (C). In each of the three sketches, the circle represents the solar disk, and the zenith is at the top.

Fig. 2 : The First-Quarter Moon for an observer in the northern hemisphere: (A) near the south, around the time of sunset; and (B) later that evening. The zenith is up.

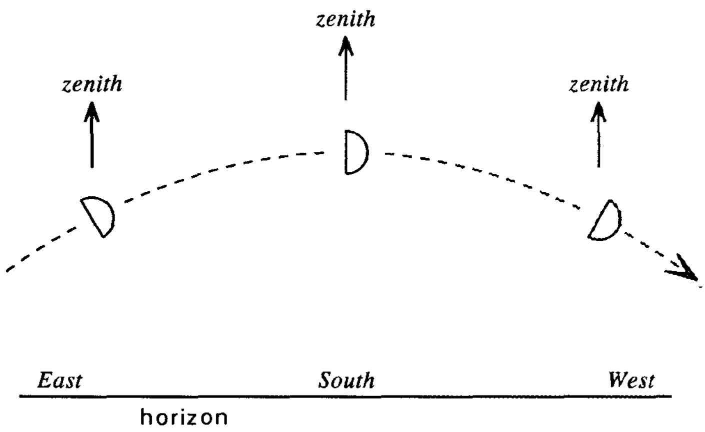


Fig. 3

N is the *north point* of the disk; the direction CN points towards the northern celestial pole.

The angle ZCN is called the *parallactic angle* and is generally designed by q. This parallactic angle has absolutely nothing to do with the parallax! The name arises from the fact that the celestial body moves along a *parallel* circle. Compare with the "parallactic" mounting of a telescope.

By convention, the angle q is negative before, and positive after the passage through the southern meridian. Exactly on the meridian, we have $q = 0°$.

The parallactic angle q can be calculated from

$$\tan q = \frac{\sin H}{\tan \varphi \cos \delta - \sin \delta \cos H} \qquad (14.1)$$

where, as in the preceding Chapter, φ is the geographical latitude of the observer, δ the declination of the celestial body, and H its hour angle at the given instant.

Exactly in the zenith, the angle q is not defined. Indeed, in that case we have $H = 0°$ and $\delta = \varphi$, so formula (14.1) yields $\tan q = 0/0$. This can be compared with somebody who is exactly at the North Pole of the Earth: his geographical longitude is not defined, because all meridians of the Earth converge to his place. For that special observer, all points of the horizon are in the southern direction!

When a celestial body passes exactly through the zenith, the parallactic angle q suddenly jumps from $-90°$ to $+90°$.

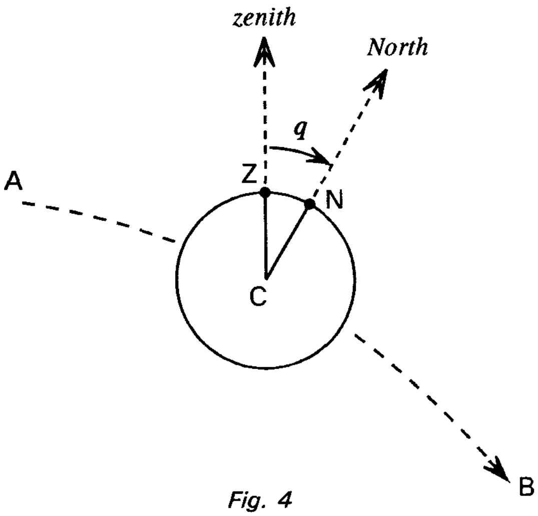


Fig. 4

If the celestial body is on the horizon (hence rising or setting), formula (14.1) simplifies greatly, namely

$$\cos q = \frac{\sin \varphi}{\cos \delta}$$

and in that case it is not necessary to know the value of the hour angle.

Ecliptic and Horizon

If ε is the obliquity of the ecliptic, φ the latitude of the observer, and θ the local sidereal time, then the longitudes of the two points of the ecliptic which are (180 degrees apart) on the horizon, are given by

$$\tan \lambda = \frac{-\cos \theta}{\sin \varepsilon \tan \varphi + \cos \varepsilon \sin \theta} \qquad (14.2)$$

The angle I between the ecliptic and the horizon is given by

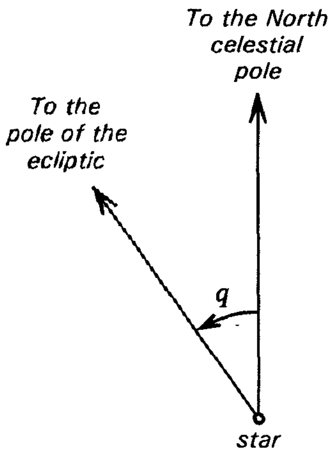
$$\cos I = \cos \varepsilon \sin \varphi - \sin \varepsilon \cos \varphi \sin \theta \qquad (14.3)$$

Note that I is *not* the angle which the daily path of the Sun makes with the horizon! In the course of one sidereal day, the angle I varies between two extreme values. For example, for latitude 48°00′ North, with ε = 23°26′, the extreme values of I are

$$90° - \varphi + \varepsilon = 65°26' \quad \text{for } \theta = 90°$$
$$90° - \varphi - \varepsilon = 18°34' \quad \text{for } \theta = 270°$$

Example 14.a — For $\varepsilon = 23\overset{\circ}{.}44$, $\varphi = +51°$, $\theta = 5^h00^m = 75°$, we find, from formula (14.2), $\tan \lambda = -0.1879$, whence $\lambda = 169°21'$ and $\lambda = 349°21'$. Formula (14.3) gives $I = 62°$.

Ecliptic and Equator



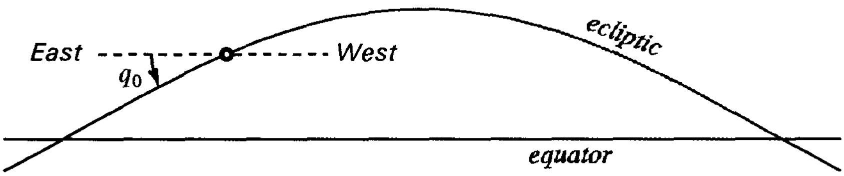
Let λ, β be the ecliptical longitude and latitude of a star, and ε the obliquity of the ecliptic. Then, the angle q between the direction of the northern celestial pole and the direction of the north pole of the ecliptic, at the star (see Figure at right), is given by

$$\tan q = \frac{\cos\lambda \tan\varepsilon}{\sin\beta \sin\lambda \tan\varepsilon - \cos\beta}$$

If in this formula we make $\beta = 0°$, then the formula reduces to

$$\tan q_0 = -\cos\lambda \tan\varepsilon$$

and q_0 is the angle between the ecliptic (at a given point of longitude λ) and the east-west direction on the celestial sphere — see the Figure below. This angle may be of importance when preparing a diagram showing the path of the Moon through the Earth's shadow during a lunar eclipse.



Diurnal path and Horizon

The angle J of the diurnal path of a celestial body (*not* the ecliptic) relative to the horizon at the time of its rising or setting can be found from

$$B = \tan\delta \tan\varphi, \qquad C = \sqrt{1 - B^2}, \qquad \tan J = C \cos\delta / \tan\varphi$$

where δ is the declination of the body, and φ the observer's latitude. In these formulae, the declination of the body is supposed to be constant, and the atmospheric refraction is neglected. When $\delta = 0°$, then $J = 90° - \varphi$.

For example, for the Sun at latitude 40° (north or south), J varies between 50° at the equinoxes and 45°31′ at the solstices.

The error in J due to neglecting the variation of the declination will be at most 4′ in the case of the Sun. For the Moon, the error can exceed 1 degree.

Chapter 15

Rising, Transit, and Setting

The local hour angle corresponding to the time of rise or set of a celestial body is obtained by putting $h = 0$ in formula (13.6). This gives

$$\cos H_0 = -\tan \varphi \tan \delta$$

However, the instant so obtained refers to the geometric rise or set of the center of the celestial body. By reason of the atmospheric refraction, the body is actually below the horizon at the instant of its apparent rise or set. The value of $0°34'$ is generally adopted for the effect of refraction at the horizon. For the Sun, the calculated times generally refer to the apparent rise or set of the upper limb of the disk; hence, $0°16'$ should be added for the semidiameter.

Actually, the amount of refraction changes with air temperature, pressure, and the elevation of the observer (see Chapter 16). A change of temperature from winter to summer can shift the times of sunrise and sunset by about 20 seconds in mid-northern and mid-southern latitudes. Similarly, observing sunrise or sunset over a range of barometric pressures leads to a variation of a dozen seconds in the times. However, in this Chapter we shall use a mean value for the atmospheric refraction at the horizon, namely, the value of $0°34'$ mentioned above.

We will use the following symbols:

L = geographic longitude of the observer in degrees, measured *positively west from Greenwich*, negatively to the east (see Chapter 13);

φ = geographic latitude of the observer, positive in the northern hemisphere, negative in the southern hemisphere;

ΔT = the difference TD − UT between Dynamical Time and Universal Time, in *seconds* of time;

h_0 = the "standard" altitude, *i.e.*, the geometric altitude of the center of the body at the time of apparent rising or setting, namely,

$$h_0 = -0°34' = -0\overset{\circ}{.}5667 \quad \text{for stars and planets;}$$
$$h_0 = -0°50' = -0\overset{\circ}{.}8333 \quad \text{for the Sun.}$$

For the Moon, the problem is more complicated because h_0 is not constant. Taking into account the variations of semidiameter and parallax, we have $h_0 = 0.7275\pi - 0°34'$, where π is the Moon's horizontal parallax. If no great accuracy is required, the mean value $h_0 = +0°.125$ can be used for the Moon.

Suppose we wish to calculate the times, in *Universal Time*, of rising, of transit (when the body crosses the local meridian at upper culmination), and of setting of a celestial body at a given place on a given date D. We take the following values from an almanac, or we calculate them ourselves with a computer program:

— the apparent sidereal time Θ_0 at 0^h *Universal Time* on day D for the meridian of Greenwich, converted into *degrees*;

— the apparent right ascensions and declinations of the body

α_1 and δ_1	on day $D-1$	at 0^h Dynamical Time
α_2 and δ_2	on day D	—
α_3 and δ_3	on day $D+1$	—

The right ascensions should be expressed in *degrees*, too.

We first calculate *approximate* times as follows.

$$\cos H_0 = \frac{\sin h_0 - \sin\varphi \sin\delta_2}{\cos\varphi \cos\delta_2} \tag{15.1}$$

Attention! First test if the second member is between -1 and $+1$ before calculating H_0. See Note 2 at the end of this Chapter.

Express H_0 in degrees. H_0 should be taken between $0°$ and $+180°$. Then we have:

$$\left.\begin{array}{lll} \text{for the transit:} & m_0 = \dfrac{\alpha_2 + L - \Theta_0}{360} \\[2ex] \text{for the rising:} & m_1 = m_0 - \dfrac{H_0}{360} \\[2ex] \text{for the setting:} & m_2 = m_0 + \dfrac{H_0}{360} \end{array}\right\} \tag{15.2}$$

These three values m are times, on day D, expressed as fractions of a day. Hence, they should be between 0 and $+1$. If one or more of them are outside of this range, add or subtract 1. For instance, $+0.3744$ should remain unchanged, but -0.1709 should be changed to $+0.8291$, and $+1.1853$ should be changed to $+0.1853$.

Now, for *each* of the three m-values *separately*, perform the following calculation.

Find the sidereal time at Greenwich, in *degrees*, from

$$\theta_0 = \Theta_0 + 360.985\,647\,m$$

where m is either m_0, m_1, or m_2.

For $n = m + \Delta T/86400$, interpolate α from α_1, α_2, α_3 and δ from δ_1, δ_2, δ_3, using the interpolation formula (3.3). For the calculation of the time of transit, δ is not needed.

Find the local hour angle of the body from $H = \theta_0 - L - \alpha$, and then the body's altitude h by means of formula (13.6). This altitude is not needed for the calculation of the time of transit.

Then the correction to m will be found as follows:

— in the case of a transit,

$$\Delta m = -\frac{H}{360}$$

where H is expressed in degrees and *must* be between -180 and $+180$ degrees. (In most cases, H will be a small angle and be between $-1°$ and $+1°$);

— in the case of a rising or a setting,

$$\Delta m = \frac{h - h_0}{360 \cos\delta \cos\varphi \sin H}$$

where h and h_0 are expressed in degrees.

The corrections Δm are small quantities, in most cases being between -0.01 and $+0.01$. The corrected value of m is then $m + \Delta m$. If necessary, a new calculation should be performed using the new value of m.

At the end of the calculation, each value of m should be converted into hours by multiplication by 24.

Example 15.a — Venus on 1988 March 20 at Boston,

$$\text{longitude} = +71°05' = +71°.0833,$$
$$\text{latitude} = +42°20' = +42°.3333.$$

From an accurate ephemeris, we take the following values:

1988 March 20, 0^h UT : $\Theta_0 = 11^h50^m58^s.10 = 177°.74208$

Coordinates of Venus at 0^h TD:

March 19	$\alpha_1 = 2^h42^m43^s.25 = 40°.68021$	$\delta_1 = +18°02'51''.4 = +18°.04761$
March 20	$\alpha_2 = 2\ 46\ 55.51 = 41.73129$	$\delta_2 = +18\ 26\ 27.3 = +18.44092$
March 21	$\alpha_3 = 2\ 51\ 07.69 = 42.78204$	$\delta_3 = +18\ 49\ 38.7 = +18.82742$

We take $h_0 = -0^\circ\!.5667$, $\Delta T = +56$ seconds, and find by formula (15.1) $\cos H_0 = -0.317\,8735$, $H_0 = 108^\circ\!.5344$, whence the approximate values:

transit :	$m_0 = -0.18035$, whence $m_0 = +0.81965$
rising :	$m_1 = m_0 - 0.30148 = +0.51817$
setting :	$m_2 = m_0 + 0.30148 = +1.12113$, whence $m_2 = +0.12113$

Calculation of more exact times:

	rising	*transit*	*setting*
m	+0.51817	+0.81965	+0.12113
θ_0	4°.79401	113°.62397	221°.46827
n	+0.51882	+0.82030	+0.12178
inter- polation { α	42°.27648	42°.59324	41°.85927
inter- polation { δ	+18°.64229		+18°.48835
H	−108°.56577	−0°.05257	+108°.52570
h	−0°.44393		−0°.52711
Δm	−0.00051	+0.00015	+0.00017
corrected m	+0.51766	+0.81980	+0.12130

A new calculation, using these new values of m, yields the new corrections $-0.000\,003$, $-0.000\,004$, and $-0.000\,004$, respectively, which can be neglected. So we have, finally:

rising :	$m_1 = +0.51766$,	$24^h \times 0.51766 = 12^h25^m$ UT
transit :	$m_0 = +0.81980$,	$24^h \times 0.81980 = 19^h41^m$ UT
setting :	$m_2 = +0.12130$,	$24^h \times 0.12130 = 2^h55^m$ UT

Notes

1. In Example 15.a we found that at Boston the time of setting was 2^h55^m UT on March 20. However, converted to *local* standard time this corresponds to an instant on the evening of the previous day! If really the time of setting on March 20 is needed in local time, the calculation should be performed using the value $m_2 = +1.12113$ first found, instead of $+0.12113$.

2. If the body is circumpolar, the second member of formula (15.1) will be larger than 1 in absolute value, and there will be no angle H_0. In such a case, the body will remain the whole day either above or below the horizon.

3. If *approximate* times are sufficient, just use the *initial* values m_0, m_1, and m_2 given by (15.2).

Chapter 16

Atmospheric Refraction

Atmospheric refraction is the bending of light while passing through the Earth's atmosphere. As a ray of light penetrates the atmosphere, it encounters layers of air of increasing density, resulting in the continuous bending of the light. As a result, a star (or the Sun's limb, etc.) will appear higher in the sky than its true position. The atmospheric refraction, which is zero in the zenith, increases towards the horizon. At an altitude of 45°, the refraction is about one arcminute; at the horizon, it amounts to about 35′. Thus, the Sun and the Moon are actually below the horizon when they appear to be rising or setting. Moreover, the rapidly changing refraction at low altitudes gives the rising or setting Sun its familiar oval appearance.

Allowance must be made for atmospheric refraction when determining positions, and one distinguishes two cases:

— the apparent altitude h_0 of a celestial body has been *measured*, and one should find the refraction R to be *subtracted* from h_0 to obtain the true altitude h;

— the true "airless" altitude h has already been *calculated* from celestial coordinates and formulae of spherical trigonometry, and we want to calculate the refraction R to be *added* to h in order to predict the apparent altitude h_0.

Almost all refraction formulae we have come across consider the first case only: they are designed for deriving true altitudes from observed ones. But here we will consider both cases.

For many purposes, "average" meteorological conditions may be assumed. However, anomalous refraction near the horizon, exemplified by distortions of the setting Sun, should remind us that rigorous exactness at very low altitudes cannot be reached.

When the altitude of the celestial body is larger than 15°, one of the following two formulae may be used, as the case may be:

$$R = 58''.294 \tan(90^\circ - h_0) - 0''.0668 \tan^3(90^\circ - h_0) \qquad (16.1)$$

$$R = 58''.276 \tan(90^\circ - h) - 0''.0824 \tan^3(90^\circ - h) \qquad (16.2)$$

The first formula was given by Smart [1], while the second one has been derived by us from the first formula. For altitudes below 15°, these expressions will give inaccurate, or even completely meaningless results.

It appears that, at high altitudes, the refraction is proportional to the tangent of the zenithal distance.

A surprisingly simple formula for refraction, with good accuracy at all altitudes from 90° to 0°, was given by G. G. Bennett of the University of New South Wales [2]. If the refraction R is expressed in *minutes* of arc, Bennett's formula is

$$R = \frac{1}{\tan\left(h_0 + \dfrac{7.31}{h_0 + 4.4}\right)} \tag{16.3}$$

where h_0 is the *apparent* altitude in degrees. According to Bennett, this formula is accurate to 0.07 arcminute for all values of h_0. The largest error, 0.07 arcminute, occurs at 12° altitude.

Note that for the zenith ($h_0 = 90°$) formula (16.3) yields $R = -0''.08$ instead of exactly zero. This can be rectified by adding +0.001 3515 to the second member of the formula.

Bennett also showed how his formula can be refined. Calculate R by means of formula (16.3); then a correction to R, expressed in minutes of arc, is

$$-0.06 \sin(14.7R + 13)$$

where the expression between parentheses is expressed in degrees. Calculated in this way, the maximum error is stated to be only 0.015 arcminute, or 0''.9, for the whole range 90°–0°. [At the zenith, one finds $R = -0''.89$, so expression (16.3), without further correction, is better in this case.]

For the inverse problem, that of calculating the effect of refraction when the *true* altitude h is known, Sæmundsson, of the University of Iceland, proposed the following formula [3]:

$$R = \frac{1.02}{\tan\left(h + \dfrac{10.3}{h + 5.11}\right)} \tag{16.4}$$

This formula is consistent with Bennett's (16.3) to within 4''. Again, it does not give exactly $R = 0$ for $h = 90°$. This can be remedied by adding +0.001 9279 to the second member.

Formulae (16.1) to (16.4) assume that the observation is made at sea level, when the atmospheric pressure is 1010 millibars, and when the air temperature is 10° Celsius. The effect of refraction increases when the pressure *increases* or when the temperature *decreases*.

If the pressure at the Earth's surface is P millibars, and the air temperature is

T degrees Celsius, then the values of R given by the formulae (16.1) to (16.4) should be multiplied by

$$\frac{P}{1010} \times \frac{283}{273 + T}$$

However, this is only approximately correct. The problem is more complicated because the refraction depends on the wave-length of the light too! The expressions given in this Chapter are for yellow light, where the human eye has maximum sensitivity.

Example 16.a — Calculate the apparent flattening of the solar disk near the horizon, when the lower limb is at an apparent altitude of exactly 0°30′. Assume a true solar diameter of exactly 0°32′, and mean conditions of air pressure and temperature.

For $h_0 = 0{.}^\circ 5$, formula (16.3) gives $R = 28'.754$, so the true altitude of the Sun's lower limb is

$$0°30' - 0°28'.754 = 0°01'.246$$

and hence the true altitude of the upper limb is

$$h = 0°01'.246 + 0°32' = 0°33'.246 = 0{.}^\circ 5541$$

For this value of h, formula (16.4) yields $R = 24'.618$, so the apparent altitude of the Sun's upper limb is $33'.246 + 24'.618 = 57'.864$, and the apparent vertical diameter of the solar disk is $57'.864 - 30' = 27'.864$.

Consequently, the ratio of the apparent vertical diameter to the horizontal diameter of the solar disk, under the conditions of this Problem, is $27.864/32 = 0.871$.

Note that, while of course the azimuth is unchanged by refraction, the *horizontal* diameter of the solar disk is very slightly contracted by reason of the refraction. This is due to the fact that the extremities of this diameter are raised along vertical circles that meet at the zenith. Danjon [4] writes that the apparent contraction of the horizontal diameter of the Sun is practically constant and independent of the altitude, and that this contraction is approximately 0″.6.

For heights of a few degrees the results of the formulae should be judged with care. Near the horizon unpredictable disturbances of the atmosphere become rather important. According to investigations by Schaefer and Liller [5], the refraction at the horizon fluctuates by $0{.}^\circ 3$ around a mean value normally, and in some cases apparently much more. Remembering our Chapter about accuracy, it should be mentioned here that giving rising or setting times of a celestial body more accurately than to the nearest minute makes no sense.

REFERENCES

1. W.M. Smart, *Text-Book on Spherical Astronomy*; Cambridge (England), University Press (1956); page 68.
2. G.G. Bennett, "The Calculation of Astronomical Refraction in Marine Navigation", *Journal of the Institute for Navigation*, Vol. 35, pages 255-259 (1982).
3. Þorsteinn Sæmundsson, *Sky and Telescope*, Vol. 72, page 70 (July 1986).
4. A. Danjon, *Astronomie Générale* (Paris, 1959); page 156.
5. B.E. Schaefer, W. Liller, "Refraction near the Horizon", *Publ. Astron. Society of the Pacific*, Vol. 102, pages 796-805 (July 1990).

Chapter 17

Angular Separation

The angular distance d between two celestial bodies whose right ascensions and declinations are known is given by the formula

$$\cos d = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos (\alpha_1 - \alpha_2) \tag{17.1}$$

where α_1 and δ_1 are the right ascension and declination of one body, and α_2 and δ_2 those of the other body. This distance is measured along the great circle joining the two bodies, which is the shortest possible arc between the two points.

The same formula may be used when the ecliptical (celestial) longitudes λ and latitudes β of the two bodies are given, provided that α_1, α_2, δ_1, and δ_2 are replaced by λ_1, λ_2, β_1, and β_2, respectively.

Formula (17.1) may not be used when d is very near to 0° or to 180° because in those cases $|\cos d|$ is nearly equal to 1 and varies very slowly with d, so that d cannot be found accurately. For instance,

$$\begin{aligned}
\cos 0°01'00'' &= 0.999\,999\,958 \\
\cos 0°00'30'' &= 0.999\,999\,989 \\
\cos 0°00'15'' &= 0.999\,999\,997 \\
\cos 0°00'00'' &= 1.000\,000\,000
\end{aligned}$$

If the angular separation is small, say less than 0°10′, then this separation may be calculated by means of the approximate formula

$$d = \sqrt{(\Delta\alpha \,.\, \cos \delta)^2 + (\Delta\delta)^2} \tag{17.2}$$

where $\Delta\alpha$ is the difference between the right ascensions, $\Delta\delta$ the difference between the declinations, while δ is the average of the declinations of the two bodies. Note that $\Delta\alpha$ and $\Delta\delta$ should be expressed in the same angular units.

If $\Delta\alpha$ is expressed in hours (and decimals), $\Delta\delta$ in degrees (and decimals), then d expressed in seconds of a degree (″) is given by

$$d = 3600 \sqrt{(15\, \Delta\alpha \,.\, \cos \delta)^2 + (\Delta\delta)^2} \tag{17.3}$$

If $\Delta\alpha$ is expressed in seconds of time (^{s}), and $\Delta\delta$ in seconds of a degree ($''$), then d expressed in $''$ is given by

$$d = \sqrt{(15\ \Delta\alpha\,.\cos\delta)^2 + (\Delta\delta)^2} \tag{17.4}$$

Formulae (17.2), (17.3), and (17.4) may be used only when d is small. However, see also the alternative formulae further in this Chapter.

Example 17.a — Calculate the angular distance between the stars Arcturus (α Boo) and Spica (α Vir).

The J2000.0 coordinates of these stars, as taken from a catalogue, are

$$\begin{array}{llll} \alpha \text{ Boo}: & \alpha_1 = 14^h15^m39^s.7 & = 213^\circ.9154 \\ & \delta_1 = +19^\circ10'57'' & = +19^\circ.1825 \\ \alpha \text{ Vir}: & \alpha_2 = 13^h25^m11^s.6 & = 201^\circ.2983 \\ & \delta_2 = -11^\circ09'41'' & = -11^\circ.1614 \end{array}$$

Formula (17.1) gives $\cos d = +0.840\,633$, whence $d = 32^\circ.7930 = 32^\circ48'$.

Of course, this distance holds only for the epoch for which the stars' coordinates are given, namely 2000.0. It varies slowly with time, by reason of the proper motions of the stars. It is, however, independent of the precession.

Exercise. — Calculate the angular distance between Aldebaran and Antares. (Answer: 169°58′).

One or both bodies may be moving objects. For example: a planet and a star, or two planets. In that case, a program may be written where first the quantities δ_1, δ_2, and $(\alpha_1 - \alpha_2)$ are interpolated, after which d is calculated by means of the formulae (17.1) or (17.2).

Exercise. — Using the following coordinates, calculate the instant and the value of the least angular separation between Mercury and Saturn.

1978 0h TD	*Mercury*		*Saturn*	
	α_1	δ_1	α_2	δ_2
	h m s	° ′ ″	h m s	° ′ ″
Sep 12	10 23 17.65	+11 31 46.3	10 33 01.23	+10 42 53.5
13	10 29 44.27	+11 02 05.9	10 33 29.64	+10 40 13.2
14	10 36 19.63	+10 29 51.7	10 33 57.97	+10 37 33.4
15	10 43 01.75	+ 9 55 16.7	10 34 26.22	+10 34 53.9
16	10 49 48.85	+ 9 18 34.7	10 34 54.39	+10 32 14.9

Answer: The least angular separation between the two planets was $0°03'44''$, on 1978 September 13 at $15^h06^m.5$ TD $= 15^h06^m$ UT.

As we see, this was a rather close conjunction. We must insist on the fact that, in such a case, first the quantities δ_1, δ_2, and $(\alpha_1 - \alpha_2)$ should be interpolated, *not* the distances themselves. The distance is to be deduced from the *interpolated coordinates*.

Suppose that, nevertheless, we try to interpolate the distances themselves. By means of formula (17.1), we find the following distances between Mercury and Saturn, in degrees and decimals, for the five given times:

1978 Sep	12.0 TD	$d_1 = 2°.5211$
	13.0	$d_2 = 0.9917$
	14.0	$d_3 = 0.5943$
	15.0	$d_4 = 2.2145$
	16.0	$d_5 = 3.8710$

It is evident that the least separation occurs between 13.0 and 14.0 September, and closer to 14.0 than to 13.0.

If we now use the *three* central values d_2, d_3, d_4 and calculate the value of the minimum by means of formula (3.4), we obtain $0°.5017 = 0°30'06''$. Taking the *five* values d_1 to d_5, formula (3.9) yields a "better" value for n_m, after which (3.8) is used to calculate the value of the function for that value of the interpolating factor n; this gives $0°.4865 = 0°29'11''$.

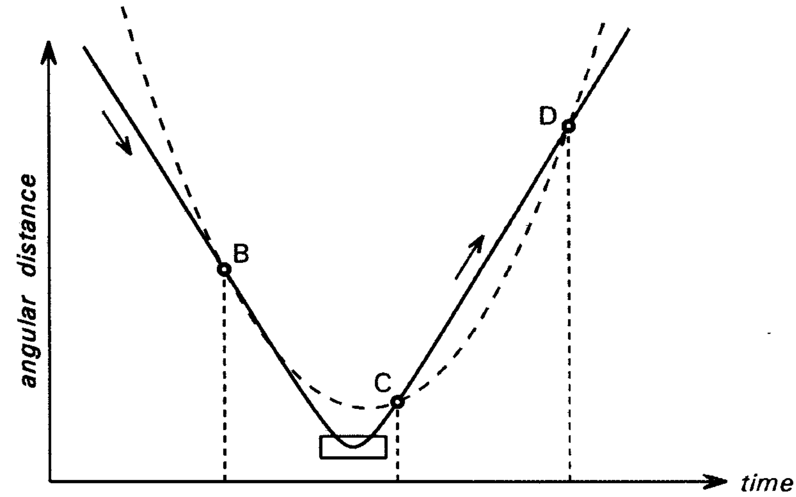
Both results are completely wrong, however. As has been mentioned above, the correct value of the least distance is only $0°03'44''$. So, what happened?

The reason is that the conjunction was a close one. Until a short time before the least distance, Mercury was moving almost exactly straight towards Saturn, and the angular distance between the two planets was decreasing almost exactly linearly with time. Similarly, some short time after the least distance, Mercury was moving almost straight away from Saturn.

In the Figure on next page, the solid curve represents the true variation of the angular separation between the two planets. Except very close to the least distance, this curve consists of two almost exactly *straight* segments (one near B, the other from C to D), and in such a case the interpolation formulae are no longer valid!

Formulae (3.3), (3.4) and (3.5), for instance, suppose that the function, in the considered part of the curve, is a *parabola*. But the curve is not a parabola, except very close to the minimum, inside the small rectangle.

If we make use of the three points B, C, D, corresponding to the three central distances d_2, d_3, d_4, then by the interpolation formula (3.3) we in fact draw a parabola through those three points; it is the dashed curve in the Figure. This parabola differs considerably from the true curve, and in particular its minimum is too high.



And it would be of no help to use the *five* values d_1 to d_5 instead of the three central ones, because the solid curve differs even considerably from a polynomial of the fourth degree!

Hence, performing an interpolation from the *distances* cannot give accurate results. As we have said, we must interpolate the original *coordinates* separately, and only then can the accurate distance for an intermediate instant be deduced. Using the interpolation formula (3.8), we so find the value of the distance for several values of the interpolating factor n:

$n =$	-0.50	distance = 0.21437 degree
	-0.45	0.14057
	-0.40	0.07790
	-0.35	0.07028
	-0.30	0.12815

The least separation occurs for n between -0.40 and -0.35, so we calculate the angular distance for three more values, at smaller intervals (but, again, from the *interpolated coordinates*):

$n =$	-0.38	distance = 0.06408 degree
	-0.37	0.06229
	-0.36	0.06448

The tabular interval is now small enough so that formulae (3.4) and (3.5) may be used. We find that the least separation is $0^\circ\!.06228 = 0^\circ03'44''$, for $n_m = -0.370502$, corresponding to September 13.629498 = September 13 at $15^h06^m\!.5$ TD, as mentioned earlier.

It is possible, however, to find the least angular separation without trying several values of the interpolating factor n, namely, by using rectangular coordinates. These coordinates u and v, in seconds of arc, can be calculated as follows [1].

Calculate the auxiliary quantity

$$K = \frac{206\,264.8062}{1 + \sin^2\delta_1 \tan \Delta\alpha \tan \frac{\Delta\alpha}{2}}$$

where 206 264.8062 is the number of arcseconds in one radian. Then

$$u = -K(1 - \tan\delta_1 \sin\Delta\delta) \cos\delta_1 \tan\Delta\alpha$$

$$v = K(\sin\Delta\delta + \sin\delta_1 \cos\delta_1 \tan\Delta\alpha \tan\frac{\Delta\alpha}{2})$$

In the above expressions, α_1, δ_1 are the right ascension and declination of the first planet, and $\Delta\alpha = \alpha_2 - \alpha_1$, $\Delta\delta = \delta_2 - \delta_1$, where α_2, δ_2 are the right ascension and declination of the second planet.

Calculate the values of u and v for three equidistant times. For any intermediate time, then, their values can be interpolated by means of formula (3.3), while their variation (in arcseconds per unit of the tabular interval) is given by

$$u' = \frac{u_3 - u_1}{2} + n(u_1 + u_3 - 2u_2)$$

where n is the interpolating factor, and u_1, u_2, u_3 are the three calculated values of u, and with a similar expression for the variation v'.

Start from any value for the interpolating factor n; a good choice is $n = 0$. For this value of n, interpolate u and v by means of formula (3.3), and find the variations u' and v'. Then the correction to n is given by

$$\Delta n = -\frac{uu' + vv'}{u'^2 + v'^2}$$

So the new value of n is $n + \Delta n$. Repeat the calculation for the new value of n until the correction Δn is a very small quantity, for instance less than 0.000 001 in absolute value.

For the final value of n, calculate u and v again. Then the least distance, in arcseconds, will be $\sqrt{u^2 + v^2}$.

Let us apply this method to the above-mentioned conjunction between Mercury and Saturn. The three chosen instants are 13.0, 14.0, and 15.0 September 1978. We find the following values for u and v, retaining one extra decimal to avoid rounding errors:

	u	v
Sept. 13.0	$-3322''.44$	$-1307''.48$
14.0	$+2088.54$	$+\ 463.66$
15.0	$+7605.36$	$+2401.71$

For $n = 0$, we have

$$u = +2088.54 \qquad u' = +5463.90$$
$$v = +\ 463.66 \qquad v' = +1854.595$$

whence $\Delta n = -0.368\,582$, and the corrected value of n is $0 - 0.368\,582 = -0.368\,582$.

For this new value of n we find

$$u = +\ 81.83 \qquad u' = +5424.89$$
$$v = -208.57 \qquad v' = +1793.07$$

whence $\Delta n = -0.002\,142$, and the new corrected value of n is $-0.368\,582 - 0.002\,142 = -0.370\,724$.

A new iteration gives $\Delta n = -0.000\,003$, so the final value of n is $-0.370\,724 - 0.000\,003 = -0.370\,727$.

[This value differs from the value $n = -0.370\,502$ found before, because in the present calculation we used the positions of the planets for only three instants instead of five. But the difference is only 0.000 225 day, or 19 seconds.]

For the value $n = -0.370\,727$, we find $u = +70''.20$, $v = -212''.42$, and consequently the least distance between the two planets is

$$\sqrt{u^2 + v^2} = 224'' = 3'44'',$$

as found before.

The same method can be used if one of the bodies is a star. The latter's coordinates are then constant, but it is important to note that the α and δ of the star should be referred to the same equinox as that of the moving body.

If the moving body is a major planet whose apparent right ascension and declination referred to the equinox of the date are given, then for the star the apparent coordinates too must be used. If one takes the star's position from a catalogue, where it is referred to a standard equinox (for instance that of 2000.0), then the apparent α and δ are found by taking into account the proper motion of the star and the effects of precession, nutation, and aberration, as explained in Chapter 23.

If the α and δ of the moving body are referred to a standard equinox (astrometric coordinates), then the α and δ of the star should be referred to this same standard equinox, the only correction being those for the proper motion of the star.

Alternative formulae

Although formula (17.1) is truly exact, mathematically speaking, its accuracy is very poor for small values of the angle d, as has been seen at the beginning of this Chapter. For this reason, several alternative methods have been proposed.

One of them [2] consists in using the old *haversine* (hav) function, which can be a great aid in certain astronomical calculations involving small angles, as it can preserve significant digits. By definition, for any angle θ, we have

$$\text{hav } \theta = \frac{1 - \cos\theta}{2}$$

The cosine formula (17.1) for angular separation is precisely equivalent to

$$\text{hav } d = \text{hav } \Delta\delta + \cos\delta_1 \cos\delta_2 \text{ hav } \Delta\alpha \tag{17.5}$$

where $\Delta\alpha = \alpha_1 - \alpha_2$, $\Delta\delta = \delta_1 - \delta_2$. To use this formula on a computer we can get the help of another identity, namely

$$\text{hav } \theta = \sin^2\left(\frac{\theta}{2}\right)$$

By means of formula (17.5), angular separations can be calculated accurately for angles from nearly 180° all the way down to exactly 0 degree!

V.J. Slabinski [3] offers another approach that can be used:

$$\sin^2 d = (\cos\delta_1 \sin\Delta\alpha)^2 + (\sin\delta_2 \cos\delta_1 \cos\Delta\alpha - \cos\delta_2 \sin\delta_1)^2$$

However, this formula cannot distinguish between supplementary angles, for instance 144° and 36°, and it has a poor accuracy when d is close to 90°.

Mr. Thierry Pauwels, of the Royal Observatory of Belgium, communicated the following method. Calculate

$$\begin{aligned}
x &= \cos\delta_1 \sin\delta_2 - \sin\delta_1 \cos\delta_2 \cos(\alpha_2 - \alpha_1) \\
y &= \cos\delta_2 \sin(\alpha_2 - \alpha_1) \\
z &= \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_2 - \alpha_1)
\end{aligned}$$

and then

$$d = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

where d should be taken between 90 and 180 degrees if z is negative.

Mathematically speaking, this method is completely identical to formula (17.1), but a computer will yield more accurate results from an arctangent than from an arccosine.

Example 17.b — Taking again the case described in Example 17.a, we find

$$x = -0.497\,404$$
$$y = -0.214\,303$$
$$z = +0.840\,633$$

from which $\tan d = 0.644\,283$, $d = 32^\circ\!.7930$, as in Example 17.a.

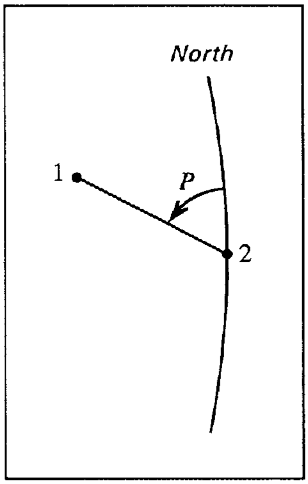
Relative Position Angle

The Position Angle P of a body (α_1, δ_1) with respect to another body (α_2, δ_2) can be found from

$$\tan P = \frac{\sin \Delta\alpha}{\cos \delta_2 \tan \delta_1 - \sin \delta_2 \cos \Delta\alpha}$$

where $\Delta\alpha = \alpha_1 - \alpha_2$.

If the denominator of the fraction is negative, then P lies in the range 90°–270°.



REFERENCES

1. A. Danjon, *Astronomie Générale*, page 36, formulae 3bis (Paris, 1959).
2. *Sky and Telescope*, Vol. 68, page 159 (August 1984).
3. *Sky and Telescope*, Vol. 69, page 158 (February 1985).

Chapter 18

Planetary Conjunctions

Given three or five ephemeris positions of two planets passing near each other, a program can be written which calculates the time of conjunction in *right ascension* and the difference in declination between the two bodies at that time. The method consists in calculating the differences $\Delta\alpha$ of the corresponding right ascensions, and then calculating the instant when $\Delta\alpha = 0$ by means of formula (3.6) or (3.7) in the case of three positions, or (3.10) or (3.11) in the case of five points. When that instant is found, direct interpolation of the differences $\Delta\delta$ of the declinations, by means of formula (3.3) or (3.8), yields the required difference in declination at the time of conjunction.

Conjunctions in celestial *longitude* can be calculated in the same way using, of course, the planets' geocentric ecliptical (celestial) longitudes and latitudes instead of their right ascensions and declinations.

Note that neither the instant of the conjunction in right ascension, nor that of the conjunction in longitude, coincides with that of the least angular separation between the two bodies. *By definition, conjunction is the phenomenon in which two bodies have the same apparent right ascension or celestial longitude as viewed from a third body (generally the Earth).*

Example 18.a — Calculate the circumstances of the Mercury–Venus conjunction of 1991 August 7, using the following apparent positions, for 0^h TD of the date, which are taken from an accurate ephemeris:

Date, 1991	*Mercury*		*Venus*	
	α	δ	α'	δ'
	h m s	° ′ ″	h m s	° ′ ″
Aug. 5	10 24 30.125	+6 26 32.05	10 27 27.175	+4 04 41.83
6	10 25 00.342	+6 10 57.72	10 26 32.410	+3 55 54.66
7	10 25 12.515	+5 57 33.08	10 25 29.042	+3 48 03.51
8	10 25 06.235	+5 46 27.07	10 24 17.191	+3 41 10.25
9	10 24 41.185	+5 37 48.45	10 22 57.024	+3 35 16.61

We first calculate the differences of the right ascensions (in seconds of time) and those of the declinations (in degrees and decimals):

Aug. 5	$\Delta\alpha = -177.050$	$\Delta\delta = +2.363\,950$
6	$-\ 92.068$	$+2.250\,850$
7	$-\ 16.527$	$+2.158\,214$
8	$+\ 49.044$	$+2.088\,006$
9	$+104.161$	$+2.042\,178$

Applying formula (3.10) to the values of $\Delta\alpha$, we find that $\Delta\alpha$ is zero for the value $n = +0.23797$ of the interpolation factor. Hence, the conjunction in right ascension took place on

$$\begin{aligned} \text{1991 August 7.23797} &= \text{1991 August 7 at } 5^h42^m\!.7 \text{ TD} \\ &= \text{1991 August 7 at } 5^h42^m \text{ UT} \end{aligned}$$

With the value of n just found, and applying formula (3.8) to the values of $\Delta\delta$, we find $\Delta\delta = +2^\circ\!.13940$ or $+2°08'$. Hence, at the time of the conjunction in right ascension, Mercury was 2°08′ north of Venus.

If the second body is a star, its coordinates may be considered as being constant during the time interval considered. We then have

$$\alpha_1' = \alpha_2' = \alpha_3' = \alpha_4' = \alpha_5'$$
$$\delta_1' = \delta_2' = \delta_3' = \delta_4' = \delta_5'$$

The program should be written in such a way that, if the second object is a star, its coordinates must be entered only once.

The important remark given on page 114 does apply here too: *the coordinates of the star and those of the moving body must be referred to the same equinox.*

As an exercise, calculate the conjunction in right ascension between the minor planet 4 Vesta and the star β Librae in February 1996. The minor planet's right ascension and declination, referred to the standard equinox of J2000.0, are as follows (from an ephemeris calculated by Edwin Goffin):

0h TD	α_{2000}	δ_{2000}
	h m s	° ′ ″
1996 Feb. 7	15 03 51.937	−8 57 34.51
12	15 09 57.327	−9 09 03.88
17	15 15 37.898	−9 17 37.94
22	15 20 50.632	−9 23 16.25
27	15 25 32.695	−9 26 01.01

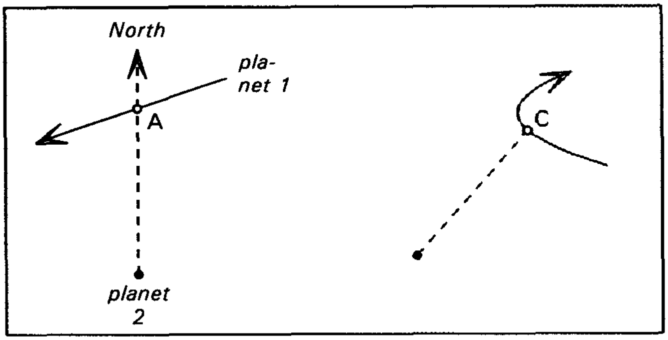
The star's coordinates for the epoch and equinox of 2000.0, taken from the FK5 star catalogue, are $\alpha' = 15^h17^m00^s.421$ and $\delta' = -9°22'58''.54$, and the centennial proper motions (that is, the proper motions per 100 years) are $-0^s.649$ in right ascension and $-1''.91$ in declination.

Consequently, from the proper motions during the −3.87 years (−0.0387 century) from 2000.0, we find that the star's position referred to the equinox of 2000.0, but for the epoch 1996.13, is

$$\alpha' = 15^h17^m00^s.446, \qquad \delta' = -9°22'58''.47$$

Now, calculate the conjunction.

Answer: Vesta passed 0°03′38″ north of β Lib on 1996 February 18 at 6^h37^m Dynamical Time.



Do not confuse conjunction with least angular separation. Two planets are in conjunction when their right ascensions (or their celestial longitudes) are equal. At left in the Figure, the motion of planet 1 with respect to planet 2 is depicted. There is conjunction when the first planet arrives in A, *and this is* not *the instant of least separation. In the drawing at right, the least angular separation occurs in* C, *but it is clear that there is no conjunction here.*

Chapter 19

Bodies in Straight Line

In this Chapter and in the next one, we shall deal with two problems which have no importance "scientifically", but which may be of value to persons interested in nice celestial events or to authors of popular articles.

Let (α_1, δ_1), (α_2, δ_2), (α_3, δ_3) be the equatorial coordinates of three heavenly bodies. These bodies are in "straight line" — that is, they lie on the same great circle of the celestial sphere — if

$$\tan \delta_1 \sin(\alpha_2 - \alpha_3) + \tan \delta_2 \sin(\alpha_3 - \alpha_1) + \tan \delta_3 \sin(\alpha_1 - \alpha_2) = 0 \tag{19.1}$$

This formula is valid for ecliptical coordinates too, provided that the right ascensions α are replaced by the longitudes λ, and the declinations δ by the latitudes β.

Do not forget that the right ascensions are generally expressed in hours, minutes, and seconds. They should be converted to hours and decimals, and then into degrees by multiplication by 15.

If one or two of the bodies are stars, then once again the important remark given on page 114 does apply: *the coordinates of the star(s) must be referred to the same equinox as that of the planets.*

Example 19.a — Find the instant when Mars was seen in straight line with Pollux and Castor in 1994.

From an ephemeris of Mars and a star atlas, it is found that the planet was in straight line with the two stars about 1994 October 1. For this date, the apparent equatorial coordinates of the stars were:

Castor (α Gem):	α_1	$= 7^h34^m16^s.40$	$= 113^\circ.56833$
	δ_1	$= +31^\circ53'51''.2$	$= +31^\circ.89756$
Pollux (β Gem):	α_2	$= 7^h45^m00^s.10$	$= 116^\circ.25042$
	δ_2	$= +28^\circ02'12''.5$	$= +28^\circ.03681$

For our problem, these values of α_1, δ_1, α_2, and δ_2 may be considered as being constant for several days.

The apparent coordinates of Mars (α_3, δ_3) are variable. Here are their values, taken from an accurate ephemeris:

TD	α_3		δ_3	
	h m s	°	° ′ ″	°
1994 Sep. 29.0	7 55 55.36	= 118.98067	+21 41 03.0	= +21.68417
30.0	7 58 22.55	= 119.59396	+21 35 23.4	= +21.58983
Oct. 1.0	8 00 48.99	= 120.20413	+21 29 38.2	= +21.49394
2.0	8 03 14.66	= 120.81108	+21 23 47.5	= +21.39653
3.0	8 05 39.54	= 121.41475	+21 17 51.4	= +21.29761

Using these values, the first member of formula (19.1) takes the following values:

Sep. 29.0	+0.001 9767
30.0	+0.001 0851
Oct. 1.0	+0.000 1976
2.0	−0.000 6855
3.0	−0.001 5641

By means of formula (3.10), we find that the value is zero for

1994 October 1.2233 = 1994 October 1, at 5^h TD (UT)

In the preceding Example, we made use of geocentric positions of Mars. For this reason the result is, strictly speaking, valid only for a geocentric observer, and for an observer for whom Mars is at the zenith. But for the present problem, it is not worthwhile to take into account the parallax of the planet, which is very small. This is no longer true in the case of the Moon, whose parallax can reach one degree. In this case, the *topocentric* position of the Moon should be used (see Chapter 40).

Straight lines on the celestial sphere

Once on a winter evening I admired the constellation Orion, when suddenly I thought about the following problem: the three stars of Orion's "Belt" (δ, ε, and ζ Orionis) are nearly on a "straight line" on the sky. But how nearly straight, precisely? Then I remembered another nearly-straight-line: when, according to the well-known rule, the line joining the stars α and β of Ursa Major is extended northward, we arrive close to the Pole Star (α Ursae Minoris). But exactly how close?

I obtained the following formulae which I give here without proof. Remember that a straight line on the celestial sphere is actually an arc of a great circle.

Consider the three stars S_1, S_2, and S_3, whose right ascensions and declinations are α_1, δ_1, α_2, δ_2, and α_3, δ_3, respectively, in such a way that S_2 is the middle star. The angle $S_1 - S_2 - S_3$, that is, the angle which the arc S_1S_2 makes with the arc S_2S_3, is equal to $C_1 + C_2$, where the angles C_1 and C_2 are given by the following formulae and should be taken between 0° and +180°:

$$\tan C_1 = \frac{\sin(\alpha_2 - \alpha_1)}{\cos\delta_2 \tan\delta_1 - \sin\delta_2 \cos(\alpha_2 - \alpha_1)}$$

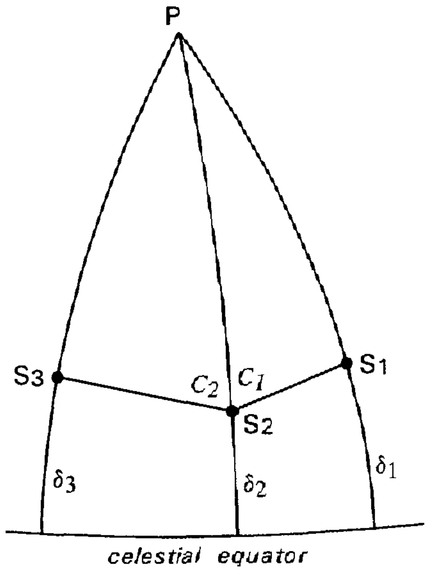
$$\tan C_2 = \frac{\sin(\alpha_3 - \alpha_2)}{\cos\delta_2 \tan\delta_3 - \sin\delta_2 \cos(\alpha_3 - \alpha_2)}$$

The drawing represents the three stars S_1, S_2, and S_3. P is the northern celestial pole. The arcs PS_1, PS_2, and PS_3 are the celestial meridians (the great circles of constant right ascension) through the three stars. The Figure also illustrates the meaning of the angles C_1 and C_2.

If the three stars are taken in increasing order of their right ascension (that is, $\alpha_1 < \alpha_2 < \alpha_3$), then $C_1 + C_2$ is the value of the *northern* angle at S_2. Of course, this angle can be larger as well as smaller than 180 degrees.

As an example, let us consider the three stars of Orion's Belt. Their positions for the epoch and equinox 2000.0 are:

	α	δ
δ Ori	$5^h32^m00^s.40$	$-0°17'56''.9$
ε Ori	$5^h36^m12^s.81$	$-1°12'07''.0$
ζ Ori	$5^h40^m45^s.52$	$-1°56'33''.3$



We find $C_1 = 49°.3622$ and $C_2 = 123°.1209$. The sum is 172.4831 degrees, or 172°29′. So the three stars of Orion's Belt indeed are not exactly aligned. They form an obtuse angle of 172½ degrees. Because $C_1 + C_2$ is smaller than 180°, and we took the stars in increasing order of their right ascensions, the middle star (ε Ori) is a little *south* of the great circle through δ and ζ Ori.

At what angular distance is ε from this great circle? This can be found as follows.

We have the two stars S_1 (α_1, δ_1) and S_2 (α_2, δ_2), and we wish to calculate the angular distance of a third star S_0 (α_0, δ_0) to the great circle $S_1 - S_2$. Calculate

$$X_1 = \cos\delta_1 \cos\alpha_1 \qquad X_2 = \cos\delta_2 \cos\alpha_2 \qquad A = Y_1 Z_2 - Z_1 Y_2$$
$$Y_1 = \cos\delta_1 \sin\alpha_1 \qquad Y_2 = \cos\delta_2 \sin\alpha_2 \qquad B = Z_1 X_2 - X_1 Z_2$$
$$Z_1 = \sin\delta_1 \qquad Z_2 = \sin\delta_2 \qquad C = X_1 Y_2 - Y_1 X_2$$

$$m = \tan\alpha_0 \qquad n = \frac{\tan\delta_0}{\cos\alpha_0}$$

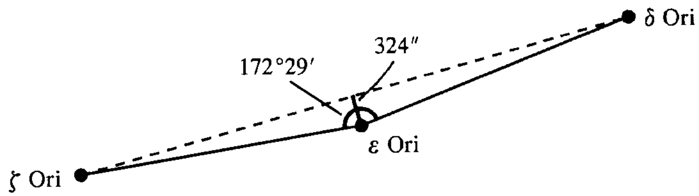
The required angular distance ω is then given by

$$\sin\omega = \frac{A + Bm + Cn}{\sqrt{A^2 + B^2 + C^2}\ \sqrt{1 + m^2 + n^2}}$$

where ω should be taken between 0° and 90°.

As an example, let us again consider the three stars of Orion's Belt. Now δ Ori and ζ Ori are the stars S_1 and S_2, respectively, and we want to calculate the angular distance of ε Ori (= star S_0) to the line δ-ζ.

Using the stars' positions mentioned above, we find $\omega = 0.^\circ 089\,876 = 324''$, or a little more than 5 arcminutes.



As an exercise, the reader can calculate the distance of the Pole Star (α UMi) to the line, extended northward, joining α and β Ursae Majoris. The 2000.0 positions of these stars are:

α UMa	$\alpha = 11^h 03^m 43.^s 666$	$\delta = +61°45'03.''22$
β UMa	$11^h 01^m 50.^s 482$	$+56°22'56.''65$
α UMi	$2^h 31^m 48.^s 704$	$+89°15'50.''72$

Answer: $\omega = 1°55'$. Hence, the line from α to β Ursae Majoris extended northward misses α Ursae Minoris by almost two degrees.

After the preceding formulae were published in the Belgian journal *Heelal* of May 1988, we received a letter from Mr. Ben Piessens, of Mechelen, Belgium, who gave another way to calculate the angle between two great circles on the celestial sphere. He wrote:

> The angle between two planes (or two great circles) as well as the angle between a straight line and a plane can easily be calculated through analytic geometry. For this, only one formula is needed, namely, the expression for the angle between two directions. The angle between two planes is equal to the angle between the perpendiculars to these planes. The angle between a straight line and a plane is the complement of the angle between that straight line and the perpendicular on that plane.

For our problem we then have, using the same symbols (α_1, etc.) as before, and O being the center of the celestial sphere, that is, the observer:

Direction numbers of the straight lines OS_1, OS_2, OS_3:

$$
\begin{array}{lll}
a_1 = \cos\delta_1 \cos\alpha_1 & b_1 = \cos\delta_1 \sin\alpha_1 & c_1 = \sin\delta_1 \\
a_2 = \cos\delta_2 \cos\alpha_2 & b_2 = \cos\delta_2 \sin\alpha_2 & c_2 = \sin\delta_2 \\
a_3 = \cos\delta_3 \cos\alpha_3 & b_3 = \cos\delta_3 \sin\alpha_3 & c_3 = \sin\delta_3
\end{array}
$$

Direction numbers of the perpendiculars to the planes OS_1S_2, OS_2S_3, OS_1S_3:

$$
\begin{array}{lll}
l_1 = b_1c_2 - b_2c_1 & m_1 = c_1a_2 - c_2a_1 & n_1 = a_1b_2 - a_2b_1 \\
l_2 = b_2c_3 - b_3c_2 & m_2 = c_2a_3 - c_3a_2 & n_2 = a_2b_3 - a_3b_2 \\
l_3 = b_1c_3 - b_3c_1 & m_3 = c_1a_3 - c_3a_1 & n_3 = a_1b_3 - a_3b_1
\end{array}
$$

With these data one can calculate the angle between any two great circles, or the angle between one of the straight lines OS_1, OS_2, OS_3 and the great circle through the two other points. Let ψ be the angle between the great circles OS_1S_2 and OS_2S_3, and ω the angle between OS_2 and the plane OS_1S_3. Then we have

$$\cos\psi = \frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\ \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\sin\omega = \frac{a_2l_3 + b_2m_3 + c_2n_3}{\sqrt{a_2^2 + b_2^2 + c_2^2}\ \sqrt{l_3^2 + m_3^2 + n_3^2}}$$

If we consider again the case of the stars δ, ε, ζ Orionis, we find $\psi = 7°31'$, in agreement with our previous result, $172°29'$. Indeed, at the crossing point of two arcs there are two angles which are supplementary: $172°29' + 7°31' = 180°$.

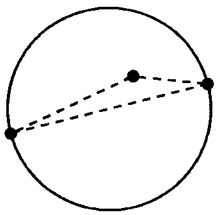
Chapter 20

Smallest Circle containing three Celestial Bodies

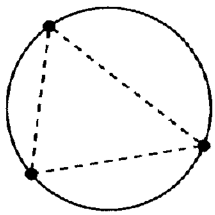
Let A, B, C be three celestial bodies situated not too far from each other on the celestial sphere, say closer than about 6 degrees. We wish to calculate the angular diameter of the smallest circle containing these three bodies. Two cases can occur:

type I : the smallest circle has as diameter the longest side of the triangle ABC, and one point is inside of the circle;

type II : the smallest circle is the circle passing through the three points A, B, C.



Type I



Type II

The diameter Δ of the smallest circle can be found as follows. Calculate the lengths (in degrees) of the sides of the triangle ABC by means of the method given in Chapter 17.

Let a be the length of the *longest* side of the triangle, and b and c the lengths of the two other sides.

If $a > \sqrt{b^2 + c^2}$, then the grouping is of type I, and $\Delta = a$;

if $a < \sqrt{b^2 + c^2}$, then the grouping is of type II, and

$$\Delta = \frac{2abc}{\sqrt{(a+b+c)\ (a+b-c)\ (b+c-a)\ (a+c-b)}} \tag{20.1}$$

Example 20.a — Calculate the diameter of the smallest circle containing Mercury, Jupiter, and Saturn on 1981 September 11 at 0^h Dynamical Time. The positions of these planets at that instant were:

Mercury	$\alpha = 12^h41^m08^s.63$	$\delta = -5°37'54''.2$
Jupiter	12 52 05.21	−4 22 26.2
Saturn	12 39 28.11	−1 50 03.7

The three angular separations are found by means of (17.1):

Mercury-Jupiter	3°.00152
Mercury-Saturn	3.82028
Jupiter-Saturn	4.04599 $= a$

Because 4.04599 is smaller than $\sqrt{(3.00152)^2 + (3.82028)^2}$, or 4.85836, we calculate Δ by means of formula (20.1). The result is

$$\Delta = 4°.26364 = 4°16'$$

This is an example of type II.

As an exercise, perform the calculation for the planets Venus, Mars, and Jupiter on 1991 June 20 at 0^h TD, using the following positions:

Venus	$\alpha = 9^h05^m41^s.44$	$\delta = +18°30'30''.0$
Mars	9 09 29.00	+17 43 56.7
Jupiter	8 59 47.14	+17 49 36.8

Show that this is a case of type I, and that $\Delta = 2°19'$.

A program can be written in which first the right ascensions and the declinations of the planets are interpolated, after which a, b, c, and finally Δ are calculated. With such a program, it is possible to calculate (by trial) the minimum value of Δ of a grouping of three planets. Indeed, Δ varies with time, and the method described in this Chapter provides the value of Δ for only one given instant.

It is important to note that, while the *positions* of the planets can be interpolated by means of the usual formulae, the values of the circle's diameter Δ cannot. The reason is that the variation of Δ generally cannot be represented by a polynomial. See, for instance, the graph in Example 20.c, on the next page.

Example 20.b — In September 1981, there was a grouping of the planets Mercury, Jupiter, and Saturn. The positions of these planets were as follows; instead of right ascensions and declinations, we will use ecliptical coordinates (longitudes and latitudes) here.

1981 0h TD	*Mercury*		*Jupiter*		*Saturn*	
	long.	*latit.*	*long.*	*latit.*	*long.*	*latit.*
	°	°	°	°	°	°
Sep. 7	186.045	−0.560	192.866	+1.117	189.324	+2.226
8	187.482	−0.696	193.069	+1.116	189.439	+2.225
9	188.897	−0.833	193.272	+1.114	189.555	+2.224
10	190.290	−0.971	193.476	+1.113	189.671	+2.223
11	191.661	−1.109	193.681	+1.112	189.788	+2.222
12	193.008	−1.246	193.886	+1.110	189.906	+2.221
13	194.332	−1.384	194.092	+1.109	190.023	+2.220
14	195.631	−1.521	194.299	+1.108	190.142	+2.219

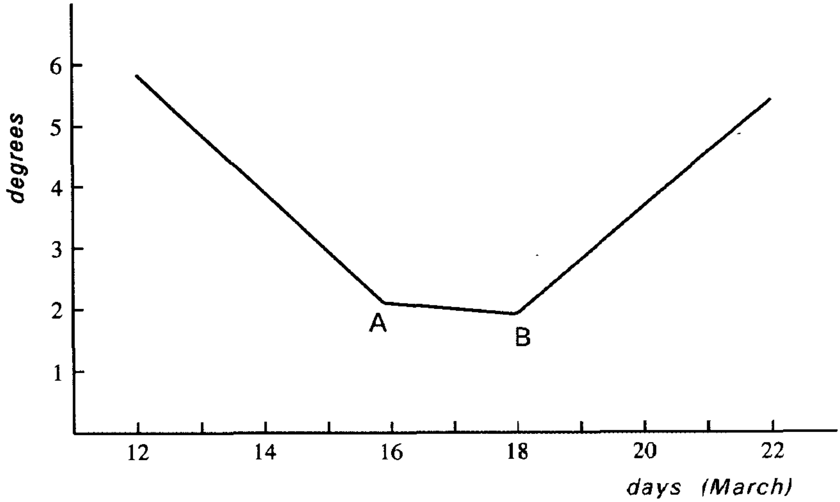
We will not give details here, and leave it as an exercise to the reader. Let us just mention that from September 7.00 to 8.81 the grouping was of type I, the diameter Δ of the smallest circle decreasing almost linearly from 7°01′ to 5°00′. From September 8.81 to 12.19, the grouping was of type II, and Δ reached a minimum value of 4°14′ on September 10.53. From September 12.19 on, the grouping was of type I again, Δ increasing almost linearly with time.

Example 20.c — Let us now consider the following fictitious case. On March 12.0, the ecliptical coordinates (in degrees) of three planets are as follows.

	longitude	*latitude*	*daily motion in longitude*
planet P1	214.23	+0.29	+0.11
planet P2	211.79	+0.48	+0.20
planet P3	208.41	+0.75	+1.08

We suppose that the latitudes are constant and that the longitudes increase at the constant rates mentioned in the last column.

Again, we leave the actual calculation as an exercise to the reader. Let us just illustrate the variation of the diameter Δ of the smallest circle (see the Figure below). Note the discontinuities at the points *A* and *B*. Except during two short periods (March 15.87 to 15.91 near *A*, and March 17.93 to 18.05 near *B*), where the grouping is of type II, we have type I. The least value of Δ, namely 1°55′, occurs at *B* on March 17.94.



If one of the bodies is a star, once again the important remark made on page 114 does apply: the coordinates of the star should be referred to the same equinox as that for the planets.

Chapter 21

Precession

The direction of the rotational axis of the Earth is not really fixed in space. Over time it undergoes a slow drift, or *precession*, much like that of a spinning top. This effect stems from the gravitational attraction of the Sun and the Moon on the Earth's equatorial bulge.

Due to the precession, the northern celestial pole (presently situated near the star α Ursae Minoris, or Polaris) slowly turns around the pole of the ecliptic with a period of about 26 000 years. As a consequence, the vernal equinox, the intersection of equator and ecliptic, regresses by about $50''$ per year along the ecliptic.

Moreover, the plane of the ecliptic itself is not fixed in space. Due to the gravitational attraction of the planets on the Earth, it slowly rotates around a "line of nodes", the speed of this rotation being presently $47''$ per century.

The plane of the ecliptic and that of the equator, and the vernal equinox, are the fundamental planes and the origin of two important coordinate systems on the celestial sphere: the ecliptical coordinates (longitude λ and latitude β) and the equatorial coordinates (right ascension α and declination δ). So, due to the precession, the coordinates of the "fixed" stars are continuously changing. Star catalogues, therefore, list the right ascensions and declinations of stars for a given epoch, such as 1900.0, or 1950.0, or 2000.0.

In this Chapter, we consider the problem of converting the right ascension α and the declination δ of a star, given for an epoch and an equinox, to the corresponding values for another epoch and equinox. Only the *mean* place of a star, and hence the effects of the precession and proper motion, will be considered here. The problem of finding the apparent place of a star will be considered in Chapter 23.

Low accuracy

If no great accuracy is required, if the two epochs are not widely separated, and if the star is not too close to one of the celestial poles, the following formulae may be used for the *annual* precession in right ascension and declination:

$$\Delta\alpha = m + n \sin\alpha \tan\delta \qquad \Delta\delta = n \cos\alpha \tag{21.1}$$

where m and n are two quantities which vary slowly with time. They are given by

$$m = 3^{s}.07496 + 0^{s}.00186\,T$$
$$n = 1^{s}.33621 - 0^{s}.00057\,T$$
$$n = 20''.0431 - 0''.0085\,T$$

T being the time measured in centuries from 2000.0 (the beginning of the year 2000). Here are the values of m and n for some epochs:

Epoch	m	n	n
	s	s	″
1700.0	3.069	1.338	20.07
1800.0	3.071	1.337	20.06
1900.0	3.073	1.337	20.05
2000.0	3.075	1.336	20.04
2100.0	3.077	1.336	20.03
2200.0	3.079	1.335	20.03

For the calculation of $\Delta\alpha$ the value of n expressed in seconds of time (^{s}) must be used. Remember that 1^{s} corresponds to $15''$ (seconds of *arc*).

In the case of a star, the effect of the proper motion should be added to the values given by formulae (21.1).

Example 21.a — The coordinates of Regulus (α Leonis) for the epoch and equinox of 2000.0 are

$$\alpha_0 = 10^{h}08^{m}22^{s}.3 \qquad \delta_0 = +11°58'02''$$

and the annual proper motions are

$-0^{s}.0169$ in right ascension,
$+0''.006$ in declination.

Reduce these coordinates to the epoch and the equinox of 1978.0.

Here we have $\alpha = 152^\circ\!.093$, $\delta = +11^\circ\!.967$, $m = 3^s\!.075$, $n = 1^s\!.336 = 20''\!.04$. From the formulae (21.1) we deduce $\Delta\alpha = +3^s\!.208$, $\Delta\delta = -17''\!.71$, to which we must add the annual proper motion, giving finally an annual variation of $+3^s\!.191$ in right ascension, and $-17''\!.70$ in declination.

Variations during -22 years (from 2000.0 to 1978.0) :

$$\text{in } \alpha: \quad +3^s\!.191 \times (-22) = -70^s\!.2 = -1^m 10^s\!.2$$
$$\text{in } \delta: \quad -17''\!.70 \times (-22) = +389'' = +6'29''$$

Required right ascension : $\alpha = \alpha_0 - 1^m 10^s\!.2 = 10^h 07^m 12^s\!.1$

Required declination : $\delta = \delta_0 + 6'29'' = +12^\circ 04' 31''$

Besselian and Julian Year

The International Astronomical Union has decided that from 1984 onwards the astronomical ephemerides should use the following system.

The new standard epoch is 2000 January 1 at 12^h TD, corresponding to JDE 2451 545.0. This epoch is designated J2000.0. For purposes of calculating positions of stars, the beginning of a "year" differs from the standard epoch J2000.0 by an integral multiple of the Julian year, or 365.25 days. For example, the epoch J1986.0 is 14 × 365.25 days before J2000.0, and hence the corresponding JDE is 2451 545.0 − 14 × 365.25 = 2446 431.50.

The letter J, in notations such as J2000.0 or J1986.0, indicates that the unit of time (for star catalogues) is the Julian year. Previously, star position catalogues used for a standard epoch the beginning of a Besselian year. The beginning of the Besselian solar year is the instant when the mean longitude of the Sun, affected by the aberration ($-20''\!.5$) and measured from the mean equinox of the date, is exactly 280°. This instant is always near the beginning of the Gregorian civil year. The length of the Besselian year, equal to that of the tropical year, was 365.242 1988 days in A.D. 1900, according to Newcomb.

To distinguish an old epoch, based on the Besselian year, from the new system, the letter B is used. For example,

B1900.0 = JDE 2415 020.3135 = 1900 January 0.8135
B1950.0 = JDE 2433 282.4235 = 1950 January 0.9235

but

J2000.0 = JDE 2451 545.00 exactly
J2050.0 = JDE 2469 807.50 exactly

and so on. The notation .0 after a year number (as in 1986.0 or 2000.0) signifies that the start of the year is meant.

Rigorous method

Let T be the time interval, in Julian centuries, between J2000.0 and the starting epoch, and let t be the interval, in the same units, between the starting epoch and the final epoch.

In other words, if $(\mathrm{JD})_0$ and (JD) are the Julian Days corresponding to the initial and the final epoch, respectively, we have

$$T = \frac{(\mathrm{JD})_0 - 2451\,545.0}{36525} \qquad\qquad t = \frac{(\mathrm{JD}) - (\mathrm{JD})_0}{36525}$$

Then the numerical expressions for the quantities ζ, z and θ which are needed for the accurate reduction of positions from one equinox to another are [1]:

$$\begin{aligned}
\zeta &= (2306''.2181 + 1''.39656T - 0''.000\,139T^2)\,t \\
&\qquad + (0''.30188 - 0''.000\,344T)\,t^2 + 0''.017\,998t^3 \\[1ex]
z &= (2306''.2181 + 1''.39656T - 0''.000\,139T^2)\,t \\
&\qquad + (1''.09468 + 0''.000\,066T)\,t^2 + 0''.018\,203t^3 \\[1ex]
\theta &= (2004''.3109 - 0''.85330T - 0''.000\,217T^2)\,t \\
&\qquad - (0''.42665 + 0''.000\,217T)\,t^2 - 0''.041\,833t^3
\end{aligned} \tag{21.2}$$

If the starting epoch is J2000.0 itself, we have $T = 0$ and the expressions (21.2) reduce to

$$\begin{aligned}
\zeta &= 2306''.2181t + 0''.30188t^2 + 0''.017\,998t^3 \\
z &= 2306''.2181t + 1''.09468t^2 + 0''.018\,203t^3 \\
\theta &= 2004''.3109t - 0''.42665t^2 - 0''.041\,833t^3
\end{aligned} \tag{21.3}$$

Then, the rigorous formulae for the reduction of the given equatorial coordinates α_0 and δ_0 of the starting epoch to the coordinates α and δ of the final epoch are:

$$\left.\begin{aligned}
A &= \cos\delta_0 \sin(\alpha_0 + \zeta) \\
B &= \cos\theta \cos\delta_0 \cos(\alpha_0 + \zeta) - \sin\theta \sin\delta_0 \\
C &= \sin\theta \cos\delta_0 \cos(\alpha_0 + \zeta) + \cos\theta \sin\delta_0 \\[1ex]
\tan(\alpha - z) &= \frac{A}{B} \qquad\qquad \sin\delta = C
\end{aligned}\right\} \tag{21.4}$$

The angle $\alpha - z$ can be obtained in the correct quadrant by applying the "second" arctangent function ATN2 to the quantities A and B, or by another procedure — see "The correct quadrant" in Chapter 1.

If the star is close to the celestial pole, one should calculate the declination by means of the formula $\cos\delta = \sqrt{A^2 + B^2}$ instead of $\sin\delta = C$.

Before making the reduction from α_0, δ_0 to α, δ, the effect of the star's proper motion should be calculated.

Example 21.b — The star θ Persei has the following mean coordinates for the epoch and equinox of J2000.0:

$$\alpha_0 = 2^h44^m11^s.986 \qquad \delta_0 = +49°13'42''.48$$

and its annual proper motions referred to that same equinox are

$+0^s.03425$ in right ascension,
$-0''.0895$ in declination.

Reduce the coordinates to the epoch and mean equinox of 2028 November 13.19 TD.

The initial epoch is J2000.0 or JD 2451 545.0. The final one is JD 2462 088.69. Hence, $t = +0.288\,670\,500$ Julian centuries, or 28.867 0500 Julian years.

We first calculate the effect of the proper motion. The variations over 28.86705 years are

$$\begin{array}{llll} +0^s.03425 \times 28.86705 & = & +0^s.989 & \text{in right ascension,} \\ -0''.0895 \times 28.86705 & = & -2''.58 & \text{in declination.} \end{array}$$

Thus the star's coordinates, for the mean equinox of J2000.0, but for the epoch 2028 November 13.19, are

$$\begin{array}{lllllll} \alpha_0 & = & 2^h44^m11^s.986 + 0^s.989 & = & 2^h44^m12^s.975 & = & +41°.054\,063 \\ \delta_0 & = & +49°13'42''.48 - 2''.58 & = & +49°13'39''.90 & = & +49°.227\,750 \end{array}$$

Since the initial equinox is that of J2000.0, we can use the expressions (21.3). With the value $t = +0.288\,670\,500$, we obtain

$$\begin{array}{lllll} \zeta & = & +665''.7627 & = & +0°.184\,9341 \\ z & = & +665''.8288 & = & +0°.184\,9524 \\ \theta & = & +578''.5489 & = & +0°.160\,7080 \end{array}$$

$$\begin{array}{lll} A & = & +0.430\,494\,05 \\ B & = & +0.488\,948\,49 \\ C & = & +0.758\,685\,86 \end{array}$$

$$\begin{array}{lllll} \alpha - z & = & +41°.362\,262 & & \\ \alpha & = & +41°.547\,214 & = & 2^h46^m11^s.331 \\ \delta & = & +49°.348\,483 & = & +49°20'54''.54 \end{array}$$

Exercise. — The equatorial coordinates of α Ursae Minoris (the Pole Star), for the epoch and mean equinox of J2000.0, are

$$\alpha = 2^h31^m48^s.704, \qquad \delta = +89°15'50''.72$$

and the star's annual proper motions for the same equinox are

$+0^s.19877$ in right ascension,
$-0''.0152$ in declination.

Find the coordinates of the star for the epochs and mean equinoxes of B1900.0, J2050.0, and J2100.0.

Answer :	B1900.0	$\alpha = 1^h22^m33^s.90$	$\delta = +88°46'26''.18$
	J2050.0	3 48 16.43	+89 27 15.38
	J2100.0	5 53 29.17	+89 32 22.18

Note that the formulae (21.2) and (21.3) are valid only for a limited period of time. If we use them for the year 32 700, for instance, we find for that epoch that α UMi will be at declination $-87°$, a completely wrong result!

Using ecliptical coordinates

If, instead of the equatorial coordinates (right ascension and declination) of a body, we use its ecliptical coordinates (longitude and latitude), the following rigorous method can be used [2].

T and t having the same meaning as before, calculate

$$\begin{aligned}
\eta &= (47''.0029 - 0''.06603\,T + 0''.000\,598\,T^2)\,t \\
&\quad + (-0''.03302 + 0''.000\,598\,T)\,t^2 + 0''.000\,060\,t^3 \\
\Pi &= 174°.876\,384 + 3289''.4789\,T + 0''.60622\,T^2 \\
&\quad - (869''.8089 + 0''.50491\,T)\,t + 0''.03536\,t^2 \qquad (21.5) \\
p &= (5029''.0966 + 2''.22226\,T - 0''.000\,042\,T^2)\,t \\
&\quad + (1''.11113 - 0''.000\,042\,T)\,t^2 - 0''.000\,006\,t^3
\end{aligned}$$

The quantity η is the angle between the ecliptic at the starting epoch and the ecliptic at the final epoch.

If the starting epoch is J2000.0, we have $T = 0$ and the expressions reduce to

$$\begin{aligned}
\eta &= 47''.0029\,t - 0''.03302\,t^2 + 0''.000\,060\,t^3 \\
\Pi &= 174°.876\,384 - 869''.8089\,t + 0''.03536\,t^2 \qquad (21.6) \\
p &= 5029''.0966\,t + 1''.11113\,t^2 - 0''.000\,006\,t^3
\end{aligned}$$

Then, the rigorous formulae for the reduction of the given ecliptical coordinates λ_0 and β_0 of the starting epoch to the coordinates λ and β of the final epoch are:

$$\left.\begin{aligned} A' &= \cos\eta \cos\beta_0 \sin(\Pi - \lambda_0) - \sin\eta \sin\beta_0 \\ B' &= \cos\beta_0 \cos(\Pi - \lambda_0) \\ C' &= \cos\eta \sin\beta_0 + \sin\eta \cos\beta_0 \sin(\Pi - \lambda_0) \\ \\ \tan(p + \Pi - \lambda) &= \frac{A'}{B'} \qquad\qquad \sin\beta = C' \end{aligned}\right\} \qquad (21.7)$$

Example 21.c — The following astrometric ecliptical coordinates of Venus have been calculated for the instant −214 June 30.0 TD, but in the reference frame J2000.0:

$$\lambda_0 = 149.^\circ48194, \qquad \beta_0 = +1.^\circ76549$$

Reduce them to the mean equinox of that date.

The date corresponds to JDE = 1643 074.5, whence

$$t = (1643\,074.5 - 2451\,545.0)/36525 = -22.134\,716$$

and we find successively:

η	$-1057''.225 = -0.^\circ293\,673$
Π	$180.^\circ22924$
p	$-110\,773''.167 = -30.^\circ770\,324$
A'	$+0.511\,1611$
B'	$+0.859\,0225$
C'	$+0.028\,1891$
$p + \Pi - \lambda$	$30.^\circ75475$
λ	$118.^\circ704$
β	$+1.^\circ615$

In the case of a star, one should take the proper motion into account. Proper motions, however, are generally given in equatorial, not in celestial (ecliptical) coordinates. The proper motions in longitude $\mu(\lambda)$ and in latitude $\mu(\beta)$ can be obtained by means of the formulae given at the top of next page, where $\mu(\alpha)$ and $\mu(\delta)$ are the proper motions in right ascension and in declinations, respectively. They should be expressed in *arcseconds*. Generally, $\mu(\alpha)$ is given in seconds of time; multiplication by 15 will convert it to arcseconds. The resulting $\mu(\lambda)$ and $\mu(\beta)$ will be in arcseconds too.

In the formulae, ε is the obliquity of the ecliptic, α the star's right ascension, δ its declination, and β its latitude.

$$\mu(\lambda) = \frac{\mu(\delta) \sin \varepsilon \cos \alpha + \mu(\alpha) \cos \delta \, (\cos \varepsilon \cos \delta + \sin \varepsilon \sin \delta \sin \alpha)}{\cos^2 \beta}$$

$$\mu(\beta) = \frac{\mu(\delta) \, (\cos \varepsilon \cos \delta + \sin \varepsilon \sin \delta \sin \alpha) - \mu(\alpha) \sin \varepsilon \cos \alpha \cos \delta}{\cos \beta}$$

TABLE 21.A

Proper motions of some stars in celestial longitude and latitude expressed in arcseconds per century for the epoch 2000.0

Star	$\mu(\lambda)$	$\mu(\beta)$	*Star*	$\mu(\lambda)$	$\mu(\beta)$
Alcyone (η Tau)	+ 0.82	− 4.90	Regulus	−23.48	− 8.13
Aldebaran	+ 3.55	− 19.68	Spica	− 2.75	− 4.15
Rigel	+ 0.04	− 0.13	Arcturus	−28.10	−226.49
Capella	+ 4.47	− 42.95	α Lib	− 8.17	− 9.48
β Tau	+ 1.20	− 17.61	π Sco	− 0.60	− 2.73
Betelgeuse	+ 2.69	+ 0.85	β Sco	− 0.18	− 1.98
μ Gem	+ 5.86	− 10.88	σ Sco	− 0.67	− 2.21
γ Gem	+ 4.51	− 3.87	Antares	− 0.63	− 2.15
ε Gem	− 0.45	− 1.38	σ Sgr	+ 0.81	− 5.52
Sirius	−55.56	−125.50	π Sgr	− 0.44	− 3.51
δ Gem	− 2.42	− 1.57	Altair	+69.67	+ 26.35
Castor	−15.57	− 12.52	β Cap	+ 4.16	− 0.82
Procyon	−54.28	−113.08	δ Cap	+14.96	− 36.73
Pollux	−61.37	− 15.67	Fomalhaut	+25.26	− 28.68

The old precessional elements

As we have said earlier, for star catalogues and for the purpose of calculating star positions, the standard epoch is now J2000.0 and the unit of time is now the *Julian* year (365.25 days) or the Julian century (36525 days). Previously the beginning of the Besselian year was taken as reference instant and the unit of time was the *tropical* year or the tropical century.

However, these are not the only differences between the old system (the FK4) and the new one, the FK5. ["FK" means *Fundamental Katalog*.]

Firstly, there is a small error (the "equinox correction") in the zero point of the right ascensions of the FK4.

Secondly, as we shall see in Chapter 23, the aberrational displacements of a star in longitude ($\Delta\lambda$) and in latitude ($\Delta\beta$) resulting from the motion of the Earth in its elliptical orbit are given by

$$\Delta\lambda = -\kappa \frac{\cos(\odot - \lambda)}{\cos\beta} + e\kappa \frac{\cos(\pi - \lambda)}{\cos\beta}$$

$$\Delta\beta = -\kappa \sin(\odot - \lambda) \sin\beta + e\kappa \sin(\pi - \lambda) \sin\beta$$

where $\odot$ is the longitude of the Sun, π the longitude of the perihelion of the Earth's orbit, e the eccentricity of this orbit, and κ the constant of aberration.

Now, the second terms in the right-hand sides of these expressions are almost constant for a given star, because e, $\pi - \lambda$, and β vary extremely slowly with time. For this reason, it has been astronomical practice to leave this part of the aberration (the so-called E-terms) in the mean positions of the stars.

Presently, the terms depending on the ellipticity of the Earth's orbit are no longer included in the mean places of stars; they are, instead, calculated in the reduction from mean to apparent places (see Chapter 23).

A procedure for performing the conversion of mean positions and proper motions of stars referred to the mean equinox and equator B1950.0 and based on Newcomb's expressions for the precession (the FK4) to the new IAU system at J2000.0 (the FK5) can be found, for instance, in the *Astronomical Almanac* for 1984 [3].

The precessional formulae (21.2) and (21.3) may be used only for the stars referred to the FK5 system. If only FK4 positions and proper motions are available, then one should proceed as follows to calculate apparent star positions in the FK5 system:

1. use must be made of Newcomb's precessional formulae (see below);
2. in the reduction from mean to apparent place, the E-terms of the aberration should be dropped;
3. to the final right ascension of the star, add the equinox correction

$$\Delta\alpha = 0^s.0775 + 0^s.0850T$$

where T is the time in Julian centuries from J2000.0.

Newcomb's precessional expressions are the following ones.

Let $(JD)_0$ and (JD) be the Julian Days corresponding to the initial and the final epoch, respectively. Then

$$T = \frac{(JD)_0 - 2415\,020.3135}{36524.2199} \qquad t = \frac{(JD) - (JD)_0}{36524.2199}$$

$$\begin{aligned}
\zeta &= (2304''.250 + 1''.396T)\,t + 0''.302t^2 + 0''.018t^3 \\
z &= \zeta + 0''.791t^2 + 0''.001t^3 \\
\theta &= (2004''.682 - 0''.853T)\,t - 0''.426t^2 - 0''.042t^3
\end{aligned}$$

If the starting epoch is B1950.0, we have $T = 0.5$, and the above expressions become

$$\begin{aligned}
\zeta &= 2304''.948t + 0''.302t^2 + 0''.018t^3 \\
z &= 2304''.948t + 1''.093t^2 + 0''.019t^3 \\
\theta &= 2004''.255t - 0''.426t^2 - 0''.042t^3
\end{aligned}$$

Motion in space

So far, we have assumed that the proper motion of a star across the sky is uniform. In other words, we considered its proper motions in right ascension and in declination to be constant. This is not correct, however. In fact, the proper motion should be combined with the radial velocity and distance to obtain the star's true motion through space relative to the Sun. Over thousands of years, the proper motion of a star will vary, as the star is approaching the Sun or is receding from it.

Let us disregard the precession here. That is, we will work in an invariable reference frame, for instance that of J2000.0. Then the method for calculating the effect of proper motion by taking into account the star's motion in space is as follows.

Let α_0, δ_0 be the star's right ascension and declination for the starting epoch, r its distance in parsecs, and Δr its radial velocity in parsecs per year (with proper sign!).

If the star's distance is given in light-years, multiply it by 0.30660 to convert it to parsecs. If, instead, the star's parallax π (in arcseconds) is given, the distance in parsecs is $1/\pi$.

Radial velocities of stars are generally given in kilometers per second. They should be divided by 977 792 in order to have them in parsecs per year.

Let $\Delta\alpha$ and $\Delta\delta$ be the proper-motion components in radians per year. They are found by dividing the *annual* proper motion $\mu(\alpha)$ listed in seconds of time by 13 751, and the annual proper motion $\mu(\delta)$ listed in seconds of arc by 206 265, respectively. Then calculate [4]

$$\begin{aligned} x &= r \cos \delta_0 \cos \alpha_0 \\ y &= r \cos \delta_0 \sin \alpha_0 \\ z &= r \sin \delta_0 \end{aligned}$$

$$\begin{aligned} \Delta x &= (x/r)\Delta r - z\Delta\delta \cos \alpha_0 - y\Delta\alpha \\ \Delta y &= (y/r)\Delta r - z\Delta\delta \sin \alpha_0 + x\Delta\alpha \\ \Delta z &= (z/r)\Delta r + r\Delta\delta \cos \delta_0 \end{aligned}$$

Then, if t is the number of years from the starting epoch, negative in the past, positive in the future,

$$\begin{aligned} x' &= x + t\Delta x \\ y' &= y + t\Delta y \\ z' &= z + t\Delta z \end{aligned}$$

The final right ascension and declination for time t, but still in the reference frame of the starting epoch, are then given by

$$\tan \alpha = \frac{y'}{x'} \qquad \text{(sin } \alpha \text{ having the same sign as } y'\text{)}$$

$$\tan \delta = \frac{z'}{\sqrt{x'^2 + y'^2}}$$

Example 21.d — Let us calculate the position (mean place) of Sirius for several epochs in the past, but still referred to the equinox of J2000.0, using the following starting values:

$$\begin{aligned} \alpha_{2000} &= 6^h45^m08^s.871 = 101^\circ.286\,962 \\ \delta_{2000} &= -16^\circ 42' 57''.99 = -16^\circ.716\,108 \end{aligned}$$

proper motions per year:

$-0^s.03847$ in right ascension
$-1''.2053$ in declination

distance = 2.64 parsecs
radial velocity = -7.6 km/second

We find $\Delta r = -0.000\,007\,773$, $\Delta\alpha = -0.000\,002\,7976$, $\Delta\delta = -0.000\,005\,8435$

Epoch	t	*This method (motion in space)*		*Using uniform proper motions*	
		α	δ	α	δ
		h m s	° ′ ″	h m s	° ′ ″
1000.0	−1000	6 45 47.16	−16 22 56.0	6 45 47.34	−16 22 52.7
0.0	−2000	6 46 25.09	−16 03 00.8	6 46 25.81	−16 02 47.4
−1000.0	−3000	6 47 02.67	−15 43 12.3	6 47 04.28	−15 42 42.9
−2000.0	−4000	6 47 39.91	−15 23 30.6	6 47 42.75	−15 22 36.8
−10000.0	−12000	6 52 25.72	−12 50 06.7	6 52 50.51	−12 41 54.4

However, an extreme accuracy cannot be obtained, because the results depend on the values adopted for the distance and the radial velocity of the star. In most cases, these values are not known with high accuracy. In the case of Sirius, if we use a radial velocity (at the epoch 2000.0) of −7.7 km/second instead of −7.6, the declination at −10000.0 becomes $-12°50'13''.0$ instead of $-12°50'06''.7$.

The "classical" method, consisting in adopting a uniform proper motion, is good for modern epochs, for instance for the calculation of occultations of stars by the Moon. Indeed, the difference between the results of the two methods varies approximately as the square of the time elapsed. Between the years 1900 and 2100, the error in the declination of Sirius, due to the fact that a uniform proper motion is adopted, is not larger than 0.04 arcsecond. And note that Sirius is only one of a *few* stars with large proper motion and close to the solar system. Therefore, the "classical" method will give no appreciable errors for epochs which are not too far from A.D. 2000.

Moreover, even the second method (taking the motion in space into account) is not valid *ad infinitum*. It will indeed give more precise results than the classical method for time lapses of many millennia, but even its validity is limited in time. Indeed, no star has a truly uniform and linear motion in space with respect to the Sun. All stars, including the Sun, describe orbits in our Galaxy system!

REFERENCES

1. *Astronomical Almanac* for the year 1984 (Washington, D.C.; 1983), page S 19.
2. *Connaissance des Temps* pour 1984 (Paris, 1983), pages XXX and XL.
3. *Astronomical Almanac* for the year 1984 (Washington, D.C.; 1983), pages S34 - S35.
   Note: Page S35 contains an error: $\Delta m = 1''.037 = 0^s.06912$ (not 0.6912).
4. A. Hirshfeld and R.W. Sinnott, *Sky Catalogue 2000.0*, Vol. 1, page xiv (Sky Publishing Corporation, Cambridge, Mass.; 1982).

Chapter 22

Nutation and the Obliquity of the Ecliptic

The nutation, discovered by the British astronomer James Bradley (1693–1762), is a periodic oscillation of the rotational axis of the Earth around its "mean" position. Due to the nutation, the instantaneous pole of rotation of the Earth oscillates around a mean pole which advances by the precession around the pole of the ecliptic.

The nutation is due principally to the action of the Moon, and can be described by a sum of periodic terms. The most important term has a period of 6798.4 days (18.6 years), but some other terms have a very short period (less than 10 days).

Nutation is conveniently partitioned into a component parallel to and one perpendicular to the ecliptic. The component along the ecliptic is denoted by $\Delta\psi$ and is called the *nutation in longitude*; it affects the celestial longitude of all heavenly bodies. The component perpendicular to the ecliptic is denoted by $\Delta\varepsilon$ and is called the *nutation in obliquity*, since it affects the obliquity of the equator to the ecliptic. The nutation does not affect the latitude of the heavenly bodies.

The quantities $\Delta\psi$ and $\Delta\varepsilon$ are needed for the calculation of the apparent place of a heavenly body and for that of the apparent sidereal time. For any given instant, $\Delta\psi$ and $\Delta\varepsilon$ can be calculated as follows.

Find the time T, measured in Julian centuries from the Epoch J2000.0 (JDE 2451 545.0),

$$T = \frac{\text{JDE} - 2451\,545}{36525} \tag{22.1}$$

where JDE is the Julian Ephemeris Day; it differs from the Julian Day (JD) by the small quantity ΔT (see Chapter 7). Then calculate the following angles expressed in degrees and decimals. These expressions are those which are provided by the International Astronomical Union [1]. They differ slightly from those used in Chapront's lunar theory (Chapter 47).

Mean elongation of the Moon from the Sun:

$$D = 297.85036 + 445\,267.111\,480\,T - 0.001\,9142\,T^2 + T^3/189\,474$$

Mean anomaly of the Sun (Earth):

$$M = 357.52772 + 35\,999.050\,340\,T - 0.000\,1603\,T^2 - T^3/300\,000$$

Mean anomaly of the Moon:

$$M' = 134.96298 + 477\,198.867\,398\,T + 0.008\,6972\,T^2 + T^3/56\,250$$

Moon's argument of latitude:

$$F = 93.27191 + 483\,202.017\,538\,T - 0.003\,6825\,T^2 + T^3/327\,270$$

Longitude of the ascending node of the Moon's mean orbit on the ecliptic, measured from the mean equinox of the date:

$$\Omega = 125.04452 - 1934.136\,261\,T + 0.002\,0708\,T^2 + T^3/450\,000$$

The nutations in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$) are then obtained by making the sum of the terms given in Table 22.A, where the coefficients are given in units of $0''.0001$. These terms are those of the "1980 IAU Theory of Nutation" [2] where, however, we have neglected the terms with a coefficient smaller than $0''.0003$. The argument of each sine (for $\Delta\psi$) and cosine (for $\Delta\varepsilon$) is a linear combination of the five fundamental arguments D, M, M', F, and Ω. For instance, the argument on the second line is $-2D + 2F + 2\Omega$.

Of course, if no great accuracy is needed, only the periodic terms with the largest coefficients can be used.

If an accuracy of $0''.5$ in $\Delta\psi$ and of $0''.1$ in $\Delta\varepsilon$ are sufficient, then we may drop the terms in T^2 and in T^3 in the above expression for Ω, and then use the following simplified expressions:

$$\begin{aligned}
\Delta\psi &= -17''.20 \sin \Omega - - 1''.32 \sin 2L - 0''.23 \sin 2L' + 0''.21 \sin 2\Omega \\
\Delta\varepsilon &= +9''.20 \cos \Omega + 0''.57 \cos 2L + 0''.10 \cos 2L' - 0''.09 \cos 2\Omega
\end{aligned}$$

where L and L' are the mean longitudes of the Sun and the Moon, respectively:

$$\begin{aligned}
L &= 280^\circ.4665 + 36\,000^\circ.7698\,T \\
L' &= 218^\circ.3165 + 481\,267^\circ.8813\,T
\end{aligned}$$

TABLE 22.A

Periodic terms for the nutation in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$). The unit is 0″.0001.

Argument multiple of					$\Delta\psi$ Coefficient of the sine of the argument		$\Delta\varepsilon$ Coefficient of the cosine of the argument	
D	M	M'	F	Ω				
0	0	0	0	1	−171996	−174.2T	+92025	+8.9T
−2	0	0	2	2	−13187	−1.6T	+5736	−3.1T
0	0	0	2	2	−2274	−0.2T	+977	−0.5T
0	0	0	0	2	+2062	+0.2T	−895	+0.5T
0	1	0	0	0	+1426	−3.4T	+54	−0.1T
0	0	1	0	0	+712	+0.1T	−7	
−2	1	0	2	2	−517	+1.2T	+224	−0.6T
0	0	0	2	1	−386	−0.4T	+200	
0	0	1	2	2	−301		+129	−0.1T
−2	−1	0	2	2	+217	−0.5T	−95	+0.3T
−2	0	1	0	0	−158			
−2	0	0	2	1	+129	+0.1T	−70	
0	0	−1	2	2	+123		−53	
2	0	0	0	0	+63			
0	0	1	0	1	+63	+0.1T	−33	
2	0	−1	2	2	−59		+26	
0	0	−1	0	1	−58	−0.1T	+32	
0	0	1	2	1	−51		+27	
−2	0	2	0	0	+48			
0	0	−2	2	1	+46		−24	
2	0	0	2	2	−38		+16	
0	0	2	2	2	−31		+13	
0	0	2	0	0	+29			
−2	0	1	2	2	+29		−12	
0	0	0	2	0	+26			
−2	0	0	2	0	−22			
0	0	−1	2	1	+21		−10	
0	2	0	0	0	+17	−0.1T		
2	0	−1	0	1	+16		−8	
−2	2	0	2	2	−16	+0.1T	+7	
0	1	0	0	1	−15		+9	

TABLE 22.A (cont.)

Argument					$\Delta\psi$	$\Delta\varepsilon$
D	*M*	*M′*	*F*	Ω	*sine*	*cosine*
−2	0	1	0	1	−13	+7
0	−1	0	0	1	−12	+6
0	0	2	−2	0	+11	
2	0	−1	2	1	−10	+5
2	0	1	2	2	−8	+3
0	1	0	2	2	+7	−3
−2	1	1	0	0	−7	
0	−1	0	2	2	−7	+3
2	0	0	2	1	−7	+3
2	0	1	0	0	+6	
−2	0	2	2	2	+6	−3
−2	0	1	2	1	+6	−3
2	0	−2	0	1	−6	+3
2	0	0	0	1	−6	+3
0	−1	1	0	0	+5	
−2	−1	0	2	1	−5	+3
−2	0	0	0	1	−5	+3
0	0	2	2	1	−5	+3
−2	0	2	0	1	+4	
−2	1	0	2	1	+4	
0	0	1	−2	0	+4	
−1	0	1	0	0	−4	
−2	1	0	0	0	−4	
1	0	0	0	0	−4	
0	0	1	2	0	+3	
0	0	−2	2	2	−3	
−1	−1	1	0	0	−3	
0	1	1	0	0	−3	
0	−1	1	2	2	−3	
2	−1	−1	2	2	−3	
0	0	3	2	2	−3	
2	−1	0	2	2	−3	

The obliquity of the ecliptic

The obliquity of the ecliptic, or inclination of the Earth's axis of rotation, is the angle between the equator and the ecliptic. One distinguishes the *mean* and the *true* obliquity, being the angles which the ecliptic makes with the mean and with the true (instantaneous) equator, respectively. In other words, the adjective *mean* indicates that the correction for nutation is not taken into account.

The mean obliquity of the ecliptic is given by the following formula, adopted by the International Astronomical Union [1]:

$$\varepsilon_0 = 23°26'21''.448 - 46''.8150\,T - 0''.00059\,T^2 + 0''.001\,813\,T^3 \qquad (22.2)$$

where, again, T is the time measured in Julian centuries from the epoch J2000.0.

The accuracy of formula (22.2) is not satisfactory over a long period of time: the error in ε_0 reaches $1''$ over a period of 2000 years, and about $10''$ over a period of 4000 years. The following improved expression is due to Laskar [3]. Here, U is the time measured in units of 10000 Julian years from J2000.0, or $U = T/100$.

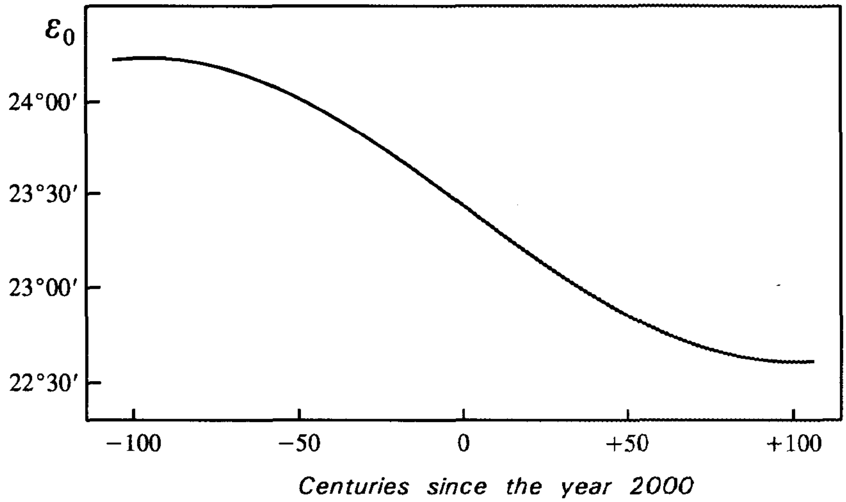
$$\begin{aligned}\varepsilon_0 = 23°26'21''.448 &- 4680''.93\,U \\ &- 1.55\,U^2 \\ &+ 1999.25\,U^3 \\ &- 51.38\,U^4 \\ &- 249.67\,U^5 \\ &- 39.05\,U^6 \\ &+ 7.12\,U^7 \\ &+ 27.87\,U^8 \\ &+ 5.79\,U^9 \\ &+ 2.45\,U^{10}\end{aligned} \qquad (22.3)$$

The accuracy of this expression is estimated at $0''.01$ after 1000 years (that is, between A.D. 1000 and 3000), and a few seconds of arc after 10000 years.

It is important to note that formula (22.3) is valid only over a period of 10000 years on each side of J2000.0, that is, for $|U| < 1$. For $U = +2.834$, for example, the formula would yield $\varepsilon_0 = 90°$, a completely wrong result!

The Figure on the next page shows the variation of ε_0 from 10000 years before to 10000 years after A.D. 2000. According to Laskar's formula, the inclination of the Earth's axis of rotation was a maximum (24°14′07″) about the year −7530. And near the year +12030 a minimum (22°36′41″) will be reached. By a mere chance we are presently approximately half-way between these extreme values, near the middle of the curve in the Figure. Here the curve is almost linear; this is the reason why in (22.3) the coefficient of U^2 is very small.

The *true* obliquity of the ecliptic is $\varepsilon = \varepsilon_0 + \Delta\varepsilon$, where $\Delta\varepsilon$ is the nutation in obliquity.



Example 22.a — Calculate $\Delta\psi$, $\Delta\varepsilon$, and the true obliquity of the ecliptic for 1987 April 10 at 0^h TD.

This date corresponds to JDE 2446 895.5, and we find

T	$-0.127\,296\,372\,348$		
D	$-56\,383^\circ.0377$	=	$136^\circ.9623$
M	$-4225^\circ.0208$	=	$94^\circ.9792$
M'	$-60\,610^\circ.7216$	=	$229^\circ.2784$
F	$-61\,416^\circ.5921$	=	$143^\circ.4079$
Ω	$371^\circ.2531$	=	$11^\circ.2531$
$\Delta\psi$	$-3''.788$		
$\Delta\varepsilon$	$+9''.443$		
ε_0	$23^\circ 26' 27''.407$		
ε	$23^\circ 26' 36''.850$		

REFERENCES

1. *Astronomical Almanac* for the year 1984 (Washington, D.C.; 1983), page S26.
2. *Ibid.*, page S23.
3. J. Laskar, *Astronomy and Astrophysics*, Vol. 157, page 68 (1986).

Chapter 23

Apparent Place of a Star

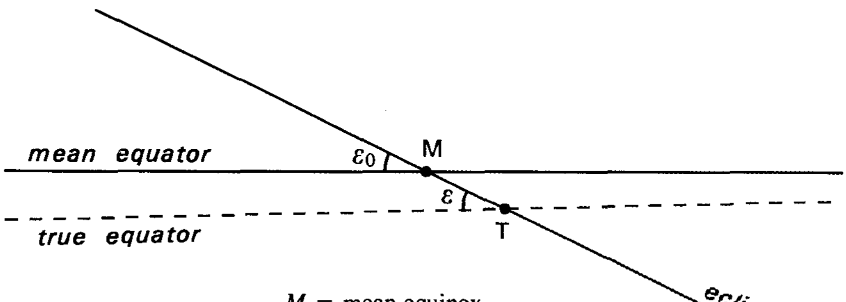
The *mean place* of a star at any time is its apparent position on the celestial sphere, as it would be seen by an observer at rest on the Sun (or, more exactly, at the barycenter of the solar system), and referred to the ecliptic and mean equinox of the date (or to the mean equator and mean equinox of the date).

The *apparent* place of a star at any time is its position on the celestial sphere as it is actually seen from the center of the moving Earth, and referred to the instantaneous equator, ecliptic, and equinox. Note that:

— the *mean equinox* is the intersection of the ecliptic of the date with the mean equator of the date;

— the *true equinox* is the intersection of the ecliptic with the true (instantaneous) equator, that is, the equator affected by the nutation;

— there is no "mean" ecliptic, because the ecliptic has a regular motion — the slow rotation mentioned on page 131.



M = mean equinox
T = true equinox
arc $MT = \Delta\psi$

The problem of the reduction of the place of a star from the mean place at one time (for instance, of a standard epoch and equinox, such as J2000.0) to the apparent place at another time involves the following corrections:

(A) The *proper motion* of the star between the two epochs. We may assume that by its proper motion each star moves on a great circle with an invariable angular speed — however, see also "Motion in space" in Chapter 21. Except when the proper motion is an important fraction of the polar distance of the star, not only the proper motion itself, but also its components in right ascension and declination *with respect to a fixed equinox* may be considered as constants during several centuries. Therefore, we start by finding the effect of the proper motion when the axes of reference remain fixed, as in Example 21.b;

(B) The effect of *precession*. This has been explained in Chapter 21;

(C) The effect of *nutation* (see below);

(D) The effect of *annual aberration* (see below);

(E) The effect of the *annual parallax*. Of course, stellar parallaxes are of fundamental importance in astronomy. As George Lovi wrote [1]:

> Parallax is the only true geometrical link between us and our nearer neighbors in that vast interstellar void. It has enabled astronomers to create and calibrate procedures to take us much farther out.

However, for the person wishing to calculate accurate star positions, the stellar parallax is a nuisance. Fortunately, stellar parallaxes never exceed $0''.8$ and they may be neglected in most cases. According to R. Burnham [2], only 13 stars brighter than magnitude 9.0 are nearer than 13 light-years (4 parsecs) and have a parallax exceeding $0''.25$. These stars are α Centauri, Lalande 21185 (in Ursa Major), Sirius, ε Eridani, 61 Cygni, Procyon, ε Indi, Σ2398 (in Draco), Groombridge 34 (in Andromeda), τ Ceti, Lacaille 9352 (in Piscis Austrinus), Cordoba 29191 (in Microscopium), and the Star of Kapteyn (in Pictor). None of these stars is near the ecliptic, and so none is involved in occultations by the Moon or in close conjunctions with planets.

For this reason, in what follows we shall neglect the effect of the annual parallax in the calculation of the apparent position of a star.

(F) The *gravitational deflection of light*. The path of light is bent by the gravitational field of the Sun in the direction toward the Sun (Einstein effect). Formulae for calculating this effect are given in [3]. However, for any elongation larger than 15° the effect is smaller than $0''.03$. For this reason, we will neglect this effect here.

The effect of nutation

The simplest and most direct method of applying the effect of nutation to mean position is to add $\Delta\psi$ to the ecliptical longitude of the objects. The ecliptic and therefore the latitude of a body is unchanged by nutation.

This procedure can profitably be used in the calculation of apparent positions of *planets*, where ecliptical coordinates are calculated first. Stellar positions, however, are generally given in the equatorial system, so we prefer to calculate the corrections in right ascension and in declination directly.

First-order corrections to a star's right ascension α and declination δ due to the nutation are

$$\left.\begin{aligned} \Delta\alpha_1 &= (\cos\varepsilon + \sin\varepsilon \sin\alpha \tan\delta)\, \Delta\psi - (\cos\alpha \tan\delta)\, \Delta\varepsilon \\ \Delta\delta_1 &= (\sin\varepsilon \cos\alpha)\, \Delta\psi + (\sin\alpha)\, \Delta\varepsilon \end{aligned}\right. \qquad (23.1)$$

These expressions are invalid if the star is close to one of the celestial poles. If this is the case, it is better to work in ecliptical coordinates and just add $\Delta\psi$ to the longitude, as mentioned above.

The quantities $\Delta\psi$ and $\Delta\varepsilon$ can be calculated by means of the method described in Chapter 22, while ε is the obliquity of the ecliptic given by formula (22.2).

The effect of aberration

Let λ and β be the star's celestial longitude and latitude, κ the constant of aberration (20''.49552), $\odot$ the true (geometric) longitude of the Sun, e the eccentricity of the Earth's orbit, and π the longitude of the perihelion of this orbit.

$\odot$ can be calculated by the method described in Chapter 25, while

$$e = 0.016\,708\,634 - 0.000\,042\,037\,T - 0.000\,000\,1267\,T^2$$

$$\pi = 102^\circ\!.93735 + 1^\circ\!.71946\,T + 0^\circ\!.00046\,T^2$$

where T is the time in Julian centuries from the epoch J2000.0, as obtained by formula (22.1).

Then the changes in longitude and in latitude of the star due to the annual aberration are

$$\left.\begin{aligned} \Delta\lambda &= \frac{-\kappa\cos(\odot - \lambda) + e\,\kappa\cos(\pi - \lambda)}{\cos\beta} \\ \Delta\beta &= -\kappa\sin\beta\,(\sin(\odot - \lambda) - e\sin(\pi - \lambda)) \end{aligned}\right\} \qquad (23.2)$$

In equatorial coordinates, the changes in the right ascension α and in the declination δ of the star due to the annual aberration are

$$\left.\begin{aligned}\Delta\alpha_2 &= -\kappa\,\frac{\cos\alpha\cos\odot\cos\varepsilon+\sin\alpha\sin\odot}{\cos\delta}\\ &\qquad + e\kappa\,\frac{\cos\alpha\cos\pi\cos\varepsilon+\sin\alpha\sin\pi}{\cos\delta}\\[2ex] \Delta\delta_2 &= -\kappa\,[\cos\odot\cos\varepsilon(\tan\varepsilon\cos\delta-\sin\alpha\sin\delta)\\ &\qquad\qquad + \cos\alpha\sin\delta\sin\odot\,]\\ &\qquad + e\kappa\,[\cos\pi\cos\varepsilon(\tan\varepsilon\cos\delta-\sin\alpha\sin\delta)\\ &\qquad\qquad + \cos\alpha\sin\delta\sin\pi\,]\end{aligned}\right\}\tag{23.3}$$

The total corrections to α and δ, due to the nutation and the aberration, are therefore $\Delta\alpha_1 + \Delta\alpha_2$ and $\Delta\delta_1 + \Delta\delta_2$, respectively. Calculated from the above formulae, both are expressed in seconds of a degree (if $\Delta\psi$, $\Delta\varepsilon$ and κ are expressed in the same units).

Important remark. — Formulae (23.2) and (23.3) are the complete expressions for the components of the aberration. They include the so-called *E*-terms and should be used for the star positions given in the FK5 [4] and in all catalogues based on it.

If, however, FK4 positions are used, those parts of formulae (23.2) and (23.3) that contain the eccentricity e of the orbit of the Earth should be dropped, as explained in Chapter 21.

Example 23.a — Calculate the apparent place of θ Persei for 2028 Nov. 13.19 TD.

The mean position of this star for that instant, including the effect of proper motion, was found in Example 21.b, namely

$$\alpha = 2^h46^m11\overset{s}{.}331 = 41\overset{\circ}{.}5472 \qquad \delta = +49°20'54''.54 = +49\overset{\circ}{.}3485$$

The nutations in longitude and in obliquity, for the same instant, can be found by means of the method given in Chapter 22. We obtain

$$\Delta\psi = +14''.861 \qquad \Delta\varepsilon = +2''.705$$

Formula (22.2) gives $\varepsilon = 23\overset{\circ}{.}436$, while the Sun's true longitude, calculated by means of the method "low accuracy" of Chapter 25, is $\odot = 231\overset{\circ}{.}328$. (An accuracy of 0.01 degree is sufficient in this case.) We further find

$$T = +0.288\,6705 \qquad e = 0.016\,696\,49 \qquad \pi = 103\overset{\circ}{.}434$$

Putting the values of α, δ, ε, $\Delta\psi$, $\Delta\varepsilon$, $\odot$, e, and π in formulae (23.1) and (23.3), one finds

$$\Delta\alpha_1 = +15''.843 \qquad \Delta\delta_1 = +6''.218$$
$$\Delta\alpha_2 = +30''.045 \qquad \Delta\delta_2 = +6''.697$$

and the total corrections in right ascension and in declination are

$$\Delta\alpha = +15''.843 + 30''.045 = 45''.888 = +3^s.059$$
$$\Delta\delta = +6''.218 + 6''.697 = +12''.91$$

Hence, the required apparent coordinates of the star are

$$\alpha = 2^h46^m11^s.331 + 3^s.059 = 2^h46^m14^s.390$$
$$\delta = +49°20'54''.54 + 12''.91 = +49°21'07''.45$$

The Ron-Vondrák expression for aberration

Expressions (23.2) and (23.3) contain the effect of the eccentricity of the Earth's orbit and will provide quite accurate results. Nevertheless, these results are not rigorously exact because the said formulae are based on an unperturbed motion of the Earth in its elliptical orbit. Actually, the Earth's motion is somewhat perturbed by the attraction of the Moon and that of the planets. And the Sun itself is slowly moving around the center of mass of the solar system, mainly due to the action of the giants Jupiter and Saturn.

If a very accurate result is required, stellar aberration must, in fact, be computed from the total velocity of the Earth referred to this barycenter. One method of performing this calculation has been presented by Ron and Vondrák [5].

If $T = (\mathrm{JD} - 2451\,545)/36525$ is, as before, the time in Julian centuries elapsed since J2000.0, then calculate, for the given instant, the following angles expressed in *radians*:

$$\begin{aligned}
L2 &= 3.176\,1467 + 1021.328\,5546\,T \\
L3 &= 1.753\,4703 + 628.307\,5849\,T \\
L4 &= 6.203\,4809 + 334.061\,2431\,T \\
L5 &= 0.599\,5465 + 52.969\,0965\,T \\
L6 &= 0.874\,0168 + 21.329\,9095\,T \\
L7 &= 5.481\,2939 + 7.478\,1599\,T \\
L8 &= 5.311\,8863 + 3.813\,3036\,T \\
L' &= 3.810\,3444 + 8399.684\,7337\,T \\
D &= 5.198\,4667 + 7771.377\,1486\,\mathrm{T} \\
M' &= 2.355\,5559 + 8328.691\,4289\,T \\
F &= 1.627\,9052 + 8433.466\,1601\,T
\end{aligned}$$

TABLE 23.A

Velocity components of the Earth with respect to the center of mass of the solar system

No.	Argument	X' sin		X' cos		Y' sin		Y' cos		Z' sin		Z' cos	
1	$L3$	−1719914	$-2T$	−25		25	$-13T$	1578089	$+156T$	10	$+32T$	684185	$-358T$
2	$2L3$	6434	$+141T$	28007	$-107T$	25697	$-95T$	−5904	$-130T$	11141	$-48T$	−2559	$-55T$
3	$L5$	715		0		6		−657		−15		−282	
4	L'	715		0		0		−656		0		−285	
5	$3L3$	486	$-5T$	−236	$-4T$	−216	$-4T$	−446	$+5T$	−94		−193	
6	$L6$	159		0		2		−147		−6		−61	
7	F	0		0		0		26		0		−59	
8	$L' + M'$	39		0		0		−36		0		−16	
9	$2L5$	33		−10		−9		−30		−5		−13	
10	$2L3 - L5$	31		1		1		−28		0		−12	
11	$3L3 - 8L4 + 3L5$	8		−28		25		8		11		3	
12	$5L3 - 8L4 + 3L5$	8		−28		−25		−8		−11		−3	
13	$2L2 - L3$	21		0		0		−19		0		−8	
14	$L2$	−19		0		0		17		0		8	
15	$L7$	17		0		0		−16		0		−7	
16	$L3 - 2L5$	16		0		0		15		1		7	
17	$L8$	16		0		1		−15		−3		−6	
18	$L3 + L5$	11		−1		−1		−10		−1		−5	
19	$2L2 - 2L3$	0		−11		−10		0		−4		0	
20	$L3 - L5$	−11		−2		−2		9		−1		4	
21	$4L3$	−7		−8		−8		6		−3		3	
22	$3L3 - 2L5$	−10		0		0		9		0		4	
23	$L2 - 2L3$	−9		0		0		−9		0		−4	
24	$2L2 - 3L3$	−9		0		0		−8		0		−4	

TABLE 23.A (cont.)

25	$2L6$	0	−9	−8	0	−3	0
26	$2L2 - 4L3$	0	−9	8	0	3	0
27	$3L3 - 2L4$	8	0	0	−8	0	−3
28	$L' + 2D - M'$	8	0	0	−7	0	−3
29	$8L2 - 12L3$	−4	−7	−6	4	−3	2
30	$8L2 - 14L3$	−4	−7	6	−4	3	−2
31	$2L4$	−6	−5	−4	5	−2	2
32	$3L2 - 4L3$	−1	−1	−2	−7	1	−4
33	$2L3 - 2L5$	4	−6	−5	−4	−2	−2
34	$3L2 - 3L3$	0	−7	−6	0	−3	0
35	$2L3 - 2L4$	5	−5	−4	−5	−2	−2
36	$L' - 2D$	5	0	0	−5	0	−2

The quantities $L2$ up to $L8$ are the mean longitudes of the planets Venus to Neptune referred to the mean equinox of J2000.0 (the effects of Mercury and Pluto are negligible), while L' is the mean longitude of the Moon.

Then the components X', Y', Z' of the velocity of the Earth with respect to the barycenter of the solar system, in the equatorial J2000.0 reference frame, are equal to the sums of the terms given in Table 23.A. Here, the argument of each sine and cosine is a linear combination of some of the angles $L2$, $L3$, etc. For instance, the terms on line 12 of the table have as argument the angle

$$A = 5L3 - 8L4 + 3L5$$

and the contributions to the velocity components are:

$$\begin{array}{ll} \text{to } X': & +8 \sin A - 28 \cos A \\ \text{to } Y': & -25 \sin A - 8 \cos A \\ \text{to } Z': & -11 \sin A - 3 \cos A \end{array}$$

The values of X', Y', Z' thus obtained are expressed in units of 10^{-8} astronomical unit per day. Let c be the velocity of light in the same units, namely

$$c = 17\,314\,463\,350.$$

Then the changes in the star's right ascension and declination due to the annual aberration are, in *radians*, given by formulae (23.4).

$$\left.\begin{aligned}\Delta\alpha &= \frac{Y'\cos\alpha - X'\sin\alpha}{c\cos\delta}\\[2ex] \Delta\delta &= -\frac{(X'\cos\alpha + Y'\sin\alpha)\sin\delta - Z'\cos\delta}{c}\end{aligned}\right\}\qquad (23.4)$$

Important: the Earth's velocity components, as calculated by means of Table 23.A, are given in a rectangular coordinate system based on the *fixed* equator and equinox of FK5 for the epoch J2000.0, *not* with respect to the mean equinox of the date. Consequently, if the Ron-Vondrák method for the calculation of the aberration is preferred instead of the formulae (23.3), then the corrections (23.4) should be performed *before* the calculation of the effects of precession and nutation. In other words, the sequence of the calculations will be: FK5 position (J2000.0), proper motion, aberration (Table 23.A and expressions 23.4), precession (expressions 21.3 and 21.4), nutation (Chapter 22 and expressions 23.1).

Example 23.b — Let us again calculate the apparent place of θ Persei for 2028 November 13.19 TD, but now using the Ron-Vondrák algorithm.

As in Example 21.b, we find that the star's coordinates for the epoch 2028 November 13.19, but referred to the mean equinox of J2000.0, are (allowing for proper motion)

$$\begin{aligned}\alpha &= 2^h44^m12^s.9747 &&= +41^\circ.054\,0613\\ \delta &= +49^\circ13'39''.896 &&= +49^\circ.227\,7489\end{aligned}$$

We keep extra decimals here, in order to avoid rounding errors. We further find

T	+0.288 670 500	L'	2428.551 5363 rad.
$L2$	298.003 5712 rad.	D	2248.565 7939
$L3$	183.127 3350	M'	2406.603 0750
$L4$	102.637 1070	F	2436.120 7984
$L5$	15.890 1621		
$L6$	7.031 3324	X'	−1363 700
$L7$	7.640 0181	Y'	+ 990 286
$L8$	6.412 6746	Z'	+ 429 285

Formulae (23.4) then give

$$\begin{aligned}\Delta\alpha &= +0.000\,145\,252 \text{ radian} &&= +0^\circ.008\,3223\\ \Delta\delta &= +0.000\,032\,723 \text{ radian} &&= +0^\circ.001\,8749\end{aligned}$$

so that the new values for α and δ, corrected for aberration but still in the J2000.0 reference frame, are

$$\begin{aligned}\alpha &= 41^\circ.054\,0613 + 0^\circ.008\,3223 &&= 41^\circ.062\,3836\\ \delta &= 49^\circ.227\,7489 + 0^\circ.001\,8749 &&= 49^\circ.229\,6238\end{aligned}$$

The effect of precession is obtained by means of formulae (21.4). The values of ζ, z, and θ, for the same instant, were found in Example 21.b. We now find

$$A = +0.430\,549\,036$$
$$B = +0.488\,867\,290$$
$$C = +0.758\,706\,993$$

$$\text{new } \alpha = 41^\circ\!.555\,5635$$
$$\text{new } \delta = 49^\circ\!.350\,3415$$

Finally, the corrections for the nutation are given by (23.1). As in Example 23.a, we have $\Delta\psi = +14''\!.861$, $\Delta\varepsilon = +2''\!.705$, and $\varepsilon = 23^\circ\!.436$. We find

$$\Delta\alpha_1 = +15''\!.844 = +0^\circ\!.004\,4011$$
$$\Delta\delta_1 = +6''\!.217 = +0^\circ\!.001\,7270$$

Hence, the required apparent right ascension and declination of the star are

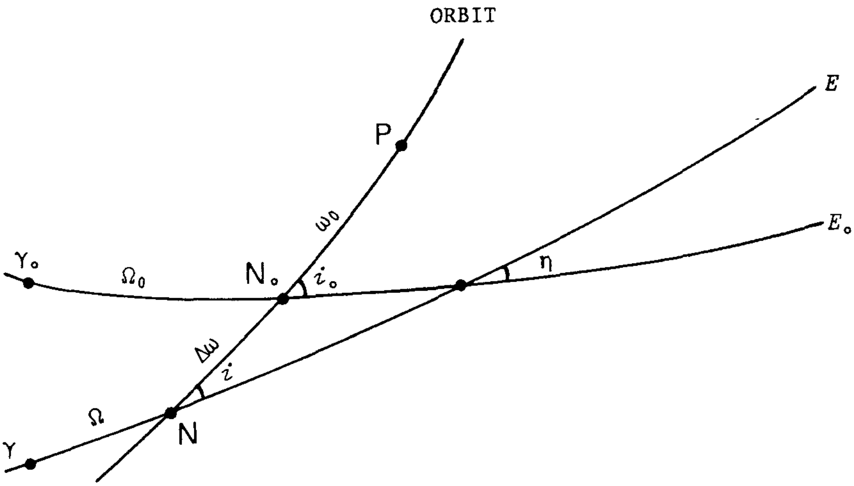
$$\alpha = 41^\circ\!.555\,5635 + 0^\circ\!.004\,4011 = 41^\circ\!.559\,9646 = 2^h46^m14^s\!.392$$

$$\delta = 49^\circ\!.350\,3415 + 0^\circ\!.001\,7270 = +49^\circ\!.352\,0685 = +49^\circ21'07''\!.45$$

Compare these results with those of Example 23.a.

REFERENCES

1. *Sky and Telescope*, Vol. 77, page 288 (March 1989).
2. Robert Burnham, *Burnham's Celestial Handbook*, Vol. III, page 2126 (Dover Publications, New York; 1978).
3. *Astronomical Almanac* for the year 1984 (Washington, D.C.; 1983), page S20.
4. Fifth Fundamental Catalogue (FK5), *Veröffentlichungen Astronomisches Rechen-Institut Heidelberg*, No. 32 (Karlsruhe, 1988).
5. C. Ron, J. Vondrák, "Expansion of Annual Aberration into Trigonometric Series", *Bull. Astron. Inst. Czechosl.*, Vol. 37, pages 96–103 (1986).


ORBIT
E
P
ω_0
E_0
Υ_0
Ω_0
N_0
i_0
η
$\Delta\omega$
i
Ω
Υ
N

Chapter 24

Reduction of Ecliptical Elements from one Equinox to another one

For some problems, it may be necessary to reduce the orbital elements of a planet, a minor planet, or a comet from one equinox to another one. Of course, the semimajor axis a and the eccentricity e do not change when the orbit is referred to another equinox, and hence only the three elements

i = inclination,
ω = argument of perihelion,
Ω = longitude of ascending node

should be taken into consideration here. Let i_0, ω_0, Ω_0 be the known values of these elements at the initial epoch, and i, ω, Ω their (unknown) values at the final epoch.

In the Figure on the preceding page, E_0 and γ_0 are the ecliptic and the (mean) vernal equinox at the initial epoch, and E and γ the ecliptic and (mean) equinox at the final epoch. The angle between the two ecliptics is denoted by η, and the orbit's perihelion by P.

As in Chapter 21, let T be the time interval, in Julian centuries, between J2000.0 and the initial epoch, and t the time interval, in the same units, between the initial epoch and the final epoch.

Then calculate the angles η, Π, and p by means of formulae (21.5) or, if the initial epoch is J2000.0, by means of (21.6).

Find $\psi = \Pi + p$. Then the quantities i and $\Omega - \psi$, and hence Ω, can be found from

$$\cos i = \cos i_0 \cos \eta + \sin i_0 \sin \eta \cos (\Omega_0 - \Pi) \tag{24.1}$$

$$\begin{aligned} \sin i \sin (\Omega - \psi) &= \sin i_0 \sin (\Omega_0 - \Pi) \\ \sin i \cos (\Omega - \psi) &= -\sin \eta \cos i_0 + \cos \eta \sin i_0 \cos (\Omega_0 - \Pi) \end{aligned} \tag{24.2}$$

Formula (24.1) should not be used when the inclination is too small.

Then $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is found from

$$\begin{aligned} \sin i \sin \Delta\omega &= -\sin \eta \sin(\Omega_0 - \Pi) \\ \sin i \cos \Delta\omega &= \sin i_0 \cos \eta - \cos i_0 \sin \eta \cos(\Omega_0 - \Pi) \end{aligned} \tag{24.3}$$

If $i_0 = 0$, then Ω_0 is not determined, and we have $i = \eta$ and $\Omega = \psi + 180°$.

It is important to note that the method described here reduces the orbital elements i, ω, and Ω from one *equinox* to another one, but the new orbital elements remain valid for the same *epoch* as the initial elements. It is, in fact, the same orbit. The calculation of the orbital elements for another *epoch* is a completely different problem (celestial mechanics!) which we cannot discuss here.

Example 24.a — In their *Catalogue Général des Orbites de Comètes de l'an −466 à 1952* (Observatoire de Paris, Section d'Astrophysique de Meudon; 1952), F. Baldet and G. De Obaldia give the following orbital elements for comet Klinkenberg (1744), referred to the mean equinox of B1744.0:

$$\begin{aligned} i_0 &= 47\overset{\circ}{.}1220 \\ \omega_0 &= 151\overset{\circ}{.}4486 \\ \Omega_0 &= 45\overset{\circ}{.}7481 \end{aligned}$$

Reduce these elements to the standard equinox of B1950.0.

The final epoch is B1950.0, or (JD) = 2433 282.4235 (see Chapter 21), and the initial epoch is 206 *tropical* years earlier (because both epochs correspond to the beginning of a Besselian year), whence

$$(\mathrm{JD})_0 = 2433\,282.4235 - (206 \times 365.242\,1988) = 2358\,042.5305.$$

We then find

$$\begin{array}{ll} T & -2.559\,958\,097 \\ t & +2.059\,956\,002 \\ \eta & +97''.0341 = +0\overset{\circ}{.}026\,954 \\ \Pi & 174\overset{\circ}{.}876\,384 - 10205''.9108 = 172\overset{\circ}{.}041\,409 \\ p & +10\,352''.7137 = +2\overset{\circ}{.}875\,754 \\ \psi & 174\overset{\circ}{.}917\,163 \end{array}$$

Then formulae (24.2) give

$$\begin{aligned} \sin i \sin(\Omega - \psi) &= -0.5906\,3831 = A \\ \sin i \cos(\Omega - \psi) &= -0.4340\,8084 = B \end{aligned}$$

from which we deduce $\sin i = \sqrt{A^2 + B^2} = 0.7329\,9372, \quad i = 47\overset{\circ}{.}1380$

$$\begin{aligned} \Omega - \psi &= \mathrm{ATN2}(A, B) = -126\overset{\circ}{.}313\,473 \\ \Omega &= 48\overset{\circ}{.}6037 \end{aligned}$$

Formulae (24.3) give $\sin i \sin \Delta\omega = +0.0003\,7917$
$\sin i \cos \Delta\omega = +0.7329\,9362$

whence $\Delta\omega = +0^\circ\!.0296$, and $\omega = 151^\circ\!.4782$.

In his *Catalogue of Cometary Orbits*, sixth edition (1989), Marsden gives the values $i = 47^\circ\!.1378$, $\omega = 151^\circ\!.4783$, $\Omega = 48^\circ\!.6030$. The discrepancy of $0^\circ\!.0007$ between the values of Ω results from the fact that the new IAU precession formulae yield for the general precession in longitude a value which is a little larger ($+1''\!.1$ per century) than that adopted by Newcomb. The effect over 206 years (from 1744 to 1950) amounts to 0.0006 degree.

If the initial equinox is that of B1950.0, and the final equinox that of J2000.0, the formulae simplify to the following ones.

$$\left.\begin{array}{l}
S = 0.000\,113\,9788 \qquad C = 0.999\,999\,9935 \\
W = \Omega_0 - 174^\circ\!.298\,782 \\
A = \sin i_0 \sin W \\
B = C \sin i_0 \cos W - S \cos i_0 \\
\sin i = \sqrt{A^2 + B^2} \qquad \tan x = \dfrac{A}{B} \\
\Omega = 174^\circ\!.997\,194 + x \\
\text{and finally} \quad \omega = \omega_0 + \Delta\omega, \quad \text{with} \\
\tan \Delta\omega = \dfrac{-S \sin W}{C \sin i_0 - S \cos i_0 \cos W}
\end{array}\right\} \qquad (24.4)$$

Care must be taken for the correct quadrant of the angles x and $\Delta\omega$. For safety, they should be calculated by means of the ATN2 function, if the latter is available in the programming language, for instance x = ATN2 (A, B). Except when the orbital inclination is *very* small, the new value of Ω should be approximately $0^\circ\!.7$ larger than the initial value Ω_0, and $\Delta\omega$ must lie near 0°, not near 180°.

Example 24.b — S. Nakano calculated the following orbital elements for the 1990 return of periodic comet Encke (*Minor Planet Circular* 12577):

Epoch = 1990 November 5.0 TD = JDE 2448 200.5

T = 1990 October 28.54502 TD

$$\begin{array}{ll}
q = 0.330\,8858 & i = 11^\circ\!.93911 \\
a = 2.209\,1404 & \Omega = 334^\circ\!.04096 \\
e = 0.850\,2196 & \omega = 186^\circ\!.24444
\end{array} \left.\vphantom{\begin{array}{l}a\\b\\c\end{array}}\right\} \ 1950.0$$

We wish to reduce i, Ω, and ω to the equinox J2000.0, and we find successively

W	$+159°.742\,178$	x	$+159°.752\,866$
A	$+0.071\,628\,4465$	Ω	$334°.75006$
B	$-0.194\,187\,3149$	$\Delta\omega$	$-0°.01092$
$\sin i$	$0.206\,9767$	ω	$186°.23352$
i	$11°.94524$		

The other orbital elements (T, q, a, e) remain unchanged, and the Epoch is still 1990 November 5.0.

However, formulae (24.4) assume that the elements i_0, ω_0, and Ω_0 are given in the FK5 system. To convert elements from B1950.0/FK4 to J2000.0/FK5, one may use the following algorithm due to Yeomans (note from D. K. Yeomans, Chairman IAU System Transition Committee, to Richard West, President of IAU Commission 20; 1990 August 10).

Let

$$\begin{aligned} L' &= 4.500\,016\,88 \text{ degrees} \\ L &= 5.198\,562\,09 \text{ degrees} \\ J &= 0.006\,519\,66 \text{ degrees} \\ W &= L + \Omega_0 \end{aligned}$$

Then we have

$$\begin{aligned} \sin(\omega - \omega_0)\sin i &= \sin J \sin W \\ \cos(\omega - \omega_0)\sin i &= \sin i_0 \cos J + \cos i_0 \sin J \cos W \\ \cos i &= \cos i_0 \cos J - \sin i_0 \sin J \cos W \\ \sin(L' + \Omega)\sin i &= \sin i_0 \sin W \\ \cos(L' + \Omega)\sin i &= \cos i_0 \sin J + \sin i_0 \cos J \cos W \end{aligned}$$

from which i, Ω, and ω can be deduced.

Example 24.c — Same starting values i_0, Ω_0, and ω_0 as in Example 24.b.

We obtain

$$\left.\begin{aligned} i &= 11°.94521 \\ \Omega &= 334°.75043 \\ \omega &= 186°.23327 \end{aligned}\right\} \text{FK5, J2000.0}$$

Chapter 25

Solar Coordinates

Low accuracy

When an accuracy of 0.01 degree is sufficient, the geocentric position of the Sun may be calculated by assuming a purely elliptical motion of the Earth; that is, the perturbations by the Moon and the planets may be neglected. The calculation can be performed as follows.

Let JD be the Julian (Ephemeris) Day, which can be calculated by means of the method described in Chapter 7. Then the time T, measured in Julian centuries of 36525 ephemeris days from the epoch J2000.0 (2000 January 1.5 TD), is given by

$$T = \frac{\mathrm{JD} - 2451\,545.0}{36525} \tag{25.1}$$

This quantity should be calculated with a sufficient number of decimals. For instance, five decimals are not sufficient (unless the Sun's longitude is required with an accuracy not better than one degree): remember that T is expressed in centuries, so that an error of 0.00001 in T corresponds to an error of 0.37 day in the time.

Then the geometric mean longitude of the Sun, referred to the mean equinox of the date, is given by

$$L_0 = 280^\circ\!.46646 + 36\,000^\circ\!.769\,83\,T + 0^\circ\!.000\,3032\,T^2 \tag{25.2}$$

The mean anomaly of the Sun is

$$M = 357^\circ\!.52911 + 35\,999^\circ\!.050\,29\,T - 0^\circ\!.000\,1537\,T^2 \tag{25.3}$$

(The mean anomaly of the Sun is the same as the mean anomaly of the Earth. For the definition of the mean anomaly, see Chapter 30.)

The eccentricity of the Earth's orbit is

$$e = 0.016\,708\,634 - 0.000\,042\,037\,T - 0.000\,000\,1267\,T^2 \tag{25.4}$$

Find the Sun's equation of the center C as follows:

$$\begin{aligned} C = & + (1^\circ\!.914\,602 - 0^\circ\!.004\,817T - 0^\circ\!.000\,014T^2) \sin M \\ & + (0^\circ\!.019\,993 - 0^\circ\!.000\,101T) \sin 2M \\ & + 0^\circ\!.000\,289 \sin 3M \end{aligned}$$

Then the Sun's true longitude is $\odot = L_0 + C$
and its true anomaly is $v = M + C$

The Sun's radius vector, or the distance between the centers of the Sun and the Earth, expressed in astronomical units, is given by

$$R = \frac{1.000\,001\,018\,(1 - e^2)}{1 + e \cos v} \tag{25.5}$$

The numerator of the fraction is a quantity which varies slowly with time. It is equal to

0.999 7190	in the year	1800
0.999 7204		1900
0.999 7218		2000
0.999 7232		2100

The Sun's longitude $\odot$, obtained by the method described above, is the true *geometric* longitude referred to the *mean* equinox of the date. This longitude is the quantity required for instance in the calculation of geocentric planetary positions.

If the *apparent* longitude λ of the Sun, referred to the *true* equinox of the date, is required, $\odot$ should be corrected for the nutation and the aberration. Unless high accuracy is required, this can be performed as follows.

$$\begin{aligned} \Omega &= 125^\circ\!.04 - 1934^\circ\!.136T \\ \lambda &= \odot - 0^\circ\!.00569 - 0^\circ\!.00478 \sin \Omega \end{aligned}$$

In some instances, for example in meteor work, it is necessary to have the Sun's longitude referred to the standard equinox of J2000.0. Between the years 1900 and 2100, this can be performed with sufficient accuracy from

$$\odot_{2000} = \odot - 0^\circ\!.01397\,(\text{year} - 2000)$$

If the Sun's longitude, referred to the standard equinox of J2000.0, should be obtained with a higher accuracy than 0.01 degree, the method given in Chapter 26 can be used.

Due to the actions of the Moon and the planets, the Sun's latitude is not exactly zero. Referred to the ecliptic of the date, it never exceeds 1.2 arcseconds. Unless high accuracy is required, this latitude may be put equal to zero. In that case, the

Sun's right ascension α and declination δ can be calculated from the following expressions where ε, the obliquity of the ecliptic, is given by (22.2).

$$\tan \alpha = \frac{\cos \varepsilon \, \sin \odot}{\cos \odot} \tag{25.6}$$

$$\sin \delta = \sin \varepsilon \, \sin \odot \tag{25.7}$$

If the *apparent* position of the Sun is required, then in formulae (25.6) and (25.7) one should use λ instead of $\odot$, and ε should be corrected by the quantity

$$+0^\circ\!.00256 \cos \Omega \tag{25.8}$$

Formula (25.6) may of course be transformed to $\tan \alpha = \cos \varepsilon \tan \odot$ but then it must be remembered that α must be in the same quadrant as $\odot$. However, if the ATN2 function is available in the programming language, it is better to leave formula (25.6) unchanged and to apply the ATN2 function to the numerator and the denominator of the fraction: $\alpha = \text{ATN2}(\cos \varepsilon \sin \odot, \cos \odot)$.

Example 25.a — Calculate the Sun's position on 1992 October 13 at 0^h TD.

This date corresponds to JDE 2448 908.5, and we find successively:

T	$-0.072\,183\,436$
L_0	$-2318^\circ\!.19280 = 201^\circ\!.807\,20$
M	$-2241^\circ\!.00603 = 278^\circ\!.993\,97$
e	$0.016\,711\,668$
C	$-1^\circ\!.897\,32$
$\odot$	$199^\circ\!.909\,88 = 199^\circ 54' 36''$
R	$0.997\,66$
Ω	$264^\circ\!.65$
λ	$199^\circ\!.908\,95 = 199^\circ 54' 32''$
ε_0	$23^\circ 26' 24''\!.83 = 23^\circ\!.440\,23$ [by (22.2)]
ε	$23^\circ\!.439\,99$

$\alpha_{app} = -161^\circ\!.619\,17 = +198^\circ\!.380\,83 = 13^h\!.225\,389 = 13^h 13^m 31^s\!.4$

$\delta_{app} = -7^\circ\!.785\,07 = -7^\circ 47' 06''$

The correct values, calculated by means of the complete VSOP87 theory (see Chapter 32), are:

geometric long., mean equinox of date :	$\odot = 199^\circ 54' 26''\!.18$
apparent longitude :	$\lambda = 199^\circ 54' 21''\!.56$
apparent latitude :	$\beta = +0''\!.72$
radius vector :	$R = 0.997\,608\,53$
apparent right ascension :	$13^h 13^m 30^s\!.749$
apparent declination :	$-7^\circ 47' 01''\!.74$

Higher accuracy

In their book *Planetary Programs and Tables from −4000 to +2800* (Wilmann-Bell, Richmond; 1986), Bretagnon and Simon give a method for the calculation of the longitude of the Sun with an accuracy that is sufficient for many applications. Their method yields an accuracy of 0.0006 degree ($2''.2$) between the years 0 and +2800, and of 0.0009 degree ($3''.2$) between −4000 and +8000, yet only 49 periodic terms are used.

A very high accuracy, better than 0.01 arcsecond, is obtained when use is made of the complete VSOP87 theory (see Chapter 32), but for the Earth this theory contains 2425 periodic terms, namely 1080 terms for the Earth's longitude, 348 for the latitude, and 997 for the radius vector. Evidently, this big amount of numerical data cannot be reproduced in this book. Instead, we give in Appendix III the most important terms from the VSOP87, allowing the calculation of the position of the Sun with an error not exceeding $1''$ between the years −2000 and +6000. The procedure is as follows.

Using from Appendix III the data for the *Earth*, calculate the latter's heliocentric longitude L, latitude B, and radius vector R for the given instant, as explained in Chapter 32. Don't forget that the time τ is measured from JDE 2451 545.0 in Julian *millennia* (365 250 days), not in centuries, and that the final values obtained for L and B are in radians.

To obtain the *geocentric* longitude $\odot$ and latitude β of the Sun, add 180° (or π radians) to L, and change the sign of B:

$$\odot = L + 180°, \qquad \beta = -B$$

Conversion to the FK5 system. — The Sun's longitude $\odot$ and latitude β obtained thus far are referred to the mean *dynamical* ecliptic and equinox of the date defined by the VSOP planetary theory of P. Bretagnon. This reference frame differs slightly from he standard FK5 system mentioned in Chapter 21. The conversion of $\odot$ and β to the FK5 system can be performed as follows, where T is the time in centuries from 2000.0, or $T = 10\tau$.

Calculate

$$\lambda' = \odot - 1^\circ.397\,T - 0^\circ.000\,31\,T^2$$

Then the corrections to $\odot$ and β are

$$\begin{aligned} \Delta\odot &= -0''.09033 \\ \Delta\beta &= +0''.03916\,(\cos\lambda' - \sin\lambda') \end{aligned} \tag{25.9}$$

These corrections are needed only for very accurate calculations. They may be dropped when use is made of the abridged version of the VSOP87 given in Appendix III.

Apparent place of the Sun. — The Sun's longitude ☉ obtained thus far is the true ("geometric") longitude of the Sun referred to the mean equinox of the date. To obtain the apparent longitude λ, the effects of nutation and aberration should be taken into account.

For the nutation, simply add to ☉ the nutation in longitude $\Delta\psi$ (Chapter 22). To take the aberration into account, apply to the Sun's geometric longitude the correction

$$-\frac{20''.4898}{R} \tag{25.10}$$

where R is the Earth's radius vector in AU. The numerator of the fraction is equal to the constant of aberration ($\kappa = 20''.49552$) multiplied by $a(1-e^2)$, the same as the numerator in formula (25.5). Therefore, the numerator of (25.10) actually varies very slowly with time, from 20".4893 in the year 0 to 20".4904 in the year +4000.

But, more important, formula (25.10) will not give a rigorously exact result, because it assumes an unperturbed motion of the Earth in its elliptical orbit. By reason of perturbations, mainly due to the Moon, the result can be up to 0.01 arcsecond in error.

When a very high accuracy is needed — this is not the case when the data of Appendix III are used for the calculation — the correction to the Sun's longitude due to the aberration can be obtained as follows. Find the variation $\Delta\lambda$ of the Sun's longitude, in arcseconds per day, as explained below. The correction for aberration is then

$$-\ 0.005\,775\,518\ R\ \Delta\lambda \tag{25.11}$$

where R is, as before the Sun's radius vector in astronomical units. The numerical constant is the light-time for unit distance, in days (= 8.3 minutes).

After the Sun's longitude has been corrected for nutation and aberration, we have obtained the Sun's *apparent longitude* λ. The apparent longitude λ and latitude β of the Sun can then be transformed into the apparent right ascension α and declination δ by means of formulae (13.3) and (13.4), where ε is the true obliquity of the ecliptic, that is, affected by the nutation in obliquity $\Delta\varepsilon$.

The variation $\Delta\lambda$ of the geocentric longitude of the Sun, in arcseconds per day, in the fixed reference frame J2000.0, can be obtained by means of the formula given on the next page, where τ is the time in millennia from J2000.0 (as in Chapter 32), and the arguments of the sines are in *degrees* and decimals.

In that expression, only the most important periodic terms have been retained. Consequently, the result will not be rigorous, but $\Delta\lambda$ will not be more than 0".1 in error. If the resulting value of $\Delta\lambda$ is used to calculate the Sun's aberration by means of (25.11), the error will be less than 0".001.

If, for some other application, the value of $\Delta\lambda$ is needed with respect to the mean equinox of the date instead of to a fixed reference frame, the constant term 3548.193 should be replaced by 3548.330.

Daily variation, in arcseconds, of the geocentric longitude of the Sun in a fixed reference frame

The time τ is measured from J2000.0 (JDE 2451 545.0) in Julian millennia.

The arguments of the sines are in degrees.

$$
\begin{aligned}
\Delta\lambda = {} & 3548.193 \\
& + 118.568 \sin(\ 87.5287 + 359\,993.7286\,\tau) \\
& + 2.476 \sin(\ 85.0561 + 719\,987.4571\,\tau) \\
& + 1.376 \sin(\ 27.8502 + 4452\,671.1152\,\tau) \\
& + 0.119 \sin(\ 73.1375 + 450\,368.8564\,\tau) \\
& + 0.114 \sin(337.2264 + 329\,644.6718\,\tau) \\
& + 0.086 \sin(222.5400 + 659\,289.3436\,\tau) \\
& + 0.078 \sin(162.8136 + 9224\,659.7915\,\tau) \\
& + 0.054 \sin(\ 82.5823 + 1079\,981.1857\,\tau) \\
& + 0.052 \sin(171.5189 + 225\,184.4282\,\tau) \\
& + 0.034 \sin(\ 30.3214 + 4092\,677.3866\,\tau) \\
& + 0.033 \sin(119.8105 + 337\,181.4711\,\tau) \\
& + 0.023 \sin(247.5418 + 299\,295.6151\,\tau) \\
& + 0.023 \sin(325.1526 + 315\,559.5560\,\tau) \\
& + 0.021 \sin(155.1241 + 675\,553.2846\,\tau) \\
& + 7.311\,\tau \sin(333.4515 + 359\,993.7286\,\tau) \\
& + 0.305\,\tau \sin(330.9814 + 719\,987.4571\,\tau) \\
& + 0.010\,\tau \sin(328.5170 + 1079\,981.1857\,\tau) \\
& + 0.309\,\tau^2 \sin(241.4518 + 359\,993.7286\,\tau) \\
& + 0.021\,\tau^2 \sin(205.0482 + 719\,987.4571\,\tau) \\
& + 0.004\,\tau^2 \sin(297.8610 + 4452\,671.1152\,\tau) \\
& + 0.010\,\tau^3 \sin(154.7066 + 359\,993.7286\,\tau)
\end{aligned}
$$

The periodic terms where τ has the coefficient 359 993.7, 719 987, or 1079 981, are due to the eccentricity of the Earth's orbit. The terms with 4452 671, 9224 660, or 4092 677 are due to the action of the Moon; those with 450 369, 225 184, 315 560, or 675 553 are due to Venus; those with 329 645, 659 289, or 299 296 are due to Jupiter; finally, the term with 337 181 is due to the action of Mars.

Example 25.b — Let us again, as in Example 25.a, calculate the position of the Sun for 1992 October 13.0 TD = JDE 2448 908.5.

Using from Appendix III the data for the Earth, we find by the method explained in Chapter 32,

$$L = -43.634\,847\,96 \text{ radians} = -2500.092\,628 \text{ degrees} = +19.907\,372 \text{ degrees}$$

$$B = -0.000\,003\,12 \text{ radian} = -0^\circ.000\,179 = -0''.644$$

$$R = 0.997\,607\,75$$

Whence

$$\odot = L + 180^\circ = 199^\circ.907\,372$$
$$\beta = +0''.644$$

Converting to the FK5 system, we find

$$\lambda' = 200^\circ.01 \qquad \Delta\odot = -0''.09033 = -0^\circ.000\,025 \qquad \Delta\beta = -0''.023$$

whence

$$\odot = 199^\circ.907\,347 = 199^\circ 54' 26''.449 \qquad \beta = +0''.62$$

The nutation is calculated by means of the method described in Chapter 22.

$$\Delta\psi = +15''.908 \qquad \Delta\varepsilon = -0''.308 \qquad \text{true } \varepsilon = 23^\circ.440\,1443$$

and by (25.10) the correction for aberration is $-20''.539$.

Hence, the Sun's apparent longitude is

$$\lambda = \odot + 15''.908 - 20''.539 = 199^\circ 54' 21''.818$$

Then, by (13.3) and (13.4),

$$\alpha = 198^\circ.378\,178 = 13^h 13^m 30^s.763$$
$$\delta = -7^\circ.783\,871 = -7^\circ 47' 01''.94$$

Resuming, the final results are

$$\odot = 199^\circ 54' 26''.45 \qquad R = 0.997\,607\,75$$
$$\lambda = 199^\circ 54' 21''.82 \qquad \alpha = 13^h 13^m 30^s.763$$
$$\beta = +0''.62 \qquad \delta = -7^\circ 47' 01''.94$$

Compare these results with the correct values mentioned at the end of Example 25.a. Our results are now much better than those obtained with the low-accuracy method.

Chapter 26

Rectangular Coordinates of the Sun

The rectangular geocentric equatorial coordinates X, Y, Z of the Sun are needed for the calculation of an ephemeris of a minor planet (see Chapter 33) or a comet. The origin of these coordinates is the center of the Earth. The X-axis is directed towards the vernal equinox (longitude 0°); the Y-axis lies in the plane of the equator too and is directed towards longitude 90°, while the Z-axis is directed towards the north celestial pole.

The values of X, Y, Z are given for each day at 0^h TD in the great astronomical almanacs; they are expressed in astronomical units. Generally they are not referred to the mean equator and mean equinox of the date, but to a standard equinox, for instance that of J2000.0.

Reference to the mean equinox of the date

Calculate the *geometric* coordinates of the Sun by means of the method "higher accuracy" described in Chapter 25, *with* the corrections (25.9) for reduction to the FK5 system, but *without* the corrections for nutation and aberration.

If $\odot$ and β are the geometric longitude and latitude of the Sun, and R its radius vector in astronomical units, then the required rectangular coordinates of the Sun, referred to the mean equator and equinox of the date, are given by

$$\begin{aligned}
X &= R \cos\beta \cos\odot \\
Y &= R\,(\cos\beta \sin\odot \cos\varepsilon - \sin\beta \sin\varepsilon) \\
Z &= R\,(\cos\beta \sin\odot \sin\varepsilon + \sin\beta \cos\varepsilon)
\end{aligned} \tag{26.1}$$

where ε is the *mean* obliquity of the ecliptic given by (22.2).

Since the Sun's latitude, referred to the ecliptic *of the date*, never exceeds 1.2 arcsecond, one may safely put $\cos\beta = 1$ in the formulae (26.1).

Example 26.a — For 1992 October 13.0 TD = JDE 2448 908.5, we have found in Example 25.b:

$$\odot = 199^\circ\!.907\,347 \qquad \beta = +0''\!.62 \qquad R = 0.997\,607\,75$$

For the same instant, formula (22.2) gives $\varepsilon = 23°26'24''\!.827 = 23^\circ\!.440\,2297$

whence, by (26.1),

$$\begin{aligned} X &= -0.937\,9952 \\ Y &= -0.311\,6544 \\ Z &= -0.135\,1215 \end{aligned}$$

Reference to the standard equinox J2000.0

As explained in Chapter 32, calculate for the given instant the Earth's heliocentric longitude L and latitude B referred to the equinox of J2000.0, and its radius vector R. For this purpose, use from Appendix III the data for the Earth, *with the following exceptions*:

- in section L1, replace the first value of the coefficient "A", namely 628 331 966 747, by 628 307 584 999;
- sections L2, L3, and L4 should be replaced by those given in Table 26.A (next page);
- drop section L5;
- for the calculation of the latitude B, use section B0 from Appendix III, but sections B1 to B4 from Table 26.A.

Obtain the geocentric longitude $\odot$ of the Sun by adding 180° (or π radians) to L, and the Sun's latitude β by changing the sign of B. That is,

$$\odot = L + 180^\circ \qquad \text{and} \qquad \beta = -B$$

At this stage, if *only* the Sun's geometric longitude referred to the standard equinox of J2000.0 is required, subtract 0″.09033 from $\odot$ in order to convert the longitude from the VSOP dynamical equinox to the FK5 equinox, as in (25.9). — Otherwise, do *not* perform this correction and proceed as follows.

Calculate

$$\begin{aligned} X &= R \cos\beta \cos\odot \\ Y &= R \cos\beta \sin\odot \\ Z &= R \sin\beta \end{aligned} \qquad (26.2)$$

Of course, these expressions are equivalent to $X = -R\cos B\cos L$, $Y = -R\cos B\sin L$, and $Z = -R\sin B$, respectively.

TABLE 26.A

EARTH J2000.0 (some terms only)

	No.	A	B	C
L2	1	8 722	1.072 5	6 283.075 8
	2	991	3.141 6	0
	3	295	0.437	12 566.152
	4	27	0.05	3.52
	5	16	5.19	26.30
	6	16	3.69	155.42
	7	9	0.30	18 849.23
	8	9	2.06	77 713.77
	9	7	0.83	775.52
	10	5	4.66	1 577.34
	11	4	1.03	7.11
	12	4	3.44	5 573.14
	13	3	5.14	796.30
	14	3	6.05	5 507.55
	15	3	1.19	242.73
	16	3	6.12	529.69
	17	3	0.30	398.15
	18	3	2.28	553.57
	19	2	4.38	5 223.69
	20	2	3.75	0.98
L3	1	289	5.842	6 283.076
	2	21	6.05	12 566.15
	3	3	5.20	155.42
	4	3	3.14	0
	5	1	4.72	3.52
	6	1	5.97	242.73
	7	1	5.54	18 849.23
L4	1	8	4.14	6 283.08
	2	1	3.28	12 566.15
B1	1	227 778	3.413 766	6 283.075 850
	2	3 806	3.370 6	12 566.151 7
	3	3 620	0	0
	4	72	3.33	18 849.23
	5	8	3.89	5 507.55
	6	8	1.79	5 223.69
	7	6	5.20	2 352.87
B2	1	9 721	5.151 9	6 283.075 85
	2	233	3.141 6	0
	3	134	0.644	12 566.152
	4	7	1.07	18 849.23
B3	1	276	0.595	6 283.076
	2	17	3.14	0
	3	4	0.12	12 566.15
B4	1	6	2.27	6 283.08
	2	1	0	0

The rectangular coordinates X, Y, Z calculated by means of (26.2) are still defined in the ecliptical dynamical reference frame (VSOP) of J2000.0. They can be transformed into the equatorial FK5 J2000.0 reference frame as follows:

$$\begin{aligned}
X_0 &= \quad X + 0.000\,000\,440\,360\,Y - 0.000\,000\,190\,919\,Z \\
Y_0 &= -0.000\,000\,479\,966\,X + 0.917\,482\,137\,087\,Y - 0.397\,776\,982\,902\,Z \\
Z_0 &= \quad 0.397\,776\,982\,902\,Y + 0.917\,482\,137\,087\,Z
\end{aligned} \tag{26.3}$$

Reference to the mean equinox of B1950.0

Proceed as above for J2000.0, except that expressions (26.3) should be replaced by the following ones.

$$\begin{aligned}
X_0 &= \quad 0.999\,925\,702\,634\,X + 0.012\,189\,716\,217\,Y + 0.000\,011\,134\,016\,Z \\
Y_0 &= -0.011\,179\,418\,036\,X + 0.917\,413\,998\,946\,Y - 0.397\,777\,041\,885\,Z \\
Z_0 &= -0.004\,859\,003\,787\,X + 0.397\,747\,363\,646\,Y + 0.917\,482\,111\,428\,Z
\end{aligned}$$

Note that the rectangular coordinates obtained in this way are referred to the mean equator and equinox of the epoch B1950.0 in the FK5 system, not in the FK4 system which is affected by the "equinox error" as mentioned in Chapter 21.

Reference to any other mean equinox

First, calculate the Sun's rectangular equatorial coordinates X_0, Y_0, Z_0 referred to the standard equinox of J2000.0 as explained above, that is, by means of the expressions (26.2) and (26.3).

Then, if JD is the Julian Day corresponding to the epoch of the given equinox, calculate

$$t = \frac{\text{JD} - 2451\,545.0}{36525}$$

and then the angles ζ, z, and θ from (21.3).

Then the required rectangular coordinates of the Sun are given by

$$\begin{aligned}
X' &= X_X X_0 + Y_X Y_0 + Z_X Z_0 \\
Y' &= X_y X_0 + Y_y Y_0 + Z_y Z_0 \\
Z' &= X_Z X_0 + Y_Z Y_0 + Z_Z Z_0
\end{aligned}$$

where

$$\begin{aligned}
X_x &= \cos\zeta \cos z \cos\theta - \sin\zeta \sin z \\
X_y &= \sin\zeta \cos z + \cos\zeta \sin z \cos\theta \\
X_z &= \cos\zeta \sin\theta \\
Y_x &= -\cos\zeta \sin z - \sin\zeta \cos z \cos\theta \\
Y_y &= \cos\zeta \cos z - \sin\zeta \sin z \cos\theta \\
Y_z &= -\sin\zeta \sin\theta \\
Z_x &= -\cos z \sin\theta \\
Z_y &= -\sin z \sin\theta \\
Z_z &= \cos\theta
\end{aligned}$$

Note that the coordinates X', Y', Z' are referred to the mean equinox of an epoch which differs from the date for which the values are calculated.

Example 26.b — For 1992 October 13.0 TD = JDE 2448 908.5, calculate the equatorial rectangular coordinates of the Sun referred to

(a) the standard equinox of J2000;
(b) that of B1950.0;
(c) the mean equinox of J2044.0.

We find successively

$$\begin{aligned}
\tau &= -0.007\,218\,343\,6003 \\
L &= -43.633\,088\,03 \text{ radians} = -2499.991\,791 \text{ degrees} \\
&= +20.008\,209 \text{ degrees} \\
B &= +0.000\,003\,86 \text{ radian} = +0^\circ\!.000\,221 = +0''\!.796 \\
R &= 0.997\,607\,75 \quad \text{(as in Example 25.b, of course)}
\end{aligned}$$

$$\left.\begin{aligned}
X &= -0.937\,395\,75 \\
Y &= -0.341\,336\,25 \\
Z &= -0.000\,003\,85
\end{aligned}\right\} \begin{array}{l}\text{ecliptic,}\\ \text{dynamical equinox,}\\ \text{J2000.0}\end{array}$$

$$\left.\begin{aligned}
X_0 &= -0.937\,395\,90 \\
Y_0 &= -0.313\,167\,93 \\
Z_0 &= -0.135\,779\,24
\end{aligned}\right\} \begin{array}{l}\text{equatorial,}\\ \text{FK5 frame,}\\ \text{J2000.0}\end{array}$$

The correct values, obtained by means of an accurate calculation using the complete VSOP87 theory, are $-0.937\,397\,07$, $-0.313\,167\,25$, and $-0.135\,778\,42$, respectively.

$$\left.\begin{aligned} X_0 &= -0.941\,487 \\ Y_0 &= -0.302\,666 \\ Z_0 &= -0.131\,214 \end{aligned}\right\} \begin{array}{l} \text{equatorial,} \\ \text{FK5 system,} \\ \text{B1950.0 frame} \end{array}$$

JD = 2467 616.0
(since the epoch J2044.0 is 44×365.25 days later than J2000.0)

$$\begin{aligned} t &= +0.440\,000 \\ \zeta &= +1014''.7959 = +0^\circ.281\,8878 \\ z &= +1014''.9494 = +0^\circ.281\,9304 \\ \theta &= +\ 881''.8106 = +0^\circ.244\,9474 \end{aligned}$$

$$\begin{array}{lll} X_x = +0.999\,9424 & Y_x = -0.009\,8403 & Z_x = -0.004\,2751 \\ X_y = +0.009\,8403 & Y_y = +0.999\,9516 & Z_y = -0.000\,0210 \\ X_z = +0.004\,2751 & Y_z = -0.000\,0210 & Z_z = +0.999\,9909 \end{array}$$

$$\left.\begin{aligned} X' &= -0.933\,680 \\ Y' &= -0.322\,374 \\ Z' &= -0.139\,779 \end{aligned}\right\} \begin{array}{l} \text{equatorial,} \\ \text{FK5 system,} \\ \text{J2044.0 frame} \end{array}$$

Chapter 27

Equinoxes and Solstices

By definition, the times of the equinoxes and solstices are the instants when the apparent geocentric longitude of the Sun (that is, calculated by including the effects of aberration and nutation) is an integer multiple of 90 degrees. (Because the latitude of the Sun is not exactly zero, the declination of the Sun is not exactly zero at the instant of an equinox.)

Approximate times can be obtained as follows. First, find the instant of the "mean" equinox or solstice, using the relevant expression in Table 27.A or in Table 27.B, on the next page. Note that Table 27.A should be used for the years -1000 to $+1000$ only, and Table 27.B for the years $+1000$ to $+3000$. In fact, Table 27.A may also be used for several centuries before the year -1000, and Table 27.B for several centuries after $+3000$; the errors will still be quite small.

Important: in the formula for Y, given at the top of each table, "year" is an *integer*; other values for "year" would give meaningless results!

Then find

$$T = \frac{\mathrm{JDE}_0 - 2451\,545.0}{36525}$$

$$W = 35\,999^\circ\!.373\,T - 2^\circ\!.47$$

$$\Delta\lambda = 1 + 0.0334 \cos W + 0.0007 \cos 2W$$

Calculate the sum S of the 24 periodic terms given in Table 27.C. Each of these terms is of the form $A \cos(B + CT)$, and the argument of each cosine is given in *degrees*. In other words,

$$\begin{aligned} S = {} & 485 \cos(324^\circ\!.96 + 1934^\circ\!.136\,T) \\ & + 203 \cos(337^\circ\!.23 + 32964^\circ\!.467\,T) \\ & + \ldots . \end{aligned}$$

The required time, expressed as a Julian Ephemeris Day (hence, in Dynamical Time), is then

$$\text{JDE} = \text{JDE}_0 + \frac{0.00001\, S}{\Delta\lambda} \text{ days}$$

This final JDE can be converted into the ordinary calendar date by means of the method described in Chapter 7. The result will be expressed in Dynamical Time.

For the years 1951–2050, the accuracy of this method is seen from Table 27.D.

TABLE 27.A *For the years −1000 to +1000* $Y = \frac{\text{year}}{1000}$

March equinox (beginning of astronomical spring):

$\text{JDE}_0 = 1721\,139.29189 + 365\,242.13740\,Y + 0.06134\,Y^2 + 0.00111\,Y^3 - 0.00071\,Y^4$

June solstice (beginning of astronomical summer):

$\text{JDE}_0 = 1721\,233.25401 + 365\,241.72562\,Y - 0.05323\,Y^2 + 0.00907\,Y^3 + 0.00025\,Y^4$

September equinox (beginning of astronomical autumn):

$\text{JDE}_0 = 1721\,325.70455 + 365\,242.49558\,Y - 0.11677\,Y^2 - 0.00297\,Y^3 + 0.00074\,Y^4$

December solstice (beginning of astronomical winter):

$\text{JDE}_0 = 1721\,414.39987 + 365\,242.88257\,Y - 0.00769\,Y^2 - 0.00933\,Y^3 - 0.00006\,Y^4$

TABLE 27.B *For the years +1000 to +3000* $Y = \frac{\text{year} - 2000}{1000}$

March equinox (beginning of astronomical spring):

$\text{JDE}_0 = 2451\,623.80984 + 365\,242.37404\,Y + 0.05169\,Y^2 - 0.00411\,Y^3 - 0.00057\,Y^4$

June solstice (beginning of astronomical summer):

$\text{JDE}_0 = 2451\,716.56767 + 365\,241.62603\,Y + 0.00325\,Y^2 + 0.00888\,Y^3 - 0.00030\,Y^4$

September equinox (beginning of astronomical autumn):

$\text{JDE}_0 = 2451\,810.21715 + 365\,242.01767\,Y - 0.11575\,Y^2 + 0.00337\,Y^3 + 0.00078\,Y^4$

December solstice (beginning of astronomical winter):

$\text{JDE}_0 = 2451\,900.05952 + 365\,242.74049\,Y - 0.06223\,Y^2 - 0.00823\,Y^3 + 0.00032\,Y^4$

TABLE 27.C

$S = \Sigma[A \cos(B + CT)]$ B and C in degrees!

A	B	C	A	B	C
485	324.96	1934.136	45	247.54	29929.562
203	337.23	32964.467	44	325.15	31555.956
199	342.08	20.186	29	60.93	4443.417
182	27.85	445267.112	18	155.12	67555.328
156	73.14	45036.886	17	288.79	4562.452
136	171.52	22518.443	16	198.04	62894.029
77	222.54	65928.934	14	199.76	31436.921
74	296.72	3034.906	12	95.39	14577.848
70	243.58	9037.513	12	287.11	31931.756
58	119.81	33718.147	12	320.81	34777.259
52	297.17	150.678	9	227.73	1222.114
50	21.02	2281.226	8	15.45	16859.074

TABLE 27.D

	Number of errors < 20 seconds	*Number of errors < 40 seconds*	*Largest error (seconds)*
March equinox	76	97	51
June solstice	80	100	39
September equinox	78	99	44
December solstice	68	99	41

Example 27.a — Find the time of the June solstice of A.D. 1962.

We find successively

$$
\begin{aligned}
Y &= -0.038 \\
\mathrm{JDE}_0 &= 2437\,837.38589 \\
T &= -0.375\,294\,021 \\
\Delta\lambda &= 0.9681 \\
S &= +635
\end{aligned}
$$

$$\mathrm{JDE} = 2437\,837.38589 + \frac{0.00635}{0.9681} = 2437\,837.39245$$

which corresponds to 1962 June 21 at $21^h25^m08^s$ TD.

The correct instant, as calculated with the complete VSOP87 theory, is $21^h24^m42^s$ Dynamical Time.

Of course, higher accuracy can be obtained by actually calculating the value of the apparent longitude of the Sun for two or three instants, and then finding by interpolation the time when that longitude is exactly 0°, or 90°, or 180°, or 270°.

One should keep in mind that the motion of the Sun along the ecliptic is only 3548 arcseconds per day, approximately. Hence, an error of 1″ in the calculated longitude of the Sun results in an error of approximately 24 seconds in the times of the equinoxes or solstices.

Alternatively, one may start from any approximate time. The value obtained from Table 27.A or 27.B is more than sufficient. For that instant, calculate the Sun's apparent longitude λ as explained in Chapter 25, including the corrections for reduction to the FK5 system, for aberration and for nutation. Then the correction to the assumed time, in *days*, is given by

$$+58 \sin(k.90^\circ - \lambda) \tag{27.1}$$

where

$$
\begin{aligned}
k = {} & 0 \text{ for the March equinox,} \\
& 1 \text{ for the June solstice,} \\
& 2 \text{ for the September equinox,} \\
& 3 \text{ for the December solstice.}
\end{aligned}
$$

The calculation is then repeated until the new correction is very small or, equivalently, until the new value for the Sun's apparent longitude is exactly $k.90°$.

Example 27.b — Let us again calculate the instant of the June solstice in 1962.

In Example 27.a, we found that the "mean" solstice took place at JDE_0 = 2437 837.38589 (from Table 27.B). Let us start from this approximate time, and calculate the Sun's apparent longitude for this instant, using the "higher accuracy" procedure (Chapter 25). We find

$$L = -234.048\,595\,59 \text{ radians} = 270^\circ.003\,272$$
$$R = 1.016\,3018$$

Nutation in longitude :	$\Delta\psi =$	$-12''.965$	(Chapter 22)
FK5 correction :		$-\ 0''.09033$	(formula (25.9))
aberration :		$-20''.161$	(formula (25.10))

Apparent longitude of the Sun:

$$\lambda = 270^\circ.003\,272 - 180^\circ - 12''.965 - 0''.09033 - 20''.161 = 89^\circ.994\,045$$

Formula (27.1) then gives the correction to the assumed value of JDE_0:

$$\text{correction} = +58 \sin(90^\circ - \lambda) = +0.00603$$

and hence the corrected time is

$$JDE = 2437\,837.38589 + 0.00603 = 2437\,837.39192$$

Repeating the calculation for this new instant, we find

$$\lambda = 89^\circ.999\,797,$$

resulting in the correction +0.00021 day. This gives the improved instant JDE = 2437 837.39213.

A final calculation, performed for this new instant, yields $\lambda = 89^\circ.999\,998$ and a correction smaller than 0.000 005 day.

Hence, the final instant is JDE = 2437 837.39213, which corresponds to 1962 June 21 at $21^h24^m40^s$ TD.

This differs by only two seconds from the correct time mentioned at the end of Example 27.a.

In 1962, the difference TD − UT was 34 seconds (see Table 10.A), so our result may be rounded to 21^h24^m Universal Time.

Table 27.E gives the times of the equinoxes and solstices for the years 1996 to 2005, to the nearest second of time.

Table 27.F gives the durations of the four astronomical seasons for some epochs. About the year −4080, the Earth was in perihelion at the beginning of the autumn, and consequently the summer had the same duration as the autumn, and the winter had the same duration as the spring. In A.D. 1246, the Earth was in perihelion at the time of the winter solstice, and consequently the spring had the same duration as the summer, and the autumn had the same duration as the winter. Since the year +1246, the winter is the shortest season; it will reach its minimum value by about A.D. 3500, and remain the shortest season till about A.D. 6427, when the Earth will be in perihelion at the time of the March equinox.

TABLE 27.E

Equinoxes and Solstices, 1996–2005, calculated by means of the complete VSOP87 theory. Instants are in Dynamical Time.

Year	March equinox		June solstice		Sept. equinox		Dec. solstice	
	d	*h m s*	*d*	*h m s*	*d*	*h m s*	*d*	*h m s*
1996	20	8 04 07	21	2 24 46	22	18 01 08	21	14 06 56
1997	20	13 55 42	21	8 20 59	22	23 56 49	21	20 08 05
1998	20	19 55 35	21	14 03 38	23	5 38 15	22	1 57 31
1999	21	1 46 53	21	19 50 11	23	11 32 34	22	7 44 52
2000	20	7 36 19	21	1 48 46	22	17 28 40	21	13 38 30
2001	20	13 31 47	21	7 38 48	22	23 05 32	21	19 22 34
2002	20	19 17 13	21	13 25 29	23	4 56 28	22	1 15 26
2003	21	1 00 50	21	19 11 32	23	10 47 53	22	7 04 53
2004	20	6 49 42	21	0 57 57	22	16 30 54	21	12 42 40
2005	20	12 34 29	21	6 47 12	22	22 24 14	21	18 36 01

TABLE 27.F

Duration of the astronomical seasons, in days

Year	Spring	Summer	Autumn	Winter
−4000	93.55	89.18	89.07	93.44
−3500	93.83	89.53	88.82	93.07
−3000	94.04	89.92	88.61	92.67
−2500	94.20	90.33	88.47	92.25
−2000	94.28	90.76	88.39	91.81
−1500	94.30	91.20	88.38	91.37
−1000	94.25	91.63	88.42	90.94
− 500	94.14	92.05	88.53	90.52
0	93.96	92.45	88.69	90.13
+ 500	93.73	92.82	88.91	89.78
1000	93.44	93.15	89.18	89.47
1500	93.12	93.42	89.50	89.20
2000	92.76	93.65	89.84	88.99
2500	92.37	93.81	90.22	88.84
3000	91.97	93.92	90.61	88.74
3500	91.57	93.96	91.01	88.71
4000	91.17	93.93	91.40	88.73
4500	90.79	93.84	91.79	88.82
5000	90.44	93.70	92.15	88.96
5500	90.11	93.50	92.49	89.15
6000	89.82	93.25	92.79	89.38
6500	89.58	92.96	93.04	89.66

Chapter 28

Equation of Time

Due to the eccentricity of its orbit, and to a much less degree due to the perturbations by the Moon and the planets, the Earth's heliocentric longitude does not vary uniformly. It follows that the Sun appears to describe the ecliptic at a non-uniform rate. Due to this, and also to the fact that the Sun is moving in the ecliptic and not along the celestial equator, its right ascension does not increase uniformly.

Consider a first fictitious Sun travelling along the *ecliptic* with a constant speed and coinciding with the true Sun at the perigee and apogee (when the Earth is in perihelion and aphelion, respectively). Then consider a second fictitious Sun travelling along the *celestial equator* at a constant speed and coinciding with the first fictitious Sun at the equinoxes. This second fictitious Sun is the *mean Sun*, and by definition its right ascension increases at a uniform rate — that is, there are no periodic terms, but its expression contains small secular terms in τ^2, τ^3, ...

When the mean Sun crosses the observer's meridian, it is mean noon there. True noon is the instant when the true Sun crosses the meridian. The *equation of time* is the difference between apparent and mean time. In other words, it is the difference between the hour angles of the true Sun and the mean Sun.

Defined in this manner, the equation of time E, at a given instant, is given by

$$E = L_0 - 0^\circ\!.0057183 - \alpha + \Delta\psi \,.\, \cos\varepsilon \qquad (28.1)$$

In this formula, L_0 is the Sun's mean longitude. According to the VSOP87 theory (see Chapter 32) we have, in degrees,

$$\begin{aligned} L_0 = 280.4664567 &+ 360007.6982779\,\tau \\ &+ 0.03032028\,\tau^2 + \tau^3/49931 \\ &- \tau^4/15300 - \tau^5/2000000 \end{aligned} \qquad (28.2)$$

where τ is the time measured in Julian millennia (365250 ephemeris days) from J2000.0 = JDE 2451545.0. L_0 should be reduced to less than 360° by adding or subtracting a convenient multiple of 360°.

In the French almanacs and in older textbooks, the equation of time is defined with opposite sign, hence being equal to mean time minus apparent time.

In formula (28.1), the constant $0^\circ.005\,7183$ is the sum of the mean value of the aberration in longitude ($-20''.49552$) and the correction for reduction to the FK5 system ($-0''.09033$); α is the apparent right ascension of the Sun, calculated by taking into account the aberration and the nutation. The quantity $\Delta\psi\,.\cos\varepsilon$, where $\Delta\psi$ is the nutation in longitude and ε the obliquity of the ecliptic, is needed to refer the apparent right ascension of the Sun to the *mean* equinox of the date, as is the mean longitude L_0.

In formula (28.1), the quantities L_0, α, and $\Delta\psi$ should be expressed in degrees. Then the equation of time E will be expressed in degrees, too; it can be converted to minutes of time by multiplication by 4.

The equation of time E can be positive or negative. If $E > 0$, the true Sun crosses the observer's meridian before the mean Sun.

The equation of time is always smaller than 20 minutes in absolute value. If $|E|$ appears to be too large, add 24 hours to or subtract it from your result.

Example 28.a — Find the equation of time on 1992 October 13 at 0^h TD.

This date corresponds to JDE = 2448 908.5, from which we deduce

$$\tau = \frac{\text{JDE} - 2451\,545.0}{365\,250} = -0.007\,218\,343\,600$$

$$L_0 = -2318^\circ.192\,807 = +201^\circ.807\,193$$

For the same instant we have, from Example 25.b,

$$\begin{aligned} \alpha &= 198^\circ.378\,178 \\ \Delta\psi &= +15''.908 = +0^\circ.004\,419 \\ \varepsilon &= 23^\circ.440\,1443 \end{aligned}$$

whence, by formula (28.1),

$$E = +3^\circ.427\,351 = +13.70940 \text{ minutes} = +13^m42^s.6$$

Alternatively, the equation of time can be obtained, with somewhat less accuracy, by means of the following formula given by Smart [1]:

$$E = y \sin 2L_0 - 2e \sin M + 4ey \sin M \cos 2L_0 - \frac{1}{2} y^2 \sin 4L_0 - \frac{5}{4} e^2 \sin 2M \tag{28.3}$$

where

$y = \tan^2 \dfrac{\varepsilon}{2}$, ε being the obliquity of the ecliptic,
L_0 = Sun's mean longitude,
e = eccentricity of the Earth's orbit,
M = Sun's mean anomaly.

The values of ε, L_0, e, and M can be found by means of the formulae (22.2), (28.2) or (25.2), (25.4), and (25.3), respectively.

The value of E given by formula (28.3) is expressed in radians. The result may be converted into degrees, and then into hours and decimals by division by 15.

Example 28.b — Find, once again, the value of the equation of time on 1992 October 13.0 TD = JDE 2448 908.5.

We find successively

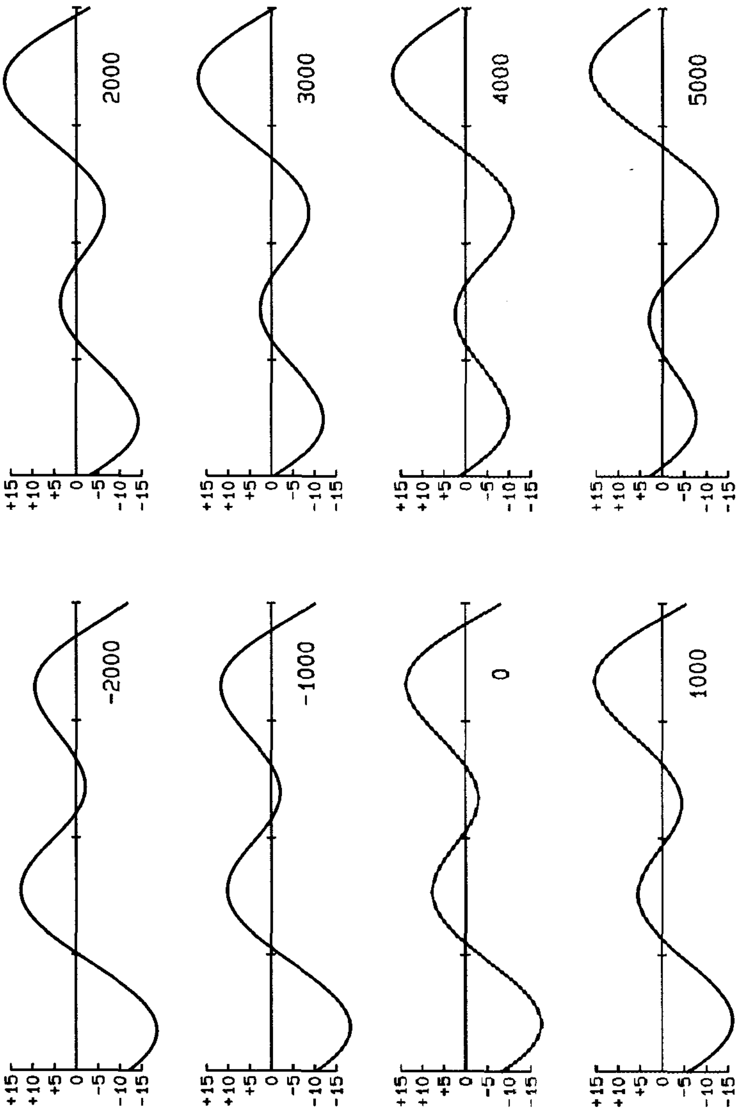
$T = -0.072\,183\,436$	$e = 0.016\,711\,668$
$\varepsilon = 23^\circ.44023$	$M = 278^\circ.99397$
$L_0 = 201^\circ.80720$	$y = 0.043\,0381$

Formula (28.3) then gives $E = +0.059\,825\,572$ radian
$= +3.427\,753$ degrees
$= +13$ minutes 42.7 seconds

The curve representing the variation of the equation of time during the year is well-known and can be found in many astronomy books. Presently, the curve has a deep minimum near February 11, a high maximum near November 3, and a secondary maximum and minimum about May 14 and July 26, respectively.

However, the curve of the equation of time is gradually changing in the course of the centuries, because the obliquity of the ecliptic, the eccentricity of the Earth's orbit, and the longitude of the perihelion of this orbit are all slowly changing. The figure on the next page shows the curve of the equation of time at intervals of 1000 years, from −2000 to +5000. On the vertical scale, the tics are given at intervals of five minutes of time; the horizontal line represents the value E = zero. The tics on this horizontal line divide the year in four periods of three months each, beginning from January 1 at left. We see, for instance, that the minimum of February will be less deep in the future.

The curve of the equation of time at intervals of 1000 years, from 2000 B.C. to A.D. 5000. For each curve, the scale is given at left, in minutes of time. The tics on the horizontal line divide the year in four quarters: January 1 at left, December 31 at right.



Between A.D. 1600 and 2100, the extreme values of the equation of time vary as shown in Table 28.A. These are "mean" values: the calculation is based on a non-perturbed elliptical motion of the Earth, and the nutation has not been taken into account.

In A.D. 1246, when the Sun's perigee coincided with the winter solstice, the curve representing the annual variation of the equation of time was exactly symmetrical with respect to the zero-line: the minimum of February was exactly as deep as the height of the November maximum, and the smaller May maximum was exactly as high as the value of the July minimum — see the last line of the Table.

TABLE 28.A

The extreme values of the equation of time in modern times

Year	*Minimum of February*	*Maximum of May*	*Minimum of July*	*Maximum of November*
	m s	*m s*	*m s*	*m s*
1600	−15 01	+4 19	−5 40	+16 03
1700	−14 50	+4 09	−5 53	+16 09
1800	−14 38	+3 59	−6 05	+16 15
1900	−14 27	+3 50	−6 18	+16 20
2000	−14 15	+3 41	−6 31	+16 25
2100	−14 03	+3 32	−6 44	+16 30
1246	−15 39	+4 58	−4 58	+15 39

REFERENCE

1. W. M. Smart, *Text-Book on Spherical Astronomy*; Cambridge (U.K.), University Press (1956); page 149.

Chapter 29

Ephemeris for Physical Observations of the Sun

The formulae given in this Chapter are based on the elements determined by Carrington (1863), which have been in use for many years. For a given instant, the required quantities are:

P = the position angle of the northern extremity of the axis of rotation, measured eastwards from the North Point of the solar disk;

B_0 = the heliographic latitude of the center of the solar disk;

L_0 = the heliographic longitude of the same point.

Although position angles are generally counted from 0° to 360° (this is the case for the Moon, the planets, double stars, etc.), in the case of the Sun it is customary to keep P, in absolute value, less than 90°, and to assign to it a plus or a minus sign: P is positive when the northern extremity of the rotation axis of the Sun is tilted to the East, negative if towards the West. Celestial and solar north can differ by up to 26 degrees. P reaches a minimum of $-26^\circ\!.3$ about April 7, a maximum of $+26^\circ\!.3$ about October 11, and is zero near January 5 and July 7.

B_0 represents the tilt of the Sun's north pole toward (+) or away (−) from Earth. It is zero about June 6 and December 7, and reaches a maximum value about March 6 ($-7^\circ\!.25$) and September 8 ($+7^\circ\!.25$).

L_0 *decreases* by about 13.2 degrees per day. The mean synodic period is 27.2752 days. The beginning of each "synodic rotation" is the instant at which L_0 passes through 0°. Rotation No. 1 commenced on 1853 November 9.

Let JD be the Julian Ephemeris Day, which can be calculated by means of the method described in Chapter 7. If the given instant is in Universal Time, add to JD the value ΔT = TD − UT expressed in days (see Chapter 10). If ΔT is expressed in seconds of time, the correction to JD will be $+\Delta T/86400$.

Then calculate the following quantities:

$$\theta = (\mathrm{JD} - 2398\,220) \times \frac{360^\circ}{25.38}$$

$$I = 7{^\circ\!\!.}25 = 7^\circ 15'$$

$$K = 73{^\circ\!\!.}6667 + 1{^\circ\!\!.}395\,8333\ \frac{\mathrm{JD} - 2396\,758}{36525}$$

where I is the inclination of the solar equator on the ecliptic, and K is the longitude of the ascending node of the solar equator on the ecliptic. In the formula for θ, 25.38 is the Sun's sidereal period of rotation in days. This value has been fixed *conventionally* by Carrington. It defines the zero meridian of the heliographic longitudes and therefore must be treated as *exact*. Strictly speaking, because the plane of the ecliptic slowly rotates (presently by 47" per century) while the rotation axis of the Sun is supposed to be fixed in space, the angle I slowly varies over time. However, it is astronomical practice to assign I the constant value $7{^\circ\!\!.}25$.

Calculate the *apparent* longitude λ of the Sun (including the effect of aberration, but *not* that of nutation) by the method described in Chapter 25, and the obliquity of the ecliptic ε (including the effect of nutation) as explained in Chapter 22. Let λ' be λ corrected for the nutation in longitude.

Then calculate the angles x and y by means of

$$\tan x = -\cos\lambda' \tan\varepsilon$$

$$\tan y = -\cos(\lambda - K)\tan I$$

where both x and y should be taken between -90° and $+90^\circ$. Then the required quantities P, B_0, and L_0 are found as follows:

$$P = x + y$$

$$\sin B_0 = \sin(\lambda - K)\sin I$$

$$\tan\eta = \frac{-\sin(\lambda - K)\cos I}{-\cos(\lambda - K)} = \tan(\lambda - K)\cos I$$

η being in the same quadrant as $\lambda - K \pm 180^\circ$,

$$L_0 = \eta - \theta, \quad \text{to be reduced to the interval 0–360 degrees.}$$

Example 29.a — Calculate P, B_0, and L_0 for 1992 October 13 at 0^h Universal Time = JD 2448 908.5.

We will use the value ΔT = +59 seconds = +0.000 68 day. Consequently the corrected JD, or Julian Ephemeris Day, is 2448 908.50068 and we find successively

$$\theta = 718\,985^\circ.8252 = 65^\circ.8252$$
$$I = 7^\circ.25$$
$$K = 75^\circ.6597$$

From Chapters 25 and 22:

$$L \text{ (Earth)} = -43.634\,836\,22 \text{ radians} = +19^\circ.908\,045$$
$$R = 0.997\,608$$
$$\Delta\psi = +15''.908 = +0^\circ.004\,419$$
$$\varepsilon = 23^\circ.440\,144$$

$$\text{correction for aberration} = -\frac{20''.4898}{R} = -0^\circ.005\,705$$

whence

$$\lambda = L + 180^\circ - 0^\circ.005\,705 = 199^\circ.902\,340$$
$$\lambda' = \lambda + \Delta\psi = 199^\circ.906\,759$$

$$\tan x = +0.407\,664 \qquad x = +22^\circ.1790$$
$$\tan y = +0.071\,584 \qquad y = +\;4^\circ.0945$$
$$P = 26^\circ.27$$
$$\sin B_0 = +0.104\,324 \qquad B_0 = +5^\circ.99$$

$$\tan \eta = \frac{-0.820\,053}{+0.562\,699} \qquad \eta = -55^\circ.5431$$

$$L_0 = -121^\circ.3683 = 238^\circ.63$$

As mentioned above, a solar "synodic rotation" begins when L_0 is equal to 0°. An approximate time for the beginning of Carrington's synodic rotation No. C is

$$\text{Julian Ephemeris Day} = 2398\,140.2270 + 27.275\,2316\,C \tag{29.1}$$

where, of course, C is an integer. The instant so obtained will be at most 0.16 day in error. However, the time obtained from the formula above can be corrected as follows. Calculate the angle M, in degrees, from

$$M = 281.96 + 26.882\,476\,C$$

Then the correction in days is

$$\begin{aligned} &+0.1454 \sin M \\ &-0.0085 \sin 2M \\ &-0.0141 \cos 2M \end{aligned} \tag{29.2}$$

Between the years 1850 and 2100, the resulting time will be less than 0.002 day in error.

Of course, a correct value for the time of the beginning of a synodic rotation can be obtained by calculating L_0 for two instants near the time given by the formula above, and then by performing an inverse interpolation to find when L_0 is zero.

Example 29.b — Find the instant of the beginning of solar rotation No. 1699.

For $C = 1699$, formula (29.1) gives JDE = 2444 480.8455.

We further find $M = 45\,955^\circ\!.287 = 235^\circ\!.287$, and the correction as given by (29.2) is -0.1225 day.

To convert from Dynamical Time to Universal Time, there is a further correction of -0.0006 day, because in 1980 the value of ΔT = TD − UT was 51 seconds.

Hence, the final instant is

$$\text{JD} = 2444\,480.8455 - 0.1225 - 0.0006 = 2444\,480.7224$$

which corresponds to 1980 August 29.22.

The *Astronomical Ephemeris* for 1980, page 359, gives the same value.

It is customary to give the times of the commencement of the Sun's synodic rotations to the nearest 0.01 day, hence in days and *decimals*, not in hours and minutes.

Chapter 30

Equation of Kepler

There are several methods for calculating the position of a body (planet, minor planet, or periodic comet) on its elliptical orbit around the Sun at a given instant:

— by numerical integration, a subject which is outside the scope of this book;

— obtaining the body's heliocentric coordinates (longitude, latitude, and radius vector) by calculating the sum of periodic terms, as will be explained in Chapter 32;

— from the orbital elements of the body, as explained in Chapter 33.

In the latter case, we need to find the true anomaly of the object. This can be achieved either by solving Kepler's equation or, when the orbital eccentricity is not too large, by using series expressions (see "The Equation of the Center" in Chapter 33).

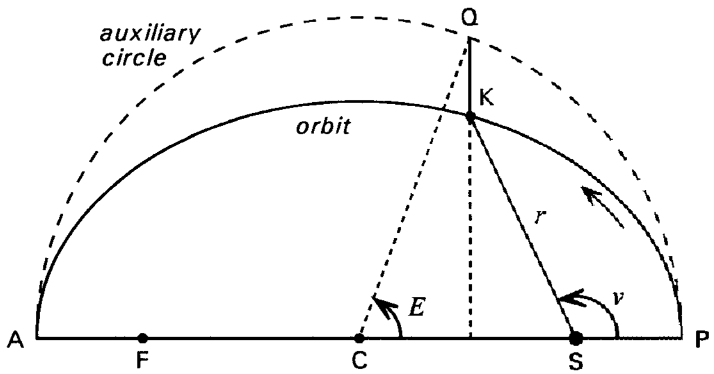


Figure 1

In Figure 1 we represent one half of an elliptical orbit PKA. The Sun is situated in the focus S; the other, empty focus of the ellipse is F. The straight line AP is the *major axis* of the orbit. The center C of the ellipse is exactly half-way between the perihelion P and the aphelion A, as well as half-way between the foci F and S.

Suppose that, at a given instant, the moving body is at K. The distance SK is the *radius vector* of the body at that instant; this distance r is expressed in astronomical units. The *true anomaly* (v) at the same instant is the angle between the directions SP and SK; it is the angle over which the object moved, as seen from the Sun, since the previous passage through the perihelion P.

The semimajor axis, CP in Figure 1, is generally designated by a and is expressed in astronomical units. By definition, the eccentricity e of the orbit is equal to the ratio of the distances CS and CP, or $e = CS/CP$. The eccentricity of an orbit is a measure of how much that orbit deviates from a circle. It takes values between 0 and 1 for an ellipse, 1 for a parabola, and larger than 1 for a hyperbola. For a perfect circle, $e = 0$.

The perihelion and aphelion distances are designated by q and Q, respectively. In the perihelion, $v = 0°$ and $r = q$, while in the aphelion we have $v = 180°$ and $r = Q$. It follows that

$$\begin{aligned}
\text{distance } CS &= ae \\
\text{distance } SP &= q &&= a(1-e) \\
\text{distance } SA &= Q &&= a(1+e) \\
\text{distance } PA &= 2a &&= q+Q
\end{aligned}$$

Let us now consider (Figure 2) a fictitious planet or comet K' describing around the Sun a circular orbit, hence with a constant velocity, with the *same period* as the real planet or comet K. Moreover, let us suppose that this fictitious body is at P', on the line SP, at the instant when the real body is at the perihelion P. Some time later, when the true body is at K, the fictitious body is at K'. As we have seen, the angle v = angle PSK is the true anomaly of the body (at the given instant). The angle PSK' at the same instant is called the *mean anomaly* and is generally designated by M.

In other words, the mean anomaly is the angular distance from perihelion which the planet would have if it moved around the Sun with a constant angular velocity. By definition, the angle M increases uniformly with time. The value of M at a given instant is easily found, for $M = 0°$ when the planet is at perihelion, and it increases by exactly 360° in the course of one complete revolution of the planet.

The problem consists in finding the true anomaly v when the mean anomaly M and the orbital eccentricity e are known. Unless use is made of series expressions such as those given in Chapter 33, one has to solve Kepler's equation.

In this connection, it is necessary to introduce an auxiliary angle E, called the *eccentric anomaly*, whose definition is illustrated in Figure 1. The exterior, dashed circle has diameter AP. We draw KQ perpendicular to AP. The angle PCQ is the eccentric anomaly.

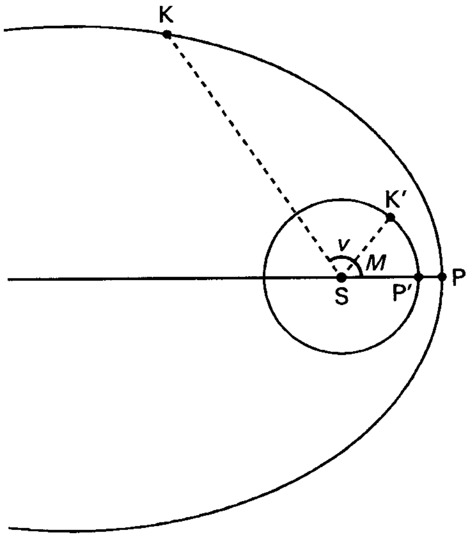


Figure 2

When the planet is at perihelion the angles v, E, and M are all zero. Near the perihelion, the true planet moves at a greater speed than the mean, fictitious planet. Hence, between perihelion and aphelion, when the planet moves away from the Sun, we have $v > M$ and, because E is always between v and M, we then have

$$0° < M < E < v < 180°.$$

In the aphelion, v, E, and M are all equal to 180°, and after aphelion passage, on its way back to perihelion, the true planet remains behind the mean planet.

When E is known, v can be obtained from

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \tag{30.1}$$

while the radius vector can be calculated from one of the following expressions:

$$r = a\,(1 - e \cos E) \tag{30.2}$$

$$r = \frac{a\,(1 - e^2)}{1 + e \cos v} \tag{30.3}$$

$$r = \frac{q\,(1 + e)}{1 + e \cos v} \tag{30.4}$$

But let us now consider the problem of finding the eccentric anomaly E. The equation of Kepler is

$$E = M + e \sin E \tag{30.5}$$

This equation must be solved for E. It is, however, a transcendental function which cannot be solved directly. Hundreds of methods of solution to the equation exist. An account of the history of solving the famous equation can be found in Colwell's book [1]. We will describe three iteration methods for finding the eccentric anomaly E, and finally give a formula which yields an approximate result.

First Method

In formula (30.5) the angles M and E should be expressed in *radians*. Hence the calculation should be performed in "radian mode", which is the case for many programming languages. If the calculation is made in "degree mode", then in (30.5) one should multiply e by $180/\pi$, or 57.295 7795, the factor for converting radians into degrees. Let e_0 be the thus "modified" eccentricity. Kepler's equation is then

$$E = M + e_0 \sin E \tag{30.6}$$

and now we can calculate with ordinary degrees.

To solve equation (30.6), give an approximate value to E in the right side of the formula. Then the formula will give a better approximation for E. This is repeated until the required accuracy is obtained. This process can be performed automatically in a computer program. For the first approximation, we may use $E = M$.

We thus have

$$\begin{aligned} E_0 &= M \\ E_1 &= M + e \sin E_0 \\ E_2 &= M + e \sin E_1 \\ E_3 &= M + e \sin E_2 \\ &\text{etc.} \end{aligned}$$

E_1, E_2, E_3, etc., are successive and better approximations for E.

Example 30.a — Solve the equation of Kepler for $e = 0.100$ and $M = 5°$, to an accuracy of 0.000 001 degree.

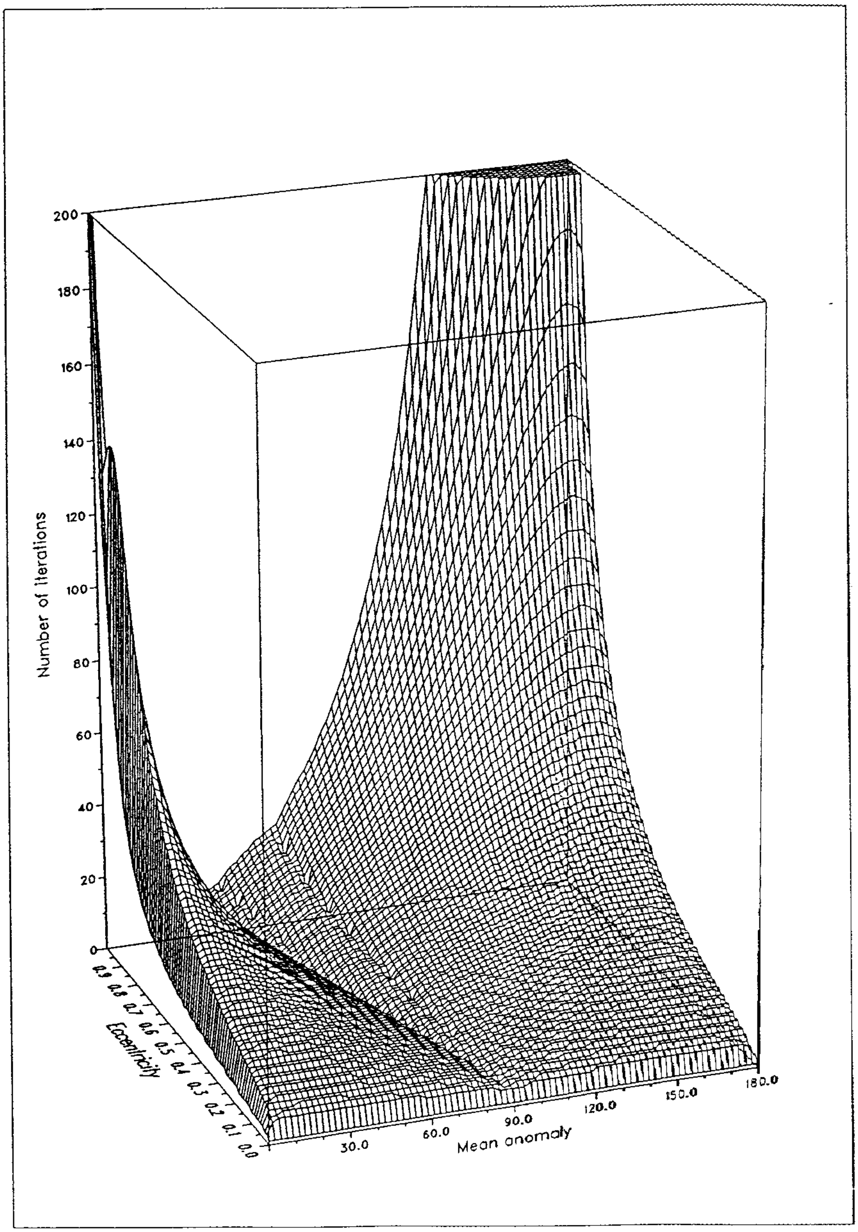
We have $e_0 = 0.100 \times 180/\pi = 5^\circ\!.729\,577\,95$, and the equation of Kepler becomes

$$E = 5 + 5.729\,577\,95 \sin E$$

where all quantities are in degrees. We must now, of course, work in degree mode. Starting with $E = M = 5°$, we obtain successively

5.499 366
5.549 093
5.554 042
5.554 535
5.554 584
5.554 589
5.554 589

Hence, the required value is $E = 5^\circ\!.554\,589$.



Number of Iterations
200
180
160
140
120
100
80
60
40
20
0
Eccentricity
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Mean anomaly
30.0
60.0
90.0
120.0
150.0
180.0

Figure 3

This method is very simple and does always converge. There will be no problems when e is small. However, the number of required iterations is generally increasing with e. For example, for $e = 0.990$ and $M = 2°$, the successive results of the iteration procedure are as follows:

2.000 000	15.168 909	24.924 579	29.813 009
3.979 598	16.842 404	25.904 408	30.200 940
5.936 635	18.434 883	26.780 556	30.533 515
7.866 758	19.937 269	27.557 863	30.817 592
9.763 644	21.341 978	28.242 483	.
11.619 294	22.643 349	28.841 471	:
13.424 417	23.837 929	29.362 399	:

After the 50th iteration, the result (32°345 452) still differs from the correct result (32.361 007) by more than 0.01 degree.

Figure 3, due to the Belgian calculator Edwin Goffin, is a three-dimensional representation of the number of iterations needed to obtain an accuracy of 10^{-9} degree, as a function of the orbital eccentricity and the mean anomaly. We see that the number of required iterations becomes large when the eccentricity approaches 1 *and* when the mean anomaly is close to 0° or to 180°. — Note that 10^{-9} degree (4 millionths of an arcsecond) is an absurdly high accuracy; it has been retained here merely as a mathematical exercise.

At the bottom of the drawing we notice a horizontal straight "valley". This valley extends from the point $e = 0$, $M = 90°$ to the point $e = 1$, $M = 32°42'$. (This latter value is equal to $\pi/2 - 1$ radians.) This means that, for any eccentricity e, there is a value M_0 of the mean anomaly for which the number of iterations (to solve Kepler's equation by the method described above) is a minimum. This "particular" mean anomaly is given by $M_0 = (\pi/2 - e)$ radians and corresponds to the solution $E = \pi/2$ radians $= 90°$ exactly.

The number of required iterations increases as M differs more from M_0, on both sides of the "valley". For instance, for $e = 0.75$ we have $M_0 = 47.03$ degrees, and the number of steps needed to obtain E with an accuracy of 0.000 001 degree is as follows:

M	*iter.*	*M*	*Iter.*
5°	51	60°	11
10°	37	70°	12
20°	23	90°	21
30°	15	110°	32
40°	9	130°	43
47°	5	150°	54
55°	8	170°	59

An interesting fact is that, when M is between M_0 and 180°, the results of the successive iterations *oscillate* while converging to the exact value: they do not constantly vary in the same direction as was the case in Example 30.a. For $e = 0.75$ and $M = 70°$, the results of the successive iterations are

70°000 000	starting value
110.380 316	larger
110.281 870	smaller
110.307 524	larger
110.300 850	smaller
110.302 587	larger
110.302 135	smaller
etc.	

Second Method

When the orbital eccentricity e is larger than 0.4 or 0.5, the convergence of the method described above can be so slow that it may be advisable to use a better iteration formula. A better value E_1 for E is

$$E_1 = E_0 + \frac{M + e \sin E_0 - E_0}{1 - e \cos E_0} \tag{30.7}$$

where E_0 is the last obtained value for E. In this formula, the angles M, E_0, and E_1 are all expressed in *radians*. If one wishes to work in "degree mode", then *in the numerator only* of the fraction the eccentricity e should be replaced by the "modified" eccentricity $e_0 = 180\,e/\pi$.

Here, again, the process should be repeated as often as is necessary.

Note the difference between formulae (30.6) and (30.7). The first one directly gives a new approximation for E. While formula (30.7) too gives a new approximation E_1 for the eccentric anomaly, the fraction in the second member is actually a *correction* to the previous value E_0.

Example 30.b — Same problem as in Example 30.a, but now using formula (30.7).

We shall work in degree mode, so in this case formula (30.7) takes the following form:

$$E_1 = E_0 + \frac{5 + 5.729\,577\,95 \sin E_0 - E_0}{1 - 0.100 \cos E_0}$$

Starting with $E_0 = M = 5°$, we obtain the following values:

E_0	*correction*	E_1
5.000 000 000	+0.554 616 193	5.554 616 193
5.554 616 193	−0.000 026 939	5.554 589 254
5.554 589 254	−0.000 000 001	5.554 589 253

In this case, an accuracy of 0.000 000 001 degree is obtained after only three iterations.

We solved Kepler's equation for some values of e and M; see Table 30.A, where the successive columns give the orbital eccentricity e, the mean anomaly M, the corresponding value of E, and the number of iterations needed by using the first (1) and the second (2) method, starting with $E = M$ as the first approximation. A computer working with twelve significant digits was used, and iterations were performed until the new value of E differed from the previous one by less than 0.000 001 degree.

It appears that, generally speaking, a larger value of e requires a larger number of iterations, for the first method as well as for the second one. But with the second method the number of these iterations is much smaller.

For small values of the eccentricity, say for $e < 0.3$, the first method still seems the best one: we may prefer to perform 5 or 10 *easy* iterations instead of two iterations with the more complicated formula (30.7). Only for larger values of the eccentricity is formula (30.7) to be preferred.

In some cases, the first method is disastrous. See the next-to-last line of the table, where no less than 150 iterations are needed to obtain E.

Finally, Table 30.A shows that the number of steps needed to obtain a given accuracy does not only depend on the value of e, but on that of M too. See the last line of the table, where the first method requires only six iterations, in spite of the large value of the orbital eccentricity, $e = 0.99$.

TABLE 30.A

e	M	E	*(1)*	*(2)*
0.1	5°	5°554589	6	2
0.2	5	6.246908	9	2
0.3	5	7.134960	12	2
0.4	5	8.313903	16	2
0.5	5	9.950063	21	2
0.6	5°	12.356653	28	3
0.7	5	16.167990	39	3
0.8	5	22.656579	52	4
0.9	5	33.344447	58	5
0.99	5	45.361023	50	11
0.99	1°	24.725822	150	8
0.99	33	89.722155	6	5

Although for large values of the eccentricity formula (30.7) is superior to (30.6), there can still be problems. We performed some calculations with formula (30.7) on the old HP-85 microcomputer, each time taking M as starting value for E. Table 30.B gives the successive "better" values of E (in degrees) for three cases.

TABLE 30.B

$e = 0.99$ $M = 2°$	$e = 0.999$ $M = 6°$	$e = 0.999$ $M = 7°$
188.700250865	930.362114752	832.86912333
90.0043959725	418.384869795	275.954959759
58.7251974236	−345.064633754	−87.610596019
41.762008288	10182.3247508	−48.5623921307
34.1821261793	1840.68260539	−11.225108839
32.4485414136	−5573.41581953	340.962715254
32.361223124	−2776.37618814	−5996.93473678
32.3610074734	−478.97469399	−2079.96780001
32.3610074722	−185.902957505	511.49423506
32.3610074722	−86.6958017962	257.391360843
	−48.9711628749	5.969894505
	−14.7148241705	1094.05946279
	168.189220986	−33606.763133
	92.1098260913	−12599.3759885
	64.2252288664	11889243.763
	52.4123211568	3642203.90477
	49.7106850572	−432120.48862
	49.5699983807	−145379.711482
	49.5696248567	142691.415319
	49.5696248539	56806.8295471
		....

In the first example ($e = 0.99$, $M = 2°$) we start with $E = 2°$. The first iteration gives $E = 188{\overset{\circ}{.}}7$, which is even farther away from the solution! But thereafter come the values 90°, 59°, 42°, and then the procedure converges rapidly: after the eighth iteration the result is reached with an accuracy of 0.000004 arcsecond.

In the second case ($e = 0.999$, $M = 6°$), the first iterations give bizarre values, almost as if by a random-number generator! There is no convergence at all, until after the 13th iteration the value 168° is obtained; seven more steps then give us the correct solution.

Third case: same eccentricity, but now $M = 7°$. Here, too, the successive results jump irregularly back and forth, and after 20 steps still nothing reasonable is reached. Not before the *47th iteration* (not shown in the table) do we obtain the correct solution, namely $52{\overset{\circ}{.}}2702615$.

It is truly remarkable that for the same eccentricity 0.999, but for $M = 7{\overset{\circ}{.}}01$ instead of $7{\overset{\circ}{.}}00$, the correct value of E is reached after only *twelve* iterations.

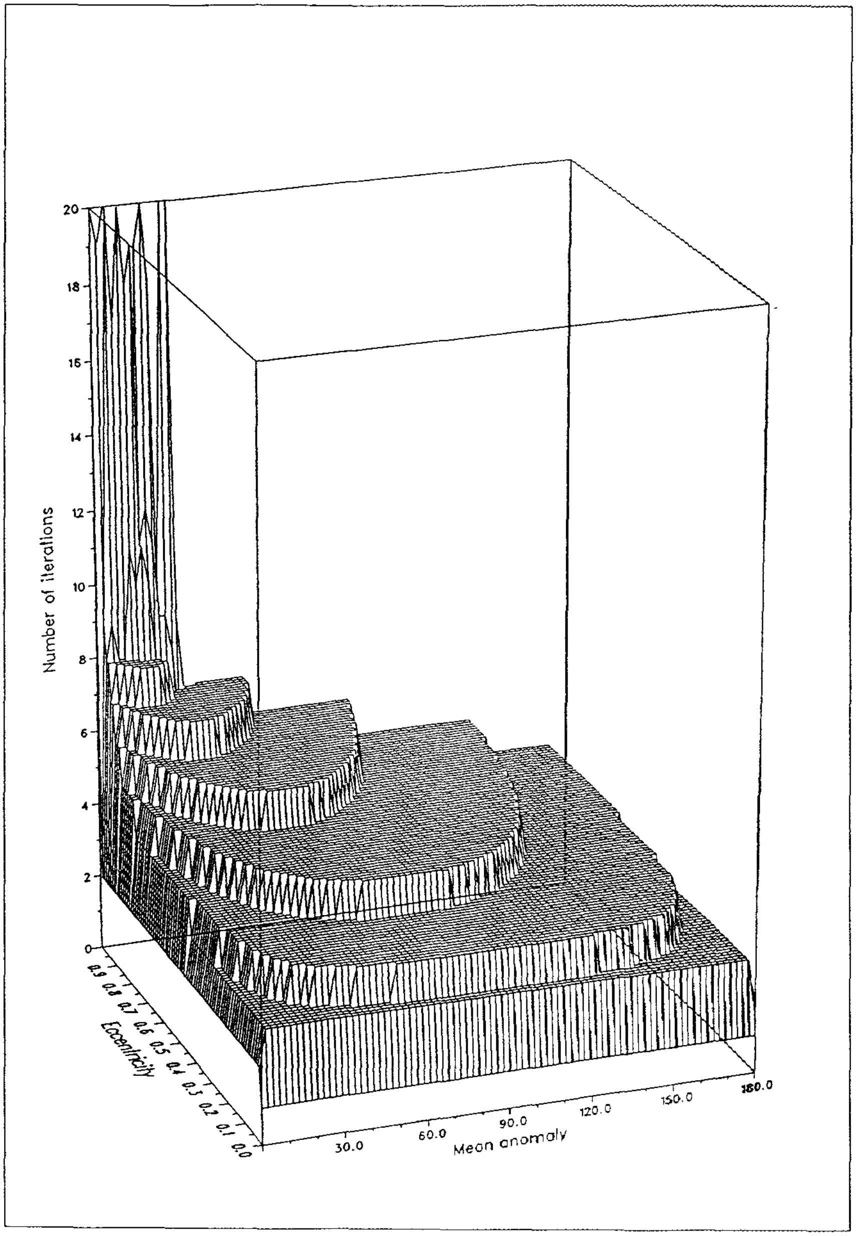

Number of iterations
0
2
4
6
8
10
12
14
16
18
20
Eccentricity
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Mean anomaly
30.0
60.0
90.0
120.0
150.0
180.0

Figure 4

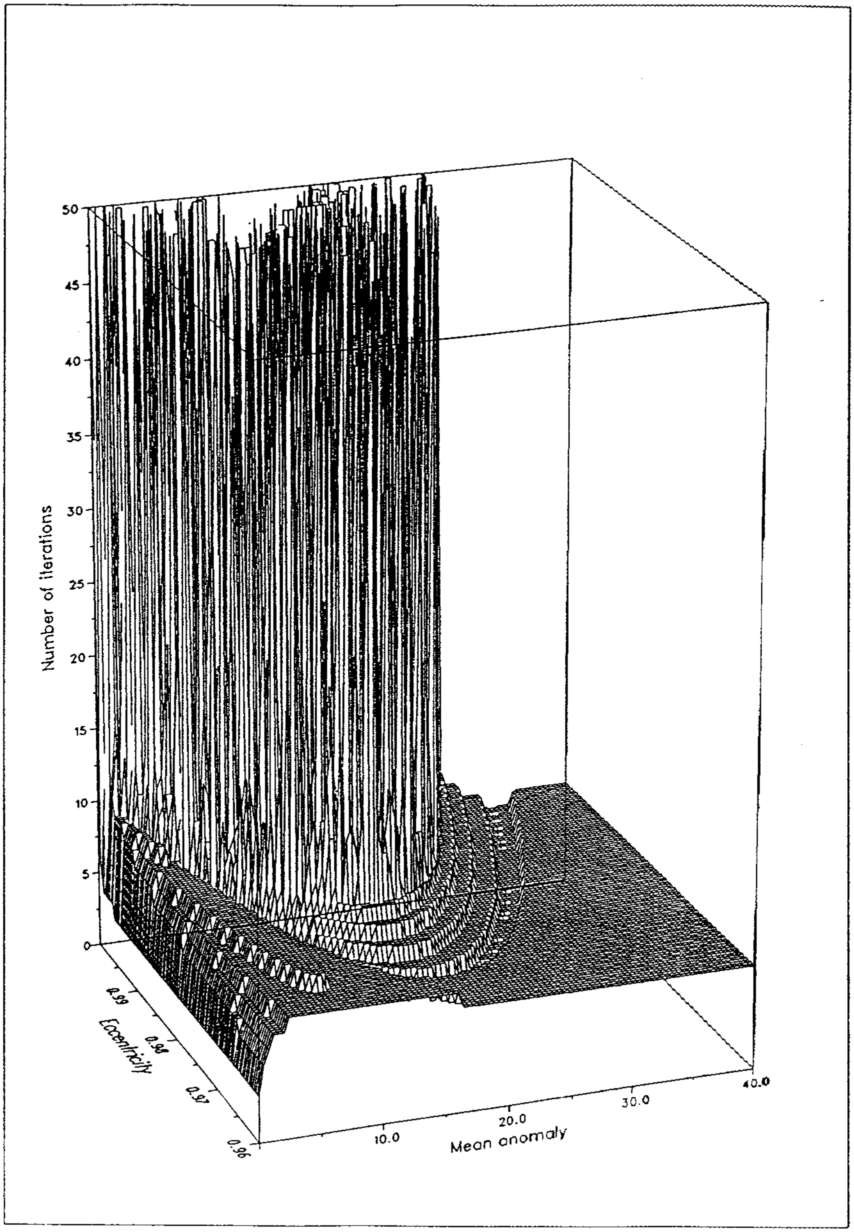

Number of iterations
50
45
40
35
30
25
20
15
10
5
0
Eccentricity
0.99
0.98
0.97
0.96
Mean anomaly
10.0
20.0
30.0
40.0

Figure 5

The HP-85 worked with 12 significant digits. If you use another computer with another programming language, the number of iterations can sometimes differ appreciably from those we mention here. When one calculates the second case ($e = 0.999$, $M = 6°$) with the HP-67 *pocket* calculator, which works with 10 significant digits, the successive results (in degrees) are

930.3621195
418.3848584
−345.0649049
10182.69391
1883.665232
−162.6729360
−85.06198931
−47.82386405
−13.18454655
211.0527629
84.65261970
60.76546811
51.35803706
49.62703439
49.56968687
49.56962485
49.56962485

It is interesting to compare these values with those of Table 30.B. After the third iteration, the difference with the value obtained with the HP-85 is still 0.00027 degree only. After the next iteration, the difference is 0°.37, and after the next one it is 43 degrees! Nevertheless, convergence to the exact value is eventually achieved.

It is evident that, when e is large, formula (30.7) guarantees only a *local* convergence. The successive results jump irregularly back and forth, and only when by chance a result falls into the "right domain" do the next results converge rapidly.

Figure 4, due to Goffin, is a three-dimensional representation of the number of steps needed to obtain E with an accuracy of 10^{-9} degree, as a function of the orbital eccentricity and the mean anomaly, when formula (30.7) is used. As before, M is used as the starting value for E. The left corner, near $e = 1$ and $M = 0°$, is the "dangerous zone". Figure 5 shows a magnification of that zone: we see a large number of peaks which are close together; the number of iterations needed to obtain the stated accuracy differs considerably even when e or M is changed very little.

Consequently, formula (30.7) is rather worrying for large values of e and small values of M. In some cases, the computer runs the risk of overflowing because the denominator of the fraction becomes almost zero. This trouble can be avoided by choosing, as a starting value for E, a better value than just M.

Mikkola [2] proposed a procedure to find such a good starting value. It was reproduced in the first edition of this book [3].

However, there are easier ways to avoid the (sometimes) many irregular jumps of the results of the successive iterations when e is large. We note that Kepler's equation can be written as $E - M = e \sin E$, the second member of which can never exceed 1 in absolute value, and has the same sign as E. Therefore, the fraction term in (30.7) should never be allowed to exceed a magnitude of 1.

One method is to take *the arcsine of the sine* of the fraction. This will result in a value which is always between $-90°$ and $+90°$. This trick was mentioned to the author by Kurt Leingärtner, of Kassel, Germany.

As an example, consider the case $e = 0.99$, $M = 0.2$ radian. We will work in radian mode. On the first step, the fraction in formula (30.7) takes the value 6.614 719 035 698 radians which, by taking the arcsine of its sine, changes to 0.331 533 728 518. The successive iterations yield the following results:

correction to E	*new value of E*
0.331 533 728 518	0.531 533 728 518
1.161 431 415 069	1.692 965 143 587
−0.455 401 365 518	1.237 563 778 069
−0.150 884 433 942	1.086 679 344 127
−0.019 368 331 549	1.067 311 012 578
−0.000 313 565 645	1.066 997 446 933
−0.000 000 081 651	1.066 997 365 282
$< 10^{-14}$	1.066 997 365 282

Hence, the final result is 1.066 997 365 282 radians, or 61.134 445 78 degrees.

Another interesting trick, which avoids the extra functions sine and arcsine, was devised by John M. Steele, of Bloomfield Hills, Michigan [4]. If the absolute value of the fraction in formula (30.7) is larger than 0.5, it is replaced by 0.5, preserving the sign. In BASIC, w being the value of the fraction:

```
IF ABS (w) > 0.5 THEN cor = 0.5 * SGN(w) ELSE cor = w
```

According to Steele, a "limit value" of 1 (instead of 0.5) works, although smaller values in the range 0.4–0.6 seem to work better.

Let us again consider the case $e = 0.99$, $M = 0.2$ radian. On the first step, the fraction in formula (30.7) takes the value 6.614 719 035 698 radians, which is changed to the "limit value" 0.5. The successive iterations yield the following results:

correction (rad)	*changed to*	*new value of E (rad)*
6.614 719 035 698	0.5	0.7
0.567 429 870 979	0.5	1.2
−0.120 513 681 086	unchanged	1.079 486 318 914
−0.012 361 504 682	unchanged	1.067 124 814 232
−0.000 127 435 465	unchanged	1.066 997 378 767
−0.000 000 013 485	unchanged	1.066 997 365 282

Third Method

Roger Sinnott [5] devised a method using a binary search to locate the correct value of E. The binary search was already mentioned at the end of Chapter 5. The procedure is absolutely foolproof, it always converges to the most exact value of which the machine is capable, and it works for any eccentricity between 0 and 1. The relevant part of Sinnott's program, in BASIC, is given below. Here, E is the orbital eccentricity, and M the mean anomaly in radians. The result of the program is the eccentric anomaly E_0 expressed in radians, too.

For a computer language with 10-digit accuracy, 33 steps are needed in the binary search. The number of loops in line 180 should be increased to 53 if you are using a 16-digit BASIC. The number of steps needed is 3.32 × the number of required digits, where 3.32 is equal to $1/\log_{10} 2$.

```
100  PI = 3.14159265359
110  F = SGN (M) : M = ABS (M) / (2 * PI)
120  M = (M - INT (M)) * 2 * PI * F
130  IF M < 0 THEN M = M + 2 * PI
140  F = 1
150  IF M > PI THEN F = -1
160  IF M > PI THEN M = 2 * PI - M
170  E0 = PI/2 : D = PI/4
180  FOR J = 1 TO 33
190  M1 = E0 - E * SIN (E0)
200  E0 = E0 + D * SGN (M - M1) : D = D/2
210  NEXT J
220  E0 = E0 * F
```

Fourth Method

The formula

$$\tan E = \frac{\sin M}{\cos M - e} \tag{30.8}$$

gives an *approximate* value for E, and is valid only for small values of the eccentricity.

For the same data as in Example 30.a, the formula (30.8) gives

$$\tan E = \frac{+0.087\,155\,74}{+0.896\,194\,70} = +0.097\,250\,90$$

whence $E = 5°.554599$, the exact value being $5°.554589$, so the error is only $0''.035$ in this case. But for the same eccentricity and $M = 82°$, the error amounts to $35''$.

The greatest error due to the use of formula (30.8) is

$0°.0327$	for $e = 0.15$
0.0783	for $e = 0.20$
0.1552	for $e = 0.25$
1.42	for $e = 0.50$
24.7	for $e = 0.99$

For the orbit of the Earth ($e = 0.0167$), the error is less than $0''.2$. In that case, formula (30.8) can safely be used unless high accuracy is needed.

REFERENCES

1. Peter Colwell, *Solving Kepler's Equation* (Willmann-Bell, 1993).
2. Seppo Mikkola, "A cubic approximation for Kepler's Equation", *Celestial Mechanics*, Vol. 40, pages 329–334 (1987).
3. Jean Meeus, *Astronomical Algorithms*, page 193 (Willmann-Bell, 1991).
4. John M. Steele, personal communication to Jean Meeus, 1994 November 20.
5. Roger W. Sinnott, *Sky and Telescope*, Vol. 70, page 159 (August 1985).

Chapter 31

Elements of the Planetary Orbits

Although Appendix III mentions the principal periodic terms needed to calculate the heliocentric positions of the planets (with explanations given in Chapter 32), it may be of interest to have information about the *mean* orbits of these bodies.

The orbital elements of the major planets can be expressed as polynomials of the form

$$a_0 + a_1 T + a_2 T^2 + a_3 T^3$$

where T is the time measured in Julian centuries of 36525 ephemeris days from the epoch J2000.0 = 2000 January 1.5 TD = JDE 2451 545.0.

In other words,

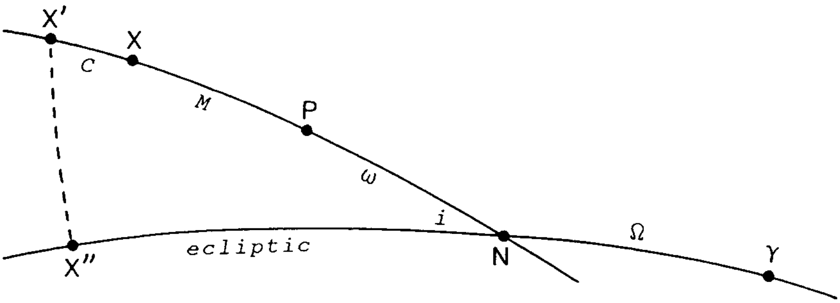
$$T = \frac{\text{JDE} - 2451\,545.0}{36525} \tag{31.1}$$

This quantity is negative before the beginning of the year 2000, positive afterwards. The orbital elements are:

L = mean longitude of the planet;
a = semimajor axis of the orbit;
e = eccentricity of the orbit;
i = inclination on the plane of the ecliptic;
Ω = longitude of the ascending node;
π = longitude of the perihelion.

Many authors denote the longitude of the perihelion by ϖ, which is a modified form of π. But this may be confusing because the *argument* of the perihelion has the symbol ω. For this reason, we prefer the symbol π for the longitude of the perihelion, and we have $\pi = \Omega + \omega$. (But don't confuse π with the parallax or with the number 3.14159...!)

Note that the angles L and π are measured in two different planes, namely from the vernal equinox along the ecliptic to the orbit's ascending node, and then from this node along the orbit. See the Figure on next page.



The arc γNX″ *is a part of the ecliptic as seen from the Sun, and* NPXX′ *is a part of the orbit of the planet (the intersection of the orbital plane with the celestial sphere).* γ *is the vernal equinox (longitude 0°),* N *the ascending node of the orbit,* P *the planet's perihelion. At a given instant, the mean planet is at* X, *the true planet at* X′. *Then we have*

Ω = *arc* γN = *longitude of the ascending node,*
ω = *arc* NP = *argument of the perihelion,*
π = *arc* γN + *arc* NP = $\Omega + \omega$ = *longitude of the perihelion,*
L = *arc* γN + *arc* NX = $\Omega + \omega + M$ = *mean longitude of the planet,*
M = *arc* PX = *planet's mean anomaly,*
C = *arc* XX′ = *equation of the center,*
v = *arc* PX′ = $M + C$ = *planet's true anomaly,*
i = *inclination of the orbit* = *angle between arcs* NP *and* NX″.

The planet's mean anomaly is given by

$$M = L - \pi$$

Table 31.A gives the coefficients a_0 to a_3 for the orbital elements of the planets Mercury to Neptune. The values for the semimajor axes are in astronomical units. Those for the angular quantities L, i, Ω, and π are expressed in degrees and decimals; they are referred to the ecliptic and mean equinox of the date.

The values have been deduced from a study by Simon *e.a.* [1]. However, in the case of the planets Mercury to Mars we added the correction $+0''.2766T$ to a_1 for the elements L, Ω, and π in order to bring them in accordance with the VSOP87 theory. The elements L, i, Ω, and π are actually referred to the mean *dynamical* ecliptic and equinox of the date, which differ very slightly from the FK5 system (see Chapter 25).

In some cases, it may be desirable to refer the elements L, i, Ω, and π to a standard equinox. This is the case, for instance, when one wishes to calculate the least distance between the orbit of a comet and that of a major planet, when the elements of the first orbit are referred to a standard equinox.

By means of Table 31.B, it is possible to calculate these elements for the major planets, referred to the standard equinox of J2000.0. The elements a and e are not modified by a change of reference frame, of course. They should be calculated by means of Table 31.A.

For the Earth, in order to avoid a discontinuity in the variation of the inclination and a jump of 180° in the longitude of the ascending node at the epoch J2000.0, the inclination on the ecliptic of 2000.0 is considered as negative before A.D. 2000.

Example 31.a — Calculate the mean orbital elements of Mercury on 2065 June 24 at 0^h TD.

We have (see Chapter 7)

$$2065 \text{ June } 24.0 = \text{JDE } 2475\,460.5$$

whence, by formula (31.1),

$$T = +0.654\,770\,704\,997$$

Consequently, from Table 31.A we find:

$$\begin{aligned} L &= 252^\circ\!.250\,906 + (149\,474^\circ\!.072\,2491 \times 0.654\,770\,704\,997) \\ &\quad + (0.000\,303\,50)\,(0.654\,770\,704\,997)^2 \\ &\quad + (0.000\,000\,018)\,(0.654\,770\,704\,997)^3 \\ &= 98\,123^\circ\!.494\,701 = 203^\circ\!.494\,701 \end{aligned}$$

$$\begin{aligned} a &= 0.387\,098\,310 & \pi &= 78^\circ\!.475\,382 \\ e &= 0.205\,645\,10 & & \text{from which we deduce} \\ i &= 7^\circ\!.006\,171 & M &= L - \pi = 125^\circ\!.019\,319 \\ \Omega &= 49^\circ\!.107\,650 & \omega &= \pi - \Omega = 29^\circ\!.367\,732 \end{aligned}$$

From Tables 31.A and 31.B it appears that the inclination of the orbit of Mercury on the ecliptic of the date is increasing, but that it is decreasing with respect to the fixed ecliptic of 2000.0. The opposite occurs for Saturn and Neptune.

Between $T = -30$ and $T = +30$, Venus' orbital inclination on the ecliptic of the date is continuously increasing, but with respect to the fixed ecliptic of 2000.0 Venus' inclination reached a maximum about the year +690.

Uranus' orbital inclination on the ecliptic of the date reached a minimum about the year +1000, but with respect to the fixed ecliptic of 2000.0 its value is continuously decreasing during the time period considered here.

The longitudes of the nodes, referred to the equinox of the date, are increasing for all planets. But with respect to the fixed equinox of 2000.0 these longitudes are decreasing, except for Jupiter and Uranus.

TABLE 31.A

Orbital Elements for the mean equinox of the date

	a_0	a_1	a_2	a_3
MERCURY				
L	252.250 906	+149 474.072 2491	+0.000 303 50	+0.000 000 018
a	0.387 098 310			
e	0.205 631 75	+0.000 020 407	−0.000 000 0283	−0.000 000 000 18
i	7.004 986	+0.001 8215	−0.000 018 10	+0.000 000 056
Ω	48.330 893	+1.186 1883	+0.000 175 42	+0.000 000 215
π	77.456 119	+1.556 4776	+0.000 295 44	+0.000 000 009
VENUS				
L	181.979 801	+58 519.213 0302	+0.000 310 14	+0.000 000 015
a	0.723 329 820			
e	0.006 771 92	−0.000 047 765	+0.000 000 0981	+0.000 000 000 46
i	3.394 662	+0.001 0037	−0.000 000 88	−0.000 000 007
Ω	76.679 920	+0.901 1206	+0.000 406 18	−0.000 000 093
π	131.563 703	+1.402 2288	−0.001 076 18	−0.000 005 678
EARTH				
L	100.466 457	+36 000.769 8278	+0.000 303 22	+0.000 000 020
a	1.000 001 018			
e	0.016 708 63	−0.000 042 037	−0.000 000 1267	+0.000 000 000 14
i	0			
π	102.937 348	+1.719 5366	+0.000 456 88	−0.000 000 018
MARS				
L	355.433 000	+19 141.696 4471	+0.000 310 52	+0.000 000 016
a	1.523 679 342			
e	0.093 400 65	+0.000 090 484	−0.000 000 0806	−0.000 000 000 25
i	1.849 726	−0.000 6011	+0.000 012 76	−0.000 000 007
Ω	49.558 093	+0.772 0959	+0.000 015 57	+0.000 002 267
π	336.060 234	+1.841 0449	+0.000 134 77	+0.000 000 536

TABLE 31.A (cont.)

	a_0	a_1	a_2	a_3
JUPITER				
L	34.351 519	+3036.302 7748	+0.000 223 30	+0.000 000 037
a	5.202 603 209	+0.000 000 1913		
e	0.048 497 93	+0.000 163 225	−0.000 000 4714	−0.000 000 002 01
i	1.303 267	−0.005 4965	+0.000 004 66	−0.000 000 002
Ω	100.464 407	+1.020 9774	+0.000 403 15	+0.000 000 404
π	14.331 207	+1.612 6352	+0.001 030 42	−0.000 004 464
SATURN				
L	50.077 444	+1223.511 0686	+0.000 519 08	−0.000 000 030
a	9.554 909 192	−0.000 002 1390	+0.000 000 004	
e	0.055 548 14	−0.000 346 641	−0.000 000 6436	+0.000 000 003 40
i	2.488 879	−0.003 7362	−0.000 015 19	+0.000 000 087
Ω	113.665 503	+0.877 0880	−0.000 121 76	−0.000 002 249
π	93.057 237	+1.963 7613	+0.000 837 53	+0.000 004 928
URANUS				
L	314.055 005	+429.864 0561	+0.000 303 90	+0.000 000 026
a	19.218 446 062	−0.000 000 0372	+0.000 000 000 98	
e	0.046 381 22	−0.000 027 293	+0.000 000 0789	+0.000 000 000 24
i	0.773 197	+0.000 7744	+0.000 037 49	−0.000 000 092
Ω	74.005 957	+0.521 1278	+0.001 339 47	+0.000 018 484
π	173.005 291	+1.486 3790	+0.000 214 06	+0.000 000 434
NEPTUNE				
L	304.348 665	+219.883 3092	+0.000 308 82	+0.000 000 018
a	30.110 386 869	−0.000 000 1663	+0.000 000 000 69	
e	0.009 455 75	+0.000 006 033	+0.000 000 0000	−0.000 000 000 05
i	1.769 953	−0.009 3082	−0.000 007 08	+0.000 000 027
Ω	131.784 057	+1.102 2039	+0.000 259 52	−0.000 000 637
π	48.120 276	+1.426 2957	+0.000 384 34	+0.000 000 020

TABLE 31.B

Orbital Elements for the standard equinox J2000.0

	a_0	a_1	a_2	a_3
MERCURY				
L	252.250 906	+149 472.674 6358	−0.000 005 36	+0.000 000 002
i	7.004 986	−0.005 9516	+0.000 000 80	+0.000 000 043
Ω	48.330 893	−0.125 4227	−0.000 088 33	−0.000 000 200
π	77.456 119	+0.158 8643	−0.000 013 42	−0.000 000 007
VENUS				
L	181.979 801	+58 517.815 6760	+0.000 001 65	−0.000 000 002
i	3.394 662	−0.000 8568	−0.000 032 44	+0.000 000 009
Ω	76.679 920	−0.278 0134	−0.000 142 57	−0.000 000 164
π	131.563 703	+0.004 8746	−0.001 384 67	−0.000 005 695
EARTH				
L	100.466 457	+35 999.372 8565	−0.000 005 68	−0.000 000 001
i	0	+0.013 0548	−0.000 009 31	−0.000 000 034
Ω	174.873 176	−0.241 0908	+0.000 042 62	+0.000 000 001
π	102.937 348	+0.322 5654	+0.000 147 99	−0.000 000 039
MARS				
L	355.433 000	+19 140.299 3039	+0.000 002 62	−0.000 000 003
i	1.849 726	−0.008 1477	−0.000 022 55	−0.000 000 029
Ω	49.558 093	−0.295 0250	−0.000 640 48	−0.000 001 964
π	336.060 234	+0.443 9016	−0.000 173 13	+0.000 000 518

TABLE 31.B (cont.)

	a_0	a_1	a_2	a_3
JUPITER				
L	34.351 519	+3034.905 6606	−0.000 085 01	+0.000 000 016
i	1.303 267	−0.001 9877	+0.000 033 20	+0.000 000 097
Ω	100.464 407	+0.176 7232	+0.000 907 00	−0.000 007 272
π	14.331 207	+0.215 5209	+0.000 722 11	−0.000 004 485
SATURN				
L	50.077 444	+1222.113 8488	+0.000 210 04	−0.000 000 046
i	2.488 879	+0.002 5514	−0.000 049 06	+0.000 000 017
Ω	113.665 503	−0.256 6722	−0.000 183 99	+0.000 000 480
π	93.057 237	+0.566 5415	+0.000 528 50	+0.000 004 912
URANUS				
L	314.055 005	+428.466 9983	−0.000 004 86	+0.000 000 006
i	0.773 197	−0.001 6869	+0.000 003 49	+0.000 000 016
Ω	74.005 957	+0.074 1431	+0.000 405 39	+0.000 000 119
π	173.005 291	+0.089 3212	−0.000 094 70	+0.000 000 414
NEPTUNE				
L	304.348 665	+218.486 2002	+0.000 000 59	−0.000 000 002
i	1.769 953	+0.000 2256	+0.000 000 23	−0.000 000 000
Ω	131.784 057	−0.006 1651	−0.000 002 19	−0.000 000 078
π	48.120 276	+0.029 1866	+0.000 076 10	+0.000 000 000

REFERENCE

1. J. L. Simon, P. Bretagnon, J. Chapront, M. Chapront-Touzé, G. Francou, J. Laskar, "Numerical expressions for precession formulae and mean elements for the Moon and the planets", *Astronomy & Astrophysics*, Vol. 282, pages 663–683 (1994).

Chapter 32

Positions of the Planets

In 1982, P. Bretagnon of the Bureau des Longitudes of Paris published his planetary theory VSOP82. The acronym VSOP means "Variations Séculaires des Orbites Planétaires". The VSOP82 consists of long series of periodic terms for each of the major planets Mercury to Neptune. When, for a given planet, the sums of these series are evaluated for a given instant, one obtains the values of the following quantities for the osculating orbit. The osculating orbit is the "instantaneous" orbit of the planet; see more about this notion in the next Chapter.

$$
\begin{aligned}
a &= \text{semimajor axis of the orbit} \\
\lambda &= \text{mean longitude of the planet} \\
h &= e \sin \pi \\
k &= e \cos \pi \\
p &= \sin \tfrac{1}{2} i \, \sin \Omega \\
q &= \sin \tfrac{1}{2} i \, \cos \Omega
\end{aligned}
$$

where e is the orbital eccentricity, π the longitude of the perihelion, i the inclination, and Ω the longitude of the ascending node.

Once a, λ, e and π (from h and k), i and Ω (from p and q) are known, the true position in space can be obtained for the given instant.

The inconvenience of the VSOP82 solution is that one does not know where the several series should be truncated when no full accuracy is required. Fortunately, in 1987 Bretagnon and Francou constructed the version called VSOP87, which gives periodic terms for calculating the planets' heliocentric coordinates directly, namely

L, the ecliptical longitude
B, the ecliptical latitude
R, the radius vector (= distance to the Sun)

Note that L is really the planet's ecliptical longitude, not the orbital longitude. In the figure on page 210, the *orbital* longitude of the planet is the sum of the arcs γN and NX' (in two different planes). Through the planet's position X', a great circle $X'X''$ is drawn perpendicularly to the ecliptic. Then the planet's *ecliptical* longitude is the measure of the arc $\gamma X''$.

Although the methods used for the construction of the VSOP82 and VSOP87 have been described in the astronomical literature (see the References 1 and 2), these theories themselves are available only on magnetic tape or on CD-ROM. By kind permission of Messrs. Bretagnon and Francou, we give in Appendix III the most important periodic terms from the VSOP87 theory. For each planet, series labelled L0, L1, L2, ..., B0, B1, ..., R0, R1, ... are provided.

The series L0, L1, ... are needed to calculate the planet's heliocentric ecliptical longitude L, the series B0, B1, ... are needed for the ecliptical latitude B, and the series R0, R1, ... are for the radius vector R.

Each horizontal line in the list represents one periodic term and contains four numbers:

— the current No. of the term in the series. It is *not* needed in the actual calculation and is given for reference purpose only;

— three numbers which we shall call here A, B, and C, respectively.

Let JDE be the Julian Ephemeris Day corresponding to the given instant. Calculate the time τ measured in Julian millennia from the epoch J2000.0

$$\tau = \frac{\mathrm{JDE} - 2451\,545.0}{365\,250} \tag{32.1}$$

The value of each term is given by

$$A \cos(B + C\tau)$$

For example, the ninth term of the series L0 for Mercury is equal to $1803 \cos(4.1033 + 5661.3320\tau)$.

In the lists of Appendix III, the quantities B and C are expressed in *radians*. The coefficients A are in units of 10^{-8} radian in the case of the longitude and the latitude, in units of 10^{-8} astronomical unit for the radius vector.

When a coefficient A has less decimals, then less decimals too are given for the corresponding B and C. This is merely done to avoid keypunching extraneous digits which do not influence the result.

To obtain the heliocentric ecliptical longitude L of a planet at a given instant, referred to the mean equinox of the date, proceed as follows. Calculate the sum *L0* of the terms of series L0, the sum *L1* of the terms of the series L1, etc. Then the required longitude in radians is given by

$$L = (L0 + L1\tau + L2\tau^2 + L3\tau^3 + L4\tau^4 + L5\tau^5) / 10^8 \tag{32.2}$$

Proceed similarly for the heliocentric latitude B and for the radius vector R.

The planet's heliocentric longitude L and latitude B, obtained thus far, are referred to the mean *dynamical* ecliptic and equinox of the date defined by Bretagnon's VSOP planetary theory. This reference frame differs very slightly from the standard FK5 system mentioned in Chapter 21. The conversion of L and B to the FK5 system can be performed as follows, where T is the time in centuries from 2000.0, or $T = 10\tau$.

Calculate

$$L' = L - 1^\circ\!.397\,T - 0^\circ\!.00031\,T^2$$

Then the corrections to L and B are

$$\begin{aligned} \Delta L &= -0''\!.09033 + 0''\!.03916\,(\cos L' + \sin L')\tan B \\ \Delta B &= +0''\!.03916\,(\cos L' - \sin L') \end{aligned} \qquad (32.3)$$

These corrections are needed only for very accurate calculations. They may be dropped when use is made of the abridged version of the VSOP87 given in Appendix III.

How to obtain the *geocentric* positions of the planets will be explained in Chapter 33.

Example 32.a — Calculate the heliocentric coordinates of Venus on 1992 Dec. 20 at 0^h Dynamical Time.

This instant corresponds to JDE 2448 976.5, from which

$$\tau = -0.007\,032\,169\,747.$$

For Venus, series L0 has 24 terms in Appendix III (there are many more in the original VSOP87 theory), L1 has 12 terms, L2 has 8 terms, L3 and L4 both have 3 terms, while L5 contains just a single term. For the sums of these series, we find

$$\begin{aligned} L0 &= +316\,402\,122 & \qquad L3 &= -56 \\ L1 &= +1\,021\,353\,038\,718 & L4 &= -109 \\ L2 &= +50\,055 & L5 &= -1 \end{aligned}$$

Hence, by formula (32.2), we find that the heliocentric longitude of Venus, for the given instant and referred to the mean equinox of the date, is

$$L = -68.659\,2582 \text{ radians} = -3933^\circ\!.88572 = +26^\circ\!.11428$$

We calculate the heliocentric latitude B and the radius vector R in the same way. Note that, in the case of Venus, the series B5 and R5 do not exist. The results are

$$B = -0.045\,7399 \text{ radian} = -2^\circ\!.62070, \qquad R = 0.724\,603 \text{ AU}$$

Accuracy of the results

When high accuracy is desired, it appears that the periodic terms in the VSOP87 solution converge rather slowly. What is the magnitude of the errors in the coordinates if one truncates the list of terms at any point? The following empirical rule has been given by Bretagnon and Francou [3]:

> If n is the number of retained terms, and A the amplitude of the smallest retained term, the accuracy of the thus truncated series is about $\eta\sqrt{n} \times A$, where η is a number smaller than 2.

As an example, consider the heliocentric longitude of Mercury. In Appendix III, series L0 for this planet contains 38 terms, and the coefficient of the smallest retained term is 100×10^{-8} radian. Therefore, we may expect that the greatest possible error in Mercury's heliocentric longitude, as calculated by means of that truncated series, is approximately

$$2 \times \sqrt{38} \times 100 \times 10^{-8} \text{ radian} = 2''.54.$$

Of course, series L1, L2, etc., are truncated too, which gives rise to additional uncertainties of the order of $0''.41\,\tau$, $0''.08\,\tau^2$, etc.

Polynomial Expressions

The giant planets Jupiter, Saturn, Uranus, and Neptune move so slowly on their orbits around the Sun, that it is possible to construct polynomial expressions giving their heliocentric coordinates, each expression being valid for one year.

We choosed polynomials of the fifth degree, so that the required value of the heliocentric longitude, latitude, or radius vector is given by

$$A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4 + A_5 t^5 \tag{32.4}$$

where t is the time (in the scale of Dynamical Time) measured from January 0.0 of the given year in units of 365 days. In other words, if d is the day of the year (with decimals, if any), then $t = d/365$. Note that even in the case of a bissextile (leap) year, the denominator in this formula is still 365.

The constants A_0 to A_5 are given in Appendix IV for the years 1998 to 2025. For each planet there are three polynomials per year: one for the heliocentric longitude (L), one for the latitude (B), and one for the radius vector (R). The coefficients are expressed in degrees for the longitude and the latitude, in astronomical units for the radius vector.

The coordinates so obtained are geometric, and they are referred to the mean equinox of the date in the FK5 reference frame.

For the years 1998 to 2012, January 0.0 corresponds to the following Julian Days:

Year	*JD*	*Year*	*JD*	*Year*	*JD*
1998	2450 813.5	2003	2452 639.5	2008	2454 465.5
1999	2451 178.5	2004	2453 004.5	2009	2454 831.5
2000	2451 543.5	2005	2453 370.5	2010	2455 196.5
2001	2451 909.5	2006	2453 735.5	2011	2455 561.5
2002	2452 274.5	2007	2454 100.5	2012	2455 926.5

Example 32.b — Calculate the heliocentric longitude of Saturn on 1999 July 26 at 0^h Dynamical Time, referred to the mean equinox of the date.

July 26 being the 207th day of the year, we have

$$d = 207 \quad \text{and} \quad t = 207/365 = 0.567\,123\,288$$

From Appendix IV we take for the longitude (L) of Saturn in 1999:

$$\begin{aligned} A_0 &= 32.578\,4232 & A_3 &= -0.010\,5762 \\ A_1 &= 12.966\,6139 & A_4 &= 0.007\,6613 \\ A_2 &= 0.129\,4965 & A_5 &= -0.003\,6652 \end{aligned}$$

whence, by formula (32.4), $l = 39\overset{\circ}{.}972\,3901 = 39°58'20''.60$

This is indeed the result obtained directly from the VSOP87 theory.

Calculated by means of these polynomial expressions, the maximum error in the heliocentric longitude will not exceed 0.05 arcsecond in the case of Jupiter, and 0.02 arcsecond for Saturn. For the much slower planets Uranus and Neptune, the error will even be less as compared with the VSOP87 theory.

REFERENCES

1. P. Bretagnon, "Théorie du mouvement de l'ensemble des planètes. Solution VSOP82", *Astronomy and Astrophysics*, Vol. 114, pages 278–288 (1982).
2. P. Bretagnon, G. Francou, "Planetary theories in rectangular and spherical variables. VSOP87 solutions", *Astronomy and Astrophysics*, Vol. 202, pages 309–315 (1988).
3. *Ibid.*, page 314.

Chapter 33

Elliptic Motion

In this Chapter we will describe two methods for the calculation of geocentric positions in the case of an elliptic orbit. In the first method, the geocentric ecliptical longitude and latitude of a major planet (Mercury to Neptune) are obtained from the heliocentric ecliptical coordinates of the planet and the Earth. In the second method, which is better suited for minor planets and periodic comets, the right ascension and declination of the body, referred to a standard equinox, are obtained directly, and use is made of the geocentric rectangular coordinates of the Sun.

First Method

We will describe how the apparent right ascension and declination of a major planet can be calculated for a given instant.

For the given instant calculate, by means of the appropriate series given in Appendix III and using the method described in Chapter 32, the heliocentric coordinates L, B, R of the planet, and the heliocentric coordinates L_0, B_0, R_0 of the Earth. Do *not* convert from the dynamical ecliptic and equinox to the FK5 ecliptic and equinox at this stage.

Then calculate

$$\begin{aligned} x &= R \cos B \cos L - R_0 \cos B_0 \cos L_0 \\ y &= R \cos B \sin L - R_0 \cos B_0 \sin L_0 \\ z &= R \sin B - R_0 \sin B_0 \end{aligned} \tag{33.1}$$

The geocentric longitude λ and latitude β of the planet are then given by

$$\tan \lambda = \frac{y}{x} \qquad \tan \beta = \frac{z}{\sqrt{x^2 + y^2}} \tag{33.2}$$

Look out for the proper quadrant of λ. One may use the "second" arctangent function, $\lambda = \text{ATN2}(y, x)$ or use the fact that, if $x < 0$, then $\cos \lambda < 0$.

However, the geocentric coordinates λ and β obtained in this way are the planet's *geometric* coordinates referred to the mean equinox of the date. If high accuracy is needed, it is necessary to take into account the apparent displacement of the planet from its true position due to the finite velocity of light. This apparent displacement includes:

(a) the effect of light-time, the planet being seen where it was when the light left it;

(b) the effect of the Earth's motion which, combined with the velocity of light, causes an apparent displacement of the object, just as the annual aberration in the case of a star.

The *combination* of the two effects is often called "planetary aberration". However, we prefer to reserve the term *aberration* to the effect (b) alone, because this effect is of the same nature as the aberration of the stars. Moreover, for some applications it is not necessary to take effect (b) into account. Suppose we want to calculate occultations of stars by planets. Then the effect of light-time *must* be taken into account in the calculation of the position of the planet; but we may drop effect (b) on the condition that the effect of aberration on the star's position is dropped too. Similarly, the effect of nutation can be neglected for *both* bodies in that particular case. The reason is evident: because the planet and the star are close together on the celestial sphere, the effects of aberration and nutation will not change their *relative* positions.

(a) *effect of* light-time: at time t, the planet is seen where it was at time $t - \tau$, hence in the direction obtained by combining the Earth's position at time t with that of the planet at time $t - \tau$, where τ is the time taken by the light to reach the Earth from the planet. This time is given by

$$\tau = 0.005\,77\,55\,183\ \Delta \text{ days} \tag{33.3}$$

where Δ is the planet's distance to the Earth in astronomical units, given by

$$\Delta = \sqrt{x^2 + y^2 + z^2} \tag{33.4}$$

(b) the *effect of aberration* can be calculated as for the stars, namely, by means of formulae (23.2), where $\odot$ is equal to $L_0 \pm 180°$.

However, *both* effects can be calculated simultaneously. To the order of accuracy that the motion of the Earth during the light-time is rectilinear and uniform, the planet's apparent position at time t is the same as its geometric position at time $t - \tau$. In other words, in this method the Earth's position at time $t - \tau$ must be combined with the planet's position at the same time $t - \tau$.

Of course, the value of the light-time τ is not known in advance because the planet's distance Δ to the Earth is not known. But this distance can be found by iteration, using for instance $\Delta = 0$ (and hence $\tau = 0$) in the first calculation.

For very accurate calculations, the planet's geocentric longitude λ and latitude β can be converted from the dynamical ecliptic and equinox to the FK5 ecliptic and equinox by means of formulae (32.3), replacing L by λ, and B by β.

To complete the calculation of the planet's apparent position, the corrections for *nutation* should be applied. This is achieved by calculating the nutation in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$), as explained in Chapter 22. Add $\Delta\psi$ to the planet's geocentric longitude, and $\Delta\varepsilon$ to the mean obliquity ε_0 of the ecliptic. The apparent right ascension and declination of the planet can then be deduced by means of formulae (13.3) and (13.4).

The *elongation* ψ of the planet, that is, its angular distance to the Sun, can be calculated from

$$\cos\psi = \cos\beta\,\cos(\lambda - \lambda_0) \tag{33.5}$$

where λ, β are the planet's apparent longitude and latitude, and λ_0 the Sun's apparent longitude. The Sun's latitude, which is always smaller than 1.2 arcsecond, may be neglected here.

Example 33.a — Calculate the apparent position of Venus on 1992 December 20 at 0^h TD = JDE 2448 976.5.

Because the planet's distance to the Earth is not known in advance, the value of the light-time is not known. Therefore, we start with the calculation of the true (geometric) position of the planet at the given time. We find the following values for the heliocentric coordinates (see Example 32.a):

$$L = 26^\circ\!.11428 \qquad B = -2^\circ\!.62070 \qquad R = 0.724\,603$$

The coordinates of the Earth are calculated in the same way:

$$L_0 = 88^\circ\!.35704 \qquad B_0 = +0^\circ\!.00014 \qquad R_0 = 0.983\,824 \qquad \text{(A)}$$

whence, by formulae (33.1), (33.4), and (33.3),

$$\begin{aligned} x &= +0.621\,746 & & \\ y &= -0.664\,810 & \qquad \Delta &= 0.910\,845 \\ z &= -0.033\,134 & \qquad \tau &= 0.005\,2606 \text{ day} \end{aligned}$$

Δ is the true distance of Venus to the Earth on 1992 December 20.0. We now repeat the calculation of Venus' heliocentric coordinates for the instant $t - \tau$, that is, for JDE = 2448 976.5 − 0.005 2606. We obtain

$$L = 26^\circ\!.10588 \qquad B = -2^\circ\!.62102 \qquad R = 0.724\,604 \qquad \text{(B)}$$

Combining these new values with the values (A) of L_0, B_0, R_0, we find

$$\begin{aligned} x &= +0.621\,794 & & \\ y &= -0.664\,905 \quad \text{(C)} & \qquad \Delta &= 0.910\,947 \\ z &= -0.033\,138 & \qquad \tau &= 0.005\,2612 \text{ day} \end{aligned}$$

If we repeat the calculation with this new value of τ, we find the same values (B) for L, B, and R again, to the given accuracy.

Hence, the final value for the light-time is $\tau = 0.005\,2612$ day, and $\Delta = 0.910\,947$ AU is the *apparent* distance of the planet on 1992 December 20 at 0^h TD. It is the distance at which we "see" the planet at that instant. In other words, it is the distance travelled by the light which left the planet at time $t - \tau$ to reach the Earth at time t.

Let us now calculate Venus' geocentric longitude and latitude. If we put the values (C) of x, y, z in formulae (33.2), we obtain

$$\lambda = 313^\circ\!.08102 \qquad \beta = -2^\circ\!.08474$$

which are corrected for light-time, but not yet for aberration.

From Chapter 23, we find $e = 0.016\,711\,589$, $\pi = 102^\circ\!.81644$

and formulae (23.2) give, for $\odot = 268^\circ\!.35704$,

$$\Delta\lambda = -14''\!.868 = -0^\circ\!.00413$$
$$\Delta\beta = -0''\!.531 = -0^\circ\!.00015$$

and the apparent longitude and latitude of Venus, not yet corrected for nutation, are

$$\lambda = 313^\circ\!.08102 - 0^\circ\!.00413 = 313^\circ\!.07689$$
$$\beta = -2^\circ\!.08474 - 0^\circ\!.00015 = -2^\circ\!.08489$$

(Alternatively, we could have corrected for the light-time and the aberration *together* at once by calculating the coordinates of the Earth for the instant $t - \tau$, which gives

$$L_0 = 88^\circ\!.35168 \qquad B_0 = +0^\circ\!.00014 \qquad R_0 = 0.983\,825$$

We now combine these values with Venus' coordinates (B). Formulae (33.1) and (33.2) then give

$$x = +0.621\,702 \qquad \lambda = 313^\circ\!.07687$$
$$y = -0.664\,903 \qquad \beta = -2^\circ\!.08489$$
$$z = -0.033138$$

or nearly the same values as before.)

The corrections for reduction to the FK5 system are, from (32.3),

$$\Delta\lambda = -0''\!.09027 = -0^\circ\!.00003$$
$$\Delta\beta = +0''\!.05535 = +0^\circ\!.00002$$

so the corrected values are

$$\lambda = 313^\circ\!.07689 - 0^\circ\!.00003 = 313^\circ\!.07686$$
$$\beta = -2^\circ\!.08489 + 0^\circ\!.00002 = -2^\circ\!.08487$$

From Chapter 22, we find

$$\Delta\psi = +16''.749 \qquad \Delta\varepsilon = -1''.933 \qquad \varepsilon = 23^\circ.439\,669$$

and the value of λ corrected for nutation is

$$\lambda = 313^\circ.07686 + 16''.749 = 313^\circ.08151$$

Finally, by (13.3) and (13.4),

apparent right ascension: $\alpha = 316^\circ.17291 = 21^h.078\,194 = 21^h04^m41^s.50$
apparent declination: $\delta = -18^\circ.88801 = -18^\circ 53' 16''.8$

The exact values, obtained by an accurate calculation using the complete VSOP87 theory, are $\alpha = 21^h04^m41^s.454$, $\delta = -18^\circ 53' 16''.84$, true distance $= 0.910\,845\,96$.

Second Method

Here we use the orbital elements referred to a standard equinox, for instance 2000.0, and the geocentric rectangular equatorial coordinates X, Y, Z of the Sun referred to the *same* equinox. These rectangular coordinates can be taken from an astronomical almanac, or they may be calculated by the method described in Chapter 26.

In this method, the heliocentric longitude and latitude of the body (minor planet or periodic comet) are not calculated. Instead, we calculate its heliocentric rectangular equatorial coordinates x, y, z, after which the right ascension, declination, and other quantities are derived by means of simple formulae.

The following orbital elements are supposed to be known. They may be taken, for instance, from the *Circulars* of the International Astronomical Union, from the *Minor Planet Circulars* of the Minor Planet Center, etc.

a = semimajor axis, in AU
e = eccentricity
i = inclination
ω = argument of perihelion
Ω = longitude of ascending node
n = mean motion, in degrees/day

where i, ω, and Ω are referred to a standard equinox.

If a or n are not given, they can be calculated from

$$a = \frac{q}{1-e} \qquad n = \frac{0.985\,607\,6686}{a\sqrt{a}} \tag{33.6}$$

where q is the perihelion distance in AU. The numerator of the second fraction is the Gaussian gravitational constant 0.017 202 098 95 converted from radians to degrees.

The inclination i can take values from 0° to 180°. If $0° \leq i < 90°$, then the body is said to have *direct* motion. This means that the body moves counter-clockwise as seen from the north pole of the ecliptic. If i is larger than 90°, the motion is said to be *retrograde* (*).

Strictly speaking, all these elements are valid only for one given instant, called the *Epoch*. Away from this time they change under influence of planetary perturbations. See, later in this Chapter, the note about *osculating elements*. Unless high accuracy is required, the elements may be considered as invariable during several weeks or even months, for instance during the whole apparition of a comet.

Besides the above-mentioned orbital elements, either the value M_0 of the mean anomaly at the Epoch, or the time T of passage through perihelion, is given. This allows the calculation of the mean anomaly M at any given instant. The mean anomaly increases by n degrees per day, and is zero at time T.

The orbital elements of a minor planet or a periodic comet being given, the geocentric position for a given instant can be calculated as follows. First, we must calculate the quantities a, b, c and the angles A, B, C, which are constant for a given orbit.

Let ε be the obliquity of the ecliptic. If the orbital elements are referred to the standard equinox of 2000.0, one should use the value $\varepsilon_{2000} = 23°26'21''.448$, from which

$$\sin \varepsilon = 0.397\,777\,156$$
$$\cos \varepsilon = 0.917\,482\,062$$

Then calculate

$$\begin{aligned} F &= \cos \Omega & P &= -\sin \Omega \cos i \\ G &= \sin \Omega \cos \varepsilon & Q &= \cos \Omega \cos i \cos \varepsilon - \sin i \sin \varepsilon \\ H &= \sin \Omega \sin \varepsilon & R &= \cos \Omega \cos i \sin \varepsilon + \sin i \cos \varepsilon \end{aligned} \qquad (33.7)$$

As a check, we can use the relations

$$F^2 + G^2 + H^2 = 1, \qquad P^2 + Q^2 + R^2 = 1,$$

but of course this is not needed in a program.

(*) Some authors call an orbit with $i < 90°$ a *prograde* orbit. While *retrograde* is a current English word, even outside astronomy (it means "going backward"), the word *prograde* is not. It appeared in some astronomical texts around 1960. I don't know who invented this neologism, nor why. The classic word, in use since more than two centuries, is *direct*.

Then the quantities a, b, c, A, B, C are given by

$$\left.\begin{aligned} \tan A &= \frac{F}{P} & a &= \sqrt{F^2 + P^2} \\ \tan B &= \frac{G}{Q} & b &= \sqrt{G^2 + Q^2} \\ \tan C &= \frac{H}{R} & c &= \sqrt{H^2 + R^2} \end{aligned}\right\} \qquad (33.8)$$

The quantities a, b, c should be taken *positive*, while the angles A, B, C should be taken in the correct quadrant, according to the following rules:

$\sin A$ has the same sign as $\cos \Omega$,

$\sin B$ and $\sin C$ have the same sign as $\sin \Omega$.

However, once again, one may use the "second" arctangent function if it is available in the programming language: A = ATN2 (F, P), etc.

Attention: do not confuse the quantity a with the semimajor axis a of the orbit!

For each required position, calculate the body's mean anomaly M, then the eccentric anomaly E (see Chapter 30), the true anomaly v by means of formula (30.1), and the radius vector r by means of (30.2). Then the heliocentric rectangular equatorial coordinates of the body are given by

$$\left.\begin{aligned} x &= r a \sin(A + \omega + v) \\ y &= r b \sin(B + \omega + v) \\ z &= r c \sin(C + \omega + v) \end{aligned}\right\} \qquad (33.9)$$

The convenience of these formulae is seen when the rectangular coordinates are required for several positions of the body. The auxiliary quantities a, b, c, A, B, C are functions only of Ω, i, and ε, and thus are constant for the whole ephemeris; for each position only the values of v and r must be calculated. However, remember that Ω, i, and ω are constant only if the body is in an unperturbed orbit.

For the same instant, calculate the Sun's rectangular coordinates X, Y, Z (Chapter 26), or take them from an astronomical almanac. The geocentric right ascension α and declination δ of the planet or comet are then found from

$$\left.\begin{aligned} &\xi = X + x \qquad \eta = Y + y \qquad \zeta = Z + z \\ &\tan \alpha = \eta / \xi \qquad \Delta^2 = \xi^2 + \eta^2 + \zeta^2 \\ &\sin \delta = \zeta / \Delta \qquad \text{or} \qquad \tan \delta = \frac{\zeta}{\sqrt{\xi^2 + \eta^2}} \end{aligned}\right\} \qquad (33.10)$$

where Δ is the distance to the Earth and thus is positive. The correct quadrant of α is indicated by the fact that $\sin\alpha$ has the same sign as η; however, once more, the second arctangent function can be used: $\alpha = \text{ATN2}(\eta, \xi)$.

If α is negative, add 360 degrees. Then transform α from degrees into hours by dividing by 15.

The equatorial coordinates α and δ of the body will be referred to the same standard equinox as the orbital elements and the Sun's rectangular coordinates X, Y, Z. However, the values of α and δ obtained in the way described above refer to the geometric (the true) position of the body in space. Just as in the "First Method" in this Chapter, the *effect of light-time* should be taken into account. This is performed as follows.

For the given time t, calculate the distance Δ of the body to the Earth as described above, and then the light-time τ by means of (33.3). Then repeat the calculation of M, E, v, x, y, z for the time $t - \tau$, but *leave* the Sun's coordinates X, Y, Y unchanged. With the new values of x, y, z, formulae (33.10) will give the corrected values of α and δ.

When allowance is made for the light-time only, that is, if *no* correction is made for aberration nor for nutation, then the values obtained for α and δ are the so-called *astrometric* right ascension and declination of the body at the given instant. The astrometric position of a minor planet or a comet is directly comparable with the mean places of stars as given in star catalogues (corrected for proper motion and annual parallax, if significant). Of course, α and δ are geocentric.

Instead of expressions (33.7) and (33.8), one may calculate the constants

$$\begin{aligned}
P_x &= \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\
P_y &= \cos\varepsilon(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) - \sin\varepsilon\sin\omega\sin i \\
P_z &= \sin\varepsilon(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) + \cos\varepsilon\sin\omega\sin i \\
Q_x &= -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\
Q_y &= \cos\varepsilon(\cos\omega\cos\Omega\cos i - \sin\omega\sin\Omega) - \sin\varepsilon\cos\omega\sin i \\
Q_z &= \sin\varepsilon(\cos\omega\cos\Omega\cos i - \sin\omega\sin\Omega) + \cos\varepsilon\cos\omega\sin i
\end{aligned}$$

and then, instead of (33.9), one should use

$$\begin{aligned}
x &= r(P_x\cos v + Q_x\sin v) \\
y &= r(P_y\cos v + Q_y\sin v) \\
z &= r(P_z\cos v + Q_z\sin v)
\end{aligned}$$

The elongation ψ to the Sun and the phase angle β (the angle Sun–body–Earth) can be calculated from

$$\cos \psi = \frac{\xi X + \eta Y + \zeta Z}{R\Delta} = \frac{R^2 + \Delta^2 - r^2}{2R\Delta} \tag{33.11}$$

$$\cos \beta = \frac{\xi x + \eta y + \zeta z}{r\Delta} = \frac{r^2 + \Delta^2 - R^2}{2r\Delta} \tag{33.12}$$

where $R = \sqrt{X^2 + Y^2 + Z^2}$ is the distance Earth–Sun. The angles ψ and β are both between 0 and +180 degrees. Do not confuse this R with the quantity R of expressions (33.1), nor with that of (33.7).

The *magnitude* of the body is then calculated as follows. In the case of a *comet*, the "total" magnitude is generally calculated from

$$m = g + 5 \log \Delta + \kappa \log r \tag{33.13}$$

where g is the absolute magnitude, and κ a constant which differs from one comet to another. In general, κ is a number between 5 and 15.

For the *minor planets*, a new magnitude system was adopted by Commission 20 of the International Astronomical Union (New Delhi, November 1985). The formula for the prediction of the apparent magnitude of a minor planet is

$$\text{magnitude} = H + 5 \log r\Delta - 2.5 \log \left[(1 - G) \Phi_1 + G\Phi_2 \right] \tag{33.14}$$

with

$$\Phi_1 = \exp \left[-3.33 \left(\tan \frac{\beta}{2}\right)^{0.63} \right]$$

$$\Phi_2 = \exp \left[-1.87 \left(\tan \frac{\beta}{2}\right)^{1.22} \right]$$

where β is the phase angle, and "exp" is the exponential function, $\text{EXP}(x) = e^x$. Formula (33.14) is valid for $0° \le \beta \le 120°$. H and G are magnitude parameters, which are different for each minor planet. H is the mean absolute *visual* magnitude, while G is called the "slope parameter". Here are the values of H and G for the brightest minor planets and for some unusual objects [1]:

		H	G			H	G
1	Ceres	3.34	0.12	15	Eunomia	5.28	0.23
2	Pallas	4.13	0.11	18	Melpomene	6.51	0.25
3	Juno	5.33	0.32	20	Massalia	6.50	0.25
4	Vesta	3.20	0.32	433	Eros	11.16	0.46
5	Astraea	6.85	0.15	1566	Icarus	16.9	0.15
6	Hebe	5.71	0.24	1620	Geographos	15.60	0.15
7	Iris	5.51	0.15	1862	Apollo	16.25	0.09
8	Flora	6.49	0.28	2060	Chiron	6.5	0.15
9	Metis	6.28	0.17	2062	Aten	16.80	0.15

In formulae (33.13) and (33.14), the distance to the Sun (r) and the distance to the Earth (Δ) are in astronomical units, and *all logarithms are to the base 10.* In many programming languages, the only available logarithmic function "LOG" is the natural logarithm (to the base $e = 2.71828...$); it can be converted to the common logarithm (base 10) by multiplication by 0.434 294 4819, which is $1/\log_e 10$.

Example 33.b — Calculate the geocentric position of periodic comet Encke for 1990 October 6.0 Dynamical Time, using the following orbital elements (see Example 24.b):

$$\begin{array}{lll} T = \text{1990 Oct. 28.54502 TD} & i = 11^\circ\!.94524 & \\ a = 2.209\,1404 \text{ AU} & \Omega = 334^\circ\!.75006 & \left.\right\} \text{ecliptic and equinox 2000.0} \\ e = 0.850\,2196 & \omega = 186^\circ\!.23352 & \end{array}$$

We first calculate the auxiliary constants of the orbit by means of (33.7) and (33.8):

$$\begin{array}{ll} F = +0.904\,455\,59 & P = +0.417\,330\,84 \\ G = -0.391\,368\,30 & Q = +0.729\,522\,09 \\ H = -0.169\,678\,93 & R = +0.541\,878\,67 \end{array}$$

$$\begin{array}{ll} A = 65^\circ\!.230\,615 & a = 0.996\,094\,85 \\ B = 331^\circ\!.787\,680 & b = 0.827\,871\,74 \\ C = 342^\circ\!.613\,052 & c = 0.567\,823\,42 \end{array}$$

From the value 2.209 1404 for the semimajor axis of the orbit, the second formula (33.6) yields $n = 0.300\,171\,252$ degree/day.

For the given date (1990 October 6.0), the time since perihelion is -22.54502 days. Hence, the mean anomaly is

$$M = -22.54502 \times 0^\circ\!.300\,171\,252 = -6^\circ\!.767\,367$$

We then find

$$\begin{array}{ll} E = -34^\circ\!.026\,714 & x = +0.250\,8066 \\ v = -94^\circ\!.163\,310 & y = +0.484\,9175 \\ r = 0.652\,4867 & z = +0.357\,3373 \end{array}$$

The Sun's geocentric rectangular equatorial coordinates for the same instant, referred to the same standard equinox (2000.0) and calculated by using the complete VSOP87 theory, are

$$X = -0.975\,6732, \qquad Y = -0.200\,3254, \qquad Z = -0.086\,8566,$$

from which $\Delta = 0.824\,3689$, and the light-time is $\tau = 0.00476$ day.

Repeating the calculation of the comet's position for $t - \tau$, that is, for 1990 October 5.99524, we find

$$\begin{array}{lll} M = -6.^\circ 768\,796 & x = +0.250\,9310 & \xi = -0.724\,7422 \\ E = -34.^\circ 031\,552 & y = +0.484\,9477 & \eta = +0.284\,6223 \\ v = -94.^\circ 171\,933 & z = +0.357\,3712 & \zeta = +0.270\,5146 \\ r = 0.652\,5755 & & \Delta = 0.824\,2811 \end{array}$$

from which we deduce the astrometric right ascension and declination, and the elongation from the Sun:

$$\begin{aligned} \alpha_{2000} &= 158.^\circ 558\,965 = 10^h 34^m 14.^s 2 \\ \delta_{2000} &= +19.^\circ 158\,496 = +19^\circ 09' 31'' \\ \psi &= 40.^\circ 51 \end{aligned}$$

Heliocentric ecliptical coordinates

For some applications, the heliocentric rectangular *ecliptical* coordinates may be needed. In that case one should use the following expressions instead of (33.9), and it is not needed to calculate the auxiliary quantities F, G, ..., A, B, etc.

$$\begin{aligned} u &= \omega + v \\ x &= r(\cos\Omega\cos u - \sin\Omega\sin u\cos i) \\ y &= r(\sin\Omega\cos u + \cos\Omega\sin u\cos i) \\ z &= r\sin i\sin u \end{aligned}$$

When these heliocentric rectangular ecliptical coordinates are known, the heliocentric longitude l and latitude b can be found from

$$\tan l = y/x \quad (l \text{ being taken between } 90^\circ \text{ and } 270^\circ \text{ if } x < 0)$$

$$\sin b = \frac{z}{r} \qquad \text{or} \qquad \tan b = \frac{z}{\sqrt{x^2 + y^2}}$$

Notes on the osculating elements

Mean orbital elements, such as those given in Chapter 31 for the major planets, represent the elements of a mean reference, slowly varying orbit.

For the periodic comets and the thousands of minor planets, however, no mean orbital elements are calculated. Instead, orbital elements are available for the "instantaneous" orbit at a given instant (the Epoch). These are the so-called *osculating elements*, and the instant for which they are valid is the Epoch of osculation.

> Osculating elements at a particular epoch are defined as the elements of an unperturbed elliptical orbit, referred to as the osculating orbit, in which the position and velocity of the planet at the epoch are identical with the actual position and velocity of the planet in its perturbed orbit at the same instant. The osculating elements therefore contain the effects of the perturbations due to other planets, so that, unlike the mean elements, they are subject to periodic variations. [2]

While the *mean* elements vary slowly with time (for instance, the eccentricity of the mean orbit of Mars was 0.093 31 in A.D. 1900 and will be 0.093 49 in 2100), the osculating elements vary rather rapidly. These changes generally do *not* reflect the real changes of the mean orbit.

As an example, let us give the following osculating elements of minor planet Ceres for two epochs separated by only 200 days. These elements are taken from the yearly *Ephemerides of Minor Planets* (Institute of Theoretical Astronomy of the Russian Academy of Sciences, St. Petersburg, Russia); the elements i, ω, and Ω are referred to the standard equinox of 2000.0.

Epoch (TD) :	1997 Dec. 18.0	1998 July 6.0
Semimajor axis (AU) :	a = 2.767 8380	a = 2.766 1801
Eccentricity :	e = 0.077 4119	e = 0.077 8872
Inclination (degrees) :	i = 10.58086	i = 10.58293
Argument of perihelion (deg.) :	ω = 73.46016	ω = 73.79924
Longitude of ascending node (deg.) :	Ω = 80.52954	Ω = 80.50163
Mean anomaly (degrees) :	M = 207.08221	M = 249.60014
Mean motion (degrees/day) :	n = 0.214 039 08	n = 0.214 231 53

From 1997 December 18 to 1998 July 6, the semimajor axis of the "instantaneous" orbit decreased by 0.00166 AU. From this, however, we may not deduce that during those 200 days the mean distance of Ceres to the Sun decreased by 248 000 kilometers!

On 1997 December 18, the "instantaneous" revolution period of Ceres was 1681.94 days (which is obtained by dividing 360° by n); 200 days later this had decreased to 1680.42 days.

Neptune provides an even better illustration. While the eccentricity of its mean orbit is presently 0.0095, that of its osculating orbit reached a maximum of 0.0124 in November 1964, a minimum of 0.0039 in October 1970, another maximum (0.0122) in December 1976, and so on. These rather large variations are not surprising: the osculating orbit of Neptune refers to the instantaneous position and velocity of the Sun, which itself oscillates around the barycenter of the solar system, mainly due to the actions of the giant planets Jupiter and Saturn. Orbital elements of Neptune referred to that barycenter (instead of to the Sun) would show much smaller variations.

Accurate ephemerides of the periodic comets and the minor planets are obtained by numerical integration, and for these calculations the osculating orbital elements provide starting values. Such a numerical integration takes into account the perturbations caused by the attraction of the planets, which tend to change the osculating elements of the orbit over time.

Osculating elements may be used to give the actual position and motion of the body at the epoch of osculation, and they provide a good approximation to its actual orbit over short periods around the Epoch. They may *not*, however, be used as an unperturbed orbit over a long period!

In order to have an idea of the increasing error of an ephemeris calculated by using an osculating orbit as an unperturbed one, we used the above-mentioned osculating elements of Ceres valid for 1998 July 6. The heliocentric longitude of Ceres, calculated in this way, was then compared with the exact one as obtained with the software package "Ceres" developed at the Institute of Theoretical Astronomy, St. Petersburg, Russia. It appears that until 280 days after the Epoch the error is smaller than 5″. During the first 50 days, the error is smaller than 1″. The error in the calculated heliocentric longitude reaches a maximum (+4″) 172 days after the Epoch, but after a few months the error $\Delta\lambda$ quickly reaches large negative values:

Number of days after 1998 July 6:	0	40	80	120	160	200	240	280	320	360	400
$\Delta\lambda$ (arcsec.):	0	+½	+2	+3	+4	+4	+1	−4	−13	−26	−44

The further evolution of the error $\Delta\lambda$ in the calculated heliocentric longitude of Ceres is shown in Figure 1. The oscillating curve represents the variation of the error as a function of time. So, in this particular case, the error does not increase continually with time, but reaches the following extreme values: +4″ in December 1998, −304″ in early November 2000, +862″ in early September 2003, −383″ in mid-May 2005, and +1105″ in mid-September 2007.

The situation is somewhat comparable with the undulating curve shown in Figure 2. The true function (the osculating orbit in the case of a minor planet) is represented by the curve C. The dashed line M is the "mean" curve (the mean orbit). If we use this mean curve, then for a given value x of the argument we obtain point A, which differs from the true value B on the true curve. However, the difference between A and B does not exceed a certain limit. At point P, the tangent T to the true curve is drawn. In the vicinity of P, this tangent gives a much better approximation to the true curve C than does the mean curve M. But if we use the tangent T at large distances from P, we obtain the very erroneous point E. In this case, the mean curve would give A, which is a better approximation to the correct value B. Unfortunately, for minor planets no mean orbital elements are available.

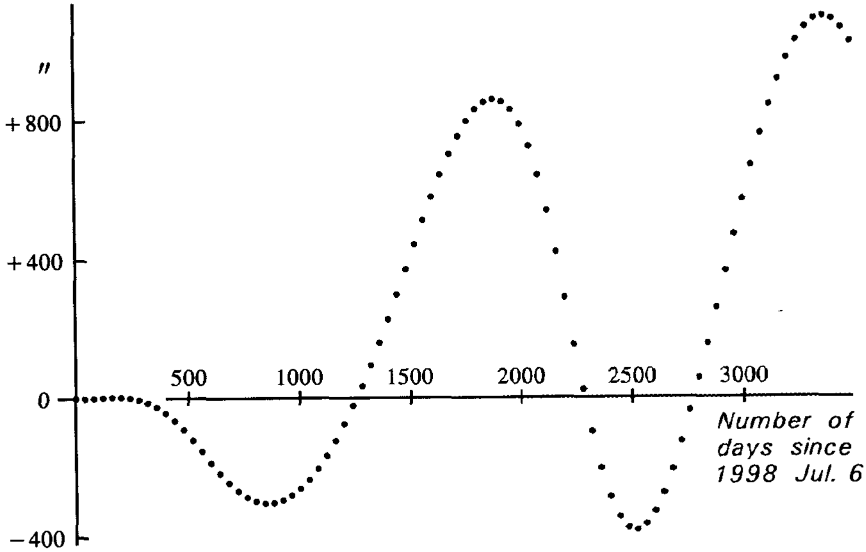


Fig. 1 : The error Δλ (in arcseconds) in the calculated heliocentric longitude of Ceres when osculating elements are used and the perturbations by the planets are ignored. The points are given at intervals of 40 days.

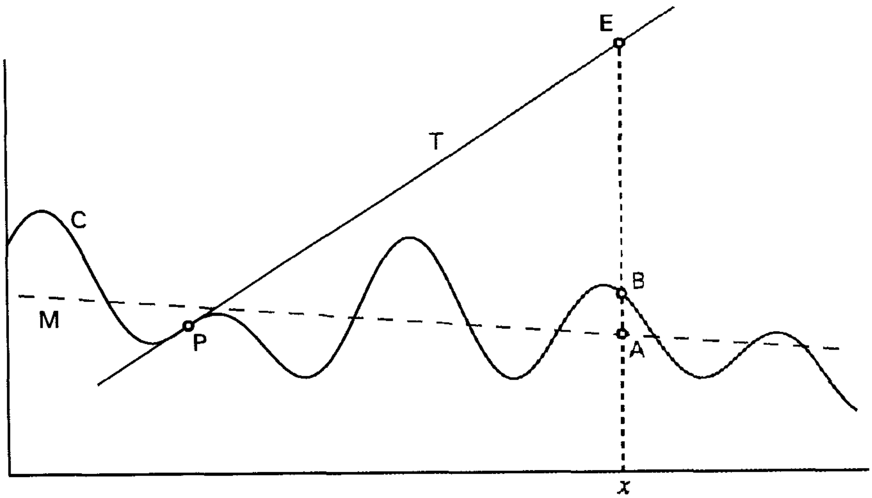


Fig. 2

The Equation of the Center

If the orbital eccentricity is small, then instead of solving the equation of Kepler (Chapter 30) and then using formula (30.1), the equation of the center C, or the difference $v - M$, can be found directly in terms of e and M by means of the following formula.

$$C = \left(2e - \frac{e^3}{4} + \frac{5}{96}e^5\right) \sin M + \left(\frac{5}{4}e^2 - \frac{11}{24}e^4\right) \sin 2M$$
$$+ \left(\frac{13}{12}e^3 - \frac{43}{64}e^5\right) \sin 3M + \frac{103}{96}e^4 \sin 4M + \frac{1097}{960}e^5 \sin 5M$$

The result is expressed in radians, and thus should be multiplied by $180/\pi$ or 57.295 779 51 in order to be converted into degrees. The formula is derived from a series expansion [3] and has been truncated after the term in e^5. Therefore it is suitable only for small values of the eccentricity. If the eccentricity is *very* small, the terms in e^4 and e^5 may be neglected.

The greatest error is

e	*The formula up to terms in e^5*	*The formula with terms e^4 and e^5 neglected*
0.03	0″.0003	0″.24
0.05	0.007	1.8
0.10	0.45	30
0.15	5	152
0.20	29	483
0.25	111	1183
0.30	331	2456

There exists a series expansion for the radius vector, too. Its terms up to the fifth power of the eccentricity are as follows:

$$\frac{r}{a} = 1 + \frac{e^2}{2} - \left(e - \frac{3}{8}e^3 + \frac{5}{192}e^5\right) \cos M$$
$$- \left(\frac{e^2}{2} - \frac{e^4}{3}\right) \cos 2M - \left(\frac{3}{8}e^3 - \frac{45}{128}e^5\right) \cos 3M$$
$$- \frac{e^4}{3} \cos 4M - \frac{125}{384}e^5 \cos 5M$$

Velocity in an elliptic orbit

In an unperturbed elliptic orbit, the instantaneous velocity of the moving body, in kilometers per second, is given by the following formula, where r is the distance of the body to the Sun, and a is the semimajor axis of the orbit, both expressed in astronomical units:

$$V = 42.1219\sqrt{\frac{1}{r} - \frac{1}{2a}}$$

If e is the orbital eccentricity, the velocities at perihelion and at aphelion, again in km/second, are respectively

$$V_p = \frac{29.7847}{\sqrt{a}}\sqrt{\frac{1+e}{1-e}} \qquad\qquad V_a = \frac{29.7847}{\sqrt{a}}\sqrt{\frac{1-e}{1+e}}$$

Example 33.c — For the 1986 return of periodic comet Halley, we have [4]

$a = 17.940\,0782 \qquad e = 0.967\,274\,26$

these osculating values being valid strictly for the Epoch 1986 February 19.0 TD.

For this orbit, the velocities at perihelion and at aphelion are $V_p = 54.52$ km/second and $V_a = 0.91$ km/second, respectively.

At the distance $r = 1$ AU from the Sun, the comet's velocity was $V = 41.53$ km/second.

Length of the ellipse

While there is an exact formula giving the area of an ellipse (area $= \pi ab$), there is no exact expression with a finite number of terms and ordinary functions for the length L (the perimeter) of an ellipse. In what follows, e is the eccentricity of the ellipse, a its semimajor axis, and b its semiminor axis given by $b = a\sqrt{1-e^2}$.

1. An approximate formula given by Ramanujan in 1914 is

$$L = \pi\left(3(a+b) - \sqrt{(a+3b)(3a+b)}\right)$$

The error is zero for $a = b$ (that is, for a circle), increasing to 0.4155 % for $e = 1$, that is, for an infinitely flat ellipse.

2. Another interesting method for finding the length of an ellipse is as follows. Let A, G, and H be the arithmetic, the geometric, and the harmonic means, respectively, of the semi-axes a and b of the ellipse. That is,

$$A = \frac{a+b}{2} \qquad G = \sqrt{ab} \qquad H = \frac{2ab}{a+b}$$

Then we have

$$L = \pi \left(\frac{21A - 2G - 3H}{8} \right)$$

with an error less than 0.001 % if $e < 0.88$, and less than 0.01 % if $e < 0.95$. But the error amounts to 1 % for $e = 0.9997$, and to 3 % for $e = 1$.

3. A formula with an infinite series expansion is

$$L = 2\pi a \left[1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1 \times 3}{2 \times 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6}\right)^2 \frac{e^6}{5} - \text{etc.} \right]$$

The expression between square brackets takes the value 0.99937 for $e = 0.05$, the value 0.99750 for $e = 0.10$, and is equal to $0.63662 = 2/\pi$ for $e = 1$.

4. More rapid convergence is obtained with the following formula, where $m = (a - b) / (a + b)$,

$$L = \frac{2\pi a}{1+m} \left[1 + \left(\frac{1}{2}\right)^2 m^2 + \left(\frac{1}{2 \times 4}\right)^2 m^4 + \left(\frac{1 \times 3}{2 \times 4 \times 6}\right)^2 m^6 \right.$$

$$\left. + \left(\frac{1 \times 3 \times 5}{2 \times 4 \times 6 \times 8}\right)^2 m^8 + \text{etc.} \right]$$

Example 33.d — Periodic comet Halley. Using the elements for the return of 1986 (see Example 33.c), we find that the length of the orbit is 77.07 astronomical units, or 11530 millions of kilometers.

REFERENCES

1. *Minor Planet Circulars* 28103-28116 (1996 November 25).
2. *Explanatory Supplement to the Astronomical Ephemeris* (London, 1961); page 114.
3. *Annales de l'Observatoire de Paris*, Vol. I, pages 202-204.
4. *Minor Planet Circular* 10634 (1986 April 24).

Chapter 34

Parabolic Motion

In this Chapter we explain how to calculate positions of a comet moving around the Sun in a parabolic orbit. We will assume that the elements of this orbit are invariable (no planetary perturbations) and that they are referred to a standard equinox, for instance that of 2000.0.

We assume that the following orbital elements are given:

T = time of passage in perihelion
q = perihelion distance, in AU
i = inclination
ω = argument of the perihelion
Ω = longitude of the ascending node

First, calculate the auxiliary constants A, B, C, a, b, c as for an elliptic orbit; see formulae (33.7) and (33.8). Then, for each required position of the comet, proceed as follows.

Let $t - T$ be the time since perihelion, in days. This quantity is negative for an instant earlier than the time of perihelion. Calculate

$$W = \frac{0.036\,491\,162\,45}{q\sqrt{q}}\,(t - T) \tag{34.1}$$

The constant in the numerator is equal to $3k/\sqrt{2}$, where k is the Gaussian gravitational constant 0.017 202 098 95.

Then the true anomaly v and the radius vector r of the comet are given by

$$\tan\frac{v}{2} = s \qquad\qquad r = q\,(1 + s^2) \tag{34.2}$$

where s is the root of the equation

$$s^3 + 3s - W = 0 \tag{34.3}$$

For an instant earlier than the time of perihelion passage the quantity s is negative and v is between -180 and 0 degrees. After the perihelion, $s > 0$ and v is between $0°$ and $+180°$. At the instant of passage through perihelion, we have $s = 0$, $v = 0°$, and $r = q$.

There are several ways to solve equation (34.3), which is called *Barker's equation*.

1. The equation can easily be solved by iteration; this algorithm has the author's preference, because the iteration formula is simple, the convergence is rapid, no trigonometric functions or cubic roots are involved, and the procedure is valid for positive as well as negative values of $t - T$, and for $t = T$ (or $s = 0$) too.

One may start from *any* value for s; a good choice is $s = 0$. A better value for s is

$$\frac{2s^3 + W}{3\,(s^2 + 1)} \tag{34.4}$$

This calculation is then repeated until the correct value of s is obtained. Note that in expression (34.4) the cube of s must be calculated. If s is negative, this operation is not possible on some calculating machines. When this is the case, calculate $s \times s \times s$ instead of s^3.

2. Instead of solving equation (34.3) by iteration, s can be obtained directly as follows (J. Bauschinger, *Tafeln zur Theoretischen Astronomie*, page 9; Leipzig, 1934):

$$\left.\begin{aligned} \tan\beta &= \frac{2}{W} = 54.807791\,\frac{q\sqrt{q}}{t-T} \\ \tan\gamma &= \sqrt[3]{\tan\frac{\beta}{2}} \\ s &= \frac{2}{\tan 2\gamma} \end{aligned}\right\} \tag{34.5}$$

The constant 54.807791 is equal to $2\sqrt{2}/3k$, where k is the Gaussian gravitational constant.

In this method, no iteration is performed, but two problems can occur:

— at the time of passage through perihelion, $t - T$ is zero, hence W is zero and $2/W$ becomes infinite. In that case we have directly $v = 0°$ and $r = q$, but the possible occurrence of this case must be anticipated in the computer program;

— before the perihelion we have $W < 0$, whence $\tan\beta$ is negative. But in this case $\tan\beta/2$ is negative too, and computers cannot calculate the cubic root of a negative quantity. This problem can be avoided by replacing W by its absolute value in the first formula (34.5). At the end of the calculation, the sign of s should then be changed accordingly. For instance, in BASIC the formulae (34.1) and (34.5) can be programmed as follows, where T stands for $t - T$, the number of days elapsed since perihelion:

```
IF T = 0 THEN .....
W = .03649116245 * T/(Q * SQR(Q))
B = ATN(2/ABS(W))
S = 2/TAN(2 * ATN(TAN(B/2)^(1/3)))
IF T < 0 THEN S = -S
```

3. The following method is easier and does not use trigonometric functions. All expressions under the root signs are positive.

$$G = \frac{W}{2} \qquad Y = \sqrt[3]{G + \sqrt{G^2 + 1}} \qquad s = Y - \frac{1}{Y} \tag{34.6}$$

When s is obtained, v and r can be found by means of (34.2), after which the calculation continues as for the elliptic motion, formulae (33.9) and (33.10), with the same precept to take the effect of light-time into account.

The first formula (34.2) will give $v/2$ between -90 and $+90$ degrees, the range of the arctangent function of the computer languages. That will give v in the correct quadrant, between $-180°$ and $+180°$, so no additional check will be required.

In the parabolic motion, $e = 1$ while a and the period of revolution are infinite; the mean daily motion is zero and therefore the mean and eccentric anomalies do not exist — in fact, they are zero.

Example 34.a — Calculate the true anomaly and the distance to the Sun of comet Stonehouse (C/1998 H1) for 1998 August 5.0 TD, using the values

T = 1998 April 14.4358 TD
q = 1.487 469

of a parabolic orbit calculated by B.G. Marsden (*Minor Planet Circular* No. 31893, 1998 June 10).

For the given instant (1998 August 5.0), the time from perihelion is $t - T = +112.5642$ days. Hence, by formula (34.1),

$$W = +2.264\,206\,862.$$

Starting from the value $s = 0$, we obtain the following successive approximations for s by means of the iteration formula (34.4):

0.000 0000
0.754 7356
0.663 4364
0.659 2441
0.659 2360
0.659 2360

Hence, $s = +0.659\,2360$, and consequently

$$v = +66^{\circ}.78862 \qquad r = 2.133\,911$$

If, instead of the iteration procedure, formulae (34.6) are used, we obtain successively

$$G = 1.132\,103\,431$$
$$Y = 1.382\,541\,577$$
$$s = Y - 1/Y = 0.659\,2360, \text{ as before.}$$

Chapter 35

Near-parabolic Motion

An eccentricity of exactly 1 means that the orbit is parabolic; in that case, it is easy to calculate the position of the body for a given instant (see Chapter 34). If the orbit has a high eccentricity (say, 0.98 to 1.1), but different from 1, it is more troublesome to deal with. An eccentricity greater than 1 means the orbit is hyperbolic.

The German astronomer Werner Landgraf has given an interesting program in BASIC [1], based on Karl Stumpff's work *Himmelsmechanik*, Vol. I (Berlin, 1959). Hereafter we give Landgraf's program, in a slightly modified form.

First, calculate

$$Q = \frac{k}{2q}\sqrt{\frac{1+e}{q}} \qquad\qquad \gamma = \frac{1-e}{1+e}$$

where, as before, k is the Gaussian gravitational constant, e is the eccentricity of the orbit, and q is the perihelion distance in astronomical units.

Then solve the following equation iteratively for s :

$$s = Qt - (1-2\gamma)\frac{s^3}{3} + \gamma\,(2-3\gamma)\frac{s^5}{5} - \gamma^2(3-4\gamma)\frac{s^7}{7} + \cdots \tag{35.1}$$

where t is the number of days before $(-)$ or after $(+)$ the perihelion. Begin by inserting into the right-hand side of the equation the value of s obtained for an orbit which would be precisely parabolic, that is, with the value of W of formula (34.1) put equal to $3Qt$. This evaluation leads to an improved s, which is used in another iteration, and so on until the value of s ceases to change.

Once the final value of s is found, the true anomaly v and the distance r to the Sun are found from

$$\tan\frac{v}{2} = s \qquad\qquad r = \frac{q\,(1+e)}{1+e\cos v}$$

The calculation of geocentric places can then be performed as for the elliptic and the parabolic motions.

Here is Landgraf's program in BASIC, slightly modified by us. It is valid for highly eccentric elliptical orbits (*e* slightly less than 1), for slightly hyperbolic orbits (*e* slightly larger than 1), as well as for an orbit that is exactly parabolic. The computer is assumed to be working in radians.

```
10  P1 = 4 * ATN (1) : R1 = 180/P1
12  K = 0.01720209895
14  D1 = 10000 : C = 1/3 : D = 1E-9
16  INPUT "PERIHELION DISTANCE = "; Q
18  INPUT "ECCENTRICITY = "; E0
20  Q1 = K * SQR ((1 + E0)/Q)/(2 * Q) : G = (1 - E0)/(1 + E0)
22  INPUT "DAYS FROM PERIHELION = "; T
24  IF T < > 0 THEN 28
26  R = Q : V = 0 : GOTO 72
28  Q2 = Q1 * T
30  S = 2 / (3 * ABS (Q2))
32  S = 2 / TAN (2 * ATN (TAN (ATN (S)/2)^C))
34  IF T < 0 THEN S = -S
36  IF E0 = 1 THEN 66
38  L = 0
40  S0 = S : Z = 1 : Y = S * S : G1 = -Y * S
42  Q3 = Q2 + 2 * G * S * Y/3
44  Z = Z + 1
46  G1 = -G1 * G * Y
48  Z1 = (Z - (Z + 1) * G) / (2 * Z + 1)
50  F = Z1 * G1
52  Q3 = Q3 + F
54  IF Z > 50 OR ABS (F) > D1 THEN 78
56  IF ABS (F) > D THEN 44
58  L = L + 1 : IF L > 50 THEN 78
60  S1 = S : S = (2 * S * S * S/3 + Q3) / (S * S + 1)
62  IF ABS (S - S1) > D THEN 60
64  IF ABS (S - S0) > D THEN 40
66  V = 2 * ATN (S)
68  R = Q * (1 + E0) / (1 + E0 * COS (V))
70  IF V < 0 THEN V = V + 2 * P1
72  PRINT "TRUE ANOMALY = "; V * R1
74  PRINT "RADIUS VECTOR (A.U.) = "; R
76  PRINT : GOTO 22
78  PRINT "NO CONVERGENCE"
80  PRINT : GOTO 22
```

Some comments about this program:

Line 10 : the first formula is a trick to obtain the number π.

Line 12 : the Gaussian gravitational constant k.

Line 14 : the number D = 10^{-9} adjusts to suit the computer's precision. If necessary, one may use 10^{-8} or 10^{-10}.

Line 26 : when $t = 0$ (the body is exactly in perihelion), then $r = q$ and $v = 0°$.

Line 36 : if the orbit is exactly parabolic, the value of s has been found.

Line 54 : if in formula (35.1) more than 50 terms are needed, or if these terms become too large, there is no convergence.

Line 56 : as long as a term of formula (35.1) is not small enough, the next term should be calculated.

Line 58 : if after 50 iterations no result has still been found, the calculation must be halted.

Lines 60 and 62 : solving equation (35.1) by iteration. This is an iteration inside of an iteration!

As an exercise, try to calculate the following cases:

Data			*Results*	
perihelion distance q *(AU)*	*eccentricity* e	*days* t	*true anomaly* v *(degrees)*	*distance to the Sun* r *(AU)*
0.921 326	1.000 00	138.4783	102.744 26	2.364 192
0.100 000	0.987 00	254.9	164.500 29	4.063 777
0.123 456	0.999 97	−30.47	221.911 90	0.965 053
3.363 943	1.057 31	1237.1	109.405 98	10.668 551
0.587 1018	0.967 2746	20	52.853 31	0.729 116
0.587 1018	0.967 2746	0	0	0.587 1018

After having calculated some cases, you will notice that the calculation time is longer as $|t|$ is larger, that is, as the body is farther away from the perihelion. The calculation time is longer too as e differs more from unity. The table on the next page mentions some calculation times on the old HP-85 microcomputer, together with a rounded value of the true anomaly v, and the number L of iterations.

q	e	t	*Calculation time in seconds*	v (°)	L
0.1	0.9	10	14	126	17
		20	47	142	30
		30	no convergence	—	—
0.1	0.987	10	4	123	7
		20	5	137	8
		30	6	143	10
		60	9	152	12
		100	14	157	16
		200	28	163	23
		400	87	167	38
		500	no convergence	—	—
0.1	0.999	100	3	156	6
		200	4	161	7
		500	5	166	8
		1 000	7	169	10
		5 000	18	174	18
1	0.999 99	100 000	2	172.5	4
		10 000 000	5	178.41	8
		14 000 000	6	178.58	9
		17 000 000	7	178.68	9
		18 000 000	no convergence	—	—

For $q = 0.1$ and $e = 0.9$, the calculation took 47 seconds for $t = 20$ days, and there was no convergence for $t = 30$ days. However, in this case the calculation could better be made using one of the methods for elliptic motion.

For $q = 0.1$ and $e = 0.999$, there is no trouble up to $t = 5000$ days.

For $q = 1$ and $e = 0.999\,99$, there is no trouble even for $t = 17$ million days. This is 465 centuries after the perihelion time; the object's distance from the Sun is then 7220 astronomical units — at least in theory!

REFERENCE

1. *Sky and Telescope*, Vol. 73, pages 535–536 (May 1987).

Chapter 36

The Calculation of some Planetary Phenomena

There are two basically different methods for calculating planetary phenomena such as the greatest elongations of Venus, or the time of an opposition of Mars:

(i) either by comparing accurate positions of the planet with those of the Sun;

(ii) or by using formulae where a mean value is corrected by a sum of periodic terms.

The first method has the advantage of giving very accurate results, because use is made of very accurate positions of the bodies. It has the inconvenience, however, of requiring the availability or the calculation of these accurate ephemerides.

With the second method, the calculation can be performed easily and rapidly for any year. The results, while not so accurate as those of the first method, are still good enough for many applications such as historical research, or even as a first approximation for a more accurate calculation. Examples of this method are found in Chapters 49 (lunar phases), 50 (perigee and apogee of the Moon), 51 (passages of the Moon through the nodes), and 52 (extreme declinations of the Moon).

In this Chapter, we provide formulae for calculating several configurations involving the planets Mercury to Neptune: oppositions and conjunctions with the Sun, greatest elongations, and stations.

Oppositions and conjunctions with the Sun

From the proper line in Table 36.A, take the values of A, B, M_0, and M_1.

Let Y be an appropriate time of the required phenomenon, expressed as *years and decimals*. For instance, 1993.0 means the beginning of the year 1993, 2028.5 denotes the middle of the year 2028, etc.

TABLE 36.A

Planet	*Event*	A	B	M_0	M_1
Mercury	Inf. conj.	2451 612.023	115.877 4771	63.5867	114.208 8742
Mercury	Sup. conj.	2451 554.084	115.877 4771	6.4822	114.208 8742
Venus	Inf. conj.	2451 996.706	583.921 361	82.7311	215.513 058
Venus	Sup. conj.	2451 704.746	583.921 361	154.9745	215.513 058
Mars	Opposition	2452 097.382	779.936 104	181.9573	48.705 244
Mars	Conjunction	2451 707.414	779.936 104	157.6047	48.705 244
Jupiter	Opposition	2451 870.628	398.884 046	318.4681	33.140 229
Jupiter	Conjunction	2451 671.186	398.884 046	121.8980	33.140 229
Saturn	Opposition	2451 870.170	378.091 904	318.0172	12.647 487
Saturn	Conjunction	2451 681.124	378.091 904	131.6934	12.647 487
Uranus	Opposition	2451 764.317	369.656 035	213.6884	4.333 093
Uranus	Conjunction	2451 579.489	369.656 035	31.5219	4.333 093
Neptune	Opposition	2451 753.122	367.486 703	202.6544	2.194 998
Neptune	Conjunction	2451 569.379	367.486 703	21.5569	2.194 998

Then find the integer k nearest to

$$\frac{365.2425\,Y + 1721\,060 - A}{B} \tag{36.1}$$

It is important to note that k must be an *integer*. Non-integer values of k would yield meaningless results. Successive values of k will provide the data for the successive events (for instance, successive oppositions of Mars), the value $k = 0$ corresponding to the first one after 2000 January 1. For years preceding A.D. 2000, k takes negative values.

Then calculate

$$\mathrm{JDE}_0 = A + kB \qquad\qquad M = M_0 + kM_1$$

JDE_0 is the Julian Ephemeris Day corresponding to the time of the *mean* planetary configuration (that is, calculated from circular orbits and uniform planetary motions), and M is the mean anomaly of the Earth at that instant.

M is an angle expressed in *degrees* and decimals. Depending on the type of the calculating machine or the programming language, it may be necessary or desirable to reduce that angle to the range 0–360 degrees by adding or subtracting a convenient multiple of 360, and to convert the result into radians.

Find the time T, expressed in centuries from the beginning of the year 2000, from

$$T = \frac{\mathrm{JDE}_0 - 2451\,545}{36525}$$

T is positive after the beginning of A.D. 2000, negative before.

For the planets Jupiter to Neptune, additional angles are required. Expressed in degrees, these angles are:

$$
\begin{array}{lrcl}
\text{for Jupiter :} & a & = & 82.74 + 40.76T \\
\text{for Saturn :} & a & = & 82.74 + 40.76T \\
 & b & = & 29.86 + 1181.36T \\
 & c & = & 14.13 + 590.68T \\
 & d & = & 220.02 + 1262.87T \\
\text{for Uranus :} & e & = & 207.83 + 8.51T \\
 & f & = & 108.84 + 419.96T \\
\text{for Neptune :} & e & = & 207.83 + 8.51T \\
 & g & = & 276.74 + 209.98T
\end{array}
$$

The time JDE of the *true* configuration is obtained by adding to JDE_0 a correction which is given in Table 36.B as a sum of periodic terms which are functions of the angle M. By reason of the secular variations of the planetary orbits, the coefficients of these periodic terms are slowly varying with time, whence the presence of terms in T and T^2 in Table 36.B.

For instance, for an inferior conjunction of Mercury, the correction (in days) is

$$
\begin{array}{l}
+\ 0.0545 + 0.0002T \\
+\ (-6.2008 + 0.0074T + 0.00003T^2) \sin M \\
+\ (-3.2750 - 0.0197T + 0.00001T^2) \cos M \\
+\ (0.4737 - 0.0052T - 0.00001T^2) \sin 2M \\
+\ \text{etc.} \ldots
\end{array}
$$

The corrected instant obtained in this way is expressed as a Julian Ephemeris Day (JDE), hence in the scale of Dynamical Time. This can be reduced to the standard Julian Day, JD, based on the Universal Time, by *subtracting* the quantity ΔT expressed in *days* (see Chapter 10). However, between the years 1500 and 2100, the correction $-\Delta T$ can be neglected for our purposes.

Finally, from the JD the corresponding calendar date can be obtained by means of standard procedures (see Chapter 7).

Example 36.a — Calculate Mercury's inferior conjunction that is nearest to 1993 October 1.

From Table 36.A, for Mercury, Inferior conjunction, we have

$$A = 2451\,612.023 \qquad M_0 = 63.5867$$
$$B = 115.877\,4771 \qquad M_1 = 114.208\,8742$$

October 1 is three quarters of a year since January 1, hence 1993 October 1 = 1993.75 = Y, and expression (36.1) yields the value -20.28, whence $k = -20$. Remember that k must be an integer! Then

$$\begin{aligned} \mathrm{JDE}_0 &= 2449\,294.473 \\ M &= -2220^\circ.5908 = +299^\circ.4092 \\ T &= -0.06162 \end{aligned}$$

The sum of the terms in the relevant part of Table 36.B (Mercury, Inferior conjunction) is $+3.171$, whence

$$\mathrm{JDE} = \mathrm{JDE}_0 + 3.171 = 2449\,297.644,$$

which corresponds to 1993 November 6, at 3^h TD.

Rounded to the nearest integer hour, this is indeed the correct instant.

Example 36.b — Find the instant of the conjunction of Saturn with the Sun in 2125.

From Table 36.A, for Saturn, Conjunction, we have

$$A = 2451\,681.124 \qquad M_0 = 131.6934$$
$$B = 378.091\,904 \qquad M_1 = 12.647\,487$$

For $Y = 2125.0$ (that is, the beginning of the year 2125), expression (36.1) gives the value $+120.39$. Because we are searching the first Saturn–Sun conjunction *after* the beginning of the year 2125, we take $k = +121$, not $+120$. Then

$$\begin{aligned} \mathrm{JDE}_0 &= 2497\,430.244 \\ M &= 1662^\circ.0393 = 222^\circ.0393 \\ T &= +1.25627 \end{aligned}$$

and for Saturn we have to calculate the following additional angles:

$$a = 133^\circ.95, \qquad b = 73^\circ.97, \qquad c = 36^\circ.18, \qquad d = 6^\circ.53.$$

The sum of the terms in the relevant part of Table 36.B (Saturn, Conjunction with the Sun) is $+7.659$, whence

$$\mathrm{JDE} = \mathrm{JDE}_0 + 7.659 = 2497\,437.903,$$

which corresponds to 2125 August 26, at 10^h TD.

The correct instant, calculated with a more accurate method, is 2125 August 26, at 11^h Dynamical Time.

Greatest elongations of Mercury and Venus

To calculate the times and the values of the greatest elongations of Mercury or Venus, *we start from the nearest inferior conjunction*. So we calculate k, JDE_0, M, and T as explained before. But we do *not* calculate the instant of the true inferior conjunction; instead, we use the periodic terms given in Table 36.C to find the correction (in days) to Mercury's or Venus' *mean* inferior conjunction, to obtain the time of greatest eastern or western elongation. In the same table, periodic terms are provided to find the value of this greatest elongation.

Do not forget that, if the planet is east from the Sun, it is visible in the evening in the west; if the elongation is west, the planet is visible in the morning in the east.

The value of the greatest elongation from the Sun is expressed in degrees and decimals. It concerns the maximum *angular distance* from the planet to the center of the Sun's disk, *not* the greatest difference between the geocentric ecliptical (celestial) longitudes of the two bodies. There is no "official" definition for the elongation of a planet to the Sun, and two different definitions could be considered:

(a) the *angular distance* between the object and the center of the solar disk;

(b) the difference between the *geocentric longitudes* of the object and the center of the solar disk.

Both definitions are used in the astronomical literature. Definition (a) has been used in the *Astronomical Ephemeris* since its beginning in 1960, and from 1981 onwards in its successor, the *Astronomical Almanac*. It is this definition we prefer. For example, for the visibility of Venus near its inferior conjunction, the important factor is not the longitude difference with the Sun, but the angular separation.

The French astronomers, however, use definition (b), for instance in their *Annuaire du Bureau des Longitudes*. On page 275 of the volume for 1990 we read: "Les plus grandes élongations des planètes inférieures: la différence des longitudes géocentriques de la planète et du Soleil est maximale."

Consequently, the results will differ somewhat according as one uses definition (a) or (b). For example, for Mercury's greatest elongation of 1990 August 11: the difference between the geocentric ecliptical longitudes of the Sun and Mercury reached its maximum value (27°22′) at 15^h UT, as mentioned on page 277 of the *Annuaire du Bureau des Longitudes* for 1990, but the maximum angular separation took place at 21^h and was equal to 27°25′.

Example 36.c — Find the instant and the value of the greatest western elongation of Mercury in November 1993.

We start from the inferior conjunction of November 1993, for which we found in Example 36.a:

$$JDE_0 = 2449\,294.473, \qquad M = 299^\circ.4092, \qquad T = -0.06162.$$

With these values of M and T, we find from the relevant part of Table 36.C (Mercury, greatest western elongation):

$$\text{correction} = +19.665 \text{ days}, \qquad \text{elongation} = 19^\circ\!.7506.$$

Hence, the time of Mercury's greatest western elongation was

$$\text{JDE} = \text{JDE}_0 + 19.665 = 2449\,314.14$$

which corresponds to 1993 November 22, at 15^h TD.

The value of this maximum elongation was $19^\circ\!.7506 = 19^\circ 45'$.

Stations in longitude

To calculate the time when a planet is stationary, we start either from the *nearest inferior conjunction* (in the case of Mercury and Venus), or from the *nearest opposition* (in the case of Mars, Jupiter, and Saturn). So we calculate k, JDE_0, M, and T as explained before. We do *not* calculate the instant of the true inferior conjunction or that of the opposition; instead, we use the periodic terms given in Table 36.D to find the correction (in days) to the *mean* inferior conjunction or to the *mean* opposition, to obtain the time when the planet is stationary.

Note that there are two stations. Station 1 is that when the planet begins to move westward (retrograde) among the stars, while Station 2 is when the planet resumes direct motion. In other words, Station 1 precedes the inferior conjunction or the opposition, while Station 2 follows it.

The stations considered here are those in *celestial longitude*, not in right ascension. The time difference between both types of stations can amount to more than one day. For instance, Mars was stationary in longitude on 1997 April 27 at 19^h UT, but its right ascension did not reach a minimum until April 29 at 6^h.

Example 36.d — Find the instant of Mars' station in longitude following the opposition of March 1997.

Starting from the opposition of March 1997, we find

$$k = -2, \qquad \text{JDE}_0 = 2450\,537.510, \qquad M = 84^\circ\!.5468, \qquad T = -0.02758.$$

With these values of M and T, we find from the relevant part of Table 36.D (Mars, Station 2): correction = +28.745 days.

Hence, the time of Mars' station in celestial longitude was

$$\text{JDE} = \text{JDE}_0 + 28.745 = 2450\,566.255$$

which corresponds to 1997 April 27, at 18^h. The correct time was 19^h.

The accuracy of the results

It is evident that the expressions given in Tables 36.B, 36.C, and 36.D are valid only for a limited period of time, namely for a few millennia before and after A.D. 2000, *not* for millions of years! Consequently, do not use the method given in this Chapter before the year −2000, nor after A.D. 4000.

For modern times, say between A.D. 1800 and 2200, the instants obtained for the phenomena involving Mercury and Venus will be less than 1 hour in error. The error can reach 2 hours in the case of Saturn, Uranus, and Neptune, 3 hours for Mars, and 4 hours for Jupiter.

It is expected that the maximum possible error will be somewhat larger near the years −2000 and +4000. On the other hand, if the calculations are performed for epochs near A.D. 2000, say between 1900 and 2100, then the terms in T^2 may safely be ignored.

Exercises

Check your computer program with the following cases; all times are in TD.

Mercury	inferior conjunction	1631 Nov. 7	7^h	(a)
Venus	inferior conjunction	1882 Dec. 6	17^h	(b)
Mars	opposition	2729 Sep. 9	3^h	(c)
Jupiter	opposition	−6 Sep. 15	7^h	(d)
Saturn	opposition	−6 Sep. 14	9^h	(d)
Uranus	opposition	1780 Dec. 17	14^h	(e)
Neptune	opposition	1846 Aug. 20	4^h	(f)

(a) the first observed transit of Mercury over the solar disk (by Gassendi, at Paris).

(b) the last transit of Venus before that of A.D. 2004.

(c) a perihelic opposition of Mars.

(d) because Jupiter and Saturn were in opposition with the Sun with a time difference less than one day, there occurred a triple conjunction between these two planets in that year.

(e) three months before Uranus' discovery by William Herschel.

(f) one month before Neptune's discovery.

TABLE 36.B

Periodic terms in days

	MERCURY Inferior conjunction	MERCURY Superior conjunction
	$+0.0545 + 0.0002T$	$-0.0548 - 0.0002T$
$\sin M$	$-6.2008 + 0.0074T + 0.00003T^2$	$+7.3894 - 0.0100T - 0.00003T^2$
$\cos M$	$-3.2750 - 0.0197T + 0.00001T^2$	$+3.2200 + 0.0197T - 0.00001T^2$
$\sin 2M$	$+0.4737 - 0.0052T - 0.00001T^2$	$+0.8383 - 0.0064T - 0.00001T^2$
$\cos 2M$	$+0.8111 + 0.0033T - 0.00002T^2$	$+0.9666 + 0.0039T - 0.00003T^2$
$\sin 3M$	$+0.0037 + 0.0018T$	$+0.0770 - 0.0026T$
$\cos 3M$	$-0.1768 + 0.00001T^2$	$+0.2758 + 0.0002T - 0.00002T^2$
$\sin 4M$	$-0.0211 - 0.0004T$	$-0.0128 - 0.0008T$
$\cos 4M$	$+0.0326 - 0.0003T$	$+0.0734 - 0.0004T - 0.00001T^2$
$\sin 5M$	$+0.0083 + 0.0001T$	$-0.0122 - 0.0002T$
$\cos 5M$	$-0.0040 + 0.0001T$	$+0.0173 - 0.0002T$
	VENUS Inferior conjunction	VENUS Superior conjunction
	$-0.0096 + 0.0002T - 0.00001T^2$	$+0.0099 - 0.0002T - 0.00001T^2$
$\sin M$	$+2.0009 - 0.0033T - 0.00001T^2$	$+4.1991 - 0.0121T - 0.00003T^2$
$\cos M$	$+0.5980 - 0.0104T + 0.00001T^2$	$-0.6095 + 0.0102T - 0.00002T^2$
$\sin 2M$	$+0.0967 - 0.0018T - 0.00003T^2$	$+0.2500 - 0.0028T - 0.00003T^2$
$\cos 2M$	$+0.0913 + 0.0009T - 0.00002T^2$	$+0.0063 + 0.0025T - 0.00002T^2$
$\sin 3M$	$+0.0046 - 0.0002T$	$+0.0232 - 0.0005T - 0.00001T^2$
$\cos 3M$	$+0.0079 + 0.0001T$	$+0.0031 + 0.0004T$
	MARS Opposition	MARS Conjunction with Sun
	$-0.3088 + 0.00002T^2$	$+0.3102 - 0.0001T + 0.00001T^2$
$\sin M$	$-17.6965 + 0.0363T + 0.00005T^2$	$+9.7273 - 0.0156T + 0.00001T^2$
$\cos M$	$+18.3131 + 0.0467T - 0.00006T^2$	$-18.3195 - 0.0467T + 0.00009T^2$
$\sin 2M$	$-0.2162 - 0.0198T - 0.00001T^2$	$-1.6488 - 0.0133T + 0.00001T^2$
$\cos 2M$	$-4.5028 - 0.0019T + 0.00007T^2$	$-2.6117 - 0.0020T + 0.00004T^2$
$\sin 3M$	$+0.8987 + 0.0058T - 0.00002T^2$	$-0.6827 - 0.0026T + 0.00001T^2$
$\cos 3M$	$+0.7666 - 0.0050T - 0.00003T^2$	$+0.0281 + 0.0035T + 0.00001T^2$
$\sin 4M$	$-0.3636 - 0.0001T + 0.00002T^2$	$-0.0823 + 0.0006T + 0.00001T^2$
$\cos 4M$	$+0.0402 + 0.0032T$	$+0.1584 + 0.0013T$
$\sin 5M$	$+0.0737 - 0.0008T$	$+0.0270 + 0.0005T$
$\cos 5M$	$-0.0980 - 0.0011T$	$+0.0433$

TABLE 36.B (cont.)

	JUPITER Opposition	JUPITER Conjunction with Sun
	$-0.1029 - 0.00009T^2$	$+0.1027 + 0.0002T - 0.00009T^2$
sin M	$-1.9658 - 0.0056T + 0.00007T^2$	$-2.2637 + 0.0163T - 0.00003T^2$
cos M	$+6.1537 + 0.0210T - 0.00006T^2$	$-6.1540 - 0.0210T + 0.00008T^2$
sin $2M$	$-0.2081 - 0.0013T$	$-0.2021 - 0.0017T + 0.00001T^2$
cos $2M$	$-0.1116 - 0.0010T$	$+0.1310 - 0.0008T$
sin $3M$	$+0.0074 + 0.0001T$	$+0.0086$
cos $3M$	$-0.0097 - 0.0001T$	$+0.0087 + 0.0002T$
sin a	$0 + 0.0144T - 0.00008T^2$	$0 + 0.0144T - 0.00008T^2$
cos a	$+0.3642 - 0.0019T - 0.00029T^2$	$+0.3642 - 0.0019T - 0.00029T^2$
	SATURN Opposition	SATURN Conjunction with Sun
	$-0.0209 + 0.0006T + 0.00023T^2$	$+0.0172 - 0.0006T + 0.00023T^2$
sin M	$+4.5795 - 0.0312T - 0.00017T^2$	$-8.5885 + 0.0411T + 0.00020T^2$
cos M	$+1.1462 - 0.0351T + 0.00011T^2$	$-1.1470 + 0.0352T - 0.00011T^2$
sin $2M$	$+0.0985 - 0.0015T$	$+0.3331 - 0.0034T - 0.00001T^2$
cos $2M$	$+0.0733 - 0.0031T + 0.00001T^2$	$+0.1145 - 0.0045T + 0.00002T^2$
sin $3M$	$+0.0025 - 0.0001T$	$-0.0169 + 0.0002T$
cos $3M$	$+0.0050 - 0.0002T$	$-0.0109 + 0.0004T$
sin a	$0 - 0.0337T + 0.00018T^2$	$0 - 0.0337T + 0.00018T^2$
cos a	$-0.8510 + 0.0044T + 0.00068T^2$	$-0.8510 + 0.0044T + 0.00068T^2$
sin b	$0 - 0.0064T + 0.00004T^2$	$0 - 0.0064T + 0.00004T^2$
cos b	$+0.2397 - 0.0012T - 0.00008T^2$	$+0.2397 - 0.0012T - 0.00008T^2$
sin c	$0 - 0.0010T$	$0 - 0.0010T$
cos c	$+0.1245 + 0.0006T$	$+0.1245 + 0.0006T$
sin d	$0 + 0.0024T - 0.00003T^2$	$0 + 0.0024T - 0.00003T^2$
cos d	$+0.0477 - 0.0005T - 0.00006T^2$	$+0.0477 - 0.0005T - 0.00006T^2$

TABLE 36.B (cont.)

	URANUS Opposition	URANUS Conjunction with Sun
	$+0.0844 - 0.0006T$	$-0.0859 + 0.0003T$
$\sin M$	$-0.1048 + 0.0246T$	$-3.8179 - 0.0148T + 0.00003T^2$
$\cos M$	$-5.1221 + 0.0104T + 0.00003T^2$	$+5.1228 - 0.0105T - 0.00002T^2$
$\sin 2M$	$-0.1428 + 0.0005T$	$-0.0803 + 0.0011T$
$\cos 2M$	$-0.0148 - 0.0013T$	$-0.1905 - 0.0006T$
$\sin 3M$	0	$+0.0088 + 0.0001T$
$\cos 3M$	$+0.0055$	0
$\cos e$	$+0.8850$	$+0.8850$
$\cos f$	$+0.2153$	$+0.2153$

	NEPTUNE Opposition	NEPTUNE Conjunction with Sun
	$-0.0140 + 0.00001T^2$	$+0.0168$
$\sin M$	$-1.3486 + 0.0010T + 0.00001T^2$	$-2.5606 + 0.0088T + 0.00002T^2$
$\cos M$	$+0.8597 + 0.0037T$	$-0.8611 - 0.0037T + 0.00002T^2$
$\sin 2M$	$-0.0082 - 0.0002T + 0.00001T^2$	$+0.0118 - 0.0004T + 0.00001T^2$
$\cos 2M$	$+0.0037 - 0.0003T$	$+0.0307 - 0.0003T$
$\cos e$	-0.5964	-0.5964
$\cos g$	$+0.0728$	$+0.0728$

TABLE 36.C

Periodic terms for greatest elongations

	MERCURY, greatest eastern elongation (evening visibility)	
	Correction (days) to the time of mean inferior conjunction	Elongation (degrees)
	$-21.6101 + 0.0002T$	22.4697
sin M	$-1.9803 - 0.0060T + 0.00001T^2$	$-4.2666 + 0.0054T + 0.00002T^2$
cos M	$+1.4151 - 0.0072T - 0.00001T^2$	$-1.8537 - 0.0137T$
sin $2M$	$+0.5528 - 0.0005T - 0.00001T^2$	$+0.3598 + 0.0008T - 0.00001T^2$
cos $2M$	$+0.2905 + 0.0034T + 0.00001T^2$	$-0.0680 + 0.0026T$
sin $3M$	$-0.1121 - 0.0001T + 0.00001T^2$	$-0.0524 - 0.0003T$
cos $3M$	$-0.0098 - 0.0015T$	$+0.0052 - 0.0006T$
sin $4M$	$+0.0192$	$+0.0107 + 0.0001T$
cos $4M$	$+0.0111 + 0.0004T$	$-0.0013 + 0.0001T$
sin $5M$	-0.0061	-0.0021
cos $5M$	$-0.0032 - 0.0001T$	$+0.0003$

	MERCURY, greatest western elongation (morning visibility)	
	Correction (days) to the time of mean inferior conjunction	Elongation (degrees)
	$+21.6249 - 0.0002T$	$22.4143 - 0.0001T$
sin M	$+0.1306 + 0.0065T$	$+4.3651 - 0.0048T - 0.00002T^2$
cos M	$-2.7661 - 0.0011T + 0.00001T^2$	$+2.3787 + 0.0121T - 0.00001T^2$
sin $2M$	$+0.2438 - 0.0024T - 0.00001T^2$	$+0.2674 + 0.0022T$
cos $2M$	$+0.5767 + 0.0023T$	$-0.3873 + 0.0008T + 0.00001T^2$
sin $3M$	$+0.1041$	$-0.0369 - 0.0001T$
cos $3M$	$-0.0184 + 0.0007T$	$+0.0017 - 0.0001T$
sin $4M$	$-0.0051 - 0.0001T$	$+0.0059$
cos $4M$	$+0.0048 + 0.0001T$	$+0.0061 + 0.0001T$
sin $5M$	$+0.0026$	$+0.0007$
cos $5M$	$+0.0037$	-0.0011

TABLE 36.C (cont.)

	VENUS, greatest eastern elongation (evening visibility)	
	Correction (days) to the time of mean inferior conjunction	Elongation (degrees)
	$-70.7600 + 0.0002T - 0.00001T^2$	$46.3173 + 0.0001T$
sin M	$+1.0282 - 0.0010T - 0.00001T^2$	$+0.6916 - 0.0024T$
cos M	$+0.2761 - 0.0060T$	$+0.6676 - 0.0045T$
sin $2M$	$-0.0438 - 0.0023T + 0.00002T^2$	$+0.0309 - 0.0002T$
cos $2M$	$+0.1660 - 0.0037T - 0.00004T^2$	$+0.0036 - 0.0001T$
sin $3M$	$+0.0036 + 0.0001T$	
cos $3M$	$-0.0011 + 0.00001T^2$	

	VENUS, greatest western elongation (morning visibility)	
	Correction (days) to the time of mean inferior conjunction	Elongation (degrees)
	$+70.7462 - 0.00001T^2$	46.3245
sin M	$+1.1218 - 0.0025T - 0.00001T^2$	$-0.5366 - 0.0003T + 0.00001T^2$
cos M	$+0.4538 - 0.0066T$	$+0.3097 + 0.0016T - 0.00001T^2$
sin $2M$	$+0.1320 + 0.0020T - 0.00003T^2$	-0.0163
cos $2M$	$-0.0702 + 0.0022T + 0.00004T^2$	$-0.0075 + 0.0001T$
sin $3M$	$+0.0062 - 0.0001T$	
cos $3M$	$+0.0015 - 0.00001T^2$	

TABLE 36.D

Periodic terms in days

	MERCURY : corrections to the time of mean inferior conjunction	
	Station 1	Station 2
	$-11.0761 + 0.0003T$	$+11.1343 - 0.0001T$
sin M	$-4.7321 + 0.0023T + 0.00002T^2$	$-3.9137 + 0.0073T + 0.00002T^2$
cos M	$-1.3230 - 0.0156T$	$-3.3861 - 0.0128T + 0.00001T^2$
sin $2M$	$+0.2270 - 0.0046T$	$+0.5222 - 0.0040T - 0.00002T^2$
cos $2M$	$+0.7184 + 0.0013T - 0.00002T^2$	$+0.5929 + 0.0039T - 0.00002T^2$
sin $3M$	$+0.0638 + 0.0016T$	$-0.0593 + 0.0018T$
cos $3M$	$-0.1655 + 0.0007T$	$-0.1733 - 0.0007T + 0.00001T^2$
sin $4M$	$-0.0395 - 0.0003T$	$-0.0053 - 0.0006T$
cos $4M$	$+0.0247 - 0.0006T$	$+0.0476 - 0.0001T$
sin $5M$	$+0.0131$	$+0.0070 + 0.0002T$
cos $5M$	$+0.0008 + 0.0002T$	$-0.0115 + 0.0001T$

	VENUS : corrections to the time of mean inferior conjunction	
	Station 1	Station 2
	$-21.0672 + 0.0002T - 0.00001T^2$	$+21.0623 - 0.00001T^2$
sin M	$+1.9396 - 0.0029T - 0.00001T^2$	$+1.9913 - 0.0040T - 0.00001T^2$
cos M	$+1.0727 - 0.0102T$	$-0.0407 - 0.0077T$
sin $2M$	$+0.0404 - 0.0023T - 0.00001T^2$	$+0.1351 - 0.0009T - 0.00004T^2$
cos $2M$	$+0.1305 - 0.0004T - 0.00003T^2$	$+0.0303 + 0.0019T$
sin $3M$	$-0.0007 - 0.0002T$	$+0.0089 - 0.0002T$
cos $3M$	$+0.0098$	$+0.0043 + 0.0001T$

	MARS : corrections to the time of mean opposition	
	Station 1	Station 2
	$-37.0790 - 0.0009T + 0.00002T^2$	$+36.7191 + 0.0016T + 0.00003T^2$
sin M	$-20.0651 + 0.0228T + 0.00004T^2$	$-12.6163 + 0.0417T - 0.00001T^2$
cos M	$+14.5205 + 0.0504T - 0.00001T^2$	$+20.1218 + 0.0379T - 0.00006T^2$
sin $2M$	$+1.1737 - 0.0169T$	$-1.6360 - 0.0190T$
cos $2M$	$-4.2550 - 0.0075T + 0.00008T^2$	$-3.9657 + 0.0045T + 0.00007T^2$
sin $3M$	$+0.4897 + 0.0074T - 0.00001T^2$	$+1.1546 + 0.0029T - 0.00003T^2$
cos $3M$	$+1.1151 - 0.0021T - 0.00005T^2$	$+0.2888 - 0.0073T - 0.00002T^2$
sin $4M$	$-0.3636 - 0.0020T + 0.00001T^2$	$-0.3128 + 0.0017T + 0.00002T^2$
cos $4M$	$-0.1769 + 0.0028T + 0.00002T^2$	$+0.2513 + 0.0026T - 0.00002T^2$
sin $5M$	$+0.1437 - 0.0004T$	$-0.0021 - 0.0016T$
cos $5M$	$-0.0383 - 0.0016T$	$-0.1497 - 0.0006T$

TABLE 36.D (cont.)

	JUPITER : corrections to the time of mean opposition	
	Station 1	Station 2
	$-60.3670 - 0.0001T - 0.00009T^2$	$+60.3023 + 0.0002T - 0.00009T^2$
sin M	$-2.3144 - 0.0124T + 0.00007T^2$	$+0.3506 - 0.0034T + 0.00004T^2$
cos M	$+6.7439 + 0.0166T - 0.00006T^2$	$+5.3635 + 0.0247T - 0.00007T^2$
sin $2M$	$-0.2259 - 0.0010T$	$-0.1872 - 0.0016T$
cos $2M$	$-0.1497 - 0.0014T$	$-0.0037 - 0.0005T$
sin $3M$	$+0.0105 + 0.0001T$	$+0.0012 + 0.0001T$
cos $3M$	-0.0098	$-0.0096 - 0.0001T$
sin a	$0 + 0.0144T - 0.00008T^2$	$0 + 0.0144T - 0.00008T^2$
cos a	$+0.3642 - 0.0019T - 0.00029T^2$	$+0.3642 - 0.0019T - 0.00029T^2$

	SATURN : corrections to the time of mean opposition	
	Station 1	Station 2
	$-68.8840 + 0.0009T + 0.00023T^2$	$+68.8720 - 0.0007T + 0.00023T^2$
sin M	$+5.5452 - 0.0279T - 0.00020T^2$	$+5.9399 - 0.0400T - 0.00015T^2$
cos M	$+3.0727 - 0.0430T + 0.00007T^2$	$-0.7998 - 0.0266T + 0.00014T^2$
sin $2M$	$+0.1101 - 0.0006T - 0.00001T^2$	$+0.1738 - 0.0032T$
cos $2M$	$+0.1654 - 0.0043T + 0.00001T^2$	$-0.0039 - 0.0024T + 0.00001T^2$
sin $3M$	$+0.0010 + 0.0001T$	$+0.0073 - 0.0002T$
cos $3M$	$+0.0095 - 0.0003T$	$+0.0020 - 0.0002T$
sin a	$0 - 0.0337T + 0.00018T^2$	$0 - 0.0337T + 0.00018T^2$
cos a	$-0.8510 + 0.0044T + 0.00068T^2$	$-0.8510 + 0.0044T + 0.00068T^2$
sin b	$0 - 0.0064T + 0.00004T^2$	$0 - 0.0064T + 0.00004T^2$
cos b	$+0.2397 - 0.0012T - 0.00008T^2$	$+0.2397 - 0.0012T - 0.00008T^2$
sin c	$0 - 0.0010T$	$0 - 0.0010T$
cos c	$+0.1245 + 0.0006T$	$+0.1245 + 0.0006T$
sin d	$0 + 0.0024T - 0.00003T^2$	$0 + 0.0024T - 0.00003T^2$
cos d	$+0.0477 - 0.0005T - 0.00006T^2$	$+0.0477 - 0.0005T - 0.00006T^2$

Chapter 37

Pluto

As for the numerous minor planets (see Chapter 33), no analytical theory for the motion of Pluto is available. However, we have constructed expressions for an accurate representation of the planet's motion (2000.0 coordinates) for the years 1885 to 2099. The coefficients of the periodic terms were determined by the least-squares method, on the basis of a numerical integration of Pluto's heliocentric motion performed by Prof. Aldo Vitagliano, of the University of Naples, Italy [1]. Perturbations by the first eight major planets and the three major asteroids were included. This numerical integration itself was based on a model and a set of starting conditions optimized through a least-squares fit on the DE 405 ephemeris calculated at the Jet Propulsion Laboratory, U.S.A.

For the calculation we used the same method as that used in an earlier investigation [2], but now referring Pluto's heliocentric longitude and latitude to the new standard equinox J2000.0. The results are given in Table 37.A.

Method of calculation

Calculate, by means of formula (22.1), the time T in Julian centuries from the epoch J2000.0, and then the following angles (in degrees):

$$\begin{aligned} J &= 34.35 + 3034.9057\,T \\ S &= 50.08 + 1222.1138\,T \\ P &= 238.96 + 144.9600\,T \end{aligned}$$

Then calculate the periodic terms given in Table 37.A. On each horizontal line, the argument α is a linear combination of the angles J, S, and P, namely

$$\alpha = iJ + jS + kP$$

where i, j, k are small integers, given in the second column of the table. The contribution of each argument is

$$A \sin \alpha + B \cos \alpha$$

For instance, on the 13th line of the table we read the numbers $i = 0$, $j = 2$, $k = -1$, so here the argument is $\alpha = 2S - P$, and for the latitude the contribution is $-122 \sin \alpha + 175 \cos \alpha$.

In Table 37.A, the numerical values of the coefficients A and B are given in units of the sixth decimal of a degree in the case of the longitude and the latitude, and in units of the seventh decimal (astronomical units) for the radius vector.

The heliocentric longitude l, latitude b (both in degrees), and the radius vector r of Pluto are then given by

$$
\begin{aligned}
l &= 238.958\,116 + 144.96T + \text{sum of periodic terms in longitude} \\
b &= -3.908\,239 + \text{sum of periodic terms in latitude} \\
r &= 40.724\,1346 + \text{sum of periodic terms in radius vector}
\end{aligned}
$$

The longitude and latitude obtained by this method are heliocentric, not barycentric, and they are referred to the standard equinox of J2000.0.

Calculated in this way, l will be less than 0".07 in error, b less than 0".02, and the radius vector less than 0.000 006 AU, with respect to Vitagliano's numerical integration on which this representation of the motion of Pluto is based. It is important to note, as has been said, that *the method given here is not valid outside the period 1885-2099.*

To find the *geocentric* astrometric 2000.0 equatorial coordinates α and δ of Pluto:

— find the geocentric 2000.0 rectangular equatorial coordinates X, Y, Z of the Sun (see Chapter 26);

— find those of Pluto by

$$
\begin{aligned}
x &= r \cos l \cos b \\
y &= r (\sin l \cos b \cos \varepsilon - \sin b \sin \varepsilon) \\
z &= r (\sin l \cos b \sin \varepsilon + \sin b \cos \varepsilon)
\end{aligned}
\qquad (37.1)
$$

where ε is the mean obliquity of the ecliptic at epoch J2000.0. We have

$$
\begin{aligned}
\sin \varepsilon &= 0.397\,777\,156 \\
\cos \varepsilon &= 0.917\,482\,062
\end{aligned}
$$

— find α and δ, and Pluto's distance Δ to the Earth, by means of formulae (33.10).

However, the effect of light-time should be taken into account. See Chapter 33 and formula (33.3). Hence, to obtain the geocentric α and δ, the values of l, b, r should be calculated for an instant which is earlier than the given instant by the light-time τ.

TABLE 37.A

Periodic terms for the heliocentric coordinates of Pluto

No.	Argument			Longitude		Latitude		Radius vector	
	J	*S*	*P*	*A*	*B*	*A*	*B*	*A*	*B*
1	0	0	1	−19799805	19850055	−5452852	−14974862	66865439	68951812
2	0	0	2	897144	−4954829	3527812	1672790	−11827535	−332538
3	0	0	3	611149	1211027	−1050748	327647	1593179	−1438890
4	0	0	4	−341243	−189585	178690	−292153	−18444	483220
5	0	0	5	129287	−34992	18650	100340	−65977	−85431
6	0	0	6	−38164	30893	−30697	−25823	31174	−6032
7	0	1	−1	20442	−9987	4878	11248	−5794	22161
8	0	1	0	−4063	−5071	226	−64	4601	4032
9	0	1	1	−6016	−3336	2030	−836	−1729	234
10	0	1	2	−3956	3039	69	−604	−415	702
11	0	1	3	−667	3572	−247	−567	239	723
12	0	2	−2	1276	501	−57	1	67	−67
13	0	2	−1	1152	−917	−122	175	1034	−451
14	0	2	0	630	−1277	−49	−164	−129	504
15	1	−1	0	2571	−459	−197	199	480	−231
16	1	−1	1	899	−1449	−25	217	2	−441
17	1	0	−3	−1016	1043	589	−248	−3359	265
18	1	0	−2	−2343	−1012	−269	711	7856	−7832
19	1	0	−1	7042	788	185	193	36	45763
20	1	0	0	1199	−338	315	807	8663	8547
21	1	0	1	418	−67	−130	−43	−809	−769
22	1	0	2	120	−274	5	3	263	−144
23	1	0	3	−60	−159	2	17	−126	32
24	1	0	4	−82	−29	2	5	−35	−16
25	1	1	−3	−36	−29	2	3	−19	−4
26	1	1	−2	−40	7	3	1	−15	8
27	1	1	−1	−14	22	2	−1	−4	12
28	1	1	0	4	13	1	−1	5	6
29	1	1	1	5	2	0	−1	3	1
30	1	1	3	−1	0	0	0	6	−2
31	2	0	−6	2	0	0	−2	2	2
32	2	0	−5	−4	5	2	2	−2	−2
33	2	0	−4	4	−7	−7	0	14	13
34	2	0	−3	14	24	10	−8	−63	13
35	2	0	−2	−49	−34	−3	20	136	−236
36	2	0	−1	163	−48	6	5	273	1065
37	2	0	0	9	−24	14	17	251	149
38	2	0	1	−4	1	−2	0	−25	−9
39	2	0	2	−3	1	0	0	9	−2
40	2	0	3	1	3	0	0	−8	7
41	3	0	−2	−3	−1	0	1	2	−10
42	3	0	−1	5	−3	0	0	19	35
43	3	0	0	0	0	1	0	10	3

The angles J, S, and P are the mean longitudes of Jupiter, Saturn, and Pluto, respectively, as adopted for our calculation of the periodic terms of Table 37.A. It may seem strange that in our solution the mean longitudes of Uranus and Neptune are not needed. The reason is that the mean motion of Uranus is almost exactly twice that of Neptune, or three times that of Pluto. As a consequence, the argument $2N - P$, for instance, where N is the mean longitude of Neptune, has almost the same period as $2P$. The small difference could not have been detected by our investigation based on the rather short interval of 214 years. Therefore, Table 37.A does not contain the argument $2N - P$; the effects of the terms with this argument are included in the terms with argument $2P$. For the same reason, there are no terms in $S - 4P$, $S - 3P$, $S - 2P$, $J - 5P$, $J - 4P$, and $2S - 3P$: they have almost the same period as $4P$, $5P$, $6P$, $2S - P$, $2S$, and $J - S + P$, respectively.

Example 37.a — For 1992 October 13.0 TD = JDE 2448 908.5, find
(1) the geometric heliocentric coordinates of Pluto;
(2) its geocentric astrometric coordinates α and δ.

(1) We find

$$\begin{aligned} T &= -0.072\,183\,4360 \\ J &= -184^\circ\!.719\,921 \\ S &= -38^\circ\!.136\,373 \\ P &= 228^\circ\!.496\,289 \end{aligned}$$

Sum of periodic terms	in longitude :	+ 4 246 306
	in latitude :	+ 18 496 056
	in radius vector :	−110 130 236

from which

$$\begin{aligned} l &= 238^\circ\!.958\,116 - 10^\circ\!.463\,711 + 4^\circ\!.246\,306 = 232^\circ\!.740\,71 \\ b &= -3^\circ\!.908\,239 + 18^\circ\!.496\,056 = +14^\circ\!.587\,82 \\ r &= 40.724\,1346 - 11.013\,0236 = 29.711\,111 \text{ AU} \end{aligned}$$

(2) For the given instant, the Sun's 2000.0 rectangular equatorial coordinates are (from Example 26.b)

$$\begin{aligned} X &= -0.937\,3959 \\ Y &= -0.313\,1679 \\ Z &= -0.135\,7792 \end{aligned}$$

Using Pluto's coordinates l, b, r found above, formulae (37.1) give

$$\begin{aligned} x &= -17.407\,9141 \\ y &= -23.973\,0804 \\ z &= -\ 2.237\,4228 \end{aligned}$$

whence, by formulae (33.10) and (33.3),

$$\Delta = 30.528\,746 \text{ AU} \qquad \text{and} \qquad \tau = 0.17632 \text{ day}$$

This value of Δ is Pluto's true distance to the Earth.

We now repeat the calculation of the planet's heliocentric coordinates for 1992 October 13.0 − 0.17632 = October 12.82368. The results are

$$\begin{aligned} l &= 232^\circ\!.73949 \\ b &= +14^\circ\!.58801 \\ r &= 29.711\,094 \end{aligned}$$

whence

$$\begin{aligned} x &= -17.408\,3780 & \qquad \Delta &= 30.528\,739 \\ y &= -23.972\,7452 & \tau &= 0.17632 \text{ day} \\ z &= -\ 2.237\,1797 \end{aligned}$$

We obtain for τ the same value as before, so no new iteration is needed.

The 2000.0 astrometric coordinates of Pluto for 1992 October 13.0 TD are then found by means of (33.10):

$$\begin{aligned} \alpha &= 232^\circ\!.93231 = 15^h31^m43^s\!.8 \\ \delta &= -4^\circ\!.45802 = -4^\circ27'29'' \end{aligned}$$

Mean orbital elements of Pluto near A.D. 2000:

$$\begin{aligned} a &= 39.543 \text{ AU} \\ e &= 0.2490 \end{aligned}$$

$$\left.\begin{aligned} i &= 17^\circ\!.140 \\ \Omega &= 110^\circ\!.307 \\ \omega &= 113^\circ\!.768 \end{aligned}\right\} 2000.0$$

REFERENCES

1. A. Vitagliano, "Numerical integration for the real time production of fundamental ephemerides over a wide time span", *Celestial Mechanics*, Vol. 66, pages 293–308 (1997).
2. E. Goffin, J. Meeus, and C. Steyaert, "An accurate representation of the motion of Pluto", *Astronomy and Astrophysics*, Vol. 155, pages 323–325 (1986).

Chapter 38

Planets in Perihelion and in Aphelion

The Julian Day corresponding to the time when a planet is in perihelion or in aphelion can be found by means of the following expressions:

Mercury	$\mathrm{JDE} = 2451\,590.257 + 87.969\,349\,63\,k - 0.000\,000\,0000\,k^2$
Venus	$\mathrm{JDE} = 2451\,738.233 + 224.700\,818\,8\,k - 0.000\,000\,0327\,k^2$
Earth	$\mathrm{JDE} = 2451\,547.507 + 365.259\,635\,8\,k + 0.000\,000\,0156\,k^2$
Mars	$\mathrm{JDE} = 2452\,195.026 + 686.995\,785\,7\,k - 0.000\,000\,1187\,k^2$
Jupiter	$\mathrm{JDE} = 2455\,636.936 + 4332.897\,065\,k + 0.000\,1367\,k^2$
Saturn	$\mathrm{JDE} = 2452\,830.12 + 10764.216\,76\,k + 0.000\,827\,k^2$
Uranus	$\mathrm{JDE} = 2470\,213.5 + 30694.8767\,k - 0.005\,41\,k^2$
Neptune	$\mathrm{JDE} = 2468\,895.1 + 60190.33\,k + 0.034\,29\,k^2$

where k is an integer for perihelion, and an integer increased by exactly 0.5 for aphelion. Any other value for k would give meaningless results!

A zero or a positive value of k will give a date after the beginning of the year 2000. If $k < 0$, one obtains a date earlier than A.D. 2000.

For example, $k = +14$ and $k = -222$ give passages through perihelion, while $k = +27.5$ and $k = -119.5$ give aphelion passages.

An *approximate* value for k can be found as follows, where the "year" should be taken with decimals, if necessary:

Mercury	$k \approx 4.15201\,(\text{year} - 2000.12)$
Venus	$k \approx 1.62549\,(\text{year} - 2000.53)$
Earth	$k \approx 0.99997\,(\text{year} - 2000.01)$
Mars	$k \approx 0.53166\,(\text{year} - 2001.78)$
Jupiter	$k \approx 0.08430\,(\text{year} - 2011.20)$
Saturn	$k \approx 0.03393\,(\text{year} - 2003.52)$
Uranus	$k \approx 0.01190\,(\text{year} - 2051.1)$
Neptune	$k \approx 0.00607\,(\text{year} - 2047.5)$

Example 38.a — Find the time of passage of Venus at perihelion nearest to 1978 October 15, that is 1978.79.

An approximate value of k is

$$1.62549\ (1978.79 - 2000.53) = -35.34$$

and, since k must be an integer (perihelion!), we take $k = -35$. Putting this value in the formula for Venus, we find

$$\text{JDE} = 2443\,873.704,$$

which corresponds to 1978 December 31.204, or 1978 December 31 at 5^h Dynamical Time.

Example 38.b — Find the time of passage of Mars through aphelion in the year 2032.

Taking "year" = 2032.0, we find $k \approx +16.07$. Since k must be an integer increased by 0.5 (aphelion!), the first aphelion of Mars after the beginning of the year 2032 occurs for $k = +16.5$.

Using the formula for Mars, this value of k gives

$$\text{JDE} = 2463\,530.456,$$

corresponding to 2032 October 24.956, or 2032 October 24 at 23^h Dynamical Time.

Important: The formulae for the calculation of JDE given on the preceding page are based on unperturbed elliptic orbits. For this reason, the instants obtained for Mars can be a few hours in error.

Due to the mutual planetary perturbations, the instants for Jupiter, calculated by the method described here, may be up to half a month in error. For Saturn, the error can be larger than one month.

For instance, putting $k = -2.5$ in the formula for Jupiter gives 1981 July 19 as the date of an aphelion passage, while the correct date is 1981 July 28. For Saturn, $k = -2$ gives 1944 July 30, while the planet actually reached perihelion on 1944 September 8.

The error can be even larger for Uranus and Neptune. For these planets, the formulae are given merely for completeness.

Accurate times can be obtained by calculating the value of the planet's distance to the Sun for several instants near the expected time, and then finding when this distance reaches a maximum or a minimum. The table on the next page gives the dates when Saturn (in the period 1920-2050) and Uranus (1750-2100) are in perihelion (P) or in aphelion (A). After the date, the distance to the Sun in astronomical units is mentioned. These data have been calculated by means of Bretagnon's complete VSOP87 theory.

Saturn			Uranus		
A	1929 Nov. 11	10.0467	A	1756 Nov. 27	20.0893
P	1944 Sep. 8	9.0288	P	1798 Mar. 3	18.2890
A	1959 May 29	10.0664	A	1841 Mar. 16	20.0976
P	1974 Jan. 8	9.0153	P	1882 Mar. 23	18.2807
A	1988 Sep. 11	10.0444	A	1925 Apr. 1	20.0973
P	2003 July 26	9.0309	P	1966 May 21	18.2848
A	2018 Apr. 17	10.0656	A	2009 Feb. 27	20.0989
P	2032 Nov. 28	9.0149	P	2050 Aug. 17	18.2830
A	2047 July 15	10.0462	A	2092 Nov. 23	20.0994

The case of Neptune is peculiar. This planet has a slow motion and a small orbital eccentricity. On the other hand, the Sun is oscillating around the barycenter of the solar system, mainly due to the actions of Jupiter and Saturn. Consequently, the distance of Neptune to the Sun (not to the barycenter of the solar system) can reach a *double* maximum or minimum.

For example, we had the following extreme values for Neptune's radius vector:

minimum	1876 Aug. 28	$r = 29.8148$ AU
maximum	1881 Dec. 12	29.8213
minimum	1886 July 11	29.8174

Half a revolution later, near the aphelion part of the orbit, we had the following extrema:

maximum	1959 July 13	$r = 30.3317$ AU
minimum	1965 Oct. 6	30.3227
maximum	1968 Nov. 21	30.3241

The maximum of 1881 was *not* an aphelion, because at that time Neptune was near the perihelion of its orbit. Similarly, the minimum of 1965 did not correspond to a perihelion. The author has coined the new terms *apheloid* (= "resembling an aphelion") and *periheloid* for these odd maximum and minimum, respectively [1]. See also Chapter 28 in my *Mathematical Astronomy Morsels* (Willmann-Bell, ed.; 1997).

Figure 1 shows the variation of the distance of Neptune to the Sun from 1954 to 1972. Note the principal aphelion (1), the periheloid (2), and the secondary aphelion (3). Half a revolution later, we have the situation pictured in Figure 2; this

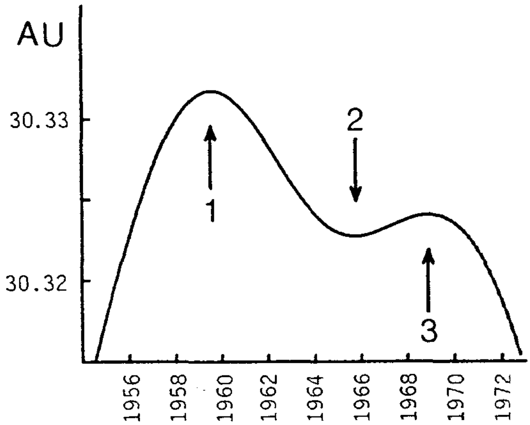


Figure 1

The variation of the distance of Neptune to the Sun, 1954 to 1972.

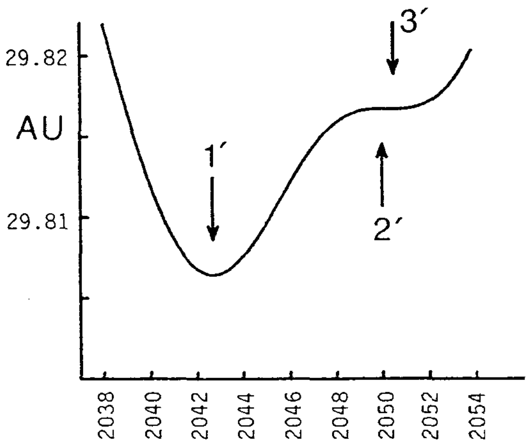


Figure 2

The variation of the distance of Neptune to the Sun, 2038 to 2054.

will be almost a "limiting case": the principal perihelion (1′) will occur in 2042, while in 2049–2050 the distance to the Sun will decrease only very slightly from the apheloid (2′) to the secondary perihelion (3′), as follows:

minimum	2042 Sep. 5	r = 29.8064 AU
maximum	2049 Oct. 24	29.816711
minimum	2050 June 25	29.816696

For the Earth, it is important to note that the formula given to calculate JDE is actually valid for the *barycenter* of the Earth-Moon system. Due to the action of the Moon, the time of least or greatest distance between the centers of Sun and Earth may differ from that for the barycenter by more than one day [2]. For instance, $k = -10$ in the formula for the Earth yields JDE = 2447 894.911, which corresponds to 1990 January 3.41, while the correct instant is 1990 January 4, at 17^h TD.

The values obtained (for the Earth only) can be corrected as follows. Calculate the following angles, in *degrees*:

$$\begin{aligned}
A_1 &= 328.41 + 132.788\,585\,k \\
A_2 &= 316.13 + 584.903\,153\,k \\
A_3 &= 346.20 + 450.380\,738\,k \\
A_4 &= 136.95 + 659.306\,737\,k \\
A_5 &= 249.52 + 329.653\,368\,k
\end{aligned}$$

Remember that k must be an integer for a perihelion, or an integer increased by 0.5 for an aphelion. Then we have the following correction terms, in days:

perihelion	*aphelion*	
+1.278	−1.352	× sin A_1
−0.055	+0.061	sin A_2
−0.091	+0.062	sin A_3
−0.056	+0.029	sin A_4
−0.045	+0.031	sin A_5

Calculated in this way, the times for the years 1980–2019 have a mean error of 3 hours. Exceptionally, the error amounts to 6 hours.

For instance, for $k = -10$, we obtain a correction of +1.261 day, so the value JDE = 2447 894.911 mentioned above is corrected to 2447 896.172, which corresponds to 1990 January 4, at 16^h TD, much closer to the exact value.

Table 38.A gives the times of the passages of the Earth in perihelion and aphelion for the years 1991 to 2010, to the nearest 0.01 hour, together with the distance in astronomical units between the centers of the Sun and the Earth. These data have been calculated accurately, using the complete VSOP87 theory, *not* the approximate method given in this Chapter.

TABLE 38.A

Perihelion and Aphelion of the Earth, 1991-2010
Instants in Dynamical Time

Year	*Perihelion*			*Aphelion*		
		h			h	
1991	Jan. 3	3.00	0.983 281	July 6	15.46	1.016 703
1992	3	15.06	324	3	12.14	740
1993	4	3.08	283	4	22.37	666
1994	2	5.92	301	5	19.30	724
1995	4	11.10	302	4	2.29	742
1996	Jan. 4	7.43	0.983 223	July 5	19.02	1.016 717
1997	1	23.29	267	4	19.34	754
1998	4	21.28	300	3	23.86	696
1999	3	13.02	281	6	22.86	718
2000	3	5.31	321	3	23.84	741
2001	Jan. 4	8.89	0.983 286	July 4	13.65	1.016 643
2002	2	14.17	290	6	3.80	688
2003	4	5.04	320	4	5.67	728
2004	4	17.72	265	5	10.90	694
2005	2	0.61	297	5	4.98	742
2006	Jan. 4	15.52	0.983 327	July 3	23.18	1.016 697
2007	3	19.74	260	6	23.89	706
2008	2	23.87	280	4	7.71	754
2009	4	15.51	273	4	1.69	666
2010	3	0.18	290	6	11.52	702

REFERENCES

1. J. Meeus, "Le centre de gravité du système solaire et le mouvement de Neptune", *Ciel et Terre* (Belgium), Vol. 68, pages 288-292 (November-December 1952).
2. J. Meeus, *Mathematical Astronomy Morsels*, Chapter 27 (Willmann-Bell, ed.; 1997). First published in *l'Astronomie* (France), Vol. 97, pp. 294-296 (June 1983).

Chapter 39

Passages through the Nodes

Given the orbital elements of a planet or a comet, the times t of passages of that body through the nodes of its orbit can easily be calculated as follows.

We have

at the ascending node :	$v = -\omega$ or $360° - \omega$
at the descending node :	$v = 180° - \omega$

where, as before, v is the true anomaly, and ω the argument of the perihelion. Then, with these values of v, proceed as follows.

Case of an elliptic orbit

Calculate the eccentric anomaly E by

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\ \tan\frac{v}{2} \tag{39.1}$$

where e is the orbital eccentricity, and the mean anomaly M by

$$M = E - e\sin E \tag{39.2}$$

In formula (39.2), E should be expressed in radians; the resulting value for M is then in radians too. If, however, E is expressed in degrees and the computer is working in degree mode, then in formula (39.2) one should replace e by its value e_0 converted from radians into degrees, that is, $e_0 = e \times 57.^{\circ}29577951$.

Express M in degrees. Then, if T is the time of perihelion passage, and n is the mean motion in degrees/day, the required time of passage through the node is given by

$$t = T + \frac{M}{n}\ \text{days} \tag{39.3}$$

The corresponding value of the radius vector is given by

$$r = a\,(1 - e \cos E) \tag{39.4}$$

where a is the semimajor axis of the orbit expressed in astronomical units.

If a and n are not given, they can be calculated from (33.6).

Case of a parabolic orbit

Calculate

$$s = \tan \frac{v}{2}$$

Then

$$t = T + 27.403\,895\,(s^3 + 3s)\,q\sqrt{q} \text{ days}$$

where the perihelion distance q is expressed in AU. The corresponding value of the radius vector is

$$r = q\,(1 + s^2)$$

Note. — The nodes refer to the ecliptic of the same epoch as that of the equinox used for the orbital elements. For example, if the orbital elements are referred to the standard equinox of 2000.0, the above-mentioned formulae give the times of passages through the nodes on the ecliptic of 2000.0, *not* on the ecliptic of the date. The difference is generally small, except when the inclination is very small or when the motion is very slow.

Example 39.a — For the 1986 return of periodic comet Halley, W. Landgraf [*Minor Planet Circular* No. 10634 (1986 April 24)] provided the following orbital elements:

T = 1986 February 9.45891 TD
ω = 111°.84644
e = 0.967 274 26
n = 0.012 970 82 degrees/day
a = 17.940 0782

the argument of perihelion ω being referred to the standard equinox of 1950.0.

For the passage at the ascending node, we have

$$v = 360° - \omega = 248°.15356$$

$$\tan \frac{E}{2} = -0.190\,6646$$

$$E = -21^\circ\!.5894332$$
$$M = -21^\circ\!.5894332 - (0.96727426 \times 57^\circ\!.29577951)\sin(-21^\circ\!.5894332)$$
$$= -1^\circ\!.1972043$$

$$t = T + \frac{-1.1972043}{0.01297082} = T - 92.2998 \text{ days}$$

Hence, the comet was at its ascending node (on the ecliptic of 1950.0) 92.2998 days before the perihelion passage, that is, on 1985 November 9.16 Dynamical Time.

Formula (39.4) then gives $r = 1.8045$ AU. So, at its ascending node the famous comet was a little outside of the orbit of Mars.

For the descending node, we find similarly:

$$v = 180^\circ - \omega = 68^\circ\!.15356$$
$$E = +9^\circ\!.9726067$$
$$M = +0^\circ\!.3749928$$
$$t = T + 28.9105 \text{ days} = 1986 \text{ March } 10.37 \text{ TD}$$
$$r = 0.8493 \text{ AU, between the orbits of Venus and Earth}$$

The fact that the comet's motion ($i = 162°$) is retrograde, is irrelevant here. Anyway, ω is measured from the ascending node in the direction of the motion of the body.

Example 39.b — For comet Helin-Roman (1989s = 1989 IX), B. G. Marsden and G. V. Williams (tenth edition of the *Catalogue of Cometary Orbits*, IAU, 1995) give the following elements of a parabolic orbit:

$$T = 1989 \text{ August } 20.2910 \text{ TD}$$
$$q = 1.324502 \text{ AU}$$
$$\omega = 154^\circ\!.9103 \quad (2000.0)$$

T is the time of passage through the perihelion, not to be confused with the T of formula (31.1)!

For the ascending node, we have	For the descending node, we have
$v = -\omega = -154^\circ\!.9103$	$v = 180^\circ - \omega = +25^\circ\!.0897$
$s = -4.4940577$	$s = +0.2225161$
$t = T - 4354.66$ days $= 1977$ September 17	$t = T + 28.3454$ days $= 1989$ September 17.636 TD
$r = 28.07$ AU	$r = 1.3901$ AU

Example 39.c — Calculate the time of passage of Venus at the ascending node nearest to the epoch 1979.0.

We use the elements given in Table 31.A. There we find for Venus

$$a = 0.723\,329\,820, \quad \text{whence } n = 1.602\,137$$
$$e = 0.006\,771\,92 - 0.000\,047\,765\,T + 0.000\,000\,0981\,T^2$$
$$\omega = \pi - \Omega = 54^\circ\!.883\,783 + 0^\circ\!.501\,1082\,T - 0^\circ\!.001\,4824\,T^2$$

The terms in T^3 can safely be dropped here. The elements e and ω vary (rather slowly) with time. Let us calculate their values for the epoch 1979.0, that is, for $T = -0.21$. We find

$$e = 0.006\,781\,95 \qquad \omega = 54^\circ\!.778\,485$$

and then, successively,

$$\begin{aligned} v &= -\omega = -54^\circ\!.778\,485 \\ E &= -54^\circ\!.461\,662 \\ M &= -54^\circ\!.145\,467 \\ t &= T - 33.7958 \text{ days} \quad (T \text{ is the time of perihelion passage}) \end{aligned}$$

In Example 38.a, we have found T = 1978 December 31.204 for the time of passage of Venus in the perihelion. Therefore, we have

t = 1978 November 27.408 or 1978 November 27, at 10^h TD.

The algorithms given in this Chapter assume that the body moves in an unperturbed orbit. To obtain full accuracy, the heliocentric latitude of the body should be calculated for three or five instants near the expected time. At the node we have, of course, latitude = zero.

Saturn reached the descending node (on the ecliptic of the date) of its orbit on 1990 September 4, and will be at its ascending node on 2005 January 8.

Uranus was at the descending node on 1984 December 21, and will go through the ascending node on 2029 May 19.

For Neptune we have

1920 June 3	ascending node
2003 Aug. 11	descending node
2084 Dec. 30	ascending node

Chapter 40

Correction for Parallax

Suppose we wish to calculate the topocentric coordinates of a body (Moon, Sun, planet, or comet) when its geocentric coordinates are known. *Geocentric* = as seen from the center of the Earth; *topocentric* = as seen from the observer's place on the Earth's surface (Greek: *topos* = place; compare with the word "topology").

In other words, we wish to find the corrections $\Delta\alpha$ and $\Delta\delta$ (the parallaxes in right ascension and in declination), in order to obtain the topocentric right ascension $\alpha' = \alpha + \Delta\alpha$ and the topocentric declination $\delta' = \delta + \Delta\delta$, when the geocentric values α and δ are known.

Let ρ be the geocentric radius and φ' the geocentric latitude of the observer. The quantities $\rho \sin \varphi'$ and $\rho \cos \varphi'$ can be calculated by the method described in Chapter 11.

Let π be the equatorial horizontal parallax of the body. For the Sun, a planet, or a comet, it is frequently more convenient to use the distance Δ (in astronomical units) to the Earth instead of the parallax. We then have

$$\sin \pi = \frac{\sin 8''.794}{\Delta}$$

or, with sufficient accuracy,

$$\pi = \frac{8''.794}{\Delta} \tag{40.1}$$

Then, if H is the geocentric hour angle of the body, the rigorous formulae are:

$$\tan \Delta\alpha = \frac{-\rho \cos \varphi' \sin \pi \sin H}{\cos \delta - \rho \cos \varphi' \sin \pi \cos H} \tag{40.2}$$

In the case of the declination we may, instead of computing $\Delta\delta$, calculate δ' directly from

$$\tan \delta' = \frac{(\sin \delta - \rho \sin \varphi' \sin \pi) \cos \Delta\alpha}{\cos \delta - \rho \cos \varphi' \sin \pi \cos H} \tag{40.3}$$

Except for the Moon, the following non-rigorous formulae may often be used instead of (40.2) and (40.3):

$$\Delta\alpha = \frac{-\pi\,\rho \cos\varphi' \sin H}{\cos\delta} \tag{40.4}$$

$$\Delta\delta = -\pi\,(\rho \sin\varphi' \cos\delta - \rho\cos\varphi' \cos H \sin\delta) \tag{40.5}$$

If π is expressed in seconds of a degree ($''$), the $\Delta\alpha$ and $\Delta\delta$ too are expressed in this unit. To express $\Delta\alpha$ in seconds of time, divide the result by 15.

Note that $\Delta\alpha$ is a small angle, always lying between $-2°$ and $+2°$ in the case of the Moon. It is, of course, much smaller in the case of a planet.

An alternative method is as follows. Calculate

$$\left.\begin{aligned} A &= \cos\delta \sin H \\ B &= \cos\delta\cos H - \rho\cos\varphi' \sin\pi \\ C &= \sin\delta - \rho\sin\varphi'\sin\pi \end{aligned}\right\} \tag{40.6}$$

$$q = \sqrt{A^2 + B^2 + C^2} > 0 \tag{40.7}$$

Then the topocentric hour angle H' and declination δ' are given by

$$\tan H' = \frac{A}{B} \qquad\qquad \sin\delta' = \frac{C}{q}$$

Example 40.a — Calculate the topocentric right ascension and declination of Mars on 2003 August 28, at $3^h17^m00^s$ Universal Time at Palomar Observatory, for which (Example 11.a)

$\rho\sin\varphi' = +0.546\,861, \qquad \rho\cos\varphi' = +0.836\,339,$
L = longitude = $+7^h47^m27^s$ (West)

Mars' geocentric apparent equatorial coordinates for the given instant, interpolated from an accurate ephemeris, are

$$\alpha = 22^h38^m07^s\!.25 = 339^\circ\!.530\,208$$
$$\delta = -15^\circ46'15''\!.9 = -15^\circ\!.771\,083$$

The planet's distance at that time is 0.37276 AU. Hence, by formula (40.1), its equatorial horizontal parallax is $\pi = 23''\!.592$.

We still need the geocentric hour angle, which is equal to $H = \theta_0 - L - \alpha$, where θ_0, the apparent sidereal time at Greenwich, can be found as indicated in Chapter 12. For the given instant, we find $\theta_0 = 1^h40^m45^s$. Consequently,

$$\begin{aligned} H &= 1^h40^m45^s - 7^h47^m27^s - 22^h38^m07^s \\ &= -28^h44^m49^s = -431^\circ\!.2042 = +288^\circ\!.7958 \end{aligned}$$

Formula (40.2) then gives

$$\tan \Delta\alpha = \frac{+0.000\,090\,557}{+0.962\,324}$$

whence

$$\Delta\alpha = +0^\circ\!.005\,3917 = +1^s\!.29$$
$$\alpha' = \alpha + \Delta\alpha = 22^h38^m08^s\!.54$$

Formula (40.3) gives

$$\tan \delta' = \frac{-0.271\,857\,13}{+0.962\,324\,47} \qquad \text{whence} \quad \delta' = -15^\circ 46' 30''\!.0$$

If, instead of (40.2) and (40.3), we chose the non-rigorous formulae (40.4) and (40.5), we find

$$\Delta\alpha = +19''\!.409 = +1^s\!.29, \quad \text{as above;}$$
$$\Delta\delta = -14''\!.1, \quad \text{whence } \delta' = \delta - 14''\!.1 = -15^\circ 46' 30''\!.0, \quad \text{as above.}$$

As an exercise, perform the calculation for the Moon, again for Palomar Observatory, using fictive values, for instance

$$\alpha = 1^h00^m00^s\!.00 = 15^\circ\!.000\,000 \qquad H = 4^h00^m00^s\!.00 = +60^\circ\!.000\,000$$
$$\delta = +5^\circ\!.000\,000 \qquad \pi = 0^\circ 59' 00''$$

First, use the formulae (40.2) and (40.3). Then do the calculation over again with (40.6) and (40.7). You should obtain the same results exactly. Compare the results with those obtained by means of the non-rigorous expressions (40.4) and (40.5).

We can consider the opposite problem: from the observed topocentric coordinates α' and δ', deduce the geocentric values α and δ. In the case of a planet or comet, the corrections $\Delta\alpha$ and $\Delta\delta$ are so small that the formulae (40.4) and (40.5) can be used also for the reduction from topocentric to geocentric coordinates, changing the signs of $\Delta\alpha$ and $\Delta\delta$, of course.

Parallax in horizontal coordinates

The parallax in azimuth is always very small. It would be zero if the Earth were exactly a sphere. At the horizon, the parallax in azimuth is always less than $\pi/300$, where π is the equatorial horizontal parallax of the body.

Due to the parallax, the apparent altitude of a celestial body is smaller than its "geocentric" altitude h. Except when high accuracy is needed, the parallax p in altitude may be calculated from $\sin p = \rho \sin \pi \cos h$.

Except in the case of the Moon, the parallax is so small that we may consider p and π to be proportional to their sines, and then we have $p = \rho\, \pi \cos h$.

The quantity ρ denotes the observer's distance to the center of the Earth, the equatorial radius being taken as unity — see Chapter 11. In many cases we may simply write $\rho = 1$.

Parallax in ecliptical coordinates

It is possible to calculate the topocentric coordinates of a celestial body (Moon or planet), from its geocentric values, directly in ecliptical coordinates. The following formulae are those given by Joseph Johann von Littrow (*Theoretische und Practische Astronomie*, Vol. I, p. 91; Wien, 1821), but in a slightly modified form. These expressions are rigorous.

Let λ = geocentric ecliptical longitude of the celestial body,
β = its geocentric ecliptical latitude,
s = its geocentric semidiameter,
λ', β', s' = the required topocentric values of the same quantities,
φ = the observer's latitude,
ε = the obliquity of the ecliptic,
θ = the local sidereal time,
π = the equatorial horizontal parallax of the body.

For the given place, calculate the quantities $\rho \sin \varphi'$ and $\rho \cos \varphi'$, as explained on page 82. For short, we shall call these quantities S and C, respectively. Then

$$N = \cos \lambda \cos \beta - C \sin \pi \cos \theta$$

$$\tan \lambda' = \frac{\sin \lambda \cos \beta - \sin \pi \, (S \sin \varepsilon + C \cos \varepsilon \sin \theta)}{N}$$

$$\tan \beta' = \frac{\cos \lambda' \, (\sin \beta - \sin \pi \, (S \cos \varepsilon - C \sin \varepsilon \sin \theta))}{N}$$

$$\sin s' = \frac{\cos \lambda' \cos \beta' \sin s}{N}$$

As an exercise, calculate λ', β', s' from the following data:

$\lambda = 181°46'22''.5$	$\varphi = +50°05'07''.8$, at sea level
$\beta = +2°17'26''.2$	$\varepsilon = 23°28'00''.8$
$\pi = 0°59'27''.7$	$\theta = 209°46'07''.9$
$s = 0°16'15''.5$	

Answer:

$\lambda' = 181°48'05''.0$
$\beta' = +1°29'07''.1$
$s' = 0°16'25''.5$

Chapter 41

Illuminated Fraction of the Disk and Magnitude of a Planet

The illuminated fraction k of the disk of a planet, as seen from the Earth, can be calculated from

$$k = \frac{1 + \cos i}{2} \tag{41.1}$$

where i is the phase angle (the angle Sun-planet-Earth), which can be found from

$$\cos i = \frac{r^2 + \Delta^2 - R^2}{2r\Delta}$$

r being the planet's distance to the Sun, Δ its distance to the Earth, and R the distance Sun-Earth, all in astronomical units. Combining these two formulae, we find

$$k = \frac{(r + \Delta)^2 - R^2}{4r\Delta} \tag{41.2}$$

If the planet's position has been obtained by the "first method" of Chapter 33, then we have, using the notations used there,

$$\cos i = \frac{R - R_0 \cos B \cos(L - L_0)}{\Delta} \tag{41.3}$$

or

$$\cos i = \frac{x \cos B \cos L + y \cos B \sin L + z \sin B}{\Delta} \tag{41.4}$$

The position angle of the mid-point of the illuminated limb of a planet can be calculated in the same way as for the Moon — see Chapter 48.

Example 41.a — Find the illuminated fraction of the disk of Venus on 1992 December 20, at 0^h TD.

In Example 33.a we have found, for that instant,

$$\begin{aligned} r &= 0.724\,604 \quad \text{(called } R \text{ there)} \\ R &= 0.983\,824 \quad \text{(called } R_0 \text{ there)} \\ \Delta &= 0.910\,947 \end{aligned}$$

whence, by formula (41.2), $k = 0.647$.

Or, using from the same Example 33.a the values L_0 and R_0 from (A), L, B, R from (B), x, y, z from (C), and $\Delta = 0.910\,947$, formulae (41.3) and (41.4) both give $\cos i = 0.29312$, whence $k = 0.647$, as above.

For Mercury and Venus, k can take all values between 0 and 1. For Mars, the illuminated fraction of the disk can never be less than approximately 0.838. In the case of Jupiter, the phase angle i is always less than 12°, whence k can vary only between 0.989 and 1. For Saturn, i is always less than 6½ degrees, so for this planet k is always between 0.997 and 1, as seen from the Earth.

In the case of Venus, an *approximate* value for k can be found as follows. Calculate T by means of formula (22.1), then

$$\begin{aligned} V &= 261^\circ\!.51 + 22518^\circ\!.443\,T \\ M &= 177^\circ\!.53 + 35999^\circ\!.050\,T \\ N &= 50^\circ\!.42 + 58517^\circ\!.811\,T \\ W &= V + 1^\circ\!.91 \sin M + 0^\circ\!.78 \sin N \\ \Delta^2 &= 1.52321 + 1.44666 \cos W \qquad (\Delta > 0) \end{aligned}$$

$$k = \frac{(0.72333 + \Delta)^2 - 1}{2.89332\,\Delta}$$

An *approximate* value of Venus' elongation ψ to the Sun is then given by

$$\cos \psi = \frac{\Delta^2 + 0.4768}{2\Delta}$$

Example 41.b — Same as in Example 41.a, but now using the approximate method described above. We find successively

JD = 2448976.5, $T = -0.070\,321\,697$, $V = -1322^\circ\!.025 = +117^\circ\!.975$,
$M = -2353^\circ\!.984 = +166^\circ\!.016$, $N = -4064^\circ\!.652 = +255^\circ\!.348$,
$W = V + 0^\circ\!.462 - 0^\circ\!.755 = 117^\circ\!.682$, $\Delta = 0.922\,575$, $k = 0.640$.

The correct value, found in Example 41.a, is 0.647.

Magnitude of the Planets

As seen from the Earth, the apparent (stellar) magnitude of a planet at a given instant depends of the planet's distance to the Earth (Δ), its distance to the Sun (r), and the phase angle (i). For Saturn, the magnitude depends also upon the aspect of the ring.

G. Müller's formulae, based on observations which he made from 1877 to 1891, are used since many years in astronomical almanacs. The numerical expressions for the visual magnitudes are as follows [1]:

Mercury :	$+1.16 + 5 \log r\Delta + 0.02838\,(i-50) + 0.000\,1023\,(i-50)^2$
Venus :	$-4.00 + 5 \log r\Delta + 0.01322\,i + 0.000\,000\,4247\,i^3$
Mars :	$-1.30 + 5 \log r\Delta + 0.01486\,i$
Jupiter :	$-8.93 + 5 \log r\Delta$
Saturn :	$-8.68 + 5 \log r\Delta + 0.044\,\lvert\Delta U\rvert - 2.60 \sin \lvert B\rvert + 1.25 \sin^2 B$
Uranus :	$-6.85 + 5 \log r\Delta$
Neptune :	$-7.05 + 5 \log r\Delta$

in which i is expressed in degrees, r and Δ are in astronomical units, and the logarithms are to the base 10. For Saturn, the quantities ΔU and B, pertaining to the ring, are defined in Chapter 45; care must be taken to have ΔU and B positive, and to express ΔU in degrees. (As an approximation, the phase angle i might be used instead of ΔU.)

Of course, Müller's expressions are not perfect. For instance, the effect of the phase is not taken into account in the case of Jupiter. In the formula for Saturn, the Sun's altitude B' above the plane of the ring is not considered (it is supposed to be equal to B); and when B and B' have opposite signs, the *dark* side of the ring is turned towards the Earth, but this case is not considered by Müller.

In any case, the calculated magnitudes should be rounded to the nearest tenth of a magnitude. Giving them to the nearest hundredth makes no sense. Mars, for instance, can differ by as much as 0.3 magnitude from the brightness it "ought" to have. Some regions of Mars have more dark markings than others, so the planet's brightness depends on which face is turned towards us; and the varying polar caps and a major dust storm can add to its magnitude. In the case of Jupiter and Saturn, there are varying atmospheric phenomena, etc.

Example 41.c — Magnitude of Venus on 1992 December 20.0 TD.

From Example 41.a, we have

$r = 0.724\,604$, $\Delta = 0.910\,947$, $\cos i = 0.29312$, whence $i = 72.96$ degrees.
Müller's formula for Venus then gives -3.8 for the magnitude.

Example 41.d — Magnitude of Saturn on 1992 December 16.0 TD.

From Example 45.a, we have

$$r = 9.867\,882, \quad \Delta = 10.464\,606, \quad B = 16^\circ.442, \quad \Delta U = 4^\circ.198.$$

Müller's formula for Saturn then gives +0.9 for the magnitude.

Since 1984, the American *Astronomical Almanac* uses other formulae for the calculation of the visual magnitudes of the planets. It has been stated [2] that these new expressions "are due to D. L. Harris". In fact, in his article [3] Harris did not provide new expressions at all. No expression for the magnitudes is "due" to Harris.

For Mercury and Venus, Harris (pages 277 and 278 of his article) just mentions expressions due to the French astronomer A. Danjon. For the outer planets, Harris discusses values of the absolute magnitude and of the phase coefficient found by others, but he himself does not propose or give new expressions.

If r and Δ (in astronomical units) and i (in degrees) have the same meanings as above, the new expressions used in the *Astronomical Almanac* since 1984 are:

Mercury :	$-0.42 + 5 \log r\Delta + 0.0380\,i - 0.000\,273\,i^2 + 0.000\,002\,i^3$
Venus :	$-4.40 + 5 \log r\Delta + 0.0009\,i + 0.000\,239\,i^2 - 0.000\,000\,65\,i^3$
Mars :	$-1.52 + 5 \log r\Delta + 0.016\,i$
Jupiter :	$-9.40 + 5 \log r\Delta + 0.005\,i$
Saturn :	$-8.88 + 5 \log r\Delta + 0.044\,\lvert\Delta U\rvert - 2.60 \sin \lvert B\rvert + 1.25 \sin^2 B$
Uranus :	$-7.19 + 5 \log r\Delta$
Neptune :	$-6.87 + 5 \log r\Delta$
Pluto :	$-1.00 + 5 \log r\Delta$

For the magnitudes of the minor planets, see Chapter 33.

REFERENCES

1. *Explanatory Supplement to the Astronomical Ephemeris* (London, 1961), page 314.
2. *Astronomical Almanac* for 1984 (Washington, D.C.), page L8; and later volumes.
3. Daniel L. Harris, "Photometry and Colorimetry of Planets and Satellites", Chapter 8 (pages 272*ff*) in *Planets and Satellites*, ed. G. P. Kuiper and B. L. Middlehurst (1961).

Chapter 42

Ephemeris for Physical Observations of Mars

In this Chapter, the following symbols will be used:

D_E = the planetocentric declination of the Earth. When it is positive, Mars' northern pole is tilted towards the Earth;

D_S = the planetocentric declination of the Sun. When it is positive, Mars' northern pole is illuminated;

P = the geocentric position angle of Mars' northern rotation pole, also called *position angle of axis*. It is the angle that the Martian meridian from the center of the disk to the northern rotation pole forms (on the geocentric celestial sphere) with the declination circle through the center. It is measured eastwards from the North Point of the disk. By definition, position angle 0° means northwards on the sky, 90° east, 180° south, and 270° west;

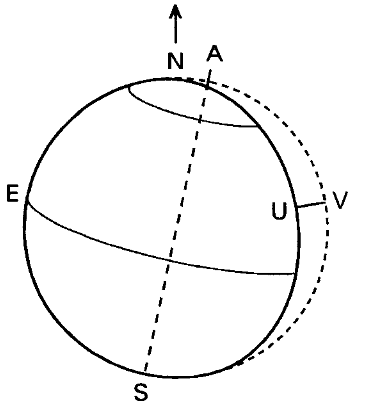
q = the angular amount of the greatest defect of illumination; it is expressed in arcseconds;

Q = the position angle of this greatest defect of illumination;

ω = the (areographic) longitude of the central meridian, as seen from the Earth. The word *areographic* means that use is made of a coordinate system on the surface of Mars. Compare with *geographic* for the Earth.

The drawing on the next page shows the appearance of Mars on 1992 Nov. 9. As seen from the Earth, the illuminated fraction of the planet's disk was 90% (k = 0.90). UV is the greatest defect of illumination. S is Mars' South Pole (just behind the limb, hence not visible), A is the northern extremity of the axis of rotation. AS is the central meridian. The arrow shows the direction of the northern celestial pole (on the celestial sphere of the *Earth*). N is the North Point of Mars' disk (not the planet's north pole!). The position angles are measured from N, towards the East. So we have

$$Q = \text{arc } NESV, \qquad P = \text{arc } NESVA.$$



In the calculation of these quantities, the effect of light-time should be taken into account. Moreover, to obtain full accuracy the aberration of the Sun as seen from *Mars* must be taken into account in the calculation of D_S; and in the calculation of P one should take into account the effect of nutation and aberration on Mars' position.

During the years, several positions for the north pole of Mars (that is, the coordinates of the point on the celestial sphere towards which the axis is directed) have been used in the astronomical almanacs.

According to Lowell and Crommelin [1], the right ascension α_0 and declination δ_0 of the north pole of Mars at the beginning of the year t, referred to the mean equinox of the date, are given by

$$\begin{aligned}\alpha_0 &= 21^h10^m + 1^s\!.565\ (t - 1905.0)\\ \delta_0 &= +54°30' + 12''\!.60\ (t - 1905.0)\end{aligned}$$

This position of the north pole was adopted in 1909. But from 1968 to 1980, the *Astronomical Ephemeris* used the position obtained by G. de Vaucouleurs [3]: at the beginning of the year t

$$\begin{aligned}\alpha_0 &= 316°\!.55 + 0°\!.006\,750\ (t - 1905.0)\\ \delta_0 &= +52°\!.85 + 0°\!.003\,479\ (t - 1905.0)\end{aligned}$$

Note the difference of 1°39′ between the two values of δ_0, for the same epoch 1905.0. Recently adopted values [4] are

$$\left.\begin{aligned}\alpha_0 &= 317°\!.342\\ \delta_0 &= +52°\!.711\end{aligned}\right\}\ \text{equinox 1950.0 and epoch J1950.0}$$

$$\left.\begin{aligned}\alpha_0 &= 317°\!.681\\ \delta_0 &= +52°\!.886\end{aligned}\right\}\ \text{equinox 2000.0 and epoch J2000.0}$$

From these values, we deduce the following expressions for the longitude and latitude of Mars' north pole, referred to the ecliptic and mean equinox of the date:

$$\begin{aligned}\lambda_0 &= 352°\!.9065 + 1°\!.17330\,T\\ \beta_0 &= +63°\!.2818 - 0°\!.00394\,T\end{aligned} \qquad (42.1)$$

where T is the time in Julian centuries from the epoch J2000.0; see formula (22.1). Formulae (42.1) take into account the precession of the rotational axes of both Earth and Mars.

For a given instant t, the values of D_E, D_S, etc., can be calculated as follows.

1. Calculate λ_0 and β_0 by means of (42.1).
2. Calculate the heliocentric longitude l_0, latitude b_0, and radius vector R of the Earth, referred to the ecliptic and mean equinox of the date, for instance by using the relevant data from Appendix III and the precepts given in Chapter 32.
3. Calculate the corresponding heliocentric coordinates l, b, r of Mars, but for the instant $t - \tau$, where τ is the light-time from Mars to the Earth, as given by (33.3). Because Mars' distance Δ is not known in advance, it should be found by iteration — see Step 4. One may use $\Delta = 0$ as a starting value.
4. Calculate

$$\begin{aligned} x &= r \cos b \cos l - R \cos l_0 \\ y &= r \cos b \sin l - R \sin l_0 \\ z &= r \sin b \qquad\quad\;\; - R \sin b_0 \end{aligned} \tag{42.2}$$

Then Mars' distance to the Earth is

$$\Delta = \sqrt{x^2 + y^2 + z^2} \quad > 0 \tag{42.3}$$

5. Calculate Mars' geocentric longitude λ and latitude β from

$$\tan \lambda = \frac{y}{x} \qquad\qquad \tan \beta = \frac{z}{\sqrt{x^2 + y^2}}$$

6. $\sin D_E = -\sin \beta_0 \sin \beta - \cos \beta_0 \cos \beta \cos(\lambda_0 - \lambda)$
7. Calculate the longitude N of the ascending node of Mars' orbit from

$$N = 49^\circ\!.5581 + 0^\circ\!.7721\, T$$

Then correct l and b for the Sun's aberration as seen from Mars:

$$\begin{aligned} l' &= l - 0^\circ\!.00697/r \\ b' &= b - 0^\circ\!.000225 \frac{\cos(l - N)}{r} \end{aligned}$$

8. $\sin D_S = -\sin \beta_0 \sin b' - \cos \beta_0 \cos b' \cos(\lambda_0 - l')$
9. If JDE is the Julian Ephemeris Day corresponding to the given time, calculate the angle W, in degrees, from

$$W = 11.504 + 350.89200025\,(\text{JDE} - \tau - 2433282.5)$$

where τ is the light-time, in days, found in steps 3 and 4.

10. Calculate the mean obliquity of the ecliptic ε_0 by means of formula (22.2). Then use expressions (13.3) and (13.4) to find the pole's equatorial coordinates α_0 and δ_0 from the ecliptical coordinates λ_0 and β_0.

11. Calculate

$$u = y \cos \varepsilon_0 - z \sin \varepsilon_0$$
$$v = y \sin \varepsilon_0 + z \cos \varepsilon_0$$

and the angles α, δ, ζ from

$$\tan \alpha = \frac{u}{x}$$

$$\tan \delta = \frac{v}{\sqrt{x^2 + u^2}}$$

$$\tan \zeta = \frac{\sin \delta_0 \cos \delta \cos (\alpha_0 - \alpha) - \sin \delta \cos \delta_0}{\cos \delta \sin (\alpha_0 - \alpha)}$$

Note that δ is between $-90°$ and $+90°$. But α and ζ can take all values from $0°$ to $360°$, and hence they should be taken in the proper quadrant.

12. Find $\omega = W - \zeta$, where ζ is expressed in degrees.

13. Calculate the nutations in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$) as explained in Chapter 22. Only the most important terms may be used here; an accuracy of, say, 0".01 is not necessary.

14. Correct λ and β for the aberration of Mars:

$$\text{correction to } \lambda: \quad +0^\circ.005\,693 \; \frac{\cos (l_0 - \lambda)}{\cos \beta}$$

$$\text{correction to } \beta: \quad +0^\circ.005\,693 \sin (l_0 - \lambda) \sin \beta$$

15. Add $\Delta\psi$ to λ_0 and to λ. Add $\Delta\varepsilon$ to ε_0 to obtain the true obliquity of the ecliptic ε.

16. Transform (λ_0, β_0) and (λ, β) to the equatorial coordinates (α_0', δ_0') and (α', δ') by means of the expressions (13.3) and (13.4), using for ε the true obliquity obtained above.

17. The position angle P is given by

$$\tan P = \frac{\cos \delta_0' \sin (\alpha_0' - \alpha')}{\sin \delta_0' \cos \delta' - \cos \delta_0' \sin \delta' \cos (\alpha_0' - \alpha')} \tag{42.4}$$

18. The position angle χ of the mid-point of the illuminated limb can be obtained as for the Moon — see Chapter 48. Then the position angle Q of the greatest defect of illumination is $\chi \pm 180°$.

19. Mars' apparent diameter d is given by $d = 9''.36/\Delta$. If k is the illuminated fraction of the planet's disk (see Chapter 41), then the greatest defect of illumination is $q = (1 - k)\,d$.

Example 42.a — Calculate the quantities concerning the appearance of Mars on 1992 November 9, at 0^h UT.

The instant corresponds to JD 2448 935.5. For the difference between Dynamical Time and Universal Time, we use the value $\Delta T = +59$ seconds, or +0.000 683 day, so that the given instant corresponds to

1992 November 9.000 683 TD = JDE 2448 935.500 683.

Step 1. $T = -0.071\,444\,1976$, $\lambda_0 = 352°.82267$, $\beta_0 = +63°.28208$

Step 2. From an accurate ephemeris, calculated by using the complete VSOP87 theory, we deduce

$$l_0 = 46°50'37''.90 = 46°.843\,861$$
$$b_0 = -0''.60 = -0°.000\,167$$
$$R = 0.990\,413\,01$$

Step 3. Geometric heliocentric coordinates of Mars, referred to the ecliptic and mean equinox of the date, taken from an accurate ephemeris:

TD	*l*	*b*	*r*
1992 Nov. 8.0	77°57′48″.45	+0°52′54″.74	1.540 3797
9.0	78 28 24.28	+0 53 46.72	1.541 6585
10.0	78 58 57.09	+0 54 38.36	1.542 9347

We use $\Delta = 0$ (hence $\tau = 0$) as a starting value. For 1992 November 9.000683 TD we find, by interpolation,

$l = 78°.473\,759$, $b = +0°.896\,321$, $r = 1.541\,6594$ AU.

Step 4. $x = -0.369\,4199$
$y = +0.787\,8856$ $\quad \Delta = 0.870\,5266$
$z = +0.024\,1192$

Step 3. With this value of Δ we obtain for the light-time the value $\tau = 0.005\,028$ day. Hence, $t - \tau$ is

1992 November 9.000683 − 0.005 028 = November 8.995 655 TD.

For this instant we find, by interpolation of the tabulated values,

$l = 78°.471\,197$, $b = +0°.896\,249$, $r = 1.541\,6529$.

Step 4. $x = -0.369\,3536$
$y = +0.787\,8654$ $\quad \Delta = 0.870\,4801$
$z = +0.024\,1172$

This new value of Δ yields for the light-time a value which differs by only 0.02 second from the preceding value, so no new iteration is needed.

Step 5. $\lambda = 115°.117\,321$, $\beta = +1°.587\,619$

Step 6. $D_E = +12°.44$

Step 7. $N = 49°.5029, \quad l' = 78°.466676, \quad b' = +0°.896121$

Step 8. $D_S = -2°.76$

Step 9. $W = 5492522°.4593 = 2°.4593$

Step 10. $\varepsilon_0 = 23°26'24''.793 = 23°.440220$
$\alpha_0 = 317°.632606$
$\delta_0 = +52°.860916$

Step 11. $u = +0.7132537 \qquad \alpha = 117°.377075$
$v = +0.3355335 \qquad \delta = +22°.672176$
$\zeta = 250°.9052$

Step 12. $\omega = -248°.45 = 111°.55$

Step 13. $\Delta\psi = +15''.42 \qquad \Delta\varepsilon = -1''.00$

Step 14. corrected $\lambda = 115°.119429$
corrected $\beta = +1°.587472$

Step 15. corrected $\lambda_0 = 352°.82695 \qquad \varepsilon = 23°.439942$
corrected $\lambda = 115°.123712$

Step 16. $\alpha_0' = 317°.63529 \qquad \alpha' = 117°.38380$
$\delta_0' = +52°.86236 \qquad \delta' = +22°.67062$

Step 17. $P = 347°.64$

Step 18. The right ascension and declination of the Sun can be obtained with sufficient accuracy from (25.6) and (25.7), with $\odot = l_0 + 180°$. We find $224°.378$ and $-16°.869$.

The equatorial coordinates of Mars being α and δ, we find by means of formula (48.5) $\chi = 99°.91$, whence $Q = 279°.91$.

Step 19. Using the values of R, r, and Δ found in Steps 2 to 4, formula (41.2) yields $k = 0.9012$. The greatest defect of illumination is

$$q = (1-k) \times 9''.36/\Delta = 1''.06.$$

Mars' apparent diameter is $9''.36/\Delta = 10''.75$.

REFERENCES

1. *Monthly Notices of the Royal Astron. Soc.*, Vol. 66, page 56 (1905). Cited in [2].
2. *Explanatory Supplement to the Astronomical Ephemeris* (London, 1961), page 334.
3. *Icarus*, Vol. 3, page 243 (1964).
4. M. E. Davies e.a., "Report of the IAU Working Group on Cartographic Coordinates and Rotational Elements of the Planets and Satellites: 1982", *Celestial Mechanics*, Vol. 29, pages 309-321 (1983).

Chapter 43

Ephemeris for Physical Observations of Jupiter

For Jupiter three rotational systems have been adopted. System I applies to features within about 10° of the planet's equator; it has an adopted sidereal rotation rate of exactly 877.90 degrees in 24 hours of mean solar time. System II, for use in higher latitudes, where the cloud features take about five minutes longer to circle the planet than those at the equator, rotates exactly 870.27 degrees per day. It follows that the planet's sidereal rotation period is $9^h50^m30^s.003$ in System I, and $9^h55^m40^s.632$ in System II.

System III, rooted deep in Jupiter's interior, applies to radio emissions of the planet. But in this Chapter we will consider only Systems I and II, which are of interest to the visual observers.

As for Mars (see Chapter 42), D_E and D_S will denote the planetocentric declinations of the Earth and the Sun, respectively, and P the position angle of Jupiter's northern rotation pole. The longitude of the Central Meridian will be denoted ω_1 for System I, and ω_2 for System II.

Because Jupiter's rotation axis is almost exactly perpendicular to the planet's orbital plane around the Sun, it is not needed to correct l and b for the Sun's aberration in the calculation of D_S. The error in D_S made by neglecting this aberration will never exceed 0″.5.

For a given instant t, the values of D_E, D_S, ω_1, ω_2, and P can be obtained as follows.

1. Calculate

$$d = \text{JDE} - 2433\,282.5$$

$$T_1 = \frac{d}{36525}$$

and then the right ascension α_0 and declination δ_0 of the north pole of Jupiter, referred to the mean equinox of the date, by the following expressions:

$$\alpha_0 = 268^\circ.00 + 0^\circ.1061\,T_1$$
$$\delta_0 = +64^\circ.50 - 0^\circ.0164\,T_1$$

2. Calculate the angles W_1 and W_2 from

$$W_1 = 17^\circ.710 + 877^\circ.900\,035\,39\,d$$
$$W_2 = 16^\circ.838 + 870^\circ.270\,035\,39\,d$$

These can be large (positive or negative) angles; they should be reduced to less than 360 degrees. The angles W_1 and W_2 are related to the longitude Systems I and II, respectively. The constant terms $17^\circ.710$ and $16^\circ.838$ have been chosen in order to maintain consistency with the Jovian longitude systems established at the end of the 19th century. The other two constants are equal to the values $877^\circ.90$ and $870^\circ.27$ mentioned at the beginning of this Chapter, increased by $0^\circ.000\,035\,39$, the daily variation of the arc of the Jovian equator from its ascending node on the celestial equator to its ascending node on the orbit.

3. Calculate the heliocentric longitude l_0, latitude b_0, and radius vector R of the Earth, referred to the ecliptic and mean equinox of the date, for instance by using the relevant data of Appendix III and the precepts given in Chapter 32.

4. For the same instant, calculate the corresponding heliocentric coordinates l, b, r of Jupiter. Do *not* take the light-time into account here.

5. Calculate x, y, z by means of formulae (42.2), and then Jupiter's distance Δ by (42.3).

6. Correct Jupiter's heliocentric longitude l (in *degrees*) for the light-time:

$$\text{correction to } l = -0^\circ.012\,990\,\Delta/r^2$$

(The correction to the heliocentric latitude can be neglected here.)

7. Using the corrected value of l, calculate x, y, z, Δ again, as in Step 5.

8. Calculate the mean obliquity of the ecliptic ε_0 by means of formula (22.2).

9. Calculate α_S and δ_S from

$$\tan\alpha_S = \frac{\cos\varepsilon_0 \sin l - \sin\varepsilon_0 \tan b}{\cos l}$$

$$\sin\delta_S = \cos\varepsilon_0 \sin b + \sin\varepsilon_0 \cos b \sin l$$

The angle α_S should be taken in the proper quadrant.

10. $\sin D_S = -\sin\delta_0 \sin\delta_S - \cos\delta_0 \cos\delta_S \cos(\alpha_0 - \alpha_S)$
The extreme values of D_S are $+3^\circ.12$ and $-3^\circ.12$.

11. Calculate u, v, α, δ, and ζ as for Mars (see Step 11 of Chapter 42).

12. $\sin D_E = -\sin\delta_0 \sin\delta - \cos\delta_0 \cos\delta \cos(\alpha_0 - \alpha)$
The extreme values of D_E are $+3^\circ.4$ and $-3^\circ.4$.

13. If ζ is expressed in degrees, and Δ in astronomical units, then

$$\omega_1 = W_1 - \zeta - 5^\circ.07033\,\Delta$$
$$\omega_2 = W_2 - \zeta - 5^\circ.02626\,\Delta$$

The last term in each formula is the amount of rotation during the light-time.

14. The values obtained for ω_1 and ω_2 should be reduced to the interval 0°–360° by adding or subtracting a convenient multiple of 360 degrees. The results refer to the geometric (the "true") disk of Jupiter. The planet actually has a very small phase, and the longitudes of the "central meridian" of the illuminated disk can be obtained by adding to ω_1 and to ω_2 the *correction for phase* C which is equal to

$$C = \pm\, 57^\circ.2958 \times \frac{2r\Delta + R^2 - r^2 - \Delta^2}{4r\Delta}$$

and has the same sign as $\sin(l - l_0)$. The angle C is always small, never exceeding $0^\circ.61$.

15. If an accuracy of 0.1 degree is sufficient for the position angle P, go to Step 18. Otherwise, calculate the nutations in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$), as explained in Chapter 22. Only the most important terms may be used; an accuracy of 0.01 arcsecond is not needed. Add $\Delta\varepsilon$ to ε_0 to obtain ε.

16. Correct α and δ for Jupiter's aberration:

correction to α:

$$+0^\circ.005\,693 \;\frac{\cos\alpha\,\cos l_0\,\cos\varepsilon \;+\; \sin\alpha\,\sin l_0}{\cos\delta}$$

correction to δ:

$$+0^\circ.005\,693\,\left[\cos l_0\,\cos\varepsilon\,(\tan\varepsilon\,\cos\delta - \sin\alpha\,\sin\delta) + \cos\alpha\,\sin\delta\,\sin l_0\right]$$

17. Correct α, δ, α_0, and δ_0 for the nutation, by means of expressions (23.1), giving α', δ', α_0', and δ_0'.

18. Obtain P by means of formula (42.4).

Example 43.a — Calculate the quantities concerning the appearance of Jupiter on 1992 December 16, at 0^h UT.

This instant corresponds to JD 2448 972.5. For the difference between Dynamical Time and Universal Time, we shall use the value $\Delta T = +59$ seconds $= +0.000\,68$ day, so that the given instant corresponds to 1992 December 16.00068 TD, or JDE 2448 972.50068.

Step 1. $d = 15690.00068$ $\alpha_0 = 268^\circ\!.04558$
$T_1 = +0.429569$ $\delta_0 = +64^\circ\!.49296$

(keeping extra decimals to minimize rounding errors)

Step 2. $W_1 = 13774269^\circ\!.8622 = 309^\circ\!.8622$
$W_2 = 13654554^\circ\!.2851 = 114^\circ\!.2851$

Steps 3-4. From accurate ephemerides, calculated by using the complete VSOP87 theory:

$l_0 = 84^\circ\!.285703$ $l = 181^\circ\!.882168$
$b_0 = +0^\circ\!.000197$ $b = +1^\circ\!.290464$
$R = 0.98412316$ $r = 5.44642320$

Step 5. $x = -5.5400914$
$y = -1.1580704$ $\Delta = 5.6611645$
$z = +0.1226552$

Step 6. $l = 181^\circ\!.882168 - 0^\circ\!.002479 = 181^\circ\!.879689$

Step 7. $x = -5.5400991$
$y = -1.1578350$ $\Delta = 5.6611239$
$z = +0.1226552$

Step 8. $\varepsilon_0 = 23^\circ 26' 24''\!.745 = 23^\circ\!.4402069$

Step 9. $\alpha_S = 182^\circ\!.237749$
$\delta_S = +0^\circ\!.436472$

Step 10. $D_S = -2^\circ\!.20$

Step 11. $u = -1.1110767$ $\alpha = 191^\circ\!.340327$
$v = -0.3480441$ $\delta = -3^\circ\!.524749$
$\zeta = 13^\circ\!.5238$

Step 12. $D_E = -2^\circ\!.48$

Step 13. $\omega_1 = 267^\circ\!.63$ $\omega_2 = 72^\circ\!.31$

These are the longitudes of the Central Meridian of the *geometric* disk in Systems I and II, respectively.

Step 14. $C = +0^\circ\!.43$. Since $\sin(l - l_0)$ is positive, so is C.

The longitudes of the Central Meridian of the *illuminated* disk are:

System I : $\omega_1 = 267^\circ\!.63 + 0^\circ\!.43 = 268^\circ\!.06$
System II : $\omega_2 = 72^\circ\!.31 + 0^\circ\!.43 = 72^\circ\!.74$

Step 15. $\Delta\psi = +16''\!.86$ $\Delta\varepsilon = -1''\!.79$ $\varepsilon = 23^\circ\!.439710$

Step 16. correction to α : $-0^\circ\!.001627$ $\alpha = 191^\circ\!.338700$
correction to δ : $+0^\circ\!.000560$ $\delta = -3^\circ\!.524189$

Step 17. $\alpha' = 191^\circ\!.34305$ $\alpha_0' = 268^\circ\!.04594$
$\delta' = -3^\circ\!.52592$ $\delta_0' = +64^\circ\!.49339$

Step 18. $P = 24^\circ\!.80$

Lower accuracy

The following, shorter method may be used when high accuracy is not needed.

For the given instant (Dynamical Time!), calculate the JDE (see Chapter 7), and then proceed as follows.

Number of days (and decimals of a day) since 2000 January 1, at 12^h TD:

$$d = \text{JDE} - 2451545.0$$

Argument for the long-period term in the motion of Jupiter:

$$V = 172^\circ\!.74 + 0^\circ\!.001\,115\,88\,d$$

Mean anomalies of Earth and Jupiter:

$$M = 357^\circ\!.529 + 0^\circ\!.985\,6003\,d$$
$$N = 20^\circ\!.020 + 0^\circ\!.083\,0853\,d + 0^\circ\!.329 \sin V$$

Difference between the mean heliocentric longitudes of Earth and Jupiter:

$$J = 66^\circ\!.115 + 0^\circ\!.902\,5179\,d - 0^\circ\!.329 \sin V$$

The angles V, M, N, and J are expressed in degrees and decimals. If necessary, reduce them to the interval 0–360 degrees; this depends on the computing language.

Equations of the center of Earth and Jupiter, in degrees:

$$A = 1.915 \sin M + 0.020 \sin 2M$$
$$B = 5.555 \sin N + 0.168 \sin 2N$$

and then

$$K = J + A - B$$

Radius vector of the Earth:

$$R = 1.00014 - 0.01671 \cos M - 0.00014 \cos 2M$$

Radius vector of Jupiter:

$$r = 5.20872 - 0.25208 \cos N - 0.00611 \cos 2N$$

Distance Earth-Jupiter:

$$\Delta = \sqrt{r^2 + R^2 - 2rR \cos K}$$

The distances R, r, and Δ are expressed in astronomical units, and Δ should of course be taken positive. The phase angle of Jupiter (that is, the angle Earth-Jupiter-Sun) is then given by

$$\sin \psi = \frac{R}{\Delta} \sin K$$

The angle ψ always lies between $-12°$ and $+12°$. Because R and Δ are always positive, the angle ψ has the same sign as $\sin K$.

The longitudes of the Central Meridian in Systems I and II are then, respectively,

$$\omega_1 = 210^\circ.98 + 877^\circ.816\,9088\left(d - \frac{\Delta}{173}\right) + \psi - B$$

$$\omega_2 = 187^\circ.23 + 870^\circ.186\,9088\left(d - \frac{\Delta}{173}\right) + \psi - B$$

where $-\Delta/173$ is the correction for the light-time in days. The denominator 173 results from the fact that the light-time for unit distance is $1/173$ day.

The values obtained for ω_1 and ω_2 should be reduced to the interval $0°$-$360°$, by adding or subtracting a convenient multiple of 360 degrees. The results refer to the geometric disk of Jupiter. The longitudes of the "central meridian" of the illuminated disk can be obtained by adding to ω_1 and ω_2 the *correction for phase* which is equal to

$$\pm\ 57^\circ.3 \sin^2 \frac{\psi}{2}$$

and the sign is opposite the sign of $\sin K$.

Calculated in this way, ω_1 and ω_2 can be up to 0.1 or 0.2 degree in error.

Find Jupiter's heliocentric longitude λ referred to the equinox of 2000.0 by the formula

$$\lambda = 34^\circ.35 + 0^\circ.083\,091\,d + 0^\circ.329 \sin V + B$$

Then we have, in degrees and decimals,

$$D_S = 3.12 \sin(\lambda + 42^\circ.8)$$

$$D_E = D_S - 2.22 \sin\psi \cos(\lambda + 22°) - 1.30\,\frac{r - \Delta}{\Delta}\,\sin(\lambda - 100^\circ.5)$$

In these expressions, $3^\circ.12$ is the inclination of the equator of Jupiter on the orbital plane, $2^\circ.22$ its inclination on the ecliptic, and $1^\circ.30$ the inclination of the orbital plane on the ecliptic.

Example 43.b — Let us take the same instant as in Example 43.a,
1992 December 16, 0^h UT
= JD 2448 972.5
= JDE 2448 972.50068.

We find successively

$$\begin{aligned}
d &= -2572.49932 \\
V &= 169°.87 \\
M &= -2177°.927 = +342°.073 \\
N &= -193°.659 \\
J &= -2255°.670 = +264°.330 \\
A &= -0°.601 \\
B &= +1°.235 \\
K &= 262°.494 \\
R &= 0.98413 \\
r &= 5.44824 \\
\Delta &= 5.66151 \\
\sin\psi &= -0.17234 \\
\psi &= -9°.924
\end{aligned}$$

$$d - \frac{\Delta}{173} = -2572.53205$$

From this we deduce, for the geometric disk of Jupiter:

$$\begin{aligned}
\omega_1 &= -2258\,012°.31 = 267°.69 \\
\omega_2 &= -2238\,407°.64 = 72°.36
\end{aligned}$$

The correct values are 267°.63 and 72°.31 (see Step 13 of Example 43.a).

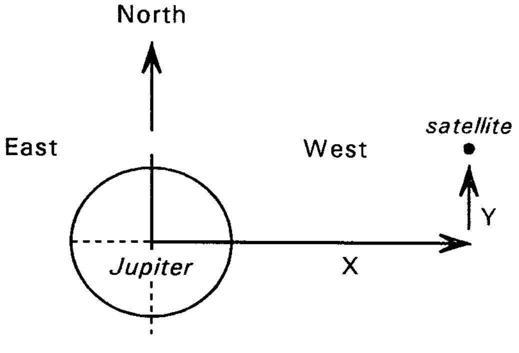
For the correction for phase we find +0°.43, exactly as in Example 43.a, Step 14.

$$\begin{aligned}
\lambda &= -178°.11 \\
D_S &= -2°.194 \\
D_E &= -2°.194 - 0°.350 + 0°.048 = -2°.50
\end{aligned}$$

Chapter 44

Positions of the Satellites of Jupiter

This Chapter gives two methods to calculate, for any given instant, the positions of the four great satellites of Jupiter with respect to the planet, as seen from the Earth. These apparent rectangular coordinates X and Y of the satellites will be measured from the center of the disk of Jupiter, in units of the planet's equatorial radius. X is measured positively to the west of Jupiter, negatively to the east, the X-axis coinciding with the equator of the planet. Y is positive to the north, negative to the south, the Y-axis coinciding with the planet's rotation axis — see the drawing.



The accuracy of the first method ("low accuracy") is sufficient for identifying the satellites at the telescope, or for drawing a wavy-line diagram showing their positions with respect to Jupiter, as given in several astronomical almanacs and magazines. The high-accuracy method is needed, for instance, to calculate the classical phenomena of the satellites (eclipses, transits, etc.) and their mutual phenomena.

Low accuracy

First, convert the date and the instant (TD) to the Julian Day, using the method described in Chapter 7. Then, obtain the following quantities as explained in Chapter 43 ("lower accuracy"): d, V, M, N, J, A, B, K, R, r, Δ, ψ, and the planetocentric declination D_E of the Earth.

For each of the four satellites, we now calculate an angle u which is measured from the inferior conjunction with Jupiter, so that $u = 0°$ corresponds to the satellite's inferior conjunction, $u = 90°$ to its greatest western elongation, $u = 180°$ to the superior conjunction, and $u = 270°$ to the greatest eastern elongation.

$$u_1 = 163.^\circ 8069 + 203.^\circ 405\,8646 \left(d - \frac{\Delta}{173} \right) + \psi - B$$

$$u_2 = 358.^\circ 4140 + 101.^\circ 291\,6335 \left(d - \frac{\Delta}{173} \right) + \psi - B$$

$$u_3 = 5.^\circ 7176 + 50.^\circ 234\,5180 \left(d - \frac{\Delta}{173} \right) + \psi - B$$

$$u_4 = 224.^\circ 8092 + 21.^\circ 487\,9800 \left(d - \frac{\Delta}{173} \right) + \psi - B$$

If necessary, these angles u should be reduced to the interval $0° - 360°$. In order to obtain more accurate values, the results can be improved as follows. Calculate the angles G and H by means of the formulae

$$G = 331.^\circ 18 + 50.^\circ 310\,482 \left(d - \frac{\Delta}{173} \right)$$

$$H = 87.^\circ 45 + 21.^\circ 569\,231 \left(d - \frac{\Delta}{173} \right)$$

Then we have the following corrections, in degrees:

correction to u_1 :	$+0.473 \sin 2(u_1 - u_2)$
correction to u_2 :	$+1.065 \sin 2(u_2 - u_3)$
correction to u_3 :	$+0.165 \sin G$
correction to u_4 :	$+0.843 \sin H$

The first correction is due to a periodic perturbation of satellite I by satellite II. The second correction is a perturbation of II by III. The two last corrections are due to the eccentricities of the orbits of satellites III and IV. (The orbits of I and II are almost circular.)

Note that here we take into account only the largest periodic terms in the motions of the satellites. There are many other (but smaller) periodic terms. For instance, satellite I is perturbed by satellite III too, satellite III by II and by IV, etc. See further the "high accuracy" method in this Chapter.

The distances of the satellites to the center of Jupiter, in units of Jupiter's equatorial radius, are given by

$$\begin{aligned}
r_1 &= 5.9057 - 0.0244 \cos 2\,(u_1 - u_2) \\
r_2 &= 9.3966 - 0.0882 \cos 2\,(u_2 - u_3) \\
r_3 &= 14.9883 - 0.0216 \cos G \\
r_4 &= 26.3627 - 0.1939 \cos H
\end{aligned}$$

where the *un*corrected values of u_1, etc, must be used. In these expressions, the periodic terms are again due to mutual perturbations of the satellites or to their orbital eccentricities.

The apparent rectangular coordinates X and Y of the satellites are then given by

$$X_1 = r_1 \sin u_1 \qquad \text{and} \qquad Y_1 = -r_1 \cos u_1 \sin D_E$$

with similar expressions for the other three satellites.

Example 44.a — Calculate the configuration of the satellites of Jupiter for 1992 December 16 at 0^h UT = JD 2448972.5 = JDE 2448972.50068. (The value $\Delta T = +59$ seconds is used.)

For this instant we have found, in Example 43.b,

$$d = -2572.49932 \qquad d - \frac{\Delta}{173} = -2572.53205$$
$$B = +1^\circ\!.235$$
$$\psi = -9^\circ\!.924 \qquad D_E = -2^\circ\!.50$$

By means of the formulae given in the present Chapter, we then find

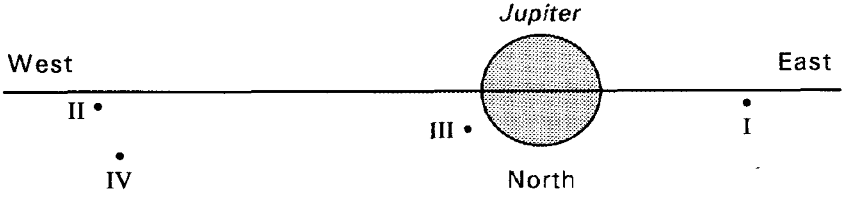
$$\begin{aligned}
u_1 &= -523\,115^\circ\!.458 = 324^\circ\!.542 & 2\,(u_1 - u_2) &= 546^\circ\!.52 = 186^\circ\!.52 \\
u_2 &= -260\,228^\circ\!.719 = 51^\circ\!.281 & 2\,(u_2 - u_3) &= 93^\circ\!.26 \\
u_3 &= -129\,235^\circ\!.349 = 4^\circ\!.651 & G &= -129\,094^\circ\!.15 = 145^\circ\!.85 \\
u_4 &= -\ 55\,064^\circ\!.867 = 15^\circ\!.133 & H &= -\ 55\,400^\circ\!.09 = 39^\circ\!.91
\end{aligned}$$

correction to u_1 : $-0^\circ\!.054$	corrected u_1 = $324^\circ\!.488$
correction to u_2 : $+1^\circ\!.063$	corrected u_2 = $52^\circ\!.344$
correction to u_3 : $+0^\circ\!.093$	corrected u_3 = $4^\circ\!.744$
correction to u_4 : $+0^\circ\!.541$	corrected u_4 = $15^\circ\!.674$

$$\begin{aligned}
r_1 &= 5.9057 + 0.0242 = 5.9299 & X_1 &= -3.44 & Y_1 &= +0.21 \\
r_2 &= 9.3966 + 0.0050 = 9.4016 & X_2 &= +7.44 & Y_2 &= +0.25 \\
r_3 &= 14.9883 + 0.0179 = 15.0062 & X_3 &= +1.24 & Y_3 &= +0.65 \\
r_4 &= 26.3627 - 0.1487 = 26.2140 & X_4 &= +7.08 & Y_2 &= +1.10
\end{aligned}$$

(It is just a coincidence that all four Y-values are positive!)

With these values of X and Y we can draw the following figure which shows the configuration of the satellites at the given time. In this drawing South is up, and West to the left, as in the field of an inverting telescope for an observer in the northern hemisphere.



The X- and Y-values resulting from an accurate calculation are mentioned in Example 44.b. The discrepancies between the Y-values are mainly due to the fact that in this simplified method the inclinations of the orbits of the satellites on the equatorial plane of Jupiter have been neglected. Actually, the four satellites can reach extreme latitudes of 0°03′, 0°31′, 0°20′, and 0°44′, respectively, with respect to the equatorial plane of the planet. As a consequence, mutual occultations cannot be calculated with certainty by means of the simplified method described above. In the case of a very close conjunction, it is even not possible to deduce which of the two satellites passes to the north of the other.

High accuracy

The following method is based on the theory "E5" of the satellites due to Lieske [1].

For the given instant, calculate the following quantities (see Chapter 25):

$\odot$ = geocentric geometric longitude of the Sun,
β = geocentric geometric latitude of the Sun,
R = radius vector of the Sun in astronomical units.

Let τ be the light-time from Jupiter to the Earth. Because the distance of Jupiter to the Earth is not known in advance, so τ is not known. The distance Δ should be found by iteration. A good starting value is $\Delta = 5$, since the extreme values of Jupiter's distance to the Earth are 3.95 and 6.5 astronomical units. The light-time is given by (33.3); a better value for Δ will be provided by formula (44.2).

Calculate the following values for the given time decreased by the light-time τ (see Chapter 32):

$$l = \text{heliocentric longitude of Jupiter,}$$
$$b = \text{heliocentric latitude of Jupiter,}$$
$$r = \text{radius vector of Jupiter, in AU.}$$

In the above, the longitudes and latitudes are referred to the ecliptic and mean equinox of the date.

Calculate the rectangular geocentric ecliptical coordinates of Jupiter

$$\begin{aligned} x &= r \cos b \cos l + R \cos \odot \\ y &= r \cos b \sin l + R \sin \odot \\ z &= r \sin b \qquad + R \sin \beta \end{aligned} \tag{44.1}$$

and its distance to the Earth

$$\Delta = \sqrt{x^2 + y^2 + z^2} \tag{44.2}$$

Calculate Jupiter's geocentric longitude λ_0 and latitude β_0 by

$$\lambda_0 = \text{ATN2}\,(y, x) \qquad \text{and} \qquad \beta_0 = \text{ATN}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

where, as mentioned earlier in this book, ATN2 is the "second" arctangent function. In other words, λ_0 is equal to ATN (y/x) taken in the proper quadrant.

Let t be the time measured in ephemeris days from 1976 August 10 at 0^h TD = JDE 2443000.5, decreased by the light-time τ. In other words, if JDE is the Julian Ephemeris Day corresponding to the given instant,

$$t = \text{JDE} - 2443000.5 - \tau$$

In the following expressions, all numerical values are expressed in degrees and decimals. The longitudes are referred to the standard equinox of 1950.0.

Mean longitudes of the satellites:

$$\begin{aligned} \ell_1 &= 106.07719 + 203.488\,955\,790\,t \\ \ell_2 &= 175.73161 + 101.374\,724\,735\,t \\ \ell_3 &= 120.55883 + 50.317\,609\,207\,t \\ \ell_4 &= 84.44459 + 21.571\,071\,177\,t \end{aligned}$$

Longitudes of the perijoves (*):

$$\begin{aligned} \pi_1 &= 97.0881 + 0.161\,385\,86\,t \\ \pi_2 &= 154.8663 + 0.047\,263\,07\,t \\ \pi_3 &= 188.1840 + 0.007\,127\,34\,t \\ \pi_4 &= 335.2868 + 0.001\,840\,00\,t \end{aligned}$$

(*) The term "periapse", used by some authors, is incorrect — see page 411.

Longitudes of the nodes on the equatorial plane of Jupiter:

$$\begin{aligned}
\omega_1 &= 312.3346 - 0.132\,793\,86\,t \\
\omega_2 &= 100.4411 - 0.032\,630\,64\,t \\
\omega_3 &= 119.1942 - 0.007\,177\,03\,t \\
\omega_4 &= 322.6186 - 0.001\,759\,34\,t
\end{aligned}$$

Principal inequality in the longitude of Jupiter:

$$\begin{aligned}
\Gamma = {} & 0.33033 \sin(163°679 + 0°001\,0512\,t) \\
& + 0.03439 \sin(34°486 - 0°016\,1731\,t)
\end{aligned}$$

There is a small libration, with a period of 2071 days, in the longitudes of the three inner satellites: when satellite II decelerates, I and III accelerate. To take this into account, we need the "phase of free libration"

$$\Phi_\lambda = 199.6766 + 0.173\,791\,90\,t$$

Longitude of the node of the equator of Jupiter on the ecliptic:

$$\psi = 316.5182 - 0.000\,002\,08\,t$$

Mean anomalies of Jupiter and Saturn:

$$\begin{aligned}
G &= 30.23756 + 0.083\,092\,5701\,t + \Gamma \\
G' &= 31.97853 + 0.033\,459\,7339\,t
\end{aligned}$$

Longitude of the perihelion of Jupiter:

$\Pi = 13.469\,942$ (considered as a constant in the E5 theory)

Periodic terms in the longitudes of the satellites

Satellite I

$+0°47259 \sin 2(\ell_1 - \ell_2)$
$-0.03478 \sin(\pi_3 - \pi_4)$
$+0.01081 \sin(\ell_2 - 2\ell_3 + \pi_3)$
$+0.00738 \sin \Phi_\lambda$
$+0.00713 \sin(\ell_2 - 2\ell_3 + \pi_2)$
$-0.00674 \sin(\pi_1 + \pi_3 - 2\Pi - 2G)$
$+0.00666 \sin(\ell_2 - 2\ell_3 + \pi_4)$
$+0.00445 \sin(\ell_1 - \pi_3)$
$-0.00354 \sin(\ell_1 - \ell_2)$
$-0.00317 \sin(2\psi - 2\Pi)$
$+0.00265 \sin(\ell_1 - \pi_4)$
$-0.00186 \sin G$
$+0.00162 \sin(\pi_2 - \pi_3)$
$+0.00158 \sin 4(\ell_1 - \ell_2)$
$-0.00155 \sin(\ell_1 - \ell_3)$
$-0.00138 \sin(\psi + \omega_3 - 2\Pi - 2G)$
$-0.00115 \sin 2(\ell_1 - 2\ell_2 + \omega_2)$
$+0.00089 \sin(\pi_2 - \pi_4)$
$+0.00085 \sin(\ell_1 + \pi_3 - 2\Pi - 2G)$
$+0.00083 \sin(\omega_2 - \omega_3)$
$+0.00053 \sin(\psi - \omega_2)$

Call $\Sigma 1$ the sum of these terms.

Satellite II

$+1^{\circ}.06476 \sin 2(\ell_2 - \ell_3)$
$+0.04256 \sin (\ell_1 - 2\ell_2 + \pi_3)$
$+0.03581 \sin (\ell_2 - \pi_3)$
$+0.02395 \sin (\ell_1 - 2\ell_2 + \pi_4)$
$+0.01984 \sin (\ell_2 - \pi_4)$
$-0.01778 \sin \Phi_\lambda$
$+0.01654 \sin (\ell_2 - \pi_2)$
$+0.01334 \sin (\ell_2 - 2\ell_3 + \pi_2)$
$+0.01294 \sin (\pi_3 - \pi_4)$
$-0.01142 \sin (\ell_2 - \ell_3)$
$-0.01057 \sin G$
$-0.00775 \sin 2(\psi - \Pi)$
$+0.00524 \sin 2(\ell_1 - \ell_2)$
$-0.00460 \sin (\ell_1 - \ell_3)$
$+0.00316 \sin (\psi - 2G + \omega_3 - 2\Pi)$
$-0.00203 \sin (\pi_1 + \pi_3 - 2\Pi - 2G)$
$+0.00146 \sin (\psi - \omega_3)$
$-0.00145 \sin 2G$
$+0.00125 \sin (\psi - \omega_4)$
$-0.00115 \sin (\ell_1 - 2\ell_3 + \pi_3)$
$-0.00094 \sin 2(\ell_2 - \omega_2)$
$+0.00086 \sin 2(\ell_1 - 2\ell_2 + \omega_2)$
$-0.00086 \sin (5G' - 2G + 52^{\circ}.225)$
$-0.00078 \sin (\ell_2 - \ell_4)$
$-0.00064 \sin (3\ell_3 - 7\ell_4 + 4\pi_4)$
$+0.00064 \sin (\pi_1 - \pi_4)$
$-0.00063 \sin (\ell_1 - 2\ell_3 + \pi_4)$
$+0.00058 \sin (\omega_3 - \omega_4)$
$+0.00056 \sin 2(\psi - \Pi - G)$
$+0.00056 \sin 2(\ell_2 - \ell_4)$
$+0.00055 \sin 2(\ell_1 - \ell_3)$
$+0.00052 \sin (3\ell_3 - 7\ell_4 + \pi_3 + 3\pi_4)$
$-0.00043 \sin (\ell_1 - \pi_3)$
$+0.00041 \sin 5(\ell_2 - \ell_3)$
$+0.00041 \sin (\pi_4 - \Pi)$
$+0.00032 \sin (\omega_2 - \omega_3)$
$+0.00032 \sin 2(\ell_3 - G - \Pi)$

Call $\Sigma 2$ the sum of these terms.

Satellite III

$+0^{\circ}.16490 \sin (\ell_3 - \pi_3)$
$+0.09081 \sin (\ell_3 - \pi_4)$
$-0.06907 \sin (\ell_2 - \ell_3)$
$+0.03784 \sin (\pi_3 - \pi_4)$
$+0.01846 \sin 2(\ell_3 - \ell_4)$
$-0.01340 \sin G$
$-0.01014 \sin 2(\psi - \Pi)$
$+0.00704 \sin (\ell_2 - 2\ell_3 + \pi_3)$
$-0.00620 \sin (\ell_2 - 2\ell_3 + \pi_2)$
$-0.00541 \sin (\ell_3 - \ell_4)$
$+0.00381 \sin (\ell_2 - 2\ell_3 + \pi_4)$
$+0.00235 \sin (\psi - \omega_3)$
$+0.00198 \sin (\psi - \omega_4)$
$+0.00176 \sin \Phi_\lambda$
$+0.00130 \sin 3(\ell_3 - \ell_4)$
$+0.00125 \sin (\ell_1 - \ell_3)$
$-0.00119 \sin (5G' - 2G + 52^{\circ}.225)$
$+0.00109 \sin (\ell_1 - \ell_2)$
$-0.00100 \sin (3\ell_3 - 7\ell_4 + 4\pi_4)$
$+0.00091 \sin (\omega_3 - \omega_4)$
$+0.00080 \sin (3\ell_3 - 7\ell_4 + \pi_3 + 3\pi_4)$
$-0.00075 \sin (2\ell_2 - 3\ell_3 + \pi_3)$
$+0.00072 \sin (\pi_1 + \pi_3 - 2\Pi - 2G)$
$+0.00069 \sin (\pi_4 - \Pi)$
$-0.00058 \sin (2\ell_3 - 3\ell_4 + \pi_4)$
$-0.00057 \sin (\ell_3 - 2\ell_4 + \pi_4)$
$+0.00056 \sin (\ell_3 + \pi_3 - 2\Pi - 2G)$
$-0.00052 \sin (\ell_2 - 2\ell_3 + \pi_1)$
$-0.00050 \sin (\pi_2 - \pi_3)$
$+0.00048 \sin (\ell_3 - 2\ell_4 + \pi_3)$
$-0.00045 \sin (2\ell_2 - 3\ell_3 + \pi_4)$
$-0.00041 \sin (\pi_2 - \pi_4)$
$-0.00038 \sin 2G$
$-0.00037 \sin (\pi_3 - \pi_4 + \omega_3 - \omega_4)$
$-0.00032 \sin (3\ell_3 - 7\ell_4 + 2\pi_3 + 2\pi_4)$
$+0.00030 \sin 4(\ell_3 - \ell_4)$
$+0.00029 \sin (\ell_3 + \pi_4 - 2\Pi - 2G)$
$-0.00028 \sin (\omega_3 + \psi - 2\Pi - 2G)$

$+0.00026 \sin (\ell_3 - \Pi - G)$
$+0.00024 \sin (\ell_2 - 3\ell_3 + 2\ell_4)$
$+0.00021 \sin 2(\ell_3 - \Pi - G)$
$-0.00021 \sin (\ell_3 - \pi_2)$
$+0.00017 \sin 2(\ell_3 - \pi_3)$

Call $\Sigma 3$ the sum of these terms.

Satellite IV

$+0\overset{\circ}{.}84287 \sin (\ell_4 - \pi_4)$
$+0.03431 \sin (\pi_4 - \pi_3)$
$-0.03305 \sin 2(\psi - \Pi)$
$-0.03211 \sin G$
$-0.01862 \sin (\ell_4 - \pi_3)$
$+0.01186 \sin (\psi - \omega_4)$
$+0.00623 \sin (\ell_4 + \pi_4 - 2G - 2\Pi)$
$+0.00387 \sin 2(\ell_4 - \pi_4)$
$-0.00284 \sin (5G' - 2G + 52\overset{\circ}{.}225)$
$-0.00234 \sin 2(\psi - \pi_4)$
$-0.00223 \sin (\ell_3 - \ell_4)$
$-0.00208 \sin (\ell_4 - \Pi)$
$+0.00178 \sin (\psi + \omega_4 - 2\pi_4)$
$+0.00134 \sin (\pi_4 - \Pi)$
$+0.00125 \sin 2(\ell_4 - G - \Pi)$
$-0.00117 \sin 2G$
$-0.00112 \sin 2(\ell_3 - \ell_4)$
$+0.00107 \sin (3\ell_3 - 7\ell_4 + 4\pi_4)$
$+0.00102 \sin (\ell_4 - G - \Pi)$
$+0.00096 \sin (2\ell_4 - \psi - \omega_4)$
$+0.00087 \sin 2(\psi - \omega_4)$
$-0.00085 \sin (3\ell_3 - 7\ell_4 + \pi_3 + 3\pi_4)$
$+0.00085 \sin (\ell_3 - 2\ell_4 + \pi_4)$
$-0.00081 \sin 2(\ell_4 - \psi)$
$+0.00071 \sin (\ell_4 + \pi_4 - 2\Pi - 3G)$
$+0.00061 \sin (\ell_1 - \ell_4)$
$-0.00056 \sin (\psi - \omega_3)$
$-0.00054 \sin (\ell_3 - 2\ell_4 + \pi_3)$
$+0.00051 \sin (\ell_2 - \ell_4)$
$+0.00042 \sin 2(\psi - G - \Pi)$
$+0.00039 \sin 2(\pi_4 - \omega_4)$
$+0.00036 \sin (\psi + \Pi - \pi_4 - \omega_4)$
$+0.00035 \sin (2G' - G + 188\overset{\circ}{.}37)$
$-0.00035 \sin (\ell_4 - \pi_4 + 2\Pi - 2\psi)$
$-0.00032 \sin (\ell_4 + \pi_4 - 2\Pi - G)$
$+0.00030 \sin (2G' - 2G + 149\overset{\circ}{.}15)$
$+0.00029 \sin (3\ell_3 - 7\ell_4 + 2\pi_3 + 2\pi_4)$
$+0.00028 \sin (\ell_4 - \pi_4 + 2\psi - 2\Pi)$
$-0.00028 \sin 2(\ell_4 - \omega_4)$
$-0.00027 \sin (\pi_3 - \pi_4 + \omega_3 - \omega_4)$
$-0.00026 \sin (5G' - 3G + 188\overset{\circ}{.}37)$
$+0.00025 \sin (\omega_4 - \omega_3)$
$-0.00025 \sin (\ell_2 - 3\ell_3 + 2\ell_4)$
$-0.00023 \sin 3(\ell_3 - \ell_4)$
$+0.00021 \sin (2\ell_4 - 2\Pi - 3G)$
$-0.00021 \sin (2\ell_3 - 3\ell_4 + \pi_4)$
$+0.00019 \sin (\ell_4 - \pi_4 - G)$
$-0.00019 \sin (2\ell_4 - \pi_3 - \pi_4)$
$-0.00018 \sin (\ell_4 - \pi_4 + G)$
$-0.00016 \sin (\ell_4 + \pi_3 - 2\Pi - 2G)$

Call $\Sigma 4$ the sum of these terms.

The true longitudes of the satellites are

$$
\begin{aligned}
L_1 &= \ell_1 + \Sigma 1 \\
L_2 &= \ell_2 + \Sigma 2 \\
L_3 &= \ell_3 + \Sigma 3 \\
L_4 &= \ell_4 + \Sigma 4
\end{aligned}
$$

Periodic terms in the latitudes of the satellites

The sum of the following terms gives the *tangent* of the satellite's latitude B_i with respect to Jupiter's equatorial plane.

Satellite I

$$+0.0006393 \sin (L_1 - \omega_1)$$
$$+0.0001825 \sin (L_1 - \omega_2)$$
$$+0.0000329 \sin (L_1 - \omega_3)$$
$$-0.0000311 \sin (L_1 - \psi)$$
$$+0.0000093 \sin (L_1 - \omega_4)$$
$$+0.0000075 \sin (3L_1 - 4\ell_2 - 1.9927\,\Sigma 1 + \omega_2)$$
$$+0.0000046 \sin (L_1 + \psi - 2\Pi - 2G)$$

Satellite II

$$+0.0081004 \sin (L_2 - \omega_2)$$
$$+0.0004512 \sin (L_2 - \omega_3)$$
$$-0.0003284 \sin (L_2 - \psi)$$
$$+0.0001160 \sin (L_2 - \omega_4)$$
$$+0.0000272 \sin (\ell_1 - 2\ell_3 + 1.0146\,\Sigma 2 + \omega_2)$$
$$-0.0000144 \sin (L_2 - \omega_1)$$
$$+0.0000143 \sin (L_2 + \psi - 2\Pi - 2G)$$
$$+0.0000035 \sin (L_2 - \psi + G)$$
$$-0.0000028 \sin (\ell_1 - 2\ell_3 + 1.0146\,\Sigma 2 + \omega_3)$$

Satellite III

$$+0.0032402 \sin (L_3 - \omega_3)$$
$$-0.0016911 \sin (L_3 - \psi)$$
$$+0.0006847 \sin (L_3 - \omega_4)$$
$$-0.0002797 \sin (L_3 - \omega_2)$$
$$+0.0000321 \sin (L_3 + \psi - 2\Pi - 2G)$$
$$+0.0000051 \sin (L_3 - \psi + G)$$
$$-0.0000045 \sin (L_3 - \psi - G)$$
$$-0.0000045 \sin (L_3 + \psi - 2\Pi)$$
$$+0.0000037 \sin (L_3 + \psi - 2\Pi - 3G)$$
$$+0.0000030 \sin (2\ell_2 - 3L_3 + 4.03\,\Sigma 3 + \omega_2)$$
$$-0.0000021 \sin (2\ell_2 - 3L_3 + 4.03\,\Sigma 3 + \omega_3)$$

Satellite IV

$$-0.0076579 \sin (L_4 - \psi)$$
$$+0.0044134 \sin (L_4 - \omega_4)$$
$$-0.0005112 \sin (L_4 - \omega_3)$$
$$+0.0000773 \sin (L_4 + \psi - 2\Pi - 2G)$$
$$+0.0000104 \sin (L_4 - \psi + G)$$
$$-0.0000102 \sin (L_4 - \psi - G)$$
$$+0.0000088 \sin (L_4 + \psi - 2\Pi - 3G)$$
$$-0.0000038 \sin (L_4 + \psi - 2\Pi - G)$$

Periodic terms for the radius vector

Satellite I

$-0.0041339 \cos 2(\ell_1 - \ell_2)$
$-0.0000387 \cos (\ell_1 - \pi_3)$
$-0.0000214 \cos (\ell_1 - \pi_4)$
$+0.0000170 \cos (\ell_1 - \ell_2)$
$-0.0000131 \cos 4(\ell_1 - \ell_2)$
$+0.0000106 \cos (\ell_1 - \ell_3)$
$-0.0000066 \cos (\ell_1 + \pi_3 - 2\Pi - 2G)$

Satellite II

$+0.0093848 \cos (\ell_1 - \ell_2)$
$-0.0003116 \cos (\ell_2 - \pi_3)$
$-0.0001744 \cos (\ell_2 - \pi_4)$
$-0.0001442 \cos (\ell_2 - \pi_2)$
$+0.0000553 \cos (\ell_2 - \ell_3)$
$+0.0000523 \cos (\ell_1 - \ell_3)$
$-0.0000290 \cos 2(\ell_1 - \ell_2)$
$+0.0000164 \cos 2(\ell_2 - \omega_2)$
$+0.0000107 \cos (\ell_1 - 2\ell_3 + \pi_3)$
$-0.0000102 \cos (\ell_2 - \pi_1)$
$-0.0000091 \cos 2(\ell_1 - \ell_3)$

Satellite III

$-0.0014388 \cos (\ell_3 - \pi_3)$
$-0.0007919 \cos (\ell_3 - \pi_4)$
$+0.0006342 \cos (\ell_2 - \ell_3)$
$-0.0001761 \cos 2(\ell_3 - \ell_4)$
$+0.0000294 \cos (\ell_3 - \ell_4)$
$-0.0000156 \cos 3(\ell_3 - \ell_4)$
$+0.0000156 \cos (\ell_1 - \ell_3)$
$-0.0000153 \cos (\ell_1 - \ell_2)$
$+0.0000070 \cos (2\ell_2 - 3\ell_3 + \pi_3)$
$-0.0000051 \cos (\ell_3 + \pi_3 - 2\Pi - 2G)$

Satellite IV

$-0.0073546 \cos (\ell_4 - \pi_4)$
$+0.0001621 \cos (\ell_4 - \pi_3)$
$+0.0000974 \cos (\ell_3 - \ell_4)$
$-0.0000543 \cos (\ell_4 + \pi_4 - 2\Pi - 2G)$
$-0.0000271 \cos 2(\ell_4 - \pi_4)$
$+0.0000182 \cos (\ell_4 - \Pi)$
$+0.0000177 \cos 2(\ell_3 - \ell_4)$
$-0.0000167 \cos (2\ell_4 - \psi - \omega_4)$
$+0.0000167 \cos (\psi - \omega_4)$
$-0.0000155 \cos 2(\ell_4 - \Pi - G)$
$+0.0000142 \cos 2(\ell_4 - \psi)$

Satellite IV (cont.)

$$
\begin{aligned}
&+0.0000105 \cos(\ell_1 - \ell_4) \\
&+0.0000092 \cos(\ell_2 - \ell_4) \\
&-0.0000089 \cos(\ell_4 - \Pi - G) \\
&-0.0000062 \cos(\ell_4 + \pi_4 - 2\Pi - 3G) \\
&+0.0000048 \cos 2(\ell_4 - \omega_4)
\end{aligned}
$$

The radius vector R_i of satellite No. i, in equatorial radii of Jupiter, is given by

$$R_i = a_i \times (1 + \text{sum of periodic terms})$$

with the following values for the mean distances:

satellite I	a_1 =	5.90569
satellite II	a_2 =	9.39657
satellite III	a_3 =	14.98832
satellite IV	a_4 =	26.36273

If JDE is the Julian Ephemeris Day corresponding to the given instant, calculate

$$T_0 = \frac{\text{JDE} - 2433\,282.423}{36525}$$

Then the precession in longitude from the epoch B1950.0 to the date, in degrees, is given by

$$P = 1.396\,6626\, T_0 + 0.000\,3088\, T_0^2$$

Add P to the four longitudes L_i and to ψ.

Inclination of Jupiter's axis of rotation on the orbital plane:

$$I = 3^\circ\!.120\,262 + 0^\circ\!.0006\, T$$

where T is the time in centuries since 1900.0.

For each of the four (i = 1 to 4) satellites, we have found the tropical longitude L_i, the equatorial latitude B_i, and the radius vector R_i in equatorial Jupiter radii. For each of them, calculate

$$
\begin{aligned}
X_i &= R_i \cos(L_i - \psi) \cos B_i \\
Y_i &= R_i \sin(L_i - \psi) \cos B_i \\
Z_i &= R_i \sin B_i
\end{aligned}
$$

Now consider a "fifth, fictitious satellite", situated at unit distance from the center of Jupiter, above the planet's north pole:

$$X_5 = 0, \qquad Y_5 = 0, \qquad Z_5 = 1.$$

This fictitious satellite will be needed later.

To obtain the apparent rectangular coordinates of the satellites as they appear on the celestial sphere, as defined at the beginning of this Chapter, several rotations must be performed. So, calculate the following for all five satellites (the four real ones and the fifth, fictitious satellite):

Rotation towards Jupiter's orbital plane:

$$\begin{aligned} A_1 &= X \\ B_1 &= Y \cos I - Z \sin I \\ C_1 &= Y \sin I + Z \cos I \end{aligned}$$

Rotation towards the ascending node of the orbit of Jupiter:

$$\begin{aligned} A_2 &= A_1 \cos \Phi - B_1 \sin \Phi \\ B_2 &= A_1 \sin \Phi + B_1 \cos \Phi \\ C_2 &= C_1 \end{aligned}$$

where $\Phi = \psi - \Omega$, Ω being the longitude of the node of Jupiter, referred to the mean equinox of the date. See in Table 31.A, under "Jupiter", the formula for Ω.

Rotation towards the plane of the ecliptic:

$$\begin{aligned} A_3 &= A_2 \\ B_3 &= B_2 \cos i - C_2 \sin i \\ C_3 &= B_2 \sin i + C_2 \cos i \end{aligned}$$

where i is the inclination of the orbit of Jupiter on the ecliptic. See in Table 31.A the expression for i.

Rotation towards the vernal equinox:

$$\begin{aligned} A_4 &= A_3 \cos \Omega - B_3 \sin \Omega \\ B_4 &= A_3 \sin \Omega + B_3 \cos \Omega \\ C_4 &= C_3 \end{aligned}$$

Then calculate

$$\begin{aligned} A_5 &= A_4 \sin \lambda_0 - B_4 \cos \lambda_0 \\ B_5 &= A_4 \cos \lambda_0 + B_4 \sin \lambda_0 \\ C_5 &= C_4 = C_3 \end{aligned}$$

$$\begin{aligned} A_6 &= A_5 \\ B_6 &= C_5 \sin \beta_0 + B_5 \cos \beta_0 \\ C_6 &= C_5 \cos \beta_0 - B_5 \sin \beta_0 \end{aligned}$$

If ξ, η are the values of A_6 and C_6 for the "fifth satellite", that is, $\xi = A_6(5)$, $\eta = C_6(5)$, then calculate

$$D = \text{ATN2}(\xi, \eta)$$

where, as mentioned earlier in this book, ATN2 is the "second" arctangent function which gives the angle D in the correct quadrant.

Calculate

$$
\begin{aligned}
X &= A_6 \cos D - C_6 \sin D \\
Y &= A_6 \sin D + C_6 \cos D \\
Z &= B_6
\end{aligned}
\tag{44.3}
$$

X and Y are the required rectangular coordinates of the satellite, as defined at the beginning of this Chapter. The quantity Z is negative if the satellite is closer to the Earth than Jupiter, positive if it is more distant than Jupiter.

However, to obtain full accuracy, the apparent coordinates X and Y just obtained should be corrected for two effects:

*1. **differential light-time:*** if a satellite is on the nearer half of its orbit, its light-time is smaller than that of Jupiter; if on the far half, its light-time is larger. The correction to be *added* to X is

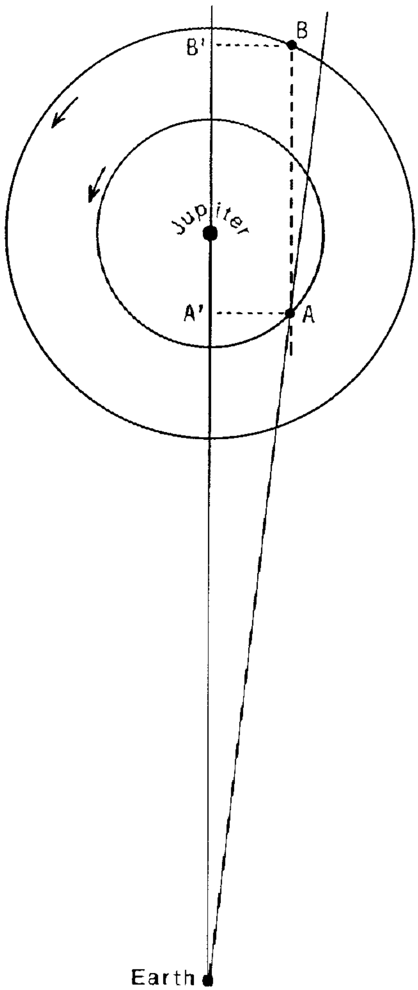
$$\frac{|Z|}{K} \sqrt{1 - (X/R)^2}$$

where

$K =$	17295	for satellite	I
	21819	—	II
	27558	—	III
	36548	—	IV

This correction is zero at the greatest elongations, and *positive* in all other cases. It is always very small, being at most 0.0003 for satellite I, and 0.0007 for satellite IV. The correction to Y is negligible. In the formula above, R is the radius vector of the satellite, while X and Z are the values given by (44.3).

*2. **the perspective effect,*** which is due to the fact that Jupiter is not situated at an infinite distance from the Earth. This is illustrated in the figure at the right, which shows the orbits of two satellites around Jupiter (not to scale!). Although the X-coordinates of satellites A and B are equal *in space* (distances AA' and BB' are equal), they are not exactly in



conjunction as seen from the Earth: their *apparent* X-coordinates are not equal. To correct for this perspective effect, the X and Y values obtained thus far should be *multiplied* by the factor

$$W = \frac{\Delta}{\Delta + Z/2095}$$

where Δ is Jupiter's distance to the Earth in astronomical units as given by (44.2), while Z is in Jupiter radii (44.3). The constant 2095 is the number of equatorial radii of Jupiter in one astronomical unit.

Example 44.b — Same instant as in Example 44.a.

We shall not give the details of the calculation. Let us just mention the values of the sums

$\Sigma 1 = -0°.00654, \quad \Sigma 2 = +1°.10011, \quad \Sigma 3 = +0°.04056, \quad \Sigma 4 = +0°.59104,$

and the final results:

	Satellite I	*Satellite II*	*Satellite III*	*Satellite IV*
X	−3.4502	+7.4418	+1.2011	+7.0720
Y	+0.2137	+0.2753	+0.5900	+1.0291

Mutual conjunctions — Two satellites are in conjunction when their X-coordinates are equal. The difference between the Y-coordinates then corresponds to the separation of the satellites. Of course, if one satellite (or both) is eclipsed or occulted by Jupiter, the conjunction is inobservable.

Conjunctions with Jupiter — A satellite is in inferior conjunction with Jupiter when its X-coordinate is zero and changing from negative to positive; its Z-coordinate is then negative. Similarly, a satellite is in superior conjunction with Jupiter when its X-coordinate, passing from positive to negative, becomes zero. Its Z-coordinate is then positive.

Exercise. — On 1988 November 23, satellites III and IV were almost simultaneously in conjunction with Jupiter. Confirm this with your program. Take the value of ΔT from Table 10.A.

Answer: Satellite III was in inferior conjunction with Jupiter on 1988 November 23, at $7^h 28^m$ UT; at that instant, its Y-value was −0.8043, so the satellite was in transit over the planet's disk.

Satellite IV was in superior conjunction that same day at $5^h 15^m$. Its Y-value was then +1.3991. Since this is larger than the polar radius of Jupiter (0.933), the satellite was not occulted, but was visible above the planet's northern polar regions.

Satellite phenomena — The X and Y coordinates are the basic data for the calculation of the satellite phenomena: occultations behind Jupiter, and transits across the planet's disk. If the calculations are made for the center of the satellite, then an occultation or a transit begins or ends when the distance d of the satellite to the center of Jupiter's disk, given by $d^2 = X^2 + Y^2$, is equal to the planet's radius ρ at the point of contact. Due to Jupiter's flattening, ρ varies between 1 (at the equator) and 0.933 (at the poles). One can avoid working with an elliptical disk by "stretching" the scale vertically: multiply the Y-values by the factor 1.071374, leaving the X-values unchanged:

$$Y_1 = 1.071374\ Y$$

Jupiter's disk then becomes exactly circular, and the condition for the beginning or end of an occultation or of a transit becomes $X^2 + Y_1^2 = 1$.

In the case of an occultation, it remains to be checked whether the satellite is visible at the time of its immersion or emersion, because it could be eclipsed in the shadow of the planet.

Eclipses and shadow transits can be calculated in the same way, except that one should replace X and Y by the apparent coordinates X_0 and Y_0 as seen from the *Sun*. These coordinates are obtained by putting $R = 0$ in expressions (44.1). Moreover, the light-time τ to the *Earth* should be *added* to the true times of the eclipses or to those of the shadow transits, because we on Earth see these events later by the amount τ. Finally, in the case of an eclipse it remains to be checked whether the disappearance or the reappearance is visible from Earth: indeed, the satellite could be occulted by Jupiter at that instant.

REFERENCE

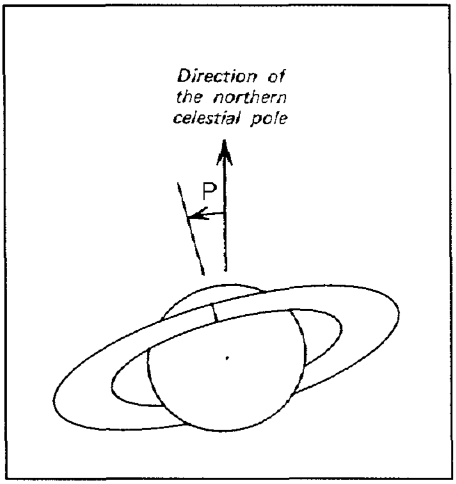
1. J. H. Lieske, *Astronomy and Astrophysics, Supplement Series*, Vol. 129, pages 205-217 (1998).

Chapter 45

The Ring of Saturn

In this Chapter, the following symbols will be used with respect to the ring of Saturn. (Of course, we know that Saturn has *many* rings. But they form one single, compact, planar system. We shall use the word *ring*, in the singular form, to denote the ring system.)

B = the Saturnicentric latitude of the Earth referred to the plane of the ring, positive towards the north; when B is positive, the visible surface of the ring is the northern one;



B' = the Saturnicentric latitude of the Sun referred to the plane of the ring, positive towards the north; when B' is positive, the illuminated surface of the ring is the northern one;

P = the geocentric position angle of the northern semiminor axis of the apparent ellipse of the ring, measured from the North towards the East (see the Figure). Because the ring is situated exactly in Saturn's equator plane, P is also the position angle of the north pole of rotation of the planet;

a, b = the major and the minor axes of the outer edge of the outer ring, in arcseconds.

In the calculation of these quantities, the effect of light-time should be taken into account. Moreover, to obtain full accuracy, the aberration of the Sun as seen from *Saturn* must be taken into account in the calculation of B' ; and in the calculation of P one should take into account the effect of the nutation and Saturn's aberration.

G. Dourneau [1] gives the following values for the inclination of the plane of the ring and the longitude of the ascending node referred to the ecliptic and mean equinox of B1950.0:

$$\begin{aligned} i &= 28^\circ\!.0817 \pm 0^\circ\!.0035 \\ \Omega &= 168^\circ\!.8112 \pm 0^\circ\!.0089 \end{aligned}$$

From these values, we deduce the following expressions to calculate i and Ω referred to the ecliptic and mean equinox of the date:

$$\begin{aligned} i &= 28^\circ\!.075\,216 - 0^\circ\!.012\,998\,T + 0^\circ\!.000\,004\,T^2 \\ \Omega &= 169^\circ\!.508\,470 + 1^\circ\!.394\,681\,T + 0^\circ\!.000\,412\,T^2 \end{aligned} \tag{45.1}$$

where T is the time from J2000.0 in Julian centuries, as given by formula (22.1). In expressions (45.1), we retained extra decimals in order to avoid loss in accuracy.

For a given instant t, the value of B, B', etc., can be calculated as follows.

1. Calculate i and Ω by means of (45.1).

2. Calculate the heliocentric longitude l_0, latitude b_0, and radius vector R of the Earth, referred to the ecliptic and mean equinox of the date, FK5 system, for instance by using the relevant data of Appendix III and the precepts given in Chapter 32.

3. Calculate the corresponding coordinates l, b, r for Saturn, but for the instant $t - \tau$, where τ is the light-time from Saturn to the Earth, as given by (33.3). Because Saturn's distance Δ is not known in advance, it should be found by iteration — see Step 4. One may use $\Delta = 9$ as a starting value, since Saturn's distance to the Earth is always between 8.0 and 11.1 astronomical units.

4. Calculate

$$\begin{aligned} x &= r \cos b \cos l - R \cos l_0 \\ y &= r \cos b \sin l - R \sin l_0 \\ z &= r \sin b \qquad\quad - R \sin b_0 \end{aligned}$$

   Then Saturn's distance Δ to the Earth is

$$\Delta = \sqrt{x^2 + y^2 + z^2} > 0$$

5. Calculate the geocentric longitude λ and latitude β of Saturn from

$$\tan \lambda = \frac{y}{x} \qquad\qquad \tan \beta = \frac{z}{\sqrt{x^2 + y^2}}$$

6. $\sin B = \sin i \cos \beta \sin(\lambda - \Omega) - \cos i \sin \beta$

$$a = \frac{375''.35}{\Delta} \qquad b = a \sin |B|$$

Factors by which the axes a and b of the outer edge of the outer ring are to be multiplied to obtain the axes of

Inner edge of outer ring :	0.8801
Outer edge of inner ring :	0.8599
Inner edge of inner ring :	0.6650
Inner edge of dusky ring :	0.5486

7. Calculate the longitude N of the ascending node of Saturn's orbit from

$$N = 113^\circ.6655 + 0^\circ.8771\,T$$

Then correct l and b for the Sun's aberration as seen from Saturn:

$$l' = l - 0^\circ.01759/r$$

$$b' = b - 0^\circ.000764\,\frac{\cos(l - N)}{r}$$

8. $\sin B' = \sin i \cos b' \sin(l' - \Omega) - \cos i \sin b'$

9. For the calculation of Saturn's magnitude (see Chapter 41), we need the quantity ΔU, the difference between the Saturnicentric longitudes of the Sun and the Earth, measured in the plane of the ring.

$$\tan U_1 = \frac{\sin i \sin b' + \cos i \cos b' \sin(l' - \Omega)}{\cos b' \cos(l' - \Omega)}$$

$$\tan U_2 = \frac{\sin i \sin \beta + \cos i \cos \beta \sin(\lambda - \Omega)}{\cos \beta \cos(\lambda - \Omega)}$$

$$\Delta U = |U_1 - U_2|, \text{ to be expressed in degrees.}$$

ΔU is a small angle, equal to at most 7°.

10. Calculate the nutations in longitude ($\Delta\psi$) and in obliquity ($\Delta\varepsilon$) and then the true obliquity of the ecliptic ε (see Chapter 22). For the nutation, only the most important terms may be used; an accuracy of, say, 0".01, is unnecessary.

11. Find the ecliptical longitude λ_0 and latitude β_0 of the northern pole of the ring plane from

$$\lambda_0 = \Omega - 90°, \qquad \beta_0 = 90° - i$$

12. Correct λ and β for the aberration of Saturn:

$$\text{correction to } \lambda : \qquad +0^\circ.005693 \, \frac{\cos(l_0 - \lambda)}{\cos\beta}$$

$$\text{correction to } \beta : \qquad +0^\circ.005693 \sin(l_0 - \lambda) \sin\beta$$

13. Add $\Delta\psi$ to λ_0 and to λ.

14. Transform (λ_0, β_0) and (λ, β) to the equatorial coordinates (α_0, δ_0) and (α, δ) by means of the formulae (13.3) and (13.4), using for ε the true obliquity obtained in Step 10.

15. The position angle P is given by

$$\tan P = \frac{\cos\delta_0 \sin(\alpha_0 - \alpha)}{\sin\delta_0 \cos\delta - \cos\delta_0 \sin\delta \cos(\alpha_0 - \alpha)}$$

Example 45.a — Calculate the quantities concerning the appearance of Saturn's ring on 1992 December 16, at 0^h UT.

This instant corresponds to JD = 2448972.5. For the difference between Dynamical Time and Universal Time, we use the value ΔT = +59 seconds = +0.00068 day, so that the instant corresponds to 1992 Dec. 16.00068 TD = JDE 2448972.50068.

Step 1.
$$T = -0.070431193$$
$$i = 28^\circ.076131$$
$$\Omega = 169^\circ.410243$$

Step 2. From an accurate ephemeris, calculated by using the complete VSOP87 theory, we deduce

$$l_0 = 84°17'08''.53 = 84^\circ.285703$$
$$b_0 = +0''.71 = +0^\circ.000197$$
$$R = 0.98412316$$

Step 3. Geometric heliocentric coordinates of Saturn, referred to the ecliptic and mean equinox of the date, taken from an accurate ephemeris:

TD	*l*	*b*	*r*
1992 Dec. 15.0	319°09'44".23	−1°04'26".52	9.8680846
16.0	319 11 36.61	−1 04 30.92	9.8678690
17.0	319 13 28.99	−1 04 35.31	9.8676534

Using $\Delta = 9$ as a first approximation for Saturn's distance, formula (33.3) yields $\tau = 0.05198$. Hence,

$$t - \tau = 1992 \text{ December } 16.00068 - 0.05198$$
$$= 1992 \text{ December } 15.94870 \text{ TD}.$$

For this instant we find, by interpolation of the values tabulated above,

$$l = 319^\circ\!.191900, \quad b = -1^\circ\!.075192, \quad r = 9.8678801.$$

Step 4. $x = +7.3697225 \qquad \Delta = 10.4646006$
$y = -7.4270295$
$z = -0.1851696$

Step 3. With this value for Δ, we obtain the new value $\tau = 0.06044$ day for the light-time; hence,

$$t - \tau = 1992 \text{ December } 16.00068 - 0.06044$$
$$= 1992 \text{ December } 15.94024 \text{ TD}$$

For this instant we find, by interpolation of the tabulated values,

$$l = 319^\circ\!.191636, \quad b = -1^\circ\!.075183, \quad r = 9.8678819.$$

Step 4. $x = +7.3696942 \qquad \Delta = 10.4646059$
$y = -7.4270651$
$z = -0.1851681$

This new value of Δ gives $\tau = 0.06044$ again, so no new iteration is needed.

Step 5. $\lambda = 314^\circ\!.777850$
$\beta = -1^\circ\!.013885$

Step 6. $B = +16^\circ\!.442$
$a = 35''\!.87$
$b = 10''\!.15$

Step 7. $N = 113^\circ\!.6037$
$l' = 319^\circ\!.189853$
$b' = -1^\circ\!.075113$

Step 8. $B' = +14^\circ\!.679$

Step 9. $U_1 = 153^\circ\!.2645$
$U_2 = 149^\circ\!.0663$
$\Delta U = 4^\circ\!.198$

Step 10. $\Delta\psi = +16''\!.86$
$\Delta\varepsilon = -1''\!.79$
$\varepsilon = 23^\circ 26' 22''\!.96 = 23^\circ\!.43971$

Step 11. $\lambda_0 = 79^\circ\!.410243$
$\beta_0 = 61^\circ\!.923869$

Step 12.	corrected $\lambda = 314°.774228$ corrected $\beta = -1°.013963$
Step 13.	corrected $\lambda_0 = 79°.414926$ corrected $\lambda = 314°.778911$
Step 14.	$\alpha_0 = 40°.36365$ $\quad \alpha = 317°.55421$ $\delta_0 = +83°.48486$ $\quad \delta = -17°.37056$
Step 15.	$P = +6°.741$

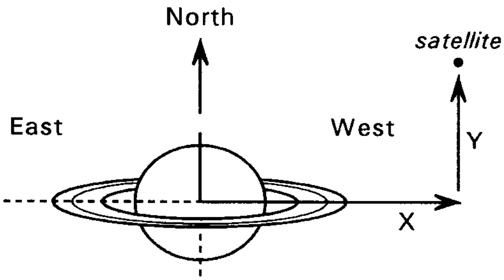
REFERENCE

1. Gérard Dourneau, *Observations et étude du mouvement des huit premiers satellites de Saturne*, Thèse de doctorat d'État, Université de Bordeaux I (1987).

Chapter 46

Positions of the Satellites of Saturn

In this Chapter a method is given to calculate, for any given instant, the positions of the eight major satellites of Saturn with respect to the planet as seen from the Earth. These apparent rectangular coordinates X and Y of the satellites will be measured from the center of the disk of Saturn, in units of the planet's equatorial radius. X will be measured positively to the west of Saturn, negatively to the east, the X-axis coinciding with the equator of the planet, and hence with the major axis of the ring. Y will be measured positively to the north, negatively to the south, the Y-axis coinciding with the planet's rotation axis — see the drawing.



The calculation method is based on the theory of the satellites due to Dourneau [1].

For the given instant, calculate the following quantities (see Chapter 25):

$\odot$ = geocentric geometric longitude of the Sun,
β = geocentric geometric latitude of the Sun,
R = radius vector of the Sun (Earth), in astronomical units.

Let τ be the light-time from Saturn to the Earth. Because the distance Δ of Saturn to the Earth is not known in advance, so is τ not known. The distance Δ should be found by iteration. A good starting value is $\Delta = 9$. The light-time is given by formula (33.3); a better value for Δ will be provided by (46.2).

TABLE 46.A

The eight major satellites of Saturn

Satellite		Year of discovery	Discoverer	Synodic period of revolution in days	Visual magnitude at mean opposition	Diameter in kilometers
I	Mimas	1789	W. Herschel	0.9425	12.9	400
II	Enceladus	1789	W. Herschel	1.3704	11.7	498
III	Tethys	1684	J. D. Cassini	1.8881	10.2	1046
IV	Dione	1684	J. D. Cassini	2.7376	10.4	1120
V	Rhea	1672	J. D. Cassini	4.5194	9.7	1528
VI	Titan	1655	Ch. Huygens	15.9691	8.3	5150
VII	Hyperion	1848	W. C. Bond	21.3188	14.2	286
VIII	Iapetus	1671	J. D. Cassini	79.9202	10.2 - 11.9	1460

Calculate the following values for the given time decreased by the light-time τ (see Chapter 32):

$$l = \text{heliocentric longitude of Saturn,}$$
$$b = \text{heliocentric latitude of Saturn,}$$
$$r = \text{radius vector of Saturn, in AU.}$$

In the above, all longitudes and latitudes are referred to the ecliptic and mean equinox of the date.

Calculate the rectangular geocentric ecliptical coordinates of Saturn

$$\begin{aligned} x &= r \cos b \cos l + R \cos \odot \\ y &= r \cos b \sin l + R \sin \odot \\ z &= r \sin b \qquad + R \sin \beta \end{aligned} \tag{46.1}$$

and its distance to the Earth

$$\Delta = \sqrt{x^2 + y^2 + z^2} \tag{46.2}$$

Calculate Saturn's geocentric longitude λ_0 and latitude β_0 by

$$\lambda_0 = \text{ATN2}\,(y, x) \qquad \text{and} \qquad \beta_0 = \text{ATN}\left(\frac{z}{\sqrt{x^2 + y^2}}\right)$$

where, as mentioned earlier in this book, ATN2 is the "second" arctangent function. In other words, λ_0 is equal to ATN (y/x) taken in the proper quadrant.

Because Dourneau constructed his theory of the satellites of Saturn in the reference frame of B1950.0, the quantities λ_0 and β_0 should be converted to that equinox. If $(\text{JD})_0$ is the Julian (Ephemeris) Day corresponding to the given instant for which the calculation is performed, and (JD) = 2433282.4235 corresponding to the epoch B1950.0, calculate T and t as explained in Chapter 21, then use the expressions (21.5) and (21.7) to convert λ_0 and β_0 to B1950.0. (We will still call λ_0 and β_0 the coordinates so converted).

Let (JDE) be the Julian Ephemeris Day corresponding to the given time decreased by the light-time τ. Then calculate the following "times" which will be needed in the calculation.

$$t_1 = (\mathrm{JDE}) - 2411093.0 \qquad t_7 = t_6/36525$$

$$t_2 = t_1/365.25 \qquad t_8 = t_6/365.25$$

$$t_3 = \frac{(\mathrm{JDE}) - 2433282.423}{365.25} + 1950.0 \qquad t_9 = \frac{(\mathrm{JDE}) - 2442000.5}{365.25}$$

$$t_4 = (\mathrm{JDE}) - 2411368.0 \qquad t_{10} = (\mathrm{JDE}) - 2409786.0$$

$$t_5 = t_4/365.25 \qquad t_{11} = t_{10}/36525$$

$$t_6 = (\mathrm{JDE}) - 2415020.0$$

We also need the following angles (in *degrees*):

$$\begin{aligned}
W0 &= 5.095\,(t_3 - 1866.39)\\
W1 &= 74.4 + 32.39\,t_2\\
W2 &= 134.3 + 92.62\,t_2\\
W3 &= 42.0 - 0.5118\,t_5\\
W4 &= 276.59 + 0.5118\,t_5\\
W5 &= 267.2635 + 1222.1136\,t_7\\
W6 &= 175.4762 + 1221.5515\,t_7\\
W7 &= 2.4891 + 0.002435\,t_7\\
W8 &= 113.35 - 0.2597\,t_7
\end{aligned}$$

and the quantities

$$s1 = \sin 28^\circ\!.0817 \qquad s2 = \sin 168^\circ\!.8112$$

$$c1 = \cos 28^\circ\!.0817 \qquad c2 = \cos 168^\circ\!.8112$$

$$e_1 = 0.05589 - 0.000346\,t_7$$

Then calculate the following quantities for the satellites. Of course, drop the data for the satellites you don't need.

Mimas (satellite I)

$$L = 127^\circ.64 + 381^\circ.994497\,t_1 - 43^\circ.57 \sin WO - 0^\circ.720 \sin 3\,WO - 0^\circ.02144 \sin 5\,WO$$

The last three terms represent a perturbation in longitude due to resonance with Tethys.

$$p = 106^\circ.1 + 365^\circ.549\,t_2$$

$$M = L - p$$

Equation of the center, in degrees:

$$C = 2^\circ.18287 \sin M + 0^\circ.025988 \sin 2M + 0^\circ.00043 \sin 3M$$

$$\lambda(1) = L + C$$

$$r(1) = \frac{3.06879}{1 + 0.01905 \cos (M + C)}$$

$$\gamma(1) = 1^\circ.563$$

$$\Omega(1) = 54^\circ.5 - 365^\circ.072\,t_2$$

WARNING

Great care should be taken with angles in degrees and radians, for Mimas as well as for all other satellites. Look at the degree symbols or at other notices.

For instance, in the expressions above, L and p are in degrees, and hence so is the mean anomaly M. However, in the formula for C the angle M must be in radians for most programming languages. The resulting C will be expressed in degrees. But in the denominator of the formula for $r(1)$ the angles M and C must again be in radians. Here we see what nuisance there is with programming languages where trigonometric functions work only in radians.

Enceladus (satellite II)

$$L = 200^\circ.317 + 262^\circ.7319002\,t_1 + 0^\circ.25667 \sin W1 + 0^\circ.20883 \sin W2$$

The last two terms represent a perturbation in longitude due to resonance with Dione.

$$p = 309^\circ.107 + 123^\circ.44121\,t_2$$

$$M = L - p$$

Equation of the center, in degrees: $C = 0^\circ.55577 \sin M + 0^\circ.00168 \sin 2M$

$$\lambda(2) = L + C$$

$$r(2) = \frac{3.94118}{1 + 0.00485 \cos(M + C)}$$

$$\gamma(2) = 0^\circ\!.0262$$

$$\Omega(2) = 348^\circ - 151^\circ\!.95\, t_2$$

Tethys (satellite III)

$$\lambda(3) = 285^\circ\!.306 + 190^\circ\!.697912 26\, t_1 + 2^\circ\!.063 \sin WO + 0^\circ\!.03409 \sin 3\,WO + 0^\circ\!.001015 \sin 5\,WO$$

The last three terms represent a perturbation in longitude due to resonance with Mimas.

The orbital eccentricity of Tethys is zero.

$$r(3) = 4.880998$$

$$\gamma(3) = 1^\circ\!.0976$$

$$\Omega(3) = 111^\circ\!.33 - 72^\circ\!.2441\, t_2$$

Dione (satellite IV)

$$L = 254^\circ\!.712 + 131^\circ\!.534931 93\, t_1 - 0^\circ\!.0215 \sin W1 - 0^\circ\!.01733 \sin W2$$

The last two terms represent a perturbation in longitude due to resonance with Enceladus.

$$p = 174^\circ\!.8 + 30^\circ\!.820\, t_2$$

$$M = L - p$$

Equation of the center, in degrees: $C = 0^\circ\!.24717 \sin M + 0^\circ\!.00033 \sin 2M$

$$\lambda(4) = L + C$$

$$r(4) = \frac{6.24871}{1 + 0.002157 \cos(M + C)}$$

$$\gamma(4) = 0^\circ\!.0139$$

$$\Omega(4) = 232^\circ - 30^\circ\!.27\, t_2$$

Rhea (satellite V)

$p' = 342°.7 + 10°.057\, t_2$

$a_1 = 0.000265 \sin p' + 0.01 \sin W4$
$a_2 = 0.000265 \cos p' + 0.01 \cos W4$

$e = \sqrt{a_1^2 + a_2^2} > 0$

$p = \text{ATN}\,(a_1/a_2)$ to be taken between 90° and 270° if $a_2 < 0$

$N = 345° - 10°.057\, t_2$

$\lambda' = 359°.244 + 79°.69004720\, t_1 + 0°.086754 \sin N$

$i = 28°.0362 + 0°.346898 \cos N + 0°.01930 \cos W3$

$\Omega = 168°.8034 + 0°.736936 \sin N + 0°.041 \sin W3$

$a = 8.725924$

Now, use the subroutine given in the box on the next page, after which

$\lambda(5) = \lambda, \quad \gamma(5) = \gamma, \quad \Omega(5) = w, \quad r(5) = r.$

Titan (satellite VI)

$L = 261°.1582 + 22°.57697855\, t_4 + 0°.074025 \sin W3$

$i' = 27°.45141 + 0°.295999 \cos W3$

$\Omega' = 168°.66925 + 0°.628808 \sin W3$

$a_1 = \sin W7 \sin(\Omega' - W8)$
$a_2 = \cos W7 \sin i' - \sin W7 \cos i' \cos(\Omega' - W8)$

$g_0 = 102°.8623$

$\psi = \text{ATN}\,(a_1/a_2)$ to be taken between 90° and 270° if $a_2 < 0$

$s = \sqrt{a_1^2 + a_2^2} > 0$

$g = W4 - \Omega' - \psi$

Calculate successive approximations to ϖ and g as follows:

$$\varpi = W4 + 0°.37515\,(\sin 2g - \sin 2g_0)$$
$$g = \varpi - \Omega' - \psi$$

This is repeated until ϖ and g no longer vary, but three iterations are always sufficient.

Subroutine ***(only for satellites V to VIII)***

From the orbital eccentricity e and the mean anomaly $M = \lambda' - p$, calculate the equation of the center C, in *radians*, from

$$\begin{aligned} C = {} & (2e - 0.25\,e^3 + 0.0520833333\,e^5)\sin M \\ & + (1.25\,e^2 - 0.458333333\,e^4)\sin 2M \\ & + (1.083333333\,e^3 - 0.671875\,e^5)\sin 3M \\ & + 1.072917\,e^4 \sin 4M + 1.142708\,e^5 \sin 5M \end{aligned}$$

and the radius vector from

$$r = \frac{a\,(1 - e^2)}{1 + e\cos(M + C)}$$

$$g = \Omega - 168^\circ\!.8112$$

$$a_1 = \sin i \sin g$$

$$a_2 = c1 \sin i \cos g - s1 \cos i$$

$$\sin\gamma = \sqrt{a_1^2 + a_2^2}, \quad \text{whence } \gamma$$

$$u = \text{ATN}\,(a_1/a_2) \quad \text{to be taken between } 90^\circ \text{ and } 270^\circ \text{ if } a_2 < 0$$

$$w = 168^\circ\!.8112 + u$$

$$h = c1 \sin i - s1 \cos i \cos g$$

$$\psi = \text{ATN}\left(\frac{s1 \sin g}{h}\right) \quad \text{to be taken between } 90^\circ \text{ and } 270^\circ \text{ if } h < 0$$

Then, if C, u, g, and ψ are in degrees, $\lambda = \lambda' + C + u - g - \psi$

$$e' = 0.029092 + 0.00019048\,(\cos 2g - \cos 2g_0)$$

$$q = 2\,(W5 - \varpi)$$

$$b_1 = \sin i' \sin(\Omega' - W8)$$

$$b_2 = \cos W7 \sin i' \cos(\Omega' - W8) - \sin W7 \cos i'$$

$$\theta = \text{ATN}\,(b_1/b_2) + W8$$

where the arctangent is to be taken between 90° and 270° if $b_2 < 0$

$$e = e' + 0.002778797\,e' \cos q$$

$$p = \varpi + 0^\circ\!.159215 \sin q$$

$$u = 2\,W5 - 2\theta + \psi$$

$$h = 0.9375\,e'^2 \sin q + 0.1875\,s^2 \sin 2\,(W5 - \theta)$$

$$\lambda' = L - 0\overset{\circ}{.}254744\ (e_1 \sin W6 + 0.75\ e_1^2 \sin 2W6 + h)$$
$$i = i' + 0\overset{\circ}{.}031843\ s \cos u$$
$$\Omega = \Omega' + \frac{0\overset{\circ}{.}031843\ s \sin u}{\sin i'}$$
$$a = 20.216193$$

Now use the subroutine given in the box on page 329 to obtain λ, γ, w, and r. Then $\lambda(6) = \lambda$, $\gamma(6) = \gamma$, $\Omega(6) = w$, $r(6) = r$.

Hyperion (satellite VII)

$$\eta = 92\overset{\circ}{.}39 + 0\overset{\circ}{.}5621071\, t_6$$
$$\zeta = 148\overset{\circ}{.}19 - 19\overset{\circ}{.}18\, t_8$$
$$\theta = 184\overset{\circ}{.}8 - 35\overset{\circ}{.}41\, t_9$$
$$\theta' = \theta - 7\overset{\circ}{.}5$$
$$a_S = 176^\circ + 12\overset{\circ}{.}22\, t_8$$
$$b_S = 8^\circ + 24\overset{\circ}{.}44\, t_8$$
$$c_S = b_S + 5^\circ$$
$$\varpi = 69\overset{\circ}{.}898 - 18\overset{\circ}{.}67088\, t_8$$
$$\varphi = 2(\varpi - W5)$$
$$\chi = 94\overset{\circ}{.}9 - 2\overset{\circ}{.}292\, t_8$$
$$a = 24.50601 - 0.08686 \cos\eta - 0.00166 \cos(\zeta + \eta) + 0.00175 \cos(\zeta - \eta)$$
$$\begin{aligned} e = {} & 0.103458 - 0.004099 \cos\eta - 0.000167 \cos(\zeta + \eta) \\ & + 0.000235 \cos(\zeta - \eta) + 0.02303 \cos\zeta - 0.00212 \cos 2\zeta \\ & + 0.000151 \cos 3\zeta + 0.00013 \cos\varphi \end{aligned}$$
$$\begin{aligned} p = {} & \varpi + 0\overset{\circ}{.}15648 \sin\chi - 0\overset{\circ}{.}4457 \sin\eta - 0\overset{\circ}{.}2657 \sin(\zeta + \eta) \\ & - 0\overset{\circ}{.}3573 \sin(\zeta - \eta) - 12\overset{\circ}{.}872 \sin\zeta + 1\overset{\circ}{.}668 \sin 2\zeta \\ & - 0\overset{\circ}{.}2419 \sin 3\zeta - 0\overset{\circ}{.}07 \sin\varphi \end{aligned}$$
$$\begin{aligned} \lambda' = {} & 177\overset{\circ}{.}047 + 16\overset{\circ}{.}91993829\, t_6 + 0\overset{\circ}{.}15648 \sin\chi + 9\overset{\circ}{.}142 \sin\eta \\ & + 0\overset{\circ}{.}007 \sin 2\eta - 0\overset{\circ}{.}014 \sin 3\eta + 0\overset{\circ}{.}2275 \sin(\zeta + \eta) \\ & + 0\overset{\circ}{.}2112 \sin(\zeta - \eta) - 0\overset{\circ}{.}26 \sin\zeta - 0\overset{\circ}{.}0098 \sin 2\zeta \\ & - 0\overset{\circ}{.}013 \sin a_S + 0\overset{\circ}{.}017 \sin b_S - 0\overset{\circ}{.}0303 \sin\varphi \end{aligned}$$
$$i = 27\overset{\circ}{.}3347 + 0\overset{\circ}{.}643486 \cos\chi + 0\overset{\circ}{.}315 \cos W3 + 0\overset{\circ}{.}018 \cos\theta - 0\overset{\circ}{.}018 \cos c_S$$
$$\begin{aligned} \Omega = {} & 168\overset{\circ}{.}6812 + 1\overset{\circ}{.}40136 \cos\chi + 0\overset{\circ}{.}68599 \sin W3 \\ & - 0\overset{\circ}{.}0392 \sin c_S + 0\overset{\circ}{.}0366 \sin\theta' \end{aligned}$$

Now use the subroutine given in the box on page 329 to obtain λ, γ, w, and r. Then $\lambda(7) = \lambda$, $\gamma(7) = \gamma$, $\Omega(7) = w$, $r(7) = r$.

Iapetus (satellite VIII)

$$L = 261\overset{\circ}{.}1582 + 22\overset{\circ}{.}57697855\, t_4$$
$$\varpi' = 91\overset{\circ}{.}796 + 0\overset{\circ}{.}562\, t_7$$
$$\psi = 4\overset{\circ}{.}367 - 0\overset{\circ}{.}195\, t_7$$
$$\theta = 146\overset{\circ}{.}819 - 3\overset{\circ}{.}198\, t_7$$
$$\varphi = 60\overset{\circ}{.}470 + 1\overset{\circ}{.}521\, t_7$$
$$\Phi = 205\overset{\circ}{.}055 - 2\overset{\circ}{.}091\, t_7$$
$$e' = 0.028298 + 0.001156\, t_{11}$$
$$\varpi_0 = 352\overset{\circ}{.}91 + 11\overset{\circ}{.}71\, t_{11}$$
$$\mu = 76\overset{\circ}{.}3852 + 4\overset{\circ}{.}53795125\, t_{10}$$
$$i' = 18\overset{\circ}{.}4602 - 0\overset{\circ}{.}9518\, t_{11} - 0\overset{\circ}{.}072\, t_{11}^2 + 0\overset{\circ}{.}0054\, t_{11}^3$$
$$\Omega' = 143\overset{\circ}{.}198 - 3\overset{\circ}{.}919\, t_{11} + 0\overset{\circ}{.}116\, t_{11}^2 + 0\overset{\circ}{.}008\, t_{11}^3$$
$$l = \mu - \varpi_0$$
$$g = \varpi_0 - \Omega' - \psi$$
$$g_1 = \varpi_0 - \Omega' - \varphi$$
$$l_S = W5 - \varpi'$$
$$g_S = \varpi' - \theta$$
$$l_T = L - W4$$
$$g_T = W4 - \Phi$$
$$u_1 = 2\,(l + g - l_S - g_S)$$
$$u_2 = l + g_1 - l_T - g_T$$
$$u_3 = l + 2\,(g - l_S - g_S)$$
$$u_4 = l_T + g_T - g_1$$
$$u_5 = 2\,(l_S + g_S)$$
$$a = 58.935028 + 0.004638 \cos u_1 + 0.058222 \cos u_2$$
$$\begin{aligned} e = e' &- 0.0014097 \cos (g_1 - g_T) + 0.0003733 \cos (u_5 - 2g) \\ &+ 0.0001180 \cos u_3 + 0.0002408 \cos l \\ &+ 0.0002849 \cos (l + u_2) + 0.0006190 \cos u_4 \end{aligned}$$
$$\begin{aligned} w = {} & 0\overset{\circ}{.}08077 \sin (g_1 - g_T) + 0\overset{\circ}{.}02139 \sin (u_5 - 2g) - 0\overset{\circ}{.}00676 \sin u_3 \\ &+ 0\overset{\circ}{.}01380 \sin l + 0\overset{\circ}{.}01632 \sin (l + u_2) + 0\overset{\circ}{.}03547 \sin u_4 \end{aligned}$$
$$p = \varpi_0 + w/e'$$
$$\begin{aligned} \lambda' = \mu &- 0\overset{\circ}{.}04299 \sin u_2 - 0\overset{\circ}{.}00789 \sin u_1 - 0\overset{\circ}{.}06312 \sin l_S \\ &- 0\overset{\circ}{.}00295 \sin 2l_S - 0\overset{\circ}{.}02231 \sin u_5 + 0\overset{\circ}{.}00650 \sin (u_5 + \psi) \end{aligned}$$
$$\begin{aligned} i = i' &+ 0\overset{\circ}{.}04204 \cos (u_5 + \psi) + 0\overset{\circ}{.}00235 \cos (l + g_1 + l_T + g_T + \varphi) \\ &+ 0\overset{\circ}{.}00360 \cos (u_2 + \varphi) \end{aligned}$$

$$w' = 0\overset{\circ}{.}04204 \sin(u_5 + \psi) + 0\overset{\circ}{.}00235 \sin(l + g_1 + l_T + g_T + \varphi) + 0\overset{\circ}{.}00358 \sin(u_2 + \varphi)$$

$$\Omega = \Omega' + w' / \sin i'$$

Now use the subroutine given in the box on page 329 to obtain λ, γ, w, and r. Then $\lambda(8) = \lambda$, $\gamma(8) = \gamma$, $\Omega(8) = w$, $r(8) = r$.

For each required satellite ($j = 1$ to 8), calculate

$$u = \lambda(j) - \Omega(j) \qquad w = \Omega(j) - 168\overset{\circ}{.}8112$$

$$X(j) = r(j)\,[\cos u \cos w - \sin u \cos \gamma(j) \sin w]$$
$$Y(j) = r(j)\,[\sin u \cos w \cos \gamma(j) + \cos u \sin w]$$
$$Z(j) = r(j) \sin u \sin \gamma(j)$$

Now consider a "ninth, fictitious satellite" situated at unit distance from the center of Saturn, above the planet's north pole:

$$X(9) = 0, \qquad Y(9) = 0, \qquad Z(9) = 1.$$

This fictitious satellite will be needed later.

To obtain the apparent rectangular coordinates of the satellites as they appear on the celestial sphere as defined at the beginning of this Chapter, several rotations must be performed. So, calculate the following for all nine satellites (the eight real ones and the ninth, fictitious satellite):

Rotation towards the plane of the ecliptic:

$$A_1 = X$$
$$B_1 = c1\,Y - s1\,Z$$
$$C_1 = s1\,Y + c1\,Z$$

Rotation towards the vernal equinox:

$$A_2 = c2\,\mathrm{A}_1 - s2\,\mathrm{B}_1$$
$$B_2 = s2\,\mathrm{A}_1 + c2\,\mathrm{B}_1$$
$$C_2 = C_1$$

Then calculate

$$A_3 = A_2 \sin \lambda_0 - B_2 \cos \lambda_0$$
$$B_3 = A_2 \cos \lambda_0 + B_2 \sin \lambda_0$$
$$C_3 = C_2 = C_1$$

$$\begin{aligned} A_4 &= A_3 \\ B_4 &= B_3 \cos \beta_0 + C_3 \sin \beta_0 \\ C_4 &= C_3 \cos \beta_0 - B_3 \sin \beta_0 \end{aligned}$$

If ξ, η are the values of A_4 and C_4 for the "ninth satellite", that is, $\xi = A_4(9)$, $\eta = C_4(9)$, then calculate the angle

$$D = \text{ATN2}\,(\xi, \eta)$$

where, as earlier in this book, ATN2 is the "second" arctangent function, which gives the angle D in the correct quadrant. Then calculate

$$\begin{aligned} X &= A_4 \cos D - C_4 \sin D \\ Y &= A_4 \sin D + C_4 \cos D \\ Z &= B_4 \end{aligned} \qquad (46.3)$$

X and Y are the required apparent rectangular coordinates of the satellite, as defined at the beginning of this Chapter. The quantity Z is negative if the satellite is closer to the Earth than Saturn, positive if it is more distant than Saturn.

However, to obtain full accuracy, the apparent coordinates X and Y just obtained should be corrected for two effects, just as for the satellites of Jupiter (see page 313):

1. differential light-time: the correction to be *added* to X is

$$\frac{|Z|}{K}\sqrt{1 - (X/r(j))^2}$$

where K =

20947	for satellite	I	35313	for satellite	V
23715	—	II	53800	—	VI
26382	—	III	59222	—	VII
29876	—	IV	91820	—	VIII

2. the perspective effect: the values X and Y obtained thus far should be *multiplied* by the factor

$$W = \frac{\Delta}{\Delta + Z/2475}$$

where Δ is Saturn's distance to the Earth in astronomical units as given by (46.2), while Z is in Saturn radii (46.3). The constant 2475 is the number of equatorial radii of Saturn in one astronomical unit.

Example 46.a — Configuration of the satellites of Saturn on 1999 September 18, at 0^h UT = JD 2451 439.5 = JDE 2451 439.50074.
(The value ΔT = +64 seconds is used.)

Using the complete VSOP87 theory, we find that at the given instant the coordinates of the Sun, referred to the mean equinox of the date, are

$$\odot = 174°.655\,278, \qquad \beta = +0°.000\,228, \qquad R = 1.005\,0057,$$

and that the true distance of Saturn to the Earth is Δ = 8.557 613 AU, so the light-time is 0.04942 day. Consequently, the geometric positions of Saturn and its satellites must be calculated for the instant

$$\text{JDE } 2451\,439.50074 - 0.04942 = 2451\,439.45132$$

For this instant, the heliocentric coordinates of Saturn, referred to the ecliptic and mean equinox of the date, are

$$l = 41°.912\,356, \qquad b = -2°.360\,096, \qquad r = 9.207\,193,$$

whence, by (46.1),

$$x = +5.845\,2457$$
$$y = +6.238\,7380$$
$$z = -0.379\,1464$$

and the "apparent" distance of Saturn to the Earth, by (46.2), is Δ = 8.557 599.

Then $\lambda_0 = 46°.865\,071$, $\beta_0 = -2°.539\,334$. Converted to the reference frame of B1950.0, these values become (Chapter 21) $\lambda_0 = 46°.170\,287$, $\beta_0 = -2°.544\,441$.

We shall not give the details of the calculation. Let us just mention the following values.

Rhea :	e = 0.010 2018,	λ' = 49°.7917,	i = 28°.2962,	Ω = 168°.2640
Titan :	e = 0.029 3386,	λ' = 273°.4387,	i = 27°.7333,	Ω = 168°.5439
Hyperion :	e = 0.118 7225,	λ' = 78°.2068,	i = 27°.2076,	Ω = 167°.5721
Iapetus :	e = 0.028 6422,	λ' = 97°.7552,	i = 17°.2486,	Ω = 138°.9121

Satellite i	$\lambda(i)$ (°)	$\gamma(i)$ (°)	$\Omega(i)$ (°)	$r(i)$
1	320.0015	1.5630	47.7110	3.1224
2	300.4638	0.0262	123.2135	3.9257
3	347.9653	1.0976	51.0615	4.8810
4	102.6463	0.0139	128.2982	6.2561
5	50.9947	0.3359	118.2589	8.7054
6	270.5289	0.3701	8.4467	19.9254
7	91.9269	1.0461	21.5992	24.1160
8	103.1318	15.5062	22.3756	58.9396

and the final results:

Satellite	*X*	*Y*
1	+ 3.102	−0.204
2	+ 3.823	+0.318
3	+ 4.027	−1.061
4	− 5.365	−1.148
5	− 1.122	−3.123
6	+14.568	+4.738
7	−18.001	−5.328
8	−48.759	+4.136

REFERENCE

1. Gérard Dourneau, *Observations et étude du mouvement des huit premiers satellites de Saturne*, Thèse de doctorat d'État, Université de Bordeaux I (1987).

Chapter 47

Position of the Moon

In order to calculate accurately the position of the Moon for a given instant, it is necessary to take into account *hundreds* of periodic terms in the Moon's longitude, latitude, and distance. Because this is outside the scope of this book, we shall limit ourselves to the most important periodic terms. The accuracy of the results will be approximately 10″ in the longitude of the Moon, and 4″ in its latitude. The interested reader can find a more accurate method in Chapront's *Lunar Tables and Programs* [2].

Using the algorithm described in this Chapter, one obtains the geocentric longitude λ and latitude β of the center of the Moon, referred to the mean equinox of the date, and the distance Δ in kilometers between the centers of Earth and Moon. The equatorial horizontal parallax π of the Moon can then be obtained from

$$\sin \pi = \frac{6378.14}{\Delta}$$

The periodic terms given in this Chapter are based on the Chapront ELP-2000/82 lunar theory [1]. However, for the mean arguments L', D, M, M', F the improved expressions given later by Chapront [3] have been used.

For the given instant (in Dynamical Time), calculate T by means of formula (22.1). Remember that T is expressed in centuries and thus should be taken with a sufficient number of decimals — at least nine, since during 0.000000001 century (approximately 3 seconds) the Moon moves over an arc of 1.7 arcseconds.

Then calculate the angles L', D, M, M', and F by means of the following expressions. The angles so calculated will be expressed in *degrees*. In order to avoid working with large angles, reduce them to less than 360°. In QuickBasic, this can be achieved by defining the function

```
DEF FNRED# (X#) = X# - 360# * INT(X# / 360#)
```

Moon's mean longitude, referred to the mean equinox of the date, and including the constant term of the effect of the light-time ($-0''.70$):

$$L' = 218.3164477 + 481267.88123421\,T - 0.0015786\,T^2 + T^3/538841 - T^4/65194000 \tag{47.1}$$

Mean elongation of the Moon:

$$D = 297.8501921 + 445267.1114034\,T - 0.0018819\,T^2 + T^3/545868 - T^4/113065000 \tag{47.2}$$

Sun's mean anomaly:

$$M = 357.5291092 + 35999.0502909\,T - 0.0001536\,T^2 + T^3/24490000 \tag{47.3}$$

Moon's mean anomaly:

$$M' = 134.9633964 + 477198.8675055\,T + 0.0087414\,T^2 + T^3/69699 - T^4/14712000 \tag{47.4}$$

Moon's argument of latitude (mean distance of the Moon from its ascending node):

$$F = 93.2720950 + 483202.0175233\,T - 0.0036539\,T^2 - T^3/3526000 + T^4/863310000 \tag{47.5}$$

Three further arguments (again, in degrees) are needed:

$$\begin{aligned} A_1 &= 119^\circ.75 + 131^\circ.849\,T \\ A_2 &= 53^\circ.09 + 479264^\circ.290\,T \\ A_3 &= 313^\circ.45 + 481266^\circ.484\,T \end{aligned}$$

Calculate the sums Σl and Σr of the terms given in Table 47.A, and the sum Σb of the terms given in Table 47.B. The argument of each sine (for Σl and Σb) and cosine (for Σr) is a linear combination of the four fundamental arguments D, M, M', and F. For instance, the argument on the eighth line of Table 47.A is $2D - M - M'$, and the contributions to Σl and Σr are $+57066 \sin(2D - M - M')$ and $-152138 \cos(2D - M - M')$, respectively.

However, the terms whose argument contains the angle M depend on the eccentricity of the Earth's orbit around the Sun, which presently is decreasing with time. For this reason, the amplitude of these terms is actually variable. To take this effect into account, multiply the terms whose argument contains M or $-M$ by E, and those containing $2M$ or $-2M$ by E^2, where

$$E = 1 - 0.002516\,T - 0.0000074\,T^2 \tag{47.6}$$

The *coefficient*, not the argument of the sine or cosine, should be multiplied by E. For example, the 8th term in the longitude is really $+57066\,E \sin(2D - M - M')$.

TABLE 47.A

Periodic terms for the longitude (Σl) and distance (Σr) of the Moon. The unit is 0.000 001 degree for Σl, and 0.001 kilometer for Σr.

Argument				Σl	Σr
Multiple of				*Coefficient of the sine of the argument*	*Coefficient of the cosine of the argument*
D	*M*	*M'*	*F*		
0	0	1	0	6 288 774	−20 905 355
2	0	−1	0	1 274 027	−3 699 111
2	0	0	0	658 314	−2 955 968
0	0	2	0	213 618	−569 925
0	1	0	0	−185 116	48 888
0	0	0	2	−114 332	−3 149
2	0	−2	0	58 793	246 158
2	−1	−1	0	57 066	−152 138
2	0	1	0	53 322	−170 733
2	−1	0	0	45 758	−204 586
0	1	−1	0	−40 923	−129 620
1	0	0	0	−34 720	108 743
0	1	1	0	−30 383	104 755
2	0	0	−2	15 327	10 321
0	0	1	2	−12 528	
0	0	1	−2	10 980	79 661
4	0	−1	0	10 675	−34 782
0	0	3	0	10 034	−23 210
4	0	−2	0	8 548	−21 636
2	1	−1	0	−7 888	24 208
2	1	0	0	−6 766	30 824
1	0	−1	0	−5 163	−8 379
1	1	0	0	4 987	−16 675
2	−1	1	0	4 036	−12 831
2	0	2	0	3 994	−10 445
4	0	0	0	3 861	−11 650
2	0	−3	0	3 665	14 403
0	1	−2	0	−2 689	−7 003
2	0	−1	2	−2 602	
2	−1	−2	0	2 390	10 056
1	0	1	0	−2 348	6 322
2	−2	0	0	2 236	−9 884

TABLE 47.A (cont.)

Argument: Multiple of D	M	M′	F	Σl Coefficient of the sine of the argument	Σr Coefficient of the cosine of the argument
0	1	2	0	−2 120	5 751
0	2	0	0	−2 069	
2	−2	−1	0	2 048	−4 950
2	0	1	−2	−1 773	4 130
2	0	0	2	−1 595	
4	−1	−1	0	1 215	−3 958
0	0	2	2	−1 110	
3	0	−1	0	−892	3 258
2	1	1	0	−810	2 616
4	−1	−2	0	759	−1 897
0	2	−1	0	−713	−2 117
2	2	−1	0	−700	2 354
2	1	−2	0	691	
2	−1	0	−2	596	
4	0	1	0	549	−1 423
0	0	4	0	537	−1 117
4	−1	0	0	520	−1 571
1	0	−2	0	−487	−1 739
2	1	0	−2	−399	
0	0	2	−2	−381	−4 421
1	1	1	0	351	
3	0	−2	0	−340	
4	0	−3	0	330	
2	−1	2	0	327	
0	2	1	0	−323	1 165
1	1	−1	0	299	
2	0	3	0	294	
2	0	−1	−2		8 752

TABLE 47.B

Periodic terms for the latitude of the Moon (Σb).
The unit is 0.000 001 degree.

Argument: Multiple of D	M	M'	F	Σb: Coefficient of the sine of the argument	Argument: Multiple of D	M	M'	F	Σb: Coefficient of the sine of the argument
0	0	0	1	5 128 122	0	0	1	−3	777
0	0	1	1	280 602	4	0	−2	1	671
0	0	1	−1	277 693	2	0	0	−3	607
2	0	0	−1	173 237	2	0	2	−1	596
2	0	−1	1	55 413	2	−1	1	−1	491
2	0	−1	−1	46 271	2	0	−2	1	−451
2	0	0	1	32 573	0	0	3	−1	439
0	0	2	1	17 198	2	0	2	1	422
2	0	1	−1	9 266	2	0	−3	−1	421
0	0	2	−1	8 822	2	1	−1	1	−366
2	−1	0	−1	8 216	2	1	0	1	−351
2	0	−2	−1	4 324	4	0	0	1	331
2	0	1	1	4 200	2	−1	1	1	315
2	1	0	−1	−3 359	2	−2	0	−1	302
2	−1	−1	1	2 463	0	0	1	3	−283
2	−1	0	1	2 211	2	1	1	−1	−229
2	−1	−1	−1	2 065	1	1	0	−1	223
0	1	−1	−1	−1 870	1	1	0	1	223
4	0	−1	−1	1 828	0	1	−2	−1	−220
0	1	0	1	−1 794	2	1	−1	−1	−220
0	0	0	3	−1 749	1	0	1	1	−185
0	1	−1	1	−1 565	2	−1	−2	−1	181
1	0	0	1	−1 491	0	1	2	1	−177
0	1	1	1	−1 475	4	0	−2	−1	176
0	1	1	−1	−1 410	4	−1	−1	−1	166
0	1	0	−1	−1 344	1	0	1	−1	−164
1	0	0	−1	−1 335	4	0	1	−1	132
0	0	3	1	1 107	1	0	−1	−1	−119
4	0	0	−1	1 021	4	−1	0	−1	115
4	0	−1	1	833	2	−2	0	1	107

Moreover, add the following additive terms to Σl and to Σb. The terms involving A_1 are due to the action of Venus, the term involving A_2 is due to Jupiter, while those involving L' are due to the flattening of the Earth.

Additive to Σl:

$$\begin{aligned}
&+3958 \sin A_1 \\
&+1962 \sin (L' - F) \\
&+\ \ 318 \sin A_2
\end{aligned}$$

Additive to Σb:

$$\begin{aligned}
&-2235 \sin L' \\
&+\ \ 382 \sin A_3 \\
&+\ \ 175 \sin (A_1 - F) \\
&+\ \ 175 \sin (A_1 + F) \\
&+\ \ 127 \sin (L' - M') \\
&-\ \ 115 \sin (L' + M')
\end{aligned}$$

The coordinates of the Moon are then given by

$$\lambda = L' + \frac{\Sigma l}{1\,000\,000} \qquad \text{(in degrees)}$$

$$\beta = \frac{\Sigma b}{1\,000\,000} \qquad \text{(in degrees)}$$

$$\Delta = 385\,000.56 + \frac{\Sigma r}{1000} \qquad \text{(in kilometers)}$$

Division of the sums by 10^6 or by 10^3 is needed because in Tables 47.A and 47.B the coefficients are given in units of 10^{-6} degree or of 10^{-3} kilometer.

Example 47.a — Calculate the geocentric longitude, latitude, distance, and equatorial horizontal parallax of the Moon for 1992 April 12, at 0^h TD.

We find successively:

JDE	= 2448724.5	A_1	= 109°.57
T	= −0.077221081451	A_2	= 123°.78
L'	= 134°.290182	A_3	= 229°.53
D	= 113°.842304	E	= 1.000194
M	= 97°.643514	Σl	= −1127527 } with the ad-
M'	= 5°.150833	Σb	= −3229126 } ditive terms
F	= 219°.889721	Σr	= −16590875

From which we deduce

$$\lambda = 134^\circ\!.290\,182 - 1^\circ\!.127\,527 = 133^\circ\!.162\,655$$
$$\beta = -3^\circ\!.229\,126 = -3^\circ 13'45''$$
$$\Delta = 385\,000.56 - 16\,590.875 = 368\,409.7 \text{ km}$$
$$\pi = \text{arcsine}\,(6378.14/368\,409.7) = 0^\circ\!.991\,990 = 0^\circ 59'31''\!.2$$

The apparent longitude of the Moon is obtained by adding to λ the nutation in longitude ($\Delta\psi$), which is equal to $+16''\!.595 = +0^\circ\!.004\,610$ (see Chapter 22). Consequently,

$$\begin{aligned} \text{apparent } \lambda &= 133^\circ\!.162\,655 + 0^\circ\!.004\,610 \\ &= 133^\circ\!.167\,265 \\ &= 133^\circ 10'02'' \end{aligned}$$

For the given instant, the true obliquity of the ecliptic is (Chapter 22)

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon = 23^\circ 26'26''\!.29 = 23^\circ\!.440\,636$$

The Moon's apparent right ascension and declination are then found by means of expressions (13.3) and (13.4):

$$\alpha = 134^\circ\!.688\,470 = 8^h 58^m 45^s\!.2$$
$$\delta = +13^\circ\!.768\,368 = +13^\circ 46'06''$$

The exact values, obtained by using the complete ELP-2000/82 theory, are

$$\begin{aligned} \lambda &= 133^\circ 10'00'' & \alpha &= 8^h 58^m 45^s\!.1 \\ \beta &= -3^\circ 13'45'' & \delta &= +13^\circ 46'06'' \\ \Delta &= 368\,405.6 \text{ km} & \pi &= 0^\circ 59'31''\!.2 \end{aligned}$$

Lunar node and lunar perigee

According to Chapront [3], the longitude of the (mean) ascending node Ω and that of the (mean) perigee Π of the lunar orbit, in degrees, are given by

$$\begin{aligned} \Omega = {} & 125.044\,5479 - 1934.136\,2891\,T + 0.002\,0754\,T^2 \\ & + T^3/467\,441 - T^4/60\,616\,000 \end{aligned} \tag{47.7}$$

$$\begin{aligned} \Pi = {} & 83^\circ\!.353\,2465 + 4069.013\,7287\,T - 0.010\,3200\,T^2 \\ & - T^3/80\,053 + T^4/18\,999\,000 \end{aligned}$$

where T has the same meaning as before. These longitudes are tropical, that is, they are measured from the mean equinox of the date.

From the formula for Ω we can find the times when the (mean) ascending or descending node of the lunar orbit coincides with the vernal equinox, that is, when Ω is equal to 0° or to 180°, respectively. During the period 1910–2110, this occurs at the following dates:

$\Omega = 0°$	$\Omega = 180°$
1913 May 27	1922 Sep. 16
1932 Jan. 6	1941 Apr. 27
1950 Aug. 17	1959 Dec. 7
1969 Mar. 29	1978 July 19
1987 Nov. 8	1997 Feb. 27
2006 June 19	2015 Oct. 10
2025 Jan. 29	2034 May 21
2043 Sep. 10	2052 Dec. 30
2062 Apr. 22	2071 Aug. 12
2080 Dec. 1	2090 Mar. 23
2099 July 13	2108 Nov. 3

The longitude of the *true* ascending node (the node of the instantaneous lunar orbit) can be deduced from Ω by the addition of periodic terms, which are given in the *Tables* of Chapront [2]. The principal of these terms are

$$\begin{aligned}
&- 1^\circ\!.4979 \sin 2(D - F) \\
&- 0^\circ\!.1500 \sin M \\
&- 0^\circ\!.1226 \sin 2D \\
&+ 0^\circ\!.1176 \sin 2F \\
&- 0^\circ\!.0801 \sin 2(M' - F)
\end{aligned}$$

See also Chapter 1 of my *Mathematical Astronomy Morsels* [4].

REFERENCES

1. M. Chapront-Touzé and J. Chapront, "The lunar ephemeris ELP 2000", *Astronomy and Astrophysics*, Vol. 124, pages 50–62 (1983). — This article gives a description of that lunar theory and discusses its accuracy. It does not give, however, the list of the many periodic terms. "ELP" means *Éphémérides Lunaires Parisiennes*, although this work is not an ephemeris (a list of calculated positions) but rather an analytical theory (a series of periodic terms).
2. M. Chapront-Touzé and J. Chapront, *Lunar Tables and Programs from 4000 B.C. to A.D. 8000*, Willmann-Bell, 1991.
3. J. Chapront, M. Chapront-Touzé, and G. Francou, *Introduction dans ELP 2000-82B de nouvelles valeurs des paramètres orbitaux de la Lune et du barycentre Terre-Lune*, Paris, January 1998.
4. J. Meeus, *Mathematical Astronomy Morsels*, Willmann-Bell, 1997.

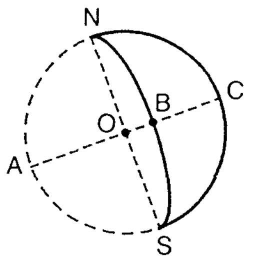
Chapter 48

Illuminated Fraction of the Moon's Disk

The illuminated fraction k of the disk of the Moon depends on the selenocentric elongation of the Earth from the Sun, called the *phase angle* (i). *Selenocentric* means "as seen from the center of the Moon". The formula is

$$k = \frac{1 + \cos i}{2} \tag{48.1}$$

and this is the value of both the ratio of the illuminated *area* of the disk to the total area, and the ratio of the illuminated *length* of the diameter perpendicular to the line of cusps to the complete diameter (see the Figure).



NCS = illuminated limb
N = northern cusp
S = southern cusp
C = midpoint of the illuminated limb
NOS = line of cusps
NBS = terminator (an ellipse)

Illuminated fraction k
= ratio of lengths BC:AC
= ratio of the areas NBSC:NASC

The phase angle i of the Moon, for a geocentric observer, can be found as follows. First, find the geocentric elongation ψ of the Moon from the Sun by means of one of the relations

$$\cos \psi = \sin \delta_0 \sin \delta + \cos \delta_0 \cos \delta \cos(\alpha_0 - \alpha)$$

$$\cos \psi = \cos \beta \cos(\lambda - \lambda_0) \tag{48.2}$$

where α_0, δ_0, λ_0 and α, δ, λ are the geocentric right ascensions, declinations, and longitudes of the Sun and the Moon, respectively, and β is the geocentric latitude of the Moon. Then we have

$$\tan i = \frac{R \sin \psi}{\Delta - R \cos \psi} \tag{48.3}$$

where R is the distance Earth-Sun, and Δ the distance Earth-Moon, both in the same units, for instance in kilometers. The angles ψ and i are always between 0 and 180 degrees. Once i is known, the illuminated fraction k can be obtained by means of formula (48.1).

Of course, for the calculation of k it is not needed to calculate the geocentric positions of the Moon and the Sun with high precision. An accuracy of, say, $1'$ will be sufficient.

If no high accuracy is required, it will suffice to put $\cos i = -\cos \psi$. The resulting error in k will never exceed 0.0014.

Lower accuracy, though still a good result, is obtained by neglecting the Moon's latitude and by calculating an approximate value of i as follows:

$$\begin{aligned} i = 180^\circ - D & - 6.^\circ289 \sin M' \\ & + 2.^\circ100 \sin M \\ & - 1.^\circ274 \sin (2D - M') \\ & - 0.^\circ658 \sin 2D \\ & - 0.^\circ214 \sin 2M' \\ & - 0.^\circ110 \sin D \end{aligned} \tag{48.4}$$

where the angles D, M, and M' can be found by means of formulae (47.2) to 47.4). In this case, the geocentric positions of the Sun and the Moon are not needed.

Position Angle of the Moon's bright limb

The position angle of the Moon's bright limb is the position angle χ of the *midpoint* of the illuminated limb of the Moon (C in the Figure on page 345), reckoned eastward from the North Point of the disk (*not* from the axis of rotation of the lunar globe). It can be obtained from

$$\tan \chi = \frac{\cos \delta_0 \sin (\alpha_0 - \alpha)}{\sin \delta_0 \cos \delta - \cos \delta_0 \sin \delta \cos (\alpha_0 - \alpha)} \tag{48.5}$$

where α_0, δ_0, α, and δ have the same meaning as before.

The angle χ is in the vicinity of 270° near First Quarter, near 90° after Full Moon. It can be found in the correct quadrant by applying the ATN2 function to the numerator and the denominator of the fraction in formula (48.5) — see "the correct quadrant" in Chapter 1.

If χ is the position angle of the (midpoint of the) bright limb, then the position angle of the cusps are $\chi - 90°$ and $\chi + 90°$. The angle χ has the advantage that it unambiguously defines the illuminated limb of the Moon.

Note that the angle χ is *not* measured from the direction of the observer's zenith. The *zenith* angle of the bright limb is $\chi - q$, where q is the parallactic angle (see Chapter 14).

Finally, note that formula (48.5) is valid in the case of a *planet* too.

Example 48.a — The Moon on 1992 April 12, at 0^h Dynamical Time.

From Example 47.a we have, for that instant,

$$\begin{aligned} \alpha &= 134.^\circ 6885 \\ \delta &= +13.^\circ 7684 \\ \Delta &= 368\,410 \text{ km} \end{aligned}$$

The apparent position and the distance of the Sun at the same instant are

$$\begin{aligned} \alpha_0 &= 1^h 22^m 37.^s 9 &&= 20.^\circ 6579 \\ \delta_0 &= +8°41'47'' &&= +8.^\circ 6964 \\ R &= 1.002\,4977 \text{ AU} &&= 149\,971\,520 \text{ km} \end{aligned}$$

The first formula (48.2) then gives $\cos \psi = -0.354\,991$, whence $\psi = 110.^\circ 7929$. Then

$$\begin{aligned} \tan i &= +2.615\,404 \quad \text{by formula (48.3)} \\ i &= 69.^\circ 0756 \end{aligned}$$

and, by formula (48.1), $k = 0.6786$, which should be rounded to 0.68.

If we use the approximate relation $\cos i = -\cos \psi$, we find $k = 0.6775$, which again rounds to 0.68.

Let us now use the approximate formula (48.4). In Example 47.a, we have found for the given instant

$$\begin{aligned} D &= 113.^\circ 8423 \\ M &= 97.^\circ 6435 \\ M' &= 5.^\circ 1508 \end{aligned}$$

Then formula (48.4) gives $i = 68.^\circ 88$ whence, by (48.1), $k = 0.6802$, which again rounds to 0.68.

Finally, formula (48.5) gives

$$\tan \chi = \frac{-0.90283}{+0.24266} \qquad \text{whence } \chi = 285.^\circ 0$$

Chapter 49

Phases of the Moon

By definition, the times of New Moon, First Quarter, Full Moon, and Last Quarter are the times at which the excess of the apparent geocentric longitude of the Moon over the apparent geocentric longitude of the Sun is 0°, 90°, 180°, and 270°, respectively.

Hence, to calculate the instants of these lunar phases, it is necessary to calculate the apparent longitudes of the Moon and the Sun separately. (However, the effect of the nutation may be neglected here, since the nutation in longitude $\Delta\psi$ will not affect the *difference* between the longitudes of Moon and Sun.)

However, if no high accuracy is required, the instants of the lunar phases can be calculated by the method described in this Chapter. The expressions are based on Chapront's ELP-2000/82 theory for the Moon (with improved expressions for the arguments M, M', etc., as mentioned in Chapter 47), and on Bretagnon's and Francou's VSOP87 theory for the Sun. The resulting times will be expressed in Julian Ephemeris Days (JDE), hence in Dynamical Time.

The times of the *mean* phases of the Moon, already affected by the Sun's aberration and by the Moon's light-time, are given by

$$\begin{aligned} \text{JDE} = 2451\,550.09766 &+ 29.530\,588\,861\,k \\ &+ 0.000\,154\,37\,T^2 \\ &- 0.000\,000\,150\,T^3 \\ &+ 0.000\,000\,000\,73\,T^4 \end{aligned} \qquad (49.1)$$

where an integer value of k gives a New Moon, an integer increased

by 0.25 gives a First Quarter,
by 0.50 gives a Full Moon,
by 0.75 gives a Last Quarter.

Any other value for k will give meaningless results!

The value $k = 0$ corresponds to the New Moon of 2000 January 6. Negative values of k give lunar phases before the year 2000.

For example,

+479.00 and −2793.00 correspond to a New Moon,
+479.25 and −2792.75 correspond to a First Quarter,
+479.50 and −2792.50 correspond to a Full Moon,
+479.75 and −2792.25 correspond to a Last Quarter.

An approximate value for k is given by

$$k \approx (\text{year} - 2000) \times 12.3685 \tag{49.2}$$

where the "year" should be taken with decimals, for example 1987.25 for the end of March 1987 (because this is 0.25 year since the beginning of the year 1987). The sign $\approx$ means "is approximately equal to".

Finally, in formula (49.1) T is the time in Julian centuries since the epoch 2000; it is obtained with sufficient accuracy from

$$T = \frac{k}{1236.85} \tag{49.3}$$

and hence is negative before the epoch 2000.0.

Calculate E by means of formula (47.6), and then the following angles, which are expressed in *degrees* and may be reduced to the interval 0–360 degrees and, if necessary, to radians before going further on.

Sun's mean anomaly at time JDE:

$$\begin{aligned} M = 2.5534 &+ 29.105\,356\,70\,k \\ &- 0.000\,0014\,T^2 \\ &- 0.000\,000\,11\,T^3 \end{aligned} \tag{49.4}$$

Moon's mean anomaly:

$$\begin{aligned} M' = 201.5643 &+ 385.816\,935\,28\,k \\ &+ 0.010\,7582\,T^2 \\ &+ 0.000\,012\,38\,T^3 \\ &- 0.000\,000\,058\,T^4 \end{aligned} \tag{49.5}$$

Moon's argument of latitude:

$$\begin{aligned} F = 160.7108 &+ 390.670\,502\,84\,k \\ &- 0.001\,6118\,T^2 \\ &- 0.000\,002\,27\,T^3 \\ &+ 0.000\,000\,011\,T^4 \end{aligned} \tag{49.6}$$

Longitude of the ascending node of the lunar orbit:

$$\begin{aligned} \Omega = 124.7746 &- 1.563\,755\,88\,k \\ &+ 0.002\,0672\,T^2 \\ &+ 0.000\,002\,15\,T^3 \end{aligned} \tag{49.7}$$

Planetary arguments, again in degrees:

$$
\begin{aligned}
A_1 &= 299.77 + 0.107\,408\,k - 0.009\,173\,T^2 \\
A_2 &= 251.88 + 0.016\,321\,k \\
A_3 &= 251.83 + 26.651\,886\,k \\
A_4 &= 349.42 + 36.412\,478\,k \\
A_5 &= 84.66 + 18.206\,239\,k \\
A_6 &= 141.74 + 53.303\,771\,k \\
A_7 &= 207.14 + 2.453\,732\,k \\
A_8 &= 154.84 + 7.306\,860\,k \\
A_9 &= 34.52 + 27.261\,239\,k
\end{aligned}
\qquad
\begin{aligned}
A_{10} &= 207.19 + 0.121\,824\,k \\
A_{11} &= 291.34 + 1.844\,379\,k \\
A_{12} &= 161.72 + 24.198\,154\,k \\
A_{13} &= 239.56 + 25.513\,099\,k \\
A_{14} &= 331.55 + 3.592\,518\,k
\end{aligned}
$$

To obtain the time of the *true* (apparent) phase, add the following corrections (in days) to the JDE obtained above.

New Moon	*Full Moon*	
-0.40720	-0.40614	$\times \sin M'$
$+0.17241 \times E$	$+0.17302 \times E$	M
$+0.01608$	$+0.01614$	$2M'$
$+0.01039$	$+0.01043$	$2F$
$+0.00739 \times E$	$+0.00734 \times E$	$M' - M$
$-0.00514 \times E$	$-0.00515 \times E$	$M' + M$
$+0.00208 \times E^2$	$+0.00209 \times E^2$	$2M$
-0.00111	-0.00111	$M' - 2F$
-0.00057	-0.00057	$M' + 2F$
$+0.00056 \times E$	$+0.00056 \times E$	$2M' + M$
-0.00042	-0.00042	$3M'$
$+0.00042 \times E$	$+0.00042 \times E$	$M + 2F$
$+0.00038 \times E$	$+0.00038 \times E$	$M - 2F$
$-0.00024 \times E$	$-0.00024 \times E$	$2M' - M$
-0.00017	-0.00017	Ω
-0.00007	-0.00007	$M' + 2M$
$+0.00004$	$+0.00004$	$2M' - 2F$
$+0.00004$	$+0.00004$	$3M$
$+0.00003$	$+0.00003$	$M' + M - 2F$
$+0.00003$	$+0.00003$	$2M' + 2F$
-0.00003	-0.00003	$M' + M + 2F$
$+0.00003$	$+0.00003$	$M' - M + 2F$
-0.00002	-0.00002	$M' - M - 2F$
-0.00002	-0.00002	$3M' + M$
$+0.00002$	$+0.00002$	$4M'$

First and Last Quarters

-0.62801		$\times \sin M'$
$+0.17172$	$\times E$	M
-0.01183	$\times E$	$M' + M$
$+0.00862$		$2M'$
$+0.00804$		$2F$
$+0.00454$	$\times E$	$M' - M$
$+0.00204$	$\times E^2$	$2M$
-0.00180		$M' - 2F$
-0.00070		$M' + 2F$
-0.00040		$3M'$
-0.00034	$\times E$	$2M' - M$
$+0.00032$	$\times E$	$M + 2F$
$+0.00032$	$\times E$	$M - 2F$
-0.00028	$\times E^2$	$M' + 2M$
$+0.00027$	$\times E$	$2M' + M$
-0.00017		Ω
-0.00005		$M' - M - 2F$
$+0.00004$		$2M' + 2F$
-0.00004		$M' + M + 2F$
$+0.00004$		$M' - 2M$
$+0.00003$		$M' + M - 2F$
$+0.00003$		$3M$
$+0.00002$		$2M' - 2F$
$+0.00002$		$M' - M + 2F$
-0.00002		$3M' + M$

Calculate, for the Quarter phases only,

$$W = 0.00306 - 0.00038\,E \cos M + 0.00026 \cos M' \\ - 0.00002 \cos(M' - M) + 0.00002 \cos(M' + M) + 0.00002 \cos 2F$$

Additional corrections: for First Quarter: $+W$
for Last Quarter : $-W$

Additional corrections for all phases:

$+\ 0.000325$	$\times \sin A_1$	$+\ 0.000056$	$\times \sin A_8$
165	A_2	047	A_9
164	A_3	042	A_{10}
126	A_4	040	A_{11}
110	A_5	037	A_{12}
062	A_6	035	A_{13}
060	A_7	023	A_{14}

Example 49.a — Calculate the instant of the New Moon which took place in February 1977.

Mid-February 1977 corresponds to 1977.13, so we find by (49.2)

$$k \approx (1977.13 - 2000) \times 12.3685 = -282.87$$

whence $k = -283$, since for the New Moon phase k should be an integer. Then, by formula (49.3), $T = -0.22881$, and then formula (49.1) gives

$$\text{JDE} = 2443\,192.94102$$

With $k = -283$ and $T = -0.22881$, we further find

$$\begin{aligned}
E &= 1.000\,5753 \\
M &= -8\,234.^\circ2625 = 45.^\circ7375 \\
M' &= -108\,984.^\circ6278 = 95.^\circ3722 \\
F &= -110\,399.^\circ0416 = 120.^\circ9584 \\
\Omega &= 567.^\circ3176 = 207.^\circ3176
\end{aligned}$$

The sum of the first group of periodic terms (for New Moon) is -0.28916, that of the 14 additional corrections is -0.00068. Consequently, the time of the true New Moon was

$$\text{JDE} = 2443\,192.94102 - 0.28916 - 0.00068 = 2443\,192.65118,$$

which corresponds to 1977 February 18.15118 TD
= 1977 February 18, at $3^h37^m42^s$ TD.

The correct value, calculated by means of the ELP-2000/82 theory, is $3^h37^m40^s$ TD.

In February 1977, the difference ΔT between Dynamical Time and Universal Time was equal to 48 seconds. Hence, the New Moon of 1977 February 18 occurred at 3^h37^m Universal Time. See also Example 10.a, on page 78.

Example 49.b — Calculate the time of the first Last Quarter of A.D. 2044.

For "year" = 2044, formula (49.2) gives $k \approx +544.21$, so we shall use the value $k = +544.75$.

Then, by formula (49.1), JDE = 2467 636.88597.

Sum of the first group of periodic terms (for Last Quarter) = -0.39153.

Additional correction for Last Quarter = $-W$ = -0.00251.

Sum of additional 14 corrections = -0.00007.

Consequently, the time of the Last Quarter is

$$2467\,636.88597 - 0.39153 - 0.00251 - 0.00007 = 2467\,636.49186$$

which corresponds to 2044 January 21, at $23^h48^m17^s$ TD.

For the period 1980 to mid-2020, we compared the results of the method described in this Chapter with the accurate times obtained with the ELP-2000/82 and VSOP87 theories.

	Mean error	*Maximum error*
New Moon :	3.6 seconds	16.4 seconds
First Quarter :	3.8	15.3
Full Moon :	3.8	17.4
Last Quarter :	3.8	13.0

Mean error of all phases = 3.72 seconds

If an error of a few minutes is not important one may, of course, drop the smallest periodic terms and the fourteen additional terms.

The *mean* time interval between two consecutive New Moons is 29.530 589 days, or 29 days 12 hours 44 minutes 03 seconds. This is the length of the (mean) synodic period of the Moon. However, mainly by reason of the perturbing action of the Sun, the actual time interval between consecutive New Moons, or *lunation*, varies greatly. See Table 49.A, taken from [1].

TABLE 49.A

The shortest and the longest lunations, 1900 to 2100

From the New Moon of	*to that of*	*Duration of the lunation*
1903 June 25	1903 July 24	29 days 06 hours 35 minutes
2035 June 6	2035 July 5	29 – 06 – 39 –
2053 June 16	2053 July 15	29 – 06 – 35 –
2071 June 27	2071 July 27	29 – 06 – 36 –
1955 Dec. 14	1956 Jan. 13	29 days 19 hours 54 minutes
1973 Dec. 24	1974 Jan. 23	29 – 19 – 55 –

REFERENCE

1. J. Meeus, "Les durées extrêmes de la lunaison", *l'Astronomie* (Société Astronomique de France), Vol. 102, pages 288–289 (July-August 1988).

Chapter 50

Perigee and apogee of the Moon

In this Chapter a method is given for the calculation of approximate times when the distance between the Earth and the Moon is a minimum (perigee) or a maximum (apogee). The resulting times will be expressed in Julian Ephemeris Days (JDE), hence in the uniform time scale of Dynamical Time. Our expressions are based on Chapront's lunar theory ELP-2000/82, with improved expressions for the arguments D, M, etc., as mentioned in Chapter 47.

First, calculate the time of the *mean* perigee or apogee by the formula

$$\begin{aligned} \text{JDE} = 2451534.6698 &+ 27.55454989\,k \\ &- 0.0006691\,T^2 \\ &- 0.000001098\,T^3 \\ &+ 0.0000000052\,T^4 \end{aligned} \tag{50.1}$$

where an integer value of k gives a perigee, and an integer increased by 0.5 an apogee. *Important :* any other value for k will give meaningless results!

The value $k = 0$ corresponds to the perigee of 1999 December 22.

So, for instance,

$k = +318$ and $k = -25$ will give a perigee,
$k = +429.5$ and $k = -1209.5$ will give an apogee,
$k = +224.87$ is an incorrect value.

An approximate value of k is given by

$$k \approx (\text{year} - 1999.97) \times 13.2555 \tag{50.2}$$

where the "year" should be taken with decimals. For instance, 2041.33 represents the end of April of the year 2041.

Finally, in formula (50.1) T is the time in Julian centuries since the epoch 2000.0. It is obtained with a sufficient accuracy from

$$T = \frac{k}{1325.55} \tag{50.3}$$

Calculate the following angles; they are expressed in degrees and may be reduced to the interval 0–360 degrees and, if necessary, converted to radians before calculating further.

Moon's mean elongation at time JDE:

$$\begin{aligned} D = 171.9179 &+ 335.910\,6046\,k \\ &- \quad 0.010\,0383\,T^2 \\ &- \quad 0.000\,011\,56\,T^3 \\ &+ \quad 0.000\,000\,055\,T^4 \end{aligned}$$

Sun's mean anomaly:

$$\begin{aligned} M = 347.3477 &+ 27.157\,7721\,k \\ &- \quad 0.000\,8130\,T^2 \\ &- \quad 0.000\,0010\,T^3 \end{aligned}$$

Moon's argument of latitude:

$$\begin{aligned} F = 316.6109 &+ 364.528\,7911\,k \\ &- \quad 0.012\,5053\,T^2 \\ &- \quad 0.000\,0148\,T^3 \end{aligned}$$

To the JDE given by (50.1), add the sum of the periodic terms of Table 50.A, taking either those for perigee or for apogee, according to the case.

The Moon's equatorial horizontal parallax is obtained by making the sum of the terms given in Table 50.B.

From Tables 50.A and 50.B it appears that:

— for the periodic terms for the instant, the sine of the argument should be taken, while for the value of the corresponding parallax the cosine must be used;

— up to a given value of the coefficient, there are more periodic terms for the perigee than for the apogee;

— the successive coefficients in the same "$2D$" series (for example the terms in $2D-M$, $4D-M$, $6D-M$, etc.) have alternate signs for the perigee, while for the apogee all have the same sign;

— the coefficient of the largest periodic term (the term with argument $2D$) is much larger in the case of the perigee than for the apogee. As a consequence, the largest possible difference between the time of the *mean* and the *true* passage is 45 hours for the perigee, but only 13 hours for the apogee. Also, the Moon's perigee distance varies in a larger interval (approximately between 356 370 and 370 350 kilometers) than does the apogee distance (404 050 to 406 720 km).

Example 50.a — The Moon's apogee of October 1988.

Because the beginning of October corresponds to 0.75 year since the beginning of the calendar year, we put the value year = 1988.75 in formula (50.2). This gives $k \approx -148.73$. We therefore take the value $k = -148.5$ (apogee!).

Formulae (50.3) and (50.1) then give

$$T = -0.112029 \qquad \text{JDE} = 2447442.8191$$

Then we find

$$\begin{aligned} D &= -49710^\circ.8070 = 329^\circ.1930 \\ M &= -3685^\circ.5815 = 274^\circ.4185 \\ F &= -53815^\circ.9147 = 184^\circ.0853 \end{aligned}$$

Sum of the terms in Table 50.A (apogee) = -0.4654 day
Sum of the terms in Table 50.B (apogee) = 3240.679

Hence, the time of the apogee is

$$\text{JDE} = 2447442.8191 - 0.4654 = 2447442.3537$$

which corresponds to 1988 October 7, at 20^h29^m TD. The corresponding value of the Moon's equatorial horizontal parallax is $3240''.679$, or $0^\circ54'00''.679$.

The exact values are 20^h30^m TD and $0^\circ54'00''.671$.

TABLE 50.A

Periodic terms for the time, in days

For the perigee			
Argument of sine	*Coefficient*	*Argument of sine*	*Coefficient*
$2D$	-1.6769	$2D - 2M$	-0.0027
$4D$	$+0.4589$	$4D - 2M$	$+0.0024$
$6D$	-0.1856	$6D - 2M$	-0.0021
$8D$	$+0.0883$	$22D$	-0.0021
$2D - M$	$-0.0773 + 0.00019\ T$	$18D - M$	-0.0021
M	$+0.0502 - 0.00013\ T$	$6D + M$	$+0.0019$
$10D$	-0.0460	$11D$	-0.0018
$4D - M$	$+0.0422 - 0.00011\ T$	$8D + M$	-0.0014
$6D - M$	-0.0256	$4D - 2F$	-0.0014
$12D$	$+0.0253$	$6D + 2F$	-0.0014
D	$+0.0237$	$3D + M$	$+0.0014$
$8D - M$	$+0.0162$	$5D + M$	-0.0014
$14D$	-0.0145	$13D$	$+0.0013$
$2F$	$+0.0129$	$20D - M$	$+0.0013$
$3D$	-0.0112	$3D + 2M$	$+0.0011$
$10D - M$	-0.0104	$4D + 2F - 2M$	-0.0011
$16D$	$+0.0086$	$D + 2M$	-0.0010
$12D - M$	$+0.0069$	$22D - M$	-0.0009
$5D$	$+0.0066$	$4F$	-0.0008
$2D + 2F$	-0.0053	$6D - 2F$	$+0.0008$
$18D$	-0.0052	$2D - 2F + M$	$+0.0008$
$14D - M$	-0.0046	$2M$	$+0.0007$
$7D$	-0.0041	$2F - M$	$+0.0007$
$2D + M$	$+0.0040$	$2D + 4F$	$+0.0007$
$20D$	$+0.0032$	$2F - 2M$	-0.0006
$D + M$	-0.0032	$2D - 2F + 2M$	-0.0006
$16D - M$	$+0.0031$	$24D$	$+0.0006$
$4D + M$	-0.0029	$4D - 4F$	$+0.0005$
$9D$	$+0.0027$	$2D + 2M$	$+0.0005$
$4D + 2F$	$+0.0027$	$D - M$	-0.0004

TABLE 50.A (cont.)

For the apogee			
Argument of sine	*Coefficient*	*Argument of sine*	*Coefficient*
$2D$	$+0.4392$	$8D - M$	$+0.0011$
$4D$	$+0.0684$	$4D - 2M$	$+0.0010$
M	$+0.0456 - 0.00011\ T$	$10D$	$+0.0009$
$2D - M$	$+0.0426 - 0.00011\ T$	$3D + M$	$+0.0007$
$2F$	$+0.0212$	$2M$	$+0.0006$
D	-0.0189	$2D + M$	$+0.0005$
$6D$	$+0.0144$	$2D + 2M$	$+0.0005$
$4D - M$	$+0.0113$	$6D + 2F$	$+0.0004$
$2D + 2F$	$+0.0047$	$6D - 2M$	$+0.0004$
$D + M$	$+0.0036$	$10D - M$	$+0.0004$
$8D$	$+0.0035$	$5D$	-0.0004
$6D - M$	$+0.0034$	$4D - 2F$	-0.0004
$2D - 2F$	-0.0034	$2F + M$	$+0.0003$
$2D - 2M$	$+0.0022$	$12D$	$+0.0003$
$3D$	-0.0017	$2D + 2F - M$	$+0.0003$
$4D + 2F$	$+0.0013$	$D - M$	-0.0003

TABLE 50.B

Terms for the parallax, in arcseconds

For the perigee

3629″.215		
+63.224	× cos	$2D$
−6.990		$4D$
+2.834 $-0.0071\,T$		$2D - M$
+1.927		$6D$
−1.263		D
−0.702		$8D$
+0.696 $-0.0017\,T$		M
−0.690		$2F$
−0.629 $+0.0016\,T$		$4D - M$
−0.392		$2D - 2F$
+0.297		$10D$
+0.260		$6D - M$
+0.201		$3D$
−0.161		$2D + M$
+0.157		$D + M$
−0.138		$12D$
−0.127		$8D - M$
+0.104		$2D + 2F$
+0.104		$2D - 2M$
−0.079		$5D$
+0.068		$14D$
+0.067	× cos	$10D - M$
+0.054		$4D + M$
−0.038		$12D - M$
−0.038		$4D - 2M$
+0.037		$7D$
−0.037		$4D + 2F$
−0.035		$16D$
−0.030		$3D + M$
+0.029		$D - M$
−0.025		$6D + M$
+0.023		$2M$
+0.023		$14D - M$
−0.023		$2D + 2M$
+0.022		$6D - 2M$
−0.021		$2D - 2F - M$
−0.020		$9D$
+0.019		$18D$
+0.017		$6D + 2F$
+0.014		$2F - M$
−0.014		$16D - M$
+0.013		$4D - 2F$
+0.012		$8D + M$
+0.011		$11D$
+0.010		$5D + M$
−0.010		$20D$

For the apogee

3245″.251		
−9.147	× cos	$2D$
−0.841		D
+0.697		$2F$
−0.656 $+0.0016\,T$		M
+0.355		$4D$
+0.159		$2D - M$
+0.127		$D + M$
+0.065		$4D - M$
+0.052	× cos	$6D$
+0.043		$2D + M$
+0.031		$2D + 2F$
−0.023		$2D - 2F$
+0.022		$2D - 2M$
+0.019		$2D + 2M$
−0.016		$2M$
+0.014		$6D - M$
+0.010		$8D$

Using the method described in this Chapter, 600 perigee and 600 apogee passages of the Moon were calculated, namely from June 1977 to August 2022. The results were compared with accurate values obtained with the ELP-2000/82 theory. The largest errors are

for the time :	31 minutes for the perigee, 3 minutes for the apogee;
for the parallax :	$0''.124$ for the perigee, $0''.051$ for the apogee.

The latter errors correspond to distance errors of 12 and 6 kilometers, respectively. The distribution of the errors of the 600 calculated times is as follows:

Number of errors less than	*Perigee*	*Apogee*
1 minute	151	478
2 minutes	264	589
3 minutes	385	599
4 minutes	460	
5 minutes	492	
10 minutes	572	

The *mean* time interval between two consecutive passages of the Moon through perigee is 27.55455 days, or 27 days 13 hours 19 minutes. This is the length of the anomalistic period of the Moon. However, mainly by reason of the perturbing action of the Sun, the actual time interval between consecutive perigees varies greatly, between the extremes 24 days 16 hours and 28 days 13 hours. Examples:

$$\left.\begin{array}{l}\text{perigee on 1997 December } 9 \text{ at } 16^h.9\\ \text{perigee on 1998 January } 3 \text{ at } 8^h.5\end{array}\right\}\ \text{diff.} = 24 \text{ days } 16 \text{ hours}$$

$$\left.\begin{array}{l}\text{perigee on 1990 December } 2 \text{ at } 10^h.8\\ \text{perigee on 1990 December } 30 \text{ at } 23^h.8\end{array}\right\}\ \text{diff.} = 28 \text{ days } 13 \text{ hours}$$

The time interval between two consecutive *apogees*, however, varies between narrower limits, namely between 26.98 and 27.90 days (26 days 23½ hours and 27 days 21½ hours).

Extreme perigee and apogee distances of the Moon

Between the years 1500 and 2500, fourteen times the Moon approaches the Earth to less than 356425 kilometers, and the same number of times the distance grows to larger than 406710 km. These cases are mentioned in Table 50.C.

For the calculation, use has been made of Chapront's lunar theory ELP-2000/82, except that we neglected all periodic terms with a coefficient less than 0.0005 km (50 centimeters).

It appears that, during the time interval of ten centuries considered here, the extreme distances between the centers of Earth and Moon are

356371 km on 2257 January 1
406720 km on 2266 January 7

The smallest perigee distance of the 20th century was that of 1912 January 4, as was already found earlier by Roger W. Sinnott, Associate Editor of *Sky and Telescope* [1]. Further, we see that these extreme perigees and apogees all occur during the winter months of the northern hemisphere, the period of the year when the Earth is closest to the Sun. It is evident that the variable Earth-Sun distance somewhat affects the Earth-Moon distance.

TABLE 50.C

Extreme perigees and apogees, A.D. 1500 to 2500 (UT dates)

perigee < 356425 km		*apogee > 406710 km*	
1548 Dec. 15	356407 km	1921 Jan. 9	406710 km
1566 Dec. 26	356399	1984 Mar. 2	406712
1771 Jan. 30	356422	2107 Jan. 23	406716
1893 Dec. 23	356396	2125 Feb. 3	406720
1912 Jan. 4	356375	2143 Feb. 14	406713
1930 Jan. 15	356397	2247 Dec. 27	406715
2052 Dec. 6	356421	2266 Jan. 7	406720
2116 Jan. 29	356403	2284 Jan. 18	406714
2134 Feb. 9	356416	2388 Nov. 29	406715
2238 Dec. 22	356406	2406 Dec. 11	406718
2257 Jan. 1	356371	2424 Dec. 21	406712
2275 Jan. 12	356378	2452 Jan. 21	406710
2461 Jan. 26	356408	2470 Feb. 1	406714
2479 Feb. 7	356404	2488 Feb. 12	406711

REFERENCES

1. Roger W. Sinnott, letter of 1981 March 4 to Jean Meeus.
2. Jean Meeus, "Extreme Perigees and Apogees of the Moon", *Sky and Telescope*, Vol. 62, pages 110-111 (August 1981).

Chapter 51

Passages of the Moon through the Nodes

When the center of the Moon passes through the ascending or through the descending node of its orbit, its geocentric latitude is zero. Approximate times of the passages through the nodes can be obtained as follows. The results will be expressed as a Julian Ephemeris Day, JDE, hence in Dynamical Time.

For a passage through the *ascending* node, take k = an integer. For a passage at the *descending* node, take for k an integer increased by 0.5. *Important :* any other value for k will give meaningless results!

Successive values of k will provide successive passages of the Moon through the nodes, the value k = zero corresponding to the passage at the ascending node of 2000 January 21. Negative values of k yield passages before this date.

For instance, $k = +223.0$ and -147.0 correspond to an ascending node, $+223.5$ and -46.5 to a descending node, while $+44.76$ is not a valid value for k.

An approximate value of k is given by

$$k \approx (\text{year} - 2000.05) \times 13.4223 \tag{51.1}$$

where "year" may be taken with decimals, for instance 2013.25. Then calculate

$$T = \frac{k}{1342.23}$$

and the following angles in *degrees* :

$$D = 183.6380 + 331.737\,356\,82\,k + 0.001\,4852\,T^2 \\ +0.000\,002\,09\,T^3 - 0.000\,000\,010\,T^4$$

$$M = 17.4006 + 26.820\,372\,50\,k + 0.000\,1186\,T^2 + 0.000\,000\,06\,T^3$$

$$M' = 38.3776 + 355.527\,473\,13\,k + 0.012\,3499\,T^2 \\ +0.000\,014\,627\,T^3 - 0.000\,000\,069\,T^4$$

$$\Omega = 123.9767 - 1.440\,989\,56\,k + 0.002\,0608\,T^2 + 0.000\,002\,14\,T^3 - 0.000\,000\,016\,T^4$$

$$V = 299.75 + 132.85\,T - 0.009\,173\,T^2$$

$$P = \Omega + 272.75 - 2.3\,T$$

The time of the passage through the node is then given by the following expression, where the terms involving M (the Sun's mean anomaly) should be multiplied by the quantity E given by formula (47.6). These terms are indicated by an asterisk.

$$
\begin{array}{rl}
\text{JDE} = 2451\,565.1619 & + \; 27.212\,220\,817\,k \\
& + \; 0.000\,2762\,T^2 \\
& + \; 0.000\,000\,021\,T^3 \\
& - \; 0.000\,000\,000\,088\,T^4 \\
& - \; 0.4721 \sin M' \\
& - \; 0.1649 \sin 2D \\
& - \; 0.0868 \sin (2D - M') \\
& + \; 0.0084 \sin (2D + M') \\
* & - \; 0.0083 \sin (2D - M) \\
* & - \; 0.0039 \sin (2D - M - M') \\
& + \; 0.0034 \sin 2M' \\
& - \; 0.0031 \sin (2D - 2M') \\
* & + \; 0.0030 \sin (2D + M) \\
* & + \; 0.0028 \sin (M - M') \\
* & + \; 0.0026 \sin M \\
& + \; 0.0025 \sin 4D \\
& + \; 0.0024 \sin D \\
* & + \; 0.0022 \sin (M + M') \\
& + \; 0.0017 \sin \Omega \\
& + \; 0.0014 \sin (4D - M') \\
* & + \; 0.0005 \sin (2D + M - M') \\
* & + \; 0.0004 \sin (2D - M + M') \\
* & - \; 0.0003 \sin (2D - 2M) \\
* & + \; 0.0003 \sin (4D - M) \\
& + \; 0.0003 \sin V \\
& + \; 0.0003 \sin P
\end{array}
$$

Example 51.a — Calculate the instant of the passage of the Moon through the ascending node in May 1987.

Since mid-May corresponds to 0.37 year since the beginning of the current year, we put year = 1987.37 in formula (51.1), which yields the approximate value -170.19 for k. For a passage through the ascending node, k should be an integer, so we take $k = -170$. Then we find

$$\begin{aligned}
T &= -0.126\,655 \\
D &= -56\,211.^\circ71264 = 308.^\circ28736 \\
M &= -4\,542.^\circ06272 = 137.^\circ93728 \\
M' &= -60\,401.^\circ29263 = 78.^\circ70737 \\
\Omega &= 368.^\circ9449 = 8.^\circ9449 \\
V &= 282.^\circ92 \\
P &= 641.^\circ99 = 281.^\circ99 \\
E &= 1.000\,319
\end{aligned}$$

The final result is JDE = 2446938.76803, which corresponds to 1987 May 23.26803, or 1987 May 23, at $6^h26^m.0$ TD.

The correct value is May 23, at $6^h25^m.6$ TD.

The table below gives an idea of the accuracy of the results obtained by means of the algorithm given in this Chapter, as compared with the times found by an accurate calculation.

Years (A.D.)	*Node*	*Number of instants*	*Greatest error in seconds*	*Number of errors < 60 sec.*	*Number of errors > 120 sec.*
1980 to 2020	ascending	551	142	487	3
1980 to 2020	descending	551	132	469	2
0 to 40	ascending	551	144	444	5
0 to 40	descending	551	135	478	2

Chapter 52

Maximum Declinations of the Moon

The plane of the Moon's orbit forms with the plane of the ecliptic an angle of 5°. Therefore, in the sky the Moon is moving *approximately* along the ecliptic, and during each revolution (27 days) it reaches its greatest northern declination (in Taurus, in Gemini, or in northern Orion), and two weeks later its greatest southern declination (in Sagittarius or in Ophiuchus).

Because the lunar orbit forms with the ecliptic an angle of 5°, and the ecliptic an angle of 23° with the celestial equator, the extreme declinations of the Moon are between 18° and 28° (North or South), approximately. When, as in 1987, the ascending node of the lunar orbit is in the vicinity of the vernal equinox (see page 344), the Moon reaches high northern and southern declinations, approximately +28½ and −28½ degrees. This situation is repeated at intervals of 18.6 years, the revolution period of the lunar nodes.

In this Chapter a method is given for the calculation of approximate times of the maximum declinations of the Moon, and the values of these extreme declinations. These data are *geocentric* and they refer to the center of the Moon's disk.

Let k be an integer, negative before the beginning of the year 2000. Successive values of k will give successive maximum northern or southern declinations of the Moon. The value $k = 0$ corresponds to January 2000. *Important:* a non-integer value of k will give meaningless results!

An approximate value of k is given by

$$k \approx (\text{year} - 2000.03) \times 13.3686 \tag{52.1}$$

where "year" can be taken with decimals. Then calculate

$$T = \frac{k}{1336.86}$$

and the following angles, in *degrees*. The quantities between square brackets should be used for *southern* declinations.

TABLE 52.A

Periodic terms (days) for the time of the Moon's maximum declination

Coefficient for declination north	Coefficient for declination south		Coefficient for declination north	Coefficient for declination south	
d	d		d	d	
+0.8975	−0.8975	$\cos F$	+0.0030	+0.0030	$\sin(2D + M')$
−0.4726	−0.4726	$\sin M'$	−0.0029	+0.0029	$\cos(M' + 2F)$
−0.1030	−0.1030	$\sin 2F$	−0.0029	−0.0029	$\sin(2D - M)$ *
−0.0976	−0.0976	$\sin(2D - M')$	−0.0027	−0.0027	$\sin(M' + F)$
−0.0462	+0.0541	$\cos(M' - F)$	+0.0024	+0.0024	$\sin(M - M')$ *
−0.0461	+0.0516	$\cos(M' + F)$	−0.0021	−0.0021	$\sin(M' - 3F)$
−0.0438	−0.0438	$\sin 2D$	+0.0019	−0.0019	$\sin(2M' + F)$
+0.0162	+0.0112	$\sin M$ *	+0.0018	−0.0006	$\cos(2D - 2M' - F)$
−0.0157	+0.0157	$\cos 3F$	+0.0018	−0.0018	$\sin 3F$
+0.0145	+0.0023	$\sin(M' + 2F)$	+0.0017	−0.0017	$\cos(M' + 3F)$
+0.0136	−0.0136	$\cos(2D - F)$	+0.0017	+0.0017	$\cos 2M'$
−0.0095	+0.0110	$\cos(2D - M' - F)$	−0.0014	+0.0014	$\cos(2D - M')$
−0.0091	+0.0091	$\cos(2D - M' + F)$	+0.0013	−0.0013	$\cos(2D + M' + F)$
−0.0089	+0.0089	$\cos(2D + F)$	+0.0013	−0.0013	$\cos M'$
+0.0075	+0.0075	$\sin 2M'$	+0.0012	+0.0012	$\sin(3M' + F)$
−0.0068	−0.0030	$\sin(M' - 2F)$	+0.0011	+0.0011	$\sin(2D - M' + F)$
+0.0061	−0.0061	$\cos(2M' - F)$	−0.0011	+0.0011	$\cos(2D - 2M')$
−0.0047	−0.0047	$\sin(M' + 3F)$	+0.0010	+0.0010	$\cos(D + F)$
−0.0043	−0.0043	$\sin(2D - M - M')$ *	+0.0010	+0.0010	$\sin(M + M')$ *
−0.0040	+0.0040	$\cos(M' - 2F)$	−0.0009	−0.0009	$\sin(2D - 2F)$
−0.0037	−0.0037	$\sin(2D - 2M')$	+0.0007	−0.0007	$\cos(2M' + F)$
+0.0031	−0.0031	$\sin F$	−0.0007	−0.0007	$\cos(3M' + F)$

$$D = 152.2029 + 333.0705546\,k - 0.0004214\,T^2 + 0.00000011\,T^3$$
[345.6676]

$$M = 14.8591 + 26.9281592\,k - 0.0000355\,T^2 - 0.00000010\,T^3$$
[1.3951]

$$M' = 4.6881 + 356.9562794\,k + 0.0103066\,T^2 + 0.00001251\,T^3$$
[186.2100]

$$F = 325.8867 + 1.4467807\,k - 0.0020690\,T^2 - 0.00000215\,T^3$$
[145.1633]

TABLE 52.B

Periodic terms (degrees) for the value of the Moon's maximum declination

Coefficient for			*Coefficient for*		
declination north	*declination south*		*declination north*	*declination south*	
°	°		°	°	
+5.1093	−5.1093	$\sin F$	+0.0038	−0.0038	$\cos(2M' - F)$
+0.2658	+0.2658	$\cos 2F$	−0.0034	+0.0034	$\cos(M' - 2F)$
+0.1448	−0.1448	$\sin(2D - F)$	−0.0029	−0.0029	$\sin 2M'$
−0.0322	+0.0322	$\sin 3F$	+0.0029	+0.0029	$\sin(3M' + F)$
+0.0133	+0.0133	$\cos(2D - 2F)$	−0.0028	+0.0028	$\cos(2D + M - F)$ *
+0.0125	+0.0125	$\cos 2D$	−0.0028	−0.0028	$\cos(M' - F)$
−0.0124	−0.0015	$\sin(M' - F)$	−0.0023	+0.0023	$\cos 3F$
−0.0101	+0.0101	$\sin(M' + 2F)$	−0.0021	+0.0021	$\sin(2D + F)$
+0.0097	−0.0097	$\cos F$	+0.0019	+0.0019	$\cos(M' + 3F)$
−0.0087	+0.0087	$\sin(2D + M - F)$ *	+0.0018	+0.0018	$\cos(D + F)$
+0.0074	+0.0074	$\sin(M' + 3F)$	+0.0017	−0.0017	$\sin(2M' - F)$
+0.0067	+0.0067	$\sin(D + F)$	+0.0015	+0.0015	$\cos(3M' + F)$
+0.0063	−0.0063	$\sin(M' - 2F)$	+0.0014	+0.0014	$\cos(2D + 2M' + F)$
+0.0060	−0.0060	$\sin(2D - M - F)$ *	−0.0012	+0.0012	$\sin(2D - 2M' - F)$
−0.0057	+0.0057	$\sin(2D - M' - F)$	−0.0012	−0.0012	$\cos 2M'$
−0.0056	−0.0056	$\cos(M' + F)$	−0.0010	+0.0010	$\cos M'$
+0.0052	−0.0052	$\cos(M' + 2F)$	−0.0010	−0.0010	$\sin 2F$
+0.0041	−0.0041	$\cos(2M' + F)$	+0.0006	+0.0037	$\sin(M' + F)$
−0.0040	−0.0040	$\cos(M' - 3F)$			

The time of greatest northern or southern declination is then

$$\begin{aligned} \text{JDE} = {} & 2451\,562.5897 + 27.321\,582\,247\,k + 0.000\,119\,804\,T^2 \\ & [2451\,548.9289] \qquad\qquad\qquad\quad - 0.000\,000\,141\,T^3 \\ & \qquad\qquad\qquad\qquad + \text{periodic terms of Table 52.A} \end{aligned}$$

In Table 52.A, the terms involving M, the Sun's mean anomaly, should be multiplied by the quantity E given by formula (47.6). These terms are indicated by an asterisk.

The value of the greatest declination, in degrees, is

$$\delta = 23.6961 - 0.013\,004\,T + \text{periodic terms of Table 52.B.}$$

Here, again, the terms indicated by an asterisk should be multiplied by E. Note that the *absolute* value of the maximum declination is obtained; in the case of a greatest southern declination, this declination thus is *not* affected by the minus sign.

Example 52.a — Greatest northern declination of the Moon in December 1988.

Inserting the value year = 1988.95 in formula (52.1), we get $k \approx -148.12$, so we take $k = -148$. We then find

$T = -0.110707$ | $M' = -52\,824^\circ\!.8411 = 95^\circ\!.1589$
$D = -49\,142^\circ\!.2392 = 177^\circ\!.7608$ | $F = 111^\circ\!.7631$
$M = -3970^\circ\!.5085 = 349^\circ\!.4915$ | $E = 1.000278$

We obtain JDE = 2447518.3347, which corresponds to 1988 December 22.8347 = 1988 Dec. 22 at 20^h02^m TD. The correct value is December 22 at 20^h01^m TD.

For the value of that maximum northern declination, we obtain $28^\circ\!.1562$, or $+28°09'22''$. The correct value is $+28°09'13''$.

Example 52.b — If we calculate the maximum southern declination for $k = +659$, we get JDE = 2469553.0834, which corresponds to 2049 April 21 at 14^h Dynamical Time, and $\delta = 22^\circ\!.1384$, so the greatest southern declination is $-22°08'$.

Example 52.c — To find the Moon's greatest northern declination of mid-March of the year -4, we have "year" = 0.20 year after the beginning of the year -4, so "year" = $-4 + 0.20 = -3.80$, and *not* -4.20!

This gives for k the approximate value -26788.40, whence $k = -26788$ (an integer!).

We then obtain JDE = 1719672.1414, which corresponds to March 16 at 15^h TD of the year -4;
greatest northern declination = $28^\circ\!.9739 = +28°58'$.

Using the method described in this Chapter, 600 maximum northern and 600 maximum southern declinations were calculated, from 1977 August to 2022 June. The maximum errors were 10 minutes for the time, and $26''$ for the value of the maximum declination. For 69% of the cases, the calculated time was less than 3 minutes in error, and in 74% of the cases the calculated declination was less than $10''$ in error.

The coefficients of the periodic terms in Tables 52.A and 52.B have been calculated using for the obliquity of the ecliptic its value for the epoch 2000.0. As a consequence, the error resulting from using these terms will increase with time, but the maximum possible error will not exceed half an hour between the years -1000 and $+5000$.

Chapter 53

Ephemeris for Physical Observations of the Moon

Optical librations

The mean period of rotation of the Moon is equal to the mean sidereal period of revolution around the Earth, and the mean plane of the lunar equator intersects the ecliptic at a constant inclination, I, in the line of nodes of the lunar orbit, with the descending node of the equator at the ascending node of the orbit.

On the average, therefore, the same hemisphere of the Moon is always turned towards the Earth. However, apparent oscillations known as *optical librations*, which are due to variations in the geometric position of the Earth relative to the lunar surface during the course of the orbital motion of the Moon, allow about 59% of the surface to be observed from the Earth.

The mean center of the Moon's apparent disk is the origin of the system of selenographic coordinates on the surface of the Moon. Selenographic longitudes are measured from the lunar meridian that passes through the mean center of the apparent disk, positive in the direction towards *Mare Crisium*, that is, towards the west on the geocentric celestial sphere. Selenographic latitudes are measured from the lunar equator, positive towards the north, that is, they are positive in the hemisphere containing *Mare Serenitatis*.

The displacement, at any time, of the mean center of the disk from the apparent center, represents the amount of libration, and is measured by the selenographic coordinates of the apparent center of the disk at that time.

The selenographic longitude and latitude of the Earth, as given in the almanacs, are the geocentric selenographic coordinates of the apparent central point of the disk. At this point on the surface of the Moon, the Earth is in the zenith. When the *libration in longitude*, that is, the selenographic longitude of the Earth, is positive, the mean central point of the disk is displaced eastwards on the celestial sphere, exposing to view a region on the west limb. When the *libration in latitude*, or the selenographic latitude of the Earth, is positive, the mean central point of the disk is displaced towards the south, and a region on the north limb is exposed to view.

The optical librations in longitude (l') and in latitude (b') can be calculated as follows. Let

I = the inclination of the mean lunar equator to the ecliptic, 1°32′32″.7 or 1°.54242. This is the value adopted by the International Astronomical Union;

λ = apparent geocentric longitude of the Moon;

β = apparent geocentric latitude of the Moon;

$\Delta\psi$ = nutation in longitude (see Chapter 22);

F = argument of latitude of the Moon, obtained from (47.5);

Ω = mean longitude of the ascending node of the lunar orbit, obtained from formula (47.7).

Then we have

$$\left.\begin{aligned} W &= \lambda - \Delta\psi - \Omega \\ \tan A &= \frac{\sin W \cos\beta \cos I - \sin\beta \sin I}{\cos W \cos\beta} \\ l' &= A - F \\ \sin b' &= -\sin W \cos\beta \sin I - \sin\beta \cos I \end{aligned}\right\} \tag{53.1}$$

In the calculation of λ, the effect of the nutation is supposed to be included, so $\lambda - \Delta\psi$ represents, in fact, the "apparent longitude of the Moon without the effect of the nutation".

Physical librations

There is an actual rotational motion of the Moon about its mean rotation; this is called the physical libration. The physical libration is much smaller than the optical libration, and can never be larger than 0.04 degree in both longitude and latitude.

The physical librations in longitude (l'') and in latitude (b'') can be calculated as follows, and the total librations are the sums of the optical and physical librations:

$$l = l' + l'', \qquad b = b' + b''.$$

Calculate the quantities ρ, σ, and τ (in degrees) by means of the following expressions from D. H. Eckhardt [1], where the angles D, M, and M' are obtained by means of expressions (47.2) to (47.4); find E by means of (47.6), and the angles K_1 and K_2 (in degrees) from

$$\begin{aligned} K_1 &= 119.75 + 131.849\ T \\ K_2 &= \ \ 72.56 + \ \ 20.186\ T \end{aligned}$$

where, as elsewhere in this book, T is the time measured in Julian centuries of 36525 days from the Epoch J2000.0 = JDE 2451 545.0.

$$\begin{aligned}
\rho = & -0.02752 \cos M' \\
& -0.02245 \sin F \\
& +0.00684 \cos (M' - 2F) \\
& -0.00293 \cos 2F \\
& -0.00085 \cos (2F - 2D) \\
& -0.00054 \cos (M' - 2D) \\
& -0.00020 \sin (M' + F) \\
& -0.00020 \cos (M' + 2F) \\
& -0.00020 \cos (M' - F) \\
& +0.00014 \cos (M' + 2F - 2D)
\end{aligned}$$

$$\begin{aligned}
\sigma = & -0.02816 \sin M' \\
& +0.02244 \cos F \\
& -0.00682 \sin (M' - 2F) \\
& -0.00279 \sin 2F \\
& -0.00083 \sin (2F - 2D) \\
& +0.00069 \sin (M' - 2D) \\
& +0.00040 \cos (M' + F) \\
& -0.00025 \sin 2M' \\
& -0.00023 \sin (M' + 2F) \\
& +0.00020 \cos (M' - F) \\
& +0.00019 \sin (M' - F) \\
& +0.00013 \sin (M' + 2F - 2D) \\
& -0.00010 \cos (M' - 3F)
\end{aligned}$$

$$\begin{aligned}
\tau = & +0.02520\, E \sin M \\
& +0.00473 \sin (2M' - 2F) \\
& -0.00467 \sin M' \\
& +0.00396 \sin K_1 \\
& +0.00276 \sin (2M' - 2D) \\
& +0.00196 \sin \Omega \\
& -0.00183 \cos (M' - F) \\
& +0.00115 \sin (M' - 2D) \\
& -0.00096 \sin (M' - D) \\
& +0.00046 \sin (2F - 2D) \\
& -0.00039 \sin (M' - F) \\
& -0.00032 \sin (M' - M - D) \\
& +0.00027 \sin (2M' - M - 2D) \\
& +0.00023 \sin K_2 \\
& -0.00014 \sin 2D \\
& +0.00014 \cos (2M' - 2F) \\
& -0.00012 \sin (M' - 2F) \\
& -0.00012 \sin 2M' \\
& +0.00011 \sin (2M' - 2M - 2D)
\end{aligned}$$

Then we have

$$\begin{aligned}
l'' &= -\tau + (\rho \cos A + \sigma \sin A) \tan b' \\
b'' &= \sigma \cos A - \rho \sin A
\end{aligned} \qquad (53.2)$$

Position Angle of Axis

The position angle of the Moon's axis of rotation, P, is defined as for the planets — see Chapters 42 and 43. It can be calculated as follows; the effect of the physical libration is taken into account.

I, Ω, $\Delta\psi$, ρ, σ, and b have the same meaning as before. Let α be the apparent geocentric right ascension of the Moon, and ε the true obliquity of the ecliptic. Then

$$V = \Omega + \Delta\psi + \frac{\sigma}{\sin I}$$

$$X = \sin(I + \rho) \sin V$$

$$Y = \sin(I + \rho) \cos V \cos \varepsilon - \cos(I + \rho) \sin \varepsilon$$

$$\tan \omega = X / Y$$

$$\sin P = \frac{\sqrt{X^2 + Y^2} \cos(\alpha - \omega)}{\cos b}$$

The angle ω can be obtained in the correct quadrant by using the "second" arctangent function: $\omega = \mathrm{ATN2}(X, Y)$. If this function is not available, divide X by Y, take the usual arctangent of the result, then add 180° if $Y < 0$.

The position angle P is to be taken in the first or in the fourth quadrant, that is, either between 0 and 90 degrees, or between 270 and 360 degrees.

Example 53.a — The Moon on 1992 April 12, at 0^h TD.

For this instant we have (see Example 47.a):

D	= 113°842304		
M	= 97°643514	λ	= 133°167265
M'	= 5°150833	β	= −3°229126
F	= 219°889721	$\lambda - \Delta\psi$	= 133°162655
$\Delta\psi$	= +0°004610	ε	= 23°440636
E	= 1.000194	α	= 134°688470

Then we obtain:

Ω	= 274°400656	l''	= −0°025
W	= 218°761999	b''	= +0°006
A	= 218°683932	l	= −1°23
l'	= −1°206	b	= +4°20
b'	= +4°194	V	= 273°820507
$K1$	= 109°57	$I + \rho$	= 1°53200
K_2	= 71°00	X	= −0.026676
ρ	= −0.01042	Y	= −0.396022
σ	= −0.01574	ω	= 183°8536
τ	= +0.02673	P	= 15°08

Topocentric librations

For precise reductions of observations, the geocentric values of the librations and position angle of the axis should be reduced to the values at the place of the observer on the surface of the Earth. For the librations, the differences may reach $1°$ and have important effects on the limb-contour.

The topocentric librations in longitude and latitude, and the topocentric position angle of the axis, may be calculated either by direct calculation or by differential corrections of the geocentric values.

a. Direct calculation. — The formulae given before are used, but the geocentric coordinates λ, β, α of the Moon are replaced by the topocentric ones. For this purpose, the topocentric right ascension and declination of the Moon are obtained by means of formulae (40.2) and (40.3); then they are transformed to the ecliptical coordinates λ and β by the usual conversion formulae (13.1) and (13.2) to obtain the topocentric longitude and latitude.

b. Differential corrections. — Let φ be the observer's latitude, δ the geocentric declination of the Moon, H the local hour angle of the Moon (calculated from the local sidereal time and the *geocentric* right ascension), and π the geocentric horizontal parallax of the Moon. Then calculate

$$\tan Q = \frac{\cos \varphi \sin H}{\cos \delta \sin \varphi - \sin \delta \cos \varphi \cos H}$$

$$\cos z = \sin \delta \sin \varphi + \cos \delta \cos \varphi \cos H$$

$$\pi' = \pi (\sin z + 0.0084 \sin 2z)$$

Then the corrections to the geocentric librations (l, b) and to the position angle (P) are

$$\Delta l = \frac{-\pi' \sin(Q - P)}{\cos b}$$

$$\Delta b = +\pi' \cos(Q - P)$$

$$\Delta P = +\Delta l \sin(b + \Delta b) - \pi' \sin Q \tan \delta$$

These formulae were given in Reference [2].

The selenographic position of the Sun

The selenographic coordinates of the Sun determine the regions of the lunar surface that are illuminated.

The selenographic longitude l_0 and latitude b_0 of the subsolar point on the lunar surface — the point where the Sun is at the zenith — are obtained by replacing, in the formulae (53.1) for the selenographic coordinates of the Earth, the geocentric ecliptical coordinates λ, β of the Moon by the *heliocentric* ecliptical coordinates λ_H, β_H of the Moon. With sufficient accuracy we have

$$\lambda_H = \lambda_0 + 180° + \frac{\Delta}{R} \times 57^\circ\!.296 \cos\beta \sin(\lambda_0 - \lambda)$$

$$\beta_H = \frac{\Delta}{R}\,\beta$$

where λ_0 is the apparent geocentric longitude of the Sun. The fraction Δ/R is the ratio of the distance Earth-Moon to the distance Earth-Sun; hence, Δ and R should be expressed in the same units, for instance kilometers. If, instead, R is expressed in astronomical units, and π is the equatorial horizontal parallax of the Moon expressed in *seconds* of arc ($''$), the fraction Δ/R is equal to $8.794/\pi R$.

Hence, to find l_0 and b_0, first calculate λ_H and β_H. Then use expressions (53.1), replacing λ by λ_H, and β by β_H; this will give l'_0 and b'_0. The quantities ρ, σ, and τ are found by the unchanged expressions, and finally l''_0 and b''_0 by (53.2), using b'_0 instead of b'. Then

$$l_0 = l'_0 + l''_0 \qquad \text{and} \qquad b_0 = b'_0 + b''_0$$

Subtracting l_0 from 90° or 450° gives the selenographic *colongitude* of the Sun (c_0), which is tabulated in some ephemerides.

The quantities l_0 (or c_0) and b_0 determine the exact position of the terminator on the surface of the Moon. The subsolar point at l_0, b_0 is the pole of the great circle on the lunar surface that bounds the illuminated hemisphere. The morning terminator, where the Sun is rising on the Moon, is at selenographic longitude $l_0 - 90°$, or $360° - c_0$. The evening terminator, where the Sun is setting, is at longitude $l_0 + 90°$, or $180° - c_0$. When $c_0 = 0°$, the Sun is rising at selenographic longitude 0°; this occurs near First Quarter. At Full Moon, Last Quarter, and New Moon, respectively, c_0 is approximately 90°, 180°, and 270°, and the morning terminator is approximately at selenographic longitudes 270°, 180°, and 90°.

Note that, while l_0 is *decreasing* with time, the colongitude c_0 is *increasing*. Their mean daily motion is equal to that of the Moon's mean elongation D, namely 12.190749 degrees.

At a point on the lunar surface at selenographic longitude η (positive towards Mare Crisium!) and latitude θ, sunrise occurs approximately when $c_0 = 360° - \eta$ or $c_0 = -\eta$, noon when $c_0 = 90° - \eta$, and sunset when $c_0 = 180° - \eta$. The altitude h of the Sun above the lunar horizon at any time may be calculated from

$$\sin h = \sin b_0 \sin \theta + \cos b_0 \cos \theta \sin (c_0 + \eta) \tag{53.3}$$

To find the time of sunrise or sunset for a given place on the Moon, calculate the Sun's altitude h for that place for an approximate time. Then the correction to the assumed time, in days, is

$$\mp \frac{h}{12.19075 \cos \theta} \tag{53.4}$$

where the upper sign is to be used for sunrise, the lower sign for sunset. The altitude h of the Sun should be taken with proper sign and be expressed in degrees. If needed, the calculation should be repeated, starting with the new time found. The time so obtained it that of the rise or set of the *center* of the solar disk.

Example 53.b — The Moon on 1992 April 12, at 0^h Dynamical Time.

For this instant we have, from accurate calculations (VSOP87 and ELP-2000/82),

$$\begin{aligned} \lambda_0 &= 22°.33978 \\ \Delta &= 368\,406 \text{ kilometers} \\ R &= 1.002\,497\,69 \text{ AU} = 149\,971\,500 \text{ km} \end{aligned}$$

The other relevant quantities were found in Example 53.a. We then find

$\lambda_H = 202°.208\,438$	$l'_0 = 67°.920$	$l_0 = 67°.89$
$\beta_H = -0°.007\,932$	$b'_0 = +1°.476$	$b_0 = +1°.46$
$W = 287°.803\,172$	$l''_0 = -0°.026$	$c_0 = 22°.11$
$A = 287°.809\,283$	$b''_0 = -0°.015$	

Example 53.c — Sunrise for the crater Copernicus in April 1992.

The selenographic coordinates of this crater are (Table 53.A) $\eta = -20°.0$, $\theta = +9°.7$. Sunrise occurs approximately when the Sun's selenographic colongitude c_0 is $-\eta$, or $+20°.0$ in the present case. This is almost the value found for 1992 April 12 at 0^h TD in Example 53.b. For this instant we found $b_0 = +1°.46$ and $c_0 = 22°.11$.

For these values, formula (53.3) gives $h = +2°.3253$ (keeping extra decimals), whence a correction of -0.1935 day by formula (53.4), giving 1992 April 11.8065.

For this improved time, a new calculation gives $b_0 = +1°.46$, $c_0 = 19°.75$, whence a value of the Sun's altitude h which is practically zero.

Hence, no other iteration is needed. The required time is 1992 April 11.8065, or 1992 April 11 at 19^h TD. This is also 19^h Universal Time.

TABLE 53.A

Selenographic coordinates of some lunar features

Name	η	θ	*Name*	η	θ
	°	°		°	°
Archimedes	− 3.9	+29.7	Lansberg	−26.6	− 0.3
Aristarchus	−47.5	+23.7	Letronne	−43	−10
Aristillus	+ 1.2	+33.9	Macrobius	+46.0	+21.2
Aristoteles	+17.3	+50.1	Manilius	+ 9.1	+14.5
Arzachel	− 1.9	−17.7	Menelaus	+16.0	+16.3
Autolycus	+ 1.5	+30.7	Messier	+47.6	− 1.9
Billy	−50.0	−13.8	Petavius	+61	−25
Birt	− 8.5	−22.3	Pico	− 8.8	+45.8
Campanus	−27.7	−28.0	Pitatus	−13.5	−29.8
Censorinus	+32.7	− 0.4	Piton	− 0.8	+40.8
Clavius	−14	−58	Plato	− 9.2	+51.4
Copernicus	−20.0	+ 9.7	Plinius	+23.6	+15.3
Delambre	+17.5	− 1.9	Posidonius	+30.0	+31.9
Dionysius	+17.3	+ 2.8	Proclus	+46.9	+16.1
Endymion	+56.4	+53.6	Ptolemaeus A	− 0.8	− 8.5
Eratosthenes	−11.3	+14.5	Pytheas	−20.6	+20.5
Eudoxus	+16.3	+44.3	Reinhold	−22.8	+ 3.2
Fracastorius	+33.2	−21.0	Riccioli	−74.3	− 3.2
Fra Mauro	−17	− 6	Schickard	−54.5	−44.0
Gassendi	−39.9	−17.5	Schiller	−39	−52
Goclenius	+45.0	−10.1	Taruntius	+46.5	+ 5.6
Grimaldi	−68.5	− 5.8	Theophilus	+26.5	−11.4
Harpalus	−43.4	+52.6	Timocharis	−13.1	+26.7
Horrocks	+ 5.9	− 4.0	Tycho	−11.0	−43.2
Kepler	−38.0	+ 8.1	Vitruvius	+31.3	+17.6
Langrenus	+60.9	− 8.9	Walter	+ 1	−33

REFERENCES

1. D. H. Eckhardt, "Theory of the Libration of the Moon", *Moon and Planets*, Vol. 25, page 3 (1981).
2. *Explanatory Supplement to the Astronomical Ephemeris* (London, 1961), page 324.

Chapter 54

Eclipses

Without too much calculation, it is possible to obtain with good accuracy the principal characteristics of an eclipse of the Sun or the Moon. For a solar eclipse, the situation is complicated by the fact that the phases of the event are different for different observers at the Earth's surface, while in the case of a lunar eclipse all observers see the same phase at the same instant.

For this reason, we will not consider here the calculation of the local circumstances of a solar eclipse. The interested reader may calculate these circumstances from the Besselian elements published yearly in the *Astronomical Almanac*. Besselian elements for all solar eclipses of the years -2003 to $+2526$ can be found in the *Canon* by Mucke and Meeus [1]. For modern times, accurate Besselian elements have been published by Meeus [2]. Besides the elements, these works provide the formulae needed for their use, together with numerical examples.

Espenak published a *Canon* [3] giving data about the paths of total and annular solar eclipses from 1986 to 2035, with beautiful world maps for all eclipses in that period. This work does not contain Besselian elements, however, so it does not provide the possibility to calculate extra data, such as local circumstances for places outside the path of total or annular phase.

Let us also mention the work by Stephenson and Houlden [4], which contains data and charts for the total and annular eclipses visible in East Asia from 1500 B.C. to A.D. 1900.

General data

First, calculate the instant (JDE) of the *mean* New or Full Moon, by means of formulae (49.1) to (49.3). Remember that k must be an integer for a New Moon (solar eclipse), and an integer increased by 0.5 for a Full Moon (lunar eclipse).

Then, calculate the values of the angles M, M', F, and Ω for that instant by means of expressions (49.4) to (49.7), and the value of E by formula (47.6).

The value of F will give the first information about the occurrence of a solar or lunar eclipse. If F differs from the nearest multiple of 180° by less than 13.9 degrees, then there is certainly an eclipse; if the difference is larger than 21°.0, there is no eclipse; between these two values, the eclipse is uncertain at this stage and the case must be examined further. Use can be made of the following rule: there is no eclipse if $|\sin F| > 0.36$.

Note that after one lunation the angle F increases by 30.6705 degrees.

If F is near 0° or 360°, the eclipse occurs near the Moon's ascending node. If F is near 180°, the eclipse takes place near the descending node of the Moon's orbit.

Calculate

$$F_1 = F - 0^\circ.02665 \sin \Omega$$
$$A_1 = 299^\circ.77 + 0^\circ.107408\,k - 0^\circ.009173\,T^2$$

Then, to obtain the *time of maximum eclipse* (for the Earth generally in the case of a solar eclipse), the following corrections (in days) should be added to the time of mean conjunction or opposition given by expression (49.1).

$$\begin{array}{lll}
-0.4075 & \times \sin & M' \\
+0.1721 \times E & & M \\
+0.0161 & & 2M' \\
-0.0097 & & 2F_1 \\
+0.0073 \times E & & M' - M \\
-0.0050 \times E & & M' + M \\
-0.0023 & & M' - 2F_1 \\
+0.0021 \times E & & 2M \\
+0.0012 & & M' + 2F_1 \\
+0.0006 \times E & & 2M' + M \\
-0.0004 & & 3M' \\
-0.0003 \times E & & M + 2F_1 \\
+0.0003 & & A_1 \\
-0.0002 \times E & & M - 2F_1 \\
-0.0002 \times E & & 2M' - M \\
-0.0002 & & \Omega
\end{array} \qquad (54.1)$$

(for the first two terms:) $\left.\begin{array}{l} M' \\ M \end{array}\right\}$ for *lunar eclipses*, change these constants to −0.4065 and +0.1727

This algorithm should not be used, of course, if high accuracy is needed. For the 221 solar eclipses of the years A.D. 1951 to 2050, the method gives a mean error of 0.36 minute, and a greatest error of 1.1 minute in the times of maximum eclipse.

Then calculate

$$
\begin{aligned}
P = \; & +0.2070 \times E \times \sin M \\
& +0.0024 \times E \quad \sin 2M \\
& -0.0392 \quad \sin M' \\
& +0.0116 \quad \sin 2M' \\
& -0.0073 \times E \quad \sin (M' + M) \\
& +0.0067 \times E \quad \sin (M' - M) \\
& +0.0118 \quad \sin 2F_1
\end{aligned}
\qquad
\begin{aligned}
Q = \; & +5.2207 \\
& -0.0048 \times E \times \cos M \\
& +0.0020 \times E \quad \cos 2M \\
& -0.3299 \quad \cos M' \\
& -0.0060 \times E \quad \cos (M' + M) \\
& +0.0041 \times E \quad \cos (M' - M)
\end{aligned}
$$

$$W = |\cos F_1|$$

$$\gamma = (P \cos F_1 + Q \sin F_1) \times (1 - 0.0048\, W)$$

$$
\begin{aligned}
u = \; & 0.0059 \\
& + 0.0046\, E \cos M \\
& - 0.0182 \cos M' \\
& + 0.0004 \cos 2M' \\
& - 0.0005 \cos (M + M')
\end{aligned}
$$

Solar eclipses

In the case of a solar eclipse, γ represents the least distance from the axis of the Moon's shadow to the center of the Earth, in units of the equatorial radius of the Earth. The quantity γ is positive or negative, depending upon the axis of the shadow passing north or south of the Earth's center. When γ is between $+0.9972$ and -0.9972, the solar eclipse is central: there exists a line of central eclipse on the Earth's surface, and for observers on this line the center of the lunar disk passes exactly over the center of the solar disk.

The quantity u denotes the radius of the Moon's *umbral* cone in the fundamental plane, again in units of the Earth's equatorial radius. The radius of the *penumbral* cone in the fundamental plane is $u + 0.5461$. The fundamental plane is the plane through the center of the Earth and perpendicular to the axis of the Moon's shadow.

If $|\gamma| > 1.5433 + u$, no eclipse is visible from the Earth's surface.

If $|\gamma|$ is between 0.9972 and $1.5433 + u$, the eclipse is not central. In most cases, it is then a partial eclipse. However, when $|\gamma|$ is between 0.9972 and 1.0260, a part of the umbral cone may touch the surface of the Earth (within the polar regions), while the axis of the cone does *not* touch the Earth. These *non-central* total or annular eclipses occur when $0.9972 < |\gamma| < 0.9972 + |u|$. Between the years 1950 and 2100, there are seven eclipses of this type:

1950 March 18	annular, not central
1957 April 30	annular, not central
1957 October 23	total, not central
1967 November 2	total, not central
2014 April 29	annular, not central
2043 April 9	total, not central
2043 October 3	annular, not central

In the case of a *central* eclipse, the type of the eclipse can be determined by the following rules: if $u < 0$, the eclipse is total; if $u > +0.0047$, it is annular; if u is between 0 and $+0.0047$, the eclipse is either annular or annular-total. In the latter case, the ambiguity is removed as follows. Calculate

$$\omega = 0.00464\sqrt{1-\gamma^2} > 0$$

Then, if $u < \omega$, the eclipse is annular-total; otherwise it is an annular one.

In the case of a *partial* solar eclipse, the greatest magnitude is attained at the point of the surface of the Earth which comes closest to the axis of shadow. The magnitude of the eclipse at that point is

$$\frac{1.5433 + u - |\gamma|}{0.5461 + 2u} \tag{54.2}$$

Lunar eclipses

In the case of a lunar eclipse, γ represents the least distance from the center of the Moon to the axis of the Earth's shadow, in units of the Earth's equatorial radius. The quantity γ is positive or negative depending upon the Moon's center passing north or south of the axis of the shadow. The radii at the distance of the Moon, again in equatorial Earth radii, are

for the penumbra : $\rho = 1.2848 + u$

for the umbra : $\sigma = 0.7403 - u$

while the magnitude of the eclipse may be found as follows:

for penumbral eclipses : $$\frac{1.5573 + u - |\gamma|}{0.5450} \tag{54.3}$$

for umbral eclipses : $$\frac{1.0128 - u - |\gamma|}{0.5450} \tag{54.4}$$

If the magnitude is negative, this indicates that there is no eclipse.

The *semidurations* of the partial and total phases in the *umbra* can be found as follows. Calculate

$$p = 1.0128 - u$$
$$t = 0.4678 - u$$
$$n = 0.5458 + 0.0400 \cos M'$$

Then the semidurations in *minutes* are

partial phase: $\dfrac{60}{n}\sqrt{p^2 - \gamma^2}$ total phase: $\dfrac{60}{n}\sqrt{t^2 - \gamma^2}$.

For the semiduration of the partial phase in the *penumbra*, find $h = 1.5573 + u$, and then the semiduration in minutes is

$$\frac{60}{n}\sqrt{h^2 - \gamma^2}$$

The semidurations are the time intervals between the beginning (or end) of the partial phase, the beginning (or end) of the total phase, or the first (or last) contact with the penumbra and the instant of *maximum eclipse*. So, for instance, in the case of a total eclipse in the umbra, the semiduration of the partial phase does include half the duration of the phase of totality.

Further, it must be noted that the contacts of the Moon with the penumbra cannot be observed, and that most penumbral eclipses (in which the Moon enters only the penumbra of the Earth) cannot be discerned visually. Only at eclipses occurring deep in the penumbra can a weak shading of the Moon's northern or southern limb be seen.

In the formulae given above, the increase of the theoretical radii of the shadow cones by the Earth's atmosphere is taken into account. However, instead of the traditional rule consisting of increasing by 1/50 the theoretical radii, we have preferred the method used since 1951 in the French almanac *Connaissance des Temps* — see for instance Reference [5]. As compared with the results of the "French rule", the magnitude of a lunar eclipse calculated by using the traditional rule is too large by about 0.005 for umbral eclipses, by about 0.026 for penumbral eclipses.

To obtain the results according to the traditional rule (1/50), the following changes should be made to the constants in the expressions given above:

replace	1.2848 by 1.2985
	0.7403 by 0.7432
	1.5573 by 1.5710
	1.0128 by 1.0157
	0.4678 by 0.4707

For the predictions of lunar eclipses, such as those published in the various almanacs, it is customary to assume the penumbra and the umbra to be exactly circular, and to use a mean radius for the Earth. In fact, the shadow differs somewhat from a circular cone as the Earth is not a true sphere. By simple geometrical considerations, it is found that the Earth's shadow, at the Moon's distance, must be *more* flattened than the terrestrial globe, the mean value for the flattening of the umbra being 1/214 [6]. The true flattening of the umbra is perhaps even larger still. Soulsby [7] finds a mean oblateness of 1/102 from observations made at 18 lunar eclipses in the period 1974-1989.

Example 54.a — Solar eclipse of 1993 May 21.

May 21 being the 141th day of the year, the given date corresponds to 1993.38. Formula (49.2) then gives $k \approx -81.88$, whence $k = -82$.

Then, by means of formulae (49.3) and (49.1), JDE = 2449 128.5894. Further,

$$\begin{aligned} M &= 135^\circ\!.9142 \\ M' &= 244^\circ\!.5757 \\ F &= 165^\circ\!.7296 \\ \Omega &= 253^\circ\!.0026 \\ F_1 &= 165^\circ\!.7550 \end{aligned}$$

Because $180^\circ - F$ is between $13^\circ\!.9$ and $21^\circ\!.0$, the eclipse is uncertain at this stage. We further find

$$\begin{aligned} P &= +0.1842 \\ Q &= +5.3589 \\ \gamma &= +1.1348 \\ u &= +0.0097 \end{aligned}$$

Because $|\gamma|$ is between 0.9972 and $1.5433 + u$, the eclipse is a partial one. Using formula (54.2), we find that the maximum magnitude is

$$\frac{1.5433 + 0.0097 - 1.1348}{0.5461 + 0.0194} = 0.740$$

Because F is near 180°, the eclipse occurs near the Moon's descending node. Because γ is positive, the eclipse is visible in the northern hemisphere of the Earth.

To obtain the time of maximum eclipse, we add to JDE the terms given by formula (54.1). This gives

$$\text{JDE} = 2449\,128.5894 + 0.5085 = 2449\,129.0979$$

which corresponds to 1993 May 21, at $14^h21^m\!.0$ TD.

The correct values, resulting from an accurate calculation [2], are $14^h20^m14^s$ Dynamical Time, $\gamma = +1.1370$, and a maximum magnitude of 0.735.

Example 54.b — Solar eclipse of 2009 July 22.

As in the preceding Example, we find:

$$
\begin{aligned}
k &= 118 \\
\text{JDE} &= 2455\,034.7071 \\
M &= 196^\circ\!.9855 \\
M' &= 7^\circ\!.9628 \\
F &= 179^\circ\!.8301 \\
F_1 &= 179^\circ\!.8531
\end{aligned}
$$

Corrected JDE $= 2455\,034.6088 =$ 2009 July 22, at 2^h37^m TD.

$$
\begin{aligned}
P &= -0.0573 \\
Q &= +4.9016 \\
\gamma &= +0.0695 \\
u &= -0.0157
\end{aligned}
$$

Because $|\gamma| < 0.9972$, the eclipse is central. Because u is negative, the eclipse is total. Because $|\gamma|$ is small, the eclipse is visible from the equatorial regions of the Earth. Because F is near 180°, the eclipse takes place near the descending node of the Moon's orbit.

Example 54.c — Lunar eclipse of June 1973. We find successively:

$$
\begin{aligned}
k &= -328.5 \\
\text{JDE} &= 2441\,849.2992 \\
M &= 161^\circ\!.4437 \\
M' &= 180^\circ\!.7018 \\
F &= 345^\circ\!.4505
\end{aligned}
$$

Corrected JDE $= 2441\,849.3687 =$ 1973 June 15, at 20^h51^m TD.

$$
\begin{aligned}
\gamma &= -1.3249 \\
u &= +0.0197
\end{aligned}
$$

The eclipse took place near the Moon's ascending node (because $F \approx 360°$) and the Moon's center passed south of the center of the Earth's umbra (because $\gamma < 0$).

According to formula (54.4), the magnitude in the umbra was -0.609. Since this is negative, there was no eclipse in the umbra. Using formula (54.3), we find that the magnitude in the penumbra was 0.462. Hence, the eclipse was a penumbral one.

According to the *Connaissance des Temps*, maximum eclipse occurred at $20^h50^m\!.7$ Dynamical Time, and the magnitude in the penumbra was 0.469.

Example 54.d — Find the first lunar eclipse after 1997 July 1.

For 1997.5, formula (49.2) gives $k \approx -30.92$, so we must try the value $k = -30.5$. This gives $F = 125^\circ.2605$, which differs more than 21 degrees from the nearest multiple of 180°, and hence gives no eclipse.

The next Full Moon, $k = -29.5$, gives $F = 155^\circ.9310$, hence again no eclipse. But it is evident that the next Full Moon will give $F \approx 187^\circ$ and thus give rise to an eclipse. We then find, as before:

$$\begin{aligned} k &= -28.5 \\ \text{JDE} &= 2450708.4759 \\ M &= 253^\circ.0507 \\ M' &= 5^\circ.7817 \\ F &= 186^\circ.6015 \end{aligned}$$

Corrected JDE = 2450708.2835 = 1997 September 16, at $18^h48^m.2$ Dynamical Time, or 18^h47^m UT (if we adopt the value $\Delta T = \text{TD} - \text{UT} = +63$ seconds).

$$\gamma = -0.3791 \qquad u = -0.0131$$

Formula (54.4) then gives a magnitude of 1.187, so the eclipse is total in the umbra.

$$p = 1.0259 \qquad t = 0.4809 \qquad h = 1.5442 \qquad n = 0.5856$$

Semiduration of partial phase:

$$\frac{60}{0.5856}\sqrt{(1.0259)^2 - (0.3791)^2} = 98 \text{ minutes}$$

Semiduration of total phase:

$$\frac{60}{0.5856}\sqrt{(0.4809)^2 - (0.3791)^2} = 30 \text{ minutes}$$

Semiduration of penumbral phase:

$$\frac{60}{0.5856}\sqrt{(1.5442)^2 - (0.3791)^2} = 153 \text{ minutes}$$

Hence, in Universal Time:

first contact with the penumbra :	$18^h47^m - 153^m$	$= 16^h14^m$
first contact with the umbra :	$18^h47^m - 98^m$	$= 17^h09^m$
beginning of total phase :	$18^h47^m - 30^m$	$= 18^h17^m$
maximum of the eclipse :		18^h47^m
end of total phase :	$18^h47^m + 30^m$	$= 19^h17^m$
last contact with the umbra :	$18^h47^m + 98^m$	$= 20^h25^m$
last contact with the penumbra :	$18^h47^m + 153^m$	$= 21^h20^m$

Notes about the accuracy

The algorithms given in this Chapter are not intended to give highly accurate results. Still, for lunar eclipses the results will be precise enough for historical research, or when high accuracy is not needed. On the other hand, as has been said at the beginning of this Chapter, accurate data for modern solar eclipses can be obtained by using our *Elements of Solar Eclipses* [2].

The formula given for γ does not yield rigorously exact results. This is quite evident, if we consider the fact that only twelve periodic terms are used to calculate the quantities P and Q, while in fact hundreds of terms are needed to obtain accurate positions of the Sun and the Moon. Even formulae (54.2), (54.3), and (54.4), and the expressions for the quantities p, t, n, and h are not rigorously exact.

For the 221 solar eclipses of the period 1951-2050, the mean error of the values of γ as calculated by using the algorithm of this Chapter is 0.00065, while the maximum error is 0.0024, which corresponds to 15 kilometers. Considering the simplicity of our formulae, this accuracy is quite satisfactory.

From what precedes, it results that *in limiting cases* the type of an eclipse will still be unknown. In such a case, an accurate calculation is needed to settle the question.

Further, in a *search procedure* for eclipses, a small safety margin should be considered in order to be sure that no eclipse will be overlooked. For instance, while the correct condition for a central solar eclipse is indeed $|\gamma| < 0.9972$ (*), a limiting value of 1.000 or even 1.005 should be used in order to find *all* possible central eclipses when use is made of the value of γ obtained with the method described in this Chapter.

Here are some examples.

For the solar eclipse of 1935 January 5 ($k = -804$), our method gives $\gamma = -1.5395$ and $u = -0.00464$, whence $|\gamma| > u + 1.5433 = 1.5387$, so we might think there was no eclipse on that date. Formula (54.4) yields the value -0.002 (*negative*!) for the maximum magnitude. The correct value of γ was -1.5383, however, so there was a very small partial solar eclipse on 1935 January 5, with a maximum magnitude of only 0.001.

For the annular solar eclipse of 1957 April 30 ($k = -528$), our algorithm yields the value $\gamma = +0.9966$, so one might think this was a central eclipse. The exact value was $\gamma = +0.9990$, so it was actually a non-central annular event.

For the lunar eclipse of 1890 November 26 ($k = -1349.5$), our algorithm gives a magnitude (in the umbra) of -0.007. In fact, it was a very small partial eclipse in the umbra.

(*) In fact, the "constant" 0.9972 may vary between 0.9970 and 0.9974 from one eclipse to another.

Exercises

Find the first solar eclipse of the year 1979, and show that it was a total one visible from the northern hemisphere.

Was the solar eclipse of April 1977 a total or an annular one?

Show that there was no eclipse of the Sun in July 1947.

Show that there are four solar eclipses in the year 2000, and that all four are partial eclipses.

Show that there will be no lunar eclipse in January 2008.

Show that there were three total eclipses of the Moon in 1982.

Find the first lunar eclipse of the year 1234. (Answer: the partial lunar eclipse of 1234 March 17).

REFERENCES

1. H. Mucke, J. Meeus, *Canon of Solar Eclipses, −2003 to +2526*; Astronomisches Büro (Wien, 1983).
2. J. Meeus, *Elements of Solar Eclipses, 1951 to 2200* (Willmann-Bell, ed.; 1989).
3. F. Espenak, *Fifty Year Canon of Solar Eclipses: 1986–2035*; NASA Reference Publication 1178 (Washington, 1987).
4. F. R. Stephenson, M. A. Houlden, *Atlas of Historical Eclipse Maps*; Cambridge University Press (1986).
5. A. Danjon, "Les éclipses de Lune par la pénombre en 1951", *l'Astronomie*, Vol. 65, pages 51–53 (February 1951).
6. J. Meeus, "Die Abplattung des Erdschattens bei Mondfinsternissen", *Die Sterne*, Vol. 45, pages 116–117 (1969).
7. B. W. Soulsby, *Journal of the British Astronomical Association*, Vol. 100, page 297 (December 1990).

Chapter 55

Semidiameters of the Sun, Moon, and Planets

Sun and Planets

The angular semidiameters s of the Sun and planets are calculated from

$$s = \frac{s_0}{\Delta}$$

where s_0 is the body's semidiameter at unit distance (1 AU), and Δ the body's distance to the Earth in AU.

For the Sun, the value adopted in the calculations is [1]

$$s_0 = 15'59''.63 = 959''.63$$

For the planets, the following values of s_0 have been used for many years [2]:

	″		″
Mercury	3.34	Saturn:	
Venus	8.41	equatorial	83.33
Mars	4.68	polar	74.57
Jupiter:		Uranus	34.28
equatorial	98.47	Neptune	36.56
polar	91.91		

(A)

Later, the following values have been adopted [3]:

	″		″	
Mercury	3.36	Saturn:		
Venus	8.34	equatorial	82.73	
		polar	73.82	
Mars	4.68			**(B)**
Jupiter:		Uranus	35.02	
equatorial	98.44	Neptune	33.50	
polar	92.06	Pluto	2.07	

Note that, according to the latter values, Neptune is smaller than Uranus.

For Venus, the value 8″.34 refers to the planet's crust, not to the top cloud level as seen from the Earth. For this reason, we prefer to use the older value 8″.41 for Venus when calculating astronomical phenomena such as transits and occultations.

In the case of Saturn, let a and b be the equatorial and the polar semidiameters at unit distance. Then, while the apparent equatorial semidiameter s_E is given by $s_E = a/\Delta$, the apparent *polar* semidiameter should be calculated from

$$s_P = s_E \sqrt{1 - k \cos^2 B}$$

where $k = 1 - (b/a)^2$, and B is the Saturnicentric latitude of the Earth (see Chapter 45).

If the older values (A) are chosen, namely $a = 83''.33$ and $b = 74''.57$, then $k = 0.199\,197$. If one adopts the values from (B), then $k = 0.203\,800$.

Strictly speaking, this procedure should also be used in the case of Jupiter. But for this planet the angle B (called D_E in Chapter 43) can never exceed 4°, so it will generally be sufficient to put $s_P = b/\Delta$ here.

Moon

Let Δ be the distance between the centers of Earth and Moon in kilometers, π the equatorial horizontal parallax of the Moon, s its geocentric semidiameter, and k the ratio of the Moon's mean radius to the equatorial radius of the Earth. In the *Astronomical Ephemeris* for the years 1963 to 1968, the value $k = 0.272\,481$ was used in eclipse calculations, and we have used this value ever since.

Then we have rigorously

$$\sin \pi = \frac{6378.14}{\Delta} \qquad \text{and} \qquad \sin s = k \sin \pi$$

but in most cases it will be sufficient to use the formula

$$s \text{ (in arcseconds)} = \frac{358\,473\,400}{\Delta}$$

which gives an error less than 0.0005 arcsecond, as compared with the result obtained by the rigorous expressions given before.

Computed in this way, the Moon's semidiameter is geocentric, that is, it applies to a fictitious observer located at the center of the Earth. The observed, topocentric semidiameter s' will be slightly larger than the geocentric semidiameter, because the observer is somewhat closer to the Moon than is the center of the Earth (except when the Moon is on the horizon). It is given by

$$\sin s' = \frac{\sin s}{q} = \frac{k}{q} \sin \pi$$

while the topocentric distance of the Moon (that is, the distance from the observer to the center of the Moon) is $\Delta' = q\Delta$, q being given by formula (40.7).

Alternatively, the topocentric semidiameter s' of the Moon can be obtained, with an accuracy which is sufficient for many purposes, by multiplying the geocentric value s by

$$1 + \sin h \sin \pi$$

where h is the altitude of the Moon above the observer's horizon.

The increase in the Moon's semidiameter, due to the fact that the observer is not geocentric, is zero when the Moon is on the horizon, and a maximum (between $14''$ and $18''$) when the Moon is at the zenith.

Asteroids

The diameter d of an asteroid, in kilometers, can be calculated from [4]

$$\log d = 3.12 - 0.2H - 0.5 \log A$$

where H is the absolute magnitude of the body (see page 231), and A the albedo, or reflective power. The logarithms are to base 10.
If the logarithms are to base e, as in most programming languages, then

$$x = 3.12 - H/5 - \frac{1}{2}\,\frac{\log A}{\log 10}$$

or $$x = 3.12 - H/5 - 0.217\,147 \log A$$

and then d (kilometers) $= 10^x$.

Many asteroids have an albedo of only about 0.04 (4 percent). According to Tedesco [5], the albedo of the first four asteroids are: Ceres 0.10, Pallas 0.14, Juno 0.22, and Vesta 0.38. Asteroid 437 Rhodia has the very high albedo 0.56.

Because many asteroids have an irregular shape, the expressions given above can yield only an approximate value of the "diameter".

If d is the diameter of an asteroid expressed in kilometers, and if its distance to the Earth is Δ astronomical units, the *apparent diameter* of the body, in arcseconds, is

$$0.001\,3788\, d/\Delta$$

REFERENCES

1. A. Auwers, *Astronomische Nachrichten*, Vol. 128, No. 3068, column 367 (1891).
2. See, for instance, the *Astronomical Ephemeris* for 1980, page 550.
3. *Astronomical Almanac* for 1984, page E43.
4. *Sky and Telescope*, Vol. 85, No. 6, page 84 (June 1993).
5. E. D. Tedesco, pages 1093 and 1098 of *Asteroids II* (University of Arizona Press, 1989).

Chapter 56

Stellar Magnitudes

Adding stellar magnitudes

If two stars have magnitudes m_1 and m_2, respectively, their combined magnitude m can be calculated as follows:

$$x = 0.4\,(m_2 - m_1)$$

$$m = m_2 - 2.5 \log\,(10^x + 1)$$

where the logarithm is to the base 10.

Example 56.a — The magnitudes of the components of Castor (α Gem) are 1.96 and 2.89. Calculate the combined magnitude.

One finds

$$x = 0.4\,(2.89 - 1.96) = 0.372$$

$$m = 2.89 - 2.5 \log\,(10^{0.372} + 1) = 1.58$$

If more than two stars are involved, with magnitudes $m_1, m_2, \ldots, m_i, \ldots,$ the combined magnitude m can better be found from

$$m = -2.5 \log \sum 10^{-0.4\,m_i}$$

where, again, the logarithm is to the base 10. The symbol Σ indicates that the sum must be made of all quantities

$$10^{-0.4\,m_i}$$

Example 56.b — The triple star β Mon has components of magnitudes 4.73, 5.22, and 5.60, respectively. Calculate the combined magnitude.

$$m = -2.5 \log \left(10^{(-0.4)(4.73)} + 10^{(-0.4)(5.22)} + 10^{(-0.4)(5.60)} \right)$$

$$= -2.5 \log (0.01282 + 0.00817 + 0.00575) = 3.93$$

Example 56.c — A star cluster consists of

4	stars of (mean)	magnitude	5.0
14	—	—	6.0
23	—	—	7.0
38	—	—	8.0

Calculate the combined magnitude.

$$4 \times 10^{(-0.4)(5)} = 0.04000$$
$$14 \times 10^{(-0.4)(6)} = 0.05574$$
$$23 \times 10^{(-0.4)(7)} = 0.03645$$
$$38 \times 10^{(-0.4)(8)} = 0.02398$$

$$\text{Sum} \quad \Sigma = 0.15617$$

Combined magnitude $= -2.5 \log 0.15647 = +2.02$

Brightness ratio

If two stars have magnitudes m_1 and m_2, respectively, the ratio I_1 / I_2 of their apparent luminosities can be found from

$$x = 0.4\,(m_2 - m_1) \qquad\qquad \frac{I_1}{I_2} = 10^x$$

If the brightness ratio I_1 / I_2 is given, the corresponding magnitude difference $\Delta m = m_2 - m_1$ can be calculated from

$$\Delta m = 2.5 \log \frac{I_1}{I_2}$$

Example 56.d — How many times is Vega (magnitude 0.14) brighter than Polaris (magnitude 2.12)?

$$x = 0.4\,(2.12 - 0.14) = 0.792$$

$$10^x = 6.19$$

Hence, Vega is 6.19 times as bright as the Pole Star.

Example 56.e — A star is 500 times as bright as another one.

The corresponding magnitude difference is

$$\Delta m = 2.5 \log 500 = 6.75$$

Distance and Absolute Magnitude

If π is a star's parallax expressed in seconds of a degree ($''$), this star's distance to us is equal to

$$\frac{1}{\pi} \text{ parsecs} \qquad \text{or} \qquad \frac{3.2616}{\pi} \text{ light-years}$$

If π is a star's parallax expressed in seconds of a degree ($''$), and m is the apparent magnitude of this star, its absolute magnitude M is given by

$$M = m + 5 + 5 \log \pi$$

where, again, the logarithm is to the base 10.

If d is the star's distance in parsecs, we have

$$M = m + 5 - 5 \log d$$

Unlike the parallaxes within the solar system (see Chapter 40), the parallax considered here is, of course, the stellar, annual parallax resulting from the orbital motion of the Earth around the Sun; so it is *not* the parallax related to the dimensions of the Earth's *globe*!

The *parsec* is the unit of length equal to the distance at which the radius of the Earth's orbit (1 AU) subtends an angle of $1''$ (parallax $= 1''$). The name is a contraction of *par*allax and *sec*ond.

$$\begin{aligned} 1 \text{ parsec} &= 3.2616 \text{ light-years} \\ &= 206\,265 \text{ astronomical units} \\ &= 30.8568 \times 10^{12} \text{ kilometers} \end{aligned}$$

The *absolute magnitude* of a star is the apparent magnitude of this star if it were located at a distance of 10 parsecs.

Chapter 57

Binary Stars

The orbital elements of a binary star are the following ones:

- P = the period of revolution expressed in mean solar years;
- T = the time of periastron passage, generally given as a year and decimals (for instance, 1945.62);
- e = the eccentricity of the true orbit;
- a = the semimajor axis expressed in seconds of a degree ($''$);
- i = the inclination of the plane of the true orbit to the plane at right angles to the line of sight. For direct motion in the apparent orbit, i ranges from 0° to 90°; for retrograde motion, i is between 90 and 180 degrees. When i is 90°, the apparent orbit is a straight line passing through the primary star;
- Ω = the position angle of the ascending node;
- ω = the longitude of the periastron; this is the angle in the plane of the true orbit measured from the ascending node to the periastron, taken always in the direction of motion.

When these orbital elements are known, the apparent position angle θ and the angular distance ρ can be calculated for any given time t, as follows.

$$n = \frac{360°}{P} \qquad\qquad M = n\,(t - T)$$

where t is expressed as a year and decimals (just as T); n is the mean annual motion of the companion, expressed in degrees and decimals, and is always positive. M is the companion's mean anomaly for the given time t.

Then solve Kepler's equation

$$E = M + e \sin E$$

by one of the methods described in Chapter 30, and then calculate the radius vector r and the true anomaly v from

$$r = a\,(1 - e\cos E)$$

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\;\tan\frac{E}{2}$$

Then find $(\theta - \Omega)$ from

$$\tan(\theta - \Omega) = \frac{\sin(v+\omega)\cos i}{\cos(v+\omega)} \qquad (57.1)$$

Of course, this formula can be written

$$\tan(\theta - \Omega) = \tan(v+\omega)\cos i$$

but in this case the correct quadrant for $(\theta - \Omega)$ is not determined. As in previous cases mentioned in this book, one may apply the ATN2 function, if it is available in the programming language, to the numerator and the denominator of the fraction in (57.1). This will place the angle $(\theta - \Omega)$ at once in the correct quadrant.

When $(\theta - \Omega)$ is found, add Ω to obtain θ. If necessary, reduce the result to the interval 0°-360°.

Remember that, by definition, position angle 0° means northward on the sky, 90° east, 180° south, and 270° west. Consequently, if θ is between 0° and 180°, the companion is "following" the primary star in the diurnal motion of the celestial sphere; if $180° < \theta < 360°$, the companion is "preceding" the primary star.

The angular separation ρ is found from

$$\rho = \frac{r\cos(v+\omega)}{\cos(\theta - \Omega)}$$

However, the possibility exists of the denominator of the fraction being equal to zero. This risky division by zero can be avoided by using the following formula for the same calculation, mentioned by Greaney [1]:

$$\rho = r\sqrt{\sin^2(v+\omega)\cos^2 i + \cos^2(v+\omega)}$$

Note that the two terms under the square root sign are the squares of the numerator and the denominator, respectively, of the fraction in formula (57.1).

Example 57.a — According to E. Silbernagel (1929), the orbital elements of η Coronae Borealis are:

$P = 41.623$ years, $T = 1934.008$, $e = 0.2763$, $a = 0''.907$,
$i = 59^\circ.025$, $\Omega = 23^\circ.717$, $\omega = 219^\circ.907$

Let us calculate θ and ρ for the epoch 1980.0. We find successively:

$$
\begin{aligned}
n &= 8.64906 \\
t - T &= 1980.0 - 1934.008 = 45.992 \\
M &= 397^\circ\!.788 = 37^\circ\!.788 \\
E &= 49^\circ\!.897 \\
r &= 0''\!.74557 \\
v &= 63^\circ\!.416
\end{aligned}
$$

$$\tan(\theta - \Omega) = \frac{-0.500\,813}{+0.230\,440}$$

$$
\begin{aligned}
\theta - \Omega &= -65^\circ\!.291 \\
\theta &= -41^\circ\!.574 = 318^\circ\!.4 \\
\rho &= 0''\!.411
\end{aligned}
$$

As an exercise, calculate an ephemeris for γ Virginis, using the following elements [2]:

$$
\begin{aligned}
P &= 168.68 \text{ years} & i &= 148^\circ\!.0 \\
T &= 2005.13 & \Omega &= 36^\circ\!.9 \quad (2000.0) \\
e &= 0.885 & \omega &= 256^\circ\!.5 \\
a &= 3''\!.697
\end{aligned}
$$

Answer. — Here is an ephemeris with an interval of four years, starting at 1980. The position angle θ decreases with time, since i is between 90 and 180 degrees. Least apparent separation ($0''\!.36$) occurs at the epoch 2005.21. The position angles θ refer to the mean equinox of 2000.0, the same as for the angle Ω.

year	θ (°)	ρ (″)
year = 1980.0	θ = 296.65	ρ = 3.78
1984.0	293.10	3.43
1988.0	288.70	3.04
1992.0	282.89	2.60
1996.0	274.41	2.08
2000.0	259.34	1.45
2004.0	208.67	0.59
2008.0	35.54	1.04
2012.0	12.72	1.87

Eccentricity of the apparent orbit

The apparent orbit of a binary star is an ellipse whose eccentricity e' is generally different from the eccentricity e of the true orbit. It may be interesting to know e', although this apparent eccentricity has no astrophysical significance.

The following formulae have been derived by the author [3]:

$$A = (1 - e^2 \cos^2\omega) \cos^2 i$$
$$B = e^2 \sin\omega \cos\omega \cos i$$
$$C = 1 - e^2 \sin^2\omega$$
$$D = (A - C)^2 + 4B^2$$

$$e'^2 = \frac{2\sqrt{D}}{A + C + \sqrt{D}}$$

It should be noted that e' is independent of the orbital elements a and Ω, and that it can be smaller as well as larger than the true eccentricity e.

Example 57.b — Find the eccentricity of the apparent orbit of η Coronae Borealis. The orbital elements are given in Example 57.a.

We find

$A = 0.25298$ $B = 0.01934$ $C = 0.96858$ $D = 0.51358$

$e' = 0.860$

Hence, for this binary the apparent orbit is much more elongated than the true orbit.

REFERENCES

1. M. P. Greaney, "The Orbit of a Binary Star", *Sky and Telescope*, Vol. 74, No. 1, pages 71-72 (July 1987).
2. W. D. Heintz, "Orbits of 15 visual binaries", *Astronomy and Astrophysics, Supplement Series*, Vol. 82, pages 65-69 (1990).
3. J. Meeus, "The eccentricity of the apparent orbit of a binary star", *Journal of the British Astronomical Association*, Vol. 89, pages 485-488 (August 1979).

Chapter 58

Calculation of a Planar Sundial

by R. Sagot and D. Savoie (*)

One wishes to draw a planar sundial of any given orientation and inclination, provided with a straight stylus of length a perpendicular to its surface. Hence, this stylus generally is *not* directed towards the celestial pole. This sundial has the following principal parameters:

— the latitude φ of the place;

— the gnomonic declination D, that is, the azimuth of the perpendicular to the sundial's plane, measured from the southern meridian towards the west, from 0 to 360 degrees. So, if $D = 0°$, the sundial is "due south"; if $D = 270°$, it is "due east"; and so on;

— the zenithal distance z of the direction defined by the straight stylus. If $z = 0°$, the sundial is horizontal; in this case, D is meaningless — but see the special case later in this Chapter. If $z = 90°$, the sundial is vertical.

The coordinates x and y of the tip of the shadow of the straight stylus of length a are measured in an orthogonal coordinate system situated in the sundial's plane. The origin of this system coincides with the footprint of the stylus. The x-axis is horizontal, while the y-axis coincides with the line of greatest slope of the sundial. In all cases, x is measured positively towards the right, while y is positive upwards.

The Sun's hour angle H is measured from the upper meridian transit (true noon); it increases by 15 degrees per hour. For example, $H = -45°$ corresponds to 9 hours a.m. (true solar time), $H = +15°$ to 1 hour p.m., etc.

(*) Robert SAGOT and Denis SAVOIE are former president and president, respectively, of the "Commission des Cadrans Solaires" (Sundials Section) of the Société Astronomique de France. Denis SAVOIE is the author of *Gnomonique moderne* (in French), published by the Société Astronomique de France (1997), in which the interested reader can find a mathematical theory of sundials.

In the following formulae, for each hour angle H the declination δ of the Sun will take the successive values (in degrees) -23.44, -20.15, -11.47, 0, $+11.47$, $+20.15$, and $+23.44$, which correspond to the dates when the longitude of the Sun is a multiple of 30°.

In the course of a day, the tip of the shadow of the stylus will describe on the sundial's plane a curve which is a conic (a circle, an ellipse, a parabola, or an hyperbola). However, if $\delta = 0°$ the curve is always a straight line.

Calculate

$$P = \sin\varphi \cos z - \cos\varphi \sin z \cos D$$

$$Q = \sin D \sin z \sin H + (\cos\varphi \cos z + \sin\varphi \sin z \cos D) \cos H + P \tan\delta$$

$$N_x = \cos D \sin H - \sin D (\sin\varphi \cos H - \cos\varphi \tan\delta)$$

$$N_y = \cos z \sin D \sin H - (\cos\varphi \sin z - \sin\varphi \cos z \cos D) \cos H - (\sin\varphi \sin z + \cos\varphi \cos z \cos D) \tan\delta$$

Then the coordinates x and y are given by

$$x = a N_x / Q \qquad\qquad y = a N_y / Q$$

For each hour angle, one obtains a series of points. By connecting these points, an hour line is created on the sundial. The point (if it exists) to which the hour lines converge, is called the *center* of the sundial; it is also the point of fixation of the *polar stylus*, which is parallel to the Earth's axis of rotation. Its coordinates x_0 and y_0 are given by

$$x_0 = \frac{a}{P} \cos\varphi \sin D, \qquad\qquad y_0 = -\frac{a}{P} (\sin\varphi \sin z + \cos\varphi \cos z \cos D)$$

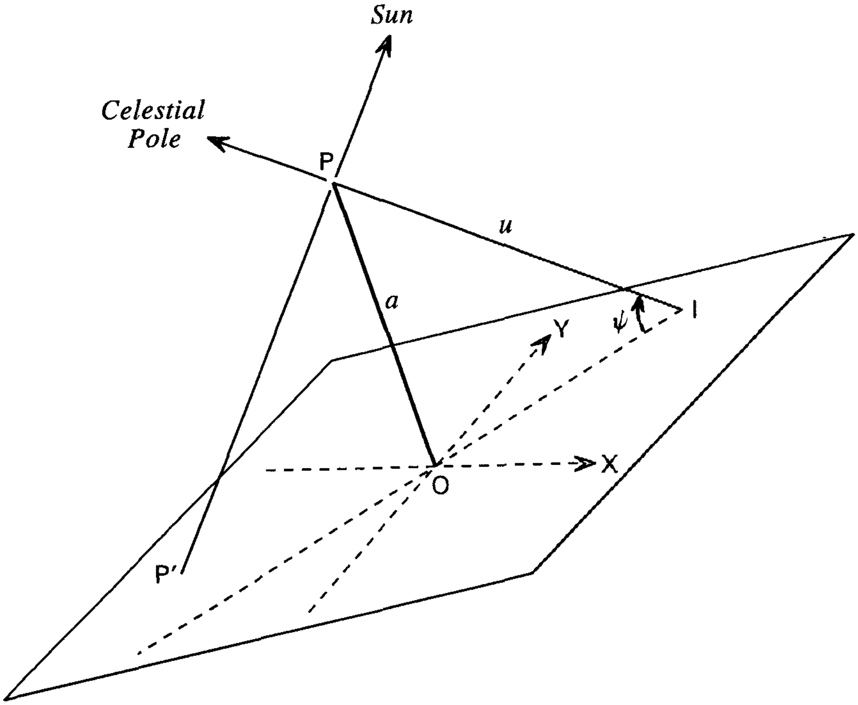
The length u of the polar stylus, from its point of fixation to the tip of the perpendicular stylus of length a, is

$$u = \frac{a}{|P|}$$

while the angle ψ which the polar stylus makes with the sundial's plane is given by

$$\sin\psi = |P|$$

The formulae for the position of the polar stylus become meaningless when $P = 0$, that is, when $\cos D \tan z = \tan\varphi$. This means that the polar stylus is then parallel to the plane of the sundial.



The plane represents the plane of the sundial. OP *is the perpendicular stylus, of length* a, *while* IP *is the polar stylus, length* u. P' *is the shadow* (x, y) *of the tip of the stylus. The point* I *is called the center of the sundial, while* O *is the origin of the* x–y *system.*

It is proper to limit the drawing of the sundial to the useful lines. For example, a vertical sundial oriented "due north" ($D = 180°$), at latitude $+40°$, can never show 11^h a.m., true solar time. At the same latitude, a vertical sundial oriented "due south" cannot indicate 19^h ($= 7^h$ p.m.) near the June solstice.

In order to make sure that the sundial really works, two conditions should be fulfilled: the Sun must be above the horizon, and the plane of the sundial must be illuminated. Consequently, it is necessary, for each calculated point (x, y), to verify whether these two conditions are satisfied simultaneously.

In practice, for a given arc of declination, the calculation should start at the moment of the geometric rise of the Sun, or at the first integer hour following that rise, and stop at the moment of the geometric sunset. The Sun's hour angle H_0 at the time of sunrise or sunset is given by

$$\cos H_0 = -\tan\varphi \tan\delta$$

with $H_0 < 0$ for sunrise, $H_0 > 0$ for sunset.

For each value of H, one should look at the sign of Q: if this quantity is negative, this means that the Sun does not illuminate the plane, and in that case one passes over to the next declination. Hence, only those values for which Q is positive must be retained.

It is possible that, on a given date, Q is at first positive, then becomes negative, and later is positive again.

Example 58.a — Consider an inclined sundial at latitude 40° North, with $D = 70°$, $z = 50°$, and $a = 1$. For $\delta = +23°.44$ (summer solstice), we have $H_0 = -111°.33$ (or 4^h35^m a.m., true solar time).

Beginning the calculations with $H = -105°$, we find $Q < 0$. This quantity is negative again for $H = -90°$, $-75°$, and $-60°$. Only from $H = -47°$ on is the sundial illuminated, and it will remain illuminated till sunset. Hence, if a step of 15 degrees has been chosen, the values of x and y should be calculated for $H = -45°$ to $+105°$.

For $H = +30°$ and $\delta = +23°.44$, we find $x = -0.0390$, $y = -0.3615$.
For $H = -15°$ and $\delta = -11°.47$, we find $x = -2.0007$, $y = -1.1069$.

The coordinates of the center are $x_0 = +3.3880$, $y_0 = -3.1102$, and we have $\psi = 12°.2672$.

Example 58.b — Consider a vertical sundial at latitude $\varphi = -35°$, with $D = 160°$, $z = 90°$, and $a = 1$.

For $\delta = 0°$ (equinox), we have $H_0 = -90°$ and $Q < 0$. Q becomes positive for $H = -57°$, so the calculations will be made for $H = -45°$ till sunset ($H_0 = +90°$).

For $H = +45°$ and $\delta = 0°$, we find $x = -0.8439$, $y = -0.9298$.

For $H = 0°$ and $\delta = +20°.15$, we find $x = +0.3640$, $y = -0.7410$.

The coordinates of the center are $x_0 = +0.3640$, $y_0 = +0.7451$, and we have $\psi = 50°.3315$.

Example 58.c — Inclined sundial at latitude 40° N, with $D = 160°$ and $z = 75°$.

For $\delta = +23°\!.44$, this sundial will be illuminated from sunrise (when $H = -111°$) until $H = -84°$. Then it will be illuminated again from $H = +2°$ until sunset ($H = +111°$). So, if a step of 15° has been chosen, the calculation will be made for $H = -105°$, $-90°$, and then for $+15°$ to $+105°$.

The formulae given above form the most general case which can occur in gnomonics. They allow the calculation of the classical hour lines of true solar time, but also the declination curves, the lines for mean time (when introducing the equation of time in the calculation of H), the lines for Universal Time or of zone time, azimuth and altitude lines, etc.

The formulae simplify greatly for some special cases, which we shall now examine briefly.

Special cases

(1) Equatorial sundial

The plane of this sundial is parallel to the plane of the equator and hence there are two sides: the northern side serves for the positive declinations (spring and summer), the southern side for the negative declinations of the Sun (autumn and winter). At a place of latitude φ, we have

for the northern side: $z = 90° - \varphi$ and $D = 180°$

for the southern side: $z = 90° + \varphi$ and $D = 0°$

The line of 12 hours ($H = 0°$) coincides with the line of greatest descending slope. Further,

$$Q = \pm \tan\delta \qquad x_0 = 0$$

$$x = -a\,\frac{\sin H}{\tan\delta} \qquad y_0 = 0$$

$$u = a$$

$$y = \mp a\,\frac{\cos H}{\tan\delta} \qquad \psi = 90°$$

where the upper sign is to be taken for the northern side, the lower sign for the southern side.

(2) Horizontal sundial

The plane of this sundial is horizontal, so $z = 0°$. The angle D is not defined and the direction of the x-axis can be chosen at will. We shall consider the case $D = 0°$, where the x-axis is directed towards the east, the y-axis towards the north. The formulae simplify to

$$Q = \cos\varphi \cos H + \sin\varphi \tan\delta \qquad x_0 = 0$$

$$x = a\,\frac{\sin H}{Q} \qquad y_0 = -\frac{a}{\tan\varphi}$$

$$y = a\,\frac{\sin\varphi \cos H - \cos\varphi \tan\delta}{Q} \qquad u = \frac{a}{|\sin\varphi|}$$

$$\psi = |\varphi|$$

(3) Vertical sundial

The plane of this sundial is vertical, so $z = 90°$. The x-axis is horizontal; the y-axis is directed towards the zenith. The formulae simplify to

$$Q = \sin D \sin H + \sin\varphi \cos D \cos H - \cos\varphi \cos D \tan\delta$$

$$x = a\,\frac{\cos D \sin H - \sin\varphi \sin D \cos H + \cos\varphi \sin D \tan\delta}{Q}$$

$$y = -a\,\frac{\cos\varphi \cos H + \sin\varphi \tan\varphi}{Q}$$

$$x_0 = -a \tan D \qquad u = \frac{a}{|\cos\varphi \cos D|}$$

$$y_0 = +a \tan\varphi / \cos D$$

General Remarks

In the case of a sundial with a perpendicular stylus, as considered here, it is the *extremity* of the umbra of that stylus which indicates the time, while in the case of a sundial with a polar stylus it is the entire umbra which gives the time.

Because we give the coordinates x_0, y_0 of the center of the sundial, it is always possible to construct the polar stylus IP, if this is wanted: the polar stylus is the straight line connecting that center with the extremity of the perpendicular stylus. See the Figure on page 403.

The advantage of the system of axes x-y used in this Chapter is that the perpendicular stylus does always exist; this is not always the case for the polar stylus.

Appendix I

Constants

Mathematical constants

$\pi = 3.14159\,26535\,89793\,23846\ldots\ldots$

$e = 2.71828\,18284\,59045\,23536\ldots\ldots$

1 radian $= 180/\pi$ degrees $= 57.295\,779\,513\,082$ degrees
$= 206\,264.806\,247$ arcseconds

1 degree $= \pi/180$ radian $= 0.017\,453\,292\,519\,943$ radian

$\log_{10} a = \log_e a / \log_e 10 = 0.434\,294\,481\,903 \log_e a$

Distances

1 *astronomical unit* (AU) $=$ 149 597 870 kilometers $=$ 499.0048 light-seconds
$=$ 8.32 light-minutes $=$ 0.005 77 55 183 light-day

1 *parsec* $= 30.8568 \times 10^{12}$ kilometers $=$ 3.2616 light-years $=$ 206 264.8 AU
$=$ the distance at which the length of one astronomical unit subtends an angle of $1''$. The name is a contraction of *par*allax and *sec*ond

1 *light-year* $= 9.4607 \times 10^{12}$ kilometers $=$ 0.30660 parsec $=$ 63 241 astron. units
$=$ the distance that light travels in one year (in vacuo)

Distance Earth–Moon (mean) $=$ 384 400 kilometers

Earth: equatorial radius $=$ 6378.14 km, polar radius $=$ 6356.76 km

Diameter of Sun $=$ 1392 000 km

Diameter of Moon $=$ 3476 km

Time

1 sidereal day = 23 hours 56 minutes 04.0905 seconds of mean solar time
= 0.997 269 5663 mean solar day

1 mean solar day = 1.002 737 909 35 sidereal days

Length of the year in mean solar days (*), for epoch 2000.0:

Tropical (equinox to equinox)	365.242 190
Sidereal (fixed star to fixed star)	365.256 363
Anomalistic (apse to apse)	365.259 636
Julian	365.25

Length of revolution period of Moon, in mean solar days (*):

Tropical (equinox to equinox)	27.321 582
Sidereal (fixed star to fixed star)	27.321 662
Anomalistic (apse to apse)	27.554 550
Draconic (node to node)	27.212 221
Synodic (New Moon to New Moon)	29.530 589

Varia

Mean obliquity of the ecliptic:
in 1900: 23°27′08″
in 1950: 23°26′45″
in 2000: 23°26′21″
in 2050: 23°25′58″

Eccentricity of Earth's orbit:
in 1900: 0.016751
in 2000: 0.016709
in 2100: 0.016666

General annual precession (in 365.25 days):
in 1900: 50″.269
in 2000: 50″.291
in 2100: 50″.313

Mean parallax of Sun = 8″.79415

Constant of aberration = 20″.4955

Flattening of the Earth = 1/298.257

Gaussian gravitational constant:
k = 0.017 202 098 95
or, converted from radians to degrees,
0.985 607 6686

Speed of light in vacuo
= 299 792.458 km/second

Earth-Moon mass ratio = 81.3007

Sun-Earth mass ratio = 332 946

(*) Or, more precisely, ephemeris days, in the uniform time scale of Dynamical Time. One ephemeris day is approximately equal to one mean solar day at epoch 1900.0.

Appendix II

Some Astronomical Terms

The following notes may be found helpful by those who are not familiar with the technical terms used in this book, but further guidance should be sought from textbooks on astronomy.

The ***celestial equator*** is the great circle that is the projection of the Earth's equator onto the celestial sphere. Its plane is perpendicular to the axis of rotation of the Earth.

The ***celestial poles*** are the poles of the celestial equator, or the intersections of the axis of rotation of the Earth with the celestial sphere.

The ***ecliptic*** is defined to be the plane of the (undisturbed) orbit of the Earth around the Sun.

The ***equinox*** or, better, the ***vernal equinox***, which is the zero point of both right ascension and celestial longitude, is defined to be in the direction of the ascending node of the ecliptic on the equator. It is that intersection of equator and ecliptic where the ecliptic runs (eastwards) from negative to positive declinations. The other intersection, which is diametrically opposite, is the ***autumnal equinox***.

The ***equinoxes*** are the instants when the Sun's apparent longitude is 0° or 180°.

Solstices: both the points on the ecliptic 90 degrees away from the equinoxes, and the instants when the apparent longitude of the Sun is 90° or 270°.

Celestial longitude, or ***ecliptical longitude***, often called simply ***longitude***, is measured (from 0° to 360°) from the vernal equinox, positive to the east, along the ecliptic.

Celestial latitude, or ***ecliptical latitude***, or simply ***latitude***, is measured (from 0° to +90° or to −90°) from the ecliptic, positive to the north, negative to the south.

Right ascension is measured (from 0 to 24 hours, sometimes from 0° to 360°) from the vernal equinox, positive to the east, along the celestial equator.

Declination is measured (from 0° to ±90°) from the equator, positive to the north, negative to the south.

Owing to the effects of ***precession*** and ***nutation***, the ecliptic and equator, and hence the equinoxes and the poles, are continuously in motion, and so the current celestial coordinates of a "fixed" direction change continuously. The motion of the equator is primarily due to the action of the Sun and the Moon, while the (much slower) motion of the ecliptic is primarily due to the perturbing action of the planets.

Mean equator: the instantaneous celestial equator exclusive of the periodic perturbations of the nutation.

Mean equator and equinox, or simply ***mean equinox***: an expression used to denote that the reference system takes into account the precession (secular effects) but not the nutation (periodic effects).

Coordinates: two (or three) numbers which define the position of a point on a surface (or in space). Examples: longitude and latitude are the two geographical coordinates of a point on the surface of the Earth; right ascension and declination; the rectangular coordinates X, Y, Z of a point in three-dimensional space.

Heliocentric: referred to the center of the Sun, for instance a heliocentric orbit, heliocentric coordinates.

Geocentric: referred to the center of the Earth, for instance a geocentric observer, geocentric coordinates.

Topocentric: referred to the observer on the Earth's surface, for example the topocentric right ascension and declination of the Moon.

Aberration is the apparent displacement of the position of an object due to the finite speed of light. The ***annual aberration*** of a star is due to the orbital motion of the Earth around the Sun (or, more exactly, around the barycenter of the solar system).

Azimuth: the angular distance measured from the South, positive to the West, along the horizon, to the vertical circle through the point in question. Navigators and meteorologists measure the azimuth from the North, positive to the East.

Ascending node: that intersection of the orbital plane with the reference plane where the latitudinal coordinate is increasing (going north). The other intersection is the ***descending node.***

Conjunction: that configuration of two celestial objects such that either their right ascensions or their celestial longitudes are equal.

Opposition: that configuration of two celestial objects such that their celestial longitudes differ by 180°. Most frequently used when one of the objects is the Sun.

Heliographic coordinate system: a coordinate system on the surface of the Sun.

Planetographic coordinate system: a coordinate system on the surface of a planet. In the case of Mars, the term ***areographic*** is generally used. For the Moon, the term is ***selenographic***. Compare with ***geographic*** for the Earth.

Epoch: a particular fixed instant used as a reference point on a time scale, such as B1950.0 or J2000.0.

A ***Julian century*** is a time interval of 36525 days.

An ***ephemeris day*** is equal to 86400 seconds in the uniform time scale known as Dynamical Time.

The ***sidereal time*** is the measure of time defined by the motion of the vernal equinox in hour angle; it is the hour angle of that equinox (at a given place and for a given instant). The ***true solar time*** is the local hour angle of the Sun. The ***mean solar time*** is the hour angle of the mean Sun, and thus is measured from mean noon. The ***civil time*** is the mean solar time increased by 12 hours, and thus is measured from mean midnight. — The expression "mean time measured from midnight" is a *contradictio in terminis*, since the mean (solar) time by definition is measured from noon. Many people erroneously use the expression "Greenwich Mean Time", when in fact Greenwich *Civil* Time is meant.

Universal Time is the civil time on the meridian of Greenwich.

The ***astronomical unit*** (AU) is a unit of length used to measure distances in the solar system. It is often called the "mean distance of the Earth to the Sun". But, rigorously, one AU is the radius of the circular orbit which a particle of negligible mass, and free of perturbations, would describe around the Sun with a period of $2\pi/k$ days, where k is the Gaussian gravitational constant, 0.01720209895. As a consequence, the semimajor axis of the elliptical orbit of the Earth is not exactly 1 AU, but 1.000001018 AU.

Radius vector: the straight line connecting a body to the central body around which it revolves, or the distance between these bodies at a given instant. The radius vector of a planet or comet is generally expressed in astronomical units.

Apsides (plural of *apse*): the points of intersection of the major axis with the orbit of a planet, a minor planet, a satellite, or a comet. These are the points of the orbit that are closest (perihelion, perigee, etc.) and farthest (aphelion, apogee, etc.) from the central body.

Perihelion: the point of the orbit (of a planet, minor planet, or comet) which is nearest to the Sun. For the corresponding point of the Moon's orbit with respect to the Earth, the term is ***perigee***. For a satellite of Jupiter with respect to this planet, the traditional term is ***perijove*** (*). For a double star, one says ***periastron***.

(*) The term *perijove* was already used by Laplace (1749-1827) and has become a classical term in astronomy. The word "periapse", used by some authors, is incorrect. The word *perihelion* means the point of the orbit that is closest to the Sun (from the Greek: *peri* = near + *helios* = Sun). Similarly, *perigee* is the point closest to the Earth (*ge* = Earth). Therefore, "periapse" would really mean the point closest to the apse; but this is ridiculous, because what is meant is the apse itself!

For the Moon, the terms *periselene* and *aposelene* seem the most appropriate; compare with *selenographic* and *selenocentric*. One should not create more neologisms, however. It would be absurd to speak of "periflore" for an orbit around minor planet Flora, or "perikosmodemyanskaya" for an orbit around minor planet 2072 Kosmodemyanskaya. For an orbit around another body than the Sun, Earth, Moon, or Jupiter, the best terms seem *periastron* and *apastron*, as for double stars.

The ***geometric*** position of a planet is the "true" position of that body at the given instant; that is, no allowance is being made for the effects of aberration and light-time.

Astrometric position: see page 230.

Anomalies. — The ***mean anomaly*** (M) of a planet is the angular distance, as seen from the Sun, between the perihelion and the mean position of the planet. The angular distance measured from the perihelion to the true position of the planet is called the ***true anomaly*** (v). The ***eccentric anomaly*** is an auxiliary quantity needed to obtain the true anomaly through solving Kepler's equation. The ***equation of the center*** is the difference between the true and the mean anomalies ($C = v - M$); it is the difference between the actual position of the body in its elliptic orbit and the position the body would have if its angular motion were uniform.

An ***ephemeris*** is a table of positions or other calculated data of a celestial body (Sun, Moon, planet, comet, etc.) for a series of (generally equidistant) instants. From the Greek ἐφημερος, *ephemeros = daily*.

Parallax: the difference in apparent direction of an object as seen from two different locations. For objects in the solar system (Sun, Moon, planet, asteroid, comet), the parallax is the difference in direction between a *topocentric* observation (by the actual observer at the Earth's surface) and a hypothetical *geocentric* observation. For the stars, the (annual) parallax is the difference between geocentric and heliocentric positions.

Arcminute (′) and ***arcsecond*** (″) are 1/60 and 1/3600, respectively, of a degree. Not to be confused with minute and second of *time* (1/60 and 1/3600 of an *hour*).

Appendix III

Planets : Periodic Terms

In this Appendix, pages 414–454, the most important periodic terms from the French planetary theory VSOP87 are given. The successive columns contain the following data:

— the name of the planet;

— the label of the series (L for the heliocentric longitude, B for the latitude, R for the radius vector);

— the current No. of the term in the series;

— the quantities A, B, and C, which all are positive (or zero).

In each series, the terms are sorted by decreasing values of A.

For example:

Planet	Series	No.	A	B	C
VENUS	R0	1	72334821	0	0
		2	489824	4.021518	10213.285546
		3	1658	4.9021	20426.5711
		4	1632	2.8455	7860.4194
		5	1378	1.1285	11790.6291
		6	498	2.587	9683.595
		7	374	1.423	3930.210
		8	264	5.529	9437.763
		9	237	2.551	15720.839
		10	222	2.013	19367.189
		11	126	2.728	1577.344
		12	119	3.020	10404.734
VENUS	R1	1	34551	0.89199	10213.28555
		2	234	1.772	20426.571
		3	234	3.142	0

For more explanation about the use of these terms, see Chapter 32.

MERCURY	L0	1	440 250 710	0	0
		2	40 989 415	1.483 020 34	26 087.903 141 57
		3	5 046 294	4.477 854 9	52 175.806 283 1
		4	855 347	1.165 203	78 263.709 425
		5	165 590	4.119 692	104 351.612 566
		6	34 562	0.779 31	130 439.515 71
		7	7 583	3.713 5	156 527.418 8
		8	3 560	1.512 0	1 109.378 6
		9	1 803	4.103 3	5 661.332 0
		10	1 726	0.358 3	182 615.322 0
		11	1 590	2.995 1	25 028.521 2
		12	1 365	4.599 2	27 197.281 7
		13	1 017	0.880 3	31 749.235 2
		14	714	1.541	24 978.525
		15	644	5.303	21 535.950
		16	451	6.050	51 116.424
		17	404	3.282	208 703.225
		18	352	5.242	20 426.571
		19	345	2.792	15 874.618
		20	343	5.765	955.600
		21	339	5.863	25 558.212
		22	325	1.337	53 285.185
		23	273	2.495	529.691
		24	264	3.917	57 837.138
		25	260	0.987	4 551.953
		26	239	0.113	1 059.382
		27	235	0.267	11 322.664
		28	217	0.660	13 521.751
		29	209	2.092	47 623.853
		30	183	2.629	27 043.503
		31	182	2.434	25 661.305
		32	176	4.536	51 066.428
		33	173	2.452	24 498.830
		34	142	3.360	37 410.567
		35	138	0.291	10 213.286
		36	125	3.721	39 609.655
		37	118	2.781	77 204.327
		38	106	4.206	19 804.827
MERCURY	L1	1	2608 814 706 223	0	0
		2	1 126 008	6.217 039 7	26 087.903 141 6
		3	303 471	3.055 655	52 175.806 283
		4	80 538	6.104 55	78 263.709 42
		5	21 245	2.835 32	104 351.612 57
		6	5 592	5.826 8	130 439.515 7
		7	1 472	2.518 5	156 527.418 8
		8	388	5.480	182 615.322
		9	352	3.052	1 109.379
		10	103	2.149	208 703.225
		11	94	6.12	27 197.28
		12	91	0.00	24 978.52
		13	52	5.62	5 661.33
		14	44	4.57	25 028.52
		15	28	3.04	51 066.43
		16	27	5.09	234 791.13
MERCURY	L2	1	53 050	0	0
		2	16 904	4.690 72	26 087.903 14
		3	7 397	1.347 4	52 175.806 3
		4	3 018	4.456 4	78 263.709 4
		5	1 107	1.262 3	104 351.612 6
		6	378	4.320	130 439.516
		7	123	1.069	156 527.419

MERCURY (cont.)	L2	8	39	4.08	182 615.32
		9	15	4.63	1 109.38
		10	12	0.79	208 703.23
MERCURY	L3	1	188	0.035	52 175.806
		2	142	3.125	26 087.903
		3	97	3.00	78 263.71
		4	44	6.02	104 351.61
		5	35	0	0
		6	18	2.78	130 439.52
		7	7	5.82	156 527.42
		8	3	2.57	182 615.32
MERCURY	L4	1	114	3.141 6	0
		2	3	2.03	26 087.90
		3	2	1.42	78 263.71
		4	2	4.50	52 175.81
		5	1	4.50	104 351.61
		6	1	1.27	130 439.52
MERCURY	L5	1	1	3.14	0
MERCURY	B0	1	11 737 529	1.983 574 99	26 087.903 141 57
		2	2 388 077	5.037 389 6	52 175.806 283 1
		3	1 222 840	3.141 592 7	0
		4	543 252	1.796 444	78 263.709 425
		5	129 779	4.832 325	104 351.612 566
		6	31 867	1.580 88	130 439.515 71
		7	7 963	4.609 7	156 527.418 8
		8	2 014	1.353 2	182 615.322 0
		9	514	4.378	208 703.225
		10	209	2.020	24 978.525
		11	208	4.918	27 197.282
		12	132	1.119	234 791.128
		13	121	1.813	53 285.185
		14	100	5.657	20 426.571
MERCURY	B1	1	429 151	3.501 698	26 087.903 142
		2	146 234	3.141 593	0
		3	22 675	0.015 15	52 175.806 28
		4	10 895	0.485 40	78 263.709 42
		5	6 353	3.429 4	104 351.612 6
		6	2 496	0.160 5	130 439.515 7
		7	860	3.185	156 527.419
		8	278	6.210	182 615.322
		9	86	2.95	208 703.23
		10	28	0.29	27 197.28
		11	26	5.98	234 791.13
MERCURY	B2	1	11 831	4.790 66	26 087.903 14
		2	1 914	0	0
		3	1 045	1.212 2	52 175.806 3
		4	266	4.434	78 263.709
		5	170	1.623	104 351.613
		6	96	4.80	130 439.52
		7	45	1.61	156 527.42
		8	18	4.67	182 615.32
		9	7	1.43	208 703.23
MERCURY	B3	1	235	0.354	26 087.903
		2	161	0	0
		3	19	4.36	52 175.81
		4	6	2.51	78 263.71

MERCURY (cont.)	B3	5	5	6.14	104 351.61
		6	3	3.12	130 439.52
		7	2	6.27	156 527.42
MERCURY	B4	1	4	1.75	26 087.90
		2	1	3.14	0
MERCURY	R0	1	39 528 272	0	0
		2	7 834 132	6.192 337 2	26 087.903 141 6
		3	795 526	2.959 897	52 175.806 283
		4	121 282	6.010 642	78 263.709 425
		5	21 922	2.778 20	104 351.612 57
		6	4 354	5.828 9	130 439.515 7
		7	918	2.597	156 527.419
		8	290	1.424	25 028.521
		9	260	3.028	27 197.282
		10	202	5.647	182 615.322
		11	201	5.592	31 749.235
		12	142	6.253	24 978.525
		13	100	3.734	21 535.950
MERCURY	R1	1	217 348	4.656 172	26 087.903 142
		2	44 142	1.423 86	52 175.806 28
		3	10 094	4.474 66	78 263.709 42
		4	2 433	1.242 3	104 351.612 6
		5	1 624	0	0
		6	604	4.293	130 439.516
		7	153	1.061	156 527.419
		8	39	4.11	182 615.32
MERCURY	R2	1	3 118	3.082 3	26 087.903 1
		2	1 245	6.151 8	52 175.806 3
		3	425	2.926	78 263.709
		4	136	5.980	104 351.613
		5	42	2.75	130 439.52
		6	22	3.14	0
		7	13	5.80	156 527.42
MERCURY	R3	1	33	1.68	26 087.90
		2	24	4.63	52 175.81
		3	12	1.39	78 263.71
		4	5	4.44	104 351.61
		5	2	1.21	130 439.52
VENUS	L0	1	317 614 667	0	0
		2	1 353 968	5.593 133 2	10 213.285 546 2
		3	89 892	5.306 50	20 426.571 09
		4	5 477	4.416 3	7 860.419 4
		5	3 456	2.699 6	11 790.629 1
		6	2 372	2.993 8	3 930.209 7
		7	1 664	4.250 2	1 577.343 5
		8	1 438	4.157 5	9 683.594 6
		9	1 317	5.186 7	26.298 3
		10	1 201	6.153 6	30 639.856 6
		11	769	0.816	9 437.763
		12	761	1.950	529.691
		13	708	1.065	775.523
		14	585	3.998	191.448
		15	500	4.123	15 720.839
		16	429	3.586	19 367.189
		17	327	5.677	5 507.553
		18	326	4.591	10 404.734

VENUS (cont.)	L0	19	232	3.163	9153.904
		20	180	4.653	1109.379
		21	155	5.570	19651.048
		22	128	4.226	20.775
		23	128	0.962	5661.332
		24	106	1.537	801.821
VENUS	L1	1	1021 352 943 053	0	0
		2	95 708	2.464 24	10 213.285 55
		3	14 445	0.516 25	20 426.571 09
		4	213	1.795	30 639.857
		5	174	2.655	26.298
		6	152	6.106	1 577.344
		7	82	5.70	191.45
		8	70	2.68	9 437.76
		9	52	3.60	775.52
		10	38	1.03	529.69
		11	30	1.25	5 507.55
		12	25	6.11	10 404.73
VENUS	L2	1	54 127	0	0
		2	3 891	0.345 1	10 213.285 5
		3	1 338	2.020 1	20 426.571 1
		4	24	2.05	26.30
		5	19	3.54	30 639.86
		6	10	3.97	775.52
		7	7	1.52	1 577.34
		8	6	1.00	191.45
VENUS	L3	1	136	4.804	10 213.286
		2	78	3.67	20 426.57
		3	26	0	0
VENUS	L4	1	114	3.141 6	0
		2	3	5.21	20 426.57
		3	2	2.51	10 213.29
VENUS	L5	1	1	3.14	0
VENUS	B0	1	5 923 638	0.267 027 8	10 213.285 546 2
		2	40 108	1.147 37	20 426.571 09
		3	32 815	3.141 59	0
		4	1 011	1.089 5	30 639.856 6
		5	149	6.254	18 073.705
		6	138	0.860	1 577.344
		7	130	3.672	9 437.763
		8	120	3.705	2 352.866
		9	108	4.539	22 003.915
VENUS	B1	1	513 348	1.803 643	10 213.285 546
		2	4 380	3.386 2	20 426.571 1
		3	199	0	0
		4	197	2.530	30 639.857
VENUS	B2	1	22 378	3.385 09	10 213.285 55
		2	282	0	0
		3	173	5.256	20 426.571
		4	27	3.87	30 639.86
VENUS	B3	1	647	4.992	10 213.286
		2	20	3.14	0
		3	6	0.77	20 426.57
		4	3	5.44	30 639.86

VENUS	B4	1	14	0.32	10 213.29
VENUS	R0	1	72 334 821	0	0
		2	489 824	4.021 518	10 213.285 546
		3	1 658	4.902 1	20 426.571 1
		4	1 632	2.845 5	7 860.419 4
		5	1 378	1.128 5	11 790.629 1
		6	498	2.587	9 683.595
		7	374	1.423	3 930.210
		8	264	5.529	9 437.763
		9	237	2.551	15 720.839
		10	222	2.013	19 367.189
		11	126	2.728	1 577.344
		12	119	3.020	10 404.734
VENUS	R1	1	34 551	0.891 99	10 213.285 55
		2	234	1.772	20 426.571
		3	234	3.142	0
VENUS	R2	1	1 407	5.063 7	10 213.285 5
		2	16	5.47	20 426.57
		3	13	0	0
VENUS	R3	1	50	3.22	10 213.29
VENUS	R4	1	1	0.92	10 213.29
EARTH	L0	1	175 347 046	0	0
		2	3 341 656	4.669 256 8	6 283.075 850 0
		3	34 894	4.626 10	12 566.151 70
		4	3 497	2.744 1	5 753.384 9
		5	3 418	2.828 9	3.523 1
		6	3 136	3.627 7	77 713.771 5
		7	2 676	4.418 1	7 860.419 4
		8	2 343	6.135 2	3 930.209 7
		9	1 324	0.742 5	11 506.769 8
		10	1 273	2.037 1	529.691 0
		11	1 199	1.109 6	1 577.343 5
		12	990	5.233	5 884.927
		13	902	2.045	26.298
		14	857	3.508	398.149
		15	780	1.179	5 223.694
		16	753	2.533	5 507.553
		17	505	4.583	18 849.228
		18	492	4.205	775.523
		19	357	2.920	0.067
		20	317	5.849	11 790.629
		21	284	1.899	796.298
		22	271	0.315	10 977.079
		23	243	0.345	5 486.778
		24	206	4.806	2 544.314
		25	205	1.869	5 573.143
		26	202	2.458	6 069.777
		27	156	0.833	213.299
		28	132	3.411	2 942.463
		29	126	1.083	20.775
		30	115	0.645	0.980
		31	103	0.636	4 694.003
		32	102	0.976	15 720.839
		33	102	4.267	7.114
		34	99	6.21	2 146.17
		35	98	0.68	155.42
		36	86	5.98	161 000.69

EARTH (cont.)	L0	37	85	1.30	6 275.96
		38	85	3.67	71 430.70
		39	80	1.81	17 260.15
		40	79	3.04	12 036.46
		41	75	1.76	5 088.63
		42	74	3.50	3 154.69
		43	74	4.68	801.82
		44	70	0.83	9 437.76
		45	62	3.98	8 827.39
		46	61	1.82	7 084.90
		47	57	2.78	6 286.60
		48	56	4.39	14 143.50
		49	56	3.47	6 279.55
		50	52	0.19	12 139.55
		51	52	1.33	1 748.02
		52	51	0.28	5 856.48
		53	49	0.49	1 194.45
		54	41	5.37	8 429.24
		55	41	2.40	19 651.05
		56	39	6.17	10 447.39
		57	37	6.04	10 213.29
		58	37	2.57	1 059.38
		59	36	1.71	2 352.87
		60	36	1.78	6 812.77
		61	33	0.59	17 789.85
		62	30	0.44	83 996.85
		63	30	2.74	1 349.87
		64	25	3.16	4 690.48
EARTH	L1	1	628 331 966 747	0	0
		2	206 059	2.678 235	6 283.075 850
		3	4 303	2.635 1	12 566.151 7
		4	425	1.590	3.523
		5	119	5.796	26.298
		6	109	2.966	1 577.344
		7	93	2.59	18 849.23
		8	72	1.14	529.69
		9	68	1.87	398.15
		10	67	4.41	5 507.55
		11	59	2.89	5 223.69
		12	56	2.17	155.42
		13	45	0.40	796.30
		14	36	0.47	775.52
		15	29	2.65	7.11
		16	21	5.34	0.98
		17	19	1.85	5 486.78
		18	19	4.97	213.30
		19	17	2.99	6 275.96
		20	16	0.03	2 544.31
		21	16	1.43	2 146.17
		22	15	1.21	10 977.08
		23	12	2.83	1 748.02
		24	12	3.26	5 088.63
		25	12	5.27	1 194.45
		26	12	2.08	4 694.00
		27	11	0.77	553.57
		28	10	1.30	6 286.60
		29	10	4.24	1 349.87
		30	9	2.70	242.73
		31	9	5.64	951.72
		32	8	5.30	2 352.87
		33	6	2.65	9 437.76
		34	6	4.67	4 690.48

EARTH	L2	1	52919	0	0
		2	8720	1.0721	6283.0758
		3	309	0.867	12566.152
		4	27	0.05	3.52
		5	16	5.19	26.30
		6	16	3.68	155.42
		7	10	0.76	18849.23
		8	9	2.06	77713.77
		9	7	0.83	775.52
		10	5	4.66	1577.34
		11	4	1.03	7.11
		12	4	3.44	5573.14
		13	3	5.14	796.30
		14	3	6.05	5507.55
		15	3	1.19	242.73
		16	3	6.12	529.69
		17	3	0.31	398.15
		18	3	2.28	553.57
		19	2	4.38	5223.69
		20	2	3.75	0.98
EARTH	L3	1	289	5.844	6283.076
		2	35	0	0
		3	17	5.49	12566.15
		4	3	5.20	155.42
		5	1	4.72	3.52
		6	1	5.30	18849.23
		7	1	5.97	242.73
EARTH	L4	1	114	3.142	0
		2	8	4.13	6283.08
		3	1	3.84	12566.15
EARTH	L5	1	1	3.14	0
EARTH	B0	1	280	3.199	84334.662
		2	102	5.422	5507.553
		3	80	3.88	5223.69
		4	44	3.70	2352.87
		5	32	4.00	1577.34
EARTH	B1	1	9	3.90	5507.55
		2	6	1.73	5223.69
EARTH	R0	1	100013989	0	0
		2	1670700	3.0984635	6283.0758500
		3	13956	3.05525	12566.15170
		4	3084	5.1985	77713.7715
		5	1628	1.1739	5753.3849
		6	1576	2.8469	7860.4194
		7	925	5.453	11506.770
		8	542	4.564	3930.210
		9	472	3.661	5884.927
		10	346	0.964	5507.553
		11	329	5.900	5223.694
		12	307	0.299	5573.143
		13	243	4.273	11790.629
		14	212	5.847	1577.344
		15	186	5.022	10977.079
		16	175	3.012	18849.228
		17	110	5.055	5486.778
		18	98	0.89	6069.78
		19	86	5.69	15720.84

EARTH (cont.)	R0	20	86	1.27	161 000.69
		21	65	0.27	17 260.15
		22	63	0.92	529.69
		23	57	2.01	83 996.85
		24	56	5.24	71 430.70
		25	49	3.25	2 544.31
		26	47	2.58	775.52
		27	45	5.54	9 437.76
		28	43	6.01	6 275.96
		29	39	5.36	4 694.00
		30	38	2.39	8 827.39
		31	37	0.83	19 651.05
		32	37	4.90	12 139.55
		33	36	1.67	12 036.46
		34	35	1.84	2 942.46
		35	33	0.24	7 084.90
		36	32	0.18	5 088.63
		37	32	1.78	398.15
		38	28	1.21	6 286.60
		39	28	1.90	6 279.55
		40	26	4.59	10 447.39
EARTH	R1	1	103 019	1.107 490	6 283.075 850
		2	1 721	1.0644	12 566.151 7
		3	702	3.142	0
		4	32	1.02	18 849.23
		5	31	2.84	5 507.55
		6	25	1.32	5 223.69
		7	18	1.42	1 577.34
		8	10	5.91	10 977.08
		9	9	1.42	6 275.96
		10	9	0.27	5 486.78
EARTH	R2	1	4 359	5.7846	6 283.0758
		2	124	5.579	12 566.152
		3	12	3.14	0
		4	9	3.63	77 713.77
		5	6	1.87	5 573.14
		6	3	5.47	18 849.23
EARTH	R3	1	145	4.273	6 283.076
		2	7	3.92	12 566.15
EARTH	R4	1	4	2.56	6 283.08
MARS	L0	1	620 347 712	0	0
		2	18 656 368	5.050 371 00	3 340.612 426 70
		3	1 108 217	5.400 998 4	6 681.224 853 4
		4	91 798	5.754 79	10 021.837 28
		5	27 745	5.970 50	3.523 12
		6	12 316	0.849 56	2 810.921 46
		7	10 610	2.939 59	2 281.230 50
		8	8 927	4.1570	0.0173
		9	8 716	6.1101	13 362.4497
		10	7 775	3.3397	5 621.8429
		11	6 798	0.3646	398.1490
		12	4 161	0.2281	2 942.4634
		13	3 575	1.6619	2 544.3144
		14	3 075	0.8570	191.4483
		15	2 938	6.0789	0.0673
		16	2 628	0.6481	3 337.0893

MARS (cont.)	L0	17	2580	0.0300	3344.1355
		18	2389	5.0390	796.2980
		19	1799	0.6563	529.6910
		20	1546	2.9158	1751.5395
		21	1528	1.1498	6151.5339
		22	1286	3.0680	2146.1654
		23	1264	3.6228	5092.1520
		24	1025	3.6933	8962.4553
		25	892	0.183	16703.062
		26	859	2.401	2914.014
		27	833	4.495	3340.630
		28	833	2.464	3340.595
		29	749	3.822	155.420
		30	724	0.675	3738.761
		31	713	3.663	1059.382
		32	655	0.489	3127.313
		33	636	2.922	8432.764
		34	553	4.475	1748.016
		35	550	3.810	0.980
		36	472	3.625	1194.447
		37	426	0.554	6283.076
		38	415	0.497	213.299
		39	312	0.999	6677.702
		40	307	0.381	6684.748
		41	302	4.486	3532.061
		42	299	2.783	6254.627
		43	293	4.221	20.775
		44	284	5.769	3149.164
		45	281	5.882	1349.867
		46	274	0.542	3340.545
		47	274	0.134	3340.680
		48	239	5.372	4136.910
		49	236	5.755	3333.499
		50	231	1.282	3870.303
		51	221	3.505	382.897
		52	204	2.821	1221.849
		53	193	3.357	3.590
		54	189	1.491	9492.146
		55	179	1.006	951.718
		56	174	2.414	553.569
		57	172	0.439	5486.778
		58	160	3.949	4562.461
		59	144	1.419	135.065
		60	140	3.326	2700.715
		61	138	4.301	7.114
		62	131	4.045	12303.068
		63	128	2.208	1592.596
		64	128	1.807	5088.629
		65	117	3.128	7903.073
		66	113	3.701	1589.073
		67	110	1.052	242.729
		68	105	0.785	8827.390
		69	100	3.243	11773.377
MARS	L1	1	334085627474	0	0
		2	1458227	3.6042605	3340.6124267
		3	164901	3.926313	6681.224853
		4	19963	4.26594	10021.83728
		5	3452	4.7321	3.5231
		6	2485	4.6128	13362.4497
		7	842	4.459	2281.230
		8	538	5.016	398.149

MARS (cont.)	L1	9	521	4.994	3 344.136
		10	433	2.561	191.448
		11	430	5.316	155.420
		12	382	3.539	796.298
		13	314	4.963	16 703.062
		14	283	3.160	2 544.314
		15	206	4.569	2 146.165
		16	169	1.329	3 337.089
		17	158	4.185	1 751.540
		18	134	2.233	0.980
		19	134	5.974	1 748.016
		20	118	6.024	6 151.534
		21	117	2.213	1 059.382
		22	114	2.129	1 194.447
		23	114	5.428	3 738.761
		24	91	1.10	1 349.87
		25	85	3.91	553.57
		26	83	5.30	6 684.75
		27	81	4.43	529.69
		28	80	2.25	8 962.46
		29	73	2.50	951.72
		30	73	5.84	242.73
		31	71	3.86	2 914.01
		32	68	5.02	382.90
		33	65	1.02	3 340.60
		34	65	3.05	3 340.63
		35	62	4.15	3 149.16
		36	57	3.89	4 136.91
		37	48	4.87	213.30
		38	48	1.18	3 333.50
		39	47	1.31	3 185.19
		40	41	0.71	1 592.60
		41	40	2.73	7.11
		42	40	5.32	20 043.67
		43	33	5.41	6 283.08
		44	28	0.05	9 492.15
		45	27	3.89	1 221.85
		46	27	5.11	2 700.72
MARS	L2	1	58 016	2.049 79	3 340.612 43
		2	54 188	0	0
		3	13 908	2.457 42	6 681.224 85
		4	2 465	2.800 0	10 021.837 3
		5	398	3.141	13 362.450
		6	222	3.194	3.523
		7	121	0.543	155.420
		8	62	3.49	16 703.06
		9	54	3.54	3 344.14
		10	34	6.00	2 281.23
		11	32	4.14	191.45
		12	30	2.00	796.30
		13	23	4.33	242.73
		14	22	3.45	398.15
		15	20	5.42	553.57
		16	16	0.66	0.98
		17	16	6.11	2 146.17
		18	16	1.22	1 748.02
		19	15	6.10	3 185.19
		20	14	4.02	951.72
		21	14	2.62	1 349.87
		22	13	0.60	1 194.45
		23	12	3.86	6 684.75

MARS (cont.)	L2	24	11	4.72	2 544.31
		25	10	0.25	382.90
		26	9	0.68	1 059.38
		27	9	3.83	20 043.67
		28	9	3.88	3 738.76
		29	8	5.46	1 751.54
		30	7	2.58	3 149.16
		31	7	2.38	4 136.91
		32	6	5.48	1 592.60
		33	6	2.34	3 097.88
MARS	L3	1	1 482	0.4443	3 340.6124
		2	662	0.885	6 681.225
		3	188	1.288	10 021.837
		4	41	1.65	13 362.45
		5	26	0	0
		6	23	2.05	155.42
		7	10	1.58	3.52
		8	8	2.00	16 703.06
		9	5	2.82	242.73
		10	4	2.02	3 344.14
		11	3	4.59	3 185.19
		12	3	0.65	553.57
MARS	L4	1	114	3.1416	0
		2	29	5.64	6 681.22
		3	24	5.14	3 340.61
		4	11	6.03	10 021.84
		5	3	0.13	13 362.45
		6	3	3.56	155.42
		7	1	0.49	16 703.06
		8	1	1.32	242.73
MARS	L5	1	1	3.14	0
		2	1	4.04	6 681.22
MARS	B0	1	3 197 135	3.7683204	3 340.6124267
		2	298 033	4.106170	6 681.224853
		3	289 105	0	0
		4	31 366	4.44651	10 021.83728
		5	3 484	4.7881	13 362.4497
		6	443	5.026	3 344.136
		7	443	5.652	3 337.089
		8	399	5.131	16 703.062
		9	293	3.793	2 281.230
		10	182	6.136	6 151.534
		11	163	4.264	529.691
		12	160	2.232	1 059.382
		13	149	2.165	5 621.843
		14	143	1.182	3 340.595
		15	143	3.213	3 340.630
		16	139	2.418	8 962.455
MARS	B1	1	350 069	5.368478	3 340.612427
		2	14 116	3.14159	0
		3	9 671	5.4788	6 681.2249
		4	1 472	3.2021	10 021.8373
		5	426	3.408	13 362.450
		6	102	0.776	3 337.089
		7	79	3.72	16 703.06
		8	33	3.46	5 621.84
		9	26	2.48	2 281.23

MARS	B2	1	16727	0.60221	3340.61243
		2	4987	3.1416	0
		3	302	5.559	6681.225
		4	26	1.90	13362.45
		5	21	0.92	10021.84
		6	12	2.24	3337.09
		7	8	2.25	16703.06
MARS	B3	1	607	1.981	3340.612
		2	43	0	0
		3	14	1.80	6681.22
		4	3	3.45	10021.84
MARS	B4	1	13	0	0
		2	11	3.46	3340.61
		3	1	0.50	6681.22
MARS	R0	1	153033488	0	0
		2	14184953	3.47971284	3340.61242670
		3	660776	3.817834	6681.224853
		4	46179	4.15595	10021.83728
		5	8110	5.5596	2810.9215
		6	7485	1.7724	5621.8429
		7	5523	1.3644	2281.2305
		8	3825	4.4941	13362.4497
		9	2484	4.9255	2942.4634
		10	2307	0.0908	2544.3144
		11	1999	5.3606	3337.0893
		12	1960	4.7425	3344.1355
		13	1167	2.1126	5092.1520
		14	1103	5.0091	398.1490
		15	992	5.839	6151.534
		16	899	4.408	529.691
		17	807	2.102	1059.382
		18	798	3.448	796.298
		19	741	1.499	2146.165
		20	726	1.245	8432.764
		21	692	2.134	8962.455
		22	633	0.894	3340.595
		23	633	2.924	3340.630
		24	630	1.287	1751.540
		25	574	0.829	2914.014
		26	526	5.383	3738.761
		27	473	5.199	3127.313
		28	348	4.832	16703.062
		29	284	2.907	3532.061
		30	280	5.257	6283.076
		31	276	1.218	6254.627
		32	275	2.908	1748.016
		33	270	3.764	5884.927
		34	239	2.037	1194.447
		35	234	5.105	5486.778
		36	228	3.255	6872.673
		37	223	4.199	3149.164
		38	219	5.583	191.448
		39	208	5.255	3340.545
		40	208	4.846	3340.680
		41	186	5.699	6677.702
		42	183	5.081	6684.748
		43	179	4.184	3333.499
		44	176	5.953	3870.303
		45	164	3.799	4136.910

MARS	R1	1	1 107 433	2.032 505 2	3 340.612 426 7
		2	103 176	2.370 718	6 681.224 853
		3	12 877	0	0
		4	10 816	2.708 88	10 021.837 28
		5	1 195	3.047 0	13 362.449 7
		6	439	2.888	2 281.230
		7	396	3.423	3 344.136
		8	183	1.584	2 544.314
		9	136	3.385	16 703.062
		10	128	6.043	3 337.089
		11	128	0.630	1 059.382
		12	127	1.954	796.298
		13	118	2.998	2 146.165
		14	88	3.42	398.15
		15	83	3.86	3 738.76
		16	76	4.45	6 151.53
		17	72	2.76	529.69
		18	67	2.55	1 751.54
		19	66	4.41	1 748.02
		20	58	0.54	1 194.45
		21	54	0.68	8 962.46
		22	51	3.73	6 684.75
		23	49	5.73	3 340.60
		24	49	1.48	3 340.63
		25	48	2.58	3 149.16
		26	48	2.29	2 914.01
		27	39	2.32	4 136.91
MARS	R2	1	44 242	0.479 31	3 340.612 43
		2	8 138	0.870 0	6 681.224 9
		3	1 275	1.225 9	10 021.837 3
		4	187	1.573	13 362.450
		5	52	3.14	0
		6	41	1.97	3 344.14
		7	27	1.92	16 703.06
		8	18	4.43	2 281.23
		9	12	4.53	3 185.19
		10	10	5.39	1 059.38
		11	10	0.42	796.30
MARS	R3	1	1 113	5.149 9	3 340.612 4
		2	424	5.613	6 681.225
		3	100	5.997	10 021.837
		4	20	0.08	13 362.45
		5	5	3.14	0
		6	3	0.43	16 703.06
MARS	R4	1	20	3.58	3 340.61
		2	16	4.05	6 681.22
		3	6	4.46	10 021.84
		4	2	4.84	13 362.45
JUPITER	L0	1	59 954 691	0	0
		2	9 695 899	5.061 917 9	529.690 965 1
		3	573 610	1.444 062	7.113 547
		4	306 389	5.417 347	1 059.381 930
		5	97 178	4.142 65	632.783 74
		6	72 903	3.640 43	522.577 42
		7	64 264	3.411 45	103.092 77
		8	39 806	2.293 77	419.484 64
		9	38 858	1.272 32	316.391 87
		10	27 965	1.784 55	536.804 51

JUPITER (cont.)	L0	11	13 590	5.774 81	1 589.072 90
		12	8 769	3.630 0	949.175 6
		13	8 246	3.582 3	206.185 5
		14	7 368	5.081 0	735.876 5
		15	6 263	0.025 0	213.299 1
		16	6 114	4.513 2	1 162.474 7
		17	5 305	4.186 3	1 052.268 4
		18	5 305	1.306 7	14.227 1
		19	4 905	1.320 8	110.206 3
		20	4 647	4.699 6	3.932 2
		21	3 045	4.316 8	426.598 2
		22	2 610	1.566 7	846.082 8
		23	2 028	1.063 8	3.181 4
		24	1 921	0.971 7	639.897 3
		25	1 765	2.141 5	1 066.495 5
		26	1 723	3.880 4	1 265.567 5
		27	1 633	3.582 0	515.463 9
		28	1 432	4.296 8	625.670 2
		29	973	4.098	95.979
		30	884	2.437	412.371
		31	733	6.085	838.969
		32	731	3.806	1 581.959
		33	709	1.293	742.990
		34	692	6.134	2 118.764
		35	614	4.109	1 478.867
		36	582	4.540	309.278
		37	495	3.756	323.505
		38	441	2.958	454.909
		39	417	1.036	2.448
		40	390	4.897	1 692.166
		41	376	4.703	1 368.660
		42	341	5.715	533.623
		43	330	4.740	0.048
		44	262	1.877	0.963
		45	261	0.820	380.128
		46	257	3.724	199.072
		47	244	5.220	728.763
		48	235	1.227	909.819
		49	220	1.651	543.918
		50	207	1.855	525.759
		51	202	1.807	1 375.774
		52	197	5.293	1 155.361
		53	175	3.730	942.062
		54	175	3.226	1 898.351
		55	175	5.910	956.289
		56	158	4.365	1 795.258
		57	151	3.906	74.782
		58	149	4.377	1 685.052
		59	141	3.136	491.558
		60	138	1.318	1 169.588
		61	131	4.169	1 045.155
		62	117	2.500	1 596.186
		63	117	3.389	0.521
		64	106	4.554	526.510
JUPITER	L1	1	52 993 480 757	0	0
		2	489 741	4.220 667	529.690 965
		3	228 919	6.026 475	7.113 547
		4	27 655	4.572 66	1 059.381 93
		5	20 721	5.459 39	522.577 42
		6	12 106	0.169 86	536.804 51
		7	6 068	4.424 2	103.092 8
		8	5 434	3.984 8	419.484 6

JUPITER (cont.)	L1	9	4238	5.8901	14.2271
		10	2212	5.2677	206.1855
		11	1746	4.9267	1589.0729
		12	1296	5.5513	3.1814
		13	1173	5.8565	1052.2684
		14	1163	0.5145	3.9322
		15	1099	5.3070	515.4639
		16	1007	0.4648	735.8765
		17	1004	3.1504	426.5982
		18	848	5.758	110.206
		19	827	4.803	213.299
		20	816	0.586	1066.495
		21	725	5.518	639.897
		22	568	5.989	625.670
		23	474	4.132	412.371
		24	413	5.737	95.979
		25	345	4.242	632.784
		26	336	3.732	1162.475
		27	234	4.035	949.176
		28	234	6.243	309.278
		29	199	1.505	838.969
		30	195	2.219	323.505
		31	187	6.086	742.990
		32	184	6.280	543.918
		33	171	5.417	199.072
		34	131	0.626	728.763
		35	115	0.680	846.083
		36	115	5.286	2118.764
		37	108	4.493	956.289
		38	80	5.82	1045.15
		39	72	5.34	942.06
		40	70	5.97	532.87
		41	67	5.73	21.34
		42	66	0.13	526.51
		43	65	6.09	1581.96
		44	59	0.59	1155.36
		45	58	0.99	1596.19
		46	57	5.97	1169.59
		47	57	1.41	533.62
		48	55	5.43	10.29
		49	52	5.73	117.32
		50	52	0.23	1368.66
		51	50	6.08	525.76
		52	47	3.63	1478.87
		53	47	0.51	1265.57
		54	40	4.16	1692.17
		55	34	0.10	302.16
		56	33	5.04	220.41
		57	32	5.37	508.35
		58	29	5.42	1272.68
		59	29	3.36	4.67
		60	29	0.76	88.87
		61	25	1.61	831.86
JUPITER	L2	1	47234	4.32148	7.11355
		2	38966	0	0
		3	30629	2.93021	529.69097
		4	3189	1.0550	522.5774
		5	2729	4.8455	536.8045
		6	2723	3.4141	1059.3819
		7	1721	4.1873	14.2271
		8	383	5.768	419.485
		9	378	0.760	515.464

JUPITER (cont.)	L2	10	367	6.055	103.093
		11	337	3.786	3.181
		12	308	0.694	206.186
		13	218	3.814	1 589.073
		14	199	5.340	1 066.495
		15	197	2.484	3.932
		16	156	1.406	1 052.268
		17	146	3.814	639.897
		18	142	1.634	426.598
		19	130	5.837	412.371
		20	117	1.414	625.670
		21	97	4.03	110.21
		22	91	1.11	95.98
		23	87	2.52	632.78
		24	79	4.64	543.92
		25	72	2.22	735.88
		26	58	0.83	199.07
		27	57	3.12	213.30
		28	49	1.67	309.28
		29	40	4.02	21.34
		30	40	0.62	323.51
		31	36	2.33	728.76
		32	29	3.61	10.29
		33	28	3.24	838.97
		34	26	4.50	742.99
		35	26	2.51	1 162.47
		36	25	1.22	1 045.15
		37	24	3.01	956.29
		38	19	4.29	532.87
		39	18	0.81	508.35
		40	17	4.20	2 118.76
		41	17	1.83	526.51
		42	15	5.81	1 596.19
		43	15	0.68	942.06
		44	15	4.00	117.32
		45	14	5.95	316.39
		46	14	1.80	302.16
		47	13	2.52	88.87
		48	13	4.37	1 169.59
		49	11	4.44	525.76
		50	10	1.72	1 581.96
		51	9	2.18	1 155.36
		52	9	3.29	220.41
		53	9	3.32	831.86
		54	8	5.76	846.08
		55	8	2.71	533.62
		56	7	2.18	1 265.57
		57	6	0.50	949.18
JUPITER	L3	1	6 502	2.5986	7.1135
		2	1 357	1.3464	529.6910
		3	471	2.475	14.227
		4	417	3.245	536.805
		5	353	2.974	522.577
		6	155	2.076	1 059.382
		7	87	2.51	515.46
		8	44	0	0
		9	34	3.83	1 066.50
		10	28	2.45	206.19
		11	24	1.28	412.37
		12	23	2.98	543.92
		13	20	2.10	639.90
		14	20	1.40	419.48

JUPITER (cont.)	L3	15	19	1.59	103.09
		16	17	2.30	21.34
		17	17	2.60	1589.07
		18	16	3.15	625.67
		19	16	3.36	1052.27
		20	13	2.76	95.98
		21	13	2.54	199.07
		22	13	6.27	426.60
		23	9	1.76	10.29
		24	9	2.27	110.21
		25	7	3.43	309.28
		26	7	4.04	728.76
		27	6	2.52	508.35
		28	5	2.91	1045.15
		29	5	5.25	323.51
		30	4	4.30	88.87
		31	4	3.52	302.16
		32	4	4.09	735.88
		33	3	1.43	956.29
		34	3	4.36	1596.19
		35	3	1.25	213.30
		36	3	5.02	838.97
		37	3	2.24	117.32
		38	2	2.90	742.99
		39	2	2.36	942.06
JUPITER	L4	1	669	0.853	7.114
		2	114	3.142	0
		3	100	0.743	14.227
		4	50	1.65	536.80
		5	44	5.82	529.69
		6	32	4.86	522.58
		7	15	4.29	515.46
		8	9	0.71	1059.38
		9	5	1.30	543.92
		10	4	2.32	1066.50
		11	4	0.48	21.34
		12	3	3.00	412.37
		13	2	0.40	639.90
		14	2	4.26	199.07
		15	2	4.91	625.67
		16	2	4.26	206.19
		17	1	5.26	1052.27
		18	1	4.72	95.98
		19	1	1.29	1589.07
JUPITER	L5	1	50	5.26	7.11
		2	16	5.25	14.23
		3	4	0.01	536.80
		4	2	1.10	522.58
		5	1	3.14	0
JUPITER	B0	1	2268616	3.5585261	529.6909651
		2	110090	0	0
		3	109972	3.908093	1059.381930
		4	8101	3.6051	522.5774
		5	6438	0.3063	536.8045
		6	6044	4.2588	1589.0729
		7	1107	2.9853	1162.4747
		8	944	1.675	426.598
		9	942	2.936	1052.268
		10	894	1.754	7.114
		11	836	5.179	103.093

JUPITER (cont.)	B0	12	767	2.155	632.784
		13	684	3.678	213.299
		14	629	0.643	1066.495
		15	559	0.014	846.083
		16	532	2.703	110.206
		17	464	1.173	949.176
		18	431	2.608	419.485
		19	351	4.611	2118.764
		20	132	4.778	742.990
		21	123	3.350	1692.166
		22	116	1.387	323.505
		23	115	5.049	316.392
		24	104	3.701	515.464
		25	103	2.319	1478.867
		26	102	3.153	1581.959
JUPITER	B1	1	177352	5.701665	529.690965
		2	3230	5.7794	1059.3819
		3	3081	5.4746	522.5774
		4	2212	4.7348	536.8045
		5	1694	3.1416	0
		6	346	4.746	1052.268
		7	234	5.189	1066.495
		8	196	6.186	7.114
		9	150	3.927	1589.073
		10	114	3.439	632.784
		11	97	2.91	949.18
		12	82	5.08	1162.47
		13	77	2.51	103.09
		14	77	0.61	419.48
		15	74	5.50	515.46
		16	61	5.45	213.30
		17	50	3.95	735.88
		18	46	0.54	110.21
		19	45	1.90	846.08
		20	37	4.70	543.92
		21	36	6.11	316.39
		22	32	4.92	1581.96
JUPITER	B2	1	8094	1.4632	529.6910
		2	813	3.1416	0
		3	742	0.957	522.577
		4	399	2.899	536.805
		5	342	1.447	1059.382
		6	74	0.41	1052.27
		7	46	3.48	1066.50
		8	30	1.93	1589.07
		9	29	0.99	515.46
		10	23	4.27	7.11
		11	14	2.92	543.92
		12	12	5.22	632.78
		13	11	4.88	949.18
		14	6	6.21	1045.15
JUPITER	B3	1	252	3.381	529.691
		2	122	2.733	522.577
		3	49	1.04	536.80
		4	11	2.31	1052.27
		5	8	2.77	515.46
		6	7	4.25	1059.38
		7	6	1.78	1066.50
		8	4	1.13	543.92
		9	3	3.14	0

JUPITER	B4	1	15	4.53	522.58
		2	5	4.47	529.69
		3	4	5.44	536.80
		4	3	0	0
		5	2	4.52	515.46
		6	1	4.20	1 052.27
JUPITER	B5	1	1	0.09	522.58
JUPITER	R0	1	520 887 429	0	0
		2	25 209 327	3.491 086 40	529.690 965 09
		3	610 600	3.841 154	1 059.381 930
		4	282 029	2.574 199	632.783 739
		5	187 647	2.075 904	522.577 418
		6	86 793	0.710 01	419.484 64
		7	72 063	0.214 66	536.804 51
		8	65 517	5.979 96	316.391 87
		9	30 135	2.161 32	949.175 61
		10	29 135	1.677 59	103.092 77
		11	23 947	0.274 58	7.113 55
		12	23 453	3.540 23	735.876 51
		13	22 284	4.193 63	1 589.072 90
		14	13 033	2.960 43	1 162.474 70
		15	12 749	2.715 50	1 052.268 38
		16	9 703	1.906 7	206.185 5
		17	9 161	4.413 5	213.299 1
		18	7 895	2.479 1	426.598 2
		19	7 058	2.181 8	1 265.567 5
		20	6 138	6.264 2	846.082 8
		21	5 477	5.657 3	639.897 3
		22	4 170	2.016 1	515.463 9
		23	4 137	2.722 2	625.670 2
		24	3 503	0.565 3	1 066.495 5
		25	2 617	2.009 9	1 581.959 3
		26	2 500	4.551 8	838.969 3
		27	2 128	6.127 5	742.990 1
		28	1 912	0.856 2	412.371 1
		29	1 611	3.088 7	1 368.660 3
		30	1 479	2.680 3	1 478.866 6
		31	1 231	1.890 4	323.505 4
		32	1 217	1.801 7	110.206 3
		33	1 015	1.386 7	454.909 4
		34	999	2.872	309.278
		35	961	4.549	2 118.764
		36	886	4.148	533.623
		37	821	1.593	1 898.351
		38	812	5.941	909.819
		39	777	3.677	728.763
		40	727	3.988	1 155.361
		41	655	2.791	1 685.052
		42	654	3.382	1 692.166
		43	621	4.823	956.289
		44	615	2.276	942.062
		45	562	0.081	543.918
		46	542	0.284	525.759
JUPITER	R1	1	1 271 802	2.649 375 1	529.690 965 1
		2	61 662	3.000 76	1 059.381 93
		3	53 444	3.897 18	522.577 42
		4	41 390	0	0
		5	31 185	4.882 77	536.804 51
		6	11 847	2.413 30	419.484 64
		7	9 166	4.759 8	7.113 5

JUPITER (cont.)	R1	8	3404	3.3469	1589.0729
		9	3203	5.2108	735.8765
		10	3176	2.7930	103.0928
		11	2806	3.7422	515.4639
		12	2677	4.3305	1052.2684
		13	2600	3.6344	206.1855
		14	2412	1.4695	426.5982
		15	2101	3.9276	639.8973
		16	1646	5.3095	1066.4955
		17	1641	4.4163	625.6702
		18	1050	3.1611	213.2991
		19	1025	2.5543	412.3711
		20	806	2.678	632.784
		21	741	2.171	1162.475
		22	677	6.250	838.969
		23	567	4.577	742.990
		24	485	2.469	949.176
		25	469	4.710	543.918
		26	445	0.403	323.505
		27	416	5.368	728.763
		28	402	4.605	309.278
		29	347	4.681	14.227
		30	338	3.168	956.289
		31	261	5.343	846.083
		32	247	3.923	942.062
		33	220	4.842	1368.660
		34	203	5.600	1155.361
		35	200	4.439	1045.155
		36	197	3.706	2118.764
		37	196	3.759	199.072
		38	184	4.265	95.979
		39	180	4.402	532.872
		40	170	4.846	526.510
		41	146	6.130	533.623
		42	133	1.322	110.206
		43	132	4.512	525.759
JUPITER	R2	1	79645	1.35866	529.69097
		2	8252	5.7777	522.5774
		3	7030	3.2748	536.8045
		4	5314	1.8384	1059.3819
		5	1861	2.9768	7.1135
		6	964	5.480	515.464
		7	836	4.199	419.485
		8	498	3.142	0
		9	427	2.228	639.897
		10	406	3.783	1066.495
		11	377	2.242	1589.073
		12	363	5.368	206.186
		13	342	6.099	1052.268
		14	339	6.127	625.670
		15	333	0.003	426.598
		16	280	4.262	412.371
		17	257	0.963	632.784
		18	230	0.705	735.877
		19	201	3.069	543.918
		20	200	4.429	103.093
		21	139	2.932	14.227
		22	114	0.787	728.763
		23	95	1.70	838.97
		24	86	5.14	323.51
		25	83	0.06	309.28
		26	80	2.98	742.99

JUPITER (cont.)	R2	27	75	1.60	956.29
		28	70	1.51	213.30
		29	67	5.47	199.07
		30	62	6.10	1 045.15
		31	56	0.96	1 162.47
		32	52	5.58	942.06
		33	50	2.72	532.87
		34	45	5.52	508.35
		35	44	0.27	526.51
		36	40	5.95	95.98
JUPITER	R3	1	3 519	6.058 0	529.691 0
		2	1 073	1.673 2	536.804 5
		3	916	1.413	522.577
		4	342	0.523	1 059.382
		5	255	1.196	7.114
		6	222	0.952	515.464
		7	90	3.14	0
		8	69	2.27	1 066.50
		9	58	1.41	543.92
		10	58	0.53	639.90
		11	51	5.98	412.37
		12	47	1.58	625.67
		13	43	6.12	419.48
		14	37	1.18	14.23
		15	34	1.67	1 052.27
		16	34	0.85	206.19
		17	31	1.04	1 589.07
		18	30	4.63	426.60
		19	21	2.50	728.76
		20	15	0.89	199.07
		21	14	0.96	508.35
		22	13	1.50	1 045.15
		23	12	2.61	735.88
		24	12	3.56	323.51
		25	11	1.79	309.28
		26	11	6.28	956.29
		27	10	6.26	103.09
		28	9	3.45	838.97
JUPITER	R4	1	129	0.084	536.805
		2	113	4.249	529.691
		3	83	3.30	522.58
		4	38	2.73	515.46
		5	27	5.69	7.11
		6	18	5.40	1 059.38
		7	13	6.02	543.92
		8	9	0.77	1 066.50
		9	8	5.68	14.23
		10	7	1.43	412.37
		11	6	5.12	639.90
		12	5	3.34	625.67
		13	3	3.40	1 052.27
		14	3	4.16	728.76
		15	3	2.90	426.60
JUPITER	R5	1	11	4.75	536.80
		2	4	5.92	522.58
		3	2	5.57	515.46
		4	2	4.30	543.92
		5	2	3.69	7.11
		6	2	4.13	1 059.38
		7	2	5.49	1 066.50

SATURN	L0	1	87 401 354	0	0
		2	11 107 660	3.962 050 90	213.299 095 44
		3	1 414 151	4.585 815 2	7.113 547 0
		4	398 379	0.521 120	206.185 548
		5	350 769	3.303 299	426.598 191
		6	206 816	0.246 584	103.092 774
		7	79 271	3.840 07	220.412 64
		8	23 990	4.669 77	110.206 32
		9	16 574	0.437 19	419.484 64
		10	15 820	0.938 09	632.783 74
		11	15 054	2.716 70	639.897 29
		12	14 907	5.769 03	316.391 87
		13	14 610	1.565 19	3.932 15
		14	13 160	4.448 91	14.227 09
		15	13 005	5.981 19	11.045 70
		16	10 725	3.129 40	202.253 40
		17	6 126	1.763 3	277.035 0
		18	5 863	0.236 6	529.691 0
		19	5 228	4.207 8	3.181 4
		20	5 020	3.177 9	433.711 7
		21	4 593	0.619 8	199.072 0
		22	4 006	2.244 8	63.735 9
		23	3 874	3.222 8	138.517 5
		24	3 269	0.774 9	949.175 6
		25	2 954	0.982 8	95.979 2
		26	2 461	2.031 6	735.876 5
		27	1 758	3.265 8	522.577 4
		28	1 640	5.505 0	846.082 8
		29	1 581	4.372 7	309.278 3
		30	1 391	4.023 3	323.505 4
		31	1 124	2.837 3	415.552 5
		32	1 087	4.183 4	2.447 7
		33	1 017	3.717 0	227.526 2
		34	957	0.507	1 265.567
		35	853	3.421	175.166
		36	849	3.191	209.367
		37	789	5.007	0.963
		38	749	2.144	853.196
		39	744	5.253	224.345
		40	687	1.747	1 052.268
		41	654	1.599	0.048
		42	634	2.299	412.371
		43	625	0.970	210.118
		44	580	3.093	74.782
		45	546	2.127	350.332
		46	543	1.518	9.561
		47	530	4.449	117.320
		48	478	2.965	137.033
		49	474	5.475	742.990
		50	452	1.044	490.334
		51	449	1.290	127.472
		52	372	2.278	217.231
		53	355	3.013	838.969
		54	347	1.539	340.771
		55	343	0.246	0.521
		56	330	0.247	1 581.959
		57	322	0.961	203.738
		58	322	2.572	647.011
		59	309	3.495	216.480
		60	287	2.370	351.817
		61	278	0.400	211.815
		62	249	1.470	1 368.660
		63	227	4.910	12.530

SATURN (cont.)	L0	64	220	4.204	200.769
		65	209	1.345	625.670
		66	208	0.483	1 162.475
		67	208	1.283	39.357
		68	204	6.011	265.989
		69	185	3.503	149.563
		70	184	0.973	4.193
		71	182	5.491	2.921
		72	174	1.863	0.751
		73	165	0.440	5.417
		74	149	5.736	52.690
		75	148	1.535	5.629
		76	146	6.231	195.140
		77	140	4.295	21.341
		78	131	4.068	10.295
		79	125	6.277	1 898.351
		80	122	1.976	4.666
		81	118	5.341	554.070
		82	117	2.679	1 155.361
		83	114	5.594	1 059.382
		84	112	1.105	191.208
		85	110	0.166	1.484
		86	109	3.438	536.805
		87	107	4.012	956.289
		88	104	2.192	88.866
		89	103	1.197	1 685.052
		90	101	4.965	269.921
SATURN	L1	1	21 354 295 596	0	0
		2	1 296 855	1.828 205 4	213.299 095 4
		3	564 348	2.885 001	7.113 547
		4	107 679	2.277 699	206.185 548
		5	98 323	1.080 70	426.598 19
		6	40 255	2.041 28	220.412 64
		7	19 942	1.279 55	103.092 77
		8	10 512	2.748 80	14.227 09
		9	6 939	0.404 9	639.897 3
		10	4 803	2.441 9	419.484 6
		11	4 056	2.921 7	110.206 3
		12	3 769	3.649 7	3.932 2
		13	3 385	2.416 9	3.181 4
		14	3 302	1.262 6	433.711 7
		15	3 071	2.327 4	199.072 0
		16	1 953	3.563 9	11.045 7
		17	1 249	2.628 0	95.979 2
		18	922	1.961	227.526
		19	706	4.417	529.691
		20	650	6.174	202.253
		21	628	6.111	309.278
		22	487	6.040	853.196
		23	479	4.988	522.577
		24	468	4.617	63.736
		25	417	2.117	323.505
		26	408	1.299	209.367
		27	352	2.317	632.784
		28	344	3.959	412.371
		29	340	3.634	316.392
		30	336	3.772	735.877
		31	332	2.861	210.118
		32	289	2.733	117.320
		33	281	5.744	2.448
		34	266	0.543	647.011

SATURN (cont.)	L1	35	230	1.644	216.480
		36	192	2.965	224.345
		37	173	4.077	846.083
		38	167	2.597	21.341
		39	136	2.286	10.295
		40	131	3.441	742.990
		41	128	4.095	217.231
		42	109	6.161	415.552
		43	98	4.73	838.97
		44	94	3.48	1 052.27
		45	92	3.95	88.87
		46	87	1.22	440.83
		47	83	3.11	625.67
		48	78	6.24	302.16
		49	67	0.29	4.67
		50	66	5.65	9.56
		51	62	4.29	127.47
		52	62	1.83	195.14
		53	58	2.48	191.96
		54	57	5.02	137.03
		55	55	0.28	74.78
		56	54	5.13	490.33
		57	51	1.46	536.80
		58	47	1.18	149.56
		59	47	5.15	515.46
		60	46	2.23	956.29
		61	44	2.71	5.42
		62	40	0.41	269.92
		63	40	3.89	728.76
		64	38	0.65	422.67
		65	38	2.53	12.53
		66	37	3.78	2.92
		67	35	6.08	5.63
		68	34	3.21	1 368.66
		69	33	4.64	277.03
		70	33	5.43	1 066.50
		71	33	0.30	351.82
		72	32	4.39	1 155.36
		73	31	2.43	52.69
		74	30	2.84	203.00
		75	30	6.19	284.15
		76	30	3.39	1 059.38
		77	29	2.03	330.62
		78	28	2.74	265.99
		79	26	4.51	340.77
SATURN	L2	1	116 441	1.179 879	7.113 547
		2	91 921	0.074 25	213.299 10
		3	90 592	0	0
		4	15 277	4.064 92	206.185 55
		5	10 631	0.257 78	220.412 64
		6	10 605	5.409 64	426.598 19
		7	4 265	1.046 0	14.227 1
		8	1 216	2.918 6	103.092 8
		9	1 165	4.609 4	639.897 3
		10	1 082	5.691 3	433.711 7
		11	1 045	4.042 1	199.072 0
		12	1 020	0.633 7	3.181 4
		13	634	4.388	419.485
		14	549	5.573	3.932
		15	457	1.268	110.206
		16	425	0.209	227.526

SATURN (cont.)	L2	17	274	4.288	95.979
		18	162	1.381	11.046
		19	129	1.566	309.278
		20	117	3.881	853.196
		21	105	4.900	647.011
		22	101	0.893	21.341
		23	96	2.91	316.39
		24	95	5.63	412.37
		25	85	5.73	209.37
		26	83	6.05	216.48
		27	82	1.02	117.32
		28	75	4.76	210.12
		29	67	0.46	522.58
		30	66	0.48	10.29
		31	64	0.35	323.51
		32	61	4.88	632.78
		33	53	2.75	529.69
		34	46	5.69	440.83
		35	45	1.67	202.25
		36	42	5.71	88.87
		37	32	0.07	63.74
		38	32	1.67	302.16
		39	31	4.16	191.96
		40	27	0.83	224.34
		41	25	5.66	735.88
		42	20	5.94	217.23
		43	18	4.90	625.67
		44	17	1.63	742.99
		45	16	0.58	515.46
		46	14	0.21	838.97
		47	14	3.76	195.14
		48	12	4.72	203.00
		49	12	0.13	234.64
		50	12	3.12	846.08
		51	11	5.92	536.80
		52	11	5.60	728.76
		53	11	3.20	1 066.50
		54	10	4.99	422.67
		55	10	0.26	330.62
		56	10	4.15	860.31
		57	9	0.46	956.29
		58	8	2.14	269.92
		59	8	5.25	429.78
		60	8	4.03	9.56
		61	7	5.40	1 052.27
		62	6	4.46	284.15
		63	6	5.93	405.26
SATURN	L3	1	16 039	5.739 45	7.113 55
		2	4 250	4.585 4	213.299 1
		3	1 907	4.760 8	220.412 6
		4	1 466	5.913 3	206.185 5
		5	1 162	5.619 7	14.227 1
		6	1 067	3.608 2	426.598 2
		7	239	3.861	433.712
		8	237	5.768	199.072
		9	166	5.116	3.181
		10	151	2.736	639.897
		11	131	4.743	227.526
		12	63	0.23	419.48
		13	62	4.74	103.09
		14	40	5.47	21.34

SATURN (cont.)	L3	15	40	5.96	95.98
		16	39	5.83	110.21
		17	28	3.01	647.01
		18	25	0.99	3.93
		19	19	1.92	853.20
		20	18	4.97	10.29
		21	18	1.03	412.37
		22	18	4.20	216.48
		23	18	3.32	309.28
		24	16	3.90	440.83
		25	16	5.62	117.32
		26	13	1.18	88.87
		27	11	5.58	11.05
		28	11	5.93	191.96
		29	10	3.95	209.37
		30	9	3.39	302.16
		31	8	4.88	323.51
		32	7	0.38	632.78
		33	6	2.25	522.58
		34	6	1.06	210.12
		35	5	4.64	234.64
		36	4	3.14	0
		37	4	2.31	515.46
		38	3	2.20	860.31
		39	3	0.59	529.69
		40	3	4.93	224.34
		41	3	0.42	625.67
		42	2	4.77	330.62
		43	2	3.35	429.78
		44	2	3.20	202.25
		45	2	1.19	1 066.50
		46	2	1.35	405.26
		47	2	4.16	223.59
		48	2	3.07	654.12
SATURN	L4	1	1 662	3.998 3	7.113 5
		2	257	2.984	220.413
		3	236	3.902	14.227
		4	149	2.741	213.299
		5	114	3.142	0
		6	110	1.515	206.186
		7	68	1.72	426.60
		8	40	2.05	433.71
		9	38	1.24	199.07
		10	31	3.01	227.53
		11	15	0.83	639.90
		12	9	3.71	21.34
		13	6	2.42	419.48
		14	6	1.16	647.01
		15	4	1.45	95.98
		16	4	2.12	440.83
		17	3	4.09	110.21
		18	3	2.77	412.37
		19	3	3.01	88.87
		20	3	0.00	853.20
		21	3	0.39	103.09
		22	2	3.78	117.32
		23	2	2.83	234.64
		24	2	5.08	309.28
		25	2	2.24	216.48
		26	2	5.19	302.16
		27	1	1.55	191.96

SATURN	L5	1	124	2.259	7.114
		2	34	2.16	14.23
		3	28	1.20	220.41
		4	6	1.22	227.53
		5	5	0.24	433.71
		6	4	6.23	426.60
		7	3	2.97	199.07
		8	3	4.29	206.19
		9	2	6.25	213.30
		10	1	5.28	639.90
		11	1	0.24	440.83
		12	1	3.14	0
SATURN	B0	1	4330678	3.6028443	213.2990954
		2	240348	2.852385	426.598191
		3	84746	0	0
		4	34116	0.57297	206.18555
		5	30863	3.48442	220.41264
		6	14734	2.11847	639.89729
		7	9917	5.7900	419.4846
		8	6994	4.7360	7.1135
		9	4808	5.4331	316.3919
		10	4788	4.9651	110.2063
		11	3432	2.7326	433.7117
		12	1506	6.0130	103.0928
		13	1060	5.6310	529.6910
		14	969	5.204	632.784
		15	942	1.396	853.196
		16	708	3.803	323.505
		17	552	5.131	202.253
		18	400	3.359	227.526
		19	319	3.626	209.367
		20	316	1.997	647.011
		21	314	0.465	217.231
		22	284	4.886	224.345
		23	236	2.139	11.046
		24	215	5.950	846.083
		25	209	2.120	415.552
		26	207	0.730	199.072
		27	179	2.954	63.736
		28	141	0.644	490.334
		29	139	4.595	14.227
		30	139	1.998	735.877
		31	135	5.245	742.990
		32	122	3.115	522.577
		33	116	3.109	216.480
		34	114	0.963	210.118
SATURN	B1	1	397555	5.332900	213.299095
		2	49479	3.14159	0
		3	18572	6.09919	426.59819
		4	14801	2.30586	206.18555
		5	9644	1.6967	220.4126
		6	3757	1.2543	419.4846
		7	2717	5.9117	639.8973
		8	1455	0.8516	433.7117
		9	1291	2.9177	7.1135
		10	853	0.436	316.392
		11	298	0.919	632.784
		12	292	5.316	853.196
		13	284	1.619	227.526
		14	275	3.889	103.093
		15	172	0.052	647.011

SATURN (cont.)	B1	16	166	2.444	199.072
		17	158	5.209	110.206
		18	128	1.207	529.691
		19	110	2.457	217.231
		20	82	2.76	210.12
		21	81	2.86	14.23
		22	69	1.66	202.25
		23	65	1.26	216.48
		24	61	1.25	209.37
		25	59	1.82	323.51
		26	46	0.82	440.83
		27	36	1.82	224.34
		28	34	2.84	117.32
		29	33	1.31	412.37
		30	32	1.19	846.08
		31	27	4.65	1066.50
		32	27	4.44	11.05
SATURN	B2	1	20630	0.50482	213.29910
		2	3720	3.9983	206.1855
		3	1627	6.1819	220.4126
		4	1346	0	0
		5	706	3.039	419.485
		6	365	5.099	426.598
		7	330	5.279	433.712
		8	219	3.828	639.897
		9	139	1.043	7.114
		10	104	6.157	227.526
		11	93	1.98	316.39
		12	71	4.15	199.07
		13	52	2.88	632.78
		14	49	4.43	647.01
		15	41	3.16	853.20
		16	29	4.53	210.12
		17	24	1.12	14.23
		18	21	4.35	217.23
		19	20	5.31	440.83
		20	18	0.85	110.21
		21	17	5.68	216.48
		22	16	4.26	103.09
		23	14	3.00	412.37
		24	12	2.53	529.69
		25	8	3.32	202.25
		26	7	5.56	209.37
		27	7	0.29	323.51
		28	6	1.16	117.32
		29	6	3.61	860.31
SATURN	B3	1	666	1.990	213.299
		2	632	5.698	206.186
		3	398	0	0
		4	188	4.338	220.413
		5	92	4.84	419.48
		6	52	3.42	433.71
		7	42	2.38	426.60
		8	26	4.40	227.53
		9	21	5.85	199.07
		10	18	1.99	639.90
		11	11	5.37	7.11
		12	10	2.55	647.01
		13	7	3.46	316.39
		14	6	4.80	632.78
		15	6	0.02	210.12

SATURN (cont.)	B3	16	6	3.52	440.83
		17	5	5.64	14.23
		18	5	1.22	853.20
		19	4	4.71	412.37
		20	3	0.63	103.09
		21	2	3.72	216.48
SATURN	B4	1	80	1.12	206.19
		2	32	3.12	213.30
		3	17	2.48	220.41
		4	12	3.14	0
		5	9	0.38	419.48
		6	6	1.56	433.71
		7	5	2.63	227.53
		8	5	1.28	199.07
		9	1	1.43	426.60
		10	1	0.67	647.01
		11	1	1.72	440.83
		12	1	6.18	639.90
SATURN	B5	1	8	2.82	206.19
		2	1	0.51	220.41
SATURN	R0	1	955 758 136	0	0
		2	52 921 382	2.392 262 20	213.299 095 44
		3	1 873 680	5.235 496 1	206.185 548 4
		4	1 464 664	1.647 630 5	426.598 190 9
		5	821 891	5.935 200	316.391 870
		6	547 507	5.015 326	103.092 774
		7	371 684	2.271 148	220.412 642
		8	361 778	3.139 043	7.113 547
		9	140 618	5.704 067	632.783 739
		10	108 975	3.293 136	110.206 321
		11	69 007	5.941 00	419.484 64
		12	61 053	0.940 38	639.897 29
		13	48 913	1.557 33	202.253 40
		14	34 144	0.195 19	277.034 99
		15	32 402	5.470 85	949.175 61
		16	20 937	0.463 49	735.876 51
		17	20 839	1.521 03	433.711 74
		18	20 747	5.332 56	199.072 00
		19	15 298	3.059 44	529.690 97
		20	14 296	2.604 34	323.505 42
		21	12 884	1.648 92	138.517 50
		22	11 993	5.980 51	846.082 83
		23	11 380	1.731 06	522.577 42
		24	9 796	5.204 8	1 265.567 5
		25	7 753	5.851 9	95.979 2
		26	6 771	3.004 3	14.227 1
		27	6 466	0.177 3	1 052.268 4
		28	5 850	1.455 2	415.552 5
		29	5 307	0.597 4	63.735 9
		30	4 696	2.149 2	227.526 2
		31	4 044	1.640 1	209.366 9
		32	3 688	0.780 2	412.371 1
		33	3 461	1.850 9	175.166 1
		34	3 420	4.945 5	1 581.959 3
		35	3 401	0.553 9	350.332 1
		36	3 376	3.695 3	224.344 8
		37	2 976	5.684 7	210.117 7
		38	2 885	1.387 6	838.969 3
		39	2 881	0.179 6	853.196 4
		40	2 508	3.538 5	742.990 1

SATURN (cont.)	R0	41	2 448	6.184 1	1 368.660 3
		42	2 406	2.965 6	117.319 9
		43	2 174	0.015 1	340.770 9
		44	2 024	5.054 1	11.045 7
SATURN	R1	1	6 182 981	0.258 435 2	213.299 095 4
		2	506 578	0.711 147	206.185 548
		3	341 394	5.796 358	426.598 191
		4	188 491	0.472 157	220.412 642
		5	186 262	3.141 593	0
		6	143 891	1.407 449	7.113 547
		7	49 621	6.017 44	103.092 77
		8	20 928	5.092 46	639.897 29
		9	19 953	1.175 60	419.484 64
		10	18 840	1.608 20	110.206 32
		11	13 877	0.758 86	199.072 00
		12	12 893	5.943 30	433.711 74
		13	5 397	1.288 5	14.227 1
		14	4 869	0.867 9	323.505 4
		15	4 247	0.393 0	227.526 2
		16	3 252	1.258 5	95.979 2
		17	3 081	3.436 6	522.577 4
		18	2 909	4.606 8	202.253 4
		19	2 856	2.167 3	735.876 5
		20	1 988	2.450 5	412.371 1
		21	1 941	6.023 9	209.366 9
		22	1 581	1.291 9	210.117 7
		23	1 340	4.308 0	853.196 4
		24	1 316	1.253 0	117.319 9
		25	1 203	1.866 5	316.391 9
		26	1 091	0.075 3	216.480 5
		27	966	0.480	632.784
		28	954	5.152	647.011
		29	898	0.983	529.691
		30	882	1.885	1 052.268
		31	874	1.402	224.345
		32	785	3.064	838.969
		33	740	1.382	625.670
		34	658	4.144	309.278
		35	650	1.725	742.990
		36	613	3.033	63.736
		37	599	2.549	217.231
		38	503	2.130	3.932
SATURN	R2	1	436 902	4.786 717	213.299 095
		2	71 923	2.500 70	206.185 55
		3	49 767	4.971 68	220.412 64
		4	43 221	3.869 40	426.598 19
		5	29 646	5.963 10	7.113 55
		6	4 721	2.475 3	199.072 0
		7	4 142	4.106 7	433.711 7
		8	3 789	3.097 7	639.897 3
		9	2 964	1.372 1	103.092 8
		10	2 556	2.850 7	419.484 6
		11	2 327	0	0
		12	2 208	6.275 9	110.206 3
		13	2 188	5.855 5	14.227 1
		14	1 957	4.924 5	227.526 2
		15	924	5.464	323.505
		16	706	2.971	95.979
		17	546	4.129	412.371
		18	431	5.178	522.577
		19	405	4.173	209.367

SATURN (cont.)	R2	20	391	4.481	216.480
		21	374	5.834	117.320
		22	361	3.277	647.011
		23	356	3.192	210.118
		24	326	2.269	853.196
		25	207	4.022	735.877
		26	204	0.088	202.253
		27	180	3.597	632.784
		28	178	4.097	440.825
		29	154	3.135	625.670
		30	148	0.136	302.165
		31	133	2.594	191.958
		32	132	5.933	309.278
SATURN	R3	1	20315	3.02187	213.29910
		2	8924	3.1914	220.4126
		3	6909	4.3517	206.1855
		4	4087	4.2241	7.1135
		5	3879	2.0106	426.5982
		6	1071	4.2036	199.0720
		7	907	2.283	433.712
		8	606	3.175	227.526
		9	597	4.135	14.227
		10	483	1.173	639.897
		11	393	0	0
		12	229	4.698	419.485
		13	188	4.590	110.206
		14	150	3.202	103.093
		15	121	3.768	323.505
		16	102	4.710	95.979
		17	101	5.819	412.371
		18	93	1.44	647.01
		19	84	2.63	216.48
		20	73	4.15	117.32
		21	62	2.31	440.83
		22	55	0.31	853.20
		23	50	2.39	209.37
		24	45	4.37	191.96
		25	41	0.69	522.58
		26	40	1.84	302.16
		27	38	5.94	88.87
		28	32	4.01	21.34
SATURN	R4	1	1202	1.4150	220.4126
		2	708	1.162	213.299
		3	516	6.240	206.186
		4	427	2.469	7.114
		5	268	0.187	426.598
		6	170	5.959	199.072
		7	150	0.480	433.712
		8	145	1.442	227.526
		9	121	2.405	14.227
		10	47	5.57	639.90
		11	19	5.86	647.01
		12	17	0.53	440.83
		13	16	2.90	110.21
		14	15	0.30	419.48
		15	14	1.30	412.37
		16	13	2.09	323.51
		17	11	0.22	95.98
		18	11	2.46	117.32
		19	10	3.14	0
		20	9	1.56	88.87

SATURN (cont.)	R4	21	9	2.28	21.34
		22	9	0.68	216.48
		23	8	1.27	234.64
SATURN	R5	1	129	5.913	220.413
		2	32	0.69	7.11
		3	27	5.91	227.53
		4	20	4.95	433.71
		5	20	0.67	14.23
		6	14	2.67	206.19
		7	14	1.46	199.07
		8	13	4.59	426.60
		9	7	4.63	213.30
		10	5	3.61	639.90
		11	4	4.90	440.83
		12	3	4.07	647.01
		13	3	4.66	191.96
		14	3	0.49	323.51
		15	3	3.18	419.48
		16	2	3.70	88.87
		17	2	3.32	95.98
		18	2	0.56	117.32
URANUS	L0	1	548 129 294	0	0
		2	9 260 408	0.891 064 2	74.781 598 6
		3	1 504 248	3.627 192 6	1.484 472 7
		4	365 982	1.899 622	73.297 126
		5	272 328	3.358 237	149.563 197
		6	70 328	5.392 54	63.735 90
		7	68 893	6.092 92	76.266 07
		8	61 999	2.269 52	2.968 95
		9	61 951	2.850 99	11.045 70
		10	26 469	3.141 52	71.812 65
		11	25 711	6.113 80	454.909 37
		12	21 079	4.360 59	148.078 72
		13	17 819	1.744 37	36.648 56
		14	14 613	4.737 32	3.932 15
		15	11 163	5.826 82	224.344 80
		16	10 998	0.488 65	138.517 50
		17	9 527	2.955 2	35.164 1
		18	7 546	5.236 3	109.945 7
		19	4 220	3.233 3	70.849 4
		20	4 052	2.277 5	151.047 7
		21	3 490	5.483 1	146.594 3
		22	3 355	1.065 5	4.453 4
		23	3 144	4.752 0	77.750 5
		24	2 927	4.629 0	9.561 2
		25	2 922	5.352 4	85.827 3
		26	2 273	4.366 0	70.328 2
		27	2 149	0.607 5	38.133 0
		28	2 051	1.517 7	0.111 9
		29	1 992	4.924 4	277.035 0
		30	1 667	3.627 4	380.127 8
		31	1 533	2.585 9	52.690 2
		32	1 376	2.042 8	65.220 4
		33	1 372	4.196 4	111.430 2
		34	1 284	3.113 5	202.253 4
		35	1 282	0.542 7	222.860 3
		36	1 244	0.916 1	2.447 7
		37	1 221	0.199 0	108.461 2
		38	1 151	4.179 0	33.679 6
		39	1 150	0.933 4	3.181 4

URANUS (cont.)	L0	40	1090	1.7750	12.5302
		41	1072	0.2356	62.2514
		42	946	1.192	127.472
		43	708	5.183	213.299
		44	653	0.966	78.714
		45	628	0.182	984.600
		46	607	5.432	529.691
		47	559	3.358	0.521
		48	524	2.013	299.126
		49	483	2.106	0.963
		50	471	1.407	184.727
		51	467	0.415	145.110
		52	434	5.521	183.243
		53	405	5.987	8.077
		54	399	0.338	415.552
		55	396	5.870	351.817
		56	379	2.350	56.622
		57	310	5.833	145.631
		58	300	5.644	22.091
		59	294	5.839	39.618
		60	252	1.637	221.376
		61	249	4.746	225.829
		62	239	2.350	137.033
		63	224	0.516	84.343
		64	223	2.843	0.261
		65	220	1.922	67.668
		66	217	6.142	5.938
		67	216	4.778	340.771
		68	208	5.580	68.844
		69	202	1.297	0.048
		70	199	0.956	152.532
		71	194	1.888	456.394
		72	193	0.916	453.425
		73	187	1.319	0.160
		74	182	3.536	79.235
		75	173	1.539	160.609
		76	172	5.680	219.891
		77	170	3.677	5.417
		78	169	5.879	18.159
		79	165	1.424	106.977
		80	163	3.050	112.915
		81	158	0.738	54.175
		82	147	1.263	59.804
		83	143	1.300	35.425
		84	139	5.386	32.195
		85	139	4.260	909.819
		86	124	1.374	7.114
		87	110	2.027	554.070
		88	109	5.706	77.963
		89	104	5.028	0.751
		90	104	1.458	24.379
		91	103	0.681	14.978
URANUS	L1	1	7502543122	0	0
		2	154458	5.242017	74.781599
		3	24456	1.71256	1.48447
		4	9258	0.4284	11.0457
		5	8266	1.5022	63.7359
		6	7842	1.3198	149.5632
		7	3899	0.4648	3.9322
		8	2284	4.1737	76.2661
		9	1927	0.5301	2.9689
		10	1233	1.5863	70.8494

URANUS (cont.)	L1	11	791	5.436	3.181
		12	767	1.996	73.297
		13	482	2.984	85.827
		14	450	4.138	138.517
		15	446	3.723	224.345
		16	427	4.731	71.813
		17	354	2.583	148.079
		18	348	2.454	9.561
		19	317	5.579	52.690
		20	206	2.363	2.448
		21	189	4.202	56.622
		22	184	0.284	151.048
		23	180	5.684	12.530
		24	171	3.001	78.714
		25	158	2.909	0.963
		26	155	5.591	4.453
		27	154	4.652	35.164
		28	152	2.942	77.751
		29	143	2.590	62.251
		30	121	4.148	127.472
		31	116	3.732	65.220
		32	102	4.188	145.631
		33	102	6.034	0.112
		34	88	3.99	18.16
		35	88	6.16	202.25
		36	81	2.64	22.09
		37	72	6.05	70.33
		38	69	4.05	77.96
		39	59	3.70	67.67
		40	47	3.54	351.82
		41	44	5.91	7.11
		42	43	5.72	5.42
		43	39	4.92	222.86
		44	36	5.90	33.68
		45	36	3.29	8.08
		46	36	3.33	71.60
		47	35	5.08	38.13
		48	31	5.62	984.60
		49	31	5.50	59.80
		50	31	5.46	160.61
		51	30	1.66	447.80
		52	29	1.15	462.02
		53	29	4.52	84.34
		54	27	5.54	131.40
		55	27	6.15	299.13
		56	26	4.99	137.03
		57	25	5.74	380.13
URANUS	L2	1	53 033	0	0
		2	2 358	2.2601	74.7816
		3	769	4.526	11.046
		4	552	3.258	63.736
		5	542	2.276	3.932
		6	529	4.923	1.484
		7	258	3.691	3.181
		8	239	5.858	149.563
		9	182	6.218	70.849
		10	54	1.44	76.27
		11	49	6.03	56.62
		12	45	3.91	2.45
		13	45	0.81	85.83
		14	38	1.78	52.69
		15	37	4.46	2.97

URANUS (cont.)	L2	16	33	0.86	9.56
		17	29	5.10	73.30
		18	24	2.11	18.16
		19	22	5.99	138.52
		20	22	4.82	78.71
		21	21	2.40	77.96
		22	21	2.17	224.34
		23	17	2.54	145.63
		24	17	3.47	12.53
		25	12	0.02	22.09
		26	11	0.08	127.47
		27	10	5.16	71.60
		28	10	4.46	62.25
		29	9	4.26	7.11
		30	8	5.50	67.67
		31	7	1.25	5.42
		32	6	3.36	447.80
		33	6	5.45	65.22
		34	6	4.52	151.05
		35	6	5.73	462.02
URANUS	L3	1	121	0.024	74.782
		2	68	4.12	3.93
		3	53	2.39	11.05
		4	46	0	0
		5	45	2.04	3.18
		6	44	2.96	1.48
		7	25	4.89	63.74
		8	21	4.55	70.85
		9	20	2.31	149.56
		10	9	1.58	56.62
		11	4	0.23	18.16
		12	4	5.39	76.27
		13	4	0.95	77.96
		14	3	4.98	85.83
		15	3	4.13	52.69
		16	3	0.37	78.71
		17	2	0.86	145.63
		18	2	5.66	9.56
URANUS	L4	1	114	3.142	0
		2	6	4.58	74.78
		3	3	0.35	11.05
		4	1	3.42	56.62
URANUS	B0	1	1 346 278	2.618 778 1	74.781 598 6
		2	62 341	5.081 11	149.563 20
		3	61 601	3.141 59	0
		4	9 964	1.616 0	76.266 1
		5	9 926	0.576 3	73.297 1
		6	3 259	1.261 2	224.344 8
		7	2 972	2.243 7	1.484 5
		8	2 010	6.055 5	148.078 7
		9	1 522	0.279 6	63.735 9
		10	924	4.038	151.048
		11	761	6.140	71.813
		12	522	3.321	138.517
		13	463	0.743	85.827
		14	437	3.381	529.691
		15	435	0.341	77.751
		16	431	3.554	213.299
		17	420	5.213	11.046
		18	245	0.788	2.969

URANUS (cont.)	B0	19	233	2.257	222.860
		20	216	1.591	38.133
		21	180	3.725	299.126
		22	175	1.236	146.594
		23	174	1.937	380.128
		24	160	5.336	111.430
		25	144	5.962	35.164
		26	116	5.739	70.849
		27	106	0.941	70.328
		28	102	2.619	78.714
URANUS	B1	1	206 366	4.123 943	74.781 599
		2	8 563	0.338 2	149.563 2
		3	1 726	2.121 9	73.297 1
		4	1 374	0	0
		5	1 369	3.068 6	76.266 1
		6	451	3.777	1.484
		7	400	2.848	224.345
		8	307	1.255	148.079
		9	154	3.786	63.736
		10	112	5.573	151.048
		11	111	5.329	138.517
		12	83	3.59	71.81
		13	56	3.40	85.83
		14	54	1.70	77.75
		15	42	1.21	11.05
		16	41	4.45	78.71
		17	32	3.77	222.86
		18	30	2.56	2.97
		19	27	5.34	213.30
		20	26	0.42	380.13
URANUS	B2	1	9 212	5.800 4	74.781 6
		2	557	0	0
		3	286	2.177	149.563
		4	95	3.84	73.30
		5	45	4.88	76.27
		6	20	5.46	1.48
		7	15	0.88	138.52
		8	14	2.85	148.08
		9	14	5.07	63.74
		10	10	5.00	224.34
		11	8	6.27	78.71
URANUS	B3	1	268	1.251	74.782
		2	11	3.14	0
		3	6	4.01	149.56
		4	3	5.78	73.30
URANUS	B4	1	6	2.85	74.78
URANUS	R0	1	1 921 264 848	0	0
		2	88 784 984	5.603 775 27	74.781 598 57
		3	3 440 836	0.328 361 0	73.297 125 9
		4	2 055 653	1.782 951 7	149.563 197 1
		5	649 322	4.522 473	76.266 071
		6	602 248	3.860 038	63.735 898
		7	496 404	1.401 399	454.909 367
		8	338 526	1.580 027	138.517 497
		9	243 508	1.570 866	71.812 653
		10	190 522	1.998 094	1.484 473
		11	161 858	2.791 379	148.078 724
		12	143 706	1.383 686	11.045 700

URANUS (cont.)	R0	13	93192	0.17437	36.64856
		14	89806	3.66105	109.94569
		15	71424	4.24509	224.34480
		16	46677	1.39977	35.16409
		17	39026	3.36235	277.03499
		18	39010	1.66971	70.84945
		19	36755	3.88649	146.59425
		20	30349	0.70100	151.04767
		21	29156	3.18056	77.75054
		22	25786	3.78538	85.82730
		23	25620	5.25656	380.12777
		24	22637	0.72519	529.69097
		25	20473	2.79640	70.32818
		26	20472	1.55589	202.25340
		27	17901	0.55455	2.96895
		28	15503	5.35405	38.13304
		29	14702	4.90434	108.46122
		30	12897	2.62154	111.43016
		31	12328	5.96039	127.47180
		32	11959	1.75044	984.60033
		33	11853	0.99343	52.69020
		34	11696	3.29826	3.93215
		35	11495	0.43774	65.22037
		36	10793	1.42105	213.29910
		37	9111	4.9964	62.2514
		38	8421	5.2535	222.8603
		39	8402	5.0388	415.5525
		40	7449	0.7949	351.8166
		41	7329	3.9728	183.2428
		42	6046	5.6796	78.7138
		43	5524	3.1150	9.5612
		44	5445	5.1058	145.1098
		45	5238	2.6296	33.6796
		46	4079	3.2206	340.7709
		47	3919	4.2502	39.6175
		48	3802	6.1099	184.7273
		49	3781	3.4584	456.3938
		50	3687	2.4872	453.4249
		51	3102	4.1403	219.8914
		52	2963	0.8298	56.6224
		53	2942	0.4239	299.1264
		54	2940	2.1464	137.0330
		55	2938	3.6766	140.0020
		56	2865	0.3100	12.5302
		57	2538	4.8546	131.4039
		58	2364	0.4425	554.0700
		59	2183	2.9404	305.3462
URANUS	R1	1	1479896	3.6720571	74.7815986
		2	71212	6.22601	63.73590
		3	68627	6.13411	149.56320
		4	24060	3.14159	0
		5	21468	2.60177	76.26607
		6	20857	5.24625	11.04570
		7	11405	0.01848	70.84945
		8	7497	0.4236	73.2971
		9	4244	1.4169	85.8273
		10	3927	3.1551	71.8127
		11	3578	2.3116	224.3448
		12	3506	2.5835	138.5175
		13	3229	5.2550	3.9322
		14	3060	0.1532	1.4845
		15	2564	0.9808	148.0787

URANUS (cont.)	R1	16	2429	3.9944	52.6902
		17	1645	2.6535	127.4718
		18	1584	1.4305	78.7138
		19	1508	5.0600	151.0477
		20	1490	2.6756	56.6224
		21	1413	4.5746	202.2534
		22	1403	1.3699	77.7505
		23	1228	1.0470	62.2514
		24	1033	0.2646	131.4039
		25	992	2.172	65.220
		26	862	5.055	351.817
		27	744	3.076	35.164
		28	687	2.499	77.963
		29	647	4.473	70.328
		30	624	0.863	9.561
		31	604	0.907	984.600
		32	575	3.231	447.796
		33	562	2.718	462.023
		34	530	5.917	213.299
		35	528	5.151	2.969
URANUS	R2	1	22440	0.69953	74.78160
		2	4727	1.6990	63.7359
		3	1682	4.6483	70.8494
		4	1650	3.0966	11.0457
		5	1434	3.5212	149.5632
		6	770	0	0
		7	500	6.172	76.266
		8	461	0.767	3.932
		9	390	4.496	56.622
		10	390	5.527	85.827
		11	292	0.204	52.690
		12	287	3.534	73.297
		13	273	3.847	138.517
		14	220	1.964	131.404
		15	216	0.848	77.963
		16	205	3.248	78.714
		17	149	4.898	127.472
		18	129	2.081	3.181
URANUS	R3	1	1164	4.7345	74.7816
		2	212	3.343	63.736
		3	196	2.980	70.849
		4	105	0.958	11.046
		5	73	1.00	149.56
		6	72	0.03	56.62
		7	55	2.59	3.93
		8	36	5.65	77.96
		9	34	3.82	76.27
		10	32	3.60	131.40
URANUS	R4	1	53	3.01	74.78
		2	10	1.91	56.62

NEPTUNE	L0	1	531188633	0	0
		2	1798476	2.9010127	38.1330356
		3	1019728	0.4858092	1.4844727
		4	124532	4.830081	36.648563
		5	42064	5.41055	2.96895
		6	37715	6.09222	35.16409
		7	33785	1.24489	76.26607
		8	16483	0.00008	491.55793
		9	9199	4.9375	39.6175
		10	8994	0.2746	175.1661
		11	4216	1.9871	73.2971
		12	3365	1.0359	33.6796
		13	2285	4.2061	4.4534
		14	1434	2.7834	74.7816
		15	900	2.076	109.946
		16	745	3.190	71.813
		17	506	5.748	114.399
		18	400	0.350	1021.249
		19	345	3.462	41.102
		20	340	3.304	77.751
		21	323	2.248	32.195
		22	306	0.497	0.521
		23	287	4.505	0.048
		24	282	2.246	146.594
		25	267	4.889	0.963
		26	252	5.782	388.465
		27	245	1.247	9.561
		28	233	2.505	137.033
		29	227	1.797	453.425
		30	170	3.324	108.461
		31	151	2.192	33.940
		32	150	2.997	5.938
		33	148	0.859	111.430
		34	119	3.677	2.448
		35	109	2.416	183.243
		36	103	0.041	0.261
		37	103	4.404	70.328
		38	102	5.705	0.112
NEPTUNE	L1	1	3837687717	0	0
		2	16604	4.86319	1.48447
		3	15807	2.27923	38.13304
		4	3335	3.6820	76.2661
		5	1306	3.6732	2.9689
		6	605	1.505	35.164
		7	179	3.453	39.618
		8	107	2.451	4.453
		9	106	2.755	33.680
		10	73	5.49	36.65
		11	57	1.86	114.40
		12	57	5.22	0.52
		13	35	4.52	74.78
		14	32	5.90	77.75
		15	30	3.67	388.47
		16	29	5.17	9.56
		17	29	5.17	2.45
		18	26	5.25	168.05

NEPTUNE	L2	1	53 893	0	0
		2	296	1.855	1.484
		3	281	1.191	38.133
		4	270	5.721	76.266
		5	23	1.21	2.97
		6	9	4.43	35.16
		7	7	0.54	2.45
NEPTUNE	L3	1	31	0	0
		2	15	1.35	76.27
		3	12	6.04	1.48
		4	12	6.11	38.13
NEPTUNE	L4	1	114	3.142	0
NEPTUNE	B0	1	3 088 623	1.441 043 7	38.133 035 6
		2	27 780	5.912 72	76.266 07
		3	27 624	0	0
		4	15 448	3.508 77	39.617 51
		5	15 355	2.521 24	36.648 56
		6	2 000	1.510 0	74.781 6
		7	1 968	4.377 8	1.484 5
		8	1 015	3.215 6	35.164 1
		9	606	2.802	73.297
		10	595	2.129	41.102
		11	589	3.187	2.969
		12	402	4.169	114.399
		13	280	1.682	77.751
		14	262	3.767	213.299
		15	254	3.271	453.425
		16	206	4.257	529.691
		17	140	3.530	137.033
NEPTUNE	B1	1	227 279	3.807 931	38.133 036
		2	1 803	1.975 8	76.266 1
		3	1 433	3.141 6	0
		4	1 386	4.825 6	36.648 6
		5	1 073	6.080 5	39.617 5
		6	148	3.858	74.782
		7	136	0.478	1.484
		8	70	6.19	35.16
		9	52	5.05	73.30
		10	43	0.31	114.40
		11	37	4.89	41.10
		12	37	5.76	2.97
		13	26	5.22	213.30
NEPTUNE	B2	1	9 691	5.571 2	38.133 0
		2	79	3.63	76.27
		3	72	0.45	36.65
		4	59	3.14	0
		5	30	1.61	39.62
		6	6	5.61	74.78
NEPTUNE	B3	1	273	1.017	38.133
		2	2	0	0
		3	2	2.37	36.65
		4	2	5.33	76.27
NEPTUNE	B4	1	6	2.67	38.13

NEPTUNE	R0	1	3 007 013 206	0	0
		2	27 062 259	1.329 994 59	38.133 035 64
		3	1 691 764	3.251 861 4	36.648 562 9
		4	807 831	5.185 928	1.484 473
		5	537 761	4.521 139	35.164 090
		6	495 726	1.571 057	491.557 929
		7	274 572	1.845 523	175.166 060
		8	135 134	3.372 206	39.617 508
		9	121 802	5.797 544	76.266 071
		10	100 895	0.377 027	73.297 126
		11	69 792	3.796 17	2.968 95
		12	46 688	5.749 38	33.679 62
		13	24 594	0.508 02	109.945 69
		14	16 939	1.594 22	71.812 65
		15	14 230	1.077 86	74.781 60
		16	12 012	1.920 62	1 021.248 89
		17	8 395	0.678 2	146.594 3
		18	7 572	1.071 5	388.465 2
		19	5 721	2.590 6	4.453 4
		20	4 840	1.906 9	41.102 0
		21	4 483	2.905 7	529.691 0
		22	4 421	1.749 9	108.461 2
		23	4 354	0.679 9	32.195 1
		24	4 270	3.413 4	453.424 9
		25	3 381	0.848 1	183.242 8
		26	2 881	1.986 0	137.033 0
		27	2 879	3.674 2	350.332 1
		28	2 636	3.097 6	213.299 1
		29	2 530	5.798 4	490.073 5
		30	2 523	0.486 3	493.042 4
		31	2 306	2.809 6	70.328 2
		32	2 087	0.618 6	33.940 2
NEPTUNE	R1	1	236 339	0.704 980	38.133 036
		2	13 220	3.320 15	1.484 47
		3	8 622	6.216 3	35.164 1
		4	2 702	1.881 4	39.617 5
		5	2 155	2.094 3	2.968 9
		6	2 153	5.168 7	76.266 1
		7	1 603	0	0
		8	1 464	1.184 2	33.679 6
		9	1 136	3.918 9	36.648 6
		10	898	5.241	388.465
		11	790	0.533	168.053
		12	760	0.021	182.280
		13	607	1.077	1 021.249
		14	572	3.401	484.444
		15	561	2.887	498.671
NEPTUNE	R2	1	4 247	5.899 1	38.133 0
		2	218	0.346	1.484
		3	163	2.239	168.053
		4	156	4.594	182.280
		5	127	2.848	35.164
NEPTUNE	R3	1	166	4.552	38.133

Appendix IV

Coefficients for the Heliocentric Coordinates, Jupiter to Neptune, 1998–2025

On the following pages coefficients are given for the calculation of the heliocentric coordinates (ecliptical longitude L, latitude B, and radius vector R) of the giant planets Jupiter to Neptune for any instant during the years 1998 to 2025. The formula is

$$A_0 + A_1t + A_2t^2 + A_3t^3 + A_4t^4 + A_5t^5$$

where $t = d/365$, and d is the number of the day in the year. In other words, the time is measured from the given Epoch in units of 365 days. The uniform time scale of Dynamical Time is used, and the heliocentric longitude and latitude are referred to the mean equinox of the date (FK5 system). Each expression is valid for one year.

See more explanations on pages 220–221.

	A_0	A_1	A_2	A_3	A_4	A_5
JUPITER	Year 1998	Epoch = JDE 2450813.5				
L	329.6821899	32.5373311	0.6251361	−0.0898353	−0.0386157	0.0083980
B	−0.9881379	−0.4836213	0.1496847	0.0343504	−0.0042712	−0.0008253
R	5.0212516	−0.0962003	0.0267709	0.0046167	0.0001739	−0.0003582
JUPITER	Year 1999	Epoch = JDE 2451178.5				
L	2.7246041	33.4053704	0.2040247	−0.1756398	−0.0005659	0.0000648
B	−1.2928206	−0.1024089	0.2189535	0.0088476	−0.0085174	0.0000270
R	4.9562546	−0.0298757	0.0376516	0.0032669	−0.0026228	0.0005117
JUPITER	Year 2000	Epoch = JDE 2451543.5				
L	36.1578582	33.2848762	−0.3202376	−0.1607197	0.0101019	0.0048032
B	−1.1759189	0.3281328	0.1949128	−0.0233461	−0.0076376	0.0015479
R	4.9651863	0.0472665	0.0371120	−0.0037156	−0.0001438	−0.0002643
JUPITER	Year 2001	Epoch = JDE 2451909.5				
L	69.0649681	32.2226338	−0.6998931	−0.0829214	0.0310692	−0.0040156
B	−0.6805959	0.6256131	0.0938187	−0.0391518	−0.0003911	0.0007610
R	5.0457384	0.1086124	0.0221681	−0.0046960	−0.0019856	0.0006987
JUPITER	Year 2002	Epoch = JDE 2452274.5				
L	100.5318411	30.6783457	−0.7959767	0.0046134	0.0241846	−0.0027664
B	0.0000541	0.6980207	−0.0179458	−0.0339429	0.0040248	−0.0001891
R	5.1705360	0.1343765	0.0032467	−0.0076620	0.0016344	−0.0005326
JUPITER	Year 2003	Epoch = JDE 2452639.5				
L	130.4402417	29.1831608	−0.6703649	0.0749318	0.0072343	−0.0006340
B	0.6500218	0.5754659	−0.0977839	−0.0188353	0.0024263	0.0001555
R	5.3015990	0.1218037	−0.0151896	−0.0040044	−0.0006998	0.0004889
JUPITER	Year 2004	Epoch = JDE 2453004.5				
L	159.0345698	28.0927741	−0.4025854	0.0863880	0.0149846	−0.0039962
B	1.1114503	0.3338606	−0.1383051	−0.0085903	0.0027221	−0.0003263
R	5.4039978	0.0790223	−0.0266881	−0.0038764	0.0019314	−0.0005949

	A_0	A_1	A_2	A_3	A_4	A_5
JUPITER	Year 2005	Epoch = JDE 2453370.5				
L	186.8977139	27.5863948	−0.0972861	0.1182676	−0.0111922	0.0041159
B	1.3009218	0.0399162	−0.1506563	−0.0005039	0.0017402	0.0000343
R	5.4538435	0.0186227	−0.0322694	−0.0003512	0.0000610	0.0002636
JUPITER	Year 2006	Epoch = JDE 2453735.5				
L	214.4980139	27.7220245	0.2346710	0.0921343	0.0115682	−0.0055378
B	1.1914523	−0.2557633	−0.1417826	0.0075267	0.0009788	0.0004777
R	5.4401702	−0.0454318	−0.0307570	0.0012036	0.0012608	−0.0003290
JUPITER	Year 2007	Epoch = JDE 2454100.5				
L	242.5528742	28.4867192	0.5229238	0.1039235	−0.0236677	0.0055436
B	0.8028896	−0.5104669	−0.1083669	0.0148232	0.0037680	−0.0003019
R	5.3661167	−0.0999313	−0.0222721	0.0033557	0.0007824	−0.0000057
JUPITER	Year 2008	Epoch = JDE 2454465.5				
L	271.6483166	29.7769198	0.7477661	0.0397082	−0.0020776	−0.0067762
B	0.2023449	−0.6691526	−0.0441376	0.0280087	0.0026931	−0.0000486
R	5.2480457	−0.1313144	−0.0081191	0.0061518	0.0001181	−0.0000020
JUPITER	Year 2009	Epoch = JDE 2454831.5				
L	302.2897525	31.3542001	0.7873094	−0.0101742	−0.0408975	0.0067920
B	−0.4821079	−0.6625902	0.0554601	0.0371385	0.0014699	−0.0011593
R	5.1145278	−0.1285893	0.0115895	0.0057933	0.0005733	−0.0002814
JUPITER	Year 2010	Epoch = JDE 2455196.5				
L	334.3869822	32.7683636	0.5760783	−0.1238230	−0.0142657	−0.0012614
B	−1.0517888	−0.4401708	0.1643785	0.0313220	−0.0042798	−0.0010832
R	5.0036133	−0.0871279	0.0290746	0.0061802	−0.0019050	0.0002955
JUPITER	Year 2011	Epoch = JDE 2455561.5				
L	7.5920741	33.4861111	0.1105793	−0.1707292	−0.0141162	0.0086056
B	−1.3016221	−0.0399523	0.2216422	0.0051585	−0.0103989	0.0009179
R	4.9501306	−0.0166052	0.0395130	0.0001809	−0.0004479	−0.0002501

	A_0	A_1	A_2	A_3	A_4	A_5
JUPITER	Year 2012	Epoch = JDE 2455926.5				
L	41.0125248	33.1813654	−0.4045897	−0.1565926	0.0274250	−0.0029575
B	−1.1242548	0.3817953	0.1837984	−0.0278906	−0.0055891	0.0010414
R	4.9725213	0.0599571	0.0345019	−0.0022206	−0.0024916	0.0006584
JUPITER	Year 2013	Epoch = JDE 2456292.5				
L	73.7448347	31.9934973	−0.7339396	−0.0653781	0.0248588	−0.0004753
B	−0.5893218	0.6489931	0.0771672	−0.0397064	0.0006613	0.0006391
R	5.0632434	0.1156997	0.0196696	−0.0073820	0.0009734	−0.0003958
JUPITER	Year 2014	Epoch = JDE 2456657.5				
L	104.9633978	30.4264402	−0.7914405	0.0238249	0.0197071	−0.0032518
B	0.0984324	0.6900469	−0.0320762	−0.0306204	0.0027843	0.0002163
R	5.1918082	0.1348517	−0.0007139	−0.0050416	−0.0011251	0.0005752
JUPITER	Year 2015	Epoch = JDE 2457022.5				
L	134.6386778	28.9775308	−0.6276824	0.0669581	0.0168200	−0.0032219
B	0.7287832	0.5462341	−0.1048855	−0.0184840	0.0039686	−0.0005057
R	5.3203546	0.1166386	−0.0168448	−0.0058563	0.0020150	−0.0006241
JUPITER	Year 2016	Epoch = JDE 2457387.5				
L	163.0690823	27.9743024	−0.3634612	0.1078701	−0.0037752	0.0019072
B	1.1551108	0.2943723	−0.1414807	−0.0066130	0.0015662	0.0001482
R	5.4156829	0.0703597	−0.0283421	−0.0018086	−0.0003652	0.0004005
JUPITER	Year 2017	Epoch = JDE 2457753.5				
L	190.8614472	27.5648846	−0.0376795	0.0950355	0.0125018	−0.0044412
B	1.3030989	−0.0022487	−0.1508309	0.0009584	0.0013231	0.0001389
R	5.4559510	0.0085834	−0.0322787	−0.0010422	0.0016684	−0.0004893
JUPITER	Year 2018	Epoch = JDE 2458118.5				
L	218.4917484	27.8027013	0.2742538	0.1168917	−0.0176575	0.0051412
B	1.1524397	−0.2950583	−0.1382465	0.0070141	0.0027396	−0.0001175
R	5.4323925	−0.0548542	−0.0297567	0.0018530	0.0004233	0.0001513

	A_0	A_1	A_2	A_3	A_4	A_5
JUPITER	Year 2019	Epoch = JDE 2458483.5				
L	246.6730789	28.6564988	0.5720391	0.0728729	0.0058592	−0.0066096
B	0.7287711	−0.5401173	−0.1020927	0.0181279	0.0018647	0.0004885
R	5.3502092	−0.1063772	−0.0206549	0.0041448	0.0007904	−0.0001698
JUPITER	Year 2020	Epoch = JDE 2458848.5				
L	275.9737393	30.0100340	0.7588199	0.0547962	−0.0346193	0.0058199
B	0.1070422	−0.6800467	−0.0317680	0.0287437	0.0041069	−0.0008004
R	5.2279426	−0.1329443	−0.0046128	0.0054240	0.0007122	−0.0001295
JUPITER	Year 2021	Epoch = JDE 2459214.5				
L	306.8551230	31.5865271	0.7714122	−0.0472867	−0.0137630	−0.0043244
B	−0.5744886	−0.6445307	0.0717685	0.0374925	0.0006703	−0.0011078
R	5.0960533	−0.1236156	0.0141271	0.0071851	−0.0009036	0.0001141
JUPITER	Year 2022	Epoch = JDE 2459579.5				
L	339.1476882	32.9113174	0.5060695	−0.1187728	−0.0354495	0.0095128
B	−1.1101959	−0.3913655	0.1767019	0.0296471	−0.0064692	−0.0004809
R	4.9929604	−0.0768694	0.0318479	0.0036877	−0.0002284	−0.0002646
JUPITER	Year 2023	Epoch = JDE 2459944.5				
L	12.4203656	33.4726048	0.0288015	−0.1829167	0.0099944	−0.0010078
B	−1.3021626	0.0227011	0.2221304	−0.0012170	−0.0087196	0.0004485
R	4.9511336	−0.0043210	0.0384200	0.0015527	−0.0026066	0.0005278
JUPITER	Year 2024	Epoch = JDE 2460309.5				
L	45.7478419	33.0167364	−0.4649779	−0.1353636	0.0149485	0.0036406
B	−1.0668191	0.4306968	0.1709431	−0.0302545	−0.0055970	0.0014732
R	4.9847066	0.0693641	0.0330463	−0.0050572	0.0000980	−0.0002446
JUPITER	Year 2025	Epoch = JDE 2460675.5				
L	78.2698292	31.7543549	−0.7507255	−0.0511148	0.0303432	−0.0048466
B	−0.4977303	0.6671174	0.0605776	−0.0387153	0.0010356	0.0005217
R	5.0822406	0.1195829	0.0156830	−0.0049544	−0.0016537	0.0006311

	A_0	A_1	A_2	A_3	A_4	A_5
SATURN	Year 1998	Epoch = JDE 2450813.5				
L	19.7448573	12.6969726	0.1368882	0.0062460	−0.0099932	0.0034523
B	−2.4797134	−0.0370230	0.0601488	0.0020497	−0.0007432	0.0001794
R	9.3669872	−0.1015298	0.0037320	0.0007303	−0.0002364	0.0001522
SATURN	Year 1999	Epoch = JDE 2451178.5				
L	32.5784232	12.9666139	0.1294965	−0.0105762	0.0076613	−0.0036652
B	−2.4551018	0.0873406	0.0635537	0.0003573	−0.0001129	−0.0001191
R	9.2698355	−0.0920465	0.0052931	0.0019792	−0.0013122	0.0004223
SATURN	Year 2000	Epoch = JDE 2451543.5				
L	45.6679535	13.2064010	0.1101519	−0.0047180	−0.0030779	0.0012521
B	−2.3040822	0.2144789	0.0629352	−0.0008924	−0.0005166	0.0000635
R	9.1841714	−0.0786804	0.0080428	−0.0004246	0.0011535	−0.0004310
SATURN	Year 2001	Epoch = JDE 2451909.5				
L	59.0146928	13.4069581	0.0860321	−0.0066438	−0.0009758	−0.0005570
B	−2.0270929	0.3362430	0.0575093	−0.0021434	−0.0007976	0.0001291
R	9.1136636	−0.0613166	0.0091754	0.0023443	−0.0018009	0.0006689
SATURN	Year 2002	Epoch = JDE 2452274.5				
L	72.4995064	13.5523675	0.0590572	−0.0190902	0.0072511	−0.0026972
B	−1.6361526	0.4422784	0.0477640	−0.0045641	0.0001316	−0.0001680
R	9.0627346	−0.0398191	0.0118501	0.0000014	0.0008616	−0.0004634
SATURN	Year 2003	Epoch = JDE 2452639.5				
L	86.0963948	13.6289016	0.0157981	−0.0062851	−0.0070380	0.0025844
B	−1.1507107	0.5238144	0.0331432	−0.0048639	−0.0008078	0.0002564
R	9.0351652	−0.0149521	0.0129003	0.0006775	−0.0007201	0.0002840
SATURN	Year 2004	Epoch = JDE 2453004.5				
L	99.7303559	13.6261946	−0.0183448	−0.0223133	0.0115296	−0.0045045
B	−0.5991684	0.5735498	0.0161432	−0.0061982	0.0002143	−0.0001050
R	9.0333550	0.0114227	0.0127670	0.0006005	−0.0004786	0.0000398

	A_0	A_1	A_2	A_3	A_4	A_5
	SATURN	Year 2005	Epoch = JDE 2453370.5			
L	113.3600304	13.5460315	−0.0593246	−0.0088776	−0.0040233	0.0022106
B	−0.0139545	0.5875677	−0.0020420	−0.0060833	0.0001101	0.0000440
R	9.0578081	0.0371069	0.0127598	−0.0013820	0.0010512	−0.0004217
	SATURN	Year 2006	Epoch = JDE 2453735.5			
L	126.8360470	13.3955444	−0.0911468	−0.0122376	0.0049237	−0.0020233
B	0.5656419	0.5658951	−0.0193918	−0.0050723	−0.0000684	0.0001223
R	9.1069223	0.0606146	0.0103141	0.0007824	−0.0014604	0.0005055
	SATURN	Year 2007	Epoch = JDE 2454100.5			
L	140.1311075	13.1861124	−0.1140389	−0.0128927	0.0055090	−0.0012513
B	1.1071268	0.5122260	−0.0337146	−0.0045624	0.0006098	−0.0001060
R	9.1776786	0.0802443	0.0090515	−0.0020322	0.0016445	−0.0007113
	SATURN	Year 2008	Epoch = JDE 2454465.5			
L	153.1945460	12.9351827	−0.1357811	0.0009022	−0.0048274	0.0020824
B	1.5815797	0.4330239	−0.0447726	−0.0028207	0.0000966	0.0001016
R	9.2658754	0.0953134	0.0060668	−0.0002510	−0.0007978	0.0003521
	SATURN	Year 2009	Epoch = JDE 2454831.5			
L	166.0267815	12.6564216	−0.1389283	−0.0113369	0.0111521	−0.0038478
B	1.9681285	0.3356251	−0.0517770	−0.0015747	0.0001886	0.0000333
R	9.3668471	0.1052642	0.0034728	−0.0009956	0.0005031	−0.0002889
	SATURN	Year 2010	Epoch = JDE 2455196.5			
L	178.5402422	12.3700943	−0.1442983	0.0062540	−0.0070508	0.0034336
B	2.2506239	0.2282659	−0.0549763	−0.0006504	0.0003647	−0.0000512
R	9.4748026	0.1097941	0.0013751	−0.0018436	0.0010008	−0.0003215
	SATURN	Year 2011	Epoch = JDE 2455561.5			
L	190.7686750	12.0889793	−0.1358846	−0.0011339	0.0056670	−0.0022230
B	2.4235765	0.1175682	−0.0552441	0.0005608	−0.0000541	0.0000983
R	9.5848075	0.1094277	−0.0018981	0.0002135	−0.0009596	0.0003480

	A_0	A_1	A_2	A_3	A_4	A_5
SATURN		Year 2012	Epoch = JDE 2455926.5			
L	202.7240798	11.8254081	−0.1238258	0.0025325	0.0002032	0.0007134
B	2.4865057	0.0090374	−0.0530558	0.0012504	−0.0000236	0.0000373
R	9.6919390	0.1041370	−0.0032098	−0.0023756	0.0019690	−0.0007545
SATURN		Year 2013	Epoch = JDE 2456292.5			
L	214.4608626	11.5891127	−0.1119922	0.0108332	−0.0055004	0.0020909
B	2.4434955	−0.0935032	−0.0489102	0.0013073	0.0003353	−0.0000939
R	9.7919646	0.0946956	−0.0058810	0.0000588	−0.0009202	0.0004417
SATURN		Year 2014	Epoch = JDE 2456657.5			
L	225.9454068	11.3858943	−0.0889779	−0.0011738	0.0074964	−0.0025684
B	2.3026308	−0.1865194	−0.0440229	0.0023516	−0.0005050	0.0002148
R	9.8803594	0.0816029	−0.0072784	−0.0011157	0.0009451	−0.0004068
SATURN		Year 2015	Epoch = JDE 2457022.5			
L	237.2460773	11.2217378	−0.0741577	0.0144744	−0.0097920	0.0041482
B	2.0741498	−0.2684658	−0.0379173	0.0018377	0.0003680	−0.0001549
R	9.9541066	0.0654485	−0.0082795	−0.0011954	0.0006345	−0.0001263
SATURN		Year 2016	Epoch = JDE 2457387.5			
L	248.4024880	11.0981886	−0.0498654	0.0036715	0.0045755	−0.0019976
B	1.7698174	−0.3380817	−0.0315546	0.0022754	−0.0001424	0.0000873
R	10.0105884	0.0472140	−0.0099693	0.0005838	−0.0008777	0.0003398
SATURN		Year 2017	Epoch = JDE 2457753.5			
L	259.4872465	11.0177308	−0.0281315	0.0082957	−0.0029458	0.0016845
B	1.4013204	−0.3946323	−0.0249619	0.0026334	−0.0002708	0.0000867
R	10.0479536	0.0271211	−0.0096701	−0.0018104	0.0017667	−0.0006354
SATURN		Year 2018	Epoch = JDE 2458118.5			
L	270.4838802	10.9829864	−0.0084115	0.0134246	−0.0058676	0.0019517
B	0.9841755	−0.4373125	−0.0176187	0.0019585	0.0004897	−0.0001736
R	10.0647254	0.0062681	−0.0108303	0.0008458	−0.0013031	0.0005968

	A_0	A_1	A_2	A_3	A_4	A_5
SATURN	Year 2019	Epoch = JDE 2458483.5				
L	281.4679638	10.9926030	0.0191397	0.0019659	0.0056652	−0.0019957
B	0.5315188	−0.4655717	−0.0106033	0.0029422	−0.0005030	0.0002128
R	10.0603027	−0.0151233	−0.0105719	−0.0006684	0.0008559	−0.0003573
SATURN	Year 2020	Epoch = JDE 2458848.5				
L	292.4853419	11.0496805	0.0373569	0.0174504	−0.0108844	0.0042565
B	0.0579959	−0.4789099	−0.0027643	0.0024092	0.0003160	−0.0001260
R	10.0344379	−0.0366309	−0.0103356	−0.0003887	0.0001631	0.0000851
SATURN	Year 2021	Epoch = JDE 2459214.5				
L	303.6137622	11.1546585	0.0657521	0.0047931	0.0046632	−0.0023548
B	−0.4223848	−0.4765430	0.0053044	0.0027169	0.0000218	0.0000308
R	9.9871735	−0.0574429	−0.0104089	0.0013747	−0.0010359	0.0004033
SATURN	Year 2022	Epoch = JDE 2459579.5				
L	314.8412743	11.3076017	0.0873280	0.0099681	−0.0049667	0.0021224
B	−0.8908538	−0.4575420	0.0137280	0.0031794	−0.0002048	0.0000870
R	9.9200638	−0.0763029	−0.0080128	−0.0008734	0.0014653	−0.0005126
SATURN	Year 2023	Epoch = JDE 2459944.5				
L	326.2433278	11.5029149	0.1044171	0.0111777	−0.0044895	0.0010461
B	−1.3316062	−0.4209363	0.0229542	0.0029128	0.0002134	−0.0000749
R	9.8358273	−0.0916158	−0.0070946	0.0020000	−0.0017546	0.0007482
SATURN	Year 2024	Epoch = JDE 2460309.5				
L	337.8583941	11.7324983	0.1249764	−0.0003094	0.0052573	−0.0021326
B	−1.7265370	−0.3658077	0.0322689	0.0032320	−0.0001439	0.0000585
R	9.7381105	−0.1031246	−0.0045061	0.0001035	0.0008145	−0.0003540
SATURN	Year 2025	Epoch = JDE 2460675.5				
L	349.7515396	11.9928729	0.1320617	0.0131542	−0.0107448	0.0037657
B	−2.0577285	−0.2916285	0.0415890	0.0032586	−0.0002831	0.0000751
R	9.6307415	−0.1103354	−0.0022551	0.0008725	−0.0004414	0.0002705

	A_0	A_1	A_2	A_3	A_4	A_5
URANUS	Year 1998	Epoch = JDE 2450813.5				
L	308.3946411	4.0234022	−0.0099074	0.0013007	−0.0021907	0.0010334
B	−0.6281267	−0.0316524	0.0015991	0.0001139	−0.0000869	0.0000281
R	19.8439293	0.0443881	−0.0017243	−0.0013796	0.0013722	−0.0004733
URANUS	Year 1999	Epoch = JDE 2451178.5				
L	312.4082792	4.0039196	−0.0113243	0.0039059	−0.0036098	0.0011560
B	−0.6581250	−0.0283220	0.0017436	−0.0001090	0.0001411	−0.0000552
R	19.8861125	0.0399427	−0.0024498	0.0011943	−0.0016195	0.0007214
URANUS	Year 2000	Epoch = JDE 2451543.5				
L	316.4023267	3.9842795	−0.0081955	−0.0029436	0.0035933	−0.0014074
B	−0.6847265	−0.0248706	0.0017191	0.0001005	−0.0001187	0.0000503
R	19.9239016	0.0357276	−0.0020273	−0.0000038	0.0001878	−0.0001152
URANUS	Year 2001	Epoch = JDE 2451909.5				
L	320.3885200	3.9664687	−0.0097794	0.0049444	−0.0054151	0.0021564
B	−0.7079044	−0.0213463	0.0017552	0.0000082	0.0000180	−0.0000143
R	19.9577580	0.0318173	−0.0013643	−0.0007121	0.0004838	−0.0000877
URANUS	Year 2002	Epoch = JDE 2452274.5				
L	324.3468951	3.9508084	−0.0074756	0.0006290	0.0007218	−0.0005844
B	−0.7274836	−0.0178115	0.0018137	−0.0000903	0.0000868	−0.0000288
R	19.9878950	0.0284669	−0.0021314	0.0014865	−0.0017320	0.0006781
URANUS	Year 2003	Epoch = JDE 2452639.5				
L	328.2909943	3.9377576	−0.0051693	0.0000154	0.0000545	0.0000916
B	−0.7435137	−0.0142493	0.0017331	0.0001269	−0.0001433	0.0000554
R	20.0146632	0.0250860	−0.0012394	−0.0010780	0.0011495	−0.0004444
URANUS	Year 2004	Epoch = JDE 2453004.5				
L	332.2237441	3.9282198	−0.0059965	0.0053021	−0.0048520	0.0016624
B	−0.7559908	−0.0107012	0.0017997	−0.0000888	0.0001159	−0.0000498
R	20.0381369	0.0217743	−0.0017986	0.0007608	−0.0012460	0.0005746

	A_0	A_1	A_2	A_3	A_4	A_5
URANUS	Year 2005	Epoch = JDE 2453370.5				
L	336.1588225	3.9209574	−0.0018016	−0.0026331	0.0039271	−0.0016943
B	−0.7649346	−0.0071424	0.0017870	−0.0000051	−0.0000173	0.0000138
R	20.0582522	0.0183255	−0.0020410	0.0005349	−0.0005401	0.0001568
URANUS	Year 2006	Epoch = JDE 2453735.5				
L	340.0775781	3.9168193	−0.0023459	0.0037264	−0.0041381	0.0016941
B	−0.7702986	−0.0035834	0.0017364	0.0000929	−0.0000886	0.0000294
R	20.0746882	0.0144546	−0.0014695	−0.0010956	0.0009112	−0.0002909
URANUS	Year 2007	Epoch = JDE 2454100.5				
L	343.9933339	3.9152077	−0.0013000	0.0021909	−0.0010142	0.0000326
B	−0.7721119	−0.0000415	0.0018199	−0.0001263	0.0001434	−0.0000556
R	20.0871980	0.0104493	−0.0026204	0.0015434	−0.0019845	0.0008009
URANUS	Year 2008	Epoch = JDE 2454465.5				
L	347.9084509	3.9152947	0.0016034	−0.0020211	0.0024112	−0.0009125
B	−0.7703719	0.0035180	0.0017552	0.0000880	−0.0001148	0.0000494
R	20.0953867	0.0058701	−0.0021983	−0.0006627	0.0007453	−0.0003326
URANUS	Year 2009	Epoch = JDE 2454831.5				
L	351.8355595	3.9176400	−0.0004279	0.0051042	−0.0052751	0.0019119
B	−0.7650567	0.0070885	0.0017676	0.0000065	0.0000153	−0.0000132
R	20.0988108	0.0008070	−0.0025051	0.0000724	−0.0005166	0.0002948
URANUS	Year 2010	Epoch = JDE 2455196.5				
L	355.7545126	3.9204849	0.0021446	−0.0022638	0.0034613	−0.0016180
B	−0.7561921	0.0106378	0.0018161	−0.0000961	0.0000892	−0.0000301
R	20.0969632	−0.0045723	−0.0032576	0.0012002	−0.0012976	0.0004622
URANUS	Year 2011	Epoch = JDE 2455561.5				
L	359.6767218	3.9238438	0.0014779	0.0014628	−0.0020231	0.0009043
B	−0.7437752	0.0141902	0.0017203	0.0001200	−0.0001449	0.0000558
R	20.0894980	−0.0103940	−0.0023728	−0.0011701	0.0012464	−0.0004592

	A_0	A_1	A_2	A_3	A_4	A_5
	URANUS	Year 2012	Epoch = JDE 2455926.5			
L	3.6023875	3.9276448	0.0003642	0.0032150	−0.0024109	0.0006078
B	−0.7278338	0.0176871	0.0017578	−0.0000982	0.0001124	−0.0000490
R	20.0763484	−0.0159219	−0.0030759	0.0015680	−0.0019150	0.0008013
	URANUS	Year 2013	Epoch = JDE 2456292.5			
L	7.5425796	3.9313963	0.0034892	−0.0034020	0.0043364	−0.0017360
B	−0.7083658	0.0211235	0.0017049	−0.0000243	−0.0000146	0.0000125
R	20.0577472	−0.0210600	−0.0025105	0.0002189	0.0000573	−0.0000987
	URANUS	Year 2014	Epoch = JDE 2456657.5			
L	11.4766635	3.9369705	0.0016703	0.0045735	−0.0046517	0.0017806
B	−0.6855638	0.0244651	0.0016003	0.0000765	−0.0000918	0.0000307
R	20.0343542	−0.0256833	−0.0017734	−0.0004195	0.0002851	−0.0000423
	URANUS	Year 2015	Epoch = JDE 2457022.5			
L	15.4170066	3.9442719	0.0041430	−0.0006173	0.0023110	−0.0012324
B	−0.6594830	0.0276789	0.0016264	−0.0001459	0.0001436	−0.0000559
R	20.0067208	−0.0295335	−0.0024078	0.0017729	−0.0019111	0.0007160
	URANUS	Year 2016	Epoch = JDE 2457387.5			
L	19.3658827	3.9538550	0.0059123	−0.0002440	0.0005465	−0.0001220
B	−0.6302359	0.0307918	0.0015029	0.0000655	−0.0001126	0.0000487
R	19.9753573	−0.0331294	−0.0012957	−0.0009381	0.0011617	−0.0004855
	URANUS	Year 2017	Epoch = JDE 2457753.5			
L	23.3366976	3.9666314	0.0051813	0.0043644	−0.0034992	0.0010928
B	−0.5978470	0.0337938	0.0014529	−0.0000100	0.0000120	−0.0000117
R	19.9405709	−0.0362844	−0.0017189	0.0010807	−0.0014608	0.0006325
	URANUS	Year 2018	Epoch = JDE 2458118.5			
L	27.3104684	3.9814942	0.0094084	−0.0035588	0.0051513	−0.0021644
B	−0.5626100	0.0366587	0.0014472	−0.0001155	0.0000929	−0.0000313
R	19.9028200	−0.0391742	−0.0017216	0.0006973	−0.0005305	0.0001099

	A_0	A_1	A_2	A_3	A_4	A_5
	URANUS	Year 2019	Epoch = JDE 2458483.5			
L	31.3007991	3.9995492	0.0087290	0.0032347	−0.0032874	0.0013338
B	−0.5245579	0.0394244	0.0013056	0.0001070	−0.0001440	0.0000560
R	19.8622009	−0.0420977	−0.0009882	−0.0008669	0.0008068	−0.0002803
	URANUS	Year 2020	Epoch = JDE 2458848.5			
L	35.3103585	4.0202030	0.0101963	0.0008671	0.0006986	−0.0006386
B	−0.4838089	0.0420579	0.0013107	−0.0001041	0.0001112	−0.0000485
R	19.8187746	−0.0448163	−0.0019781	0.0017314	−0.0020253	0.0007742
	URANUS	Year 2021	Epoch = JDE 2459214.5			
L	39.3527615	4.0428802	0.0129105	−0.0025246	0.0030871	−0.0011659
B	−0.4403595	0.0445777	0.0012393	−0.0000335	−0.0000110	0.0000109
R	19.7723296	−0.0478387	−0.0014164	−0.0005968	0.0008328	−0.0004044
	URANUS	Year 2022	Epoch = JDE 2459579.5			
L	43.4079488	4.0677519	0.0109318	0.0041260	−0.0039755	0.0013748
B	−0.3945761	0.0469666	0.0011124	0.0000701	−0.0000958	0.0000318
R	19.7229060	−0.0511244	−0.0016867	0.0003213	−0.0006636	0.0003205
	URANUS	Year 2023	Epoch = JDE 2459944.5			
L	47.4881579	4.0928916	0.0134868	−0.0035210	0.0048241	−0.0021126
B	−0.3464911	0.0491749	0.0011049	−0.0001601	0.0001422	−0.0000562
R	19.6700732	−0.0545761	−0.0023104	0.0011868	−0.0010745	0.0003239
	URANUS	Year 2024	Epoch = JDE 2460309.5			
L	51.5937268	4.1181401	0.0122956	0.0008849	−0.0013419	0.0006425
B	−0.2962854	0.0511951	0.0009292	0.0000420	−0.0001118	0.0000481
R	19.6136229	−0.0583266	−0.0013417	−0.0011207	0.0013155	−0.0005128
	URANUS	Year 2025	Epoch = JDE 2460675.5			
L	55.7356990	4.1433080	0.0111440	0.0018686	−0.0009665	0.0000679
B	−0.2440376	0.0529757	0.0008056	−0.0000329	0.0000075	−0.0000102
R	19.5534677	−0.0616438	−0.0020415	0.0016863	−0.0018582	0.0007321

	A_0	A_1	A_2	A_3	A_4	A_5
NEPTUNE		Year 1998	Epoch = JDE 2450813.5			
L	299.5407539	2.1902131	0.0002159	−0.0000101	−0.0007024	0.0004022
B	0.3746913	−0.0657516	−0.0003038	0.0001425	−0.0001276	0.0000450
R	30.1452426	−0.0114799	0.0000256	−0.0016443	0.0017726	−0.0006393
NEPTUNE		Year 1999	Epoch = JDE 2451178.5			
L	301.7308726	2.1898447	−0.0014772	0.0025844	−0.0027128	0.0009371
B	0.3086958	−0.0662200	−0.0001651	−0.0001133	0.0001518	−0.0000615
R	30.1332773	−0.0124411	−0.0005349	0.0010464	−0.0014170	0.0006547
NEPTUNE		Year 2000	Epoch = JDE 2451543.5			
L	303.9200489	2.1884417	0.0000758	−0.0019024	0.0022339	−0.0008929
B	0.2422877	−0.0665872	−0.0001798	0.0000780	−0.0000946	0.0000426
R	30.1205855	−0.0128018	0.0000027	−0.0000216	0.0002692	−0.0001503
NEPTUNE		Year 2001	Epoch = JDE 2451909.5			
L	306.1139979	2.1874304	−0.0010030	0.0025367	−0.0029947	0.0012256
B	0.1753634	−0.0668803	−0.0001554	0.0000482	−0.0000188	0.0000002
R	30.1078496	−0.0125395	0.0008452	−0.0009250	0.0007678	−0.0001989
NEPTUNE		Year 2002	Epoch = JDE 2452274.5			
L	308.3011927	2.1871600	−0.0001927	0.0006350	0.0000066	−0.0001904
B	0.1083572	−0.0671220	−0.0000564	−0.0000988	0.0001133	−0.0000401
R	30.0957992	−0.0115299	0.0000961	0.0015161	−0.0018195	0.0007336
NEPTUNE		Year 2003	Epoch = JDE 2452639.5			
L	310.4886113	2.1877683	0.0012570	−0.0008609	0.0008138	−0.0002414
B	0.0411532	−0.0672759	−0.0001019	0.0001349	−0.0001418	0.0000566
R	30.0847956	−0.0104405	0.0009560	−0.0010546	0.0011607	−0.0004588
NEPTUNE		Year 2004	Epoch = JDE 2453004.5			
L	312.6773480	2.1898129	−0.0000225	0.0035948	−0.0037652	0.0013826
B	−0.0261749	−0.0673619	−0.0000062	−0.0000484	0.0000897	−0.0000402
R	30.0749585	−0.0093289	0.0006114	0.0002615	−0.0007761	0.0004204

	A_0	A_1	A_2	A_3	A_4	A_5
NEPTUNE	Year 2005	Epoch = JDE 2453370.5				
L	314.8743572	2.1923691	0.0019779	−0.0014836	0.0021474	−0.0009828
B	−0.0937264	−0.0673602	0.0000461	−0.0000126	0.0000113	0.0000024
R	30.0661238	−0.0083386	0.0000697	0.0007833	−0.0009169	0.0003301
NEPTUNE	Year 2006	Epoch = JDE 2453735.5				
L	317.0683853	2.1956170	0.0016596	0.0010143	−0.0015197	0.0006827
B	−0.1610394	−0.0672478	0.0000391	0.0001174	−0.0001019	0.0000363
R	30.0580514	−0.0078999	0.0007450	−0.0014202	0.0012833	−0.0004389
NEPTUNE	Year 2007	Epoch = JDE 2454100.5				
L	319.2658391	2.1993302	0.0008806	0.0022165	−0.0020477	0.0006061
B	−0.2281964	−0.0670460	0.0001690	−0.0000905	0.0001259	−0.0000496
R	30.0503208	−0.0077131	−0.0003349	0.0010779	−0.0015939	0.0006915
NEPTUNE	Year 2008	Epoch = JDE 2454465.5				
L	321.4668247	2.2025604	0.0023563	−0.0020028	0.0021281	−0.0008550
B	−0.2950876	−0.0667215	0.0001770	0.0000796	−0.0000857	0.0000387
R	30.0424482	−0.0081002	−0.0001917	−0.0005047	0.0005724	−0.0002591
NEPTUNE	Year 2009	Epoch = JDE 2454831.5				
L	323.6770543	2.2055920	0.0003305	0.0027771	−0.0034717	0.0013507
B	−0.3617811	−0.0662780	0.0002330	0.0000464	−0.0000133	−0.0000003
R	30.0339404	−0.0090077	−0.0001287	−0.0006463	0.0002930	0.0000011
NEPTUNE	Year 2010	Epoch = JDE 2455196.5				
L	325.8836328	2.2074198	0.0005140	−0.0004733	0.0008905	−0.0005519
B	−0.4277933	−0.0657282	0.0003431	−0.0000674	0.0000842	−0.0000292
R	30.0244517	−0.0100091	−0.0010648	0.0015176	−0.0017270	0.0006758
NEPTUNE	Year 2011	Epoch = JDE 2455561.5				
L	328.0914319	2.2078669	0.0004037	−0.0007458	0.0003201	−0.0000318
B	−0.4931907	−0.0650512	0.0003298	0.0001162	−0.0001192	0.0000476
R	30.0138443	−0.0111563	0.0000344	−0.0012071	0.0014028	−0.0005328

	A_0	A_1	A_2	A_3	A_4	A_5
NEPTUNE	Year 2012	Epoch = JDE 2455926.5				
L	330.2992450	2.2076168	−0.0016178	0.0032781	−0.0033868	0.0011936
B	−0.5578675	−0.0642839	0.0004196	−0.0000338	0.0000656	−0.0000294
R	30.0023853	−0.0117358	−0.0002383	0.0009535	−0.0013463	0.0006305
NEPTUNE	Year 2013	Epoch = JDE 2456292.5				
L	332.5123744	2.2065963	0.0001903	−0.0018869	0.0025870	−0.0011286
B	−0.6219032	−0.0634276	0.0004650	−0.0000083	0.0000041	0.0000041
R	29.9906169	−0.0116051	−0.0000613	0.0007836	−0.0006659	0.0002076
NEPTUNE	Year 2014	Epoch = JDE 2456657.5				
L	334.7187325	2.2061042	−0.0004344	0.0017465	−0.0020850	0.0008747
B	−0.6848660	−0.0624850	0.0004570	0.0000952	−0.0000878	0.0000322
R	29.9792758	−0.0110295	0.0010366	−0.0011668	0.0011044	−0.0003637
NEPTUNE	Year 2015	Epoch = JDE 2457022.5				
L	336.9249386	2.2065131	−0.0003495	0.0016632	−0.0010552	0.0001916
B	−0.7468544	−0.0614776	0.0005549	−0.0000662	0.0000906	−0.0000355
R	29.9688569	−0.0098315	0.0001907	0.0015577	−0.0019486	0.0007998
NEPTUNE	Year 2016	Epoch = JDE 2457387.5				
L	339.1319019	2.2075407	0.0015261	−0.0015291	0.0018420	−0.0007186
B	−0.8077883	−0.0603801	0.0005615	0.0000596	−0.0000680	0.0000312
R	29.9596250	−0.0086143	0.0008367	−0.0006548	0.0008027	−0.0003576
NEPTUNE	Year 2017	Epoch = JDE 2457753.5				
L	341.3466171	2.2098703	0.0001906	0.0033414	−0.0035370	0.0013188
B	−0.8677463	−0.0591918	0.0005982	0.0000475	−0.0000266	0.0000068
R	29.9516174	−0.0074637	0.0007698	−0.0001344	−0.0002736	0.0002032
NEPTUNE	Year 2018	Epoch = JDE 2458118.5				
L	343.5578014	2.2126870	0.0017473	−0.0009114	0.0017482	−0.0008936
B	−0.9263123	−0.0579263	0.0006869	−0.0000502	0.0000639	−0.0000221
R	29.9447188	−0.0063999	−0.0000426	0.0012655	−0.0014809	0.0005456

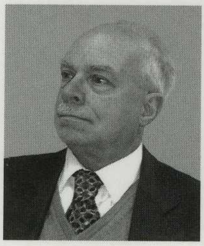
	A_0	A_1	A_2	A_3	A_4	A_5
	NEPTUNE	Year 2019	Epoch = JDE 2458483.5			
L	345.7721788	2.2160304	0.0019492	0.0001302	−0.0004086	0.0002250
B	−0.9835601	−0.0565564	0.0006838	0.0000946	−0.0000980	0.0000402
R	29.9386064	−0.0059103	0.0006087	−0.0011338	0.0010820	−0.0004058
	NEPTUNE	Year 2020	Epoch = JDE 2458848.5			
L	347.9901050	2.2198549	0.0006474	0.0027839	−0.0026574	0.0008337
B	−1.0393959	−0.0550979	0.0007576	−0.0000038	0.0000301	−0.0000148
R	29.9328472	−0.0057668	−0.0003583	0.0011605	−0.0017321	0.0007507
	NEPTUNE	Year 2021	Epoch = JDE 2459214.5			
L	350.2176582	2.2230216	0.0022203	−0.0023881	0.0028083	−0.0011985
B	−1.0938714	−0.0535429	0.0008173	−0.0000087	0.0000111	−0.0000001
R	29.9268841	−0.0062024	−0.0005211	0.0002793	−0.0002570	0.0000426
	NEPTUNE	Year 2022	Epoch = JDE 2459579.5			
L	352.4421218	2.2256340	0.0002719	0.0021329	−0.0028021	0.0011115
B	−1.1465947	−0.0518896	0.0008209	0.0000933	−0.0000896	0.0000341
R	29.9202255	−0.0072339	−0.0001205	−0.0008035	0.0005720	−0.0001362
	NEPTUNE	Year 2023	Epoch = JDE 2459944.5			
L	354.6684701	2.2269149	−0.0002141	0.0003069	0.0001917	−0.0003173
B	−1.1976256	−0.0501577	0.0009063	−0.0000366	0.0000579	−0.0000238
R	29.9125033	−0.0082600	−0.0010453	0.0016981	−0.0019454	0.0007683
	NEPTUNE	Year 2024	Epoch = JDE 2460309.5			
L	356.8953523	2.2266074	0.0003203	−0.0017139	0.0016708	−0.0006067
B	−1.2468797	−0.0483417	0.0009265	0.0000376	−0.0000455	0.0000218
R	29.9037191	−0.0092326	−0.0000327	−0.0007969	0.0010798	−0.0004546
	NEPTUNE	Year 2025	Epoch = JDE 2460675.5			
L	359.1277282	2.2258347	−0.0018769	0.0034605	−0.0036112	0.0013105
B	−1.2944081	−0.0464433	0.0009423	0.0000657	−0.0000560	0.0000191
R	29.8942558	−0.0096242	0.0000726	0.0003999	−0.0006964	0.0003620

Index

The numbers refer to the pages

O

About the Author



Jean Meeus, born in 1928, studied mathematics at the University of Louvain (Leuven) in Belgium, where he received the Degree of Licentiate in 1953. From then until his retirement in 1993, he was a meteorologist at Brussels Airport. His special interest is spherical and mathematical astronomy. He is a member of several astronomical associations and the author of many scientific papers. He is the co-author of *Canon of Solar Eclipses* (1966), the *Canon of Lunar Eclipses* (1979) and the *Canon of Solar Eclipses* (1983). His *Astronomical Formulae for Calculators* (1979, 1982, 1985 and 1988) has been widely acclaimed by both amateur and professional astronomers. Further works, published by Willmann-Bell, Inc., are *Elements of Solar Eclipses 1951–2200* (1989), *Transits* (1989), *Astronomical Tables of the Sun, Moon and Planets* (1983 and 1995) and *Mathematical Astronomy Morsels* (1997). For his numerous contributions to astronomy the International Astronomical Union announced in 1981 the naming of asteroid 2213 Meeus in his honor.

About This Book From Roger Sinnott's Introduction

In the field of celestial calculations, Jean Meeus has enjoyed wide acclaim and respect since long before microcomputers and pocket calculators appeared on the market. When he brought out his *Astronomical Formulae for Calculators* in 1979, it was practically the only book of its genre. It quickly became the "source among sources," even for other writers in the field. Many of them have warmly acknowledged their debt (or should have), citing the unparalleled clarity of his instructions and the rigor of his methods.

And now this Belgian astronomer has outdone himself yet again! Virtually every previous handbook on celestial calculations (including his own earlier work) was forced to rely on formulae for the Sun, Moon, and planets that were developed in the last century — or at least before 1920. The past 10 years, however, have seen a stunning revolution in how the world's major observatories produce their almanacs. The Jet Propulsion Laboratory in California and the U.S. Naval Observatory in Washington, D.C., have perfected powerful new machine methods for modeling the motions and interactions of bodies within the solar system. At the same time in Paris, the Bureau des Longitudes has been a beehive of activity aimed at describing these motions analytically, in the form of explicit equations.

Yet until now the fruits of this exciting work have remained mostly out of reach of ordinary people. The details have existed mainly on reels of magnetic tape in a form comprehensible only to the largest brains, human or electronic. But *Astronomical Algorithms* changes all that. With his special knack for computations of all sorts, the author has made the essentials of these modern techniques available to us all.

From a Review

There are times when an amateur astronomer wants to perform the computations that support his or her observations. *Astronomical Algorithms* is the reference to have for this. Jean Meeus' concise volume collects most of the algorithms and computational techniques an observer might want—covering coordinate transformations, the apparent place of a star, the positions of solar system bodies, eclipse predictions, and much more. Discussions are complete enough to make the equations fully understandable to the novice, and virtually every algorithm includes a fully worked numerical example. …This is a very handy reference, well worth owning, even if you never have to perform a specific calculation. The text along is helpful for understanding how the theories of celestial mechanics are applied in practice.

Sky & Telescope magazine

…There is no doubt that the book is very good value for money … computer-minded astronomers will never want to be without it.

The Observatory

ISBN 0-943396-61-1

9 780943 396613

Willmann-Bell, Inc.
Publishers and Booksellers Serving Astronomers Worldwide Since 1973

P.O. Box 35025
Richmond, Virginia 23235
☎ (804) 320-7016
FAX (804) 272-5920
www.willbell.com