Stochastic Operations Research Case Study

Juan Pablo Morande

1. Introduction

In this report the problem presented in the third case study of the stochastic operations research is presented and solved. In this problem, a call-center with ingoing and outgoing calls is modelled with a continuous Markov chain. The main objective of this study is to explore the different configurations for the value of a threshold to start making outgoing calls (*m*) and the number of agents available for the system (*n*). For this experimentation, parameters such the arrival rates and the available channels of the call-center as fixed and given.

The main findings in this case study were, for different possibilities of *m* and *n,* the probability of no wait for the customers, the expected number of rejected calls, the expected number of immediate hangs-ups, the expected number of not immediate hangs-ups, the long-run average profit and the discounted profit of the system.

The rest of this report is divided in the following section: a methodology review, a results and analysis section and finally a conclusions and recommendations section.

2. Methodology and Literature Review

As mentioned in the previous section, the methodology to solve this inventory problem was based on the modelling of continuous-time Markov chains. Using this methodology, the specific behavior of the systema was modelled to calculate the required metrics. With this model, the proportion of time in each state can be estimated and several specific counter (such as the rejected calls) can be predicted.

Most of the relationships used in this case study are found in [1], which describes in detail the methods and proofs for the matrixial computations concerning the continuous-time Markov chains (CTMC). Also, [2] is an alternative to the concepts and main components of CTMCs. Finally, in [3] a complete review of relevant aspects and a review of call center literature is available.

3. Results and Analysis

3.1 CTMC modelling

The first bullet of this case study required the modelling of the system as a Continuous-time Markov Chain (CTMC). The systems consider three sections of the state space. The first one where the states are lower than the threshold m and outgoing calls are made with a rate of kv, where v is the rate of the outgoing calls and k the probability of the call being answered. In the second section the system behaves as $M/M/n$ queue where n is the number of agents. Finally, on the last section, a probability y of immediately hanging up when entering the queue and a rate η (that depends on the number of customers on-hold) of leaving the queue. Also, it must be mentioned that the system has a fixed capacity c and that the states behave as birth and death process with rate λ and service time μ. In the following figure the model is shown.
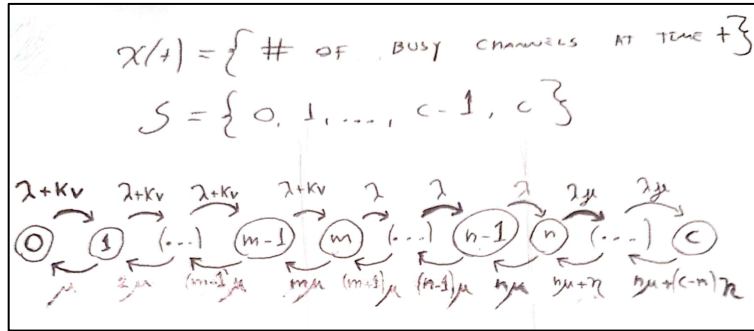

Figure 1: Rate diagram for the model

Also, the generator matrix is presented in the following figure (with given values of 2 and 4 for m and n respectively).

Figure 2: Generator matrix for the system with given values

3.2 Probabilities of no wait times

In this subsection the probabilities of no wait times for the incoming customers after exactly 10 minutes are presented. As required by the case study, different values for m (from 5 to 8) and for n (from 10 to 20) were tested. These values are the tested values for the rest of the results of this case study. To calculate these probabilities the relationship $P(t) = e^{Qt}$ was used, where Q is the generator matrix of the CTMC, and t is the time passed. To calculate this, the *scipy* library was used to evaluate the e^{Qt} expression. Then the probabilities (from an empty system as a starting point) were summed only considering the states where no wait times are incurred by the customers. The following table shows the results of these computations.

Table 1: Probability of no wait time for each strategy

m/n	10	11	12	13	14	15	16	17	18	19	20
5	0.209	0.312	0.428	0.545	0.655	0.751	0.830	0.890	0.934	0.965	0.986
6	0.202	0.306	0.421	0.539	0.650	0.747	0.827	0.888	0.933	0.965	0.986
7	0.192	0.296	0.413	0.532	0.644	0.742	0.823	0.886	0.932	0.964	0.985
8	0.178	0.283	0.401	0.522	0.636	0.736	0.819	0.883	0.930	0.963	0.985

3.3 Expected metrics

Following the last bullet, three metrics are required by the case study: the expected number rejected calls per hour, the expected number of calls that are immediately hang-up when put on hold, and the expected number of calls that are hang-up but not immediately.

To calculate these metrics the stationary distribution was calculated. Given that the CTMC is finite then a stationary distribution exists and can be found with the relationship $p = e_N \widetilde{Q}^{-1}$, where $\widetilde{Q}$ is the generator matrix with the last column set as a vector of ones and e_N is a vector of zeros but a one in the last position. With this stationary distribution the expected rejected calls were calculating only by multiplying the proportion of time in the last state (when there are no channels left) with the rate of arrival. The following results were obtained.

Table 2: Expected rejected calls for each strategy

m/n	10	11	12	13	14	15	16	17	18	19	20
5	14.177	10.692	8.203	6.493	5.352	4.619	4.178	3.959	3.921	4.047	4.341
6	14.201	10.713	8.221	6.507	5.365	4.629	4.188	3.968	3.930	4.057	4.351
7	14.251	10.758	8.258	6.539	5.391	4.652	4.208	3.988	3.949	4.076	4.372
8	14.344	10.841	8.329	6.597	5.440	4.694	4.247	4.024	3.985	4.113	4.411

For the expected calls that are immediately hang-up, the proportion of time where this can happen (when there are channels available but no agents) are summed and then multiplied by the arrival rate and the probability of the hang-up. These computation's results are shown in the following table.

Table 3: Expected number of calls that are immediately hang-up for each strategy

m/n	10	11	12	13	14	15	16	17	18	19	20
5	16.422	14.980	12.972	10.661	8.315	6.141	4.262	2.731	1.542	0.652	0
6	16.449	15.009	13.000	10.685	8.334	6.155	4.272	2.737	1.545	0.654	0
7	16.507	15.071	13.059	10.736	8.375	6.185	4.293	2.751	1.553	0.657	0
8	16.615	15.188	13.170	10.832	8.451	6.242	4.332	2.776	1.567	0.663	0

Finally, for the expected hang-ups that are not immediate was calculated by summing the proportion of time of the states that allow a transition triggered by the hang-up rate η (which will be multiplied by the number of customers waiting on hold). The following table shows the results for this computation.

Table 4: Expected hang-ups that are not immediate for each strategy

m/n	10	11	12	13	14	15	16	17	18	19	20
5	22.945	16.467	11.319	7.466	4.718	2.836	1.595	0.812	0.349	0.101	0
6	22.984	16.500	11.343	7.483	4.729	2.842	1.599	0.814	0.349	0.101	0
7	23.064	16.568	11.395	7.519	4.752	2.856	1.606	0.818	0.351	0.102	0
8	23.215	16.696	11.491	7.586	4.795	2.882	1.621	0.826	0.354	0.103	0

3.4 Long-run average costs

The next required computation in this case study was the long-run average hourly profit of the system. To compute this, the stationary distribution was used to obtain the proportion of time in which the system spends in each of the states. With the stationary distribution, the long-run average profit was computed simply by calculating a weighted average with the cost vector, which had a fixed cost (the amount of agents) and a profit that depends on the serving customers. The results for this computation are shown in the next table.

Table 5: Long-run average profit for each strategy

m/n	10	11	12	13	14	15	16	17	18	19	20
5	56.37	58.56	58.86	57.27	53.92	49.04	42.83	35.53	27.34	18.41	8.90
6	56.48	58.72	59.06	57.50	54.17	49.29	43.10	35.81	27.61	18.69	9.18
7	56.67	58.99	59.41	57.91	54.62	49.77	43.60	36.31	28.13	19.21	9.70
8	56.94	59.41	59.95	58.55	55.34	50.54	44.40	37.14	28.97	20.06	10.55

From these results, the threshold doesn't seem relevant enough for an additional analysis. In the other hand, the main factor for the results is the number of agents, which decreases the profit (probably driven by additional cost in resources that are not used).

3.5 Expected discounted costs

Finally, the last metric to compute in this case study is the total discounted profit. For this, the discounted cost vector was computed with the relationship $\phi = (\alpha I - Q)^{-1} c$, where α is the discount rate, Q the generator matrix and c the cost vector. Selecting an initial state of an empty system (state 0), the following discounted profits were obtained for each strategy.

Table 6: Expected discounted profits for each strategy with an empty initial state

m/n	10	11	12	13	14	15	16	17	18	19	20
5	56361	58551	58855	57262	53915	49026	42822	35522	27325	18403	8893
6	56475	58710	59052	57488	54161	49285	43090	35795	27601	18680	9171
7	56667	58987	59404	57897	54611	49763	43586	36302	28115	19198	9691
8	56937	59401	59947	58543	55332	50535	44391	37130	28956	20046	10542

Following the conclusion of the long-run average profits section, these results also confirm the idea that the more agents, the less profit (for the given interval). Also, the threshold number still doesn't show relevant difference at first glance.

4. Conclusions

Given the results of shown in this report, CTMCs can be useful for modelling real systems and to obtain valuables conclusions. In this case, the modelling of a call center and its strategies provided insightful conclusions in terms of the costs and of the probabilities of events of interest. Defining objectives, the decision makers can use these tools to make decisions that improve the system, saving costs and improving, in this case, the customer service.

References

[1]   Kulkarni, V. (2017). *Modeling and Analysis of Stochastic Systems.* 3rd ed. North Carolina: Chapman and Hall.

[2]   S. Ross (1996). *Wiley series in probability and statistics: Probability and statistics* Wiley.

[3]   Noah Gans, Ger Koole, Avishai Mandelbaum, (2003) Telephone Call Centers: Tutorial, Review, and Research Prospects. Manufacturing & Service Operations Management 5(2):79-141. https://doi.org/10.1287/msom.5.2.79.16071