Logic-Based Explainable Artificial Intelligence

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ESSLLI, Bochum, Germany, July 2025

Lecture 02

Recapitulate first lecture

- ML models: classification & regression

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- Glimpse of heuristic XAI

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- Answers to Why? questions as logic rules

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- ML models: classification & regression
- Glimpse of heuristic XAI
- Answers to Why? questions as logic rules
- Logic-based reasoning of ML models
- Apparent difficulties with explaining interpretable models

Plan for this course

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Unit #02

Principles of Symbolic XAI – Feature Selection

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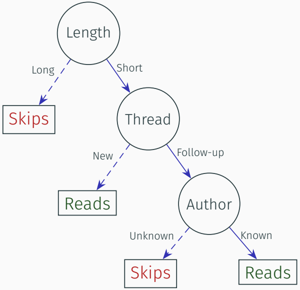
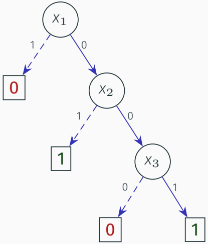
Original DT [PM17]



Rewritten DT



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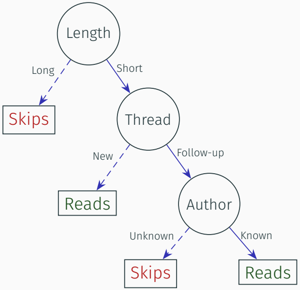

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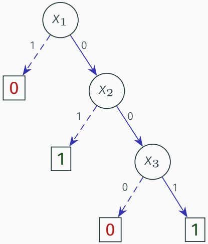

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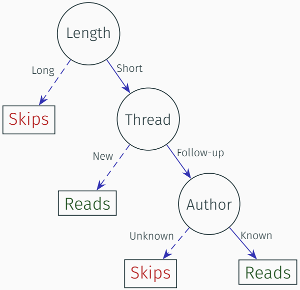

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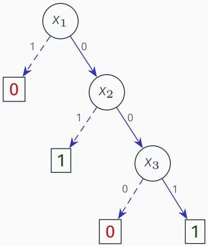

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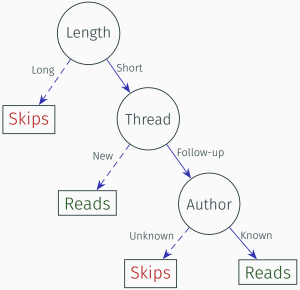

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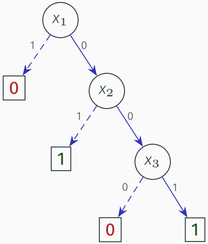

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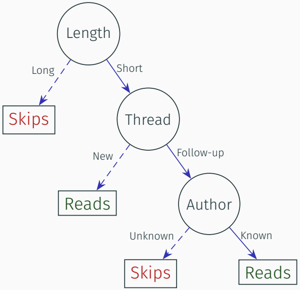

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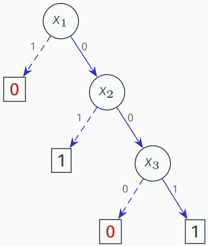

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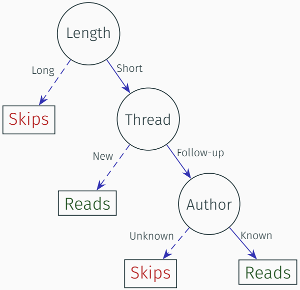

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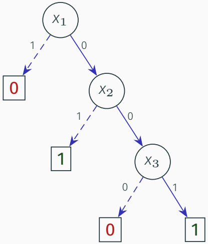

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    - One possible explanation is $\{\neg x_1, \neg x_2, x_3\}$ or simply $\{1, 2, 3\}$

The similarity predicate

[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
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- Similarity predicate: $\sigma : \mathbb{F} \to \{\top, \bot\}$
  - Classification: $\sigma(\mathbf{x}) := [\kappa(\mathbf{x}) = \kappa(\mathbf{v})]$
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- Bottom line:
  Reason about symbolic explainability by abstracting away type of ML model

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- Finding one AXp (example algorithm; many more exist): [MM20]
  - Let $\mathcal{X} = \mathcal{F}$, i.e. fix all features
  - Invariant: $\mathsf{WAXp}(\mathcal{X})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathsf{WAXp}(\mathcal{X} \setminus \{i\})$ holds, then remove i from $\mathcal{X}$, i.e. i becomes free

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Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

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- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
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  - **Obs:** for some classes of classifiers, poly-time algorithms exist

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More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i\in\mathcal{S}} (x_i = v_i)$$

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- Using probabilities, non-real-valued features: [WMHK21, IHI+22, ABOS22, IHI+23]

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- Finding one CXp: [MM20]
  - Let $\mathcal{Y} = \mathcal{F}$, i.e. free all features
  - Invariant: $\mathsf{WCXp}(\mathcal{Y})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathsf{WCXp}(\mathcal{Y}\backslash\{i\})$ holds, then remove i from $\mathcal{Y}$, i.e. i is becomes fixed

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- **Obs:** AXp is MHS of CXp and vice-versa...

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- Definition of CXp remains unchanged

Detour: global explanations

[INM19b]

- AXps and CXps are defined locally (because of $\mathbf{v}$) but hold globally
  - Localized explanations
  - Can be viewed as attempt at formalizing local explanations [RSG16, LL17, RSG18]
- One can define explanations without picking a given point in feature space
  - Let $q \in \mathbb{T}$, and refefine the similarity predicate:
    - Classification: $\sigma(\mathbf{x}) = [\kappa(\mathbf{x}) = q]$
    - Regression: $\sigma(\mathbf{x}) = [|\kappa(\mathbf{x}) - q| \leqslant \delta]$, δ is user-specified
  - Let $\mathbb{L} = \{(x_i = v_i) \,|\, i \in \mathcal{F} \land v_i \in \mathbb{V}\}$
  - Let $\mathcal{S} \subsetneq \mathbb{L}$ be a subset of literals that does not repeat features, i.e. $\mathcal{S}$ is not inconsistent
  - Then, $\mathcal{S}$ is a global AXp if,

$$\forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{(x_i=v_i)\in\mathcal{S}} (x_i = v_i) \rightarrow (\sigma(\mathbf{x}))$$

  - Counterexamples are minimal hitting sets of global AXps and vice-versa [INM19b]

Outline – Unit #02

Definitions of Explanations

Duality Properties

Computational Problems

Duality in explainability – basic results

[INAM20, Mar22]

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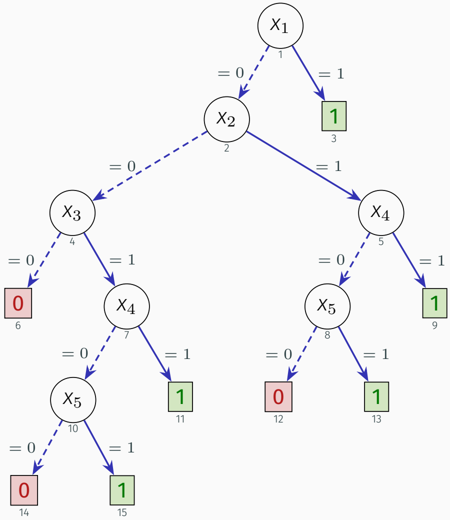

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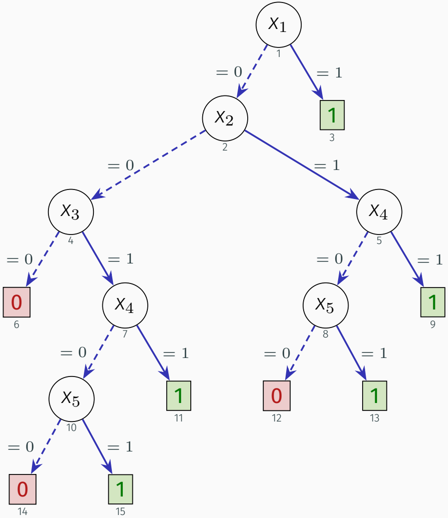

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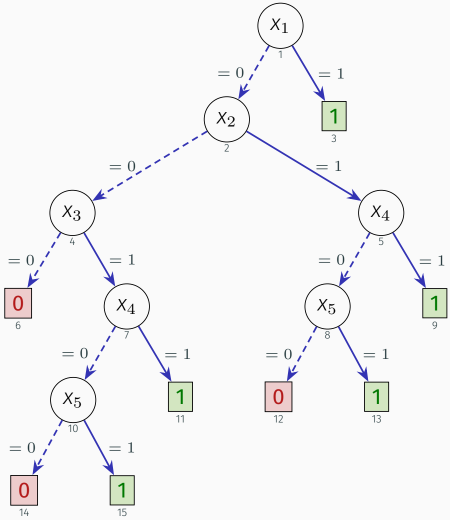

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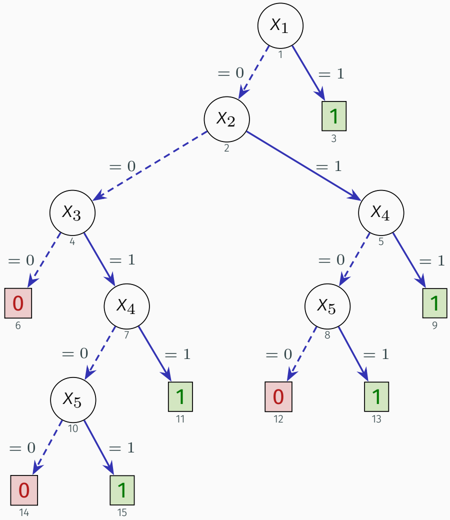

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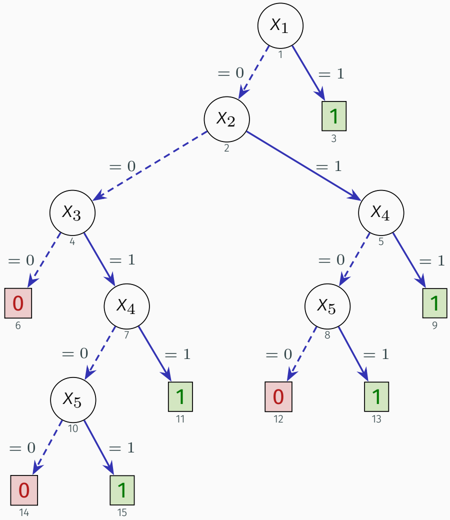
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:
  - AXps: $\{\{3,5\}\}$
  - CXps: $\{\{3\},\{5\}\}$



[INAM20, Mar22]

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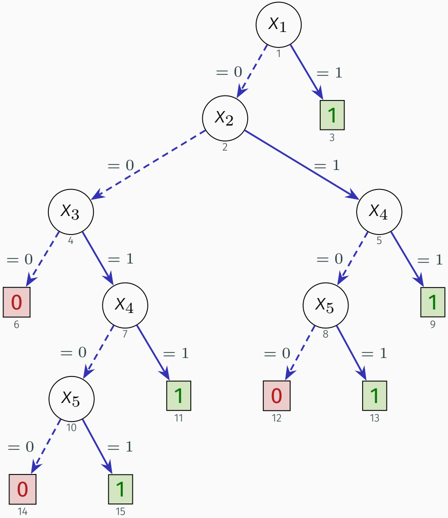

Duality in explainability – basic results

[INAM20, Mar22]

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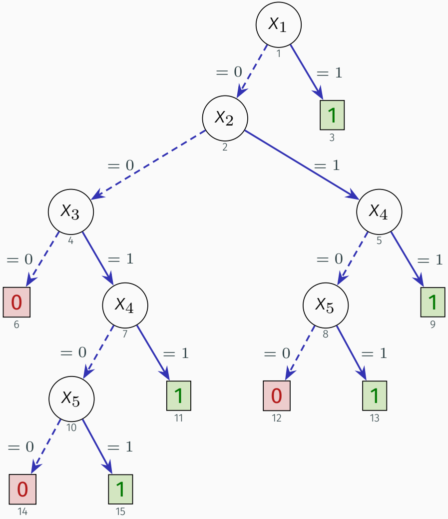
  - Each CXp is an MHS of the set of AXps
  - BTW,
    - $\{2,5\}$ is not a CXp
    - $\{1,2,3,4,5\}$, $\{1,2,3,5\}$ and $\{1,3,5\}$ are not AXps



Duality in explainability – basic results

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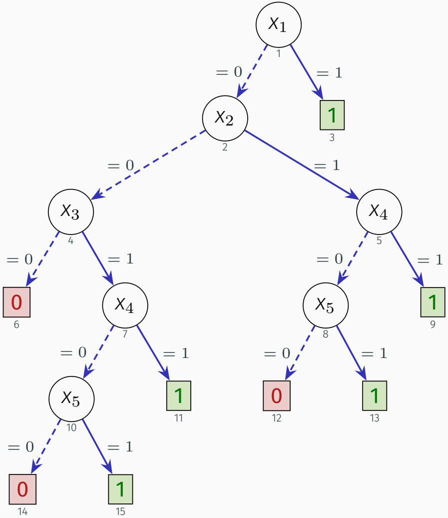
  - Each CXp is an MHS of the set of AXps
  - BTW,
    - $\{2,5\}$ is not a CXp
    - $\{1,2,3,4,5\}$, $\{1,2,3,5\}$ and $\{1,3,5\}$ are not AXps
    - Why?



Outline – Unit #02

Definitions of Explanations

Duality Properties

Computational Problems

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

- Enumerate all abductive/contrastive explanations

Computational problems in (formal) explainability

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Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner

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- Monotone predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg\,\mathsf{CO}\left(\llbracket(\textstyle\bigwedge_{i\in\mathcal{S}}(x_i = v_i)) \land (\neg\sigma(\mathbf{x}))\rrbracket\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \mathsf{CO}\left(\left\llbracket\left(\textstyle\bigwedge_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right\rrbracket\right)$$

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Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

```
procedure oneXP(ℙ)
    𝒮 ← ℱ                          ▷ Initialization: ℙ(𝒮) holds
    for i ∈ ℱ do                   ▷ Loop invariant: ℙ(𝒮) holds
        if ℙ(𝒮\{i}) then
            𝒮 ← 𝒮\{i}              ▷ Update 𝒮 only if ℙ(𝒮\{i}) holds
    return 𝒮                       ▷ Returned set 𝒮: ℙ(𝒮) holds
```

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For **AXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds
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Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}, \mathcal{M}$
Output: One XP $\mathcal{S}$

1: **procedure** oneXP($\mathbb{P}$)
2: $\quad \mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
3: $\quad$ **for** $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
4: $\quad\quad$ **if** $\mathbb{P}(\mathcal{S}\setminus\{i\})$ **then**
5: $\quad\quad\quad \mathcal{S} \leftarrow \mathcal{S}\setminus\{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ holds
6: $\quad$ **return** $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

Exploiting MSMP, i.e. basic algorithm used for different problems.

Detour: More Connections with Automated Reasoning

Prime implicants & implicates

- A conjunction of literals π (which will be viewed as a set of literals where convenient) is a prime implicant of some function φ if,
  1. $\pi \vDash \varphi$
  2. For any $\pi' \subsetneq \pi$, $\pi' \nvDash \varphi$

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  1. $\pi \vDash \varphi$
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  - Example:
    - $\mathbb{F} = \{0,1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \land x_2 \lor x_1 \land x_3$
    - Clearly, $x_1 \land x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$

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    - Clearly, $x_1 \wedge x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$

- A disjunction of literals η (also viewed as a set of literals where convenient) is a prime implicate of some function φ if
  1. $\varphi \vDash \eta$
  2. For any $\eta' \subsetneq \eta$, $\varphi \nvDash \eta'$

Reasoning about inconsistency

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \vee x_2), (x_1 \vee \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$

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- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \models \bot$
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Reasoning about inconsistency

[MM20]

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- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]

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- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]
- Variants:
  - Smallest(-cost) MCS, i.e. complement of maximum(-cost) satisfiability (MaxSAT)
  - Smallest(-cost) MUS

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{array}{rcl} \mathrm{WAXp}(\mathcal{X}) & := & \forall(\mathbf{x}\in\mathbb{F}).\bigwedge_{j\in\mathcal{X}}(x_j=v_j)\rightarrow(\sigma(\mathbf{x})) \\ \mathrm{WCXp}(\mathcal{Y}) & := & \exists(\mathbf{x}\in\mathbb{F}).\bigwedge_{j\notin\mathcal{Y}}(x_j=v_j)\land(\neg\sigma(\mathbf{x})) \end{array}$$

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$$\begin{aligned} \mathsf{WAXp}(\mathcal{X}) \quad &:= \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\ \mathsf{WCXp}(\mathcal{Y}) \quad &:= \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x})) \end{aligned}$$

- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \land_{i \in \mathcal{F}} (s_i \rightarrow (x_i = v_i)) \land \mathsf{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

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- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \wedge_{i\in\mathcal{F}}\,(s_i\rightarrow(x_i=v_i))\land\mathsf{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

  - Soft constraints: $\mathcal{S}=\{s_i\,|\,i\in\mathcal{F}\}$

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  - Soft constraints: $\mathcal{S}=\{s_i\,|\,i\in\mathcal{F}\}$
- **Claim**: Each MUS of $(\mathcal{B},\mathcal{S})$ is an AXp & each MCS of $(\mathcal{B},\mathcal{S})$ is a CXp
  - Can use MUS/MCS algorithms for AXps/CXps

Unit #03

Tractability in Symbolic XAI

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

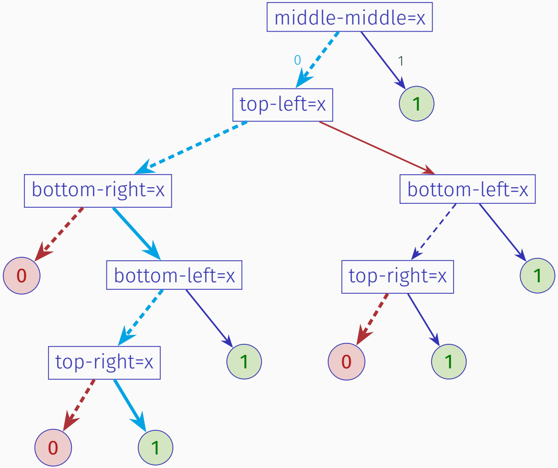
Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

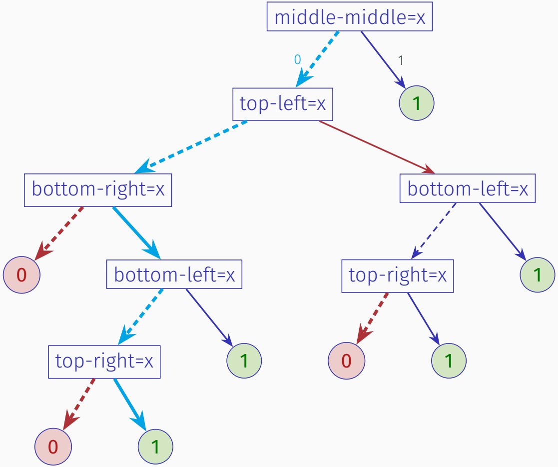
DT explanations

[IIM20]



DT explanations

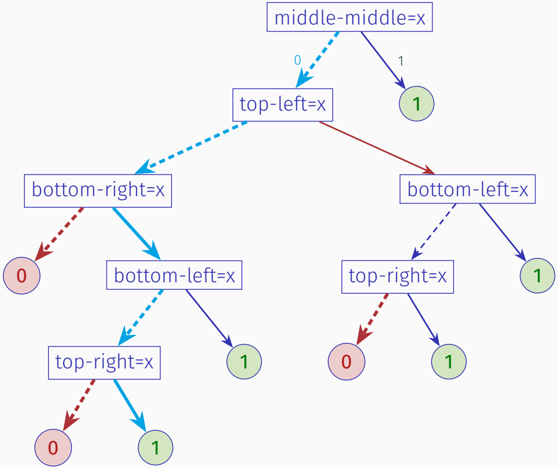
[IIM20]



- Run PI-explanation algorithm based on NP-oracles
    - Worst-case exponential time

DT explanations

[IIM20]



- Run PI-explanation algorithm based on NP-oracles
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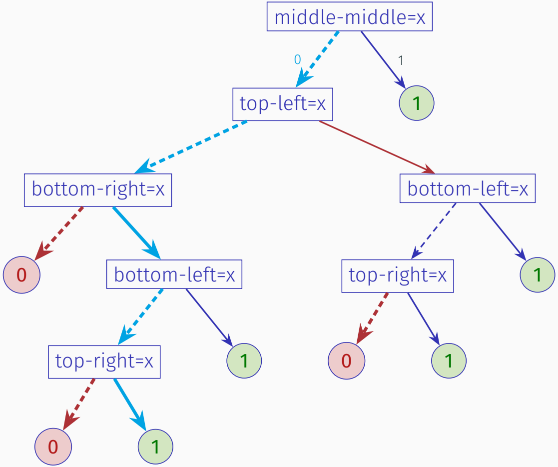
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent

DT explanations in polynomial time

[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction **0** remain inconsistent
  - I.e. find a subset-minimal hitting set of all **0** paths; these are the features to keep
    - E.g. BR and TR suffice for prediction
  - Well-known to be solvable in polynomial time [EG95]

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
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- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

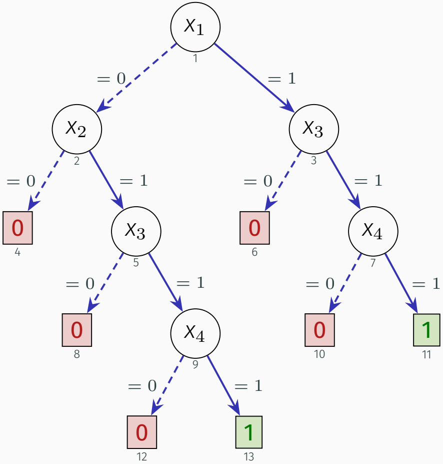
- Finding one CXp in polynomial-time

- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time

- Practically efficient enumeration of AXps – later

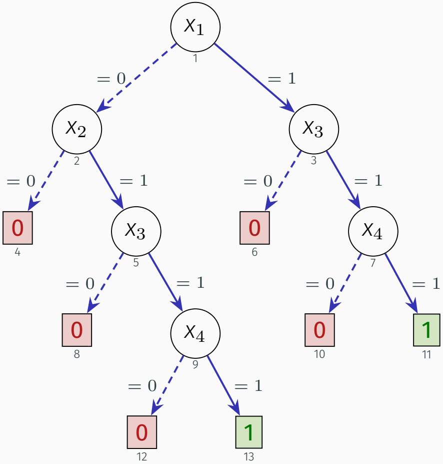
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \varnothing$



Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:



Finding all CXps in polynomial-time

- Basic algorithm:
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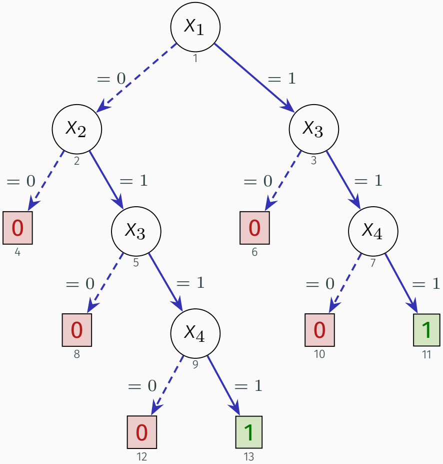
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$



Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
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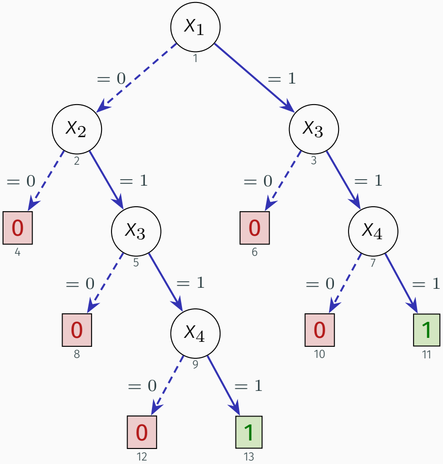
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$



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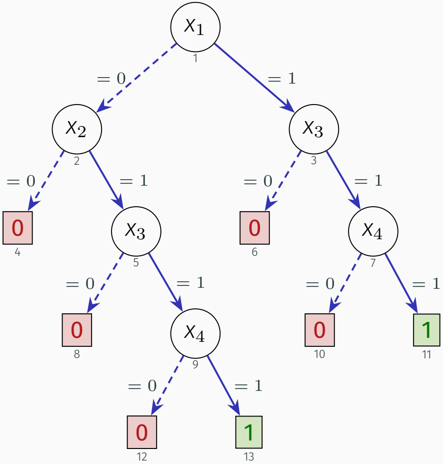

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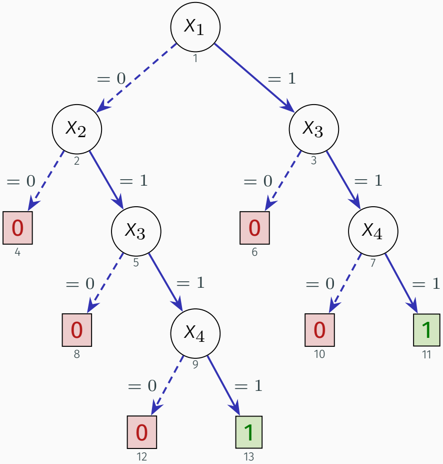

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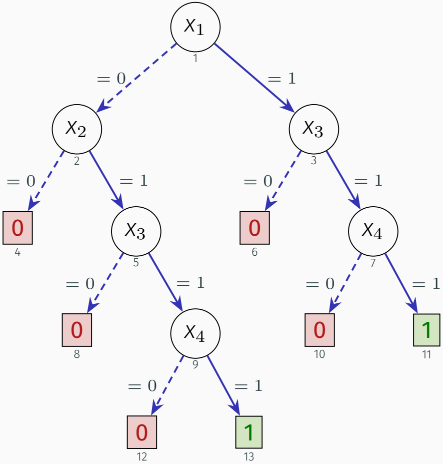

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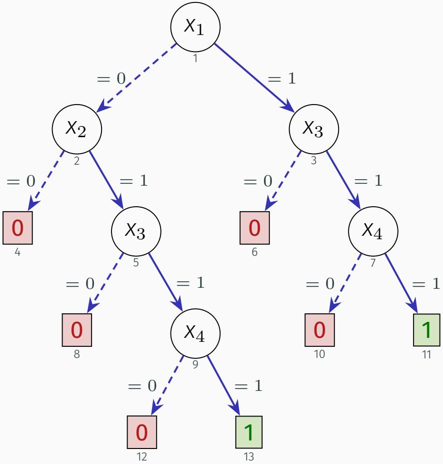

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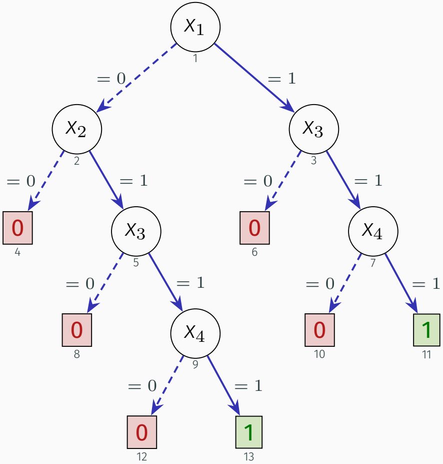

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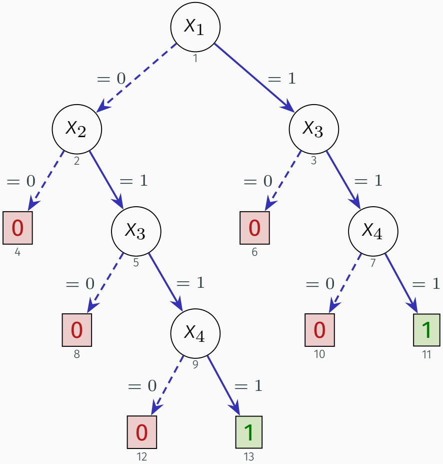

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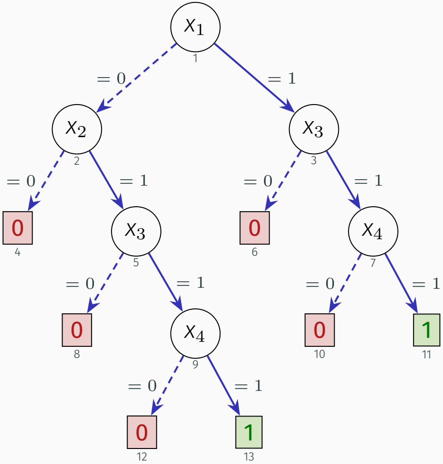

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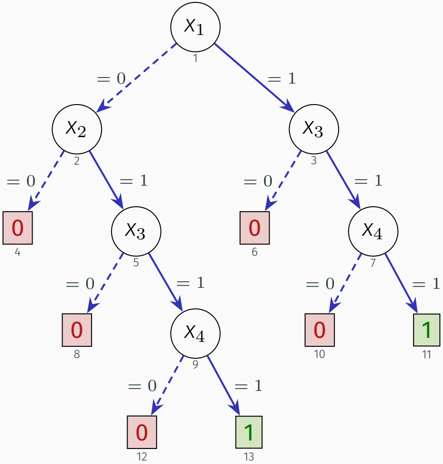

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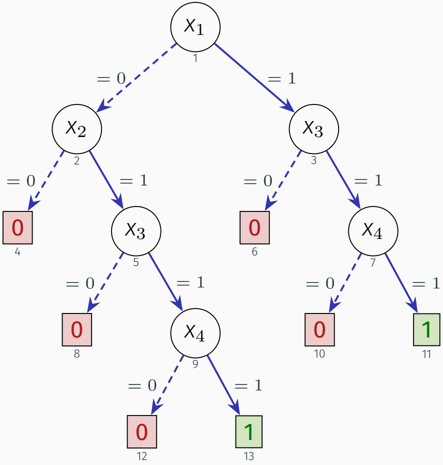

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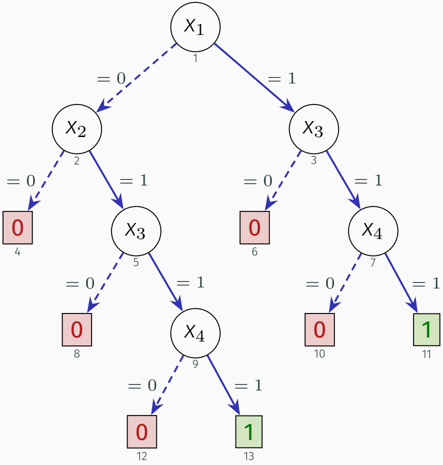

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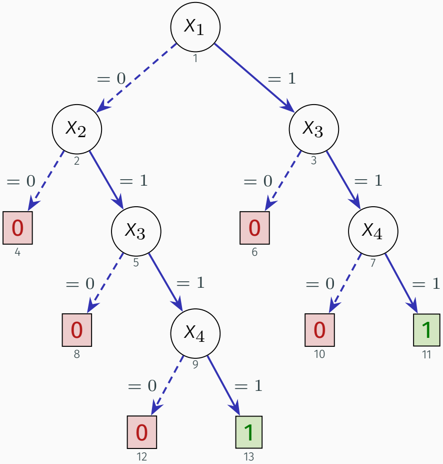
  - CXps: $\{\{1,2\},\{3\},\{4\}\}$
  - AXps: $\{\{1,3,4\},\{2,3,4\}\}$, by computing all MHSes



Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

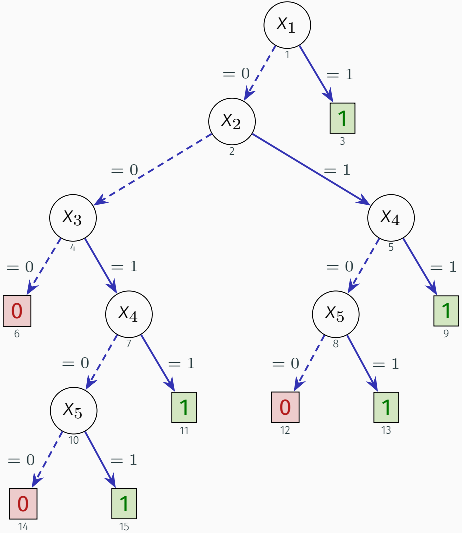
Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

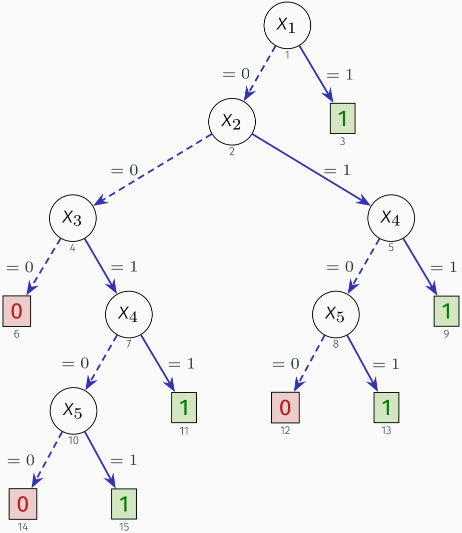
Explanations for Monotonic Classifiers

Review examples

Are *interpretable* models really interpretable? – DTs



- Case of **optimal** decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?



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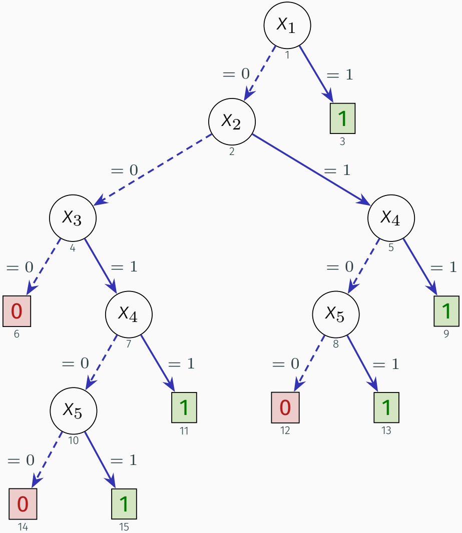

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x_3	x_5	x_1	x_2	x_4	$\kappa(\mathbf{x})$
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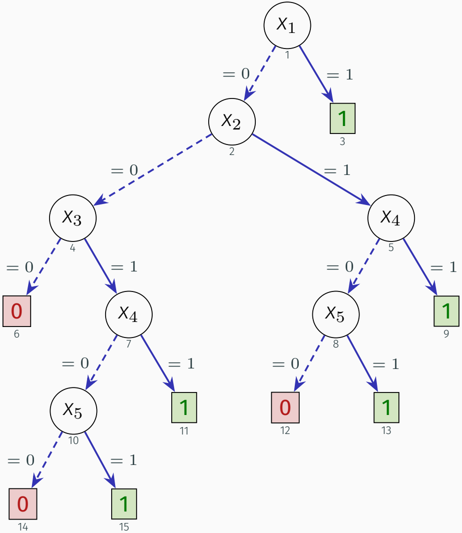

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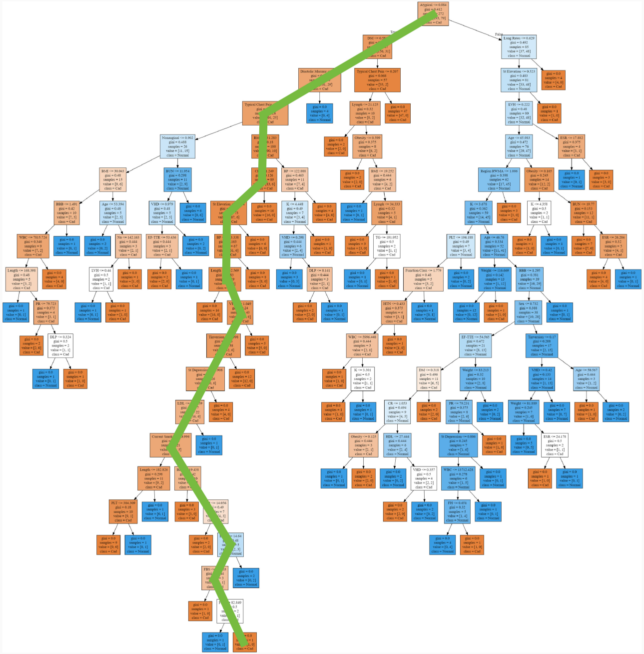
$\therefore$ one AXp is $\{3, 5\}$

Compare with $\{1, 2, 3, 4, 5\}$...

Are *interpretable* models really interpretable? – large DTs

[GZM20]



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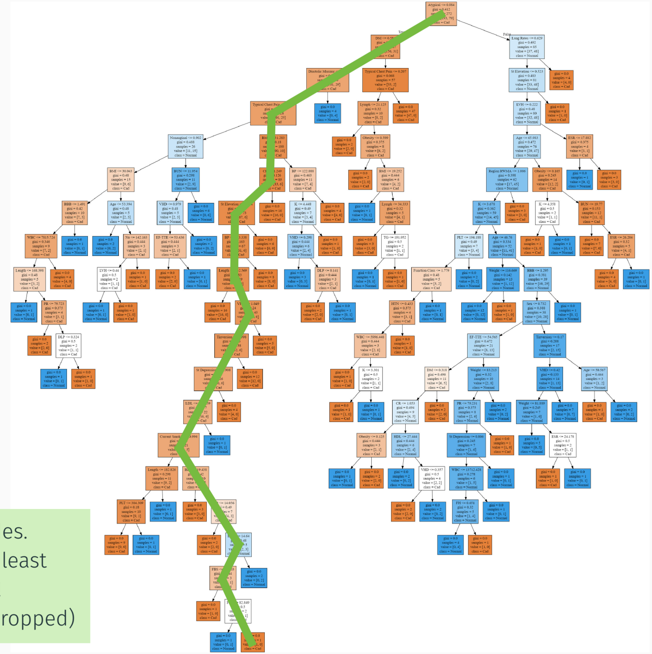

Path with 19 internal nodes.
By manual inspection, at least
10 literals are redundant!
(And at least 9 features dropped)

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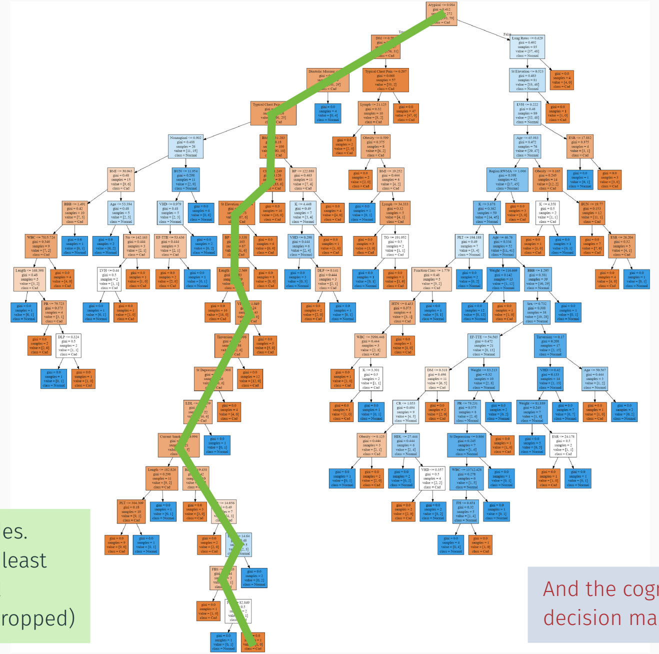

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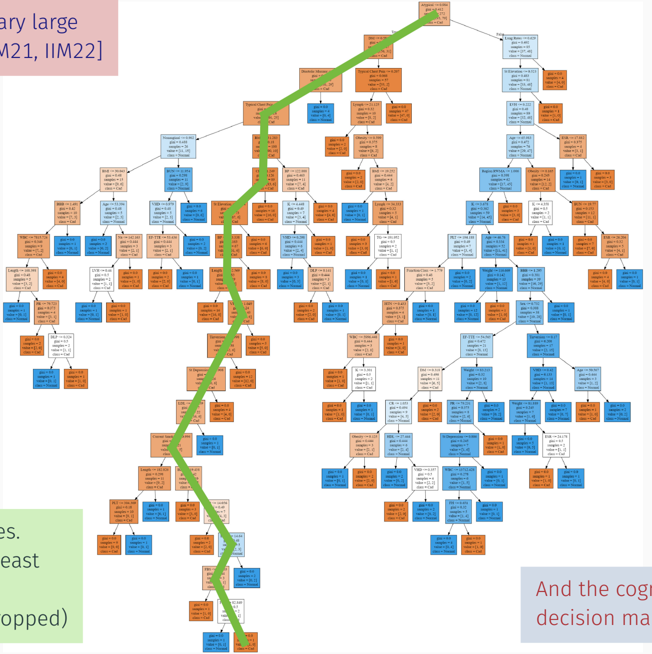
Are *interpretable* models really interpretable? – large DTs

Redundancy can be arbitrary large on path length [IIM20, HIIM21, IIM22]



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Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

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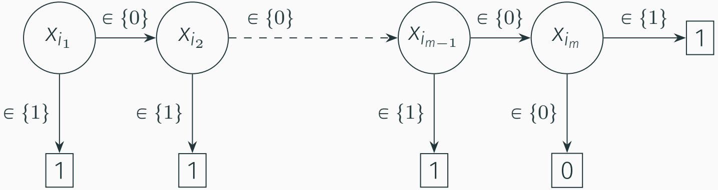

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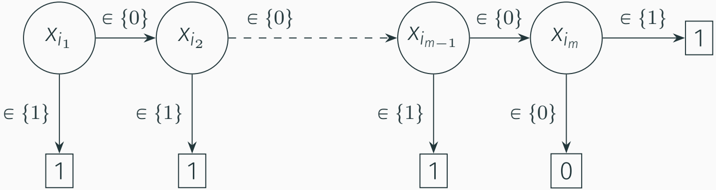

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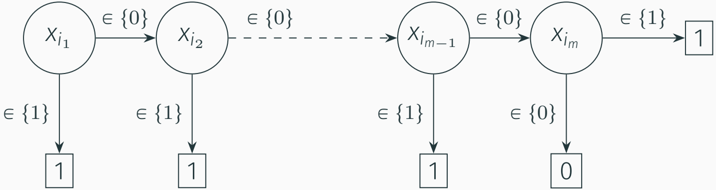

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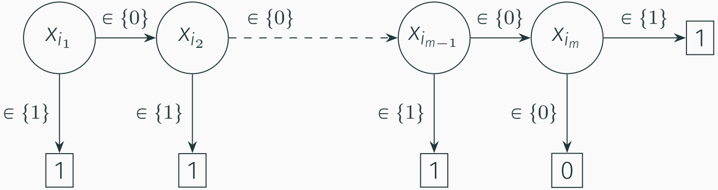

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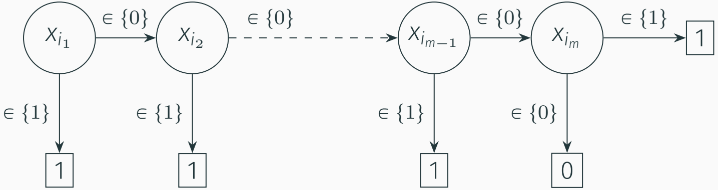

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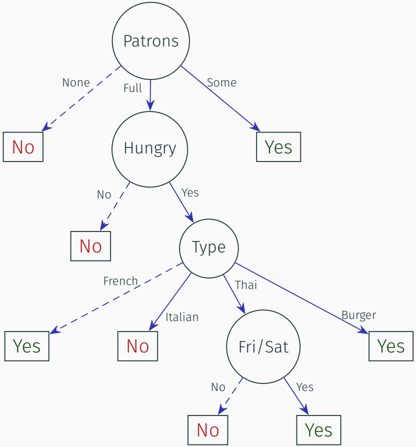
- **AXp's can be arbitrarily smaller than paths in (optimal) DTs!** [IIM20, IIM22]

Explanation redundancy in DTs is ubiquitous – published DT examples

DT Ref	D	#N	#P	%R	%C	%m	%M	%avg
[Alp14, Ch. 09, Fig. 9.1]	2	5	3	33	25	50	50	50
[Alp16, Ch. 03, Fig. 3.2]	2	5	3	33	25	50	50	50
[Bra20, Ch. 01, Fig. 1.3]	4	9	5	60	25	25	50	36
[BA97, Figure 1]	3	12	7	14	8	33	33	33
[BBHK10, Ch. 08, Fig. 8.2]	3	7	4	25	12	50	50	50
[BFOS84, Ch. 01, Fig. 1.1]	3	7	4	50	25	33	33	33
[DL01, Ch. 01, Fig. 1.2a]	2	5	3	33	25	33	33	33
[DL01, Ch. 01, Fig. 1.2b]	2	5	3	33	25	33	33	33
[KMND20, Ch. 04, Fig. 4.14]	3	7	4	25	12	50	50	50
[KMND20, Sec. 4.7, Ex. 4]	2	5	3	33	25	50	50	50
[Qui93, Ch. 01, Fig. 1.3]	3	12	7	28	17	33	50	41
[RM08, Ch. 01, Fig. 1.5]	3	9	5	20	12	33	33	33
[RM08, Ch. 01, Fig. 1.4]	3	7	4	50	25	33	33	33
[WFHP17, Ch. 01, Fig. 1.2]	3	7	4	25	12	50	50	50
[VLE$^+$16, Figure 4]	6	39	20	65	63	20	40	33
[Fla12, Ch. 02, Fig. 2.1(right)]	2	5	3	33	25	50	50	50
[Kot13, Figure 1]	3	10	6	33	11	33	33	33
[Mor82, Figure 1]	3	9	5	80	75	33	50	41
[PM17, Ch. 07, Fig. 7.4]	3	7	4	50	25	33	33	33
[RN10, Ch. 18, Fig. 18.6]	4	12	8	25	6	25	33	29
[SB14, Ch. 18, Page 212]	2	5	3	33	25	50	50	50
[Zho12, Ch. 01, Fig. 1.3]	2	5	3	33	25	33	33	33
[BHO09, Figure 1b]	4	13	7	71	50	33	50	36
[Zho21, Ch. 04, Fig. 4.3]	4	14	9	11	2	25	25	25

Many DTs have paths that are not minimal XPs – Russell&Norvig's book

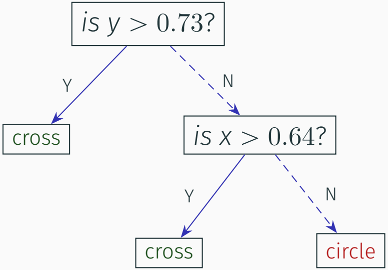
[RN10]



- Explanation for $(P, H, T, W) = (\text{Full}, \text{Yes}, \text{Thai}, \text{No})$?

Many DTs have paths that are not minimal XPs – Zhou's book

[Zho12]

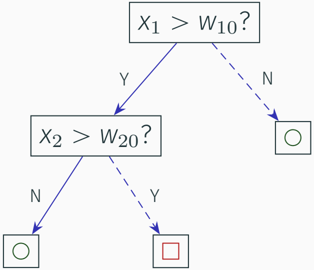


- Explanation for $(x, y) = (1.25, -1.13)$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

Many DTs have paths that are not minimal XPs – Alpaydin's book

[Alp14]



- Explanation for $(x_1, x_2) = (\alpha, \beta)$, with $\alpha > w_{10}$ and $\beta \leqslant w_{20}$?

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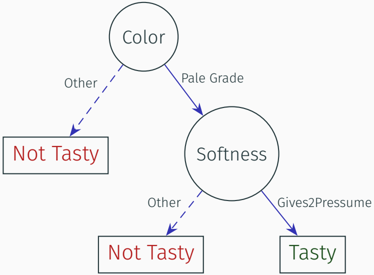
Many DTs have paths that are not minimal XPs – S.-S.&B.-D.'s book

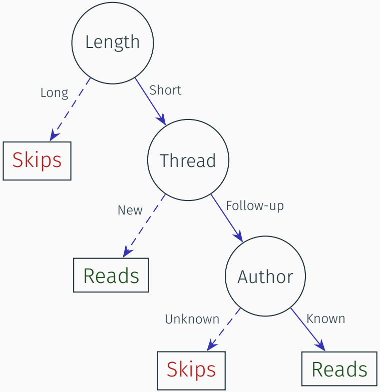
[SB14]



- Explanation for $(\text{color}, \text{softness}) = (\text{Pale Grade}, \text{Other})$?

Many DTs have paths that are not minimal XPs – Poole&Mackworth's book

[PM17]



- Explanation for $(\mathrm{L}, \mathrm{T}, \mathrm{A}) = (\mathrm{Short}, \mathrm{Follow\text{-}Up}, \mathrm{Unknown})$?
- Explanation for $(\mathrm{L}, \mathrm{T}, \mathrm{A}) = (\mathrm{Short}, \mathrm{Follow\text{-}Up}, \mathrm{Known})$?

Explanation redundancy in DTs is ubiquitous – DTs from datasets

[IIM20, HIIM21, IIM22]

Dataset	(#F	#S)	IAI									ITI								
			D	#N	%A	#P	%R	%C	%m	%M	%avg	D	#N	%A	#P	%R	%C	%m	%M	%avg
adult	(12	6061)	6	83	78	42	33	25	20	40	25	17	509	73	255	75	91	10	66	22
anneal	(38	886)	6	29	99	15	26	16	16	33	21	9	31	100	16	25	4	12	20	16
backache	(32	180)	4	17	72	9	33	39	25	33	30	3	9	91	5	80	87	50	66	54
bank	(19	36293)	6	113	88	57	5	12	16	20	18	19	1467	86	734	69	64	7	63	27
biodegradation	(41	1052)	5	19	65	10	30	1	25	50	33	8	71	76	36	50	8	14	40	21
cancer	(9	449)	6	37	87	19	36	9	20	25	21	5	21	84	11	54	10	25	50	37
car	(6	1728)	6	43	96	22	86	89	20	80	45	11	57	98	29	65	41	16	50	30
colic	(22	357)	6	55	81	28	46	6	16	33	20	4	17	80	9	33	27	25	25	25
compas	(11	1155)	6	77	34	39	17	8	16	20	17	15	183	37	92	66	43	12	60	27
contraceptive	(9	1425)	6	99	49	50	8	2	20	60	37	17	385	48	193	27	32	12	66	21
dermatology	(34	366)	6	33	90	17	23	3	16	33	21	7	17	95	9	22	0	14	20	17
divorce	(54	150)	5	15	90	8	50	19	20	33	24	2	5	96	3	33	16	50	50	50
german	(21	1000)	6	25	61	13	38	10	20	40	29	10	99	72	50	46	13	12	40	22
heart-c	(13	302)	6	43	65	22	36	18	20	33	22	4	15	75	8	87	81	25	50	34
heart-h	(13	293)	6	37	59	19	31	4	20	40	24	8	25	77	13	61	60	20	50	32
kr-vs-kp	(36	3196)	6	49	96	25	80	75	16	60	33	13	67	99	34	79	43	7	70	35
lending	(9	5082)	6	45	73	23	73	80	16	50	25	14	507	65	254	69	80	12	75	25
letter	(16	18668)	6	127	58	64	1	0	20	20	20	46	4857	68	2429	6	7	6	25	9
lymphography	(18	148)	6	61	76	31	35	25	16	33	21	6	21	86	11	9	0	16	16	16
mortality	(118	13442)	6	111	74	56	8	14	16	20	17	26	865	76	433	61	61	7	54	19
mushroom	(22	8124)	6	39	100	20	80	44	16	33	24	5	23	100	12	50	31	20	40	25
pendigits	(16	10992)	6	121	88	61	0	0	—	—	—	38	937	85	469	25	86	6	25	11
promoters	(58	106)	1	3	90	2	0	0	—	—	—	3	9	81	5	20	14	33	33	33
recidivism	(15	3998)	6	105	61	53	28	22	16	33	18	15	611	51	306	53	38	9	44	16
seismic_bumps	(18	2578)	6	37	89	19	42	19	20	33	24	8	39	93	20	60	79	20	60	42
shuttle	(9	58000)	6	63	99	32	28	7	20	33	23	23	159	99	80	33	9	14	50	30
soybean	(35	623)	6	63	88	32	9	5	25	25	25	16	71	89	36	22	1	9	12	10
spambase	(57	4210)	6	63	75	32	37	12	16	33	19	15	143	91	72	76	98	7	58	25
spect	(22	228)	6	45	82	23	60	51	20	50	35	6	15	86	8	87	98	50	83	65
splice	(2	3178)	3	7	50	4	0	0	—	—	—	88	177	55	89	0	0	—	—	—

Are *interpretable* models really interpretable? – DLs

[MSI23]

$$
\begin{array}{llcll}
\mathrm{R}_1: & \mathrm{IF} & (x_1 \land x_3) & \mathrm{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_2: & \mathrm{ELSE\ IF} & (x_2 \land x_4 \land x_6) & \mathrm{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_3: & \mathrm{ELSE\ IF} & (\neg x_1 \land x_3) & \mathrm{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_4: & \mathrm{ELSE\ IF} & (x_4 \land x_6) & \mathrm{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_5: & \mathrm{ELSE\ IF} & (\neg x_1 \land \neg x_3) & \mathrm{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_6: & \mathrm{ELSE\ IF} & (x_6) & \mathrm{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_{\mathrm{DEF}}: & \mathrm{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0,1,0,1,0,1),0)$, i.e. rule R_2 fires

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R_4: & \text{ELSE IF} & (x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_5: & \text{ELSE IF} & (\neg x_1 \land \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
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[MSI23]

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R_5 :	ELSE IF	$(\neg x_1 \land \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
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- What is the abductive explanation?
- Recall: one AXp is $\{3,4,6\}$
  - **Why?**
    - We need 3 (or 1) so that R_1 cannot fire
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  - **Some questions**:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?

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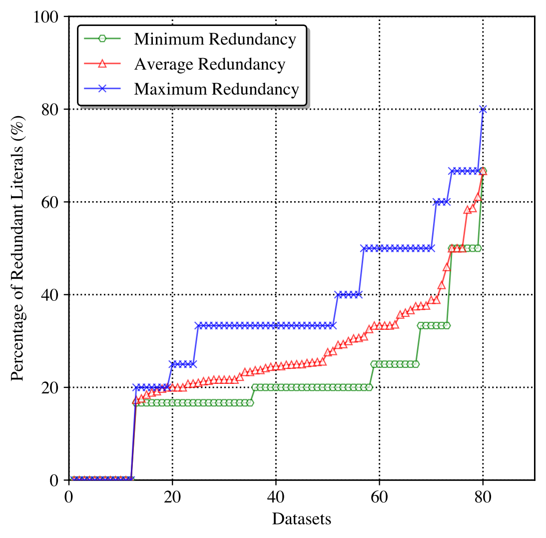
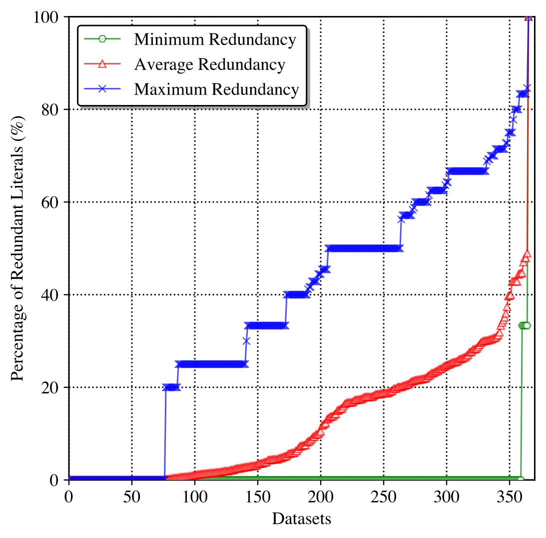
  - **Some questions**:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?
      (BTW, **we have proved that computing one AXp for DLs is computationally hard...**)

[IM21, MSI23]



DTs learned with Interpretable AI, max depth 6



DLs learned with CN2

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

From DTs to explained DSs

[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules

From DTs to explained DSs

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From DTs to explained DSs

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- And explaining DSs is computationally hard...
- One can extract explained DSs from DTs

From DTs to explained DSs

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- Decision sets raise a number of issues:
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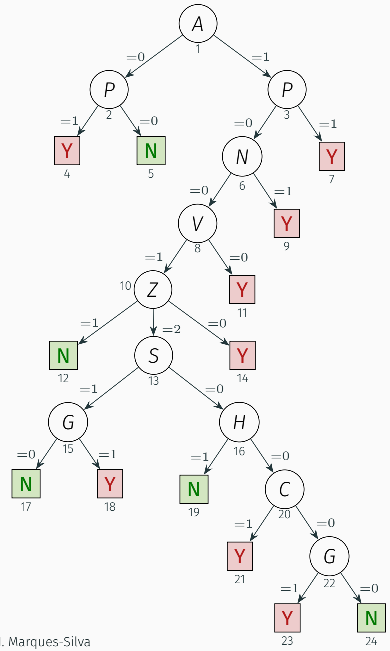
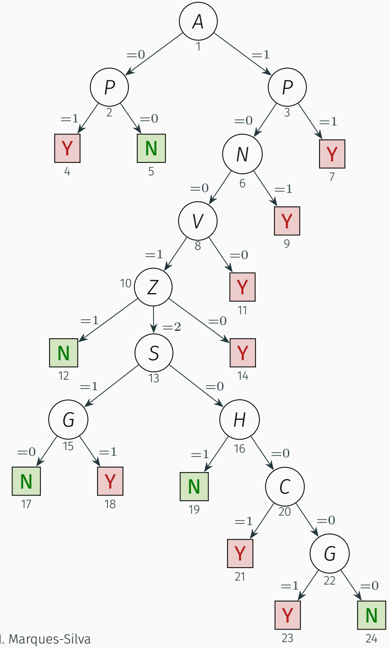
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...
- One can extract explained DSs from DTs
  - Extract one AXp (viewed as a logic rule) from each path in DT
  - Resulting rules are non-overlapping, and cover feature space

Example



A
1
=0
=1
P
2
P
3
=1
=0
=0
=1
Y
4
N
5
N
6
Y
7
=0
=1
V
8
Y
9
=1
=0
10
Z
Y
11
=1
=2
=0
N
12
S
13
Y
14
=1
=0
G
15
H
16
=0
=1
=1
=0
N
17
Y
18
N
19
C
20
=1
=0
Y
21
G
22
=1
=0
Y
23
N
24

Example



R_{01}: IF $[P]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{02}: IF $[\overline{A} \land \overline{P}]$THEN $\kappa(\cdot) = \mathbf{N}$

R_{03}: IF $[\overline{P} \land \overline{N} \land V \land Z = 1]$ THEN $\kappa(\cdot) = \mathbf{N}$

R_{04}: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land S \land \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R_{05}: IF $[A \land Z = 2 \land S \land G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{06}: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land \overline{S} \land H]$ THEN $\kappa(\cdot) = \mathbf{N}$

R_{07}: IF $[A \land Z = 2 \land \overline{S} \land \overline{H} \land C]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{08}: IF $[A \land Z = 2 \land \overline{H} \land G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{09}: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land \overline{C} \land \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R_{10}: IF $[A \land Z = 0]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{11}: IF $[A \land \overline{V}]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R_{12}: IF $[A \land N]$ THEN $\kappa(\cdot) = \mathbf{Y}$

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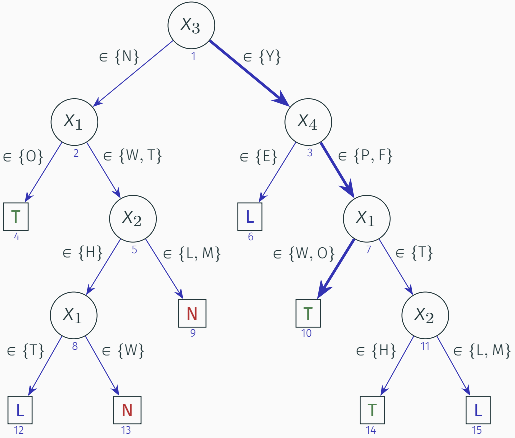
Explanation graphs – overview of results

[HIIM21]

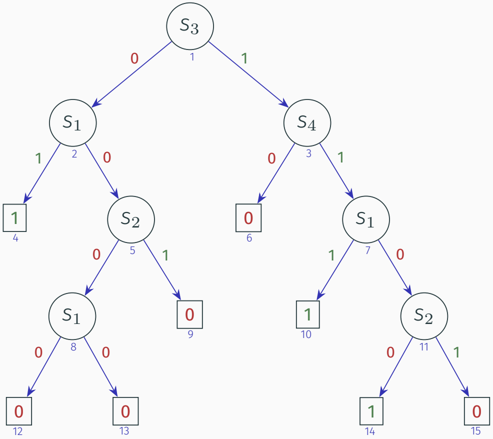
- Concept of explanation graph (XpG)
- Explanations of decision trees reducible to XpG's
- Explanations of decision graphs reducible to XpG's
- Explanations of OBDDs reducible to XpG's
- Explanations of OMDDs reducible to XpG's
- Explanations (AXp's and CXp's) of XpG's computed in polynomial time

Example of XpG – DTs

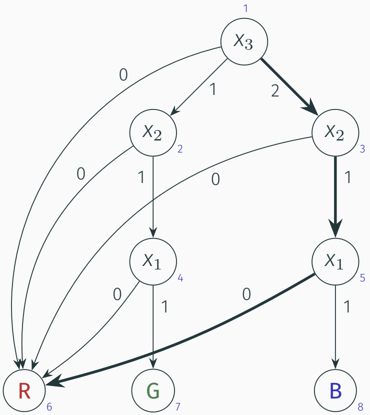
- DT; point: (O, L, Y, P); prediction T:



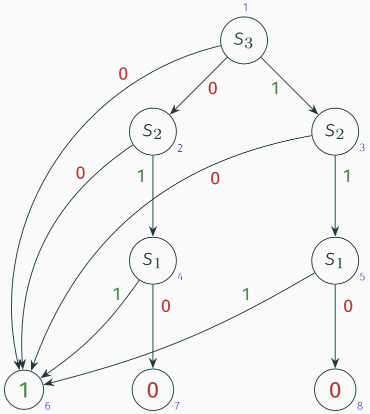
- XpG:



- OMBBD; point: $(0, 1, 2)$; prediction R:



- XpG:

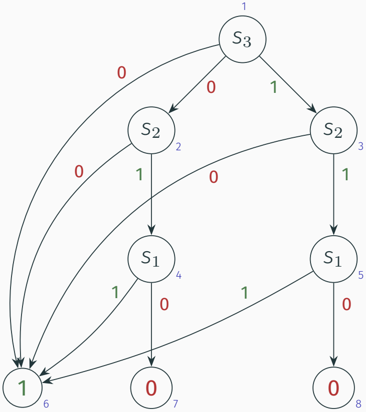


- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature i in $\mathcal{F}$

- XpG:



Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

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  Drop feature i from $\mathcal{S}$, i.e. i is free

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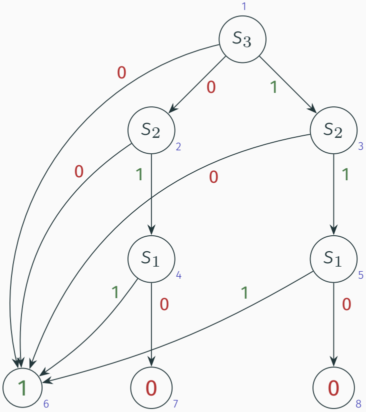

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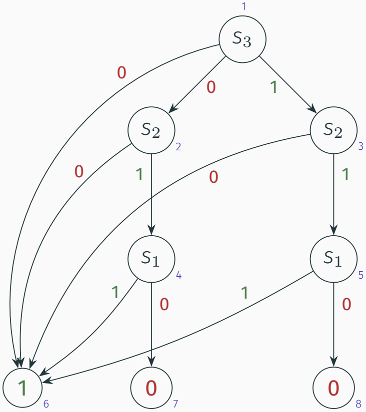

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  Return $\mathcal{S}$

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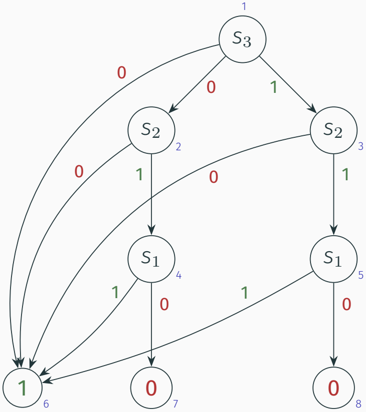

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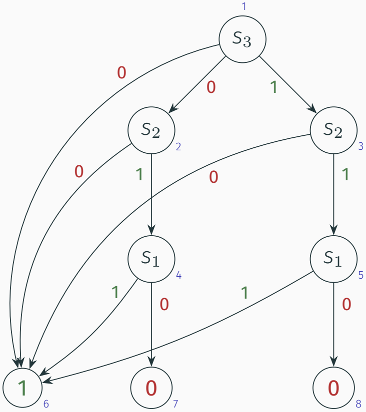
  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$

- XpG:



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    - $\mathcal{S} = \{1, 2, 3\}$
    - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$

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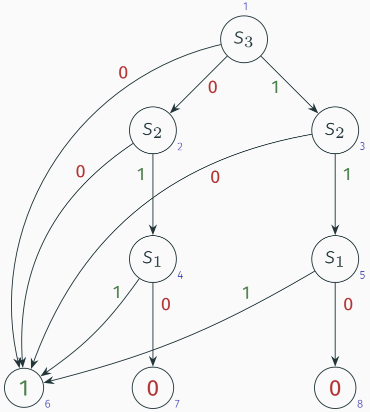

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  For each feature *i* in $\mathcal{F}$

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  Add feature *i* back to $\mathcal{S}$

  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
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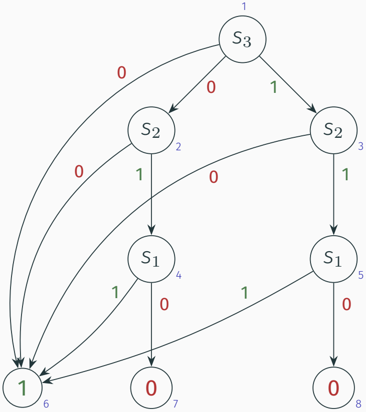

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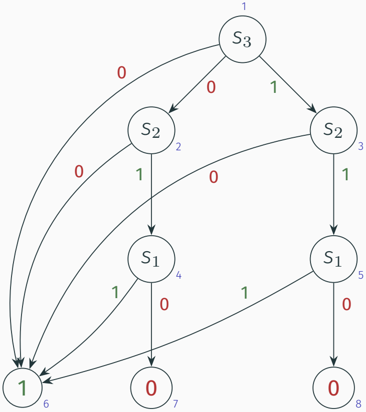
  - Both features 2 and 3 dropped from $\mathcal{S}$
  - Return $\mathcal{S} = \{1\}$

- XpG:



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Explanations for Monotonic Classifiers

Review examples

Example monotonic classifier – $(\mathbf{v}, c) = ((10, 10, 5, 0), A)$

[MGC+21]

Variable	Meaning		Range
$\kappa(\cdot) \triangleq M$	Student grade		$\in \{A, B, C, D, E, F\}$
S	Final score		$\in \{0, \ldots, 10\}$
Feat. id	Feat. var.	Feat. name	Domain
1	Q	Quiz	$\{0, \ldots, 10\}$
2	X	Exam	$\{0, \ldots, 10\}$
3	H	Homework	$\{0, \ldots, 10\}$
4	R	Project	$\{0, \ldots, 10\}$

$$M = \text{ITE}(S \geqslant 9, A, \text{ITE}(S \geqslant 7, B, \text{ITE}(S \geqslant 5, C, \text{ITE}(S \geqslant 4, D, \text{ite}(S \geqslant 2, E, F)))))$$

$$S = \max\left[0.3 \times Q + 0.6 \times X + 0.1 \times H, R\right]$$

Also, $F \preccurlyeq E \preccurlyeq D \preccurlyeq C \preccurlyeq B \preccurlyeq A$

And, $\kappa(\mathbf{x_1}) \preccurlyeq \kappa(\mathbf{x_2})$ if $\mathbf{x}_1 \leqslant \mathbf{x}_2$

Explaining monotonic classifiers

- Instance $(\mathbf{v}, c)$
- Domain for $i \in \mathcal{F}$: $\lambda(i) \leqslant x_i \leqslant \mu(i)$
- Idea: refine lower and upper bounds on the prediction
  - $\mathbf{v}_\mathsf{L}$ and $\mathbf{v}_\mathsf{U}$
- Utilities:
  - FixAttr(i):

$$
\begin{array}{l}
\mathbf{v}_\mathsf{L} \leftarrow (v_{\mathsf{L}_1}, \ldots, v_i, \ldots, v_{\mathsf{L}_N}) \\
\mathbf{v}_\mathsf{U} \leftarrow (v_{\mathsf{U}_1}, \ldots, v_i, \ldots, v_{\mathsf{U}_N}) \\
(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\}) \\
\textbf{return } (\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{A}, \mathcal{B})
\end{array}
$$

  - FreeAttr(i):

$$
\begin{array}{l}
\mathbf{v}_\mathsf{L} \leftarrow (v_{\mathsf{L}_1}, \ldots, \lambda(i), \ldots, v_{\mathsf{L}_N}) \\
\mathbf{v}_\mathsf{U} \leftarrow (v_{\mathsf{U}_1}, \ldots, \mu(i), \ldots, v_{\mathsf{U}_N}) \\
(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\}) \\
\textbf{return } (\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{A}, \mathcal{B})
\end{array}
$$

Computing one AXp

$\mathbf{v}_\mathsf{L} \leftarrow (v_1, \ldots, v_N)$
$\mathbf{v}_\mathsf{U} \leftarrow (v_1, \ldots, v_N)$ ▷ Ensures: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$
$(\mathcal{C}, \mathcal{D}, \mathcal{P}) \leftarrow (\mathcal{F}, \emptyset, \emptyset)$ ▷ $\mathcal{S}$: Some possible seed
for all $i \in \mathcal{S}$ **do**
  $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D}) \leftarrow \mathsf{FreeAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D})$ ▷ Require: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$, given $\mathcal{S}$
for all $i \in \mathcal{F} \setminus \mathcal{S}$ **do** ▷ Loop inv.: $\kappa(\mathbf{v}_\mathsf{L}) = \kappa(\mathbf{v}_\mathsf{U})$
  $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D}) \leftarrow \mathsf{FreeAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{C}, \mathcal{D})$
  if $\kappa(\mathbf{v}_\mathsf{L}) \neq \kappa(\mathbf{v}_\mathsf{U})$ **then** ▷ If invariant broken, fix it
    $(\mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{D}, \mathcal{P}) \leftarrow \mathsf{FixAttr}(i, \mathbf{v}, \mathbf{v}_\mathsf{L}, \mathbf{v}_\mathsf{U}, \mathcal{D}, \mathcal{P})$
return $\mathcal{P}$

- **Obs:** $\mathcal{S} = \emptyset$ for computing a single AXp/CXp

Computing one AXp – example

- $\lambda(i) = 0$ and $\mu(i) = 10$
- $\mathbf{v} = (10, 10, 5, 0)$, with $\kappa(\mathbf{v}) = A$
- Q: find one AXp (CXp is similar)

Feat.	Initial values		Changed values		Predictions		Dec.	Resulting values	
	$\mathbf{v}_{\mathrm{L}}$	$\mathbf{v}_{\mathrm{U}}$	$\mathbf{v}_{\mathrm{L}}$	$\mathbf{v}_{\mathrm{U}}$	$\kappa(\mathbf{v}_{\mathrm{L}})$	$\kappa(\mathbf{v}_{\mathrm{U}})$		$\mathbf{v}_{\mathrm{L}}$	$\mathbf{v}_{\mathrm{U}}$
1	(10,10,5,0)	(10,10,5,0)	(0,10,5,0)	(10,10,5,0)	C	A	✓	(10,10,5,0)	(10,10,5,0)
2	(10,10,5,0)	(10,10,5,0)	(10,0,5,0)	(10,10,5,0)	E	A	✓	(10,10,5,0)	(10,10,5,0)
3	(10,10,5,0)	(10,10,5,0)	(10,10,0,0)	(10,10,10,0)	A	A	✗	(10,10,0,0)	(10,10,10,0)
4	(10,10,0,0)	(10,10,10,0)	(10,10,0,0)	(10,10,10,10)	A	A	✗	(10,10,0,0)	(10,10,10,10)

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Recap computation of (W)AXps/(W)CXps

$$\begin{array}{rcl} \mathrm{WAXp}(\mathcal{X}) & := & \forall(\mathbf{x} \in \mathbb{F}).\, \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\ \mathrm{WCXp}(\mathcal{Y}) & := & \exists(\mathbf{x} \in \mathbb{F}).\, \bigwedge_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x})) \end{array}$$

Recap computation of (W)AXps/(W)CXps

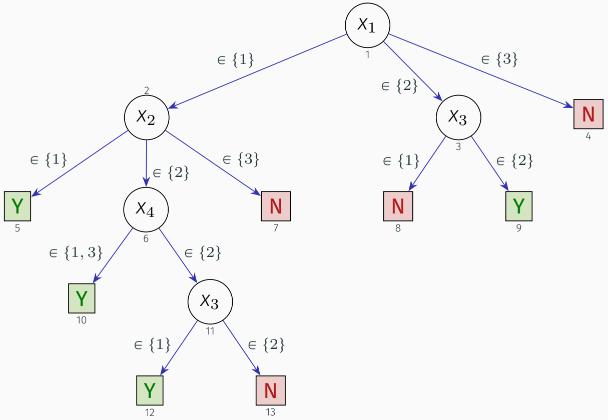
$$\begin{aligned}\mathsf{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\rightarrow(\sigma(\mathbf{x}))\\ \mathsf{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\end{aligned}$$

Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

```
1: procedure oneXP(ℙ)
2:     S ← F                      ▷ Initialization: ℙ(S) holds
3:     for i ∈ F do               ▷ Loop invariant: ℙ(S) holds
4:         if ℙ(S\{i}) then
5:             S ← S\{i}          ▷ Update S only if ℙ(S\{i}) holds
6:     return S                   ▷ Returned set S: ℙ(S) holds
```

Review exercise – one AXp for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



Review exercise – one AXp for example DT

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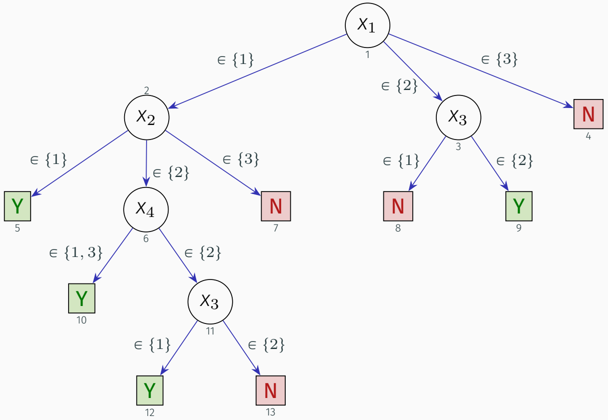
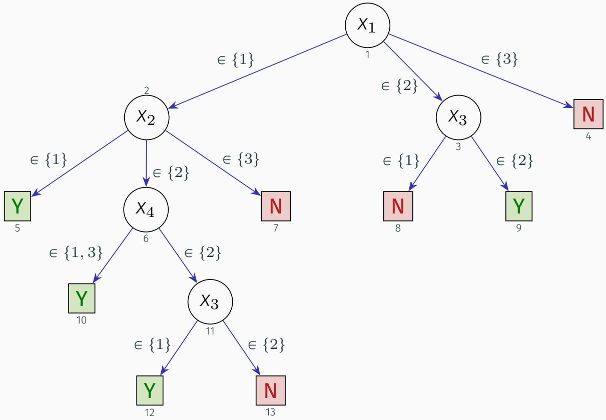


- Finding on AXp:

Review exercise – one AXp for example DT

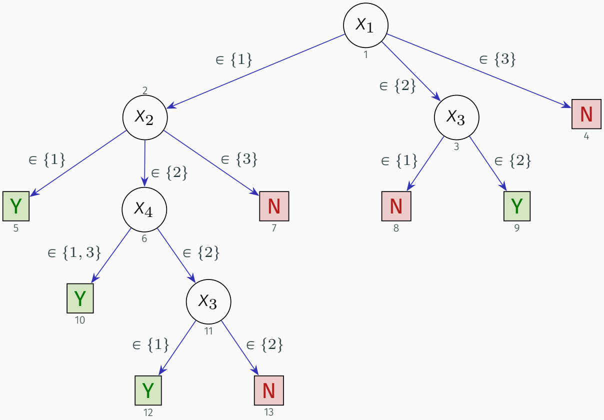
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$

Review exercise – one AXp for example DT

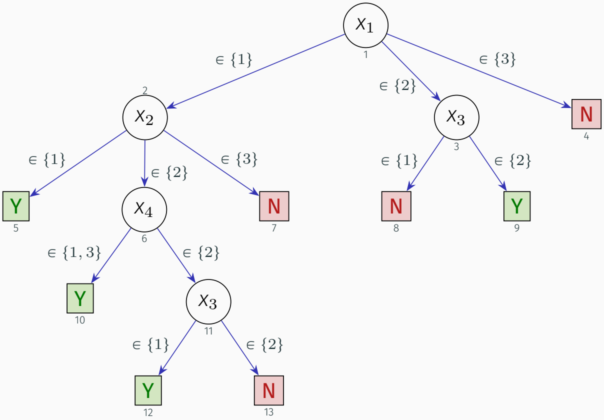
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$
  - 2nd path inconsistent: $\mathcal{H}_2 = \{2\}$

Review exercise – one AXp for example DT

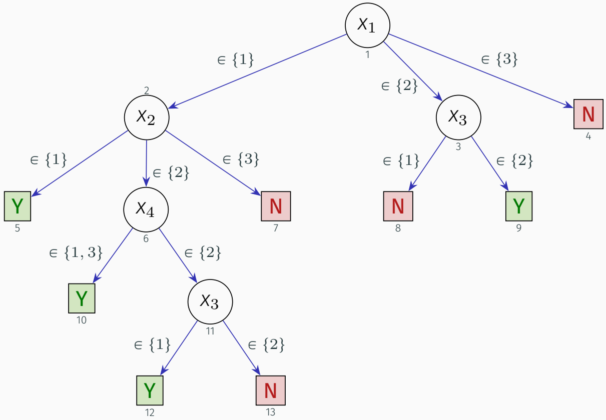
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$
  - 3rd path inconsistent: $H_3 = \{1\}$

Review exercise – one AXp for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



- Finding on AXp:
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Review exercise – one AXp for example DT

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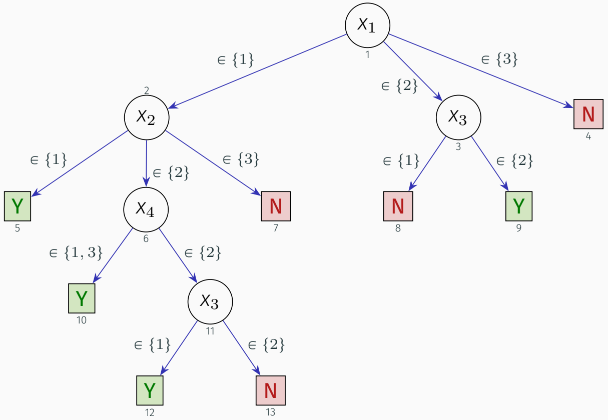
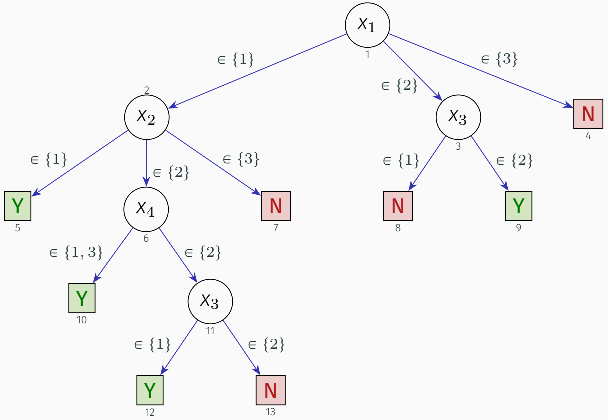


- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$
  - 2nd path inconsistent: $\mathcal{H}_2 = \{2\}$
  - 3rd path inconsistent: $\mathcal{H}_3 = \{1\}$
  - 4th path inconsistent: $\mathcal{H}_4 = \{1\}$
- AXp is MHS of $\mathcal{H}_j$ sets: $\{1,2,3\}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



Review exercise – all CXps & AXps for example DT

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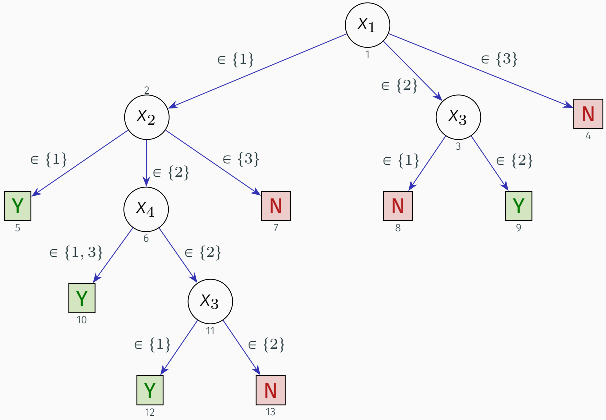

- Finding CXps:

Review exercise – all CXps & AXps for example DT

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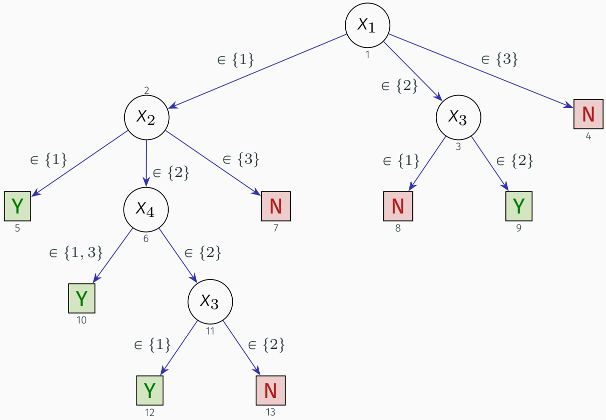
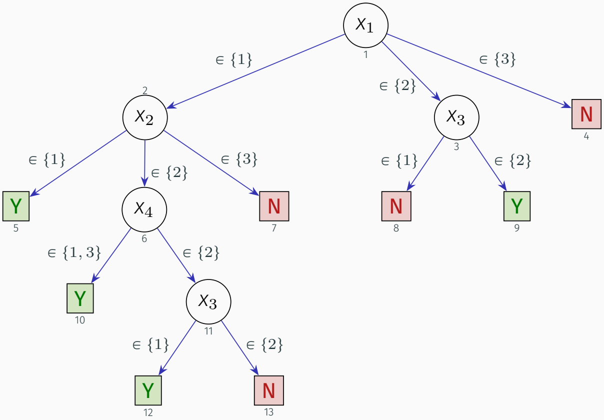


- Finding CXps:
  - 1st path: $I_1 = \{3\}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \text{Y})$



- Finding CXps:
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Review exercise – all CXps & AXps for example DT

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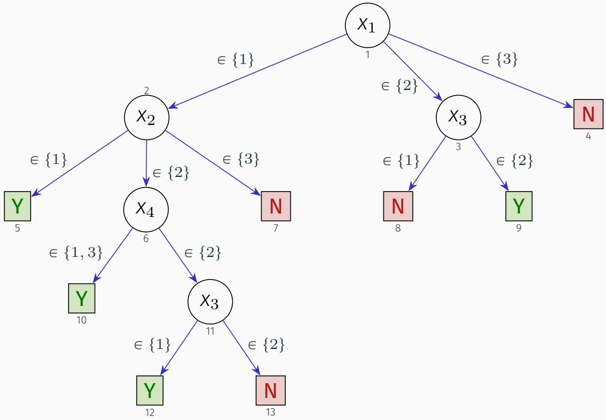

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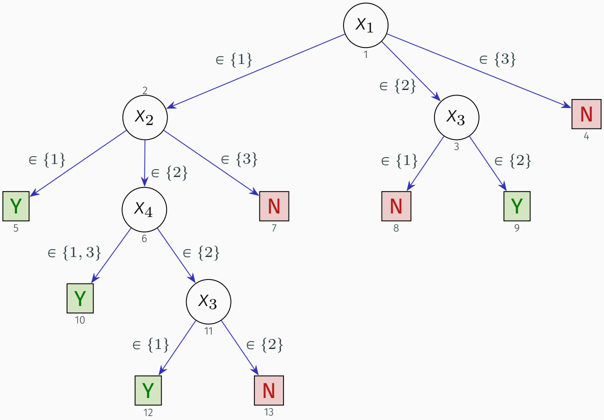

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Review exercise – all CXps & AXps for example DT

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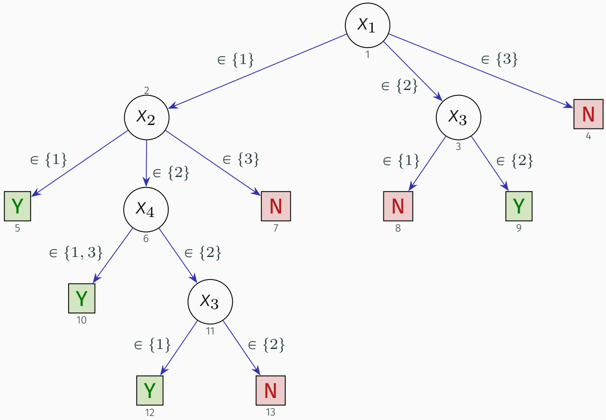
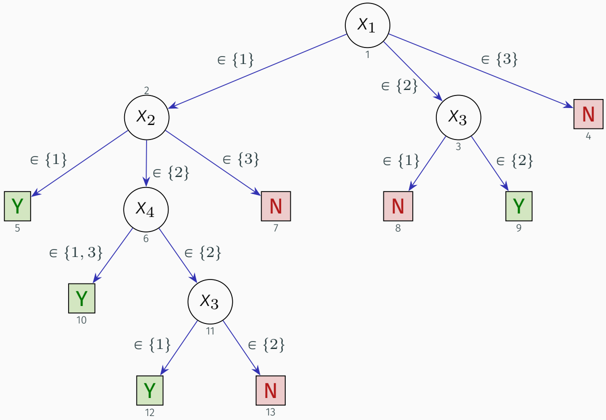


- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



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  - 1st path: $I_1 = \{3\}$
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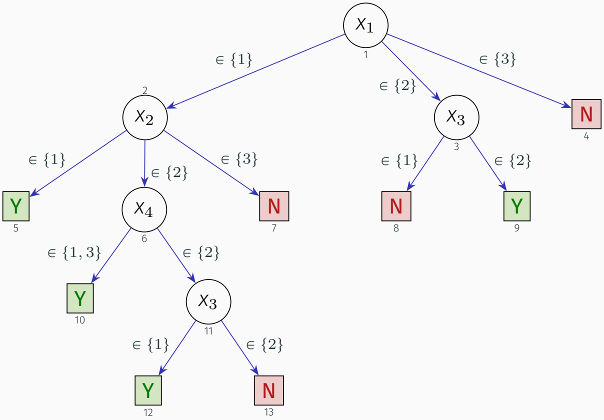
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\},\{2\},\{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$
  - $\mathbb{A} = \{\{1,2,3\}\}$

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \wedge x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \wedge x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \wedge x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \wedge \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
$\mathrm{R}_{\mathrm{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 0$

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llll}
R_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
R_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_7: & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
R_{\text{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_7: & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & \qquad\qquad \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp:

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llll}
\mathsf{R}_1: & \text{IF} & (x_1 \wedge x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_1 \wedge x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (x_2 \wedge x_4) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_1 \wedge x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
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\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3

- AXp: $\{1, 2\}$

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} \quad \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} \quad \kappa(\mathbf{x}) = 1 \\
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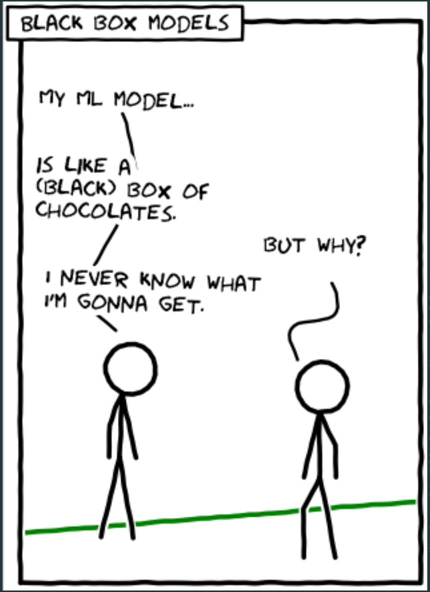
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & \qquad\qquad \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp: $\{1, 2\}$
- Quiz: write down the constraints and confirm AXp with SAT solver

Questions?



https://arxiv.org/abs/1901.01686 & http://cmx.io/edit/

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