# Logic-Based Explainable Artificial Intelligence

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ICREA & Univ. Lleida, Catalunya, Spain

ESSLLI, Bochum, Germany, July 2025

# Lecture 02

# Recapitulate first lecture

- ML models: classification & regression

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- Glimpse of heuristic XAI

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- ML models: classification & regression
- Glimpse of heuristic XAI
- Answers to Why? questions as logic rules
- Logic-based reasoning of ML models
- Apparent difficulties with explaining interpretable models

# Plan for this course

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Unit #02

# Principles of Symbolic XAI – Feature Selection

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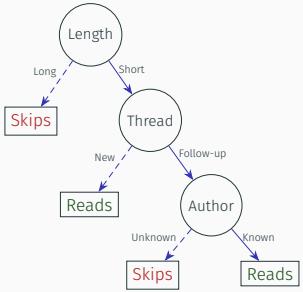
Original DT [PM17]

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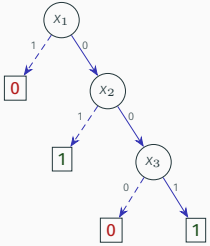


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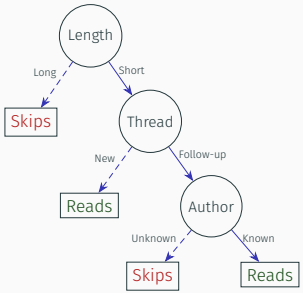


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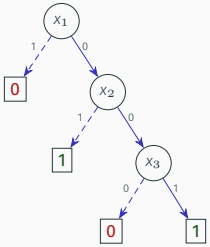


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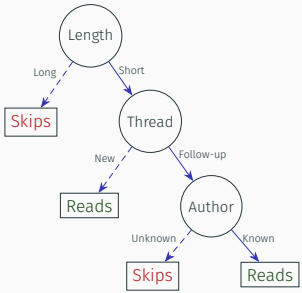


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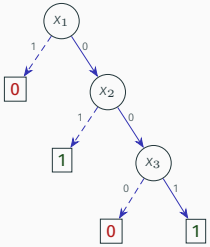


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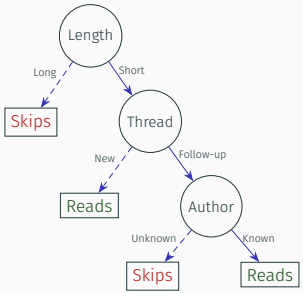


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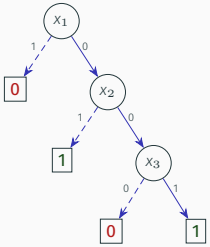


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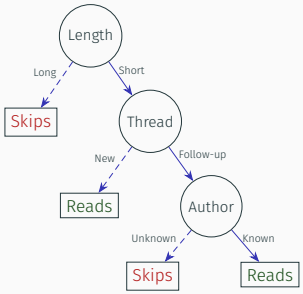


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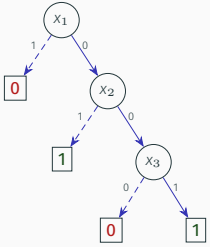


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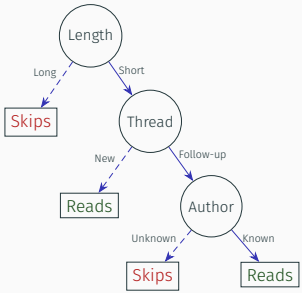


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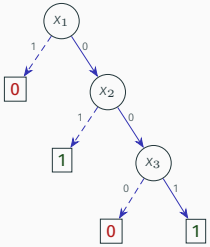


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    - One possible explanation is $\{\neg x_1, \neg x_2, x_3\}$ or simply $\{1, 2, 3\}$

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[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
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- Bottom line:
  Reason about symbolic explainability by abstracting away type of ML model

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- Finding one AXp (example algorithm; many more exist): [MM20]
  - Let $\mathcal{X} = \mathcal{F}$, i.e. fix all features
  - Invariant: $\mathsf{WAXp}(\mathcal{X})$ must hold. **Why?**
  - Analyze features in any order, one feature $i$ at a time
    - If $\mathsf{WAXp}(\mathcal{X}\backslash\{i\})$ holds, then remove $i$ from $\mathcal{X}$, i.e. $i$ becomes free

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  - **Obs:** for some classes of classifiers, poly-time algorithms exist

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- Finding one CXp: [MM20]
  - Let $\mathcal{Y} = \mathcal{F}$, i.e. free all features
  - Invariant: $\text{WCXp}(\mathcal{Y})$ must hold. Why?
  - Analyze features in any order, one feature $i$ at a time
    - If $\text{WCXp}(\mathcal{Y}\backslash\{i\})$ holds, then remove $i$ from $\mathcal{Y}$, i.e. $i$ is becomes fixed

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- CXp $\mathcal{Y} = \{4\}$
- **Obs:** AXp is MHS of CXp and vice-versa...

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- Definition of CXp remains unchanged

# Detour: global explanations

[INM19b]

- AXps and CXps are defined locally (because of $\mathbf{v}$) but hold globally
  - Localized explanations
  - Can be viewed as attempt at formalizing local explanations [RSG16, LL17, RSG18]
- One can define explanations without picking a given point in feature space
  - Let $q \in \mathbb{T}$, and refefine the similarity predicate:
    - Classification: $\sigma(\mathbf{x}) = [\kappa(\mathbf{x}) = q]$
    - Regression: $\sigma(\mathbf{x}) = [|\kappa(\mathbf{x}) - q| \leqslant \delta]$, $\delta$ is user-specified
  - Let $\mathbb{L} = \{(x_i = v_i) \,|\, i \in \mathcal{F} \land v_i \in \mathbb{V}\}$
  - Let $\mathcal{S} \subsetneq \mathbb{L}$ be a subset of literals that does not repeat features, i.e. $\mathcal{S}$ is not inconsistent
  - Then, $\mathcal{S}$ is a global AXp if,

$$\forall(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{(x_i = v_i) \in \mathcal{S}} (x_i = v_i) \rightarrow (\sigma(\mathbf{x}))$$

  - Counterexamples are minimal hitting sets of global AXps and vice-versa [INM19b]

# Outline – Unit #02

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[INAM20, Mar22]

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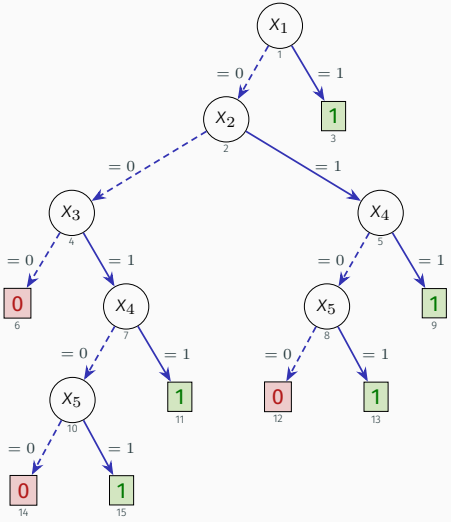

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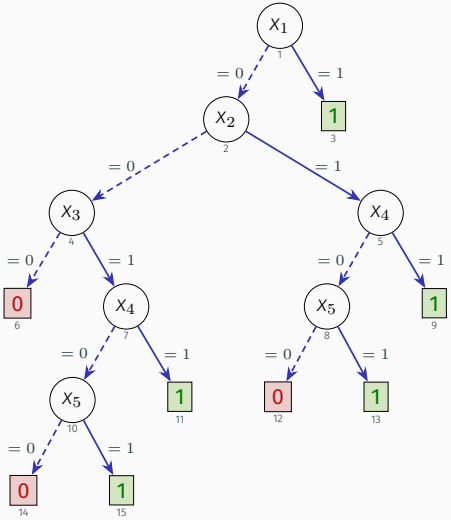

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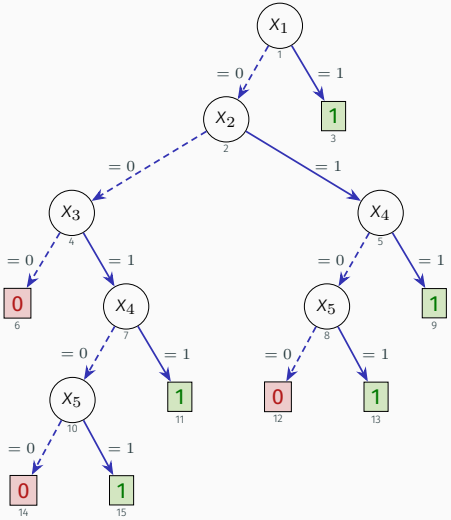

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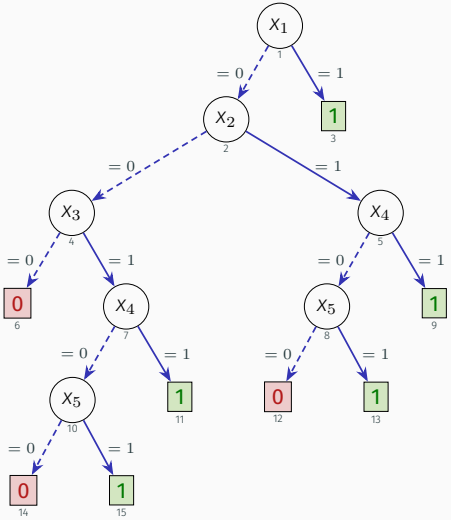

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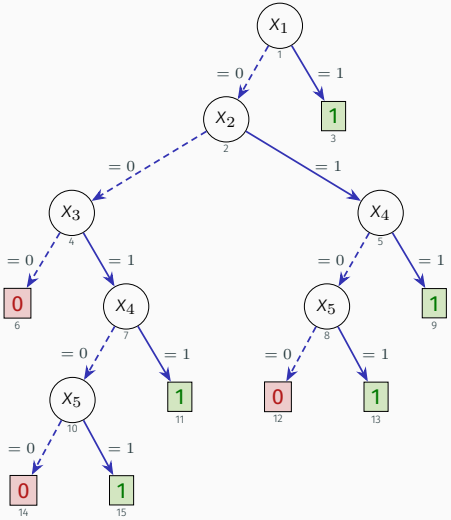

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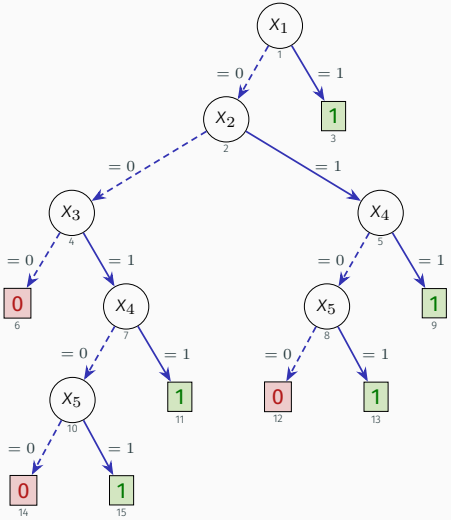

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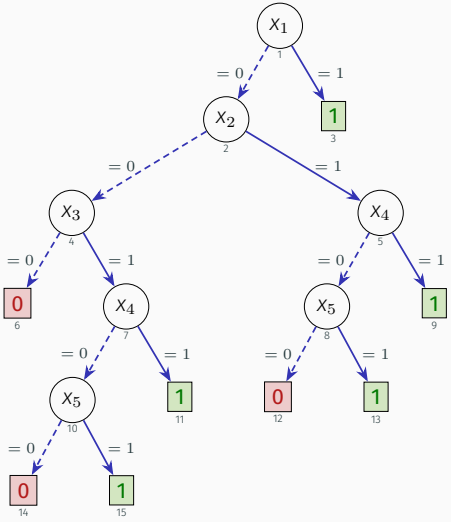
  - BTW,
    - $\{2, 5\}$ is not a CXp
    - $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 5\}$ and $\{1, 3, 5\}$ are not AXps



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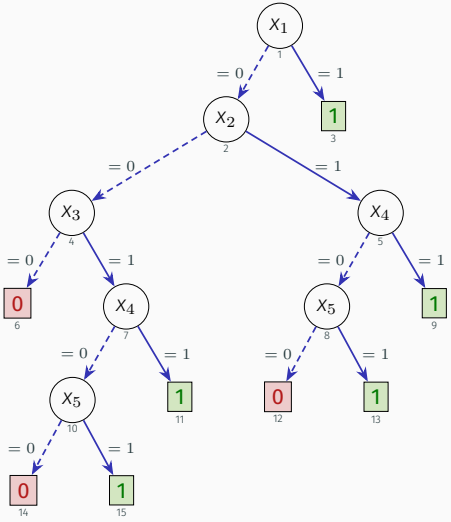
    - Why?



# Outline – Unit #02

# Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

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- Compute one abductive/contrastive explanation
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- Monotone predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg \textsf{CO}\left([\![(\bigwedge\nolimits_{i\in\mathcal{S}}(x_i = v_i)) \land (\neg\sigma(\mathbf{x}))]\!]\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \textsf{CO}\left([\![\left(\bigwedge\nolimits_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))]\!]\right)$$

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**Input**: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
**Output**: One XP $\mathcal{S}$

1: **procedure** oneXP($\mathbb{P}$)
2: $\quad\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
3: $\quad$**for** $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
4: $\quad\quad$**if** $\mathbb{P}(\mathcal{S}\setminus\{i\})$ **then**
5: $\quad\quad\quad\mathcal{S} \leftarrow \mathcal{S}\setminus\{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ holds
6: $\quad$**return** $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

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```
1: procedure oneXP(ℙ)
2:     𝒮 ← ℱ                                   ▷ Initialization: ℙ(𝒮) holds
3:     for i ∈ ℱ do                            ▷ Loop invariant: ℙ(𝒮) holds
4:         if ℙ(𝒮\{i}) then
5:             𝒮 ← 𝒮\{i}                       ▷ Update 𝒮 only if ℙ(𝒮\{i}) holds
6:     return 𝒮                                ▷ Returned set 𝒮: ℙ(𝒮) holds
```

Exploiting MSMP, i.e. basic algorithm used for different problems.

# Detour: More Connections with Automated Reasoning

# Prime implicants & implicates

- A conjunction of literals $\pi$ (which will be viewed as a set of literals where convenient) is a prime implicant of some function $\varphi$ if,
  1. $\pi \models \varphi$
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  - Example:
    - $\mathbb{F} = \{0,1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \wedge x_2 \vee x_1 \wedge x_3$
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# Reasoning about inconsistency

[MM20]

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \models \bot$
  - E.g. $\mathcal{B} = \{(x_1 \vee x_2), (x_1 \vee \neg x_3)\}, \mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$

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- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]
- Variants:
  - Smallest(-cost) MCS, i.e. complement of maximum(-cost) satisfiability (MaxSAT)
  - Smallest(-cost) MUS

# Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}
\mathrm{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\mathrm{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{aligned}$$

# Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}
\mathsf{WAXp}(\mathcal{X}) &:= \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j)\land(\neg\sigma(\mathbf{x}))\right]\\
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- **Claim**: Each MUS of $(\mathcal{B}, \mathcal{S})$ is an AXp & each MCS of $(\mathcal{B}, \mathcal{S})$ is a CXp
  - Can use MUS/MCS algorithms for finding AXps/CXps

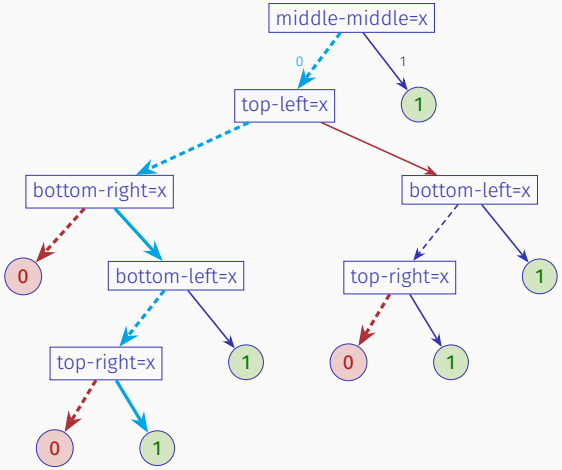
Unit #03

# Tractability in Symbolic XAI

# Outline – Unit #03

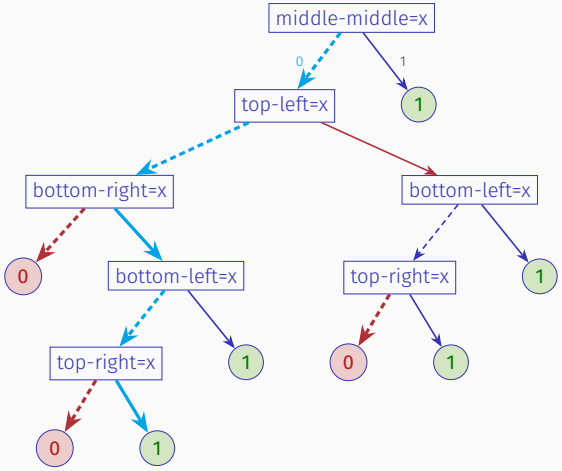
# DT explanations

[IIM20]



# DT explanations

[IIM20]




- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time

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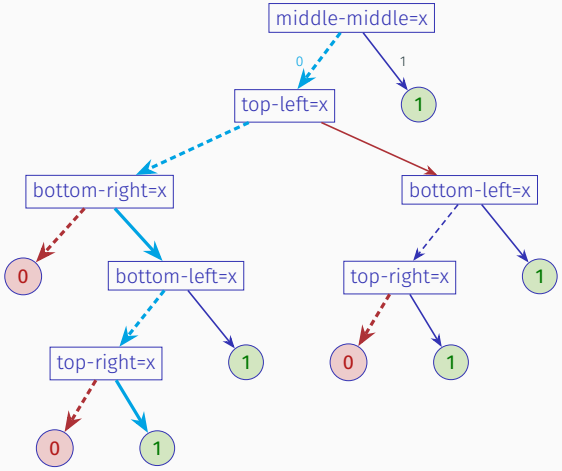


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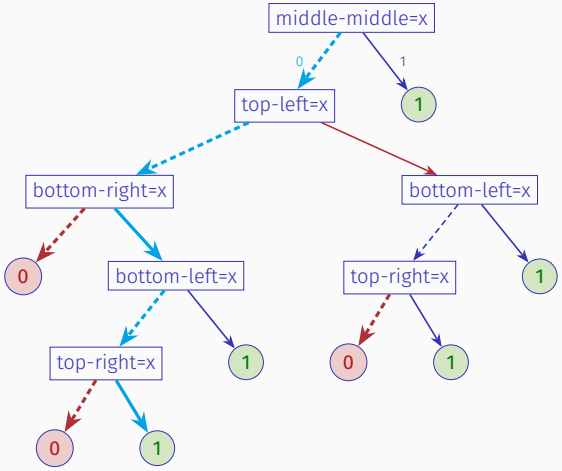
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent

# DT explanations in polynomial time

[IIM20]




- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent
  - I.e. find a subset-minimal hitting set of all 0 paths; these are the features to keep
    - E.g. BR and TR suffice for prediction
  - Well-known to be solvable in polynomial time [EG95]

# Outline – Unit #03

# Answering queries in DTs

- Finding one AXp in polynomial-time – covered

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- Finding one AXp in polynomial-time – covered
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- Finding one AXp in polynomial-time – covered
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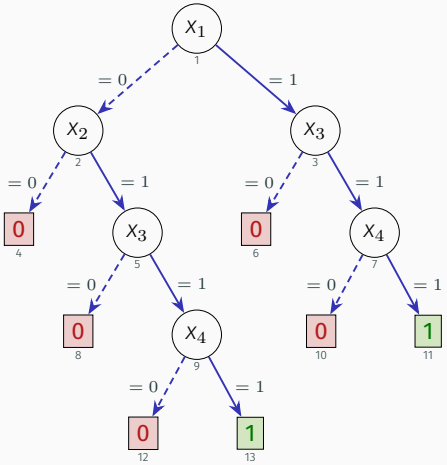
- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time
- Practically efficient enumeration of AXps – later

# Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \varnothing$



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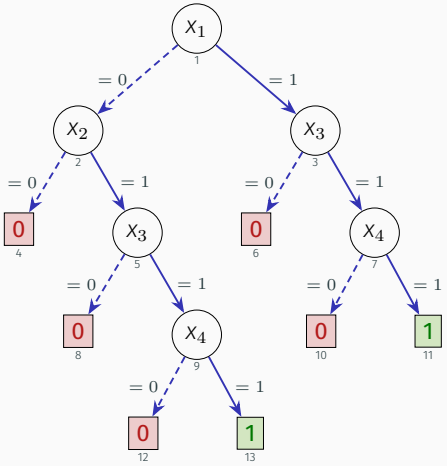

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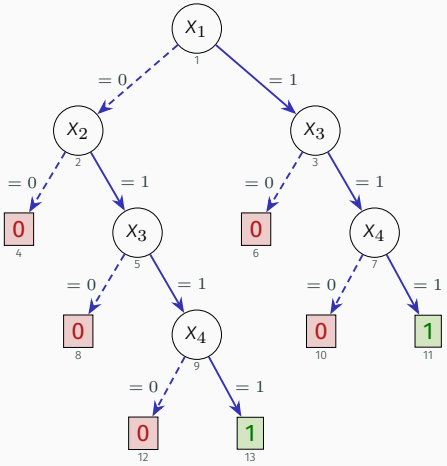

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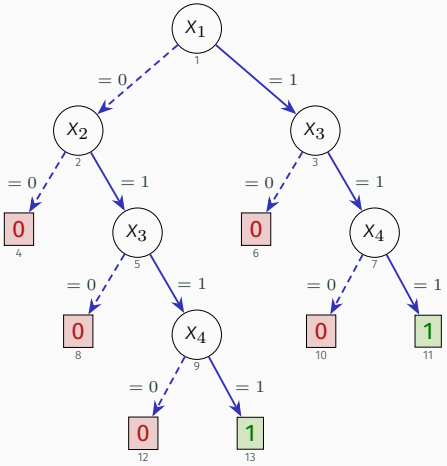

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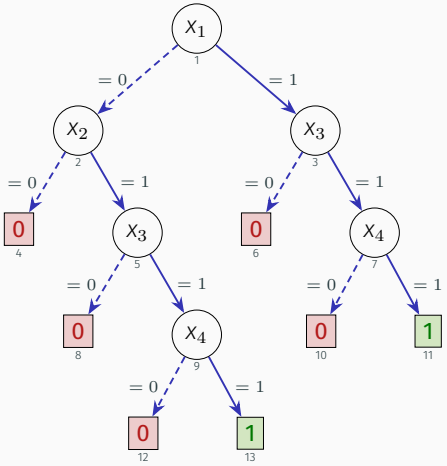

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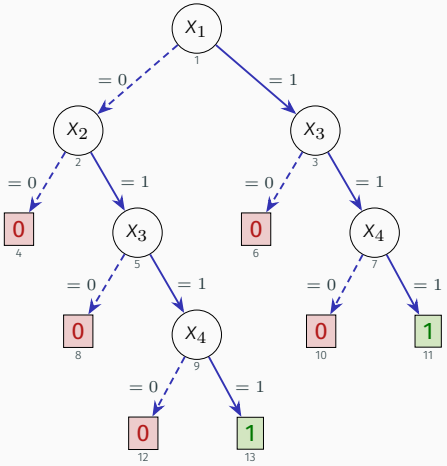

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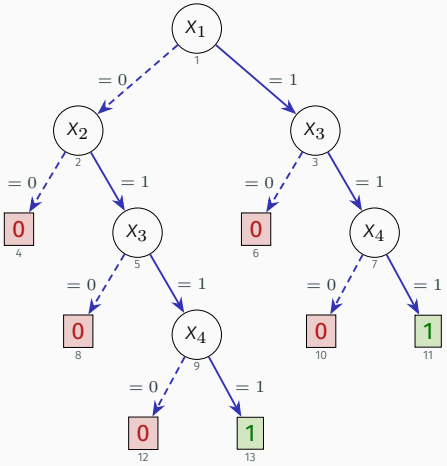

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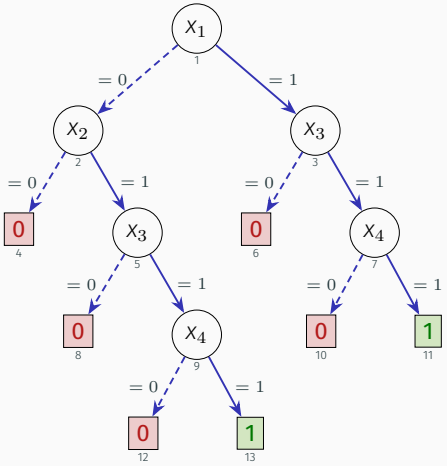

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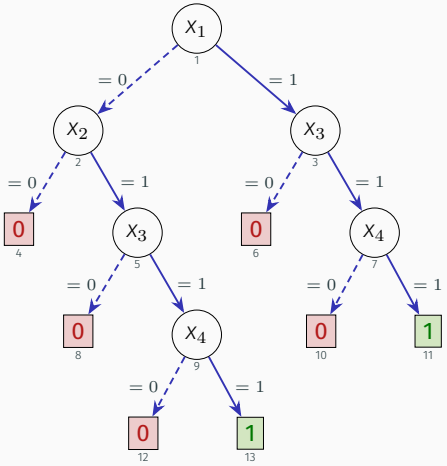

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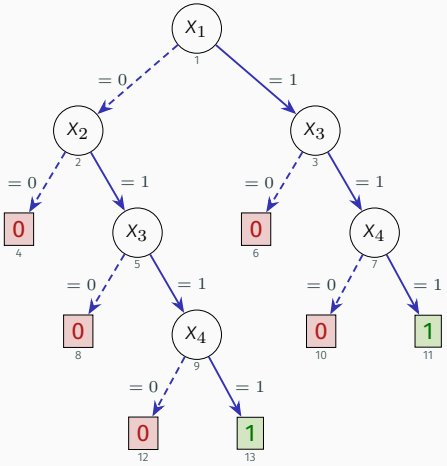

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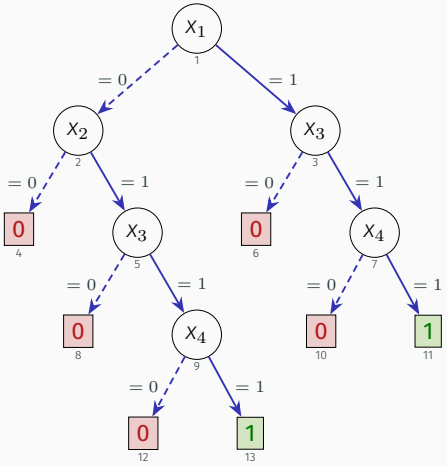

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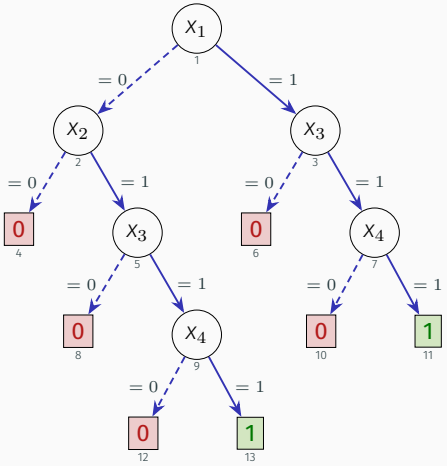

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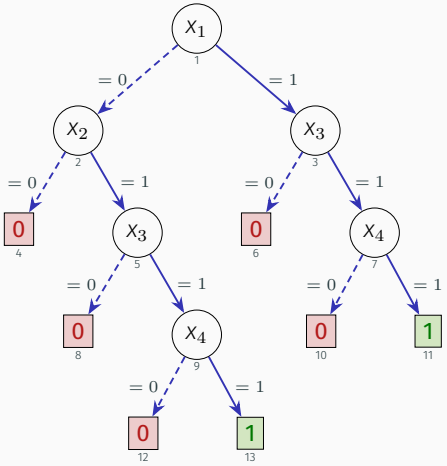

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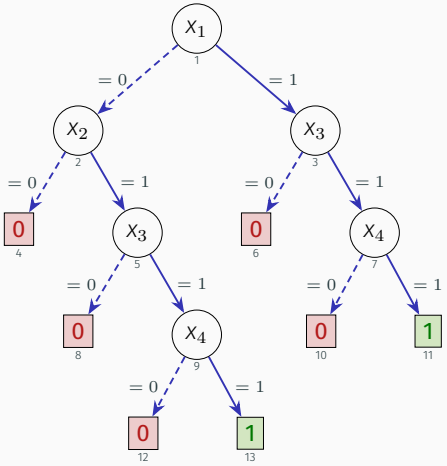

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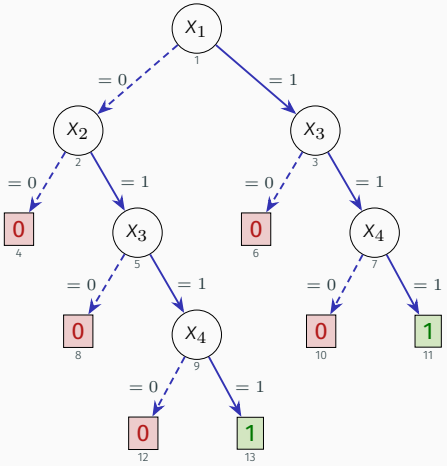
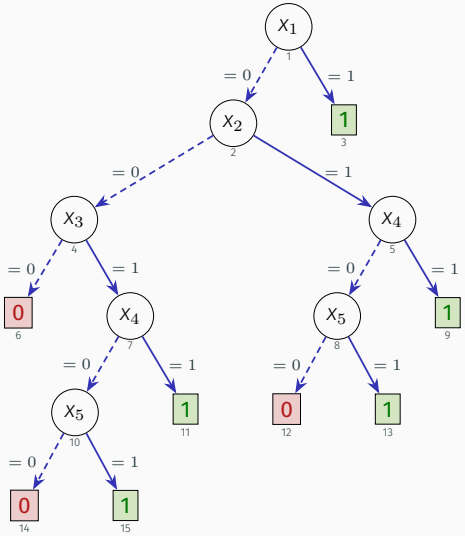
  - AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, by computing all MHSes



# Outline – Unit #03

# Are *interpretable* models really interpretable? – DTs

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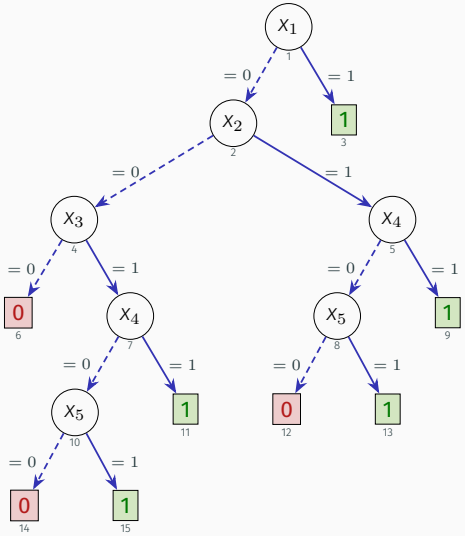


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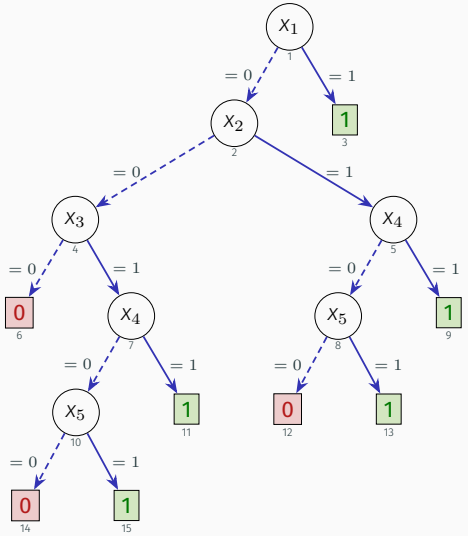


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  - But, $x_1$, $x_2$, $x_4$ are irrelevant for the prediction:

| $x_3$ | $x_5$ | $x_1$ | $x_2$ | $x_4$ | $\kappa(\mathbf{x})$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 |
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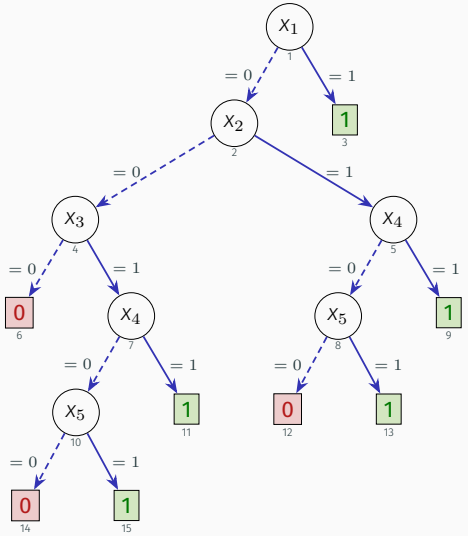


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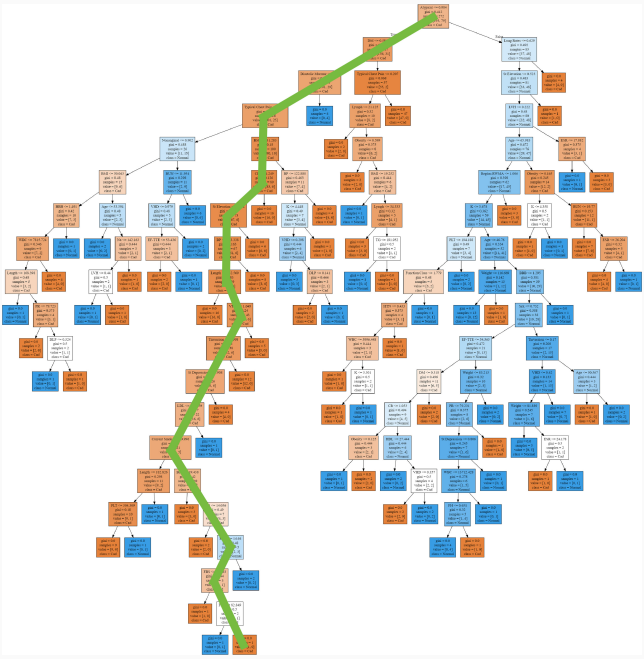
$\therefore$ one AXp is $\{3, 5\}$

Compare with $\{1, 2, 3, 4, 5\}$...

# Are *interpretable* models really interpretable? – large DTs



[GZM20]

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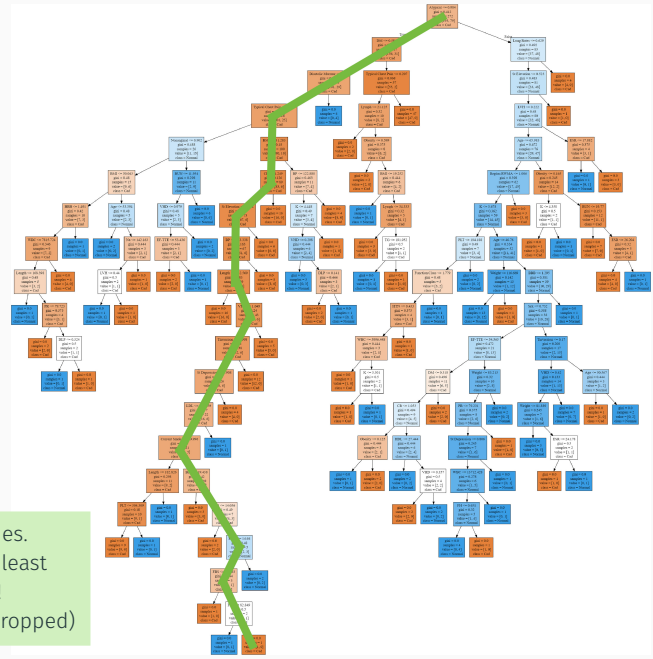

Path with 19 internal nodes.
By manual inspection, at least 10 literals are redundant!
(And at least 9 features dropped)

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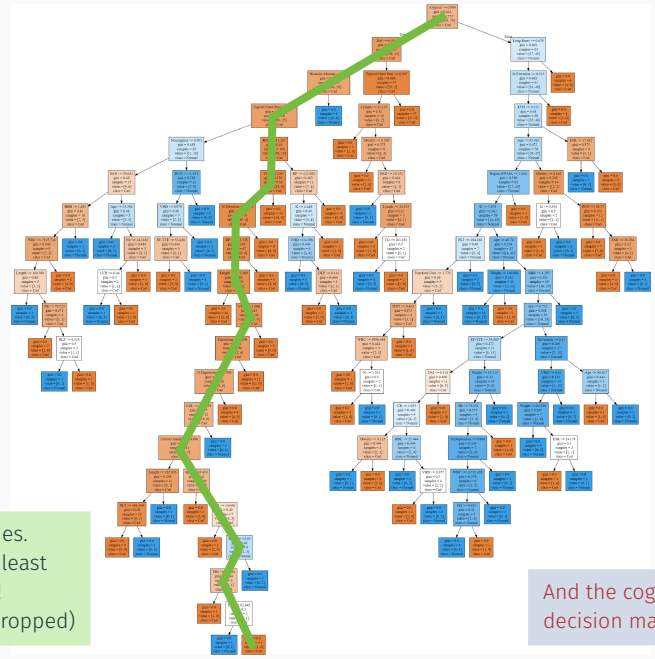

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And the cognitive limits of human decision makers are well-known [Mil56]

# Are *interpretable* models really interpretable? – large DTs

Redundancy can be arbitrary large on path length [IIM20, HIIM21, IIM22]

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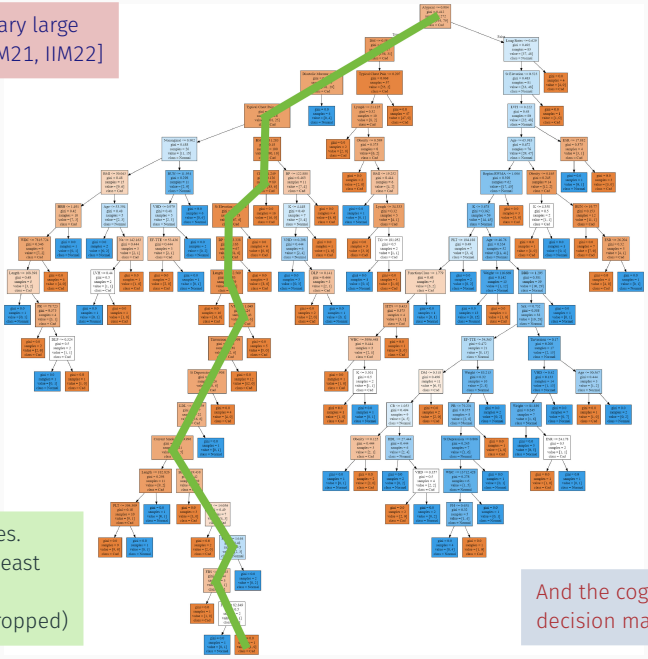

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- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

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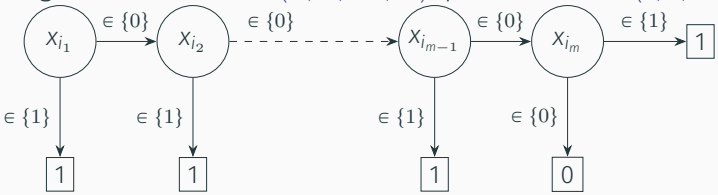
- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:



# Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

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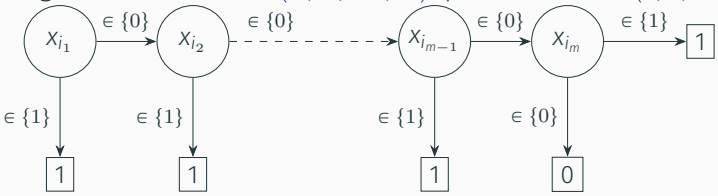


- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1

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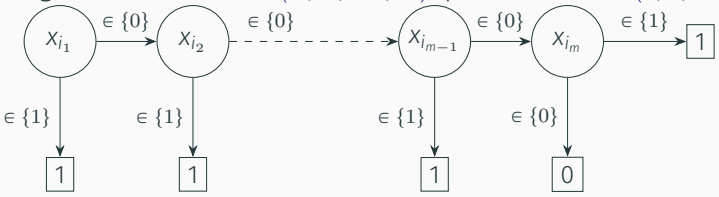


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- Explanation using path in DT: $\{i_1, i_2, \ldots, i_m\}$, i.e.

$$(x_{i_1} = 0) \wedge (x_{i_2} = 0) \wedge \ldots \wedge (x_{i_{m-1}} = 0) \wedge (x_{i_m} = 1) \rightarrow \kappa(x_1, \ldots, x_m)$$

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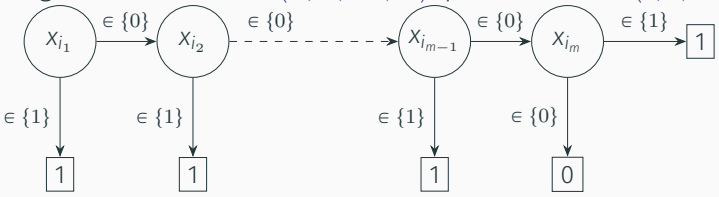


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- But $\{i_m\}$ suffices for prediction, i.e. $\forall(\mathbf{x} \in \{0, 1\}^m).(x_{i_m}) \rightarrow \kappa(\mathbf{x})$

# Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

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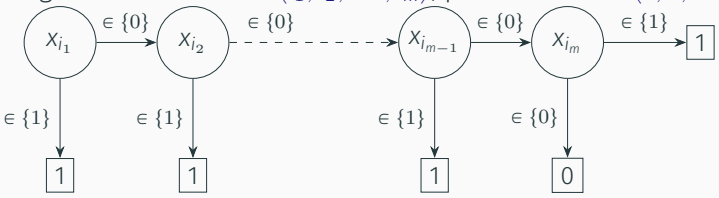



- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1
- Explanation using path in DT: $\{i_1, i_2, \ldots, i_m\}$, i.e.

$$(x_{i_1} = 0) \land (x_{i_2} = 0) \land \ldots \land (x_{i_{m-1}} = 0) \land (x_{i_m} = 1) \rightarrow \kappa(x_1, \ldots, x_m)$$

- But $\{i_m\}$ suffices for prediction, i.e. $\forall(\mathbf{x} \in \{0, 1\}^m).(x_{i_m}) \rightarrow \kappa(\mathbf{x})$
- AXp's can be arbitrarily smaller than paths in (optimal) DTs! [IIM20, IIM22]

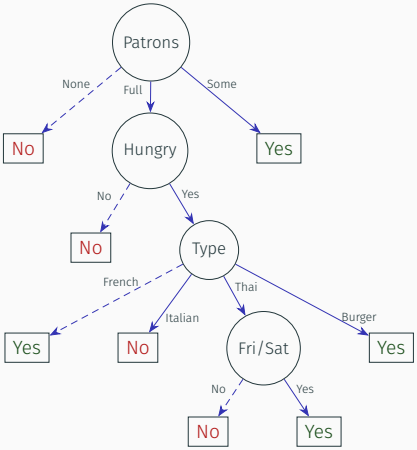
# Explanation redundancy in DTs is ubiquitous – published DT examples

[IIM22]

| DT Ref | D | #N | #P | %R | %C | %m | %M | %avg |
|---|---|---|---|---|---|---|---|---|
| [Alp14, Ch. 09, Fig. 9.1] | 2 | 5 | 3 | 33 | 25 | 50 | 50 | 50 |
| [Alp16, Ch. 03, Fig. 3.2] | 2 | 5 | 3 | 33 | 25 | 50 | 50 | 50 |
| [Bra20, Ch. 01, Fig. 1.3] | 4 | 9 | 5 | 60 | 25 | 25 | 50 | 36 |
| [BA97, Figure 1] | 3 | 12 | 7 | 14 | 8 | 33 | 33 | 33 |
| [BBHK10, Ch. 08, Fig. 8.2] | 3 | 7 | 4 | 25 | 12 | 50 | 50 | 50 |
| [BFOS84, Ch. 01, Fig. 1.1] | 3 | 7 | 4 | 50 | 25 | 33 | 33 | 33 |
| [DL01, Ch. 01, Fig. 1.2a] | 2 | 5 | 3 | 33 | 25 | 33 | 33 | 33 |
| [DL01, Ch. 01, Fig. 1.2b] | 2 | 5 | 3 | 33 | 25 | 33 | 33 | 33 |
| [KMND20, Ch. 04, Fig. 4.14] | 3 | 7 | 4 | 25 | 12 | 50 | 50 | 50 |
| [KMND20, Sec. 4.7, Ex. 4] | 2 | 5 | 3 | 33 | 25 | 50 | 50 | 50 |
| [Qui93, Ch. 01, Fig. 1.3] | 3 | 12 | 7 | 28 | 17 | 33 | 50 | 41 |
| [RM08, Ch. 01, Fig. 1.5] | 3 | 9 | 5 | 20 | 12 | 33 | 33 | 33 |
| [RM08, Ch. 01, Fig. 1.4] | 3 | 7 | 4 | 50 | 25 | 33 | 33 | 33 |
| [WFHP17, Ch. 01, Fig. 1.2] | 3 | 7 | 4 | 25 | 12 | 50 | 50 | 50 |
| [VLE+16, Figure 4] | 6 | 39 | 20 | 65 | 63 | 20 | 40 | 33 |
| [Fla12, Ch. 02, Fig. 2.1(right)] | 2 | 5 | 3 | 33 | 25 | 50 | 50 | 50 |
| [Kot13, Figure 1] | 3 | 10 | 6 | 33 | 11 | 33 | 33 | 33 |
| [Mor82, Figure 1] | 3 | 9 | 5 | 80 | 75 | 33 | 50 | 41 |
| [PM17, Ch. 07, Fig. 7.4] | 3 | 7 | 4 | 50 | 25 | 33 | 33 | 33 |
| [RN10, Ch. 18, Fig. 18.6] | 4 | 12 | 8 | 25 | 6 | 25 | 33 | 29 |
| [SB14, Ch. 18, Page 212] | 2 | 5 | 3 | 33 | 25 | 50 | 50 | 50 |
| [Zho12, Ch. 01, Fig. 1.3] | 2 | 5 | 3 | 33 | 25 | 33 | 33 | 33 |
| [BHO09, Figure 1b] | 4 | 13 | 7 | 71 | 50 | 33 | 50 | 36 |
| [Zho21, Ch. 04, Fig. 4.3] | 4 | 14 | 9 | 11 | 2 | 25 | 25 | 25 |

# Many DTs have paths that are not minimal XPs – Russell&Norvig's book

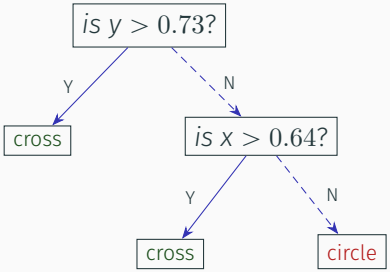
[RN10]




- Explanation for $(P, H, T, W) = (\text{Full}, \text{Yes}, \text{Thai}, \text{No})$?

# Many DTs have paths that are not minimal XPs – Zhou's book

[Zho12]

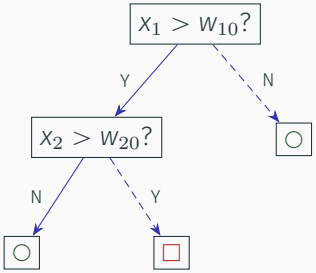



- Explanation for $(x, y) = (1.25, -1.13)$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

# Many DTs have paths that are not minimal XPs – Alpaydin's book

[Alp14]

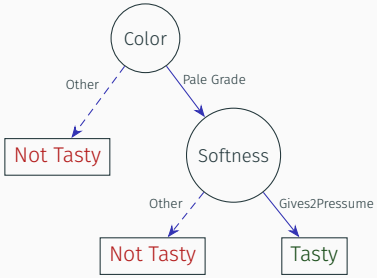



- Explanation for $(x_1, x_2) = (\alpha, \beta)$, with $\alpha > w_{10}$ and $\beta \leqslant w_{20}$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

# Many DTs have paths that are not minimal XPs – S.-S.&B.-D.'s book

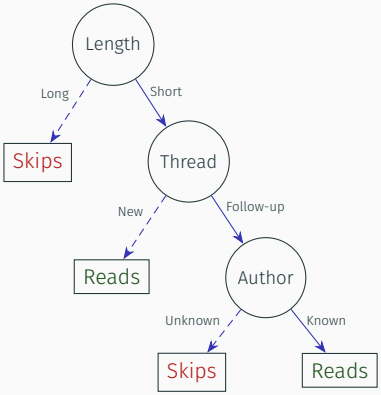
[SB14]




- Explanation for $(\text{color}, \text{softness}) = (\text{Pale Grade}, \text{Other})$?

# Many DTs have paths that are not minimal XPs – Poole&Mackworth's book

[PM17]




- Explanation for $(L, T, A) = (\text{Short}, \text{Follow-Up}, \text{Unknown})$?
- Explanation for $(L, T, A) = (\text{Short}, \text{Follow-Up}, \text{Known})$?

# Explanation redundancy in DTs is ubiquitous – DTs from datasets

[IIM20, HIIM21, IIM22]

| Dataset | (#F | #S) | IAI | | | | | | | | | ITI | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | D | #N | %A | #P | %R | %C | %m | %M | %avg | D | #N | %A | #P | %R | %C | %m | %M | %avg |
| adult | ( 12 | 6061) | 6 | 83 | 78 | 42 | 33 | 25 | 20 | 40 | 25 | 17 | 509 | 73 | 255 | 75 | 91 | 10 | 66 | 22 |
| anneal | ( 38 | 886) | 6 | 29 | 99 | 15 | 26 | 16 | 16 | 33 | 21 | 9 | 31 | 100 | 16 | 25 | 4 | 12 | 20 | 16 |
| backache | ( 32 | 180) | 4 | 17 | 72 | 9 | 33 | 39 | 25 | 33 | 30 | 3 | 9 | 91 | 5 | 80 | 87 | 50 | 66 | 54 |
| bank | ( 19 | 36293) | 6 | 113 | 88 | 57 | 5 | 12 | 16 | 20 | 18 | 19 | 1467 | 86 | 734 | 69 | 64 | 7 | 63 | 27 |
| biodegradation | ( 41 | 1052) | 5 | 19 | 65 | 10 | 30 | 1 | 25 | 50 | 33 | 8 | 71 | 76 | 36 | 50 | 8 | 14 | 40 | 21 |
| cancer | ( 9 | 449) | 6 | 37 | 87 | 19 | 36 | 9 | 20 | 25 | 21 | 5 | 21 | 84 | 11 | 54 | 10 | 25 | 50 | 37 |
| car | ( 6 | 1728) | 6 | 43 | 96 | 22 | 86 | 89 | 20 | 80 | 45 | 11 | 57 | 98 | 29 | 65 | 41 | 16 | 50 | 30 |
| colic | ( 22 | 357) | 6 | 55 | 81 | 28 | 46 | 6 | 16 | 33 | 20 | 4 | 17 | 80 | 9 | 33 | 27 | 25 | 25 | 25 |
| compas | ( 11 | 1155) | 6 | 77 | 34 | 39 | 17 | 8 | 16 | 20 | 17 | 15 | 183 | 37 | 92 | 66 | 43 | 12 | 60 | 27 |
| contraceptive | ( 9 | 1425) | 6 | 99 | 49 | 50 | 8 | 2 | 20 | 60 | 37 | 17 | 385 | 48 | 193 | 27 | 32 | 12 | 66 | 21 |
| dermatology | ( 34 | 366) | 6 | 33 | 90 | 17 | 23 | 3 | 16 | 33 | 21 | 7 | 17 | 95 | 9 | 22 | 0 | 14 | 20 | 17 |
| divorce | ( 54 | 150) | 5 | 15 | 90 | 8 | 50 | 19 | 20 | 33 | 24 | 2 | 5 | 96 | 3 | 33 | 16 | 50 | 50 | 50 |
| german | ( 21 | 1000) | 6 | 25 | 61 | 13 | 38 | 10 | 20 | 40 | 29 | 10 | 99 | 72 | 50 | 46 | 13 | 12 | 40 | 22 |
| heart-c | ( 13 | 302) | 6 | 43 | 65 | 22 | 36 | 18 | 20 | 33 | 22 | 4 | 15 | 75 | 8 | 87 | 81 | 25 | 50 | 34 |
| heart-h | ( 13 | 293) | 6 | 37 | 59 | 19 | 31 | 4 | 20 | 40 | 24 | 8 | 25 | 77 | 13 | 61 | 60 | 20 | 50 | 32 |
| kr-vs-kp | ( 36 | 3196) | 6 | 49 | 96 | 25 | 80 | 75 | 16 | 60 | 33 | 13 | 67 | 99 | 34 | 79 | 43 | 7 | 70 | 35 |
| lending | ( 9 | 5082) | 6 | 45 | 73 | 23 | 73 | 80 | 16 | 50 | 25 | 14 | 507 | 65 | 254 | 69 | 80 | 12 | 75 | 25 |
| letter | ( 16 | 18668) | 6 | 127 | 58 | 64 | 1 | 0 | 20 | 20 | 20 | 46 | 4857 | 68 | 2429 | 6 | 7 | 6 | 25 | 9 |
| lymphography | ( 18 | 148) | 6 | 61 | 76 | 31 | 35 | 25 | 16 | 33 | 21 | 6 | 21 | 86 | 11 | 9 | 0 | 16 | 16 | 16 |
| mortality | (118 | 13442) | 6 | 111 | 74 | 56 | 8 | 14 | 16 | 20 | 17 | 26 | 865 | 76 | 433 | 61 | 61 | 7 | 54 | 19 |
| mushroom | ( 22 | 8124) | 6 | 39 | 100 | 20 | 80 | 44 | 16 | 33 | 24 | 5 | 23 | 100 | 12 | 50 | 31 | 20 | 40 | 25 |
| pendigits | ( 16 | 10992) | 6 | 121 | 88 | 61 | 0 | 0 | — | — | — | 38 | 937 | 85 | 469 | 25 | 86 | 6 | 25 | 11 |
| promoters | ( 58 | 106) | 1 | 3 | 90 | 2 | 0 | 0 | — | — | — | 3 | 9 | 81 | 5 | 20 | 14 | 33 | 33 | 33 |
| recidivism | ( 15 | 3998) | 6 | 105 | 61 | 53 | 28 | 22 | 16 | 33 | 18 | 15 | 611 | 51 | 306 | 53 | 38 | 9 | 44 | 16 |
| seismic_bumps | ( 18 | 2578) | 6 | 37 | 89 | 19 | 42 | 19 | 20 | 33 | 24 | 8 | 39 | 93 | 20 | 60 | 79 | 20 | 60 | 42 |
| shuttle | ( 9 | 58000) | 6 | 63 | 99 | 32 | 28 | 7 | 20 | 33 | 23 | 23 | 159 | 99 | 80 | 33 | 9 | 14 | 50 | 30 |
| soybean | ( 35 | 623) | 6 | 63 | 88 | 32 | 9 | 5 | 25 | 25 | 25 | 16 | 71 | 89 | 36 | 22 | 1 | 9 | 12 | 10 |
| spambase | ( 57 | 4210) | 6 | 63 | 75 | 32 | 37 | 12 | 16 | 33 | 19 | 15 | 143 | 91 | 72 | 76 | 98 | 7 | 58 | 25 |
| spect | ( 22 | 228) | 6 | 45 | 82 | 23 | 60 | 51 | 20 | 50 | 35 | 6 | 15 | 86 | 8 | 87 | 98 | 50 | 83 | 65 |
| splice | ( 2 | 3178) | 3 | 7 | 50 | 4 | 0 | 0 | — | — | — | 88 | 177 | 55 | 89 | 0 | 0 | — | — | — |

$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \wedge x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_2 \wedge x_4 \wedge x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (\neg x_1 \wedge x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_4 \wedge x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_1 \wedge \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_{\text{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule $\mathsf{R}_2$ fires

$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \wedge x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
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\mathsf{R}_3: & \text{ELSE IF} & (\neg x_1 \wedge x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_4 \wedge x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_1 \wedge \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_{\text{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule $\mathsf{R}_2$ fires
- What is the abductive explanation?

$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_2 \land x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (\neg x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_1 \land \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule $\mathsf{R}_2$ fires
- What is the abductive explanation?
- Recall: one AXp is $\{3, 4, 6\}$

$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_2 \land x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (\neg x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_1 \land \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule $\mathsf{R}_2$ fires
- What is the abductive explanation?
- Recall: one AXp is $\{3, 4, 6\}$
  - Why?
    - We need 3 (or 1) so that $\mathsf{R}_1$ cannot fire
    - With 3, we do not need 2, since with 4 and 6 fixed, then $\mathsf{R}_4$ is guaranteed to fire
  - Some questions:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?

# Are *interpretable* models really interpretable? – DLs

[MSI23]

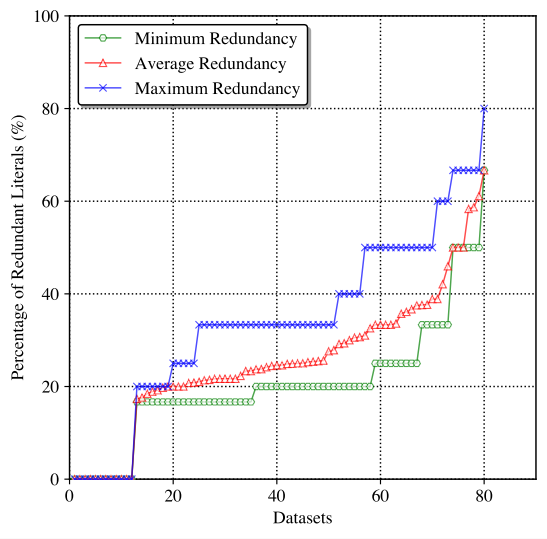
$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_2 \land x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (\neg x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_1 \land \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule $\mathsf{R}_2$ fires
- What is the abductive explanation?
- Recall: one AXp is $\{3, 4, 6\}$
  - Why?
    - We need 3 (or 1) so that $\mathsf{R}_1$ cannot fire
    - With 3, we do not need 2, since with 4 and 6 fixed, then $\mathsf{R}_4$ is guaranteed to fire
  - Some questions:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?
      (BTW, **we have proved that computing one AXp for DLs is computationally hard...**)

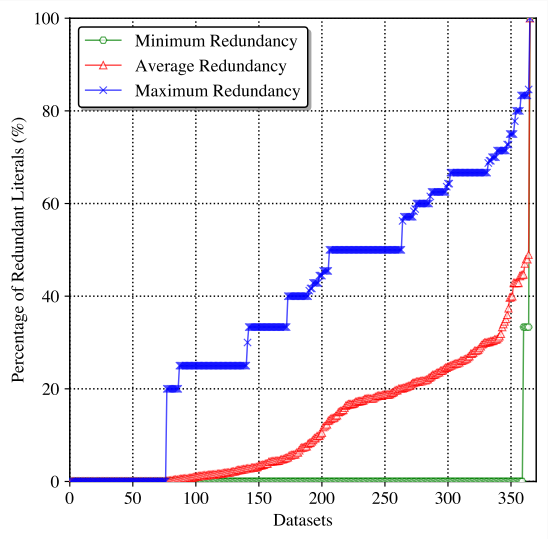
[IM21, MSI23]

# Are *interpretable* models really interpretable? – DTs/DLs in practice

[MSI23]




DTs learned with Interpretable AI, max depth 6




DLs learned with CN2

# Outline – Unit #03

# From DTs to explained DSs

[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules

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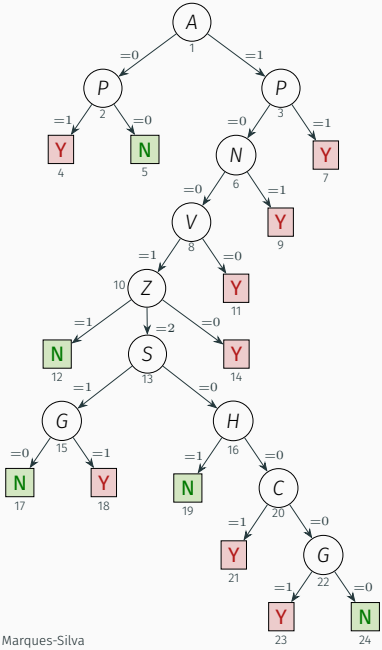
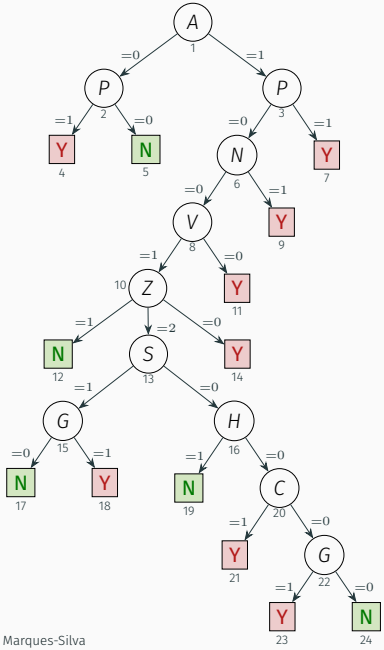
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...

- One can extract explained DSs from DTs
  - Extract one AXp (viewed as a logic rule) from each path in DT
  - Resulting rules are non-overlapping, and cover feature space

# Example


A
1
=0
=1
P
2
P
3
=1
=0
=0
=1
Y
4
N
5
N
6
Y
7
=0
=1
V
8
Y
9
=1
=0
10
Z
Y
11
=1
=2
=0
N
12
S
13
Y
14
=1
=0
G
15
H
16
=0
=1
=1
=0
N
17
Y
18
N
19
C
20
=1
=0
Y
21
G
22
=1
=0
Y
23
N
24

# Example




$R_{01}$: IF $[P]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{02}$: IF $[\overline{A} \wedge \overline{P}]$THEN $\kappa(\cdot) = \mathbf{N}$

$R_{03}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 1]$ THEN $\kappa(\cdot) = \mathbf{N}$

$R_{04}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge S \wedge \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

$R_{05}$: IF $[A \wedge Z = 2 \wedge S \wedge G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{06}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge \overline{S} \wedge H]$ THEN $\kappa(\cdot) = \mathbf{N}$

$R_{07}$: IF $[A \wedge Z = 2 \wedge \overline{S} \wedge \overline{H} \wedge C]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{08}$: IF $[A \wedge Z = 2 \wedge \overline{H} \wedge G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{09}$: IF $[\overline{P} \wedge \overline{N} \wedge V \wedge Z = 2 \wedge \overline{C} \wedge \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

$R_{10}$: IF $[A \wedge Z = 0]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{11}$: IF $[A \wedge \overline{V}]$ THEN $\kappa(\cdot) = \mathbf{Y}$

$R_{12}$: IF $[A \wedge N]$ THEN $\kappa(\cdot) = \mathbf{Y}$

# Outline – Unit #03

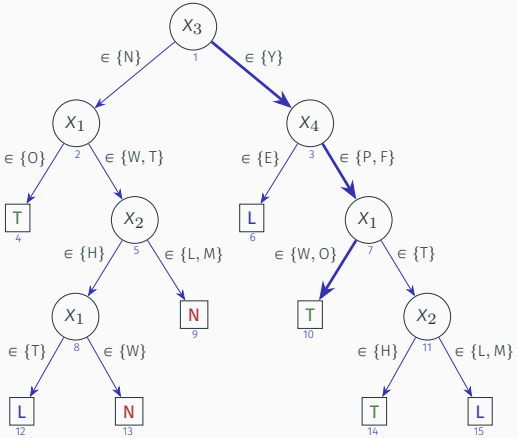
# Explanation graphs – overview of results

[HIIM21]

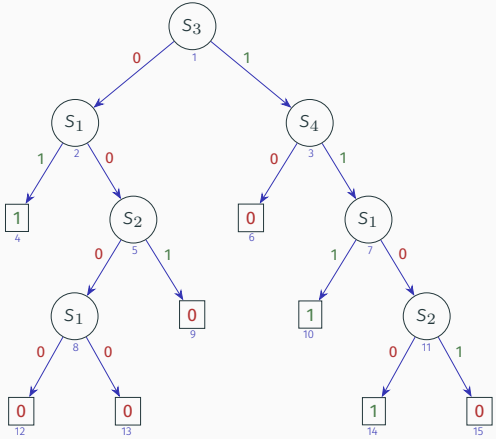
- Concept of explanation graph (XpG)
- Explanations of decision trees reducible to XpG's
- Explanations of decision graphs reducible to XpG's
- Explanations of OBDDs reducible to XpG's
- Explanations of OMDDs reducible to XpG's
- Explanations (AXp's and CXp's) of XpG's computed in polynomial time

# Example of XpG – DTs

- DT; point: $(O, L, Y, P)$; prediction $T$:

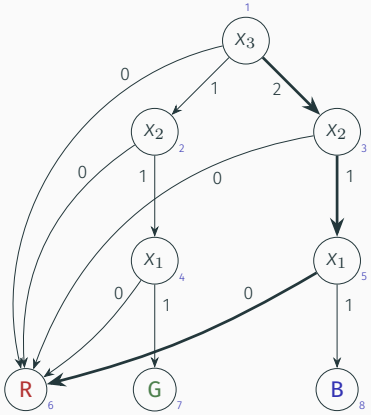



- XpG:

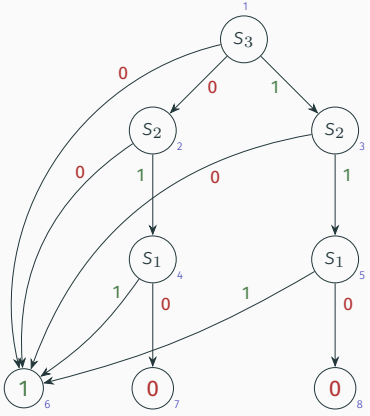


# Example of XpG – OMDDs

- OMBBD; point: $(0, 1, 2)$; prediction *R*:




- XpG:



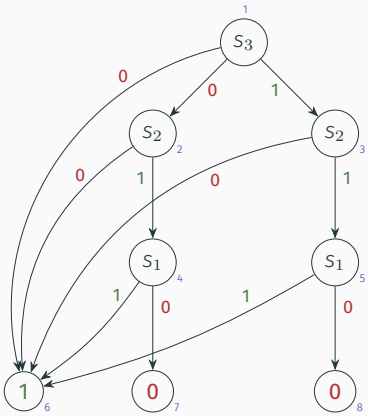
# Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature $i$ in $\mathcal{F}$

- XpG:



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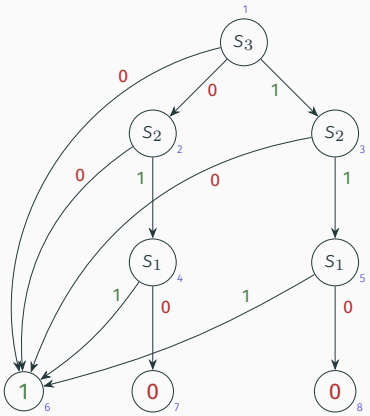

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  Add feature $i$ back to $\mathcal{S}$

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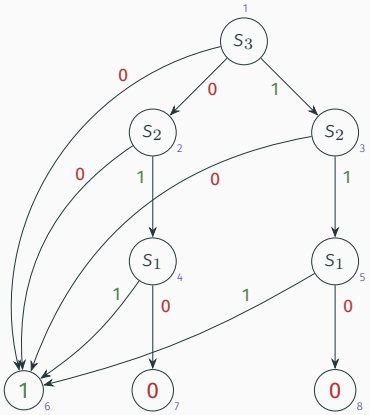

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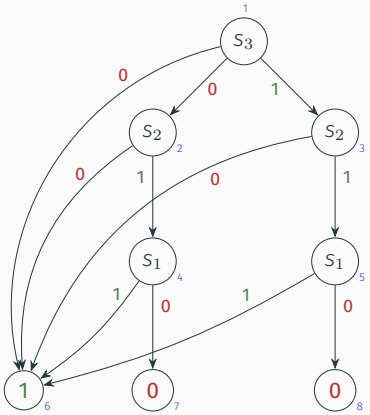

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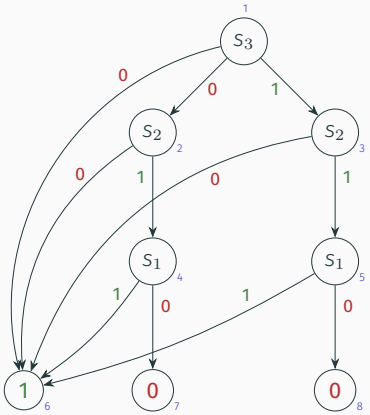
  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$

- XpG:



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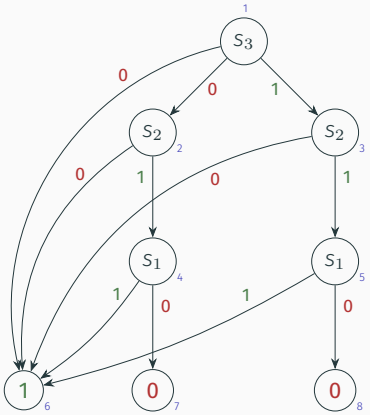
- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g.
    $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$

- XpG:



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- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
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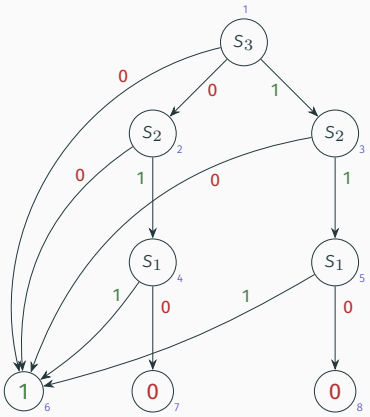
  - Both features 2 and 3 dropped from $\mathcal{S}$

- XpG:



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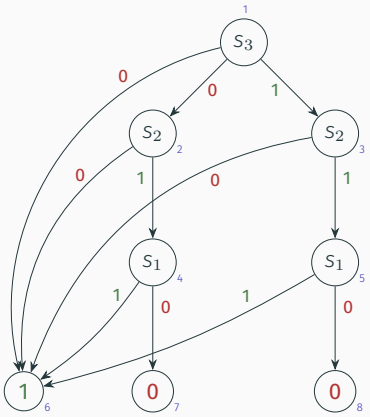
  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g.
    $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$
  - Both features 2 and 3 dropped from $\mathcal{S}$
  - Return $\mathcal{S} = \{1\}$

- XpG:



# Outline – Unit #03

# Example monotonic classifier – $(\mathbf{v}, c) = ((10, 10, 5, 0), A)$

[MGC+21]

| Variable | Meaning | Range |
|---|---|---|
| $\kappa(\cdot) \triangleq M$ | Student grade | $\in \{A, B, C, D, E, F\}$ |
| $S$ | Final score | $\in \{0, \ldots, 10\}$ |

| Feat. id | Feat. var. | Feat. name | Domain |
|---|---|---|---|
| 1 | $Q$ | Quiz | $\{0, \ldots, 10\}$ |
| 2 | $X$ | Exam | $\{0, \ldots, 10\}$ |
| 3 | $H$ | Homework | $\{0, \ldots, 10\}$ |
| 4 | $R$ | Project | $\{0, \ldots, 10\}$ |

$$M = \text{ITE}(S \geqslant 9, A, \text{ITE}(S \geqslant 7, B, \text{ITE}(S \geqslant 5, C, \text{ITE}(S \geqslant 4, D, \text{ite}(S \geqslant 2, E, F)))))$$

$$S = \max\left[0.3 \times Q + 0.6 \times X + 0.1 \times H, R\right]$$

Also, $F \preccurlyeq E \preccurlyeq D \preccurlyeq C \preccurlyeq B \preccurlyeq A$

And, $\kappa(\mathbf{x_1}) \preccurlyeq \kappa(\mathbf{x}_2)$ if $\mathbf{x}_1 \leqslant \mathbf{x}_2$

# Explaining monotonic classifiers

- Instance $(\mathbf{v}, c)$
- Domain for $i \in \mathcal{F}$: $\lambda(i) \leqslant x_i \leqslant \mu(i)$
- Idea: refine lower and upper bounds on the prediction
  - $\mathbf{v}_{\mathsf{L}}$ and $\mathbf{v}_{\mathsf{U}}$
- Utilities:
  - FixAttr($i$):

    $\mathbf{v}_{\mathsf{L}} \leftarrow (v_{\mathsf{L}_1}, \ldots, v_i, \ldots, v_{\mathsf{L}_N})$
    $\mathbf{v}_{\mathsf{U}} \leftarrow (v_{\mathsf{U}_1}, \ldots, v_i, \ldots, v_{\mathsf{U}_N})$
    $(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})$
    **return** $(\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{A}, \mathcal{B})$
  - FreeAttr($i$):

    $\mathbf{v}_{\mathsf{L}} \leftarrow (v_{\mathsf{L}_1}, \ldots, \lambda(i), \ldots, v_{\mathsf{L}_N})$
    $\mathbf{v}_{\mathsf{U}} \leftarrow (v_{\mathsf{U}_1}, \ldots, \mu(i), \ldots, v_{\mathsf{U}_N})$
    $(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})$
    **return** $(\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{A}, \mathcal{B})$

# Computing one AXp

1: $\mathbf{v}_{\mathsf{L}} \leftarrow (v_1, \ldots, v_N)$
2: $\mathbf{v}_{\mathsf{U}} \leftarrow (v_1, \ldots, v_N)$ ▷ Ensures: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$
3: $(\mathcal{C}, \mathcal{D}, \mathcal{P}) \leftarrow (\mathcal{F}, \varnothing, \varnothing)$ ▷ $\mathcal{S}$: Some possible seed
4: **for all** $i \in \mathcal{S}$ **do**
5: $\quad (\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D}) \leftarrow \text{FreeAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D})$ ▷ Require: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$, given $\mathcal{S}$
6: **for all** $i \in \mathcal{F} \backslash \mathcal{S}$ **do** ▷ Loop inv.: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$
7: $\quad (\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D}) \leftarrow \text{FreeAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D})$
8: $\quad$ **if** $\kappa(\mathbf{v}_{\mathsf{L}}) \neq \kappa(\mathbf{v}_{\mathsf{U}})$ **then** ▷ If invariant broken, fix it
9: $\quad\quad (\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{D}, \mathcal{P}) \leftarrow \text{FixAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{D}, \mathcal{P})$
10: **return** $\mathcal{P}$

- **Obs:** $\mathcal{S} = \varnothing$ for computing a single AXp/CXp

# Computing one AXp – example

- $\lambda(i) = 0$ and $\mu(i) = 10$
- $\mathbf{v} = (10, 10, 5, 0)$, with $\kappa(\mathbf{v}) = A$
- Q: find one AXp (CXp is similar)

| Feat. | Initial values | | Changed values | | Predictions | | Dec. | Resulting values | |
|---|---|---|---|---|---|---|---|---|---|
| | $\mathbf{v}_L$ | $\mathbf{v}_U$ | $\mathbf{v}_L$ | $\mathbf{v}_U$ | $\kappa(\mathbf{v}_L)$ | $\kappa(\mathbf{v}_U)$ | | $\mathbf{v}_L$ | $\mathbf{v}_U$ |
| 1 | (10,10,5,0) | (10,10,5,0) | (0,10,5,0) | (10,10,5,0) | $C$ | $A$ | ✓ | (10,10,5,0) | (10,10,5,0) |
| 2 | (10,10,5,0) | (10,10,5,0) | (10,0,5,0) | (10,10,5,0) | $E$ | $A$ | ✓ | (10,10,5,0) | (10,10,5,0) |
| 3 | (10,10,5,0) | (10,10,5,0) | (10,10,0,0) | (10,10,10,0) | $A$ | $A$ | ✗ | (10,10,0,0) | (10,10,10,0) |
| 4 | (10,10,0,0) | (10,10,10,0) | (10,10,0,0) | (10,10,10,10) | $A$ | $A$ | ✗ | (10,10,0,0) | (10,10,10,10) |

# Outline – Unit #03

# Recap computation of (W)AXps/(W)CXps

$$\begin{aligned}
\mathsf{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{j \in \mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\mathsf{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{aligned}$$

# Recap computation of (W)AXps/(W)CXps

$$\text{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$$

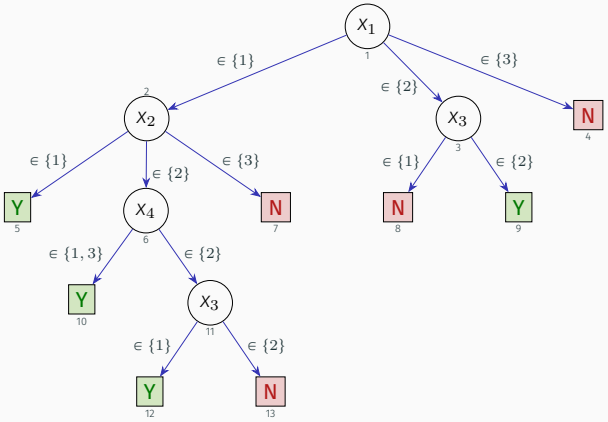
$$\text{WCXp}(\mathcal{Y}) \quad := \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))$$

**Input**: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
**Output**: One XP $\mathcal{S}$

1: **procedure** oneXP($\mathbb{P}$)
2: $\quad \mathcal{S} \leftarrow \mathcal{F}$ $\quad \triangleright$ Initialization: $\mathbb{P}(\mathcal{S})$ holds
3: $\quad$ **for** $i \in \mathcal{F}$ **do** $\quad \triangleright$ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
4: $\quad\quad$ **if** $\mathbb{P}(\mathcal{S} \backslash \{i\})$ **then**
5: $\quad\quad\quad \mathcal{S} \leftarrow \mathcal{S} \backslash \{i\}$ $\quad \triangleright$ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S} \backslash \{i\})$ holds
6: $\quad$ **return** $\mathcal{S}$ $\quad \triangleright$ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

# Review exercise – one AXp for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



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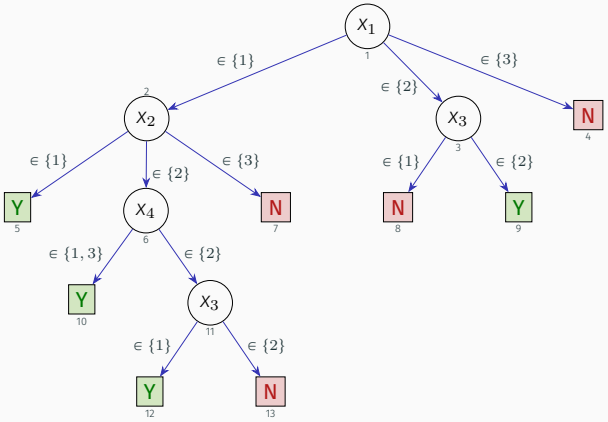
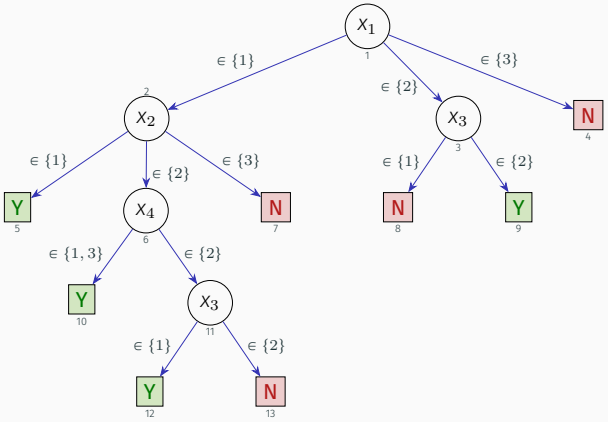



- Finding on AXp:

# Review exercise – one AXp for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$




- Finding on AXp:
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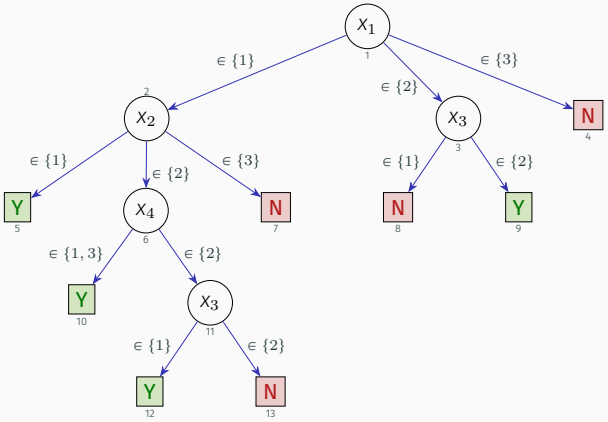


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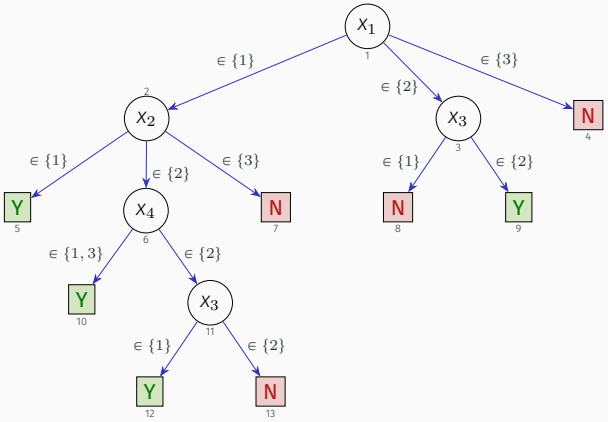


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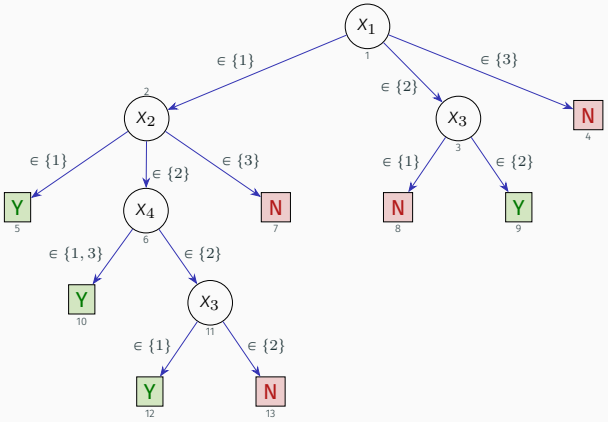


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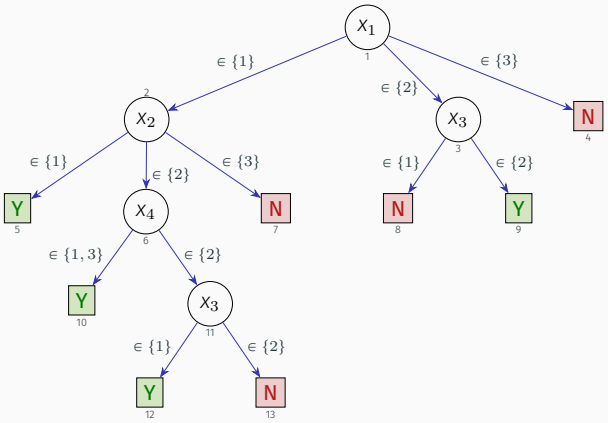


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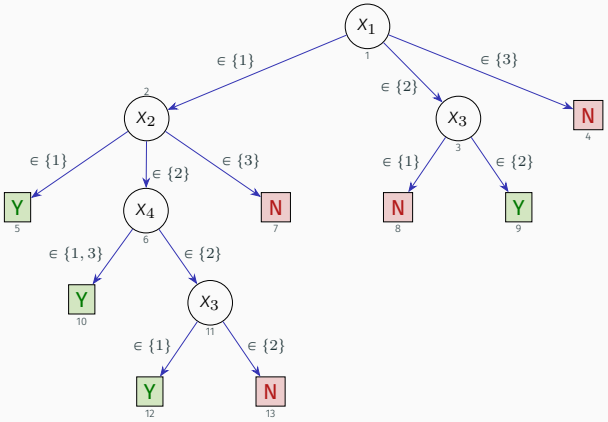
- AXp is MHS of $H_j$ sets: $\{1, 2, 3\}$

# Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



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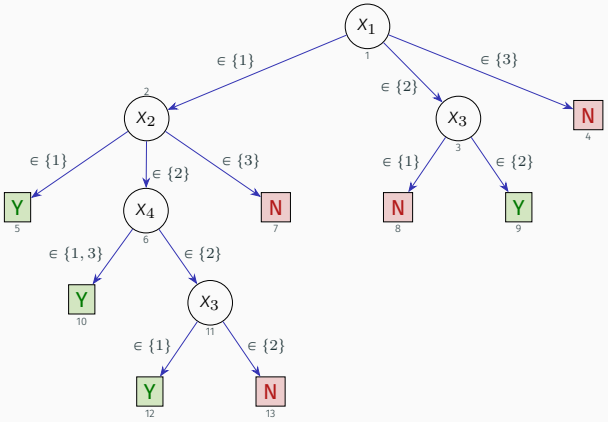


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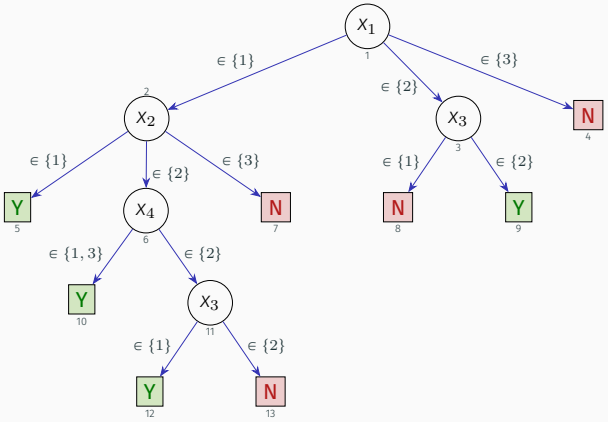


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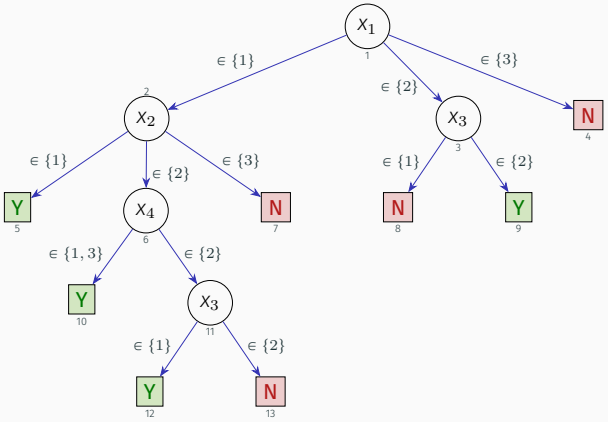


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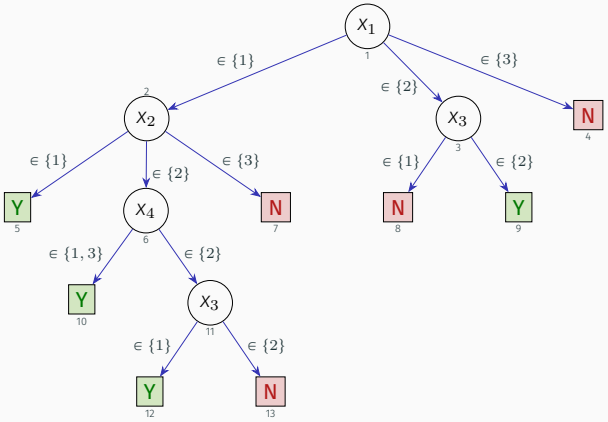


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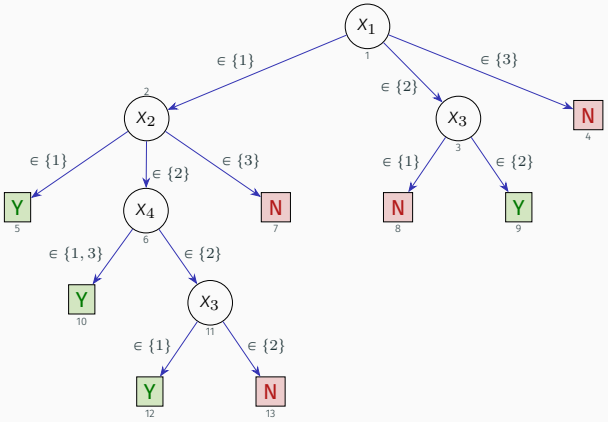


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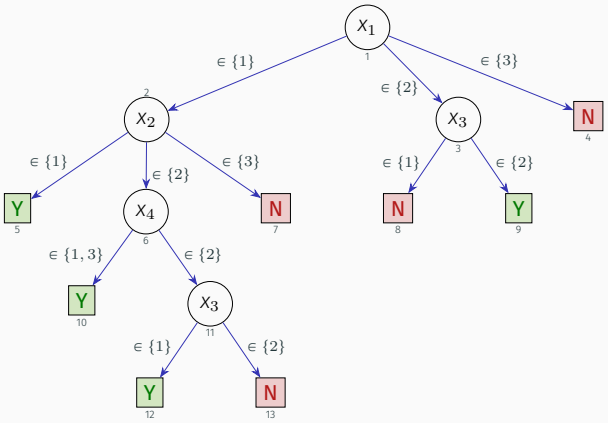


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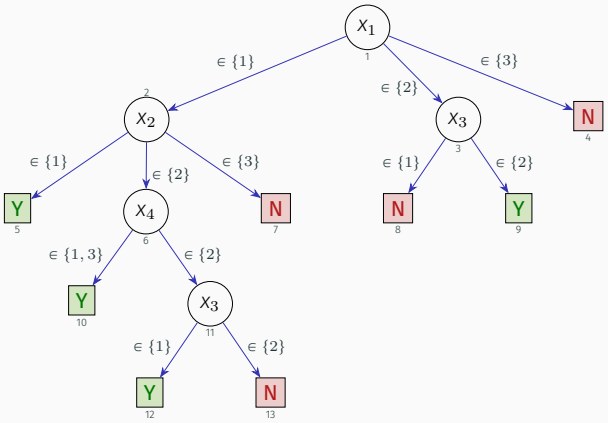


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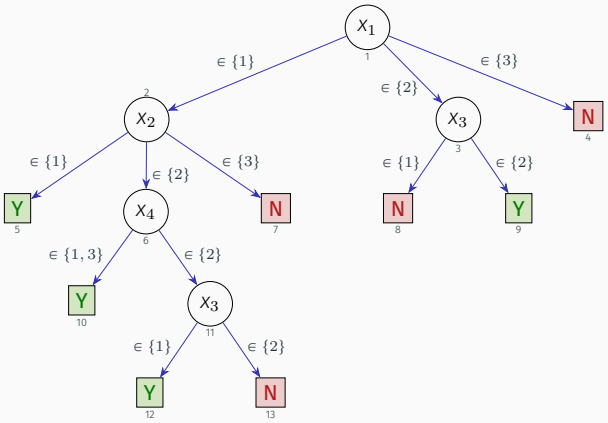


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- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$
  - $\mathbb{A} = \{\{1, 2, 3\}\}$

# Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{lllll}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
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\mathsf{R}_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
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- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to $\mathsf{R}_3$

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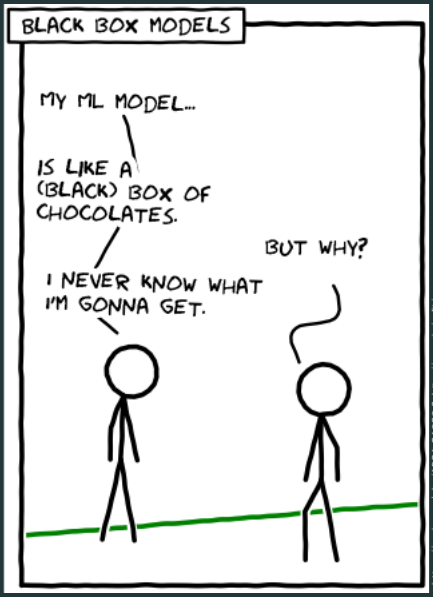
- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to $\mathsf{R}_3$
- AXp: $\{1, 2\}$
- Quiz: write down the constraints and confirm AXp with SAT solver

Questions?


BLACK BOX MODELS
MY ML MODEL...
IS LIKE A (BLACK) BOX OF CHOCOLATES.
I NEVER KNOW WHAT I'M GONNA GET.
BUT WHY?
https://arxiv.org/abs/1901.01686 & http://cmx.io/edit/

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