Logic-Based Explainable Artificial Intelligence

Joao Marques-Silva

ICREA & Univ. Lleida, Catalunya, Spain

ESSLLI, Bochum, Germany, July 2025

Lecture 02

Recapitulate first lecture

- ML models: classification & regression

Recapitulate first lecture

- ML models: classification & regression

- Glimpse of heuristic XAI

Recapitulate first lecture

- ML models: classification & regression
- Glimpse of heuristic XAI
- Answers to Why? questions as logic rules

Recapitulate first lecture

- ML models: classification & regression
- Glimpse of heuristic XAI
- Answers to Why? questions as logic rules
- Logic-based reasoning of ML models

Recapitulate first lecture

- ML models: classification & regression
- Glimpse of heuristic XAI
- Answers to Why? questions as logic rules
- Logic-based reasoning of ML models
- Apparent difficulties with explaining interpretable models

Plan for this course

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

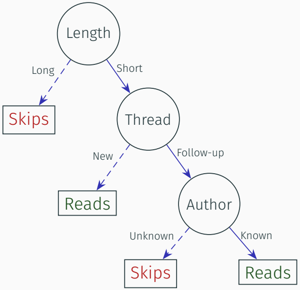
Unit #02

Principles of Symbolic XAI – Feature Selection

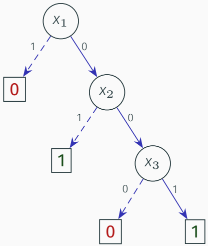
What is an explanation?

- Notation:

Original DT [PM17]



Rewritten DT



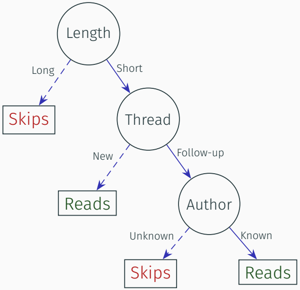
Mapping
$x_1 = 1$ iff Length = Long
$x_2 = 1$ iff Thread = New
$x_3 = 1$ iff Author = Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

- What is an explanation?

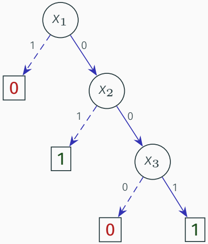
What is an explanation?

- Notation:

Original DT [PM17]



Rewritten DT



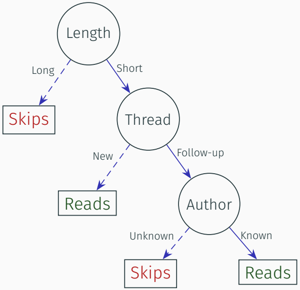
Mapping
$x_1 = 1$ iff Length $=$ Long
$x_2 = 1$ iff Thread $=$ New
$x_3 = 1$ iff Author $=$ Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

- What is an explanation?
  - Answer to question "**Why** (the prediction)?" is a **rule**: IF <COND> THEN $\kappa(\mathbf{x}) = c$

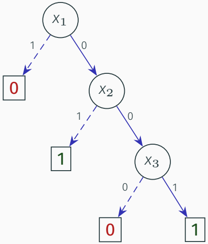
What is an explanation?

- Notation:

Original DT [PM17]



Rewritten DT

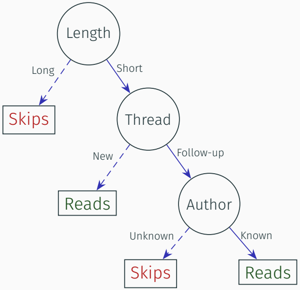


Mapping
$x_1 = 1$ iff Length $=$ Long
$x_2 = 1$ iff Thread $=$ New
$x_3 = 1$ iff Author $=$ Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

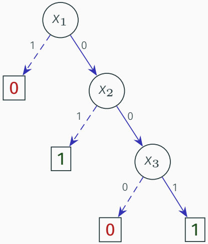
- What is an explanation?
  - Answer to question "**Why** (the prediction)?" is a **rule**: IF <COND> THEN $\kappa(\mathbf{x}) = c$
  - **Explanation**: set of literals (or just features) in **<COND>**; **irreducibility matters!**

- Notation:

Original DT [PM17]



Rewritten DT



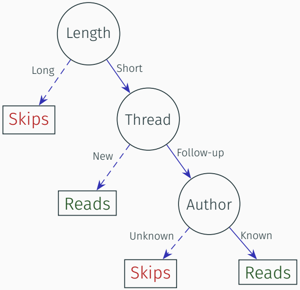
Mapping
$x_1 = 1$ iff Length $=$ Long
$x_2 = 1$ iff Thread $=$ New
$x_3 = 1$ iff Author $=$ Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

- What is an explanation?
  - Answer to question "**Why** (the prediction)?" is a **rule**: IF <COND> THEN $\kappa(\mathbf{x}) = c$
  - **Explanation**: set of literals (or just features) in <COND>; **irreducibility matters!**
  - **E.g.**: explanation for $\mathbf{v} = (\neg x_1, \neg x_2, x_3)$?

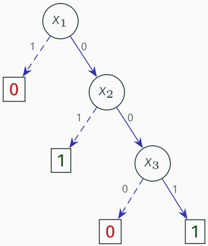
What is an explanation?

- Notation:

Original DT [PM17]



Rewritten DT

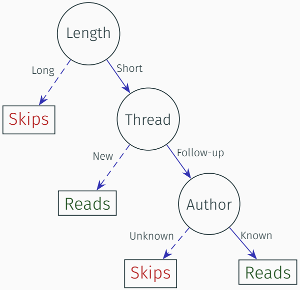


Mapping
$x_1 = 1$ iff Length $=$ Long
$x_2 = 1$ iff Thread $=$ New
$x_3 = 1$ iff Author $=$ Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

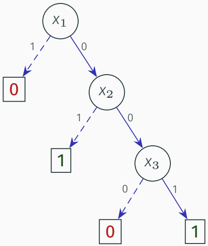
- What is an explanation?
  - Answer to question "**Why** (the prediction)?" is a **rule**: IF <COND> THEN $\kappa(\mathbf{x}) = c$
  - **Explanation**: set of literals (or just features) in <COND>; **irreducibility matters!**
  - **E.g.**: explanation for $\mathbf{v} = (\neg x_1, \neg x_2, x_3)$?
    - It is the case that, IF $\neg x_1 \land \neg x_2 \land x_3$ THEN $\kappa(\mathbf{x}) = 1$

- Notation:

Original DT [PM17]



Rewritten DT



Mapping
$x_1 = 1$ iff Length $=$ Long
$x_2 = 1$ iff Thread $=$ New
$x_3 = 1$ iff Author $=$ Known
$\kappa(\cdot) = 1$ iff $\kappa'(\cdots) =$ Reads
$\kappa(\cdot) = 0$ iff $\kappa'(\cdots) =$ Skips

- What is an explanation?
  - Answer to question "**Why** (the prediction)?" is a **rule**: IF <COND> THEN $\kappa(\mathbf{x}) = c$
  - **Explanation**: set of literals (or just features) in <COND>; **irreducibility matters!**
  - **E.g.**: explanation for $\mathbf{v} = (\neg x_1, \neg x_2, x_3)$?
    - It is the case that, IF $\neg x_1 \land \neg x_2 \land x_3$ THEN $\kappa(\mathbf{x}) = 1$
    - One possible explanation is $\{\neg x_1, \neg x_2, x_3\}$ or simply $\{1, 2, 3\}$

The similarity predicate

[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
  - Regression: $\mathcal{M}_R = (\mathcal{F}, \mathbb{F}, \mathbb{V}, \rho)$
  - General: $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathbb{T}, \tau)$

The similarity predicate

[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
  - Regression: $\mathcal{M}_R = (\mathcal{F}, \mathbb{F}, \mathbb{V}, \rho)$
  - General: $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathbb{T}, \tau)$

- Similarity predicate: $\sigma : \mathbb{F} \to \{\top, \bot\}$
  - Classification: $\sigma(\mathbf{x}) := [\kappa(\mathbf{x}) = \kappa(\mathbf{v})]$
    - **Obs:** For boolean classifiers, no need for σ
  - Regression: $\sigma(\mathbf{x}) := [|\rho(\mathbf{x}) - \rho(\mathbf{v})| \leqslant \delta]$, where δ is user-specified

The similarity predicate

[Mar24]

- Recall ML models for classification & regression:
  - Classification: $\mathcal{M}_C = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$
  - Regression: $\mathcal{M}_R = (\mathcal{F}, \mathbb{F}, \mathbb{V}, \rho)$
  - General: $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathbb{T}, \tau)$

- Similarity predicate: $\sigma : \mathbb{F} \to \{\top, \bot\}$
  - Classification: $\sigma(\mathbf{x}) := [\kappa(\mathbf{x}) = \kappa(\mathbf{v})]$
    - **Obs:** For boolean classifiers, no need for σ
  - Regression: $\sigma(\mathbf{x}) := [|\rho(\mathbf{x}) - \rho(\mathbf{v})| \leqslant \delta]$, where δ is user-specified

- Bottom line:
  Reason about symbolic explainability by abstracting away type of ML model

Abductive explanations – answering Why? questions

- Instance $(\mathbf{v}, q)$, i.e. $c = \tau(\mathbf{v})$

Abductive explanations – answering Why? questions

- Instance $(\mathbf{v}, q)$, i.e. $c = \tau(\mathbf{v})$
- Abductive explanation (AXp, PI-explanation): [SCD18, INM19a]
  - Subset-minimal set of features $\mathcal{X} \subseteq \mathcal{F}$ sufficient for ensuring prediction

$$\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$$

Abductive explanations – answering Why? questions

- Instance $(\mathbf{v}, q)$, i.e. $c = \tau(\mathbf{v})$
- Abductive explanation (AXp, PI-explanation): [SCD18, INM19a]
  - Subset-minimal set of features $\mathcal{X} \subseteq \mathcal{F}$ sufficient for ensuring prediction

$$\mathsf{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$$

- Defining AXp (from weak AXps, WAXps):

$$\mathsf{AXp}(\mathcal{X}) := \mathsf{WAXp}(\mathcal{X}) \land \forall(\mathcal{X}' \subsetneq \mathcal{X}).\neg\mathsf{WAXp}(\mathcal{X}')$$

Abductive explanations – answering Why? questions

- Instance $(\mathbf{v}, q)$, i.e. $c = \tau(\mathbf{v})$
- Abductive explanation (AXp, PI-explanation): [SCD18, INM19a]
    - Subset-minimal set of features $\mathcal{X} \subseteq \mathcal{F}$ sufficient for ensuring prediction

$$\mathsf{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$$

- Defining AXp (from weak AXps, WAXps):

$$\mathsf{AXp}(\mathcal{X}) := \mathsf{WAXp}(\mathcal{X}) \wedge \forall(\mathcal{X}' \subsetneq \mathcal{X}).\neg\mathsf{WAXp}(\mathcal{X}')$$

- But, WAXp is monotone; hence,

$$\mathsf{AXp}(\mathcal{X}) := \mathsf{WAXp}(\mathcal{X}) \wedge \forall(t \in \mathcal{X}).\neg\mathsf{WAXp}(\mathcal{X} \setminus \{t\})$$

Abductive explanations – answering Why? questions

- Instance $(\mathbf{v}, q)$, i.e. $c = \tau(\mathbf{v})$
- **Abductive explanation** (AXp, PI-explanation): [SCD18, INM19a]
  - **Subset-minimal** set of features $\mathcal{X} \subseteq \mathcal{F}$ sufficient for **ensuring** prediction

$$\mathrm{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j\in\mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$$

- Defining AXp (from weak AXps, WAXps):

$$\mathrm{AXp}(\mathcal{X}) := \mathrm{WAXp}(\mathcal{X}) \land \forall(\mathcal{X}' \subsetneq \mathcal{X}).\neg\mathrm{WAXp}(\mathcal{X}')$$

- But, WAXp is **monotone**; hence,

$$\mathrm{AXp}(\mathcal{X}) := \mathrm{WAXp}(\mathcal{X}) \land \forall(t \in \mathcal{X}).\neg\mathrm{WAXp}(\mathcal{X}\backslash\{t\})$$

- Finding one AXp (example algorithm; many more exist): [MM20]
  - Let $\mathcal{X} = \mathcal{F}$, i.e. fix all features
  - Invariant: $\mathrm{WAXp}(\mathcal{X})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathrm{WAXp}(\mathcal{X}\backslash\{i\})$ holds, then remove i from $\mathcal{X}$, i.e. i becomes free

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0,0,0,1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1,2,3,4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$?

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$?

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$?

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1,x_2,x_3,x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0,0,0,1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1,2,3,4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1,x_2,x_3,x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1,x_2,x_3,x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).x_4 \rightarrow \kappa(x_1,x_2,x_3,x_4)$? Yes

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\top \rightarrow \kappa(x_1, x_2, x_3, x_4)$?

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\top \rightarrow \kappa(x_1, x_2, x_3, x_4)$? No

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\top \rightarrow \kappa(x_1, x_2, x_3, x_4)$? No
- AXp $\mathcal{X} = \{4\}$

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\forall(\mathbf{x} \in \{0,1\}^4).\top \rightarrow \kappa(x_1, x_2, x_3, x_4)$? No
- AXp $\mathcal{X} = \{4\}$

- In general, validity/consistency checked with SAT/SMT/MILP/CP reasoners

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

A simple example – AXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$. AXp?
- Define $\mathcal{X} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_2 \land \neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\neg x_3 \land x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).x_4 \rightarrow \kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\forall(\mathbf{x} \in \{0, 1\}^4).\top \rightarrow \kappa(x_1, x_2, x_3, x_4)$? No
- AXp $\mathcal{X} = \{4\}$

- In general, **validity/consistency checked with SAT/SMT/MILP/CP reasoners**
  - **Obs:** for some classes of classifiers, poly-time algorithms exist

Recap weak AXp: $\forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x}))$

More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i \in \mathcal{S}} (x_i = v_i)$$

More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i \in \mathcal{S}} (x_i = v_i)$$

- Definition of $\Upsilon(\mathcal{S})$:

$$\Upsilon(\mathcal{S}) \quad := \quad \{\mathbf{x} \in \mathbb{F} \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i \in \mathcal{S}} (x_i = v_i)$$

- Definition of $\Upsilon(\mathcal{S})$:

$$\Upsilon(\mathcal{S}) \quad := \quad \{\mathbf{x} \in \mathbb{F} \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

- Expected value, non-real-valued features:

$$\mathbf{E}[\tau(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad := \quad {}^{1}/_{|\Upsilon(\mathcal{S};\mathbf{v})|} \sum\nolimits_{\mathbf{x} \in \Upsilon(\mathcal{S};\mathbf{v})} \tau(\mathbf{x})$$

More notation

- Notation $\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}$:

$$[\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad \equiv \quad \bigwedge_{i\in\mathcal{S}} (x_i = v_i)$$

- Definition of $\Upsilon(\mathcal{S})$:

$$\Upsilon(\mathcal{S}) \quad := \quad \{\mathbf{x} \in \mathbb{F} \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

- Expected value, non-real-valued features:

$$\mathbf{E}[\tau(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad := \quad {}^1/_{|\Upsilon(\mathcal{S};\mathbf{v})|} \sum\nolimits_{\mathbf{x}\in\Upsilon(\mathcal{S};\mathbf{v})} \tau(\mathbf{x})$$

- Expected value, real-valued features:

$$\mathbf{E}[\tau(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] \quad := \quad {}^1/_{|\Upsilon(\mathcal{S};\mathbf{v})|} \int_{\Upsilon(\mathcal{S};\mathbf{v})} \tau(\mathbf{x}) d\mathbf{x}$$

Other definitions of WAXps/AXps

- Using probabilities, non-real-valued features: [WMHK21, IHI+22, ABOS22, IHI+23]

$$\mathsf{WAXp}(\mathcal{S}) \quad := \quad \mathsf{Pr}(\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) = 1$$

Other definitions of WAXps/AXps

- Using probabilities, non-real-valued features: [WMHK21, IHI+22, ABOS22, IHI+23]

$$\text{WAXp}(\mathcal{S}) \quad := \quad \text{Pr}(\sigma(\mathbf{x})\,|\,\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) = 1$$

- Using expected values:

$$\text{WAXp}(\mathcal{S}) \quad := \quad \mathbf{E}[\sigma(\mathbf{x})\,|\,\mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] = 1$$

Other definitions of WAXps/AXps

- Using probabilities, non-real-valued features: [WMHK21, IHI+22, ABOS22, IHI+23]

$$\mathsf{WAXp}(\mathcal{S}) \quad := \quad \mathsf{Pr}(\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) = 1$$

- Using expected values:

$$\mathsf{WAXp}(\mathcal{S}) \quad := \quad \mathbf{E}[\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] = 1$$

- Definition of AXp remains unchanged
  - This is true when comparing against 1

Constrastive explanations – answering Why not? questions

- Instance $(\mathbf{v}, c)$, i.e. $c = \kappa(\mathbf{v})$

Constrastive explanations – answering Why not? questions

- Instance $(\mathbf{v}, c)$, i.e. $c = \kappa(\mathbf{v})$
- Contrastive explanation (CXp): [Mil19, INAM20]
  - Subset-minimal set of features $\mathcal{Y} \subseteq \mathcal{F}$ sufficient for **changing** prediction

$$\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}} (x_j = v_j) \wedge (\neg\sigma(\mathbf{x}))$$

Constrastive explanations – answering Why not? questions

- Instance $(\mathbf{v}, c)$, i.e. $c = \kappa(\mathbf{v})$
- Contrastive explanation (CXp): [Mil19, INAM20]
  - Subset-minimal set of features $\mathcal{Y} \subseteq \mathcal{F}$ sufficient for **changing** prediction

$$\mathsf{WCXp}(\mathcal{Y}) \quad := \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))$$

- Defining CXp:

$$\mathsf{CXp}(\mathcal{Y}) := \mathsf{WCXp}(\mathcal{Y}) \land \forall(\mathcal{Y}' \subsetneq \mathcal{Y}).\neg\mathsf{WCXp}(\mathcal{Y}')$$

Constrastive explanations – answering Why not? questions

- Instance $(\mathbf{v}, c)$, i.e. $c = \kappa(\mathbf{v})$
- **Contrastive explanation** (CXp): [Mil19, INAM20]
  - **Subset-minimal** set of features $\mathcal{Y} \subseteq \mathcal{F}$ sufficient for **changing** prediction

$$\mathsf{WCXp}(\mathcal{Y}) \quad := \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))$$

- Defining CXp:

$$\mathsf{CXp}(\mathcal{Y}) := \mathsf{WCXp}(\mathcal{Y}) \land \forall(\mathcal{Y}' \subsetneq \mathcal{Y}).\neg\mathsf{WCXp}(\mathcal{Y}')$$

- But, WCXp is also **monotone**; hence,

$$\mathsf{CXp}(\mathcal{Y}) := \mathsf{WCXp}(\mathcal{Y}) \land \forall(t \in \mathcal{Y}).\neg\mathsf{WCXp}(\mathcal{Y} \backslash \{t\})$$

Constrastive explanations – answering Why not? questions

- Instance $(\mathbf{v}, c)$, i.e. $c = \kappa(\mathbf{v})$
- Contrastive explanation (CXp): [Mil19, INAM20]
  - **Subset-minimal** set of features $\mathcal{Y} \subseteq \mathcal{F}$ sufficient for **changing** prediction

$$\mathsf{WCXp}(\mathcal{Y}) \quad := \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \wedge (\neg\sigma(\mathbf{x}))$$

- Defining CXp:

$$\mathsf{CXp}(\mathcal{Y}) := \mathsf{WCXp}(\mathcal{Y}) \wedge \forall(\mathcal{Y}' \subsetneq \mathcal{Y}).\neg\mathsf{WCXp}(\mathcal{Y}')$$

- But, WCXp is also **monotone**; hence,

$$\mathsf{CXp}(\mathcal{Y}) := \mathsf{WCXp}(\mathcal{Y}) \wedge \forall(t \in \mathcal{Y}).\neg\mathsf{WCXp}(\mathcal{Y} \setminus \{t\})$$

- Finding one CXp: [MM20]
  - Let $\mathcal{Y} = \mathcal{F}$, i.e. free all features
  - Invariant: $\mathsf{WCXp}(\mathcal{Y})$ must hold. **Why?**
  - Analyze features in any order, one feature i at a time
    - If $\mathsf{WCXp}(\mathcal{Y} \setminus \{i\})$ holds, then remove i from $\mathcal{Y}$, i.e. i is becomes fixed

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$?

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$?

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}).\bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \wedge \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \wedge \neg x_2 \wedge \neg\kappa(x_1, x_2, x_3, x_4)$? Yes

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \wedge (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land \neg\kappa(x_1, x_2, x_3, x_4)$?

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0,1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0,1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\exists(\mathbf{x} \in \{0,1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\exists(\mathbf{x} \in \{0,1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land x_4 \land \neg\kappa(x_1, x_2, x_3, x_4)$?

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land x_4 \land \neg\kappa(x_1, x_2, x_3, x_4)$? No

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

A simple example – CXp's

- Classifier:

$$\kappa(x_1, x_2, x_3, x_4) = \bigvee_{i=1}^{4} x_i$$

- Point $\mathbf{v} = (0, 0, 0, 1)$ with prediction $\kappa(\mathbf{v}) = 1$
- Define $\mathcal{Y} = \{1, 2, 3, 4\} = \mathcal{F}$
- Can feature 1 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 2 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 3 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land \neg\kappa(x_1, x_2, x_3, x_4)$? Yes
- Can feature 4 be removed, i.e. $\exists(\mathbf{x} \in \{0, 1\}^4).\neg x_1 \land \neg x_2 \land \neg x_3 \land x_4 \land \neg\kappa(x_1, x_2, x_3, x_4)$? No
- CXp $\mathcal{Y} = \{4\}$
- **Obs:** AXp is MHS of CXp and vice-versa...

Recap weak CXp: $\exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))$

Other definitions of WCXps/CXps

- Using probabilities, non-real-valued features:

$$\mathsf{WCXp}(\mathcal{S}) \quad := \quad \mathsf{Pr}(\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) < 1$$

Other definitions of WCXps/CXps

- Using probabilities, non-real-valued features:

$$\mathsf{WCXp}(\mathcal{S}) \quad := \quad \mathsf{Pr}(\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) < 1$$

- Using expected values:

$$\mathsf{WCXp}(\mathcal{S}) \quad := \quad \mathbf{E}[\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] < 1$$

Other definitions of WCXps/CXps

- Using probabilities, non-real-valued features:

$$\mathsf{WCXp}(\mathcal{S}) \quad := \quad \mathsf{Pr}(\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}) < 1$$

- Using expected values:

$$\mathsf{WCXp}(\mathcal{S}) \quad := \quad \mathbf{E}[\sigma(\mathbf{x}) \,|\, \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] < 1$$

- Definition of CXp remains unchanged

Detour: global explanations

[INM19b]

- AXps and CXps are defined locally (because of $\mathbf{v}$) but hold globally
  - Localized explanations
  - Can be viewed as attempt at formalizing local explanations [RSG16, LL17, RSG18]
- One can define explanations without picking a given point in feature space
  - Let $q \in \mathbb{T}$, and refefine the similarity predicate:
    - Classification: $\sigma(\mathbf{x}) = [\kappa(\mathbf{x}) = q]$
    - Regression: $\sigma(\mathbf{x}) = [|\kappa(\mathbf{x}) - q| \leqslant \delta]$, δ is user-specified
  - Let $\mathbb{L} = \{(x_i = v_i) \,|\, i \in \mathcal{F} \wedge v_i \in \mathbb{V}\}$
  - Let $\mathcal{S} \subsetneq \mathbb{L}$ be a subset of literals that does not repeat features, i.e. $\mathcal{S}$ is not inconsistent
  - Then, $\mathcal{S}$ is a global AXp if,

$$\forall (\mathbf{x} \in \mathbb{F}). \bigwedge_{(x_i = v_i) \in \mathcal{S}} (x_i = v_i) \rightarrow (\sigma(\mathbf{x}))$$

  - Counterexamples are minimal hitting sets of global AXps and vice-versa [INM19b]

Outline – Unit #02

Definitions of Explanations

Duality Properties

Computational Problems

Duality in explainability – basic results

[INAM20, Mar22]

Duality in explainability – basic results

[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps

Duality in explainability – basic results

[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps

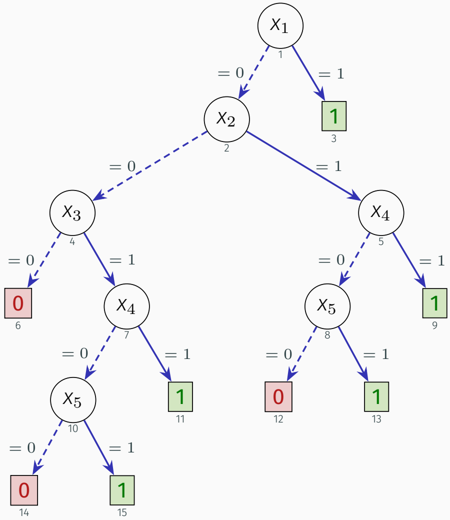
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps

[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps

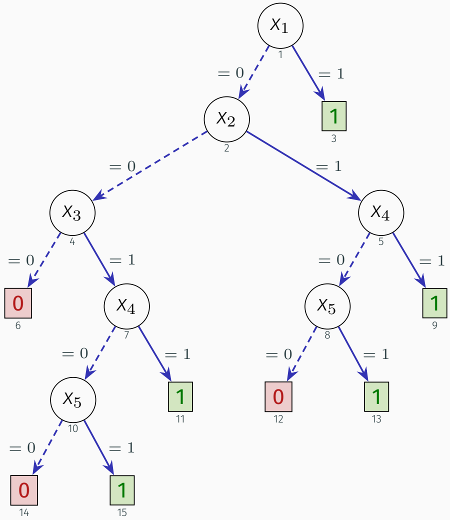
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:



Duality in explainability – basic results

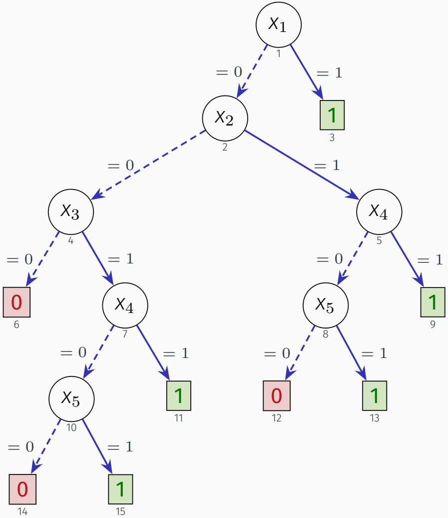
[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:
  - AXps:



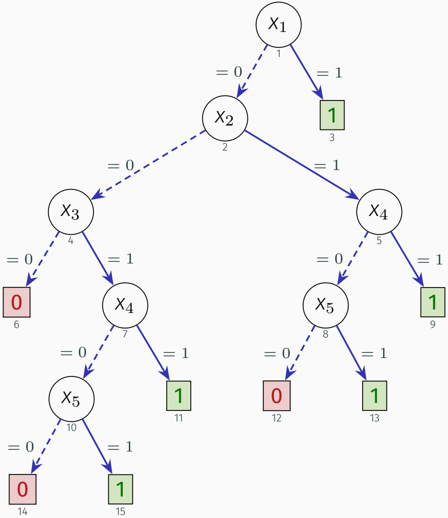
[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:
  - AXps: $\{\{3,5\}\}$



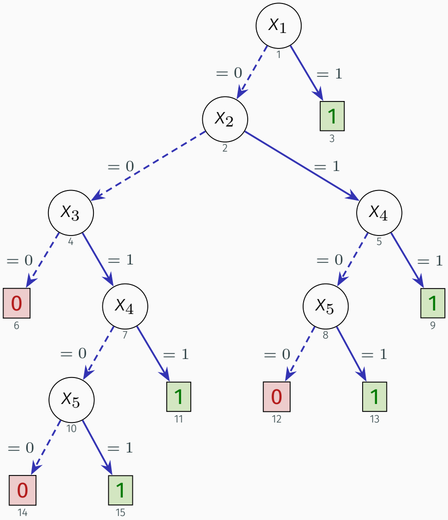
[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0, 0, 1, 0, 1), 1)$:
  - AXps: $\{\{3, 5\}\}$
  - CXps:



[INAM20, Mar22]

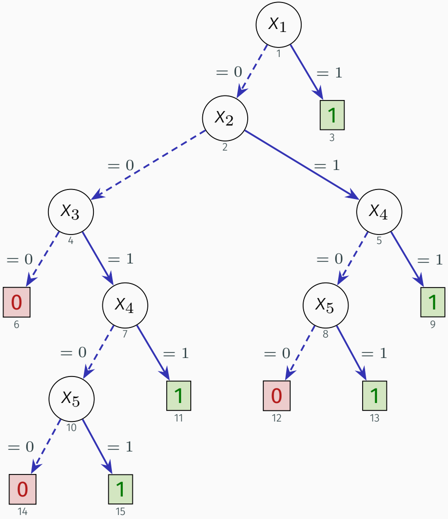
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0, 0, 1, 0, 1), 1)$:
  - AXps: $\{\{3, 5\}\}$
  - CXps: $\{\{3\}, \{5\}\}$



Duality in explainability – basic results

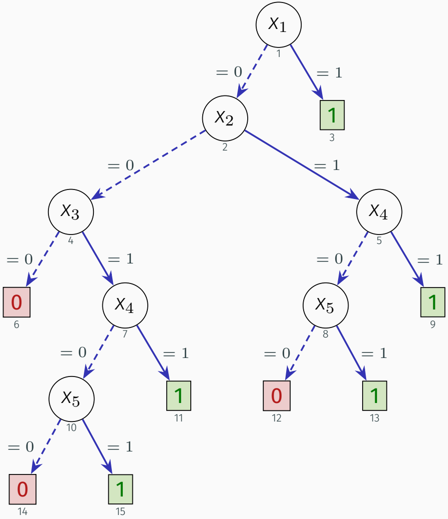
[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:
  - AXps: $\{\{3,5\}\}$
  - CXps: $\{\{3\},\{5\}\}$
  - Each AXp is an MHS of the set of CXps
  - Each CXp is an MHS of the set of AXps



[INAM20, Mar22]

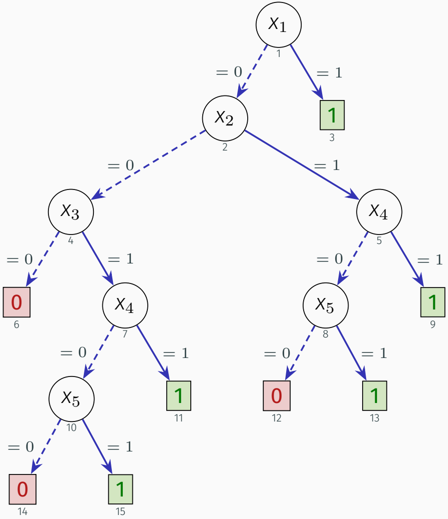
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1), 1)$:
  - AXps: $\{\{3,5\}\}$
  - CXps: $\{\{3\},\{5\}\}$
  - Each AXp is an MHS of the set of CXps
  - Each CXp is an MHS of the set of AXps
  - BTW,
    - $\{2,5\}$ is not a CXp
    - $\{1,2,3,4,5\}$, $\{1,2,3,5\}$ and $\{1,3,5\}$ are not AXps



Duality in explainability – basic results

[INAM20, Mar22]

- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is an AXp iff it is a minimal hitting set (MHS) of the set of CXps
- **Claim:**
  $\mathcal{S} \subseteq \mathcal{F}$ is a CXp iff it is a minimal hitting set (MHS) of the set of AXps
- An example, $(\mathbf{v}, c) = ((0,0,1,0,1),1)$:
  - AXps: $\{\{3,5\}\}$
  - CXps: $\{\{3\},\{5\}\}$
  - Each AXp is an MHS of the set of CXps
  - Each CXp is an MHS of the set of AXps
  - BTW,
    - $\{2,5\}$ is not a CXp
    - $\{1,2,3,4,5\}$, $\{1,2,3,5\}$ and $\{1,3,5\}$ are not AXps
    - Why?



Outline – Unit #02

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation

- Enumerate all abductive/contrastive explanations

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation
- Enumerate all abductive/contrastive explanations
- Decide whether feature included in all abductive/contrastive explanations

Computational problems in (formal) explainability

- Compute one abductive/contrastive explanation
- Enumerate all abductive/contrastive explanations
- Decide whether feature included in all abductive/contrastive explanations
- Decide whether feature included in some abductive/contrastive explanation

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For AXp: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For AXp: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds
- For CXp: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. fix) features from $\mathcal{S}$ while WCXp condition holds

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For **AXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds
- For **CXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. fix) features from $\mathcal{S}$ while WCXp condition holds
- Monotone predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg\,\mathsf{CO}\left(\left[\!\left[\left(\bigwedge_{i\in\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \mathsf{CO}\left(\left[\!\left[\left(\bigwedge_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right)$$

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For **AXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds
- For **CXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. fix) features from $\mathcal{S}$ while WCXp condition holds
- **Monotone** predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg\,\mathsf{CO}\left([\![(\textstyle\bigwedge_{i\in\mathcal{S}}(x_i = v_i)) \land (\neg\sigma(\mathbf{x}))]\!]\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \mathsf{CO}\left(\left[\!\!\left[\left(\textstyle\bigwedge_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\!\right]\right)$$

Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

procedure oneXP($\mathbb{P}$)
  $\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
  for $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
    if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ **then**
      $\mathcal{S} \leftarrow \mathcal{S}\setminus\{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ holds
  return $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

Computing one AXp/CXp

- Encode classifier into suitable logic representation $\mathcal{T}$ & pick suitable reasoner
- For **AXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. free) features from $\mathcal{S}$ while WAXp condition holds
- For **CXp**: start from $\mathcal{S} = \mathcal{F}$ and drop (i.e. fix) features from $\mathcal{S}$ while WCXp condition holds
- **Monotone** predicates for WAXp & WCXp:

$$\mathbb{P}_{\text{axp}}(\mathcal{S}) \triangleq \neg\,\mathsf{CO}\left(\left[\!\left[\left(\bigwedge_{i\in\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right) \qquad \mathbb{P}_{\text{cxp}}(\mathcal{S}) \triangleq \mathsf{CO}\left(\left[\!\left[\left(\bigwedge_{i\in\mathcal{F}\setminus\mathcal{S}}(x_i = v_i)\right) \land (\neg\sigma(\mathbf{x}))\right]\!\right]\right)$$

Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

procedure oneXP($\mathbb{P}$)
  $\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
  for $i \in \mathcal{F}$ **do** ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
    if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ **then**
      $\mathcal{S} \leftarrow \mathcal{S}\setminus\{i\}$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S}\setminus\{i\})$ holds
  return $\mathcal{S}$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

Exploiting MSMP, i.e. basic algorithm used for different problems.

Detour: More Connections with Automated Reasoning

Prime implicants & implicates

- A conjunction of literals π (which will be viewed as a set of literals where convenient) is a prime implicant of some function φ if,
  1. $\pi \vDash \varphi$
  2. For any $\pi' \subsetneq \pi$, $\pi' \nvDash \varphi$

Prime implicants & implicates

- A conjunction of literals π (which will be viewed as a set of literals where convenient) is a prime implicant of some function φ if,
  1. $\pi \vDash \varphi$
  2. For any $\pi' \subsetneq \pi$, $\pi' \nvDash \varphi$

  - Example:
    - $\mathbb{F} = \{0,1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \land x_2 \lor x_1 \land x_3$
    - Clearly, $x_1 \land x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$

Prime implicants & implicates

- A conjunction of literals π (which will be viewed as a set of literals where convenient) is a prime implicant of some function φ if,
  1. $\pi \vDash \varphi$
  2. For any $\pi' \subsetneq \pi$, $\pi' \nvDash \varphi$
  - Example:
    - $\mathbb{F} = \{0, 1\}^3$
    - $\varphi(x_1, x_2, x_3) = x_1 \land x_2 \lor x_1 \land x_3$
    - Clearly, $x_1 \land x_2 \vDash \varphi$
    - Also, $x_1 \nvDash \varphi$ and $x_2 \nvDash \varphi$
- A disjunction of literals η (also viewed as a set of literals where convenient) is a prime implicate of some function φ if
  1. $\varphi \vDash \eta$
  2. For any $\eta' \subsetneq \eta$, $\varphi \nvDash \eta'$

Reasoning about inconsistency

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \lor x_2), (x_1 \lor \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$

Reasoning about inconsistency

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \lor x_2), (x_1 \lor \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$
- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \vDash \bot$
  - E.g. $\mathcal{U} = \{(\neg x_1), (\neg x_2)\}$

Reasoning about inconsistency

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \perp$
  - E.g. $\mathcal{B} = \{(x_1 \vee x_2), (x_1 \vee \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$
- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \vDash \perp$
  - E.g. $\mathcal{U} = \{(\neg x_1), (\neg x_2)\}$
- Minimal correction subset (MCS):
  - Subset-minimal set $\mathcal{C} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup (\mathcal{S} \setminus \mathcal{C}) \nvDash \perp$
  - E.g. $\mathcal{C} = \{(\neg x_1)\}$

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \lor x_2), (x_1 \lor \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$
- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \vDash \bot$
  - E.g. $\mathcal{U} = \{(\neg x_1), (\neg x_2)\}$
- Minimal correction subset (MCS):
  - Subset-minimal set $\mathcal{C} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup (\mathcal{S} \backslash \mathcal{C}) \nvDash \bot$
  - E.g. $\mathcal{C} = \{(\neg x_1)\}$
- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]

Reasoning about inconsistency

[MM20]

- Formula $\mathcal{T} = \mathcal{B} \cup \mathcal{S}$, with
  - $\mathcal{B}$: background knowledge (base), i.e. hard constraints
  - $\mathcal{S}$: additional (inconsistent) knowledge, i.e. soft constraints
  - And, $\mathcal{T} \vDash \bot$
  - E.g. $\mathcal{B} = \{(x_1 \lor x_2), (x_1 \lor \neg x_3)\}$, $\mathcal{S} = \{(\neg x_1), (\neg x_2), (x_3)\}$
- Minimal unsatisfiable subset (MUS):
  - Subset-minimal set $\mathcal{U} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup \mathcal{U} \vDash \bot$
  - E.g. $\mathcal{U} = \{(\neg x_1), (\neg x_2)\}$
- Minimal correction subset (MCS):
  - Subset-minimal set $\mathcal{C} \subseteq \mathcal{S}$, s.t. $\mathcal{B} \cup (\mathcal{S} \setminus \mathcal{C}) \nvDash \bot$
  - E.g. $\mathcal{C} = \{(\neg x_1)\}$
- Duality:
  - MUSes are minimal-hitting sets (MHSes) of the MCSes, and vice-versa [Rei87]
- Variants:
  - Smallest(-cost) MCS, i.e. complement of maximum(-cost) satisfiability (MaxSAT)
  - Smallest(-cost) MUS

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$
\begin{array}{rcl}
\mathrm{WAXp}(\mathcal{X}) & := & \forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\mathrm{WCXp}(\mathcal{Y}) & := & \exists(\mathbf{x} \in \mathbb{F}). \bigwedge_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{array}
$$

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}
\mathsf{WAXp}(\mathcal{X}) \quad &:= \quad \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j)\land(\neg\sigma(\mathbf{x}))\right] \\
\mathsf{WCXp}(\mathcal{Y}) \quad &:= \quad \exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j = v_j)\land(\neg\sigma(\mathbf{x}))
\end{aligned}$$

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{array}{rcl} \mathrm{WAXp}(\mathcal{X}) & := & \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge_{j\in\mathcal{X}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\right] \\ \mathrm{WCXp}(\mathcal{Y}) & := & \exists(\mathbf{x}\in\mathbb{F}).\bigwedge_{j\notin\mathcal{Y}}(x_j=v_j)\land(\neg\sigma(\mathbf{x})) \end{array}$$

- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \land_{i\in\mathcal{F}}\,(s_i \rightarrow (x_i=v_i)) \land \mathrm{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}\mathsf{WAXp}(\mathcal{X}) &:= \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\right]\\ \mathsf{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\end{aligned}$$

- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B}:=\land_{i\in\mathcal{F}}(s_i\rightarrow(x_i=v_i))\land\mathsf{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

  - Soft constraints: $\mathcal{S}=\{s_i\,|\,i\in\mathcal{F}\}$

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}
\mathsf{WAXp}(\mathcal{X}) \quad &:= \quad \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))\right] \\
\mathsf{WCXp}(\mathcal{Y}) \quad &:= \quad \exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{aligned}$$

- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \land_{i\in\mathcal{F}}\,(s_i \rightarrow (x_i = v_i)) \land \mathsf{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

  - Soft constraints: $\mathcal{S} = \{s_i \,|\, i\in\mathcal{F}\}$

- **Claim**: Each MUS of $(\mathcal{B},\mathcal{S})$ is an AXp & each MCS of $(\mathcal{B},\mathcal{S})$ is a CXp

Computing AXps (resp. CXps) as MUSes (resp. MCSes)

- Recap:

$$\begin{aligned}\mathrm{WAXp}(\mathcal{X}) &:= \neg\left[\exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\right]\\ \mathrm{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x}\in\mathbb{F}).\bigwedge\nolimits_{j\notin\mathcal{Y}}(x_j=v_j)\land(\neg\sigma(\mathbf{x}))\end{aligned}$$

- Let,
  - Hard constraints, $\mathcal{B}$:

$$\mathcal{B} := \land_{i\in\mathcal{F}}\,(s_i \rightarrow (x_i=v_i)) \land \mathrm{Encode}_{\mathcal{T}}(\neg\sigma(\mathbf{x}))$$

  - Soft constraints: $\mathcal{S}=\{s_i\,|\,i\in\mathcal{F}\}$

- **Claim**: Each MUS of $(\mathcal{B},\mathcal{S})$ is an AXp & each MCS of $(\mathcal{B},\mathcal{S})$ is a CXp
  - Can use MUS/MCS algorithms for finding AXps/CXps

Unit #03

Tractability in Symbolic XAI

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

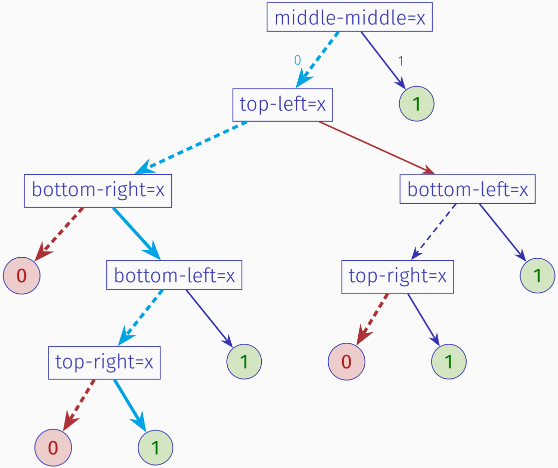
Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

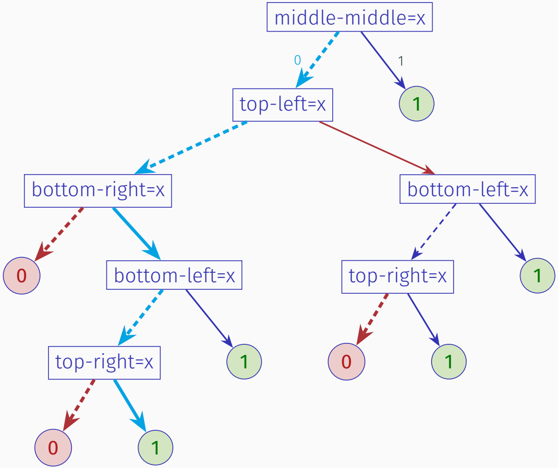
DT explanations

[IIM20]



DT explanations

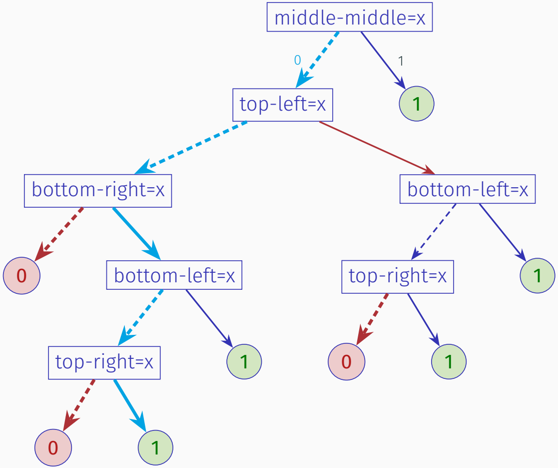
[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time

DT explanations

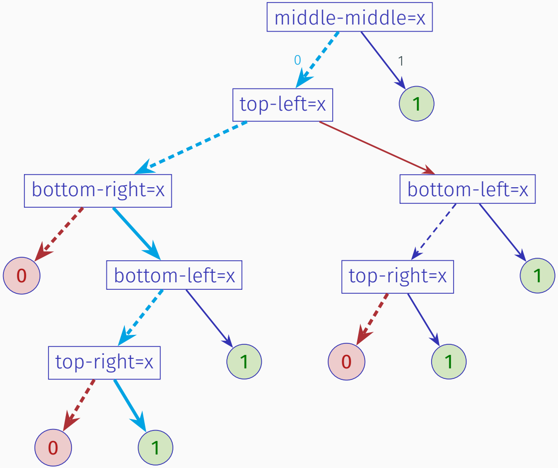
[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction 0 remain inconsistent

DT explanations in polynomial time

[IIM20]



- Run PI-explanation algorithm based on NP-oracles
  - Worst-case exponential time
- For prediction 1, it suffices to ensure all paths with prediction **0** remain inconsistent
  - I.e. find a subset-minimal hitting set of all **0** paths; these are the features to keep
    - E.g. BR and TR suffice for prediction
  - Well-known to be solvable in polynomial time [EG95]

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

- Finding one CXp in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time

Answering queries in DTs

- Finding one AXp in polynomial-time – covered

- Finding one CXp in polynomial-time

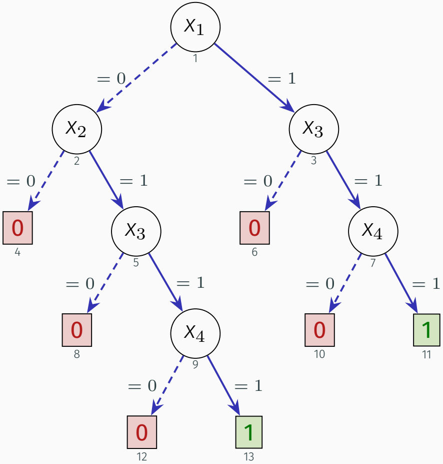
- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time

Answering queries in DTs

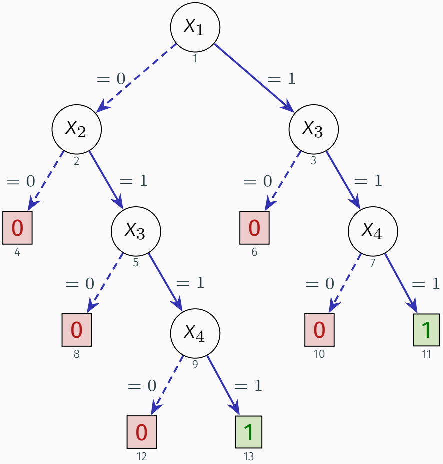
- Finding one AXp in polynomial-time – covered
- Finding one CXp in polynomial-time
- Finding all CXps in polynomial-time; hence, finding one CXp also in polynomial-time
- Practically efficient enumeration of AXps – later

Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \varnothing$

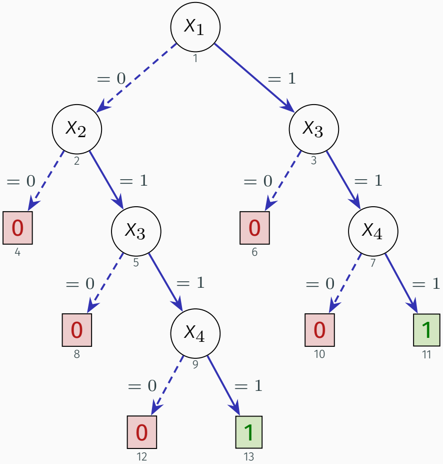


- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:



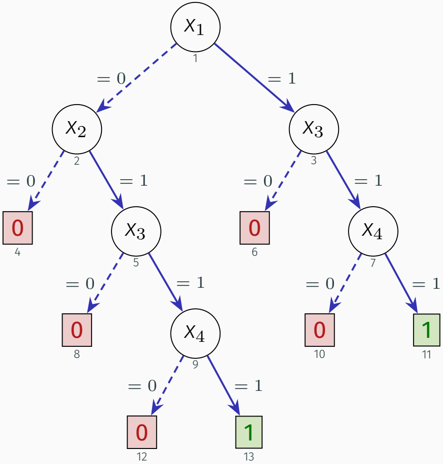
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$



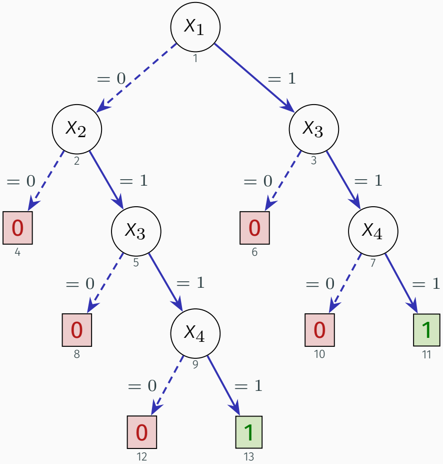
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$



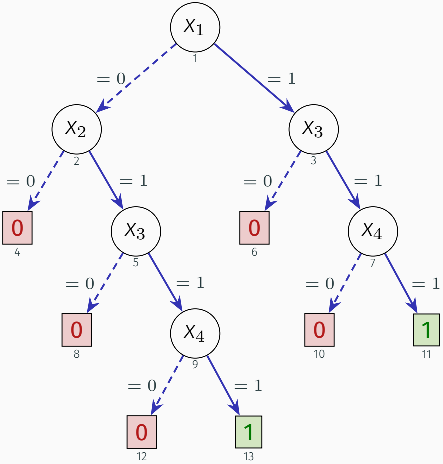
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets



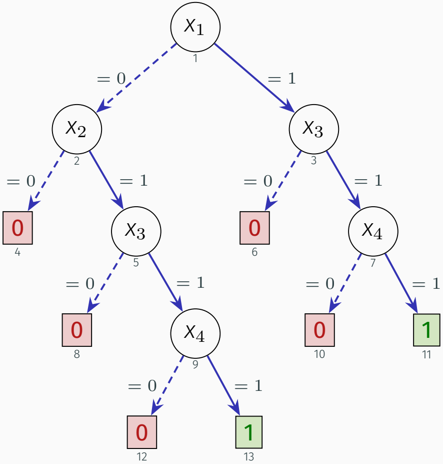
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT



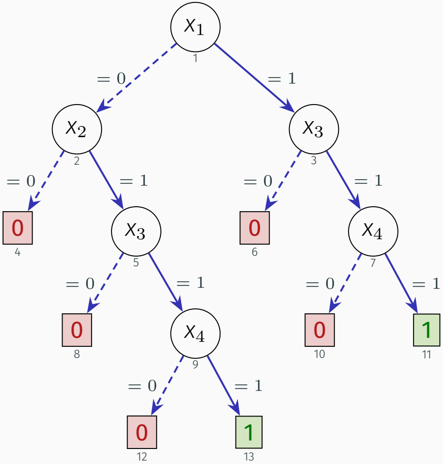
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$



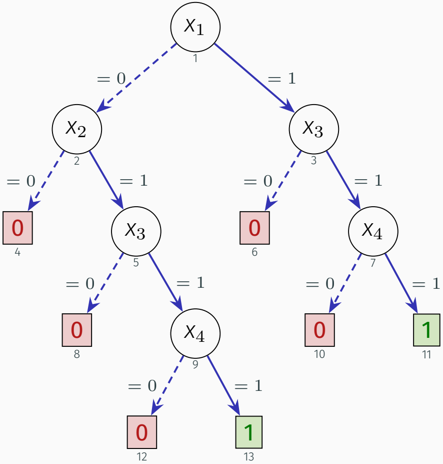
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$



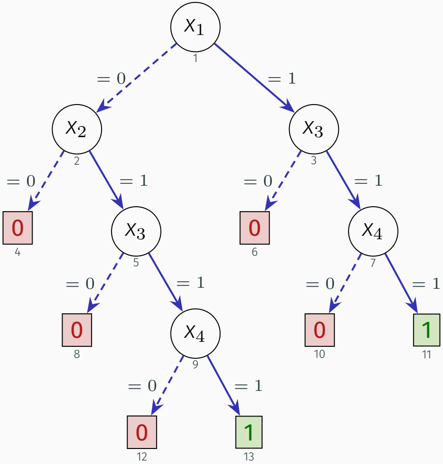
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$
  - Add $\{1,3\}$ to $\mathcal{L}$



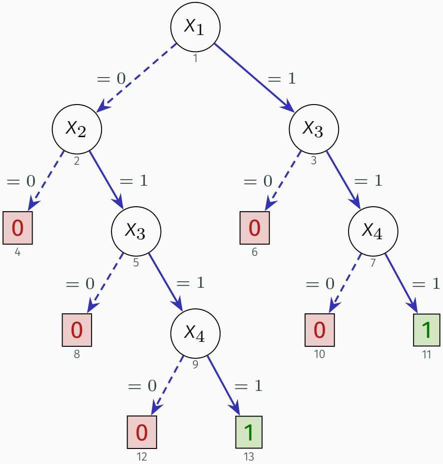
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1, 1, 1, 1), 1)$
  - Add $\{1, 2\}$ to $\mathcal{L}$
  - Add $\{1, 3\}$ to $\mathcal{L}$
  - Add $\{1, 4\}$ to $\mathcal{L}$



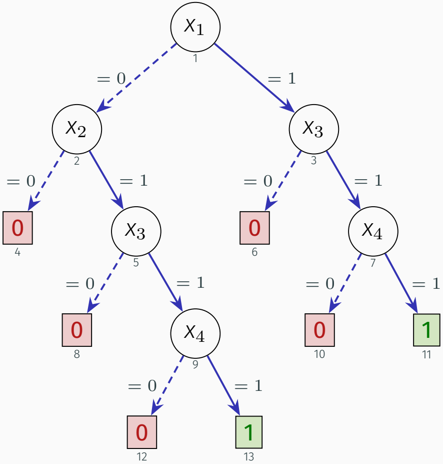
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$
  - Add $\{1,3\}$ to $\mathcal{L}$
  - Add $\{1,4\}$ to $\mathcal{L}$
  - Add $\{3\}$ to $\mathcal{L}$



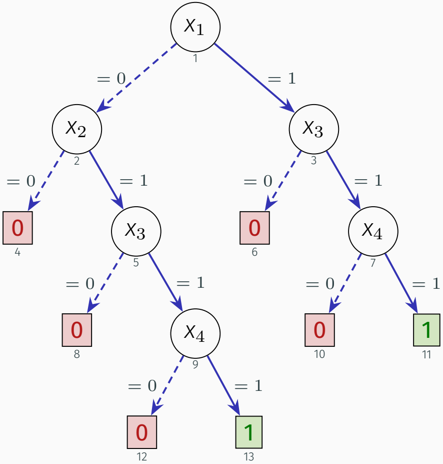
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$
  - Add $\{1,3\}$ to $\mathcal{L}$
  - Add $\{1,4\}$ to $\mathcal{L}$
  - Add $\{3\}$ to $\mathcal{L}$
  - Add $\{4\}$ to $\mathcal{L}$



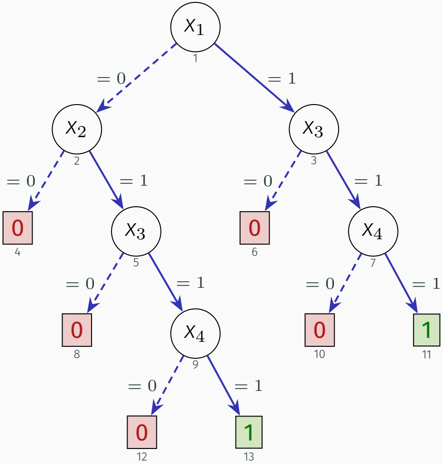
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$
  - Add $\{1,3\}$ to $\mathcal{L}$
  - Add $\{1,4\}$ to $\mathcal{L}$
  - Add $\{3\}$ to $\mathcal{L}$
  - Add $\{4\}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$: $\{1,3\}$ and $\{1,4\}$



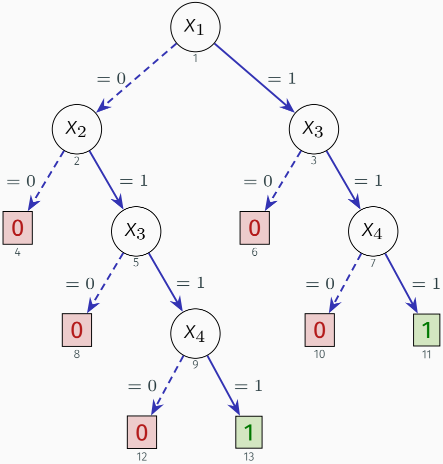
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1, 1, 1, 1), 1)$
  - Add $\{1, 2\}$ to $\mathcal{L}$
  - Add $\{1, 3\}$ to $\mathcal{L}$
  - Add $\{1, 4\}$ to $\mathcal{L}$
  - Add $\{3\}$ to $\mathcal{L}$
  - Add $\{4\}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$: $\{1, 3\}$ and $\{1, 4\}$
  - CXps: $\{\{1, 2\}, \{3\}, \{4\}\}$



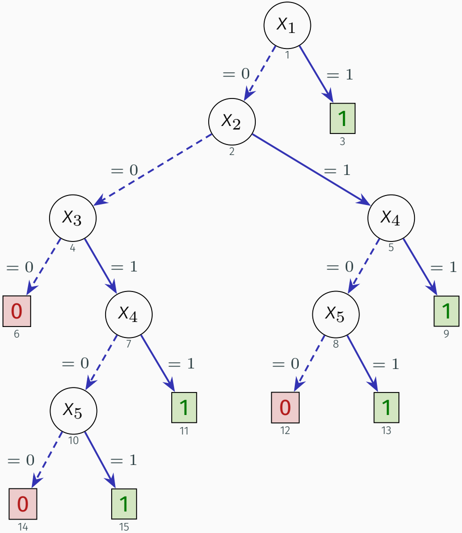
Finding all CXps in polynomial-time

- Basic algorithm:
  - $\mathcal{L} = \emptyset$
  - For each leaf node not predicting q:
    - $\mathcal{I}$: features with literals inconsistent with $\mathbf{v}$
    - Add $\mathcal{I}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$ non-minimal sets
  - $\mathcal{L}$ contains all the CXps of the DT
- Example: instance is $((1,1,1,1),1)$
  - Add $\{1,2\}$ to $\mathcal{L}$
  - Add $\{1,3\}$ to $\mathcal{L}$
  - Add $\{1,4\}$ to $\mathcal{L}$
  - Add $\{3\}$ to $\mathcal{L}$
  - Add $\{4\}$ to $\mathcal{L}$
  - Remove from $\mathcal{L}$: $\{1,3\}$ and $\{1,4\}$
  - CXps: $\{\{1,2\},\{3\},\{4\}\}$
  - AXps: $\{\{1,3,4\},\{2,3,4\}\}$, by computing all MHSes

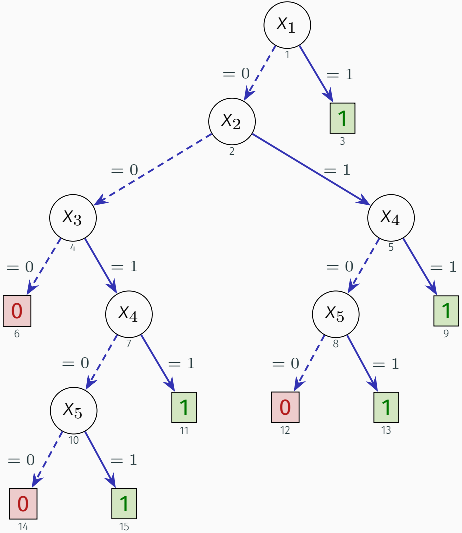


Outline – Unit #03

Are *interpretable* models really interpretable? – DTs

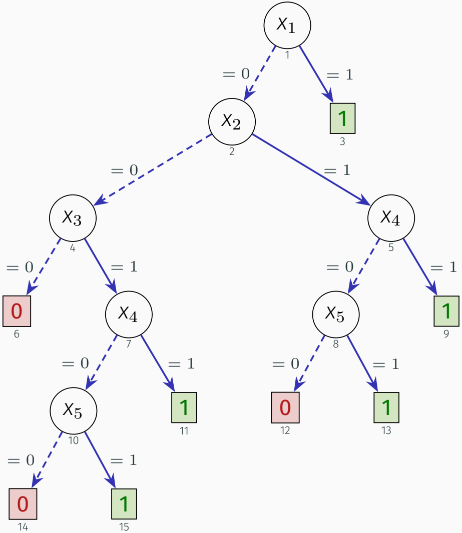


- Case of optimal decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?



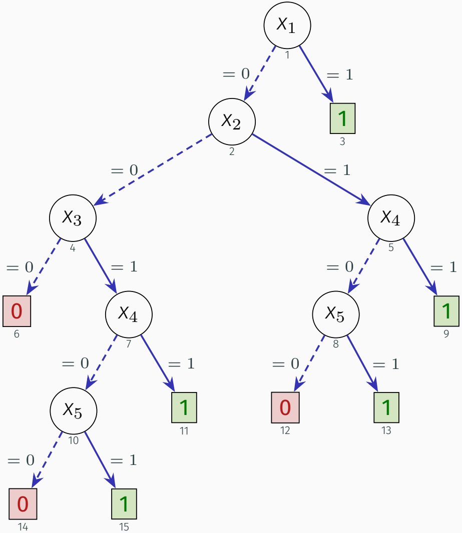
- Case of **optimal** decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?
  - Clearly, IF $\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5$ THEN $\kappa(\mathbf{x}) = 1$

Are *interpretable* models really interpretable? – DTs



- Case of **optimal** decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?
  - Clearly, IF $\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5$ THEN $\kappa(\mathbf{x}) = 1$
  - But, x_1, x_2, x_4 are irrelevant for the prediction:

x_3	x_5	x_1	x_2	x_4	$\kappa(\mathbf{x})$
1	1	0	0	0	1
1	1	0	0	1	1
1	1	0	1	0	1
1	1	0	1	1	1
1	1	1	0	0	1
1	1	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1



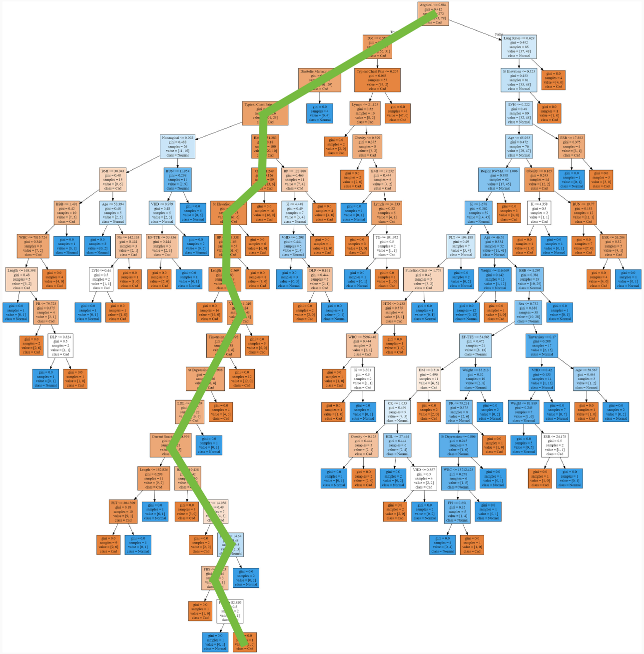
- Case of **optimal** decision tree (DT) [HRS19]
- Explanation for $(0, 0, 1, 0, 1)$, with prediction 1?
  - Clearly, IF $\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5$ THEN $\kappa(\mathbf{x}) = 1$
  - But, x_1, x_2, x_4 are irrelevant for the prediction:

x_3	x_5	x_1	x_2	x_4	$\kappa(\mathbf{x})$
1	1	0	0	0	1
1	1	0	0	1	1
1	1	0	1	0	1
1	1	0	1	1	1
1	1	1	0	0	1
1	1	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

$\therefore$ one AXp is $\{3, 5\}$
Compare with $\{1, 2, 3, 4, 5\}$...

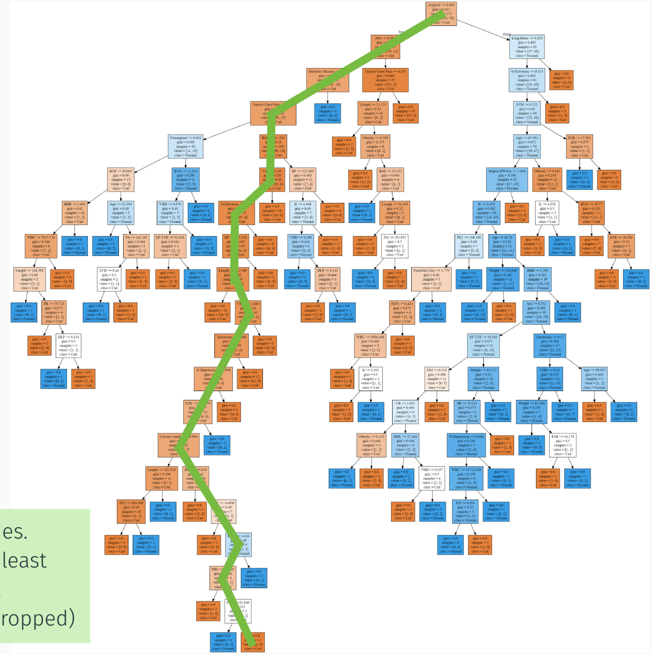
Are *interpretable* models really interpretable? – large DTs

[GZM20]



Are *interpretable* models really interpretable? – large DTs

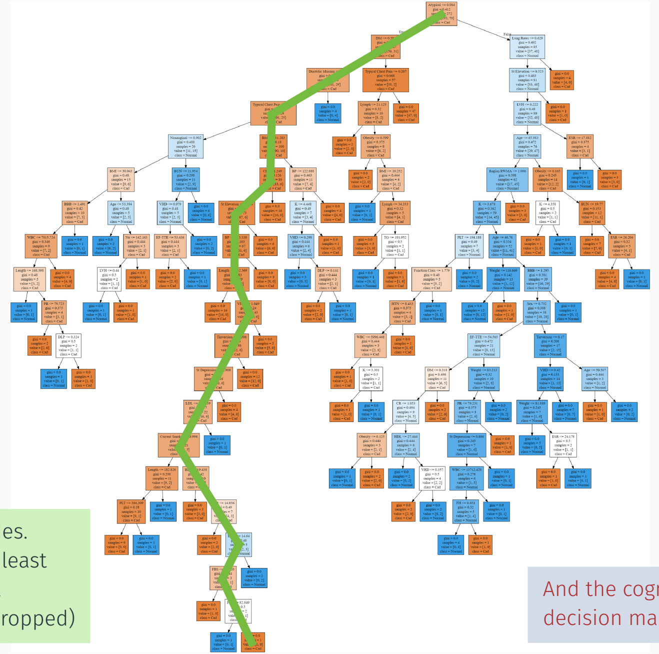
[GZM20]



Path with 19 internal nodes.
By manual inspection, at least
10 literals are redundant!
(And at least 9 features dropped)

Are *interpretable* models really interpretable? – large DTs

[GZM20]



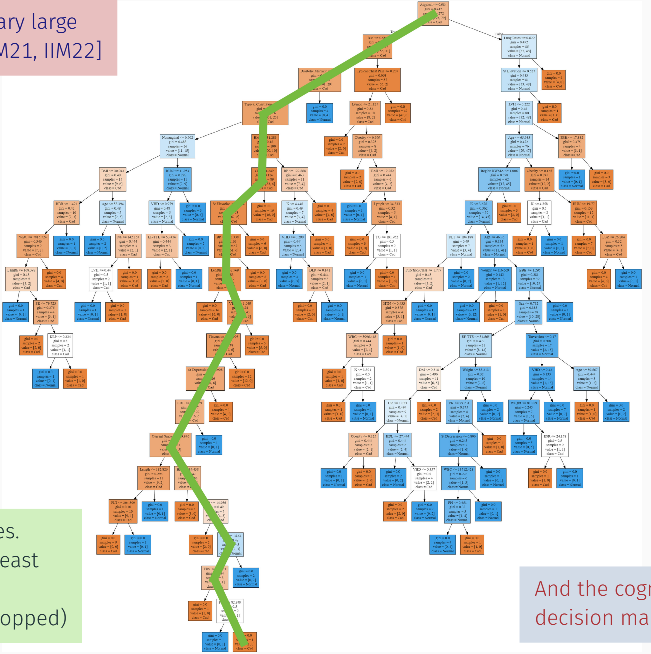
Path with 19 internal nodes.
By manual inspection, at least 10 literals are redundant!
(And at least 9 features dropped)

And the cognitive limits of human decision makers are well-known [Mil56]

Are *interpretable* models really interpretable? – large DTs

Redundancy can be arbitrary large on path length [IIM20, HIIM21, IIM22]

[GZM20]



Path with 19 internal nodes.
By manual inspection, at least 10 literals are redundant!
(And at least 9 features dropped)

And the cognitive limits of human decision makers are well-known [Mil56]

Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

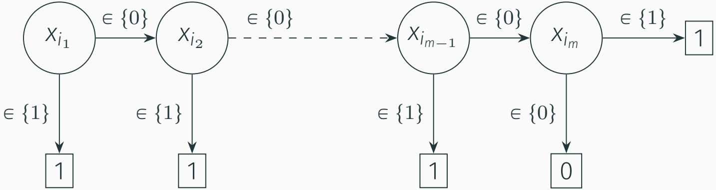
$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:

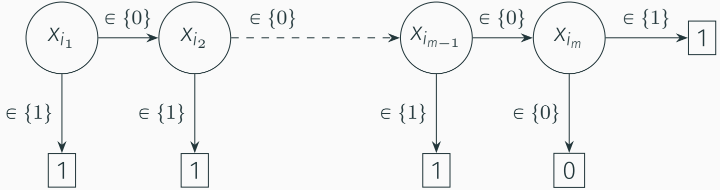


Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0,1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:



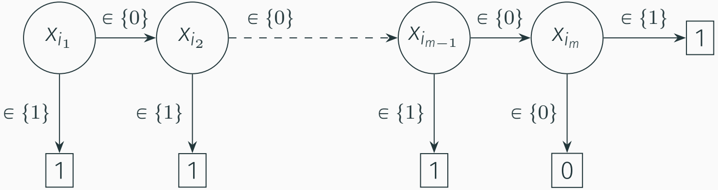
- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1

Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:



- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1
- Explanation using path in DT: $\{i_1, i_2, \ldots, i_m\}$, i.e.

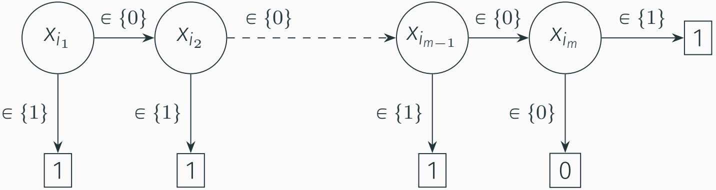
$$(x_{i_1} = 0) \land (x_{i_2} = 0) \land \ldots \land (x_{i_{m-1}} = 0) \land (x_{i_m} = 1) \rightarrow \kappa(x_1, \ldots, x_m)$$

Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0,1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:



- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1
- Explanation using path in DT: $\{i_1, i_2, \ldots, i_m\}$, i.e.

$$(x_{i_1} = 0) \land (x_{i_2} = 0) \land \ldots \land (x_{i_{m-1}} = 0) \land (x_{i_m} = 1) \rightarrow \kappa(x_1, \ldots, x_m)$$

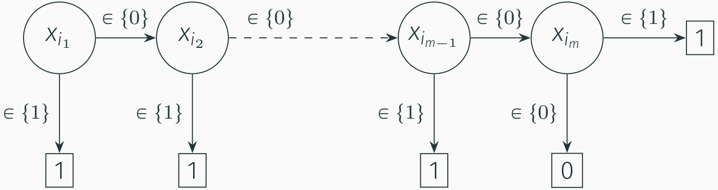
- **But $\{i_m\}$ suffices for prediction, i.e.** $\forall (\mathbf{x} \in \{0,1\}^m).(x_{i_m}) \rightarrow \kappa(\mathbf{x})$

Are *interpretable* models really interpretable? – arbitrary redundancy [IIM20, HIIM21, IIM22]

- Classifier, with $x_1, \ldots, x_m \in \{0, 1\}$:

$$\kappa(x_1, x_2, \ldots, x_{m-1}, x_m) = \bigvee_{i=1}^{m} x_i$$

- Build DT, by picking variables in order $\langle i_1, i_2, \ldots, i_m \rangle$, permutation of $\langle 1, 2, \ldots, m \rangle$:



- Point: $(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_m}) = (0, 0, \ldots, 0, 1)$, and prediction 1
- Explanation using path in DT: $\{i_1, i_2, \ldots, i_m\}$, i.e.

$$(x_{i_1} = 0) \land (x_{i_2} = 0) \land \ldots \land (x_{i_{m-1}} = 0) \land (x_{i_m} = 1) \rightarrow \kappa(x_1, \ldots, x_m)$$

- **But $\{i_m\}$ suffices for prediction, i.e. $\forall(\mathbf{x} \in \{0, 1\}^m).(x_{i_m}) \rightarrow \kappa(\mathbf{x})$**

- **AXp's can be arbitrarily smaller than paths in (optimal) DTs!** [IIM20, IIM22]

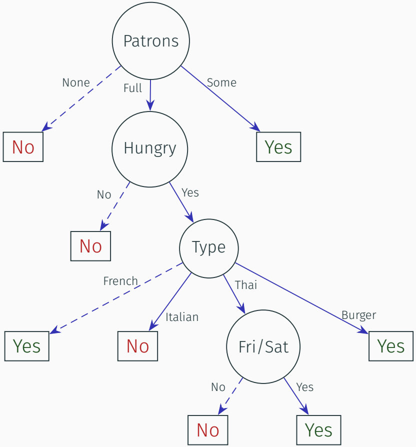
Explanation redundancy in DTs is ubiquitous – published DT examples

[IIM22]

DT Ref	D	#N	#P	%R	%C	%m	%M	%avg
[Alp14, Ch. 09, Fig. 9.1]	2	5	3	33	25	50	50	50
[Alp16, Ch. 03, Fig. 3.2]	2	5	3	33	25	50	50	50
[Bra20, Ch. 01, Fig. 1.3]	4	9	5	60	25	25	50	36
[BA97, Figure 1]	3	12	7	14	8	33	33	33
[BBHK10, Ch. 08, Fig. 8.2]	3	7	4	25	12	50	50	50
[BFOS84, Ch. 01, Fig. 1.1]	3	7	4	50	25	33	33	33
[DL01, Ch. 01, Fig. 1.2a]	2	5	3	33	25	33	33	33
[DL01, Ch. 01, Fig. 1.2b]	2	5	3	33	25	33	33	33
[KMND20, Ch. 04, Fig. 4.14]	3	7	4	25	12	50	50	50
[KMND20, Sec. 4.7, Ex. 4]	2	5	3	33	25	50	50	50
[Qui93, Ch. 01, Fig. 1.3]	3	12	7	28	17	33	50	41
[RM08, Ch. 01, Fig. 1.5]	3	9	5	20	12	33	33	33
[RM08, Ch. 01, Fig. 1.4]	3	7	4	50	25	33	33	33
[WFHP17, Ch. 01, Fig. 1.2]	3	7	4	25	12	50	50	50
[VLE+16, Figure 4]	6	39	20	65	63	20	40	33
[Fla12, Ch. 02, Fig. 2.1(right)]	2	5	3	33	25	50	50	50
[Kot13, Figure 1]	3	10	6	33	11	33	33	33
[Mor82, Figure 1]	3	9	5	80	75	33	50	41
[PM17, Ch. 07, Fig. 7.4]	3	7	4	50	25	33	33	33
[RN10, Ch. 18, Fig. 18.6]	4	12	8	25	6	25	33	29
[SB14, Ch. 18, Page 212]	2	5	3	33	25	50	50	50
[Zho12, Ch. 01, Fig. 1.3]	2	5	3	33	25	33	33	33
[BHO09, Figure 1b]	4	13	7	71	50	33	50	36
[Zho21, Ch. 04, Fig. 4.3]	4	14	9	11	2	25	25	25

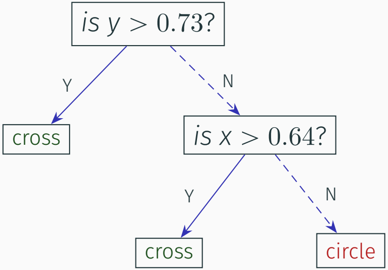
Many DTs have paths that are not minimal XPs – Russell&Norvig's book

[RN10]



- Explanation for $(\mathsf{P}, \mathsf{H}, \mathsf{T}, \mathsf{W}) = (\mathsf{Full}, \mathsf{Yes}, \mathsf{Thai}, \mathsf{No})$?

Many DTs have paths that are not minimal XPs – Zhou's book

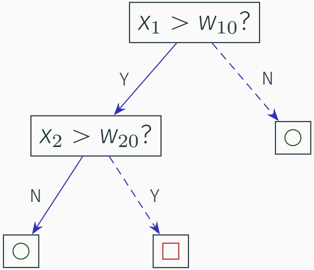


[Zho12]

- Explanation for $(x, y) = (1.25, -1.13)$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

[Alp14]

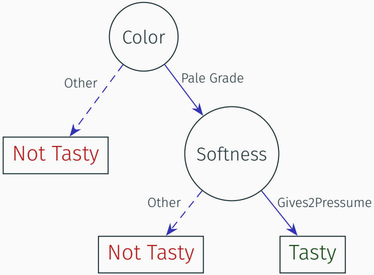


- Explanation for $(x_1, x_2) = (\alpha, \beta)$, with $\alpha > w_{10}$ and $\beta \leqslant w_{20}$?

Obs: True explanations can be computed for categorical, integer or real-valued features !

Many DTs have paths that are not minimal XPs – S.-S.&B.-D.'s book

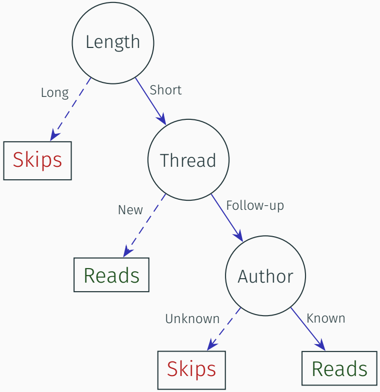
[SB14]



- Explanation for $(\text{color}, \text{softness}) = (\text{Pale Grade}, \text{Other})$?

Many DTs have paths that are not minimal XPs – Poole&Mackworth's book

[PM17]



- Explanation for $(\mathrm{L}, \mathrm{T}, \mathrm{A}) = (\mathrm{Short}, \mathrm{Follow\text{-}Up}, \mathrm{Unknown})$?
- Explanation for $(\mathrm{L}, \mathrm{T}, \mathrm{A}) = (\mathrm{Short}, \mathrm{Follow\text{-}Up}, \mathrm{Known})$?

Explanation redundancy in DTs is ubiquitous – DTs from datasets

Dataset	(#F	#S)	IAI									ITI								
			D	#N	%A	#P	%R	%C	%m	%M	%avg	D	#N	%A	#P	%R	%C	%m	%M	%avg
adult	(12	6061)	6	83	78	42	33	25	20	40	25	17	509	73	255	75	91	10	66	22
anneal	(38	886)	6	29	99	15	26	16	16	33	21	9	31	100	16	25	4	12	20	16
backache	(32	180)	4	17	72	9	33	39	25	33	30	3	9	91	5	80	87	50	66	54
bank	(19	36293)	6	113	88	57	5	12	16	20	18	19	1467	86	734	69	64	7	63	27
biodegradation	(41	1052)	5	19	65	10	30	1	25	50	33	8	71	76	36	50	8	14	40	21
cancer	(9	449)	6	37	87	19	36	9	20	25	21	5	21	84	11	54	10	25	50	37
car	(6	1728)	6	43	96	22	86	89	20	80	45	11	57	98	29	65	41	16	50	30
colic	(22	357)	6	55	81	28	46	6	16	33	20	4	17	80	9	33	27	25	25	25
compas	(11	1155)	6	77	34	39	17	8	16	20	17	15	183	37	92	66	43	12	60	27
contraceptive	(9	1425)	6	99	49	50	8	2	20	60	37	17	385	48	193	27	32	12	66	21
dermatology	(34	366)	6	33	90	17	23	3	16	33	21	7	17	95	9	22	0	14	20	17
divorce	(54	150)	5	15	90	8	50	19	20	33	24	2	5	96	3	33	16	50	50	50
german	(21	1000)	6	25	61	13	38	10	20	40	29	10	99	72	50	46	13	12	40	22
heart-c	(13	302)	6	43	65	22	36	18	20	33	22	4	15	75	8	87	81	25	50	34
heart-h	(13	293)	6	37	59	19	31	4	20	40	24	8	25	77	13	61	60	20	50	32
kr-vs-kp	(36	3196)	6	49	96	25	80	75	16	60	33	13	67	99	34	79	43	7	70	35
lending	(9	5082)	6	45	73	23	73	80	16	50	25	14	507	65	254	69	80	12	75	25
letter	(16	18668)	6	127	58	64	1	0	20	20	20	46	4857	68	2429	6	7	6	25	9
lymphography	(18	148)	6	61	76	31	35	25	16	33	21	6	21	86	11	9	0	16	16	16
mortality	(118	13442)	6	111	74	56	8	14	16	20	17	26	865	76	433	61	61	7	54	19
mushroom	(22	8124)	6	39	100	20	80	44	16	33	24	5	23	100	12	50	31	20	40	25
pendigits	(16	10992)	6	121	88	61	0	0	—	—	—	38	937	85	469	25	86	6	25	11
promoters	(58	106)	1	3	90	2	0	0	—	—	—	3	9	81	5	20	14	33	33	33
recidivism	(15	3998)	6	105	61	53	28	22	16	33	18	15	611	51	306	53	38	9	44	16
seismic_bumps	(18	2578)	6	37	89	19	42	19	20	33	24	8	39	93	20	60	79	20	60	42
shuttle	(9	58000)	6	63	99	32	28	7	20	33	23	23	159	99	80	33	9	14	50	30
soybean	(35	623)	6	63	88	32	9	5	25	25	25	16	71	89	36	22	1	9	12	10
spambase	(57	4210)	6	63	75	32	37	12	16	33	19	15	143	91	72	76	98	7	58	25
spect	(22	228)	6	45	82	23	60	51	20	50	35	6	15	86	8	87	98	50	83	65
splice	(2	3178)	3	7	50	4	0	0	—	—	—	88	177	55	89	0	0	—	—	—

$$
\begin{array}{llccc}
\mathrm{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_2: & \text{ELSE IF} & (x_2 \land x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_3: & \text{ELSE IF} & (\neg x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_4: & \text{ELSE IF} & (x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_5: & \text{ELSE IF} & (\neg x_1 \land \neg x_3) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_6: & \text{ELSE IF} & (x_6) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_{\mathrm{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 1
\end{array}
$$

- Instance: $((0,1,0,1,0,1),0)$, i.e. rule R_2 fires

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \land x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \land \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 1$

- Instance: $((0, 1, 0, 1, 0, 1), 0)$, i.e. rule R_2 fires
- What is the abductive explanation?

Are *interpretable* models really interpretable? – DLs

R_1 :	IF	$(x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \wedge x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \wedge \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
R_{DEF} :	ELSE			$\kappa(\mathbf{x}) = 1$

- Instance: $((0,1,0,1,0,1),0)$, i.e. rule R_2 fires
- What is the abductive explanation?
- Recall: one AXp is $\{3,4,6\}$

Are *interpretable* models really interpretable? – DLs

[MSI23]

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \land x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \land \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
$\mathsf{R}_{\mathsf{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 1$

- Instance: $((0,1,0,1,0,1),0)$, i.e. rule R_2 fires
- What is the abductive explanation?
- Recall: one AXp is $\{3,4,6\}$
  - **Why?**
    - We need 3 (or 1) so that R_1 cannot fire
    - With 3, we do not need 2, since with 4 and 6 fixed, then R_4 is guaranteed to fire
  - **Some questions**:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?

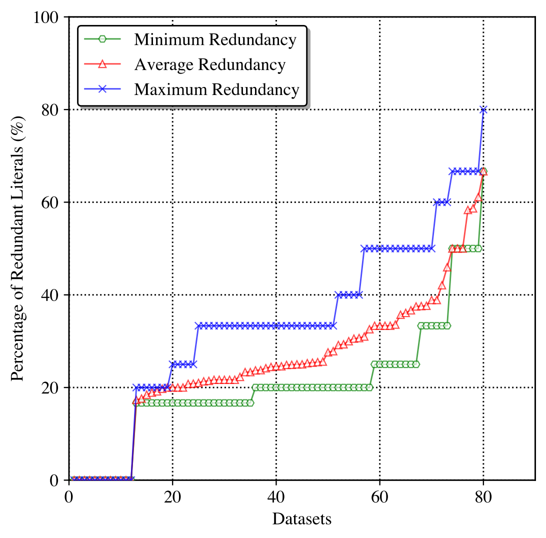
Are *interpretable* models really interpretable? – DLs

R_1 :	IF	$(x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_2 :	ELSE IF	$(x_2 \wedge x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(\neg x_1 \wedge x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_4 \wedge x_6)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_1 \wedge \neg x_3)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	(x_6)	THEN	$\kappa(\mathbf{x}) = 0$
$\mathrm{R}_{\mathrm{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 1$

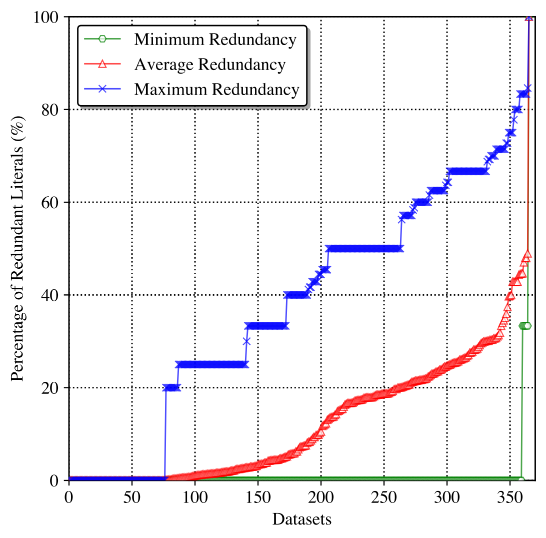
- Instance: $((0,1,0,1,0,1),0)$, i.e. rule R_2 fires
- What is the abductive explanation?
- Recall: one AXp is $\{3,4,6\}$
  - Why?
    - We need 3 (or 1) so that R_1 cannot fire
    - With 3, we do not need 2, since with 4 and 6 fixed, then R_4 is guaranteed to fire
  - Some questions:
    - Would average human decision maker be able to understand the AXp?
    - Would he/she be able to compute one AXp, by manual inspection?
      (BTW, **we have proved that computing one AXp for DLs is computationally hard...**) [IM21, MSI23]

Are *interpretable* models really interpretable? – DTs/DLs in practice

[MSI23]



DTs learned with Interpretable AI, max depth 6



DLs learned with CN2

Outline – Unit #03

- Explanations for Decision Trees
- XAI Queries for DTs
- Myth #01: Intrinsic Interpretability
- Detour: From Decision Trees to Explained Decision Sets
- Explanations for Decision Graphs
- Explanations for Monotonic Classifiers
- Review examples

From DTs to explained DSs

[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules

From DTs to explained DSs

[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules
  - A DS without overlap and complete coverage computes a classification function

From DTs to explained DSs

[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...

From DTs to explained DSs

[HM23]

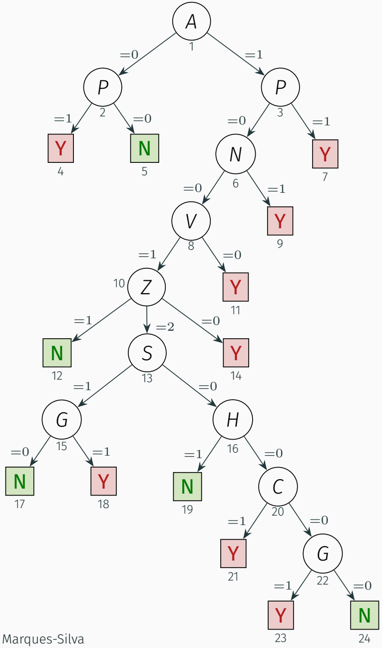
- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...
- One can extract explained DSs from DTs

From DTs to explained DSs

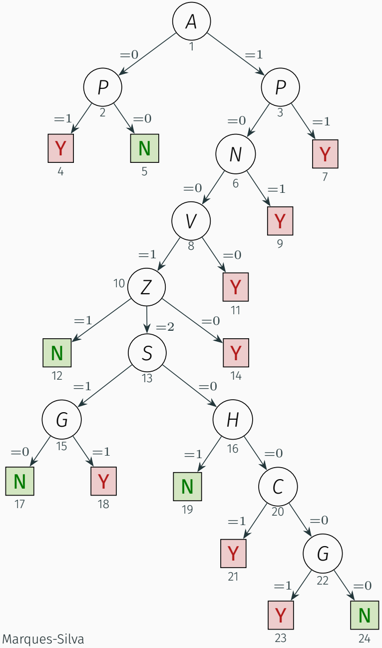
[HM23]

- Decision sets raise a number of issues:
  - Overlap: Two rules with different predictions can fire on the same input
  - Incomplete coverage: For some inputs, no rule may fire
    - A default rule defeats the purpose of unordered rules
  - A DS without overlap and complete coverage computes a classification function
- And explaining DSs is computationally hard...
- One can extract explained DSs from DTs
  - Extract one AXp (viewed as a logic rule) from each path in DT
  - Resulting rules are non-overlapping, and cover feature space

Example


A
1
=0
=1
P
2
P
3
=1
=0
=0
=1
Y
4
N
5
N
6
Y
7
=0
=1
V
8
Y
9
=1
=0
10
Z
Y
11
=1
=2
=0
N
12
S
13
Y
14
=1
=0
G
15
H
16
=0
=1
=1
=0
N
17
Y
18
N
19
C
20
=1
=0
Y
21
G
22
=1
=0
Y
23
N
24

Example



R$_{01}$: IF $[P]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{02}$: IF $[\overline{A} \land \overline{P}]$THEN $\kappa(\cdot) = \mathbf{N}$

R$_{03}$: IF $[\overline{P} \land \overline{N} \land V \land Z = 1]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{04}$: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land S \land \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{05}$: IF $[A \land Z = 2 \land S \land G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{06}$: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land \overline{S} \land H]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{07}$: IF $[A \land Z = 2 \land \overline{S} \land \overline{H} \land C]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{08}$: IF $[A \land Z = 2 \land \overline{H} \land G]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{09}$: IF $[\overline{P} \land \overline{N} \land V \land Z = 2 \land \overline{C} \land \overline{G}]$ THEN $\kappa(\cdot) = \mathbf{N}$

R$_{10}$: IF $[A \land Z = 0]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{11}$: IF $[A \land \overline{V}]$ THEN $\kappa(\cdot) = \mathbf{Y}$

R$_{12}$: IF $[A \land N]$ THEN $\kappa(\cdot) = \mathbf{Y}$

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

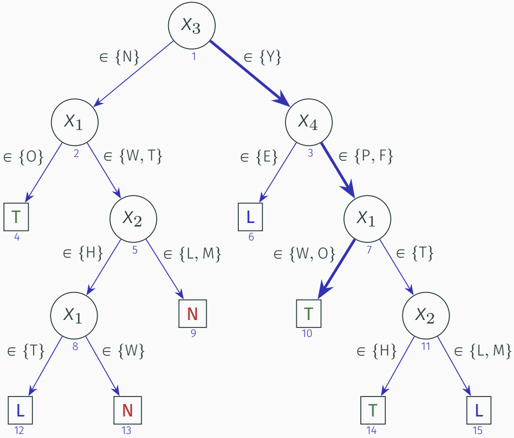
Review examples

Explanation graphs – overview of results

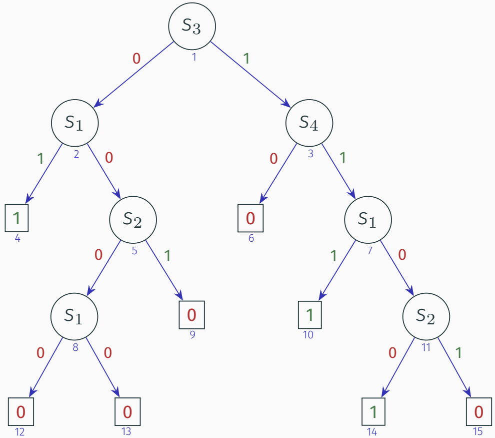
[HIIM21]

- Concept of explanation graph (XpG)
- Explanations of decision trees reducible to XpG's
- Explanations of decision graphs reducible to XpG's
- Explanations of OBDDs reducible to XpG's
- Explanations of OMDDs reducible to XpG's
- Explanations (AXp's and CXp's) of XpG's computed in polynomial time

- DT; point: (O, L, Y, P); prediction T:

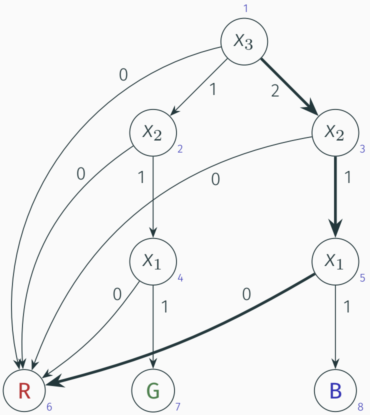


- XpG:

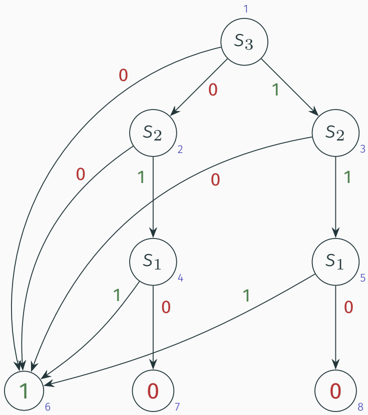


Example of XpG – OMDDs

- OMBBD; point: $(0, 1, 2)$; prediction R:



- XpG:



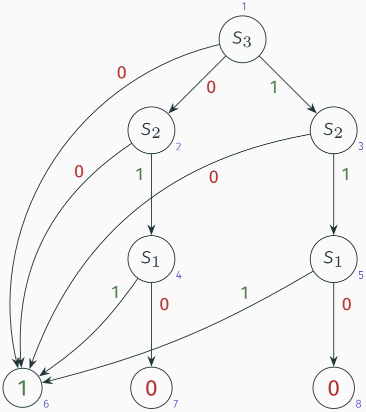
Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature i in $\mathcal{F}$

- XpG:



Finding one AXp for XpGs – polynomial time

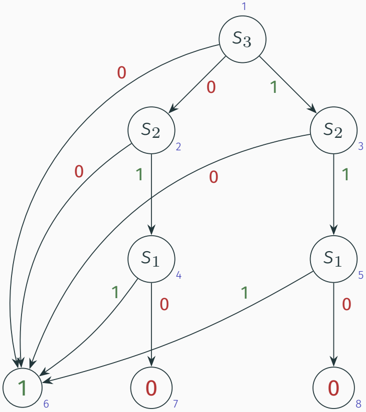
- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature i in $\mathcal{F}$

  Drop feature i from $\mathcal{S}$, i.e. i is free

- XpG:



Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

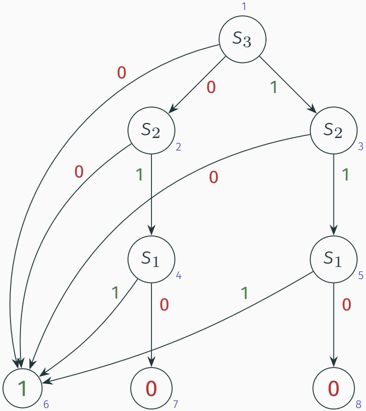
  For each feature i in $\mathcal{F}$

  Drop feature i from $\mathcal{S}$, i.e. i is free

  If path to some **0** not blocked by 0-valued literals, then

  Add feature i back to $\mathcal{S}$

- XpG:



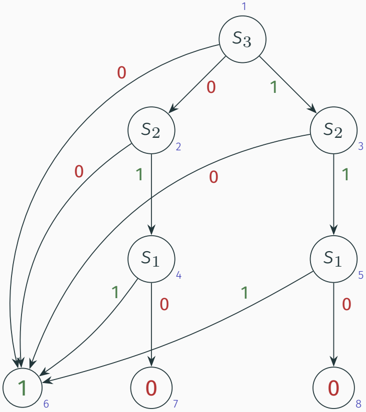
- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$
  For each feature i in $\mathcal{F}$
  Drop feature i from $\mathcal{S}$, i.e. i is free
  If path to some **0** not blocked by 0-valued literals, then

    Add feature i back to $\mathcal{S}$

  Return $\mathcal{S}$

- XpG:



Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

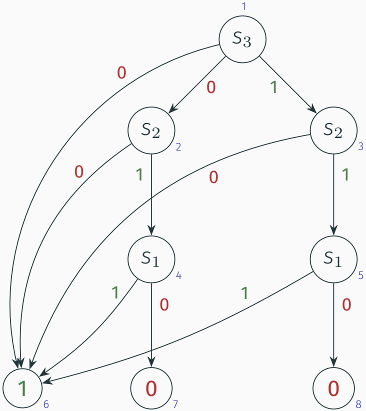
  $\mathcal{S} \leftarrow \mathcal{F}$
  For each feature i in $\mathcal{F}$
  Drop feature i from $\mathcal{S}$, i.e. i is free
  If path to some **0** not blocked by 0-valued literals, then

  Add feature i back to $\mathcal{S}$

  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$

- XpG:



Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature *i* in $\mathcal{F}$

  Drop feature *i* from $\mathcal{S}$, i.e. *i* is free

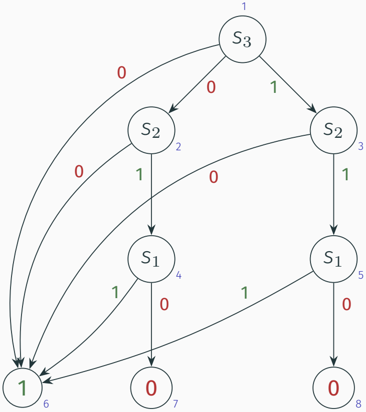
  If path to some **0** not blocked by 0-valued literals, then

  Add feature *i* back to $\mathcal{S}$

  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$

- XpG:



Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$

  For each feature *i* in $\mathcal{F}$

  Drop feature *i* from $\mathcal{S}$, i.e. *i* is free

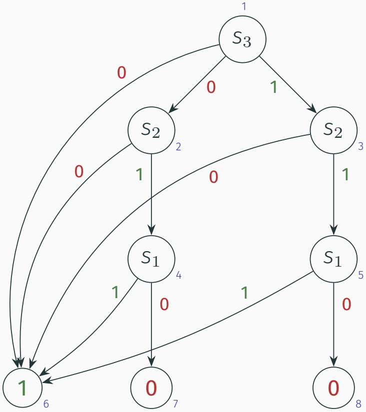
  If path to some **0** not blocked by 0-valued literals, then

  Add feature *i* back to $\mathcal{S}$

  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$
  - Both features 2 and 3 dropped from $\mathcal{S}$

- XpG:



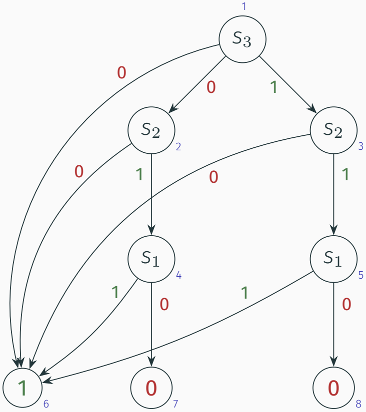
Finding one AXp for XpGs – polynomial time

- Algorithm (with no inconsistent paths):

  $\mathcal{S} \leftarrow \mathcal{F}$
  For each feature *i* in $\mathcal{F}$
  Drop feature *i* from $\mathcal{S}$, i.e. *i* is free
  If path to some **0** not blocked by 0-valued literals, then
  Add feature *i* back to $\mathcal{S}$
  Return $\mathcal{S}$

- Example:
  - $\mathcal{S} = \{1, 2, 3\}$
  - Feature 1 cannot be dropped, e.g. $s_3 \rightarrow s_2 \rightarrow s_1 \rightarrow 0$
  - Both features 2 and 3 dropped from $\mathcal{S}$
  - Return $\mathcal{S} = \{1\}$

- XpG:



Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Example monotonic classifier – $(\mathbf{v}, c) = ((10, 10, 5, 0), A)$

[MGC+21]

Variable	Meaning		Range
$\kappa(\cdot) \triangleq M$	Student grade		$\in \{A, B, C, D, E, F\}$
S	Final score		$\in \{0, \ldots, 10\}$
Feat. id	Feat. var.	Feat. name	Domain
1	Q	Quiz	$\{0, \ldots, 10\}$
2	X	Exam	$\{0, \ldots, 10\}$
3	H	Homework	$\{0, \ldots, 10\}$
4	R	Project	$\{0, \ldots, 10\}$

$$
\begin{aligned}
M &= \mathsf{ITE}(S \geqslant 9, A, \mathsf{ITE}(S \geqslant 7, B, \mathsf{ITE}(S \geqslant 5, C, \mathsf{ITE}(S \geqslant 4, D, \mathsf{ite}(S \geqslant 2, E, F))))) \\
S &= \max\left[0.3 \times Q + 0.6 \times X + 0.1 \times H, R\right]
\end{aligned}
$$

Also, $F \preccurlyeq E \preccurlyeq D \preccurlyeq C \preccurlyeq B \preccurlyeq A$

And, $\kappa(\mathbf{x_1}) \preccurlyeq \kappa(\mathbf{x}_2)$ if $\mathbf{x}_1 \leqslant \mathbf{x}_2$

Explaining monotonic classifiers

- Instance $(\mathbf{v}, c)$
- Domain for $i \in \mathcal{F}$: $\lambda(i) \leqslant x_i \leqslant \mu(i)$
- Idea: refine lower and upper bounds on the prediction
  - $\mathbf{v}_{\mathsf{L}}$ and $\mathbf{v}_{\mathsf{U}}$
- Utilities:
  - FixAttr(i):

$$
\begin{aligned}
&\mathbf{v}_{\mathsf{L}} \leftarrow (v_{\mathsf{L}_1}, \ldots, v_i, \ldots, v_{\mathsf{L}_N})\\
&\mathbf{v}_{\mathsf{U}} \leftarrow (v_{\mathsf{U}_1}, \ldots, v_i, \ldots, v_{\mathsf{U}_N})\\
&(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})\\
&\textbf{return } (\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{A}, \mathcal{B})
\end{aligned}
$$

  - FreeAttr(i):

$$
\begin{aligned}
&\mathbf{v}_{\mathsf{L}} \leftarrow (v_{\mathsf{L}_1}, \ldots, \lambda(i), \ldots, v_{\mathsf{L}_N})\\
&\mathbf{v}_{\mathsf{U}} \leftarrow (v_{\mathsf{U}_1}, \ldots, \mu(i), \ldots, v_{\mathsf{U}_N})\\
&(\mathcal{A}, \mathcal{B}) \leftarrow (\mathcal{A} \backslash \{i\}, \mathcal{B} \cup \{i\})\\
&\textbf{return } (\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{A}, \mathcal{B})
\end{aligned}
$$

Computing one AXp

$\mathbf{v}_{\mathsf{L}} \leftarrow (v_1, \ldots, v_N)$

$\mathbf{v}_{\mathsf{U}} \leftarrow (v_1, \ldots, v_N)$ $\qquad\rhd$ Ensures: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$

$(\mathcal{C}, \mathcal{D}, \mathcal{P}) \leftarrow (\mathcal{F}, \emptyset, \emptyset)$ $\qquad\rhd$ $\mathcal{S}$: Some possible seed

for all $i \in \mathcal{S}$ **do**

$\quad(\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D}) \leftarrow \mathsf{FreeAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D})$ $\qquad\rhd$ Require: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$, given $\mathcal{S}$

for all $i \in \mathcal{F} \setminus \mathcal{S}$ **do** $\qquad\rhd$ Loop inv.: $\kappa(\mathbf{v}_{\mathsf{L}}) = \kappa(\mathbf{v}_{\mathsf{U}})$

$\quad(\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D}) \leftarrow \mathsf{FreeAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{C}, \mathcal{D})$

$\quad$**if** $\kappa(\mathbf{v}_{\mathsf{L}}) \neq \kappa(\mathbf{v}_{\mathsf{U}})$ **then** $\qquad\rhd$ If invariant broken, fix it

$\qquad(\mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{D}, \mathcal{P}) \leftarrow \mathsf{FixAttr}(i, \mathbf{v}, \mathbf{v}_{\mathsf{L}}, \mathbf{v}_{\mathsf{U}}, \mathcal{D}, \mathcal{P})$

return $\mathcal{P}$

- **Obs:** $\mathcal{S} = \emptyset$ for computing a single AXp/CXp

Computing one AXp – example

- $\lambda(i) = 0$ and $\mu(i) = 10$
- $\mathbf{v} = (10, 10, 5, 0)$, with $\kappa(\mathbf{v}) = A$
- Q: find one AXp (CXp is similar)

Feat.	Initial values		Changed values		Predictions		Dec.	Resulting values	
	$\mathbf{v}_\mathsf{L}$	$\mathbf{v}_\mathsf{U}$	$\mathbf{v}_\mathsf{L}$	$\mathbf{v}_\mathsf{U}$	$\kappa(\mathbf{v}_\mathsf{L})$	$\kappa(\mathbf{v}_\mathsf{U})$		$\mathbf{v}_\mathsf{L}$	$\mathbf{v}_\mathsf{U}$
1	(10,10,5,0)	(10,10,5,0)	(0,10,5,0)	(10,10,5,0)	C	A	✓	(10,10,5,0)	(10,10,5,0)
2	(10,10,5,0)	(10,10,5,0)	(10,0,5,0)	(10,10,5,0)	E	A	✓	(10,10,5,0)	(10,10,5,0)
3	(10,10,5,0)	(10,10,5,0)	(10,10,0,0)	(10,10,10,0)	A	A	✗	(10,10,0,0)	(10,10,10,0)
4	(10,10,0,0)	(10,10,10,0)	(10,10,0,0)	(10,10,10,10)	A	A	✗	(10,10,0,0)	(10,10,10,10)

Outline – Unit #03

Explanations for Decision Trees

XAI Queries for DTs

Myth #01: Intrinsic Interpretability

Detour: From Decision Trees to Explained Decision Sets

Explanations for Decision Graphs

Explanations for Monotonic Classifiers

Review examples

Recap computation of (W)AXps/(W)CXps

$$\begin{aligned}
\text{WAXp}(\mathcal{X}) &:= \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\text{WCXp}(\mathcal{Y}) &:= \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \land (\neg\sigma(\mathbf{x}))
\end{aligned}$$

Recap computation of (W)AXps/(W)CXps

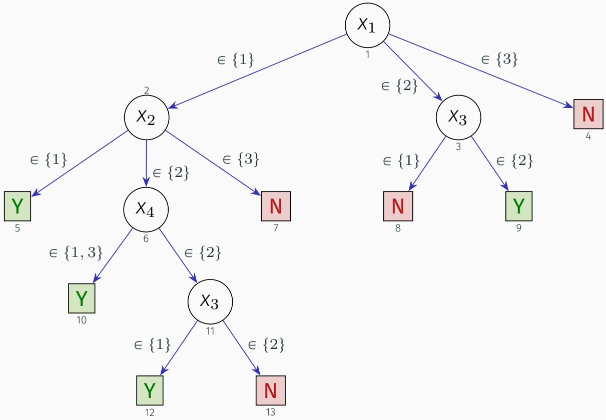
$$
\begin{aligned}
\mathsf{WAXp}(\mathcal{X}) \quad &:= \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\sigma(\mathbf{x})) \\
\mathsf{WCXp}(\mathcal{Y}) \quad &:= \quad \exists(\mathbf{x} \in \mathbb{F}). \bigwedge\nolimits_{j \notin \mathcal{Y}} (x_j = v_j) \wedge (\neg\sigma(\mathbf{x}))
\end{aligned}
$$

Input: Predicate $\mathbb{P}$, parameterized by $\mathcal{T}$, $\mathcal{M}$
Output: One XP $\mathcal{S}$

1: **procedure** oneXP($\mathbb{P}$)
2: $\quad \mathcal{S} \leftarrow \mathcal{F}$ $\qquad$ ▷ Initialization: $\mathbb{P}(\mathcal{S})$ holds
3: $\quad$ **for** $i \in \mathcal{F}$ **do** $\qquad$ ▷ Loop invariant: $\mathbb{P}(\mathcal{S})$ holds
4: $\quad\quad$ **if** $\mathbb{P}(\mathcal{S} \setminus \{i\})$ **then**
5: $\quad\quad\quad \mathcal{S} \leftarrow \mathcal{S} \setminus \{i\}$ $\qquad$ ▷ Update $\mathcal{S}$ only if $\mathbb{P}(\mathcal{S} \setminus \{i\})$ holds
6: $\quad$ **return** $\mathcal{S}$ $\qquad$ ▷ Returned set $\mathcal{S}$: $\mathbb{P}(\mathcal{S})$ holds

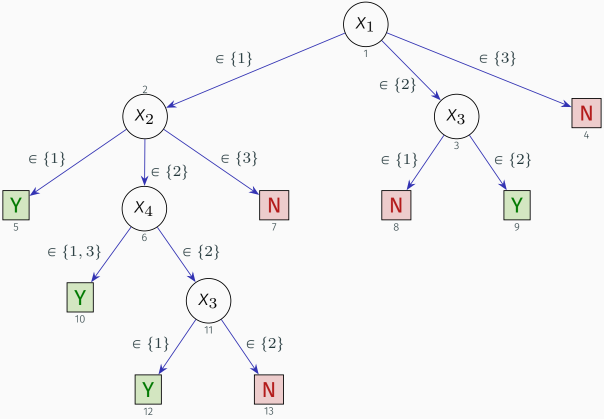
Review exercise – one AXp for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



Review exercise – one AXp for example DT

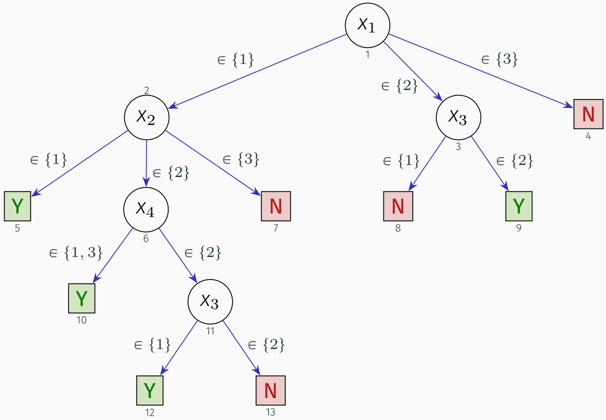
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



- Finding on AXp:

Review exercise – one AXp for example DT

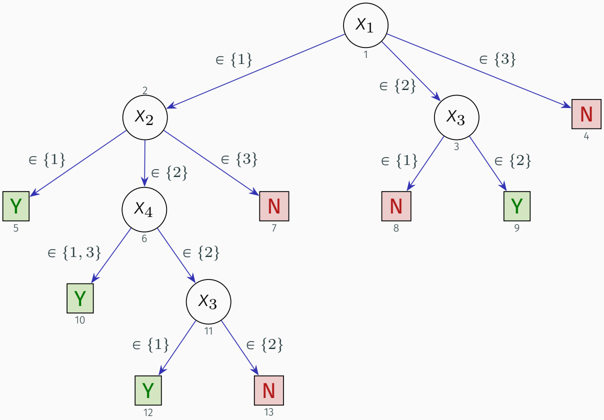
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$

Review exercise – one AXp for example DT

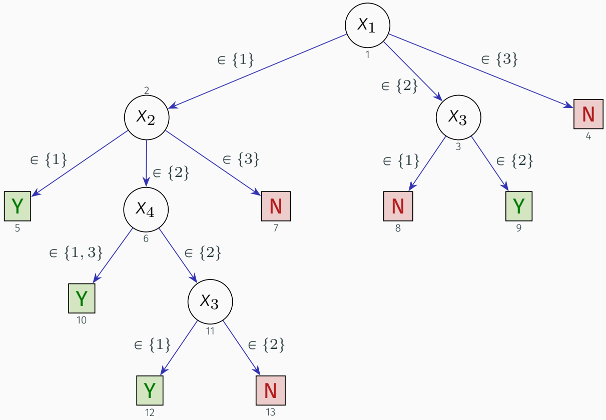
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$

Review exercise – one AXp for example DT

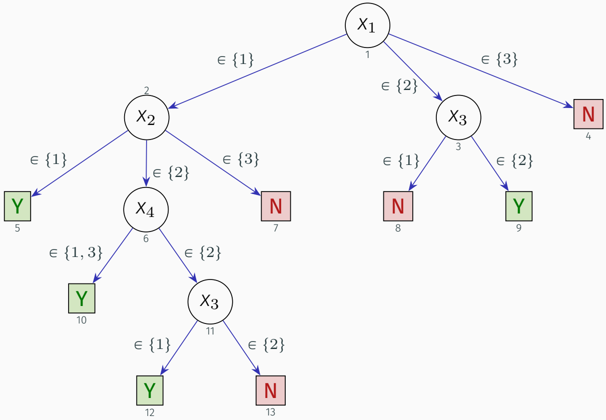
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$
  - 3rd path inconsistent: $H_3 = \{1\}$

Review exercise – one AXp for example DT

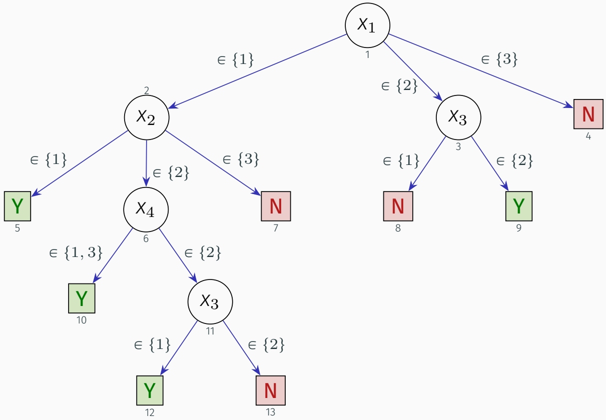
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



- Finding on AXp:
  - 1st path inconsistent: $H_1 = \{3\}$
  - 2nd path inconsistent: $H_2 = \{2\}$
  - 3rd path inconsistent: $H_3 = \{1\}$
  - 4th path inconsistent: $H_4 = \{1\}$

Review exercise – one AXp for example DT

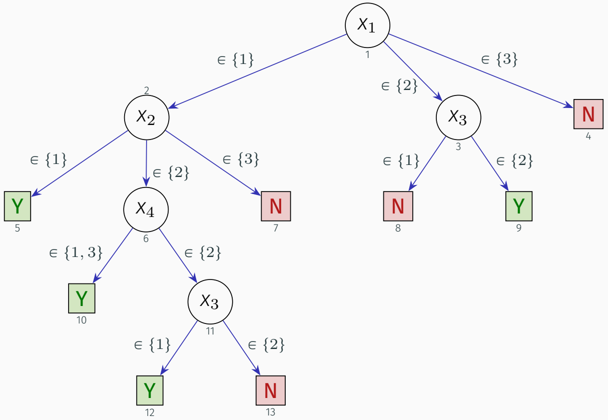
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



- Finding on AXp:
  - 1st path inconsistent: $\mathcal{H}_1 = \{3\}$
  - 2nd path inconsistent: $\mathcal{H}_2 = \{2\}$
  - 3rd path inconsistent: $\mathcal{H}_3 = \{1\}$
  - 4th path inconsistent: $\mathcal{H}_4 = \{1\}$
- AXp is MHS of $\mathcal{H}_j$ sets: $\{1, 2, 3\}$

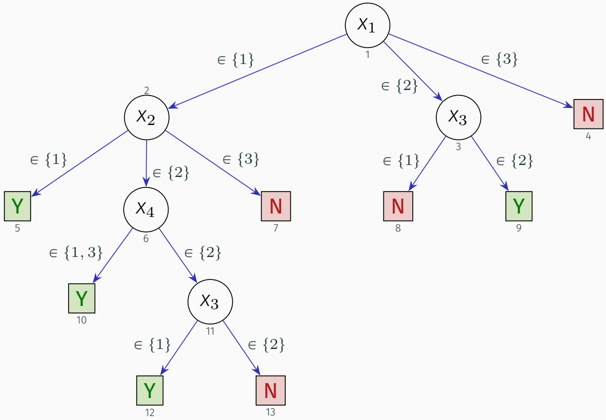
Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



Review exercise – all CXps & AXps for example DT

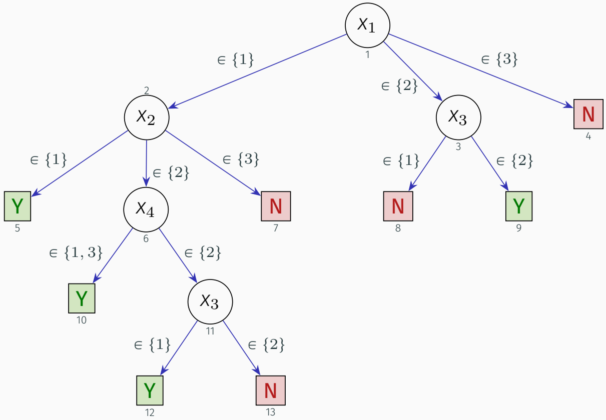
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:

Review exercise – all CXps & AXps for example DT

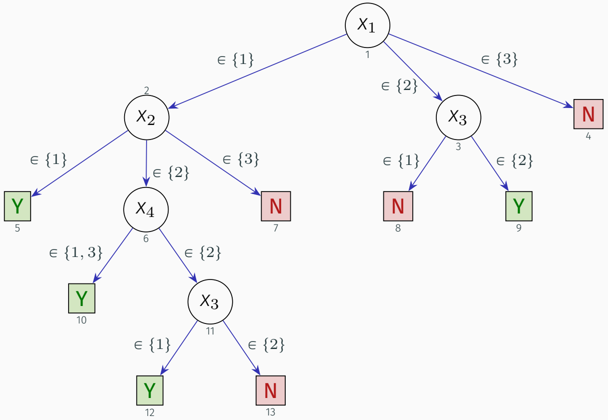
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathrm{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$

Review exercise – all CXps & AXps for example DT

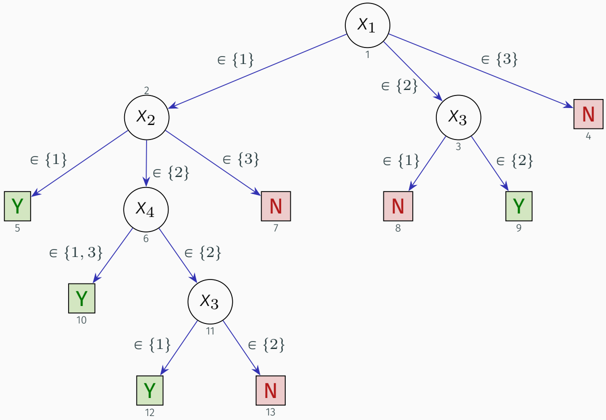
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$

Review exercise – all CXps & AXps for example DT

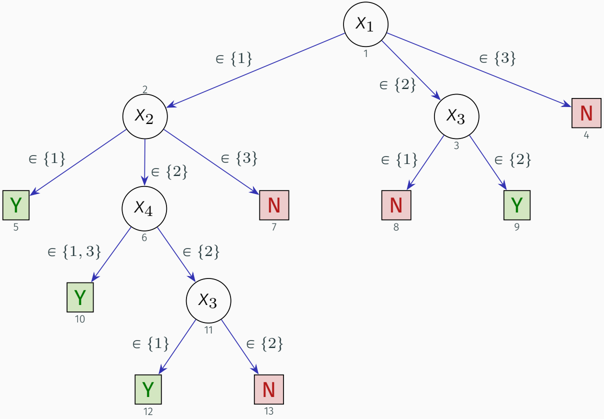
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$

Review exercise – all CXps & AXps for example DT

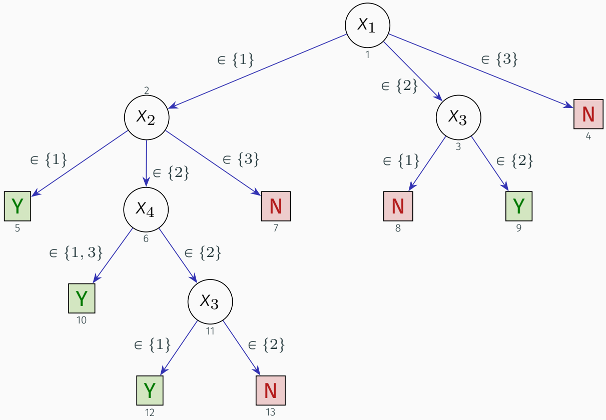
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \text{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$

Review exercise – all CXps & AXps for example DT

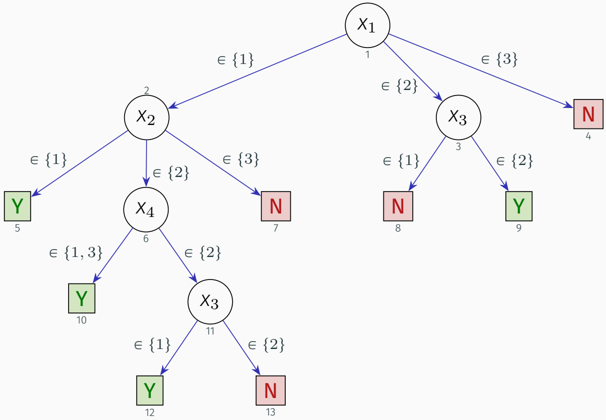
- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$

Review exercise – all CXps & AXps for example DT

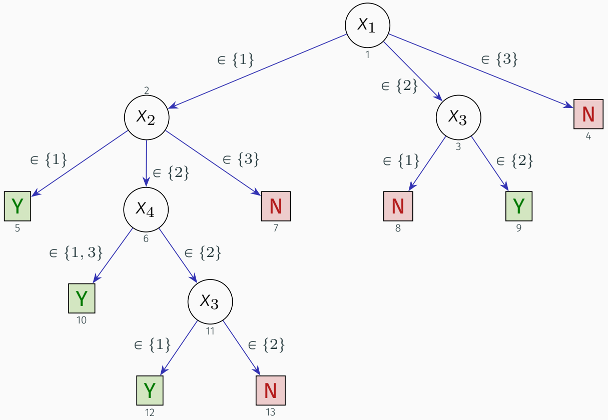
- Instance: $(\mathbf{v}, c) = ((1, 2, 1, 2), \mathsf{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$

Review exercise – all CXps & AXps for example DT

- Instance: $(\mathbf{v}, c) = ((1,2,1,2), \mathsf{Y})$



- Finding CXps:
  - 1st path: $I_1 = \{3\}$
  - 2nd path: $I_2 = \{2\}$
  - 3rd path: $I_3 = \{1\}$
  - 4th path: $I_4 = \{1\}$
  - $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\} = \mathbb{C}$
- Finding AXps:
  (i.e. all MHSes of sets in $\mathbb{C}$
  - $\mathbb{A} = \{\{1,2,3\}\}$

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \land x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \land x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \land x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \land \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
$\mathrm{R}_{\mathrm{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 0$

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \land x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \land x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \land x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \land \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
$\mathsf{R}_{\mathsf{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 0$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3

Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llllc}
\mathsf{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathsf{R}_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_7: & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathsf{R}_{\mathsf{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp:

Another review exercise – one AXp for example DL

- DL:

R_1 :	IF	$(x_1 \land x_3)$	THEN	$\kappa(\mathbf{x}) = 0$
R_2 :	ELSE IF	$(x_1 \land x_5)$	THEN	$\kappa(\mathbf{x}) = 0$
R_3 :	ELSE IF	$(x_2 \land x_4)$	THEN	$\kappa(\mathbf{x}) = 1$
R_4 :	ELSE IF	$(x_1 \land x_7)$	THEN	$\kappa(\mathbf{x}) = 0$
R_5 :	ELSE IF	$(\neg x_4 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_6 :	ELSE IF	$(\neg x_4 \land \neg x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
R_7 :	ELSE IF	$(\neg x_2 \land x_6)$	THEN	$\kappa(\mathbf{x}) = 1$
$\mathsf{R}_{\mathsf{DEF}}$:	ELSE			$\kappa(\mathbf{x}) = 0$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp: $\{1, 2\}$

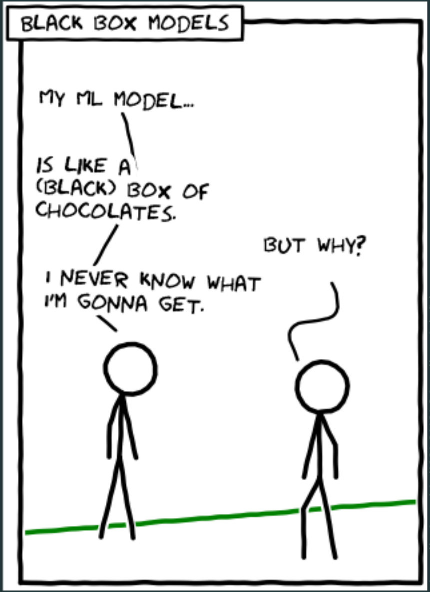
Another review exercise – one AXp for example DL

- DL:

$$
\begin{array}{llccc}
\mathrm{R}_1: & \text{IF} & (x_1 \land x_3) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_2: & \text{ELSE IF} & (x_1 \land x_5) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_3: & \text{ELSE IF} & (x_2 \land x_4) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_4: & \text{ELSE IF} & (x_1 \land x_7) & \text{THEN} & \kappa(\mathbf{x}) = 0 \\
\mathrm{R}_5: & \text{ELSE IF} & (\neg x_4 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_6: & \text{ELSE IF} & (\neg x_4 \land \neg x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_7: & \text{ELSE IF} & (\neg x_2 \land x_6) & \text{THEN} & \kappa(\mathbf{x}) = 1 \\
\mathrm{R}_{\mathrm{DEF}}: & \text{ELSE} & & & \kappa(\mathbf{x}) = 0
\end{array}
$$

- Instance: $\mathbf{v} = (0, 1, 0, 1, 0, 1, 0)$
  - The prediction is 1, due to R_3
- AXp: $\{1, 2\}$
- Quiz: write down the constraints and confirm AXp with SAT solver

Questions?



https://arxiv.org/abs/1901.01686 & http://cmx.io/edit/

References i

[ABOS22] Marcelo Arenas, Pablo Barceló, Miguel A. Romero Orth, and Bernardo Subercaseaux.
On computing probabilistic explanations for decision trees.
In *NeurIPS*, 2022.

[Alp14] Ethem Alpaydin.
Introduction to machine learning.
MIT press, 2014.

[Alp16] Ethem Alpaydin.
Machine Learning: The New AI.
MIT Press, 2016.

[BA97] Leonard A. Breslow and David W. Aha.
Simplifying decision trees: A survey.
Knowledge Eng. Review, 12(1):1–40, 1997.

[BBHK10] Michael R. Berthold, Christian Borgelt, Frank Höppner, and Frank Klawonn.
***Guide to Intelligent Data Analysis - How to Intelligently Make Sense of Real Data*, volume 42 of *Texts in Computer Science*.**
Springer, 2010.

[BFOS84] Leo Breiman, J. H. Friedman, R. A. Olshen, and C. J. Stone.
Classification and Regression Trees.
Wadsworth, 1984.

[BHO09] Christian Bessiere, Emmanuel Hebrard, and Barry O'Sullivan.
Minimising decision tree size as combinatorial optimisation.
In *CP*, pages 173–187, 2009.

[Bra20] Max Bramer.
Principles of Data Mining, 4th Edition.
Undergraduate Topics in Computer Science. Springer, 2020.

[DL01] Sašo Džeroski and Nada Lavrač, editors.
Relational data mining.
Springer, 2001.

[EG95] Thomas Eiter and Georg Gottlob.
Identifying the minimal transversals of a hypergraph and related problems.
SIAM J. Comput., 24(6):1278–1304, 1995.

[Fla12] Peter A. Flach.
Machine Learning - The Art and Science of Algorithms that Make Sense of Data.
Cambridge University Press, 2012.

[GZM20] Mohammad M. Ghiasi, Sohrab Zendehboudi, and Ali Asghar Mohsenipour.
Decision tree-based diagnosis of coronary artery disease: CART model.
Comput. Methods Programs Biomed., 192:105400, 2020.

[HIIM21] Xuanxiang Huang, Yacine Izza, Alexey Ignatiev, and Joao Marques-Silva.
On efficiently explaining graph-based classifiers.
In *KR*, November 2021.
Preprint available from `https://arxiv.org/abs/2106.01350`.

[HM23] Xuanxiang Huang and João Marques-Silva.
From decision trees to explained decision sets.
In *ECAI*, pages 1100–1108, 2023.

[HRS19] Xiyang Hu, Cynthia Rudin, and Margo Seltzer.
Optimal sparse decision trees.
In *NeurIPS*, pages 7265–7273, 2019.

[IHI+22] Yacine Izza, Xuanxiang Huang, Alexey Ignatiev, Nina Narodytska, Martin C. Cooper, and João Marques-Silva.
On computing probabilistic abductive explanations.
CoRR, abs/2212.05990, 2022.

[IHI+23] Yacine Izza, Xuanxiang Huang, Alexey Ignatiev, Nina Narodytska, Martin C. Cooper, and João Marques-Silva.
On computing probabilistic abductive explanations.
Int. J. Approx. Reason., 159:108939, 2023.

[IIM20] Yacine Izza, Alexey Ignatiev, and Joao Marques-Silva.
On explaining decision trees.
CoRR, abs/2010.11034, 2020.

[IIM22] Yacine Izza, Alexey Ignatiev, and João Marques-Silva.
On tackling explanation redundancy in decision trees.
J. Artif. Intell. Res., 75:261–321, 2022.

[IM21] Alexey Ignatiev and Joao Marques-Silva.
SAT-based rigorous explanations for decision lists.
In *SAT*, pages 251–269, July 2021.

[INAM20] Alexey Ignatiev, Nina Narodytska, Nicholas Asher, and João Marques-Silva.
From contrastive to abductive explanations and back again.
In *AIxIA*, pages 335–355, 2020.

[INM19a] Alexey Ignatiev, Nina Narodytska, and Joao Marques-Silva.
Abduction-based explanations for machine learning models.
In *AAAI*, pages 1511–1519, 2019.

References v

[INM19b] Alexey Ignatiev, Nina Narodytska, and Joao Marques-Silva.
On relating explanations and adversarial examples.
In *NeurIPS*, pages 15857–15867, 2019.

[KMND20] John D Kelleher, Brian Mac Namee, and Aoife D'arcy.
Fundamentals of machine learning for predictive data analytics: algorithms, worked examples, and case studies.
MIT Press, 2020.

[Kot13] Sotiris B. Kotsiantis.
Decision trees: a recent overview.
Artif. Intell. Rev., 39(4):261–283, 2013.

[LL17] Scott M. Lundberg and Su-In Lee.
A unified approach to interpreting model predictions.
In *NIPS*, pages 4765–4774, 2017.

[Mar22] João Marques-Silva.
Logic-based explainability in machine learning.
In *Reasoning Web*, pages 24–104, 2022.

[Mar24] Joao Marques-Silva.
Logic-based explainability: Past, present & future.
CoRR, abs/2406.11873, 2024.

[MGC+21] Joao Marques-Silva, Thomas Gerspacher, Martinc C. Cooper, Alexey Ignatiev, and Nina Narodytska.
Explanations for monotonic classifiers.
In *ICML*, pages 7469–7479, July 2021.

[Mil56] George A Miller.
The magical number seven, plus or minus two: Some limits on our capacity for processing information.
Psychological review, 63(2):81–97, 1956.

[Mil19] Tim Miller.
Explanation in artificial intelligence: Insights from the social sciences.
Artif. Intell., 267:1–38, 2019.

[MM20] João Marques-Silva and Carlos Mencía.
Reasoning about inconsistent formulas.
In *IJCAI*, pages 4899–4906, 2020.

[Mor82] Bernard M. E. Moret.
Decision trees and diagrams.
ACM Comput. Surv., 14(4):593–623, 1982.

[MSI23] Joao Marques-Silva and Alexey Ignatiev.
No silver bullet: interpretable ml models must be explained.
Frontiers in Artificial Intelligence, 6, 2023.

[PM17] David Poole and Alan K. Mackworth.
Artificial Intelligence - Foundations of Computational Agents.
CUP, 2017.

[Qui93] J Ross Quinlan.
C4.5: programs for machine learning.
Morgan-Kaufmann, 1993.

[Rei87] Raymond Reiter.
A theory of diagnosis from first principles.
Artif. Intell., 32(1):57–95, 1987.

[RM08] Lior Rokach and Oded Z Maimon.
Data mining with decision trees: theory and applications.
World scientific, 2008.

[RN10] Stuart J. Russell and Peter Norvig.
Artificial Intelligence - A Modern Approach.
Pearson Education, 2010.

[RSG16] Marco Túlio Ribeiro, Sameer Singh, and Carlos Guestrin.
”why should I trust you?”: Explaining the predictions of any classifier.
In *KDD*, pages 1135–1144, 2016.

[RSG18] Marco Túlio Ribeiro, Sameer Singh, and Carlos Guestrin.
Anchors: High-precision model-agnostic explanations.
In *AAAI*, pages 1527–1535. AAAI Press, 2018.

[SB14] Shai Shalev-Shwartz and Shai Ben-David.
Understanding Machine Learning - From Theory to Algorithms.
Cambridge University Press, 2014.

[SCD18] Andy Shih, Arthur Choi, and Adnan Darwiche.
A symbolic approach to explaining bayesian network classifiers.
In *IJCAI*, pages 5103–5111, 2018.

[VLE+16] Gilmer Valdes, José Marcio Luna, Eric Eaton, Charles B Simone, Lyle H Ungar, and Timothy D Solberg.
MediBoost: a patient stratification tool for interpretable decision making in the era of precision medicine.
Scientific reports, 6(1):1–8, 2016.

References ix

[WFHP17] Ian H Witten, Eibe Frank, Mark A Hall, and Christopher J Pal.
Data Mining.
Morgan Kaufmann, 2017.

[WMHK21] Stephan Wäldchen, Jan MacDonald, Sascha Hauch, and Gitta Kutyniok.
The computational complexity of understanding binary classifier decisions.
J. Artif. Intell. Res., 70:351–387, 2021.

[Zho12] Zhi-Hua Zhou.
Ensemble methods: foundations and algorithms.
CRC press, 2012.

[Zho21] Zhi-Hua Zhou.
Machine Learning.
Springer, 2021.