Logic-Based Explainable Artificial Intelligence

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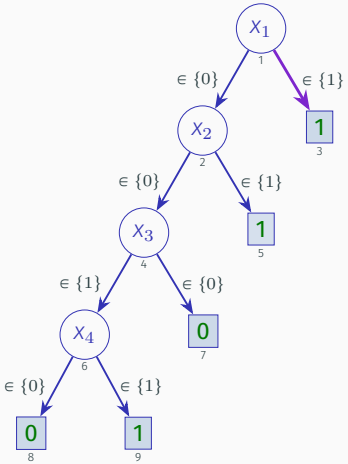
Lecture 04

Recapitulate third lecture

- Logic encoding for explaining DLs
  - And status of (in)tractability in logic-based XAI
- Query: enumeration of explanations
- Query: feature necessity, AXp & CXp
- Query: feature relevancy

Recap example

- Instance $(\mathbf{v}, c) = ((0, 0, 0, 0), 0)$



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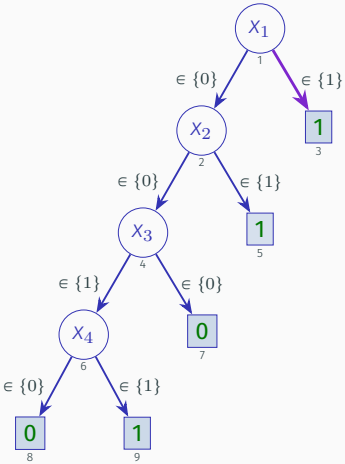

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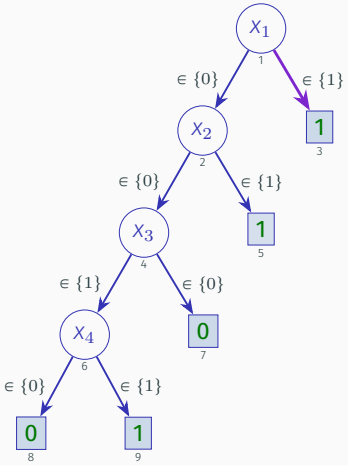

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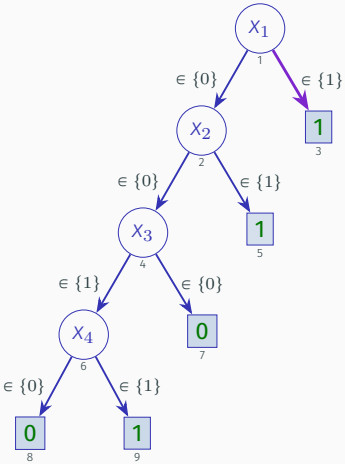

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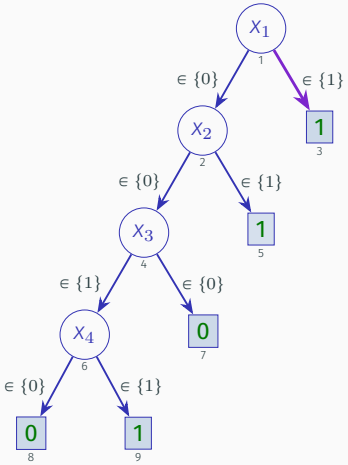

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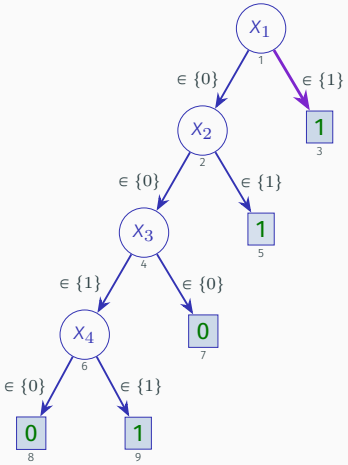

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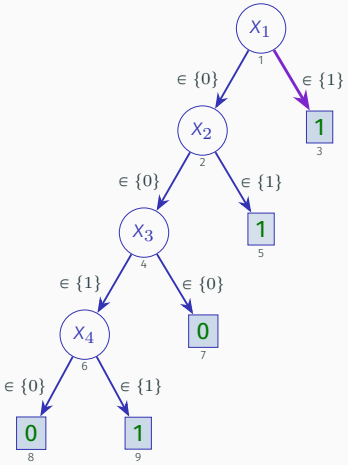

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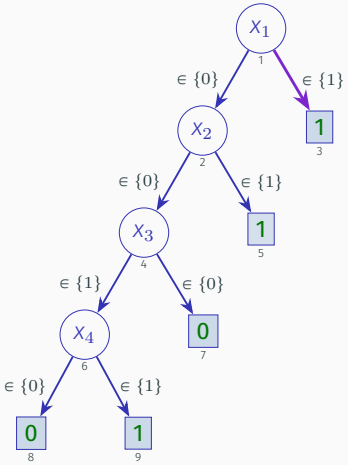

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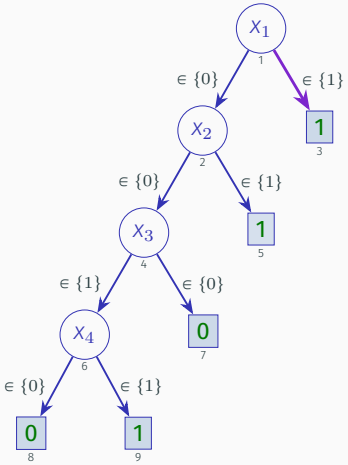

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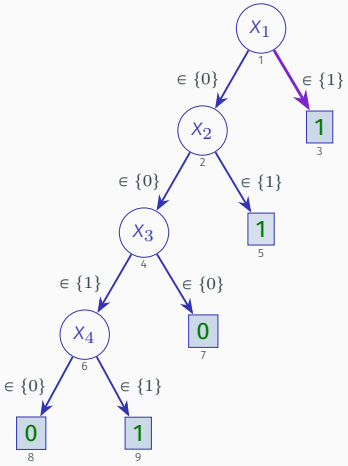
- Confirmation:
  - CXps: $\{\{1\}, \{2\}, \{3, 4\}\}$ (2 is also AXp-necessary)
  - AXps: $\{\{1, 2, 3\}, \{1, 2, 4\}\}$



Recap example – a different instance

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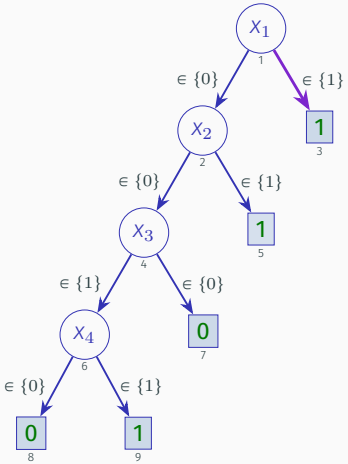

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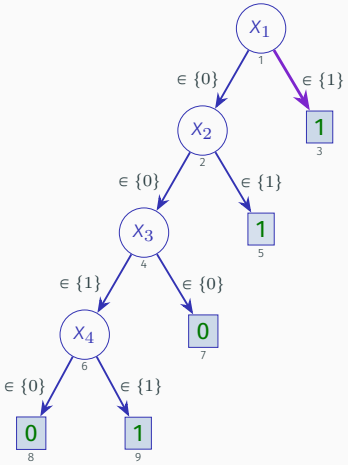

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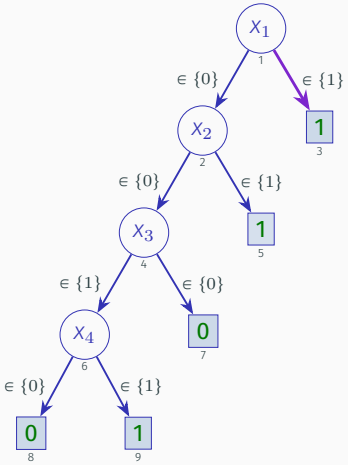

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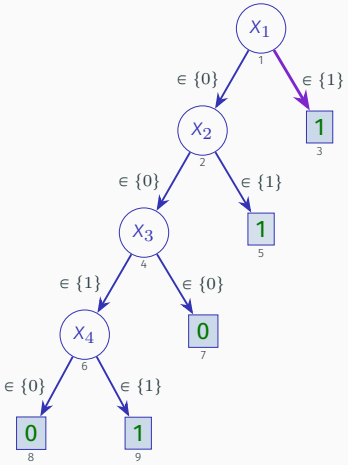

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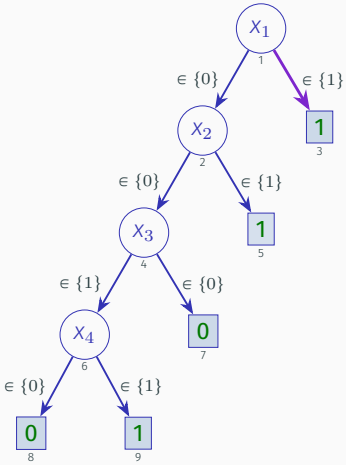

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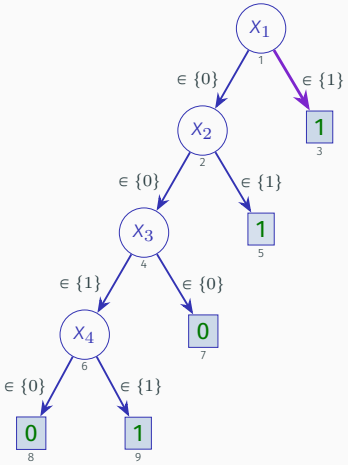

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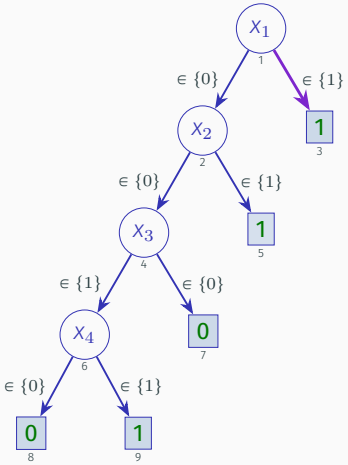

Another example – feature necessity & relevancy

- Classifier: $\mathcal{F} = \{1, 2, 3, 4, 5\}$; $\mathcal{D}_i = \{0, 1\}$, $i = 1, \ldots, 5$; $\mathcal{K} = \{0, 1\}$

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- Partially enumerate AXps/CXps, exploiting bias in enumeration

Plan for this course

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Detour: Monotonic Classification & Voting Power

Monotonically increasing boolean classifiers

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- Monotonic classifier $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$, such that each $\mathcal{D}_i = \{0, 1\}$ and $\mathcal{K} = \{0, 1\}$ are ordered (i.e. $0 < 1$), and
  - $\kappa(\mathbf{1}) = 1$;
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- Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}$ be such that $\kappa(\mathbf{v}_1) = \kappa(\mathbf{v}_2) = 1$, and $\mathbf{v}_1 \leqslant \mathbf{v}_2$
  Define the explanation problems:
  - $\mathcal{E}_1 = (\mathcal{M}, (\mathbf{v}_1, 1))$
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- Then,
  - If WAXp$(\mathcal{S}; \mathcal{E}_1)$ holds, then WAXp$(\mathcal{S}; \mathcal{E}_2)$ holds; in particular:
  - $\mathbb{A}(\mathcal{E}_{\mathbb{1}})$ contains **all** the AXps of **any** instance of the form $(\mathbf{v}_r, 1)$
    - Why?
      - Pick any explanation problem $\mathcal{E}_r$ with instance $(\mathbf{v}_r, 1)$
      - Start from $\mathbb{1} = (1, 1, \ldots, 1)$
      - Remove features that take value 0 in $\mathbf{v}_r$; we still have an WAXp
      - Then compute any AXp starting from features taking value 1 in $\mathbf{v}_r$

      $\therefore$ **Suffices to find explanations for $\mathcal{E}_{\mathbb{1}}$** (or alternatively, the global explanations for prediction 1)

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    - AXp: $\{2, 3, 4, 5\}$; Q: Is feature 6 relevant?

All AXps & all CXps...

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  - Quota q is the sum of votes required for a proposal to be approved
    - Coalitions leading to sums not less than q are **winning** coalitions

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  - Quota q is the sum of votes required for a proposal to be approved
    - Coalitions leading to sums not less than q are **winning** coalitions
  - A weighted voting game (WVG) is a tuple $[q; n_1, \ldots, n_m]$
    - Example: $[12; 4, 4, 4, 2, 2, 1]$

What is a priori voting power?

- General set-up of weighted voting games:
  - Assembly $\mathcal{A}$ of voters, with $m = |\mathcal{A}|$
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  - Problem: find a measure of importance of each voter !
    - I.e. measure the a priori voting power of each voter

An example – EEC (EU) members voting power in 1958

Coutry	Acronym	# Votes
France	F	4
Germany	D	4
Italy	I	4
Belgium	B	2
Netherlands	N	2
Luxembourg	L	1
Quota:	12	

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- WVG: $[12; 4, 4, 4, 2, 2, 1]$
- **Q**: What should be the voting power of Luxembourg?
- Can Luxembourg (L) ***matter*** for some winning coalition?
- Perhaps surprisingly, answer is **No**!
  - In 1958, Luxembourg was a **dummy** voter/player

Understanding weighted voting games

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- The corresponding classifier is:

$$\kappa(x_1, x_2, x_3, x_4, x_5, x_6) \quad := \quad \begin{cases} 1 & \text{IF } (4x_1 + 4x_2 + 4x_3 + 2x_4 + 2x_5 + x_6 \geqslant 12) \\ 0 & \text{otherwise} \end{cases}$$

which we have seen before! E.g. $\{2, 3, 4, 5\}$ is an AXp & feature 6 (L) is irrelevant

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- WVG: [21; 12, 9, 4, 4, 1, 1, 1]

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- Recently, we have devised ways of correcting SHAP scores [LHMS24]
- In turn, this revealed novel connections between logic-based XAI and a priori voting power [LHAMS24]
- Homework:
  - Create your own weighted voting games;
  - Compute the sets of AXps and CXps; and
  - Assess the importance of features and how they compare to each other

Unit #06

Advanced Topics

Recurring challenge: how to explain highly-complex ML models?

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  - And training data is nothing but a sample
    - From which ML models are learned!
  - Here is an idea:
    - Adopt symbolic (and so, rigorous) sample-based XAI

[Amg23, CA23, ACD24]

An example...

Sample:

x_1	x_2	x_3	x_4	$\kappa(\cdot)$
0	0	0	0	0
0	0	1	0	0
0	0	1	1	0
0	1	1	0	0
1	0	0	0	1
1	1	0	0	1
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Instance: $((1, 1, 1, 1), 1)$

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- How to explain prediction given only the sample
- If $x_1 = 1$, then prediction is 1 (given the sample)
- Sample-based AXp (sbAXp): $\{1\}$

Definitions in sample-based XAI – replace feature space with sample...

[MSLLM25]

- Let $\mathbb{S} \subseteq \mathbb{F}$ denote a sample, and let instance be $(\mathbf{v}, c)$
- Then, for $\mathcal{X} \subseteq \mathcal{F}$,

$$\text{sbWAXp}(\mathcal{X}) \;\coloneqq\; \forall(\mathbf{x} \in \mathbb{S}).\, \left(\bigwedge\nolimits_{i \in \mathcal{X}} x_i = v_i\right) \rightarrow (\kappa(\mathbf{x}) = c)$$

- And, for $\mathcal{Y} \subseteq \mathcal{F}$,

$$\text{sbWCXp}(\mathcal{Y}) \;\coloneqq\; \exists(\mathbf{x} \in \mathbb{S}).\, \left(\bigwedge\nolimits_{i \in \mathcal{F} \setminus \mathcal{Y}} x_i = v_i\right) \land (\kappa(\mathbf{x}) \neq c)$$

- sbAXps (resp. sbCXps) are the subset-minimal sets that respect the above definition for sbWAXp (resp. sbWCXp)

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  - Rigorous alternative to Anchor & variants

Approach for computing sbCXps & sbAXps

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- Set of sbAXps (by MHS duality): $\mathbb{A} = \{\{1\},\{2,4\}\}$

Some recent results

[MSLLM25]

- MHS duality holds for sample-based explanations:
  - $\mathcal{Y} \subseteq \mathcal{F}$ is sbCXp iff it is a MHS of set of sbAXps
  - $\mathcal{X} \subseteq \mathcal{F}$ is sbAXp iff it is a MHS of set of sbCXps
- Number of sb(W)CXps is linear on $|\mathbb{S}|$
- Number of sb(W)AXps can be exponentially large on $|\mathbb{S}|$
- Additional results:

Problem	Complexity	
	Total	Given sbCXps
All sbCXps	$\mathcal{O}(mn^2)$	—
One sbCXp	$\mathcal{O}(mn)$	$\mathcal{O}(1)$
One (smallest) sbCXp	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$
One sbAXp	$\mathcal{O}(mn)$	$\mathcal{O}(mn)$
Feature relevancy	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$
sbAXp-necessity	$\mathcal{O}(mn^2)$	$\mathcal{O}(mn)$
sbCXp-necessity	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$

- Complexity-wise:
  - Deciding the existence of an sbAXp of size no larger than k is NP-complete.
  - sbAXp enumeration corresponds to hypergraph transversal

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- An example:
  - Sample:

Entry	x_1	x_2	$\kappa(\cdot)$
1	0	1	0
2	1	0	1
3	0	0	2

  - Instance 1: $((0,1),0)$
  - Instance 2: $((1,0),1)$

AXp(s) for $((0, 1), 0)$

Sample:

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1	0	1
0	0	2

Instance: $((0, 1), 0)$

Points $\mathbf{x}$ w/ $\kappa(\mathbf{x}) \neq c$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Weak sbCXps:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

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  - Meaning: IF $(x_2 = 1)$ THEN $\kappa(\mathbf{x}) = 0$

AXp(s) for $((1, 0), 1)$

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The problem of lack of coherency

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  - Points consistent with AXp: $\{(0, 1), (1, 1)\}$
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- Open topic of research...

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Additional Topics

General definition of prediction sufficiency

- Instance $(\mathbf{v}, c)$
- Let $\mathcal{S} \subseteq \mathcal{F}$:
  - Recall,

$$\Upsilon(\mathcal{S}; \mathbf{v}) = \{\mathbf{x} \in \mathbb{F} \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

- $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Upsilon(\mathcal{S}; \mathbf{v})) \rightarrow (\sigma(\mathbf{x}))$$

  - **Obs:** a WAXp is just one possible example
- But there are other ways to study prediction sufficiency:
  - One can envision defining other sets of points Γ, parameterized by $\mathcal{E} = (\mathcal{M}, (\mathbf{v}, c))$; $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Gamma(\mathcal{S}; \mathcal{E})) \rightarrow (\sigma(\mathbf{x}))$$

    - And one can also envision generalizations of σ!

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Towards more expressive explanations – inflated explanations

[IISM24]

- Recall:

$$\text{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}).\ \bigwedge\nolimits_{j \in \mathcal{X}} (x_j = v_j) \rightarrow (\kappa(\mathbf{x}) = c)$$

  - For non-boolean features, use of $=$ may convey little information, e.g. with real-valued features, having $x_1 = 1.157$ does not help in understanding what values of feature 1 are also acceptable

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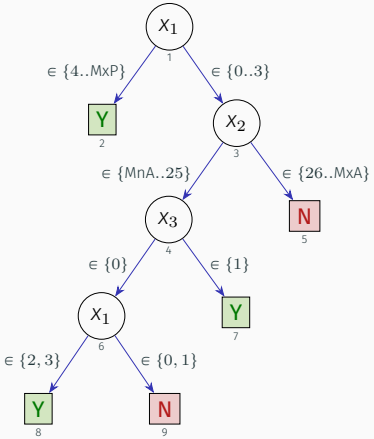
- Inflated explanations allow for more expressive literals, i.e. $=$ replaced with $\in$, and individual values replaced by ranges of values
  - Operational definition: Given an AXp, expand set of values of each feature, in some chosen order, such that the set of picked features remains unchanged

Inflated explanations – an example

[IIM22]

- Explanation for $((2, 20, 0), \mathsf{Y})$? (Obs: $\mathsf{MnA} = 18; \mathsf{MxP} > 4$)



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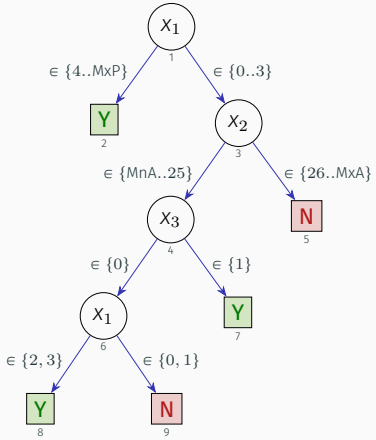

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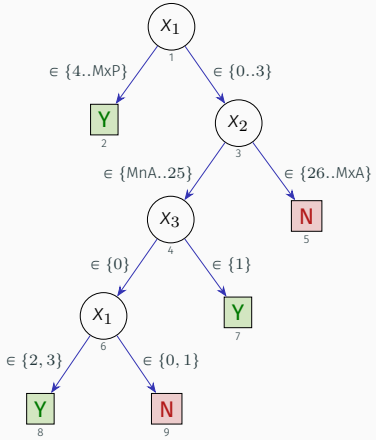

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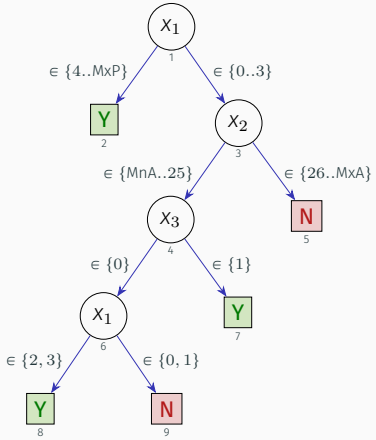

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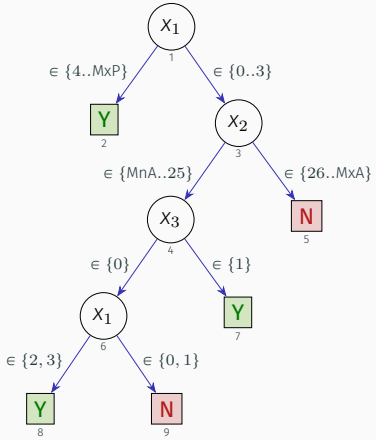

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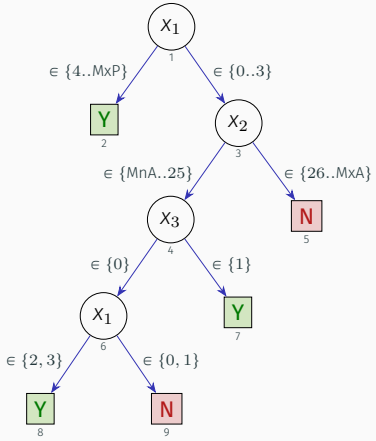

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Approach

- Compute AXp $\mathcal{X}$
- For each feature:
  - Categorical: iteratively add elements to literal
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- **Obs:** More complex alternative is to find AXp and expand domains simultaneously
  - This is conjectured to change the complexity class of finding one explanation

Another fun example (from 2019) – (inflated) AXp?

[CdPA+19]

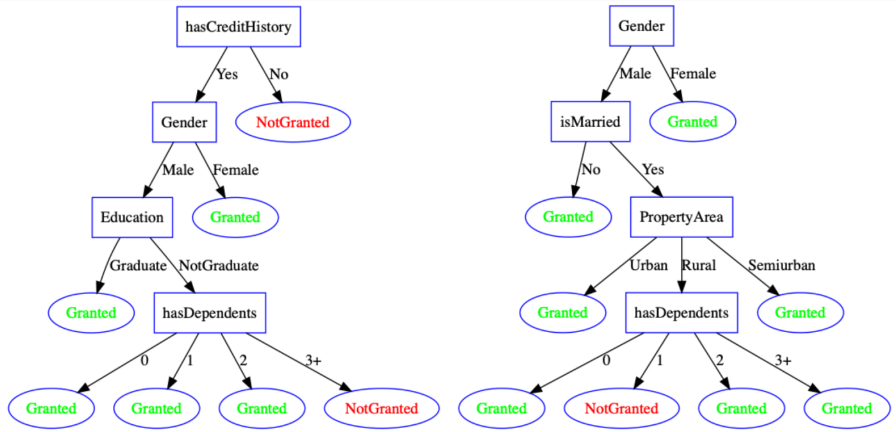

Fig. 2: Decision trees of size 'small' in the loan domain, extracted without (left) and with (right) a domain ontology. As it can be seen the features used in the creation of the conditions in the split nodes are different.

Instance: Gender=Male, Education=NotGraduate, hasCreditHistory=Yes, isMarried=Yes, PropertyArea=Rural, isSelfEmployed=No, hasDependents=2

Another fun example (now from 2020/21) – (inflated) AXp?

[CWBdPM21]

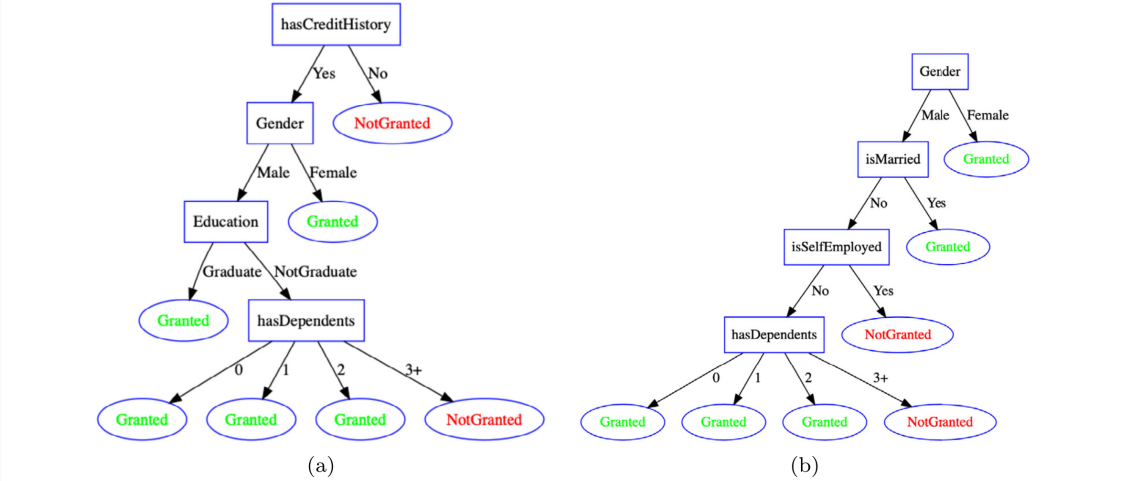

Fig. 2. Decision trees of size 'small' in the loan domain, extracted without (a) and with (b) a domain ontology. It can be seen that the use of an ontology leads to different features appearing in the split nodes. For instance, in the ontology used to build tree (b) the concept *Gender* is more abstract than *isMarried* and *isSelf Employed* has thus a lower information content according to Definition 3.1. Concepts with lower information content are favoured as conditions for split nodes in the tree according to Definition 3.2, which leads to *Gender* being used first by TREPAN-Reloaded when it generated the split nodes of tree (b). Furthermore, the ontology does not include concepts associated to *hasCreditHistory* and *Education*, which are therefore not considered in the construction of tree (b).

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Additional Topics

Not all inputs may be possible – input constraints

[GR22, YIS+23]

- The (implicit) assumption that all inputs are possible is often unrealistic
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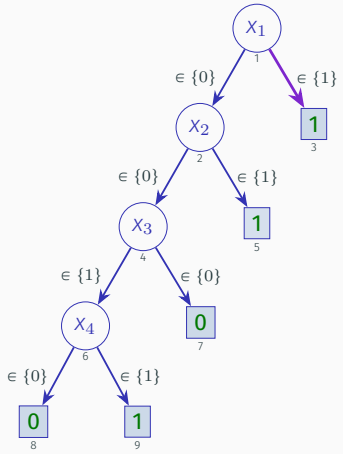
- Constrained AXps/CXps find other applications!

An example

- Instance: $((1, 1, 1, 1), 1)$
- Unconstrained AXps:



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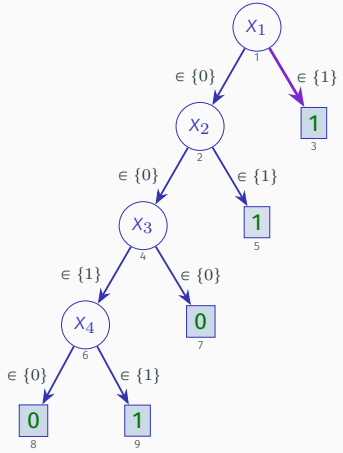

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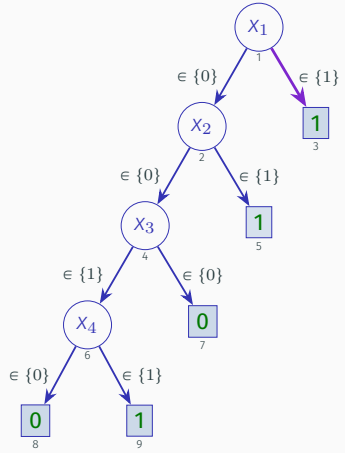

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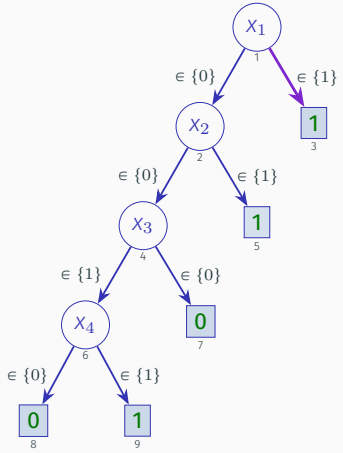

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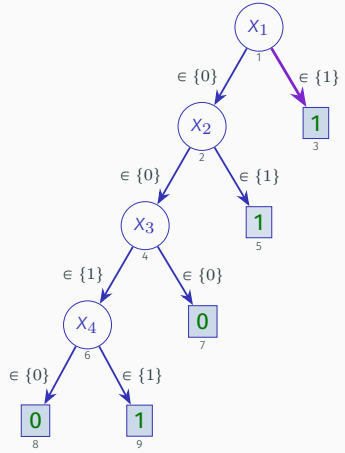

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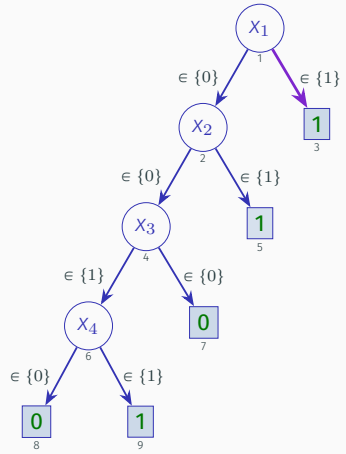

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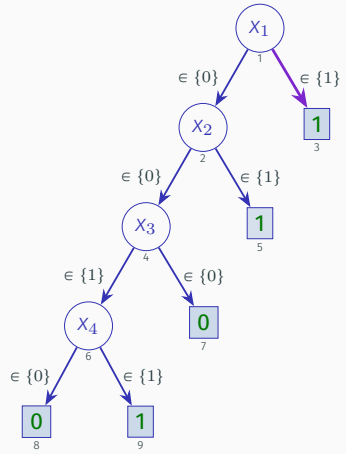

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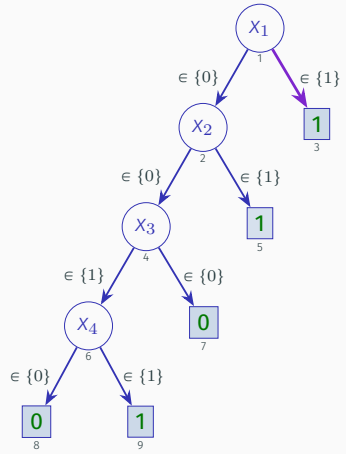

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Changing Assumptions (in Plain Logic-Based XAI)

Inflated Explanations

Constrained Explanations

Distance-Restricted Explanations

Probabilistic Explanations

Additional Topics

How to tackle poor performance on NNs?

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  - Q: can we relate AXps with adversarial examples?
  - Obs: we already proved some basic (duality) properties for global explanations [INM19b]
- Change definition of WAXp/WCXp to account for l_p distance to $\mathbf{v}$:

$$\forall(\mathbf{x} \in \mathbb{F}).\ \left[\bigwedge\nolimits_{j \in \mathcal{X}}(x_j = v_j) \land \left(\|\mathbf{x} - \mathbf{v}\|_{l_p} \leqslant \epsilon\right)\right] \rightarrow (\sigma(\mathbf{x}))$$

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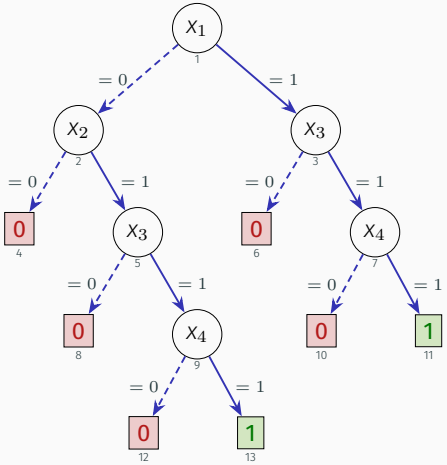
  - Norm l_p is arbitrary, e.g. Hamming, Manhattan, Euclidean, etc.
  - Distance-restricted explanations: $\eth$AXp/$\eth$CXp

An example – DT & instance $((1, 1, 1, 1), 1)$



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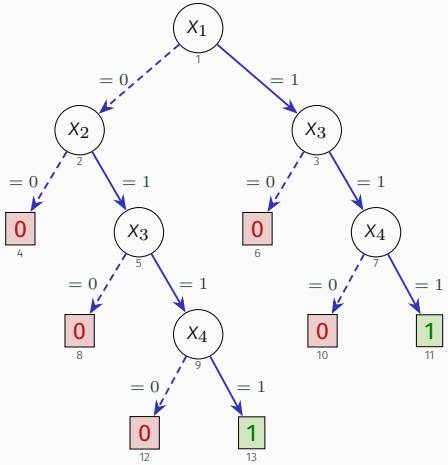

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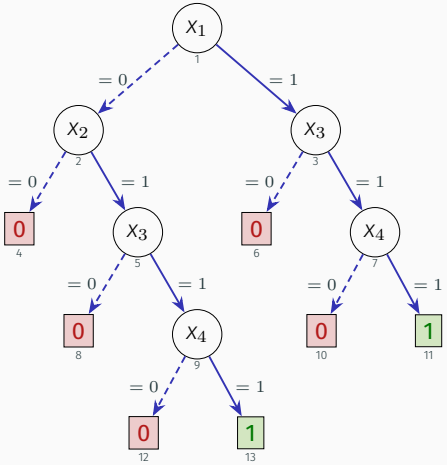

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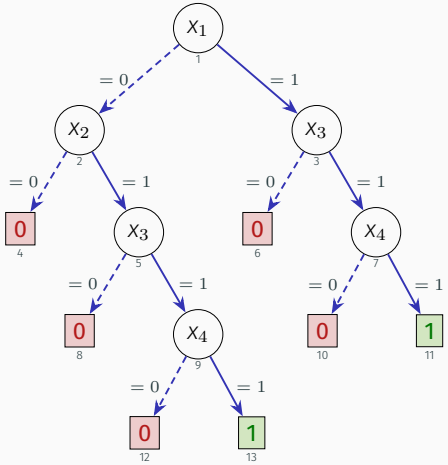

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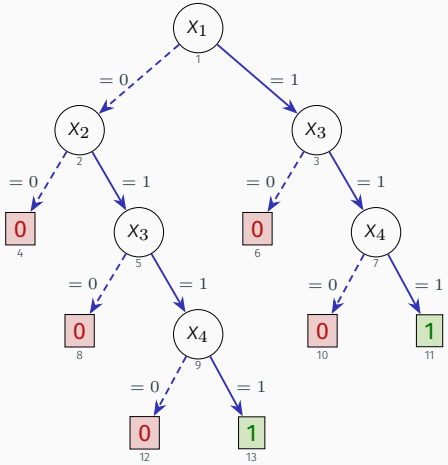

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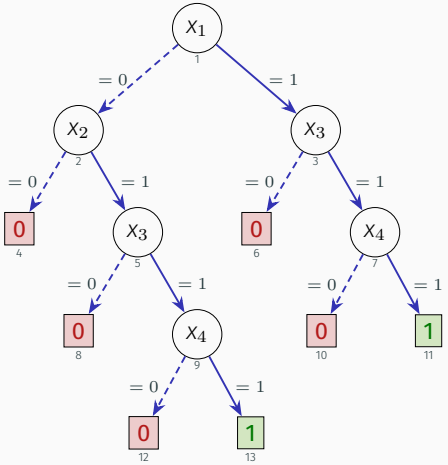

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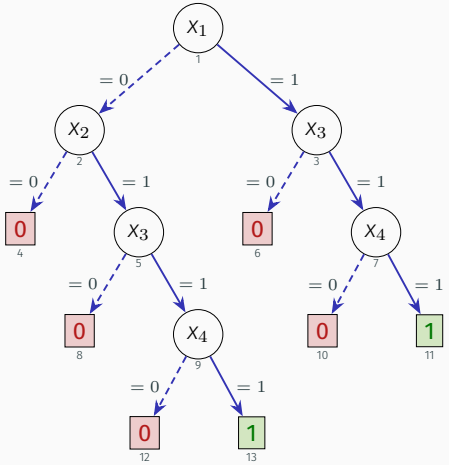

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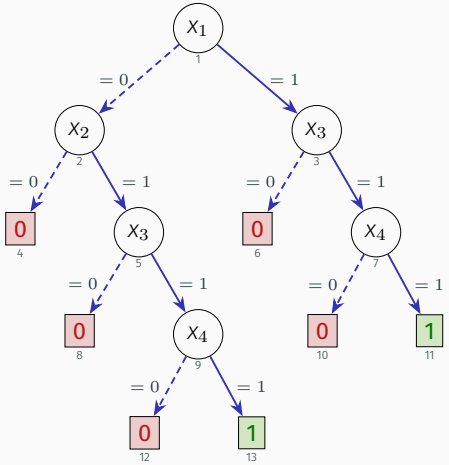

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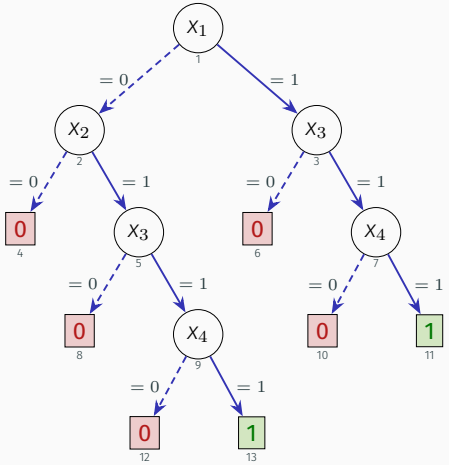

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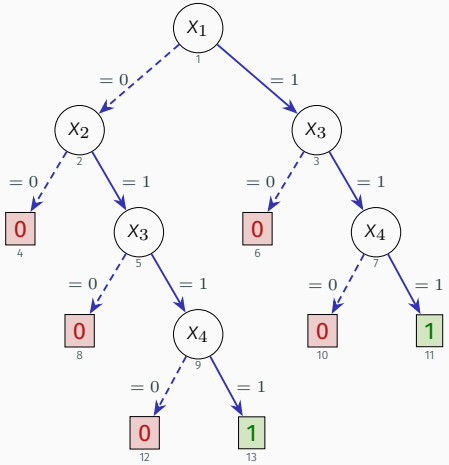

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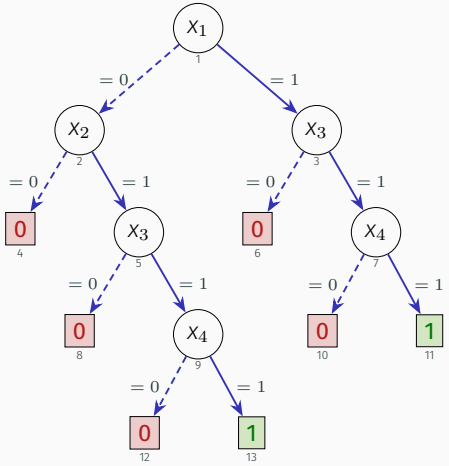

- Given ϵ, larger adversarial examples are excluded

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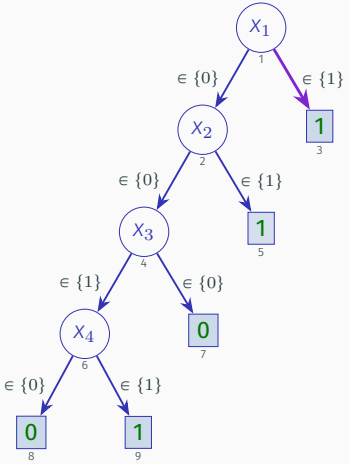

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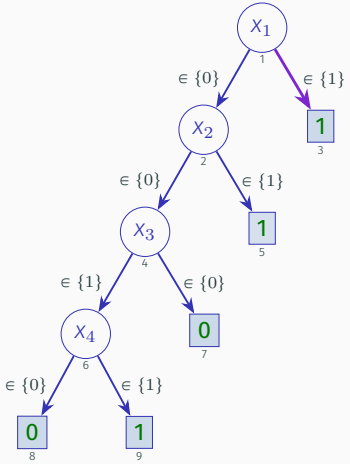

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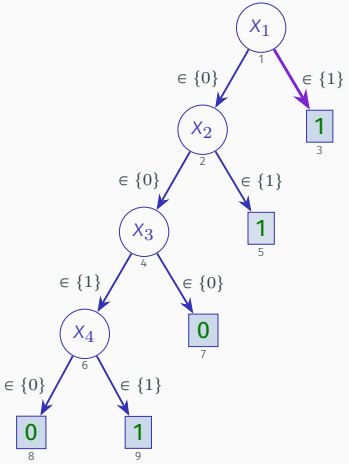

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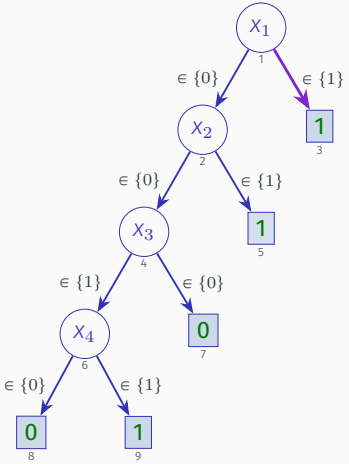

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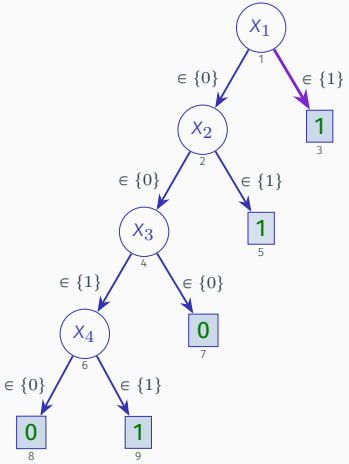

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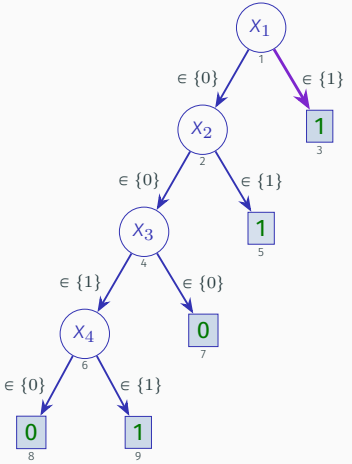

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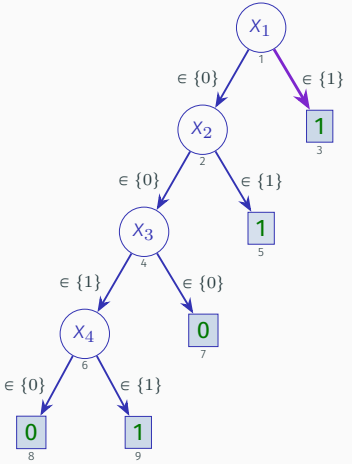

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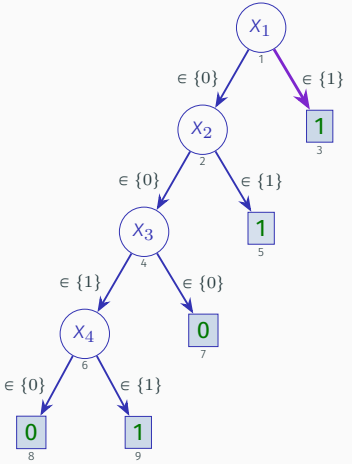

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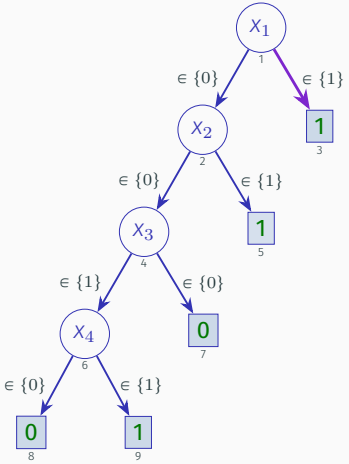

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- Clear scalability improvements for explaining NNs (see next) [HM23a, WWB23, IHM+24a, IHM+24b]

Basic algorithm

[HM23a]

Input: Arguments: ϵ; Parameters: $\mathcal{E}$, p
Output: One $\mathfrak{d}$AXp $\mathcal{S}$

1: **function** FindAXpDel($\epsilon; \mathcal{E}, p$)
2: $\quad \mathcal{S} \leftarrow \mathcal{F}$ $\quad \triangleright$ Initially, no feature is allowed to change
3: $\quad$ **for** $i \in \mathcal{F}$ **do** $\quad \triangleright$ Invariant: $\mathfrak{d}$WAXp($\mathcal{S}$)
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- **Limitation:** Running time grows with number of features

Results for NNs in 2023 (using Marabou [KHI+19])

[HM23a]

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
		$\epsilon = 0.1$				$\epsilon = 0.05$			
ACASXU_1_5	#1	3	5	185.9	0	2	5	113.8	0
	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
	#2	4	5	311.3	0	1	5	6958.6	1
	#3	2	5	7756.5	1	1	5	7807.6	1
ACASXU_4_1	#1	2	5	12413.0	2	1	5	5090.5	1
	#2	1	5	5035.1	1	0	5	2335.6	0
	#3	4	5	1237.3	0	4	5	1143.4	0
ACASXU_4_2	#1	4	5	15.9	0	4	5	12.1	0
	#2	3	5	1507.6	0	1	5	111.3	0
	#3	2	5	5641.6	2	0	5	1639.1	0

Results for NNs in 2023 (using Marabou [KHI+19])

[HM23a]

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
		$\epsilon = 0.1$				$\epsilon = 0.05$			
ACASXU_1_5	#1	3	5	185.9	0	2	5	113.8	0
	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
	#2	4	5	311.3	0	1	5	6958.6	1
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ACASXU_4_1	#1	2	5	12413.0	2	1	5	5090.5	1
	#2	1	5	5035.1	1	0	5	2335.6	0
	#3	4	5	1237.3	0	4	5	1143.4	0
ACASXU_4_2	#1	4	5	15.9	0	4	5	12.1	0
	#2	3	5	1507.6	0	1	5	111.3	0
	#3	2	5	5641.6	2	0	5	1639.1	0

Scales to a few **hundred** neurons

Recent improvements

Input: Arguments: ϵ; Parameters: $\mathcal{E}$, p
Output: One $\mathfrak{d}$AXp $\mathcal{S}$

1: **function** FindAXpDel($\epsilon; \mathcal{E}, p$)
2: $\quad \mathcal{S} \leftarrow \mathcal{F}$ ▷ Initially, no feature is allowed to change
3: $\quad$ **for** $i \in \mathcal{F}$ **do** ▷ Invariant: $\mathfrak{d}$WAXp($\mathcal{S}$)
4: $\quad\quad \mathcal{S} \leftarrow \mathcal{S} \backslash \{i\}$
5: $\quad\quad$ outc $\leftarrow$ FindAdvEx($\epsilon, \mathcal{S}; \mathcal{E}, p$)
6: $\quad\quad$ **if** outc **then**
7: $\quad\quad\quad \mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$
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  - Exploit parallelization for other algorithms, e.g. dichotomic search [IHM+24b]
- However, to decide whether $\mathcal{S}$ is an AXp, we can exploit parallelization:
  - Recall: $\text{AXp}(\mathcal{X}) := \text{WAXp}(\mathcal{X}) \land \forall(t \in \mathcal{X}).\neg\text{WAXp}(\mathcal{X}\backslash\{t\})$
  - Each $\neg$WAXp($\cdot$) (and also WAXp($\cdot$)) check can be run in parallel!
  - Do this opportunistically, i.e. when set $\mathcal{S}$ is expected to be AXp [IHM+24b]

More recent results (from 2024)...

[IHM+24a, IHM+24b]

Model	Deletion							SwiftXplain						
	avgC	nCalls	Len	Mn	Mx	avg	TO	avgC	nCalls	Len	FD%	Mn	Mx	avg
gtsrb-dense	0.06	1024	448	52.0	76.3	63.1	0	0.23	54	447	77.4	10.8	14.0	12.2
gtsrb-convSmall	0.06	1024	309	59.2	82.6	65.1	0	0.22	74	313	39.7	15.1	19.5	16.2
gtsrb-conv	—	—	—	—	—	—	100	96.49	45	174	33.2	3858.7	6427.7	4449.4
mnist-denseSmall	0.28	784	177	190.9	420.3	220.4	0	0.77	111	180	15.5	77.6	104.4	85.1
mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

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Largest for MNIST: **10142** neurons

Largest for GSTRB: **94308** neurons

Outline – Unit #06

Sample-Based Explanations

Changing Assumptions (in Plain Logic-Based XAI)

Inflated Explanations

Constrained Explanations

Distance-Restricted Explanations

Probabilistic Explanations

Additional Topics

Probabilistic (formal) explanations

[WMHK21, IIN+22, IHI+22, ABOS22, IHI+23, IMM24]

- Explanation size is critical for human understanding [Mil56]
- Probabilistic explanations provide smaller explanations, by trading off rigor of explanation by explanation size

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- Definition of weak probabilistic AXp $\mathcal{X} \subseteq \mathcal{F}$:

$$\text{WPAXp}(\mathcal{X}) \quad := \quad \mathbf{Pr}(\kappa(\mathbf{x}) = c) \,|\, \mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}) \geqslant \delta$$

  - Obs: $\mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}$ requires points $\mathbf{x} \in \mathbb{F}$ to match the values of $\mathbf{v}$ for the features dictated by $\mathcal{X}$
  - Obs: for $\delta = 1$ we obtain a WAXp

Definitions

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- Weak probabilistic AXp (WPAXp):

$$\text{WeakPAXp}(\mathcal{X}; \mathbb{F}, \kappa, \mathbf{v}, c, \delta) := \Pr_{\mathbf{x}}(\kappa(\mathbf{x}) = c \,|\, \mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}) \geqslant \delta := \frac{|\{\mathbf{x} \in \mathbb{F} : \kappa(\mathbf{x}) = c \land (\mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}})\}|}{|\{\mathbf{x} \in \mathbb{F} : (\mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}})\}|} \geqslant \delta$$

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- Locally-minimal PAXp (LmPAXp): – may differ from PAXp due to non-monotonicity

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  - Recent dedicated algorithms for simple ML models [IHI+23]
  - Recent approximate algorithms for complex ML models [IMM24]

Results for decision trees

Dataset	DT		Path			δ	MinPAXp					LmPAXp						Anchor						
							Length			Prec	Time	Length			Prec	$m_{\subseteq}$	Time	D	Length				Prec	Time
	N	A	M	m	avg		M	m	avg	avg	avg	M	m	avg	avg		avg		M	m	avg	$F_{\notin P}$	avg	avg
adult	1241	89	14	3	10.7	100	11	3	6.8	100	2.34	11	3	6.9	100	100	0.00	d	12	2	7.0	26.8	76.8	0.96
						95	11	3	6.2	98.4	5.36	11	3	6.3	98.6	99.0	0.01	u	12	3	10.0	29.4	93.7	2.20
						90	11	2	5.6	94.6	4.64	11	2	5.8	95.2	96.4	0.01							
dermatology	71	100	13	1	5.1	100	12	1	4.4	100	0.35	12	1	4.4	100	100	0.00	d	31	1	4.8	58.1	32.9	3.10
						95	12	1	4.1	99.7	0.37	12	1	4.1	99.7	99.3	0.00	u	34	1	13.1	43.2	87.2	25.13
						90	11	1	4.0	98.8	0.35	11	1	4.0	98.8	100	0.00							
kr-vs-kp	231	100	14	3	6.6	100	12	2	4.8	100	0.93	12	2	4.9	100	100	0.00	d	36	2	7.9	44.8	69.4	1.94
						95	11	2	3.9	98.1	0.97	11	2	4.0	98.1	100	0.00	u	12	2	3.6	16.6	97.3	1.81
						90	10	2	3.2	95.4	0.92	10	2	3.3	95.4	99.0	0.00							
letter	3261	93	14	4	11.8	100	12	4	8.2	100	16.06	11	4	8.2	100	100	0.00	d	16	3	13.2	43.1	71.3	12.22
						95	12	4	8.0	99.6	18.28	11	4	8.0	99.5	100	0.00	u	16	3	13.7	47.3	66.3	10.15
						90	12	4	7.7	97.7	16.35	10	4	7.8	97.8	100	0.00							
soybean	219	100	16	3	7.3	100	14	3	6.4	100	0.92	14	3	6.5	100	100	0.00	d	35	2	8.6	55.4	33.6	5.43
						95	14	3	6.4	99.8	0.95	14	3	6.4	99.8	100	0.00	u	35	3	19.2	66.0	75.0	38.96
						90	14	3	6.1	98.1	0.94	14	3	6.1	98.2	98.5	0.00							
spambase	141	99	14	3	8.5	0	12	3	7.4	100	1.23	12	3	7.5	100	100	0.01	d	38	2	6.3	65.3	63.3	24.12
						95	9	1	3.7	96.1	2.16	9	1	3.8	96.5	100	0.01	u	57	3	28.0	86.2	65.3	834.70
						90	6	1	2.4	92.4	2.15	8	1	2.4	92.2	100	0.01							

Results for naive Bayes classifiers

Dataset	(#F	#I)	NBC	AXp		LmPAXp$_{\leqslant 9}$				LmPAXp$_{\leqslant 7}$				LmPAXp$_{\leqslant 4}$			
			A%	Length	δ	Length	Precision	W%	Time	Length	Precision	W%	Time	Length	Precision	W%	Time
adult	(13	200)	81.37	6.8± 1.2	98	6.8± 1.1	100± 0.0	100	0.003	6.3± 0.9	99.61± 0.6	96	0.023	4.8± 1.3	98.73± 0.5	48	0.059
					95	6.8± 1.1	99.99± 0.2	100	0.074	5.9± 1.0	98.87± 1.8	99	0.058	3.9± 1.0	96.93± 1.1	80	0.071
					93	6.8± 1.1	99.97± 0.4	100	0.104	5.7± 1.3	98.34± 2.6	100	0.086	3.4± 0.9	95.21± 1.6	90	0.093
					90	6.8± 1.1	99.95± 0.6	100	0.164	5.5± 1.4	97.86± 3.4	100	0.100	3.0± 0.8	93.46± 1.5	94	0.103
agaricus	(23	200)	95.41	10.3± 2.5	98	7.7± 2.7	99.12± 0.8	92	0.593	6.4± 3.0	98.75± 0.6	87	0.763	6.0± 3.1	98.67± 0.5	29	0.870
					95	6.9± 3.1	97.62± 2.1	95	0.954	5.3± 3.2	96.59± 1.6	92	1.273	4.8± 3.3	96.24± 1.2	55	1.217
					93	6.5± 3.1	96.65± 2.8	95	1.112	4.8± 3.1	95.38± 1.9	93	1.309	4.3± 3.1	94.92± 1.3	64	1.390
					90	5.9± 3.3	94.95± 4.1	96	1.332	4.0± 3.0	92.60± 2.8	95	1.598	3.6± 2.8	92.08± 1.7	76	1.830
chess	(37	200)	88.34	12.1± 3.7	98	8.1± 4.1	99.27± 0.6	64	0.383	5.9± 4.9	98.70± 0.4	64	0.454	5.7± 5.0	98.65± 0.4	46	0.457
					95	7.7± 3.8	98.51± 1.4	68	0.404	5.5± 4.4	97.90± 0.9	64	0.483	5.3± 4.5	97.85± 0.8	46	0.478
					93	7.3± 3.5	97.56± 2.4	68	0.419	5.0± 4.1	96.26± 2.2	64	0.485	4.8± 4.1	96.21± 2.1	64	0.493
					90	7.3± 3.5	97.29± 2.9	70	0.413	4.9± 4.0	95.99± 2.6	64	0.483	4.8± 4.0	95.93± 2.5	64	0.543
vote	(17	81)	89.66	5.3± 1.4	98	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.95± 0.2	100	0.007	4.6± 1.1	99.60± 0.4	64	0.014
					95	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.93± 0.3	100	0.008	4.1± 1.0	98.25± 1.7	64	0.018
					93	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.1± 0.9	98.10± 1.9	64	0.018
					90	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.0± 1.2	97.24± 3.1	64	0.022
kr-vs-kp	(37	200)	88.07	12.2± 3.9	98	7.8± 4.2	99.19± 0.5	64	0.387	6.5± 4.7	98.99± 0.4	64	0.427	6.1± 4.9	98.88± 0.3	43	0.457
					95	7.3± 3.9	98.29± 1.4	64	0.416	6.0± 4.3	97.89± 1.1	64	0.453	5.5± 4.5	97.79± 0.9	43	0.462
					93	6.9± 3.5	97.21± 2.5	69	0.422	5.6± 3.8	96.82± 2.2	64	0.448	5.2± 4.0	96.71± 2.1	43	0.468
					90	6.8± 3.5	96.65± 3.1	69	0.418	5.4± 3.8	95.69± 3.0	64	0.468	5.0± 4.0	95.59± 2.8	61	0.487
mushroom	(23	200)	95.51	10.7± 2.3	98	7.5± 2.4	98.99± 0.7	90	0.641	6.5± 2.6	98.74± 0.5	83	0.751	6.3± 2.7	98.70± 0.4	18	0.828
					95	6.5± 2.6	97.35± 1.8	96	1.011	5.1± 2.5	96.52± 1.0	90	1.130	5.0± 2.5	96.39± 0.8	54	1.113
					93	5.8± 2.8	95.77± 2.7	96	1.257	4.4± 2.5	94.67± 1.6	94	1.297	4.2± 2.4	94.48± 1.3	65	1.324

Results for decision diagrams

Dataset	#I	#F	OMDD		δ	MinPAXp					LmPAXp					
						Length			Prec	Time	Length			Prec	$m_{\subseteq}$	Time
			#N	A%		M	m	avg	avg	avg	M	m	avg	avg		avg
lending	100	9	1103	81.7	100	9	6	8.0	100	24.24	9	6	7.9	100	100	1.57
					95	9	5	7.8	99.7	21.48	9	6	7.8	99.8	100	1.49
					90	9	4	7.2	96	24.65	9	5	7.4	97.0	100	1.48
monk2	100	6	70	79.3	100	6	4	5.1	100	0.10	6	4	5.1	100	100	0.03
					95	6	4	5.1	100	0.09	6	4	5.1	100	100	0.03
					90	6	3	4.8	98.1	0.09	6	3	4.8	98.1	100	0.03
postoperative	74	8	109	80	100	8	4	6.1	100	0.26	8	4	6.2	100	100	0.04
					95	8	2	6.0	99.3	0.25	8	2	6.0	99.3	100	0.04
					90	8	2	5.3	95.9	0.23	8	2	5.4	96.6	94.6	0.04
tic_tac_toe	100	9	424	70.3	100	9	5	7.7	100	3.60	9	5	7.8	100	100	0.38
					95	9	5	7.5	99.5	3.24	9	5	7.7	99.6	99.0	0.38
					90	9	3	7.3	98.3	4.06	9	3	7.5	98.6	98.0	0.38
xd6	100	9	76	83.1	100	9	4	4.6	100	0.10	9	4	4.6	100	100	0.03
					95	9	3	3.8	97	0.09	9	3	3.8	97.0	99.0	0.03
					90	9	3	3.3	94.8	0.10	9	3	3.4	94.6	100	0.03

Remarks on LmPAXps

[IHI+23]

- LmPAXps ignore non-monotonicity, and so overapproximate PAXps
  - Theoretical guarantees, but may be reducible
- For DTs, computation of LmPAXps is in P
- Experimental results confirm LmPAXps match PAXps in most cases
- Recent results on approximating LmPAXps for RFs [IMM24]

Outline – Unit #06

Sample-Based Explanations

Changing Assumptions (in Plain Logic-Based XAI)

Inflated Explanations

Constrained Explanations

Distance-Restricted Explanations

Probabilistic Explanations

Additional Topics

Surrogate models in logic-based XAI

[BAMT21]

- Motivation:
  - Logic-based XAI does not yet scale for highly complex ML models
  - Surrogate models find many uses in ML, for approximating complex models

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  - Compute rigorous explanation: plain AXp, probabilistic AXp,
- Report computed explanation as explanation for the complex ML model

Certified explainer (for monotonic classification)

[HM23c]

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- Even comprehensive testing of implemented algorithms does not guarantee correctness

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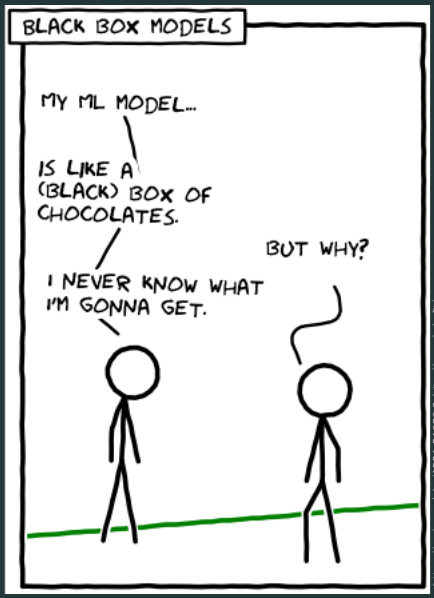
- Certification envisioned for **any** explainability algorithm

Plan for this course – light at the end of the tunnel...

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Questions?


BLACK BOX MODELS
MY ML MODEL...
IS LIKE A (BLACK) BOX OF CHOCOLATES.
I NEVER KNOW WHAT I'M GONNA GET.
BUT WHY?
https://arxiv.org/abs/1901.01686 & http://cmx.io/edit/

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