

LOGIC-BASED EXPLAINABLE ARTIFICIAL INTELLIGENCE

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ICREA & Univ. Lleida, Catalunya, Spain

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Lecture 05

Recapitulate fourth lecture

- Monotonic classifiers vs. weighted voting games
- Advanced topics:
 - Sample-based explanations
 - Inflated explanations
 - Probabilistic explanations
 - Constrained explanations
 - Distance-restricted explanations
 - Explanations using surrogate models
 - Certified explainability

Monotonicity & WCGs

- Every WVG \mathcal{G} , described by $[q; n_1, \dots, n_m]$, can be represented as a monotonically increasing boolean classifier $\mathcal{M} = (\mathcal{F}, \{0, 1\}^m, \{0, 1\}, \kappa)$, such that:
 - Each voter i is mapped to a boolean feature i , such that feature i takes value 1 if voter i votes Yes; otherwise it takes value 0;
 - The classification function $\kappa : \mathbb{F} \rightarrow \{0, 1\}$ is defined by:

$$\kappa(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^m n_i x_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

- The target instance is $(1, 1)$; and
- Each minimal winning coalition \mathcal{C} corresponds to an AXp of $\mathcal{E} = (\mathcal{M}, (1, 1))$

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∴ WVGs can be analyzed by studying the AXps/CXps of monotonically increasing boolean classifiers

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- WVG: [25; 10, 9, 7, 1, 1, 1, 1, 1, 1]

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 - Winning coalitions must include both 1 and 2
 - We can pick 3 or, alternatively, all the other ones

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- CXps:
 - $\mathbb{C} = \{\{1\}, \{2\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{3, 8\}, \{3, 9\}, \}$

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- Q: How should features be ranked in terms of importance?

Plan for this course – light at the end of the tunnel...

- Lecture 01 – unit(s):
 - #01: Foundations
- Lecture 02 – unit(s):
 - #02: Principles of symbolic XAI – **feature selection**
 - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
 - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
 - #05: Explainability queries
- Lecture 04 – unit(s):
 - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
 - #07: Principles of symbolic XAI – **feature attribution** (& myth of Shapley values in XAI)
 - #08: Corrected feature attribution – nuSHAP
 - #09: Conclusions & research directions

Unit #07

Principles of Symbolic XAI – Feature Attribution

Detour: Standard SHAP Intro (from another course...)

What are Shapley values?

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[Sha53]

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 - Used for feature attribution, i.e. [relative feature importance](#)

[Sha53]

[LC01, SK10, SK14, DSZ16, LL17, ABBM21, VLSS21, VLSS22, ABBM23]

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- Shapley values are becoming ubiquitous in XAI... – E.g. see slides from other XAI course...

[Sha53]

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🛡️ 🔍 https://en.wikipedia.org/wiki/Shapley_value

📋 ⭐

Accessed 2023/06/14

In machine learning [edit]

The Shapley value provides a principled way to explain the predictions of nonlinear models common in the field of [machine learning](#). By interpreting a model trained on a set of features as a value function on a coalition of players, Shapley values provide a natural way to compute which features contribute to a prediction.^[17] This unifies several other methods including Locally Interpretable Model-Agnostic Explanations (LIME),^[18] DeepLIFT,^[19] and Layer-Wise Relevance Propagation.^[20]

17. ^ Lundberg, Scott M.; Lee, Su-In (2017). "A Unified Approach to Interpreting Model Predictions". *Advances in Neural Information Processing Systems*. 30: 4765–4774. arXiv:1705.07874. Retrieved 2021-01-30.

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- **Q:** Do Shapley values for XAI **really** provide a rigorous measure of feature importance?

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- $\phi: 2^{\mathcal{F}} \rightarrow \mathbb{R}$ defined by,

$$\phi(\mathcal{S}) = 1/2^{|\mathcal{F} \setminus \mathcal{S}|} \sum_{\mathbf{x} \in \Upsilon(\mathcal{S})} \kappa(\mathbf{x}) = v_e(\mathcal{S})$$

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$$\text{Sc}(i) = \sum_{\mathcal{S} \subseteq (\mathcal{F} \setminus \{i\})} \frac{|\mathcal{S}|!(|\mathcal{F}| - |\mathcal{S}| - 1)!}{|\mathcal{F}|!} \times (\phi(\mathcal{S} \cup \{i\}) - \phi(\mathcal{S}))$$

For all subsets of features, excluding i , compute the expected value of the classifier, with and without i fixed, weighted by $\frac{1}{n} \binom{n}{|\mathcal{S}|}^{-1}$

- Obs:** Uniform distribution assumed; it suffices for our purposes

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Marginal contribution
(in SHAP lingo)!

[ABBM21, ABBM23]

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How are Shapley values computed in practice?

- Exact evaluation is computationally (very) hard [VLSS21, ABBM21, VLSS22, ABBM23, HMS24]
- SHAP proposes a sample-based approach; with **no** guarantees of rigor [LL17]
 - Recent experiments revealed little to **no** correlation between Shapley values and SHAP's results [HM23a]

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- **Polynomial-time** algorithm for deterministic decomposable boolean circuits [ABBM21]
- **Polynomial-time** algorithm for boolean functions represented with a truth-table [HM23a]

What do Shapley values tell in terms of feature importance?

- [SK10] reads:

“According to the 2nd axiom, if two features values have an identical influence on the prediction they are assigned contributions of equal size. The 3rd axiom says that if a feature has no influence on the prediction it is assigned a contribution of 0.”

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- Obs: Shapley values are defined **axiomatically**, i.e. **no** immediate relationship with AXp's/CXp's or with feature (ir)relevancy
 - Qs: can we have **irrelevant** features with a non-zero Shapley value, and/or **relevant** features with a Shapley of zero?
 - Recall: **relevant** features occur in some AXp/CXp; **irrelevant** features do **not** occur in **any** AXp/CXp

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

A Correct New SHAP – nuSHAP

Voting Power & Power Indices

Feature Importance Scores

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[HM23a, HM23b, HM23c, MH23, HMS24, MSH24]

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Any of these issues is a cause
of (serious) concern per se!

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Some stats – all boolean functions with 4 variables

[HM23a, HM23b, HM23c, MH23, HMS24, MSH24]

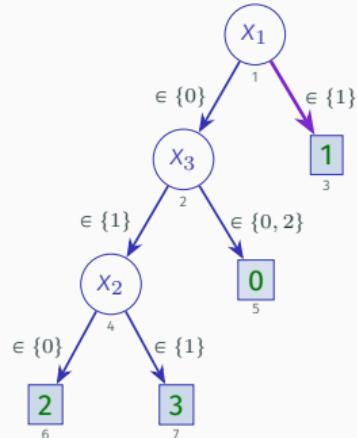
Issue-related metric	Value	Recap issue
# of functions	65536	
# number of instances	1048576	
# of I1 issues	781696	
# of functions with I1 issues	65320	
% I1 issues / function	99.67	$[\text{Irrelevant}(i) \wedge (\text{Sv}(i) \neq 0)]$
# of I2 issues	105184	
# of functions with I2 issues	40448	
% I2 issues / function	61.72	$[\text{Irrelevant}(i_1) \wedge \text{Relevant}(i_2) \wedge (\text{Sv}(i_1) > \text{Sv}(i_2))]$
# of I3 issues	43008	
# of functions with I3 issues	7800	
% I3 issues / function	11.90	$[\text{Relevant}(i) \wedge (\text{Sv}(i) = 0)]$
# of I4 issues	5728	
# of functions with I4 issues	2592	
% I4 issues / function	3.96	$[\text{Irrelevant}(i_1) \wedge (\text{Sv}(i_1) \neq 0)] \wedge [\text{Relevant}(i_2) \wedge (\text{Sv}(i_2) = 0)]$
# of I5 issues	1664	
# of functions with I5 issues	1248	
% I5 issues / function	1.90	$[\text{Irrelevant}(i) \wedge \forall_{1 \leq j \leq m, j \neq i} (\text{Sv}(j) < \text{Sv}(i))]$

Previous results do matter! Let's go non-boolean...

row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
1	0	0	0	0	0
2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
8	1	0	1	1	1
9	1	0	2	1	1
10	1	1	0	1	1
11	1	1	1	1	1
12	1	1	2	1	1

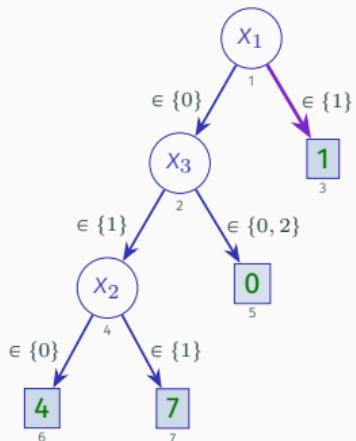
DT1

Tabular representations



DT2

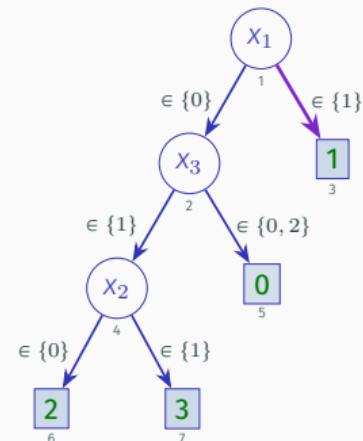
Instance $((1, 1, 2), 1)$ – which feature matters the most for prediction 1?



DT1

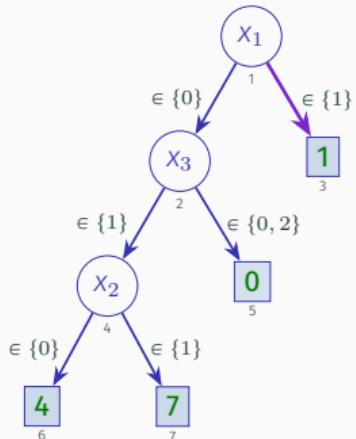
row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
1	0	0	0	0	0
2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
8	1	0	1	1	1
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10	1	1	0	1	1
11	1	1	1	1	1
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Tabular representations



DT2

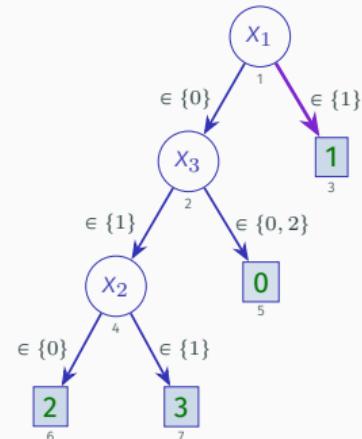
Computing XPs – make sense...



DT1

row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
1	0	0	0	0	0
2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
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12	1	1	2	1	1

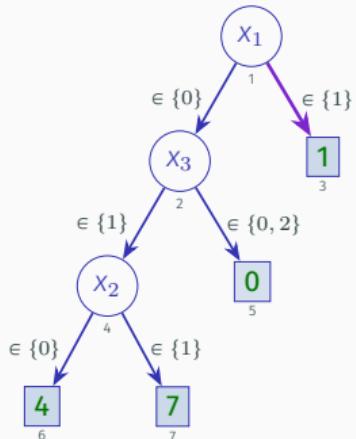
Tabular representations



DT2

XPs: AXps/CXps		
DT	AXps	CXps
DT1	{1}	{1}
DT2	{1}	{1}

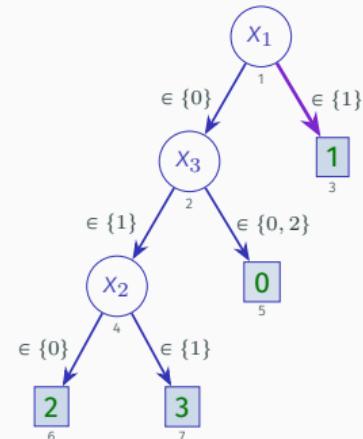
Computing XPs, AEs – also make sense...



DT1

row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
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Tabular representations

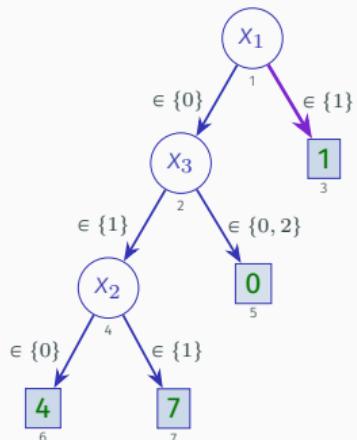


DT2

XPs: AXps/CXps		
DT	AXps	CXps
DT1	{1}	{1}
DT2	{1}	{1}

Adversarial Examples	
DT	l_0 -minimal AEs
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DT2	{1}

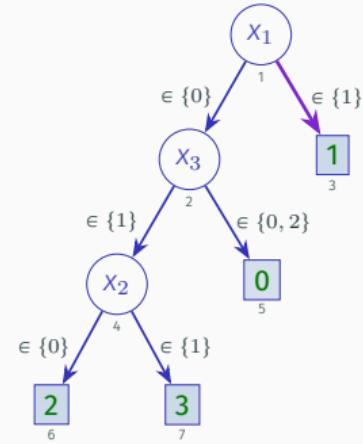
Computing XPs, AEs & Svs



DT1

row #	x_1	x_2	x_3	$\kappa_1(\mathbf{x})$	$\kappa_2(\mathbf{x})$
1	0	0	0	0	0
2	0	0	1	4	2
3	0	0	2	0	0
4	0	1	0	0	0
5	0	1	1	7	3
6	0	1	2	0	0
7	1	0	0	1	1
8	1	0	1	1	1
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10	1	1	0	1	1
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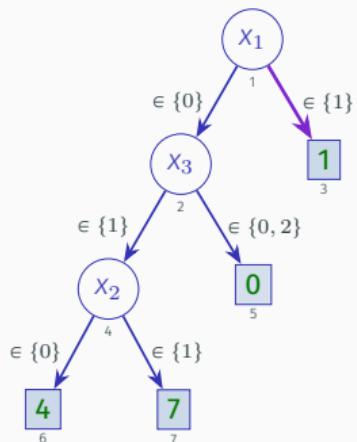
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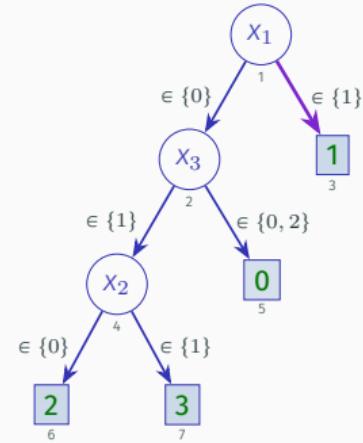
Shapley values			
DT	Sc(1)	Sc(2)	Sc(3)
DT1	0.000	0.083	-0.500
DT2	0.278	0.028	-0.222

Computing XPs, AEs & Svs – what???



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DT2

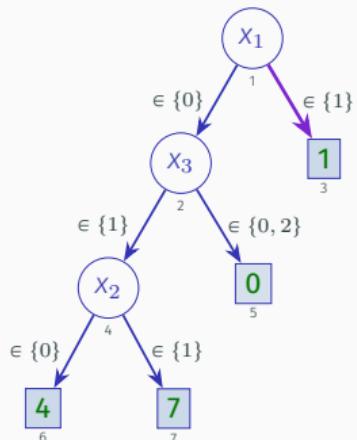
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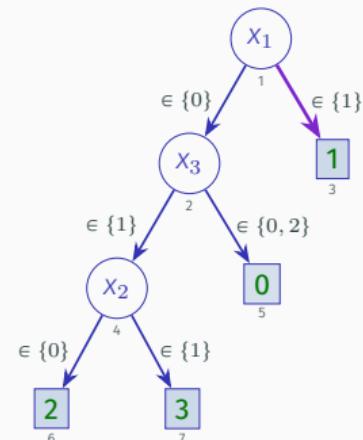
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Tabular representations



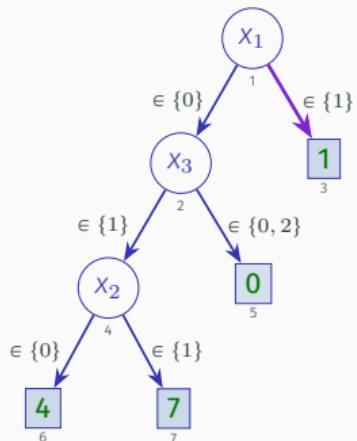
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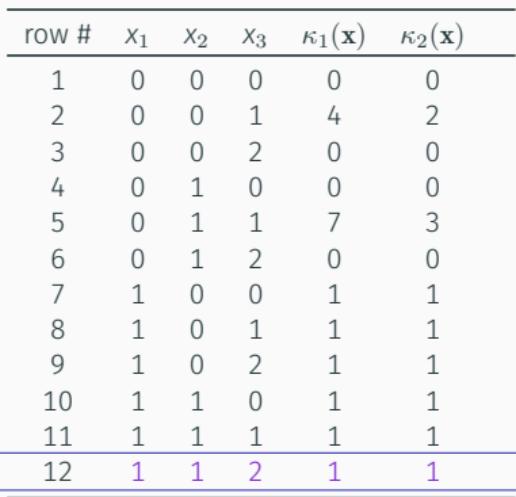
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Computing XPs, AEs & Svs – what???



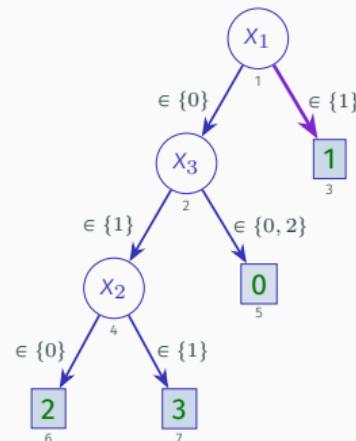
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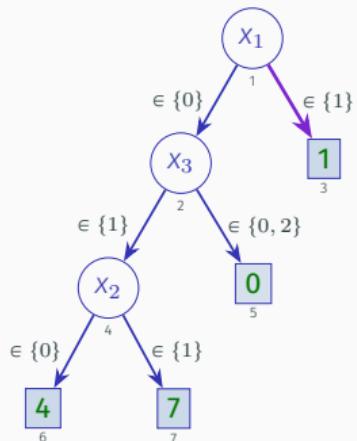


PT2

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∴ Shapley values can mislead
human decision-makers !

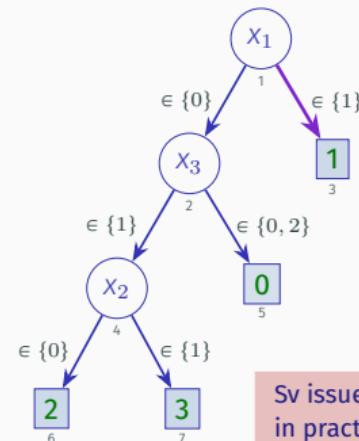
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Tabular representations



DT2

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Sv issues also occur in practice [HM23c]

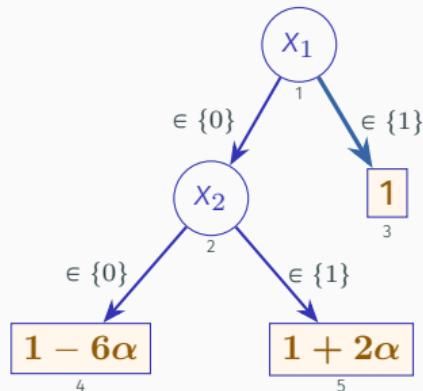
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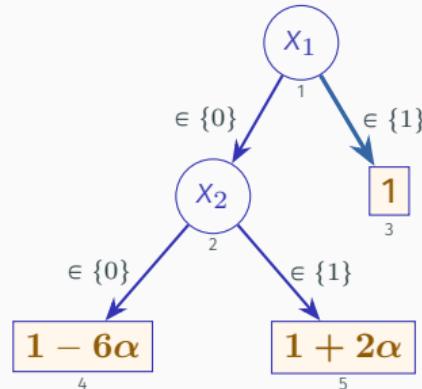
Another example – arbitrary mistakes!

[LHAMS24]



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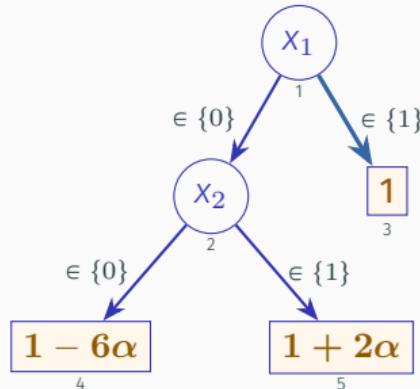
[LHAMS24]



- Instance: $((1, 1), 1)$
- Obs: $\alpha \neq 0$

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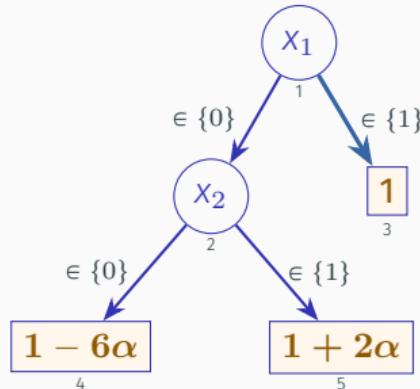
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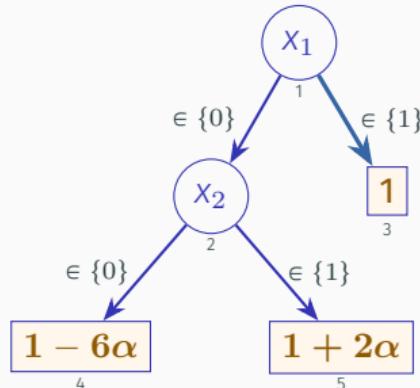
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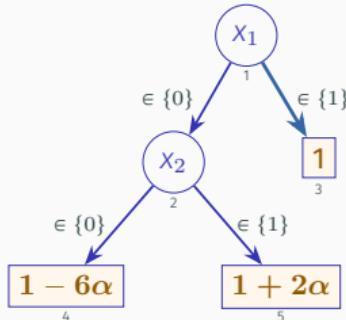


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Example devised by O. Letoffe, PhD student at IRIT

More detail

row	x_1	x_2	$\rho(\mathbf{x})$	$\rho_a(\mathbf{x})$ $\alpha = 1/2$	$\rho_b(\mathbf{x})$ $\alpha = 1/4$
1	0	0	$1 - 6\alpha$	-2	$-1/2$
2	0	1	$1 + 2\alpha$	2	$3/2$
3	1	0	1	1	1
4	1	1	1	1	1



\mathcal{S}	$\text{rows}(\mathcal{S})$	$v_e(\mathcal{S})$
\emptyset	1, 2, 3, 4	$1 - \alpha$
$\{x_1\}$	3, 4	1
$\{x_2\}$	2, 4	$1 + \alpha$
$\{x_1, x_2\}$	4	1

$i = 1$					
\mathcal{S}	$v_e(\mathcal{S})$	$v_e(\mathcal{S} \cup \{1\})$	$\Delta_1(\mathcal{S})$	$\varsigma(\mathcal{S})$	$\varsigma(\mathcal{S}) \times \Delta_1(\mathcal{S})$
\emptyset	$1 - \alpha$	1	α	$1/2$	$\alpha/2$
$\{2\}$	$1 + \alpha$	1	$-\alpha$	$1/2$	$-\alpha/2$
$\text{Sc}_E(1) = 0$					
$i = 2$					
\mathcal{S}	$v_e(\mathcal{S})$	$v_e(\mathcal{S} \cup \{2\})$	$\Delta_2(\mathcal{S})$	$\varsigma(\mathcal{S})$	$\varsigma(\mathcal{S}) \times \Delta_2(\mathcal{S})$
\emptyset	$1 - \alpha$	$1 + \alpha$	2α	$1/2$	α
$\{1\}$	1	1	0	$1/2$	0
$\text{Sc}_E(2) = \alpha$					

SHAP scores also fail with regression models

[LHM24]

- Let $\mathcal{F} = \{1, 2\}$, $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{D} = [-1/2, 3/2]$, $\mathbb{F} = \mathbb{D} \times \mathbb{D}$
 - Also, let $\mathbb{D}^+ = [1/2, 3/2]$ and $\mathbb{D}^- = \mathbb{D} \setminus \mathbb{D}^+$
- Regression model maps to real values, i.e. $\mathbb{K} = \mathbb{R}$:

$$\rho_2(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 \in \mathbb{D}^+ \\ x_2 - 2 & \text{if } x_1 \notin \mathbb{D}^+ \wedge x_2 \notin \mathbb{D}^+ \\ x_2 + 1 & \text{if } x_1 \notin \mathbb{D}^+ \wedge x_2 \in \mathbb{D}^+ \end{cases}$$

- Average values unchanged (wrt previous example), and so $Sc(1) = 0$ and $Sc(2) = \alpha$

SHAP scores even fail when Lipschitz continuity holds!

$$\rho_3(x_1, x_2) = \begin{cases} x_1 & \text{if } x_2 \leq 1 \wedge \alpha x_1 \leq \alpha \\ (1 + 4|\alpha|)x_1 - 4|\alpha| & \text{if } x_2 \leq 1 \wedge \alpha x_1 \geq \alpha \\ 28|\alpha|x_1x_2 + (1 - 28|\alpha|)x_1 - 28|\alpha|x_2 + 28|\alpha| & \text{if } x_2 \geq 1 \wedge \alpha x_1 \leq \alpha \\ -4|\alpha|x_1x_2 + (1 + 8|\alpha|)x_1 + 4|\alpha|x_2 - 8|\alpha| & \text{if } x_2 \geq 1 \wedge \alpha x_1 \geq \alpha \end{cases}$$

[LHM24]

- As before, average values unchanged, and so $\text{Sc}(1) = 0$ and $\text{Sc}(2) = \alpha$

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

A Correct New SHAP – nuSHAP

Voting Power & Power Indices

Feature Importance Scores

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

- Is the theory of Shapley values **incorrect?**

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

- Is the theory of Shapley values **incorrect?** **No!**

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

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- What is inadequate is the **characteristic function** used in XAI
 - In XAI: characteristic function uses the **expected value**
 - This defines the *marginal contribution* in SHAP lingo...

[SK10, SK14, LL17]

Corrected SHAP scores & feature importance scores

[LHMS24, LHAMS24]

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- Replace characteristic function based on **expected values** by new characteristic function based on **AXps/WAXps**
 - Resulting scores are (**still**) Shapley values & identified issues no longer observed

[SK10, SK14, LL17]

[LHMS24]

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[LHMS24, LHAMS24]

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[LHMS24, LHAMS24]

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 - **Feature importance scores** (more later):
 - Generalize recent axiomatic aggregations
 - Adapt best known power indices
 - Devise new scores for XAI

[SK10, SK14, LL17]

[LHMS24]

[LHAMS24]

[BIL⁺24]

An initial compromise

[LHAMS24]

- Replace the characteristic function used for SHAP scores:

$$v_e(\mathcal{S}) := \mathbf{E}[\tau(\mathbf{x}) \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}]$$

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[LHAMS24]

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$$\sigma(\mathbf{x}) = \begin{cases} 1 & \text{if } (\kappa(\mathbf{x}) = \kappa(\mathbf{v})) \\ 0 & \text{otherwise} \end{cases}$$

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- Issues with non-boolean classifiers **disappear**; issues with boolean classifiers **remain**
- Developed SSHAP prototype using SHAP's code base

[LHMS24]

Fixing the known issues of SHAP scores

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- Recall: $\mathbf{E}[\sigma(\mathbf{x}) \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] = 1$ holds iff \mathcal{S} is a WAXp!
- Known issues of SHAP scores guaranteed **not** to occur
- **Corrected** SHAP scores reveal tight connection between XAI by feature selection (i.e. WAXps) and feature attribution

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

A Correct New SHAP – nuSHAP

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A glimpse of NuSHAP

- Shapley values estimated with well-known algorithm – CGT
 - Strong theoretical guarantees on approximation of Shapley values
 - With enough sampling, given by ϵ and α :

[CGT09]

$$\text{Prob}[|\hat{Sc}(i) - Sc(i)| \leq \epsilon] \geq 1 - \alpha$$

- E.g. we used $\epsilon = 0.0015$ and $\alpha = 0.015$

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[CGT09]

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- E.g. we used $\epsilon = 0.0015$ and $\alpha = 0.015$
- Replace expected value with novel characteristic function: test for WAXp
 - New approach for computing symbolic explanations, targeting scalability for complex models
 - Replace WAXp check with sbWAXp check, i.e. polynomial-time check
 - Flexibility in the sample chosen, e.g. training data

NuSHAP vs. *SHAP

- Compute SHAP scores with NuSHAP and SHAP
- Compare features' rankings using Rank-Biased Overlap (RBO) [WMZ10]
 - Give weight to differences in first k elements of ranking
 - E.g. $\text{RBO}([0, 1, 2, 3, 4, 5, 6, 7, 8, 9], [0, 1, 2, 3, 4, 5, 6, 7, 8, 9], k = 5) = 1.0$
- Results for different datasets and ML models (using scikit, XGBoost):
 - LR: (adult, corral); DT: (iris, mux6); KNN: (connect_4, spambase, spectf);
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[WMZ10]

		adult	corral	iris	mux6	connect_4	spambase	spectf	clean1	coil2000	dna	MNIST
Min	nuSHAP vs. SHAP	0.08	0.17	0.31	0.32	0.0	0.01	0.0	0.0	0.0	0.0	0.0
	nuSHAP vs. SHAP	0.05	0.12	0.27	0.32	0.0	0.05	0.0	0.0	0.0	0.03	0.0
Max	nuSHAP vs. SHAP	0.96	0.96	0.94	0.97	0.9	0.94	0.91	0.69	0.69	0.88	0.06
	nuSHAP vs. SHAP	0.88	0.97	0.94	0.95	0.77	0.94	0.91	0.88	0.69	0.88	0.06
Mean	nuSHAP vs. SHAP	0.37	0.53	0.84	0.7	0.21	0.41	0.2	0.12	0.05	0.17	0.0
	nuSHAP vs. SHAP	0.31	0.5	0.84	0.69	0.19	0.42	0.19	0.17	0.08	0.43	0.0

NuSHAP vs. *SHAP – run times

- Compute SHAP scores with NuSHAP and SHAP
- Compare features' rankings using Rank-Biased Overlap (RBO)
 - Give weight to differences in first k elements of ranking
 - E.g. $\text{RBO}([0, 1, 2, 3, 4, 5, 6, 7, 8, 9], [5, 6, 7, 8, 9, 0, 1, 2, 3, 4], k = 5) = 0.0$
- Results for different datasets and ML models (using scikit, XGBoost):
 - LR: (adult, corral); DT: (iris, mux6); KNN: (connect_4, spambase, spectf);
BT: (clean1, coil2000, dna); CNN: (MNIST)

		adult	corral	iris	mux6	connect_4	spambase	spectf	clean1	coil2000	dna	MNIST
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Tool SHAP produces results of very poor quality!

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	adult	corral	iris	mux6	connect_4	spambase	spectf	clean1	coil2000	dna	MNIST
SHAP	3.4	0.1	0.0	0.0	21.7	0.5	0.7	6.8	28.2	23.0	281.3
nuSHAP	1.9	1.5	1.5	1.5	4.5	2.7	2.9	2.7	1.7	4.5	48.9
#Samples	68045.5	63.7	955.4	63.9	9202.5	13654.1	36459.5	3929.6	2960.8	2756.8	2929.9

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

A Correct New SHAP – nuSHAP

Voting Power & Power Indices

Feature Importance Scores

Recap: weighted voting games

- General set up of weighted voting games:
 - Assembly \mathcal{A} of voters, with $m = |\mathcal{A}|$
 - Each voter $i \in \mathcal{A}$ votes **Yes** with n_i votes; otherwise no votes are counted (and he/she votes **No**)
 - A coalition is a subset of voters, $\mathcal{C} \subseteq \mathcal{A}$
 - Quota q is the sum of votes required for a proposal to be approved
 - Coalitions leading to sums not less than q are **winning** coalitions
 - A **weighted voting game (WVG)** is a tuple $[q; n_1, \dots, n_m]$
 - Example: $[12; 4, 4, 4, 2, 2, 1]$
 - Problem: find a measure of importance of each voter !
 - I.e. measure the **a priori voting power** of each voter

What are power indices?

- Power indices assign a measure of importance to each voter

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- Many power indices proposed over the years:

- Penrose [Pen46]
- Shapley-Shubik [SS54]
- Banzhaf [Bil65]
- Coleman [Col71]
- Johnston [Joh78]
- Deegan-Packel [DP78]
- Holler-Packel [HP83]
- Andjiga [ACL03]
- Responsability* [CH04, BIL⁺24]
- ...

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 - Andjiga [ACL03]
 - Responsability* [CH04, BIL⁺24]
 - ...
- What characterizes power indices?
 - Account for the cases when voter is *critical* for a winning coalition
 - E.g. in previous example, Luxembourg is never critical for a winning coalition
 - Account for whether coalition is subset-minimal or cardinality-minimal

Towards defining power indices

- Understanding **criticality** (used at least since 1954):

[SS54]

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 - This means that a voter i is **critical** when:
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- Understanding (subset-)minimal winning coalitions:
 - A winning coalition is subset-minimal if removing any single voter results in a losing coalition
 - A winning coalition is cardinality-minimal if it has the smallest cardinality among subset-minimal winning coalitions
 - Recall that minimal winning coalitions can be obtained by computing the AXps of a monotonically increasing boolean classifier

Example power indices I

[LHAMS24]

- Necessary definitions (using formal XAI notation...):

$$\mathbb{W}\mathbb{A}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \mid \text{WAXp}(\mathcal{S}; \mathcal{E}) \wedge i \in \mathcal{S}\}$$

$$\mathbb{W}\mathbb{C}_i(\mathcal{E}) = \{\mathcal{S} \subseteq \mathcal{F} \mid \text{WCXp}(\mathcal{S}; \mathcal{E}) \wedge i \in \mathcal{S}\}$$

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- Power indices of Holler-Packel and Deegan-Packel:

[HP83, DP78]

$$\text{Sc}_H(i; \mathcal{E}) = \sum_{\mathcal{S} \in \mathbb{A}_i(\mathcal{E})} (1/|\mathbb{A}(\mathcal{E})|)$$

$$\text{Sc}_D(i; \mathcal{E}) = \sum_{\mathcal{S} \in \mathbb{A}_i(\mathcal{E})} (1/(|\mathcal{S}| \times |\mathbb{A}(\mathcal{E})|))$$

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- **Obs:** One *only* needs the **AXps**

Example power indices II

- Additional definitions:

$$\text{Crit}(i, \mathcal{S}; \mathcal{E}) := \text{WAXp}(\mathcal{S}; \mathcal{E}) \wedge \neg \text{WAXp}(\mathcal{S} \setminus \{i\}; \mathcal{E})$$

Example power indices II

- Additional definitions:

$$\text{Crit}(i, \mathcal{S}; \mathcal{E}) := \text{WAXp}(\mathcal{S}; \mathcal{E}) \wedge \neg \text{WAXp}(\mathcal{S} \setminus \{i\}; \mathcal{E})$$

- Power indices of Shapley-Shubik, Banzhaf and Johnston:

[SS54, BI65, Joh78]

$$Sc_S(i; \mathcal{E}) = \sum_{\mathcal{S} \subseteq \mathcal{F} \wedge \text{Crit}(i, \mathcal{S}; \mathcal{E})} \left(1 / \binom{|\mathcal{F}| - 1}{|\mathcal{S}| - 1} \right)$$

$$Sc_B(i; \mathcal{E}) = \sum_{\mathcal{S} \subseteq \mathcal{F} \wedge \text{Crit}(i, \mathcal{S}; \mathcal{E})} \left(1 / 2^{|\mathcal{F}| - 1} \right)$$

$$Sc_J(i; \mathcal{E}) = \sum_{\mathcal{S} \subseteq \mathcal{F} \wedge \text{Crit}(i, \mathcal{S}; \mathcal{E})} \left(1 / \Delta(\mathcal{S}) \right)$$

Example power indices II

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- One needs the **WAXps** to find critical voters...

Example #01

- WVG: [9; 9, 2, 2, 2, 2, 1, 1]

Example #01

- WVG: [9; 9, 2, 2, 2, 2, 1, 1]
- AXps:

1					
2	3	4	5	6	
2	3	4	5	7	

Example #01

- WVG: [9; 9, 2, 2, 2, 2, 1, 1]
- AXps:

1					
2	3	4	5	6	
2	3	4	5	7	

- Holler-Packel scores: $\langle 0.333, 0.667, 0.667, 0.667, 0.667, 0.333, 0.333 \rangle$
- Banzhaf scores (normalized): $\langle 0.813, 0.040, 0.040, 0.040, 0.040, 0.013, 0.013 \rangle$
- Shapley-Shubik scores: $\langle 0.810, 0.043, 0.043, 0.043, 0.043, 0.010, 0.010 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #02

- WVG: [16; 10, 6, 4, 2, 2]

Example #02

- WVG: [16; 10, 6, 4, 2, 2]
- AXps:

1	2	
1	3	4
1	3	5

Example #02

- WVG: [16; 10, 6, 4, 2, 2]
- AXps:

1	2	
1	3	4
1	3	5

- Deegan-Packel scores: $\langle 0.389, 0.167, 0.222, 0.111, 0.111 \rangle$
- Banzhaf scores (normalized): $\langle 0.524, 0.238, 0.143, 0.048, 0.048 \rangle$
- Shapley-Shubik scores: $\langle 0.617, 0.200, 0.117, 0.033, 0.033 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #03

- WVG: [6; 4, 2, 1, 1, 1, 1]

Example #03

- WVG: [6; 4, 2, 1, 1, 1, 1]
- AXps:

2	3	4	5	6
1	3	4		
1	4	5		
1	4	6		
1	3	6		
1	5	6		
1	2			
1	3	5		

Example #03

- WVG: [6; 4, 2, 1, 1, 1, 1]
- AXps:

2	3	4	5	6
1	3	4		
1	4	5		
1	4	6		
1	3	6		
1	5	6		
1	2			
1	3	5		

- Deegan-Packel scores: $\langle 0.312, 0.087, 0.150, 0.150, 0.150, 0.150 \rangle$
- Banzhaf scores (normalized): $\langle 0.542, 0.125, 0.083, 0.083, 0.083, 0.083 \rangle$
- Shapley-Shubik scores: $\langle 0.533, 0.133, 0.083, 0.083, 0.083, 0.083 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Example #04

- WVG: [21; 12, 9, 4, 4, 1, 1, 1]

Example #04

- WVG: [21; 12, 9, 4, 4, 1, 1, 1]
- AXps:

1	2		
1	3	4	5
1	3	4	6
1	3	4	7

Example #04

- WVG: [21; 12, 9, 4, 4, 1, 1, 1]
- AXps:

1	2		
1	3	4	5
1	3	4	6
1	3	4	7

- Deegan-Packel scores: $\langle 0.312, 0.125, 0.188, 0.188, 0.062, 0.062, 0.062 \rangle$
- Banzhaf scores (normalized): $\langle 0.481, 0.309, 0.086, 0.086, 0.012, 0.012, 0.012 \rangle$
- Shapley-Shubik scores: $\langle 0.574, 0.257, 0.074, 0.074, 0.007, 0.007, 0.007 \rangle$
- Different relative orders of voter importance... which ones seem more realistic?

Outline – Unit #07

Exact Shapley Values for XAI

Myth #03: Shapley Values for XAI

Corrected SHAP Scores

A Correct New SHAP – nuSHAP

Voting Power & Power Indices

Feature Importance Scores

From power indices to feature importance scores

- A **Feature Importance Score** (FIS) is a measure of feature importance in XAI, parameterizable on an **explanation problem** and a chosen **characteristic function**
 - Explanation problem: $(\mathcal{M}, (\mathbf{v}, q))$
 - Define characteristic function using explanation problem (more next slide)
- Obs: Can adapt (generalized) power indices as templates for feature importance scores
- Obs: Can devise new templates and/or new FISs

Some examples (1 of 2)

- More notation:

$$\Delta_i(\mathcal{S}; \mathcal{E}, v) = v(\mathcal{S}; \mathcal{E}) - v(\mathcal{S} \setminus \{i\}; \mathcal{E})$$

- Can use **any** characteristic function, including those presented earlier in this lecture

Some examples (1 of 2)

- More notation:

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- Can use **any** characteristic function, including those presented earlier in this lecture

- Some templates:

- Shapley-Shubik:

$$\text{TSc}_S(i; \mathcal{E}, v) := \sum_{\mathcal{S} \in \{\mathcal{T} \subseteq \mathcal{F} \mid i \in \mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S}; \mathcal{E}, v)}{|\mathcal{F}| \times \binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}} \right)$$

- Banzhaf:

$$\text{TSc}_B(i; \mathcal{E}, v) := \sum_{\mathcal{S} \in \{\mathcal{T} \subseteq \mathcal{F} \mid i \in \mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S}; \mathcal{E}, v)}{2^{|\mathcal{F}|-1}} \right)$$

Some examples (1 of 2)

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- Some templates:

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- Can use other templates

Some examples (1 of 2)

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- Can use other templates
- Can devise FISs without exploiting existing templates

Some examples (2 of 2)

- Recall WAXp-based characteristic function:

$$v_a(\mathcal{S}) := \begin{cases} 1 & \text{if } \mathbf{E}[\sigma(\mathbf{x}) \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}] = 1 \\ 0 & \text{otherwise} \end{cases}$$

Some examples (2 of 2)

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- Some FISs:

- Shapley-Shubik:

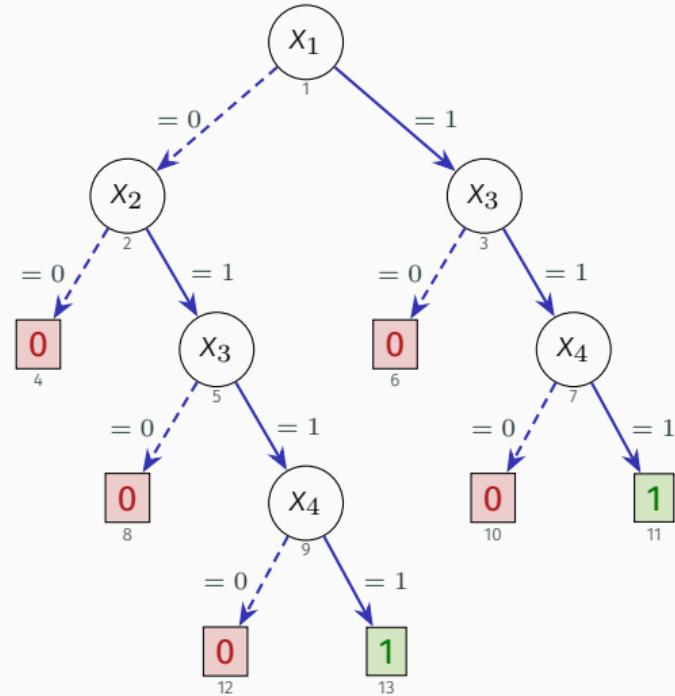
$$\text{Sc}_S(i; \mathcal{E}) := \text{TSc}_S(i; \mathcal{E}, v_a) := \sum_{\mathcal{S} \in \{\mathcal{T} \subseteq \mathcal{F} \mid i \in \mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S}; \mathcal{E}, v_a)}{|\mathcal{F}| \times \binom{|\mathcal{F}|-1}{|\mathcal{S}|-1}} \right)$$

- Banzhaf:

$$\text{Sc}_B(i; \mathcal{E}) := \text{TSc}_B(i; \mathcal{E}, v_a) := \sum_{\mathcal{S} \in \{\mathcal{T} \subseteq \mathcal{F} \mid i \in \mathcal{T}\}} \left(\frac{\Delta_i(\mathcal{S}; \mathcal{E}, v_a)}{2^{|\mathcal{F}|-1}} \right)$$

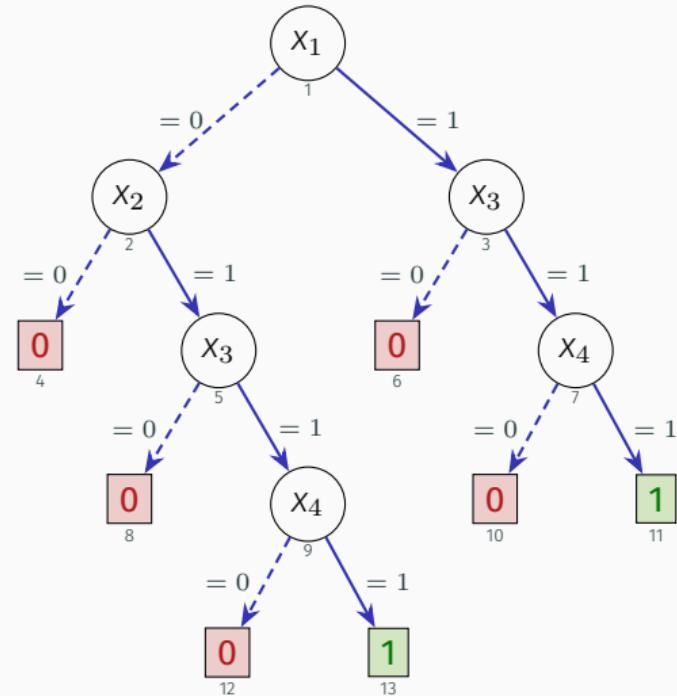
A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
- Feature attribution:



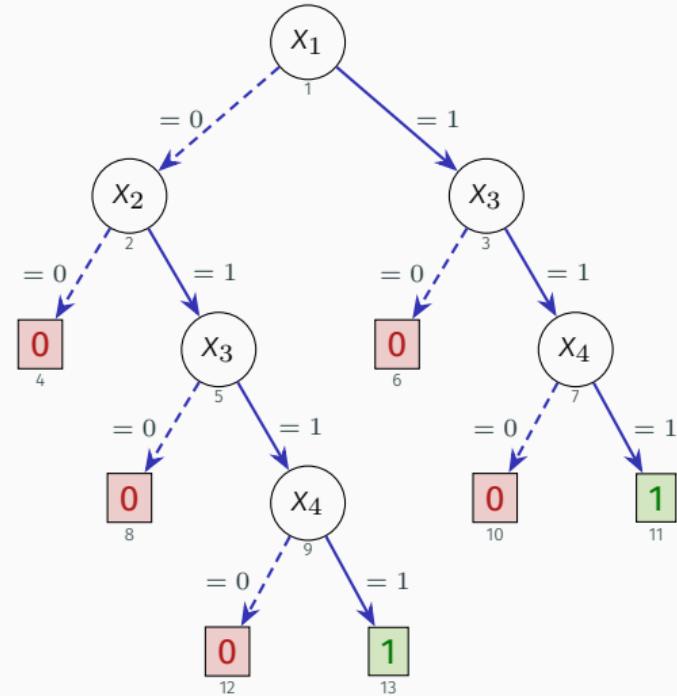
A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
- Feature attribution:
 - SS: $\langle 0.083, 0.083, 0.417, 0.417 \rangle$



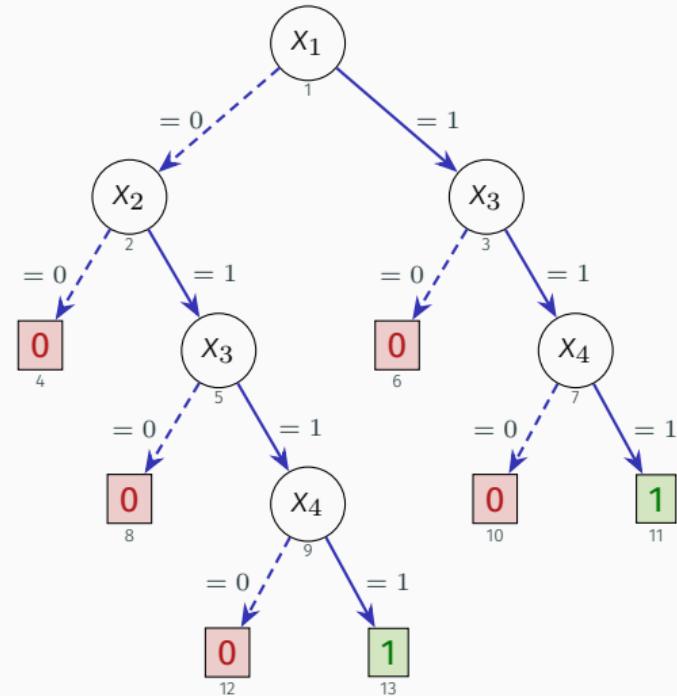
A concrete example

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- Feature attribution:
 - SS: $\langle 0.083, 0.083, 0.417, 0.417 \rangle$
 - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375 \rangle$



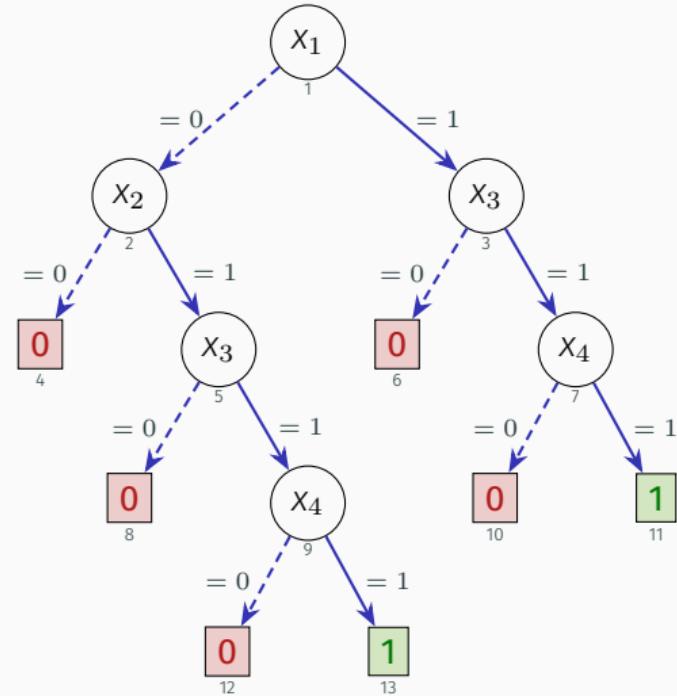
A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
- Feature attribution:
 - SS: $\langle 0.083, 0.083, 0.417, 0.417 \rangle$
 - B (norm.): $\langle 0.125, 0.125, 0.375, 0.375 \rangle$
 - J (norm.): $\langle 0.111, 0.111, 0.389, 0.389 \rangle$



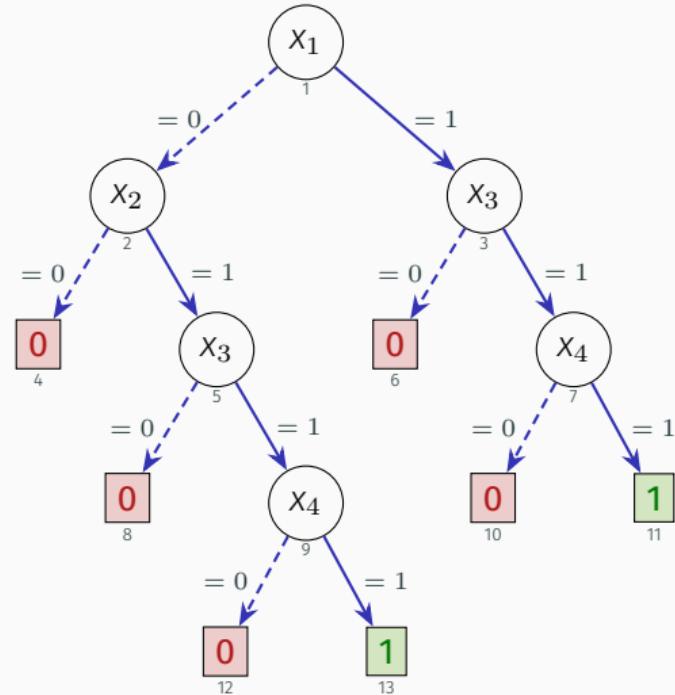
A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
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 - HP: $\langle 0.167, 0.167, 0.333, 0.333 \rangle$



A concrete example

- AXps: $\{\{1, 3, 4\}, \{2, 3, 4\}\}$
- Feature attribution:
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 - HP: $\langle 0.167, 0.167, 0.333, 0.333 \rangle$
 - DP: $\langle 0.167, 0.167, 0.333, 0.333 \rangle$



Questions?

Unit #08

Conclusions & Research Directions

Outline – Unit #08

Some Words of Concern

Conclusions & Research Directions

Can non-symbolic XAI's myths be stopped?

SHAP on 2023/05/31:

The screenshot shows a Google Scholar search results page. The search query is "A unified approach to interpreting model predictions". The top result is a paper by SM Lundberg and SI Lee, titled "A unified approach to interpreting model predictions". The paper is from Advances in neural information ..., 2017 - proceedings.neurips.cc. The abstract discusses the tension between accuracy and interpretability in complex models. The result includes a PDF link, the authors' names, the journal, the year, and a summary. Below the main result, there are filters for time (Any time, Since 2023, Since 2022, Since 2019, Custom range...), sorting options (Sort by relevance, Sort by date), and a "Any type" section with a "Review articles" link. At the bottom, there are checkboxes for "include patents" and "include citations".

scholar.google.com/scholar

Google Scholar

A unified approach to interpreting model predictions

Articles

My profile

Any time

Since 2023

Since 2022

Since 2019

Custom range...

Sort by relevance

Sort by date

Any type

Review articles

include patents

include citations

[PDF] neurips.cc

SM Lundberg, SI Lee - Advances in neural information ..., 2017 - proceedings.neurips.cc

Understanding why a model makes a certain prediction can be as crucial as the prediction's accuracy in many applications. However, the highest accuracy for large modern datasets is often achieved by complex models that even experts struggle to interpret, such as ensemble or deep learning models, creating a tension between accuracy and interpretability. In response, various methods have recently been proposed to help users interpret the predictions of complex models, but it is often unclear how these methods are related and ...

☆ Save 99 Cite Cited by 13080 Related articles All 17 versions

Showing the best result for this search. See all results

Can non-symbolic XAI's myths be stopped?

SHAP on 2024/09/15:

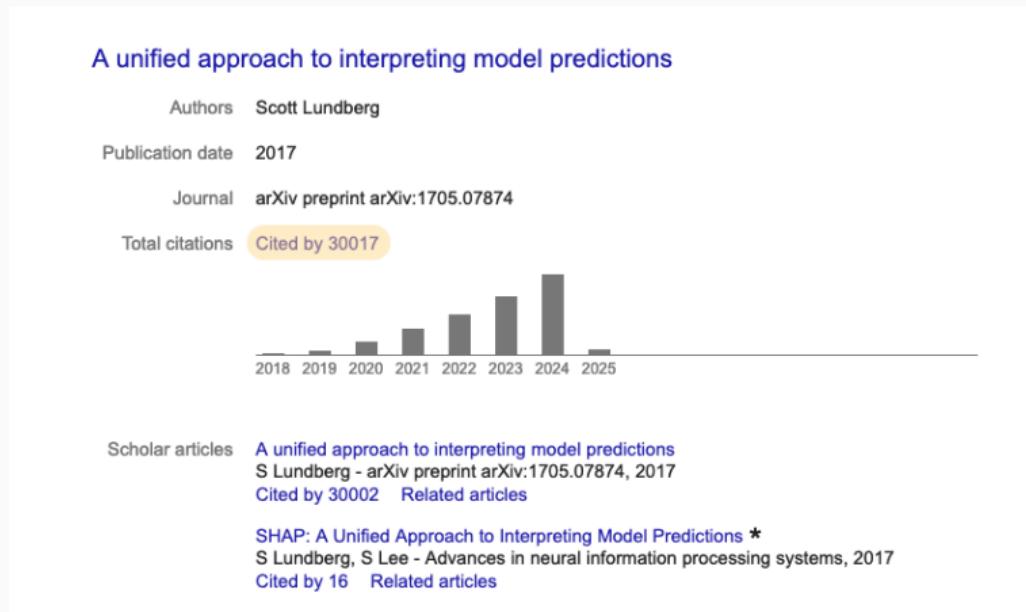
The screenshot shows a Google Scholar search results page. The search query is "A unified approach to interpreting model predictions". The results list two papers:

- [CITATION] A unified approach to interpreting model predictions**
S Lundberg - arXiv preprint arXiv:1705.07874, 2017
☆ Save ⚡ Cite Cited by 25778 Related articles
- [CITATION] A unified approach to interpreting model predictions**
M Scott, L Su-In - Advances in neural information ..., 2017 - Curran Associates, Inc
☆ Save ⚡ Cite Cited by 108 Related articles

Below the results, it says "Showing the best results for this search. See all results". On the left sidebar, there are filters for time (Any time, Since 2024, Since 2023, Since 2020, Custom range...), sorting options (Sort by relevance, Sort by date), and document types (Any type, Review articles). At the bottom, there are checkboxes for "include patents" (unchecked) and "include citations" (checked).

Can non-symbolic XAI's myths be stopped?

SHAP on 2025/01/12:



Can non-symbolic XAI's myths be stopped?

SHAP on 2025/07/14:

A unified approach to interpreting model predictions

Authors Scott M Lundberg, Su-In Lee

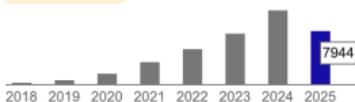
Publication date 2017

Journal Advances in neural information processing systems

Volume 30

Description Understanding why a model makes a certain prediction can be as crucial as the prediction's accuracy in many applications. However, the highest accuracy for large modern datasets is often achieved by complex models that even experts struggle to interpret, such as ensemble or deep learning models, creating a tension between accuracy and interpretability. In response, various methods have recently been proposed to help users interpret the predictions of complex models, but it is often unclear how these methods are related and when one method is preferable over another. To address this problem, we present a unified framework for interpreting predictions, SHAP (SHapley Additive exPlanations). SHAP assigns each feature an importance value for a particular prediction. Its novel components include:(1) the identification of a new class of additive feature importance measures, and (2) theoretical results showing there is a unique solution in this class with a set of desirable properties. The new class unifies six existing methods, notable because several recent methods in the class lack the proposed desirable properties. Based on insights from this unification, we present new methods that show improved computational performance and/or better consistency with human intuition than previous approaches.

Total citations Cited by 37527



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SHAP on 2025/07/14:

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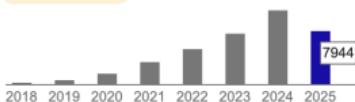
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Total citations

Cited by 37527



But,

Theoretical SHAP scores can mislead;
& results of tool SHAP of poor quality!

Many, many high-risk uses of SHAP & clones

He et al. *Journal of Translational Medicine* (2024) 22:686
https://doi.org/10.1186/s12967-024-05417-y

Journal of Translational
Medicine

RESEARCH

Open Access



Predictive models for personalized precision medical intervention in spontaneous regression stages of cervical precancerous lesions

XGBoost-SHAP-based interpretable diagnostic framework for alzheimer's disease

Hypothesis-free discovery of novel cancer predictors using machine learning

Explainable AI-based Deep-SHAP for mapping the multivariate relationships between regional neuroimaging biomarkers and cognition

npj Digital Medicine

Interpretable prediction of 30-day mortality in patients with acute pancreatitis based on machine learning and SHAP

scientific reports

scientific reports

OPEN

An explainable machine learning framework for lung cancer hospital length of stay prediction

© J. Marques-Silva

PLOS ONE

RESEARCH ARTICLE

Combining explainable machine learning, demographic and multi-omic data to inform precision medicine strategies for inflammatory bowel disease

scientific reports

OPEN

Explanatory predictive model for COVID-19 severity risk employing machine learning, shapley addition, and LIME



ARTICLE OPEN

Comparing machine learning algorithms for predicting ICU admission and mortality in COVID-19



scientific reports

OPEN

The importance of interpreting machine learning models for blood glucose prediction in diabetes: an analysis using SHAP



OPEN SHAP based predictive modeling for 1 year all-cause readmission risk in elderly heart failure patients: feature selection and model interpretation



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What next?

What next? Massive retraction (of tens of thousands) of papers??



thanks to ChatGPT...

What next? Massive retraction (of tens of thousands) of papers??



& beware of high-risk uses of SHAP! 😞

thanks to ChatGPT...

What's the bottom line?

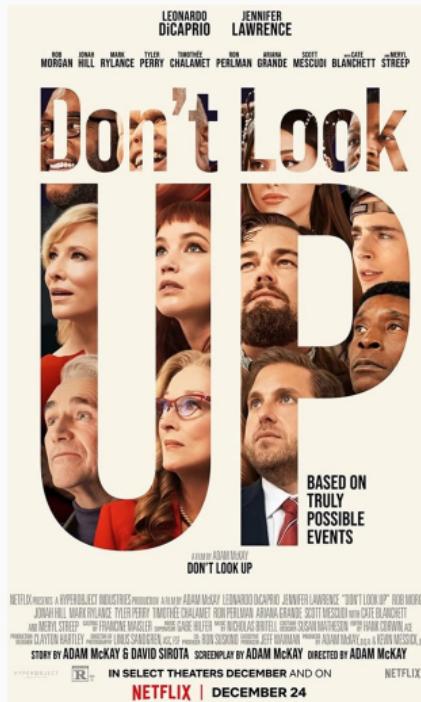
What's the bottom line?

- Non-symbolic XAI research experiences a persistent “*Don’t Look Up*” moment...



What's the bottom line?

- Non-symbolic XAI research experiences a persistent “*Don’t Look Up*” moment...



BTW, there are a multitude
of proposed uses of
LIME/Shap in medicine... ⚠

Some unsettling works...

- For DTs:
 - One AXp in polynomial-time [IIM20, HIIM21, IIM22]
 - All CXps in polynomial-time [HIIM21, IIM22]

Some unsettling works...

- For DTs:
 - One AXp in polynomial-time [IIM20, HIIM21, IIM22]
 - All CXps in polynomial-time [HIIM21, IIM22]

Declarative Reasoning on Explanations Using Constraint Logic Programming

Abstract. Explaining opaque Machine Learning (ML) models is an increasingly relevant problem. Current explanation in AI (XAI) methods suffer several shortcomings, among others an insufficient incorporation of background knowledge, and a lack of abstraction and interactivity with the user. We propose REASONX, an explanation method based on Constraint Logic Programming (CLP). REASONX can provide declarative, interactive explanations for decision trees, which can be the ML models under analysis or global/local surrogate models of any black-box model. Users can express background or common sense knowledge using linear constraints and MILP optimization over features of factual and contrastive instances, and interact with the answer constraints at different levels of abstraction through constraint projection. We present here the architecture of REASONX, which consists of a Python layer, closer to the user, and a CLP layer. REASONX's core execution engine is a Prolog meta-program with declarative semantics in terms of logic theories.

arXiv:2309.00422v1 [cs.AI] 1 Sep 2023

Some unsettling works...

- For DTs:
 - One AXp in polynomial-time
 - All CXps in polynomial-time

[IIM20, HIIM21, IIM22]

[HIIM21, IIM22]

HHAI 2024: Hybrid Human AI Systems for the Social Good
F. Lorig et al. (Eds.)
© 2024 The Authors.

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of the Creative Commons Attribution Non-Commercial License 4.0 (CC BY-NC 4.0).
doi:10.3233/FAIA240183*

Exploring Large Language Models Capabilities to Explain Decision Trees

Some unsettling works...

- For DTs:
 - One AXp in polynomial-time
 - All CXps in polynomial-time

[IIM20, HIIM21, IIM22]

[HIIM21, IIM22]

Explainable Artificial Intelligence for Academic Performance Prediction. An Experimental Study on the Impact of Accuracy and Simplicity of Decision Trees on Causability and Fairness Perceptions

FAccT '24, June 03–06, 2024, Rio de Janeiro, Brazil
© 2024 Copyright held by the owner/author(s).
ACM ISBN 979-8-4007-0450-5/24/06
<https://doi.org/10.1145/3630106.3658953>

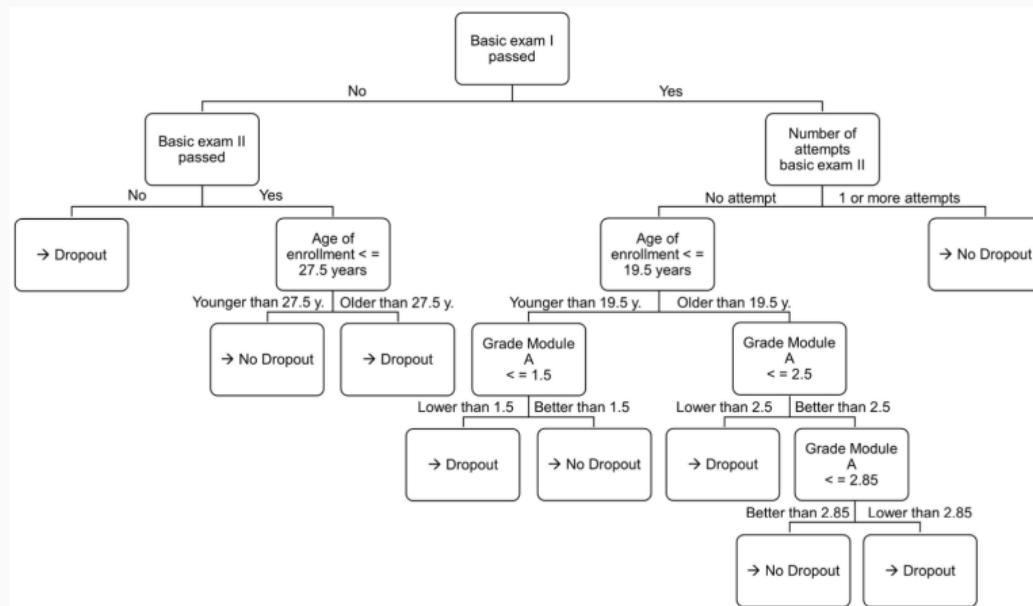
Some unsettling works...

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[IIM20, HIIM21, IIM22]

[HIIM21, IIM22]



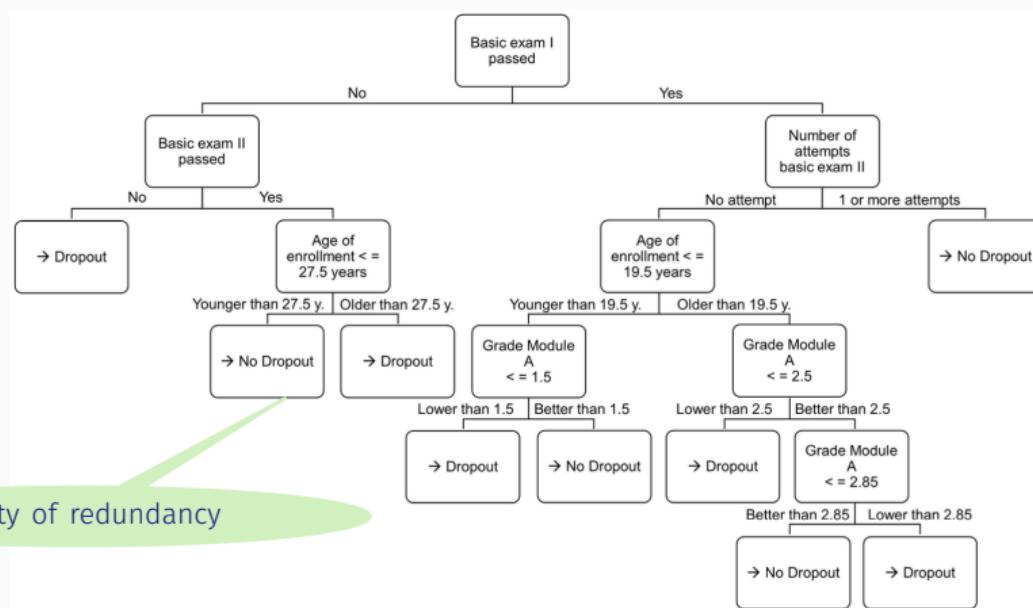
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[HIIM21, IIM22]



Outline – Unit #08

Some Words of Concern

Conclusions & Research Directions

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- Covered logic-based (aka symbolic, aka formal) XAI & its recent progress:
 - Abductive & contrastive explanations
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 - Duality & enumeration
 - Other explainability queries – feature necessity & relevancy

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- Demonstrated tight connection between (rigorous) feature selection and (rigorous) feature attribution in XAI
- Symbolic XAI exhibits links with many fields of research:
machine learning, artificial intelligence, formal methods, automated reasoning, optimization, computational social choice (& game theory), etc.

Research directions

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- Scalability, scalability, and scalability

Research directions

- Scalability, scalability, and scalability
- Sample-based explanations & coherency

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- Sample-based explanations & coherency
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- Rigorous feature attribution
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- Any ideas from you?

Research directions

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- Distance-restricted explanations
- Probabilistic explanations
- Any ideas from you?
- ... And trying to curb the **massive** momentum of (heuristic) XAI **myths!**

What this course covered

- Lecture 01 – unit(s):
 - #01: Foundations
- Lecture 02 – unit(s):
 - #02: Principles of symbolic XAI – **feature selection**
 - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
 - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
 - #05: Explainability queries
- Lecture 04 – unit(s):
 - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
 - #07: Principles of symbolic XAI – **feature attribution** (& myth of Shapley values in XAI)
 - #08: Corrected feature attribution – nuSHAP
 - #09: Conclusions & research directions

Some food for thought...

"All truths are easy to understand once they are discovered; the point is to discover them."

(G. Galilei)

"Beware of false knowledge; it is more dangerous than ignorance."

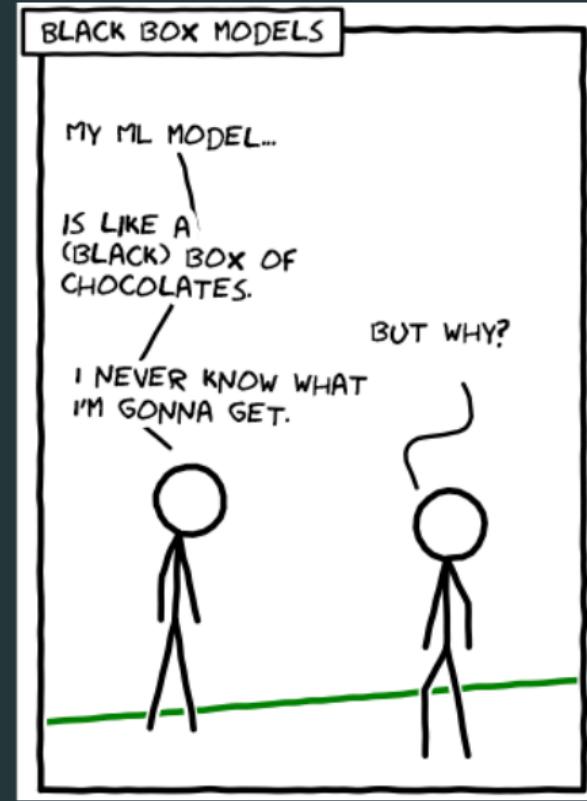
(G. B. Shaw)

"He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast."

(L. da Vinci)

Q & A

Acknowledgment: joint work with X. Huang, Y. Izza, O. Létoffé, A. Ignatiev, N. Narodytska, M. Cooper, N. Asher, A. Morgado, J. Planes, et al.



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