Logic-Based Explainable Artificial Intelligence

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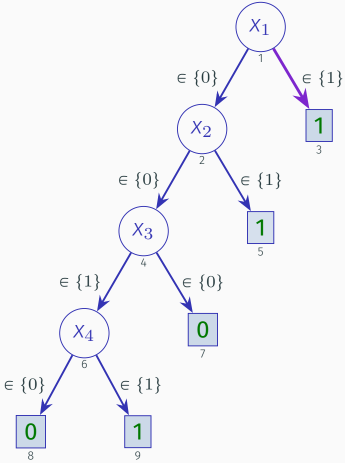
Lecture 04

Recapitulate third lecture

- Logic encoding for explaining DLs
  - And status of (in)tractability in logic-based XAI
- Query: enumeration of explanations
- Query: feature necessity, AXp & CXp
- Query: feature relevancy

Recap example

- Instance $(\mathbf{v}, c) = ((0, 0, 0, 0), 0)$



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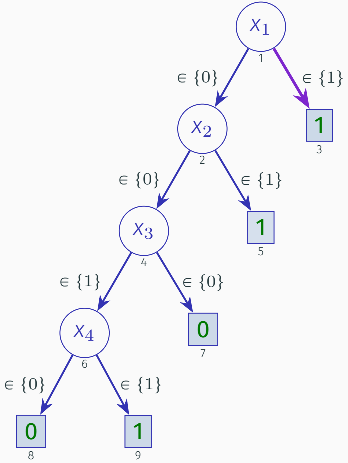
- Instance $(\mathbf{v}, c) = ((0,0,0,0),0)$
- Is feature 1 AXp-necessary?



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- Instance $(\mathbf{v}, c) = ((0,0,0,0), 0)$
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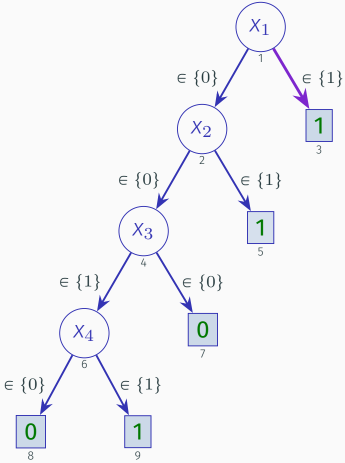

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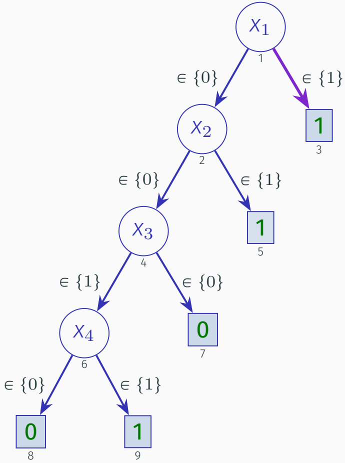

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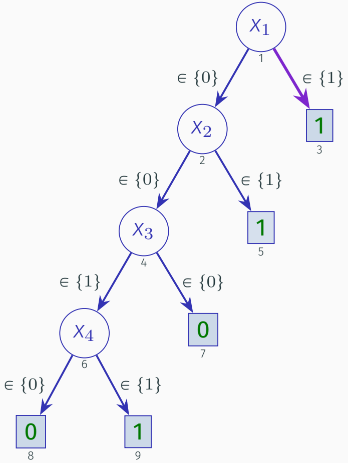

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- Is feature 3 AXp-necessary?
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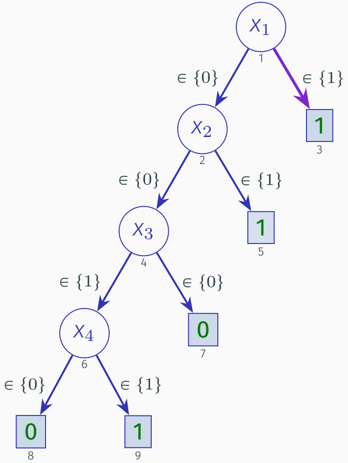

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- Is feature 3 AXp-necessary?
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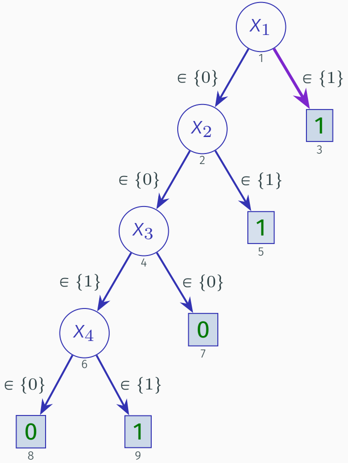
  - **No!** Thus, feature 3 is not AXp-necessary



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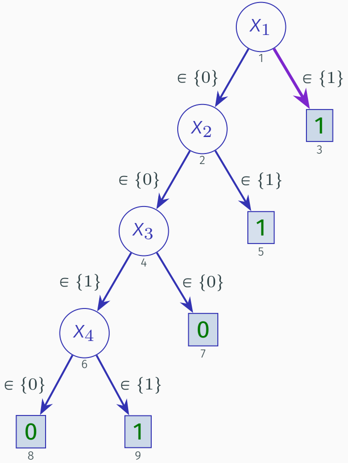
- Are there CXp-necessary features?
  - **No!** There are no singleton AXps



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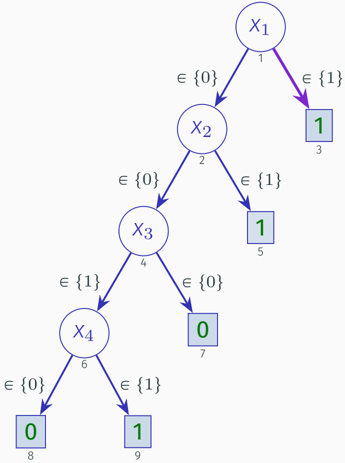

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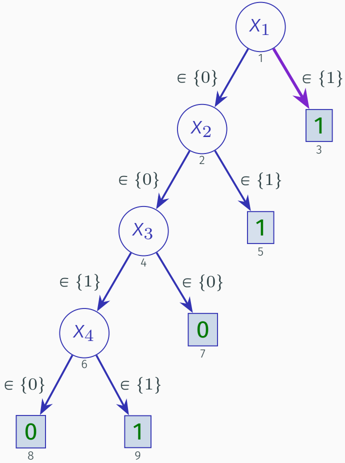
- Are there CXp-necessary features?
  - No! There are no singleton AXps
- Confirmation:
  - CXps: $\{\{1\}, \{2\}, \{3, 4\}\}$ (2 is also AXp-necessary)
  - AXps: $\{\{1, 2, 3\}, \{1, 2, 4\}\}$



Recap example – a different instance

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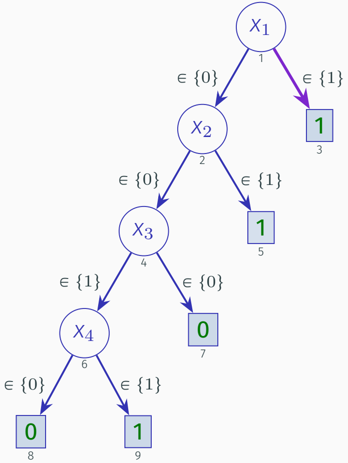

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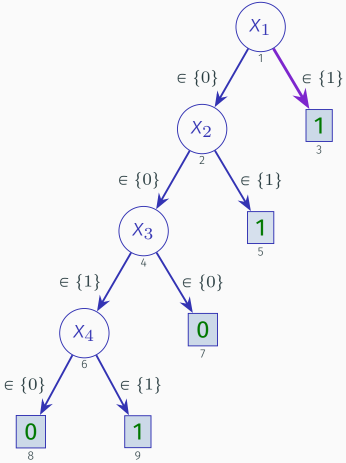

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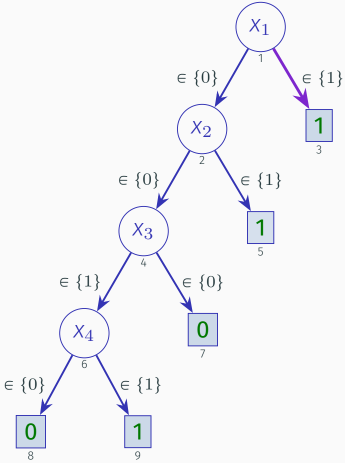
  - **Yes!** Features 1 and 2 (i.e. singleton AXps)



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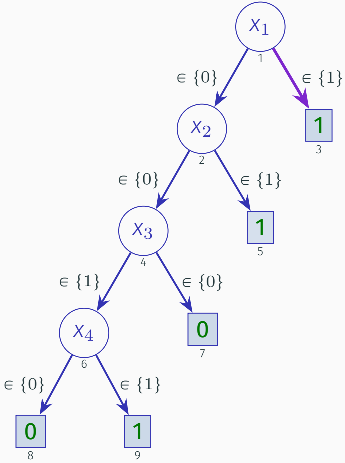

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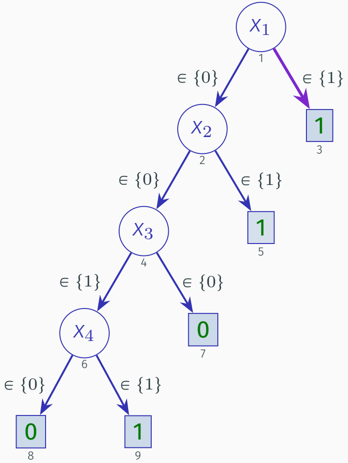

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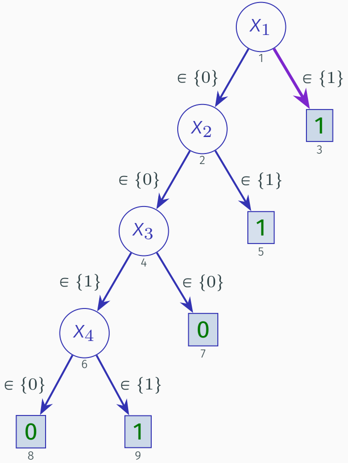

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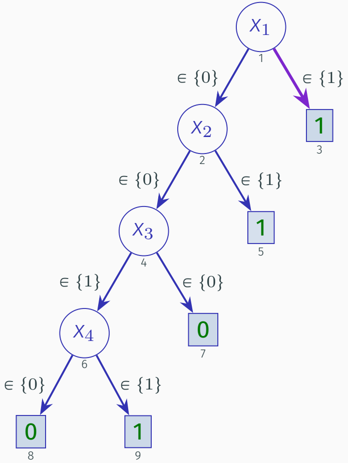

Another example – feature necessity & relevancy

- Classifier: $\mathcal{F} = \{1, 2, 3, 4, 5\}$; $\mathcal{D}_i = \{0, 1\}$, $i = 1, \ldots, 5$; $\mathcal{K} = \{0, 1\}$

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- Partially enumerate AXps/CXps, exploiting bias in enumeration

Plan for this course

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Detour: Monotonic Classification & Voting Power

Monotonically increasing boolean classifiers

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- Monotonic classifier $\mathcal{M} = (\mathcal{F}, \mathbb{F}, \mathcal{K}, \kappa)$, such that each $\mathcal{D}_i = \{0, 1\}$ and $\mathcal{K} = \{0, 1\}$ are ordered (i.e. $0 < 1$), and
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- Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}$ be such that $\kappa(\mathbf{v}_1) = \kappa(\mathbf{v}_2) = 1$, and $\mathbf{v}_1 \leqslant \mathbf{v}_2$
  Define the explanation problems:
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- Then,
  - If WAXp$(\mathcal{S}; \mathcal{E}_1)$ holds, then WAXp$(\mathcal{S}; \mathcal{E}_2)$ holds; in particular:
  - $\mathbb{A}(\mathcal{E}_{\mathbb{1}})$ contains **all** the AXps of **any** instance of the form $(\mathbf{v}_r, 1)$
    - Why?
      - Pick any explanation problem $\mathcal{E}_r$ with instance $(\mathbf{v}_r, 1)$
      - Start from $\mathbb{1} = (1, 1, \ldots, 1)$
      - Remove features that take value 0 in $\mathbf{v}_r$; we still have an WAXp
      - Then compute any AXp starting from features taking value 1 in $\mathbf{v}_r$

      $\therefore$ **Suffices to find explanations for $\mathcal{E}_{\mathbb{1}}$** (or alternatively, the global explanations for prediction 1)

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  - Problem: find a measure of importance of each voter !
    - I.e. measure the a priori voting power of each voter

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- WVG: $[12; 4, 4, 4, 2, 2, 1]$
- **Q**: What should be the voting power of Luxembourg?

An example – EEC (EU) members voting power in 1958

Coutry	Acronym	# Votes
France	F	4
Germany	D	4
Italy	I	4
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- **Q**: What should be the voting power of Luxembourg?
- Can Luxembourg (L) ***matter*** for some winning coalition?
- Perhaps surprisingly, answer is **No**!
  - In 1958, Luxembourg was a **dummy** voter/player

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- The corresponding classifier is:

$$\kappa(x_1, x_2, x_3, x_4, x_5, x_6) \quad := \quad \begin{cases} 1 & \text{IF } (4x_1 + 4x_2 + 4x_3 + 2x_4 + 2x_5 + x_6 \geqslant 12) \\ 0 & \text{otherwise} \end{cases}$$

which we have seen before! E.g. $\{2, 3, 4, 5\}$ is an AXp & feature 6 (L) is irrelevant

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- WVG: [21; 12, 9, 4, 4, 1, 1, 1]

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- Homework:
  - Create your own weighted voting games;
  - Compute the sets of AXps and CXps; and
  - Assess the importance of features and how they compare to each other

Unit #06

Advanced Topics

Recurring challenge: how to explain highly-complex ML models?

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  - **And training data is nothing but a sample**
    - From which ML models are learned!
  - Here is an idea:
    - Adopt **symbolic** (and so, rigorous) sample-based XAI

[Amg23, CA23, ACD24]

An example...

Sample:

x_1	x_2	x_3	x_4	$\kappa(\cdot)$
0	0	0	0	0
0	0	1	0	0
0	0	1	1	0
0	1	1	0	0
1	0	0	0	1
1	1	0	0	1
1	1	1	1	1

Instance: $((1, 1, 1, 1), 1)$

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- How to explain prediction given only the sample
- If $x_1 = 1$, then prediction is 1 (given the sample)
- Sample-based AXp (sbAXp): $\{1\}$

Definitions in sample-based XAI – replace feature space with sample...

[MSLLM25]

- Let $\mathbb{S} \subseteq \mathbb{F}$ denote a sample, and let instance be $(\mathbf{v}, c)$
- Then, for $\mathcal{X} \subseteq \mathcal{F}$,

$$\mathsf{sbWAXp}(\mathcal{X}) \;:=\; \forall(\mathbf{x} \in \mathbb{S}).\left(\bigwedge\nolimits_{i\in\mathcal{X}} x_i = v_i\right) \rightarrow (\kappa(\mathbf{x}) = c)$$

- And, for $\mathcal{Y} \subseteq \mathcal{F}$,

$$\mathsf{sbWCXp}(\mathcal{Y}) \;:=\; \exists(\mathbf{x} \in \mathbb{S}).\left(\bigwedge\nolimits_{i\in\mathcal{F}\setminus\mathcal{Y}} x_i = v_i\right) \land (\kappa(\mathbf{x}) \neq c)$$

- sbAXps (resp. sbCXps) are the subset-minimal sets that respect the above definition for sbWAXp (resp. sbWCXp)

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- sbAXps (resp. sbCXps) are the subset-minimal sets that respect the above definition for sbWAXp (resp. sbWCXp)
  - Rigorous alternative to Anchor & variants

Approach for computing sbCXps & sbAXps

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- Set of sbAXps (by MHS duality): $\mathbb{A} = \{\{1\},\{2,4\}\}$

Some recent results

- MHS duality holds for sample-based explanations:
  - $\mathcal{Y} \subseteq \mathcal{F}$ is sbCXp iff it is a MHS of set of sbAXps
  - $\mathcal{X} \subseteq \mathcal{F}$ is sbAXp iff it is a MHS of set of sbCXps
- Number of sb(W)CXps is linear on $|\mathbb{S}|$
- Number of sb(A)CXps can be exponentially large on $|\mathbb{S}|$
- Additional results:

Problem	Complexity	
	Total	Given sbCXps
All sbCXps	$\mathcal{O}(mn^2)$	—
One sbCXp	$\mathcal{O}(mn)$	$\mathcal{O}(1)$
One (smallest) sbCXp	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$
One sbAXp	$\mathcal{O}(mn)$	$\mathcal{O}(mn)$
Feature relevancy	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$
sbAXp-necessity	$\mathcal{O}(mn^2)$	$\mathcal{O}(mn)$
sbCXp-necessity	$\mathcal{O}(mn^2)$	$\mathcal{O}(n)$

- Complexity-wise:
  - Deciding the existence of an sbAXp of size no larger than k is NP-complete.
  - sbAXp enumeration corresponds to hypergraph transversal

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- An example:
  - Sample:

Entry	x_1	x_2	$\kappa(\cdot)$
1	0	1	0
2	1	0	1
3	0	0	2

  - Instance 1: $((0,1),0)$
  - Instance 2: $((1,0),1)$

AXp(s) for $((0,1),0)$

Sample:

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- Set of sbAXps (by MHS duality): $\mathbb{A} = \{\{2\}\}$
  - Meaning: IF $(x_2 = 1)$ THEN $\kappa(\mathbf{x}) = 0$

AXp(s) for $((1,0),1)$

Sample:

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- Set of sbAXps (by MHS duality): $\mathbb{A} = \{\{1\}\}$
  - Meaning: IF $(x_1 = 1)$ THEN $\kappa(\mathbf{x}) = 1$

The problem of lack of coherency

- Instance $((0,1),0)$ with AXp $\{2\}$, for $x_2 = 1$:
  - Points consistent with AXp: $\{(0,1),(1,1)\}$
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- Open topic of research...

Outline – Unit #06

General definition of prediction sufficiency

- Instance $(\mathbf{v}, c)$
- Let $\mathcal{S} \subseteq \mathcal{F}$:
  - Recall,

$$\Upsilon(\mathcal{S}; \mathbf{v}) = \{\mathbf{x} \in \mathbb{F} \mid \mathbf{x}_{\mathcal{S}} = \mathbf{v}_{\mathcal{S}}\}$$

- $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Upsilon(\mathcal{S}; \mathbf{v})) \rightarrow (\sigma(\mathbf{x}))$$

  - **Obs:** a WAXp is just one possible example
- But there are other ways to study prediction sufficiency:
  - One can envision defining other sets of points Γ, parameterized by $\mathcal{E} = (\mathcal{M}, (\mathbf{v}, c))$; $\mathcal{S} \subseteq \mathcal{F}$ suffices for prediction c if:

$$\forall(\mathbf{x} \in \mathbb{F}).(\mathbf{x} \in \Gamma(\mathcal{S}; \mathcal{E})) \rightarrow (\sigma(\mathbf{x}))$$

    - And one can also envision generalizations of σ!

Outline – Unit #06

Towards more expressive explanations – inflated explanations

[IISM24]

- Recall:

$$\mathsf{WAXp}(\mathcal{X}) \quad := \quad \forall(\mathbf{x} \in \mathbb{F}). \bigwedge_{j\in\mathcal{X}}(x_j = v_j) \rightarrow (\kappa(\mathbf{x}) = c)$$

  - For non-boolean features, use of $=$ may convey little information, e.g. with real-valued features, having $x_1 = 1.157$ does not help in understanding what values of feature 1 are also acceptable

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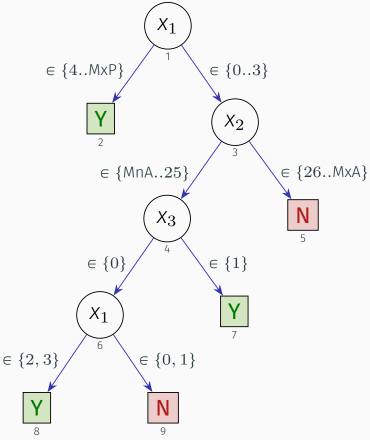
- Inflated explanations allow for more expressive literals, i.e. $=$ replaced with $\in$, and individual values replaced by ranges of values
  - Operational definition: Given an AXp, expand set of values of each feature, in some chosen order, such that the set of picked features remains unchanged

[IIM22]



- Explanation for $((2, 20, 0), \mathsf{Y})$? (Obs: $\mathsf{MnA} = 18; \mathsf{MxP} > 4$)

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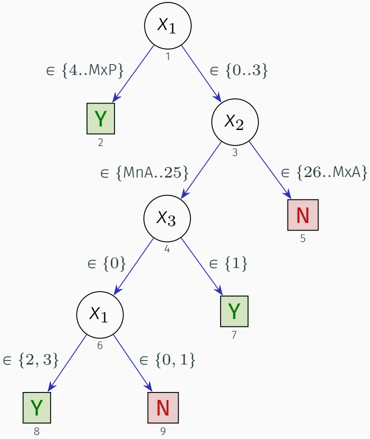

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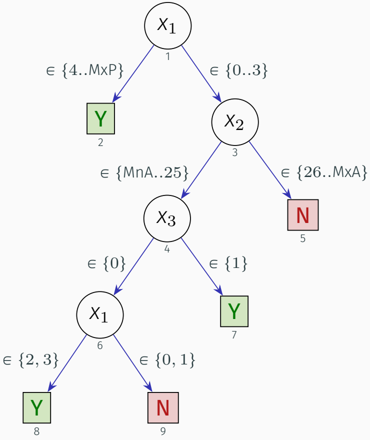

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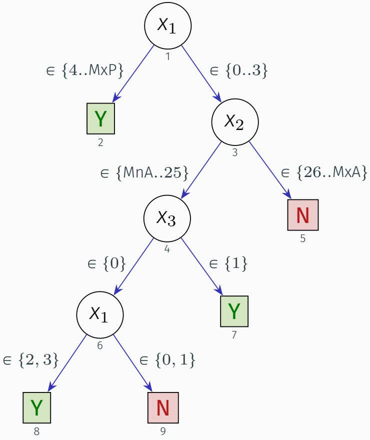

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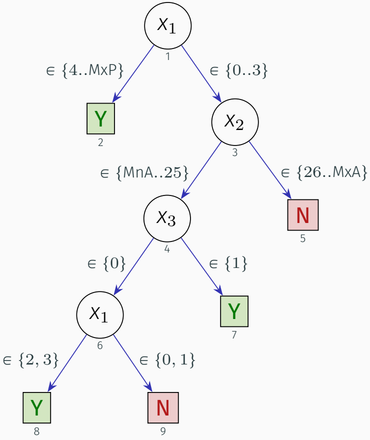

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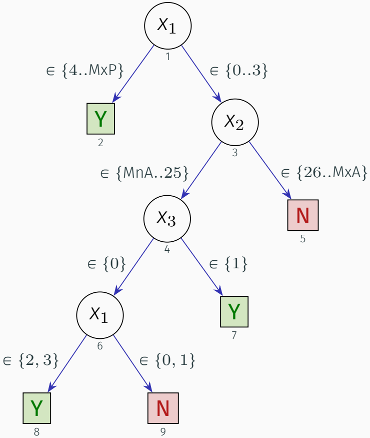

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Approach

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- **Obs:** More complex alternative is to find AXp and expand domains simultaneously
  - This is conjectured to change the complexity class of finding one explanation

Another fun example (from 2019) – (inflated) AXp?

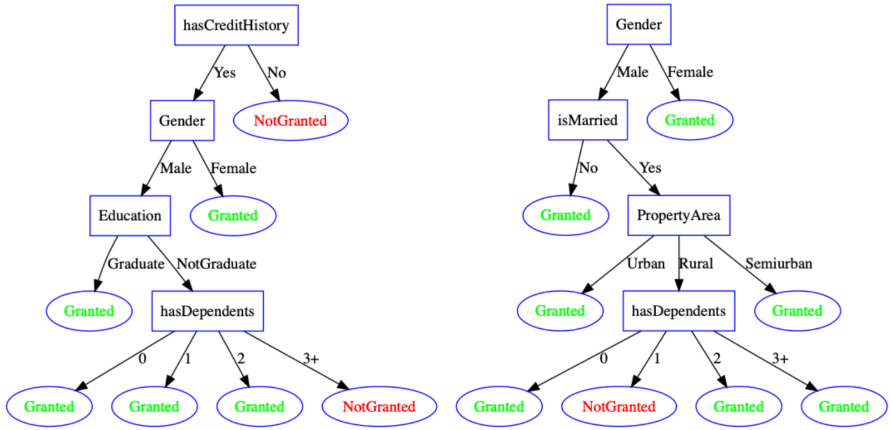

Fig. 2: Decision trees of size 'small' in the loan domain, extracted without (left) and with (right) a domain ontology. As it can be seen the features used in the creation of the conditions in the split nodes are different.

Instance: Gender=Male, Education=NotGraduate, hasCreditHistory=Yes, isMarried=Yes, PropertyArea=Rural, isSelfEmployed=No, hasDependents=2

Another fun example (now from 2020/21) – (inflated) AXp?

[CWBdPM21]

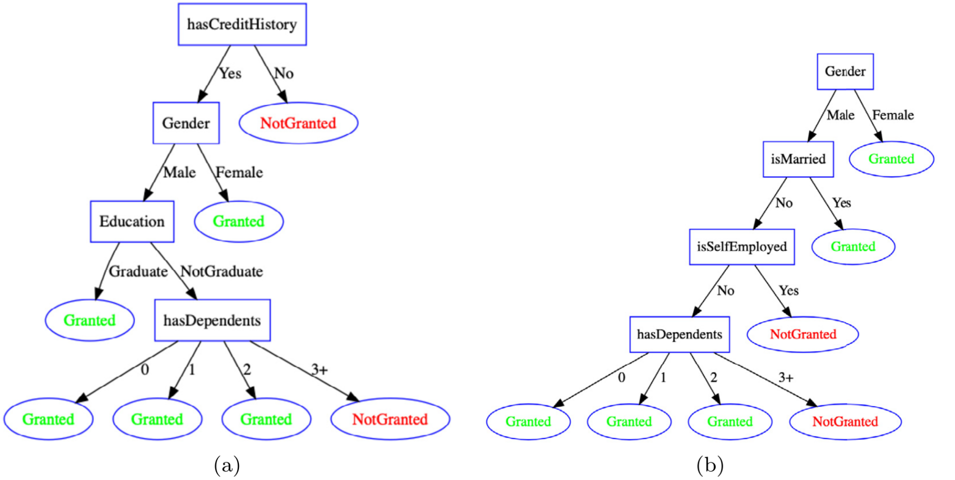

Fig. 2. Decision trees of size 'small' in the loan domain, extracted without (a) and with (b) a domain ontology. It can be seen that the use of an ontology leads to different features appearing in the split nodes. For instance, in the ontology used to build tree (b) the concept *Gender* is more abstract than *isMarried* and *isSelfEmployed* has thus a lower information content according to Definition 3.1. Concepts with lower information content are favoured as conditions for split nodes in the tree according to Definition 3.2, which leads to *Gender* being used first by TREPAN-Reloaded when it generated the split nodes of tree (b). Furthermore, the ontology does not include concepts associated to *hasCreditHistory* and *Education*, which are therefore not considered in the construction of tree (b).

Outline – Unit #06

Not all inputs may be possible – input constraints

[GR22, YIS+23]

- The (implicit) assumption that all inputs are possible is often unrealistic
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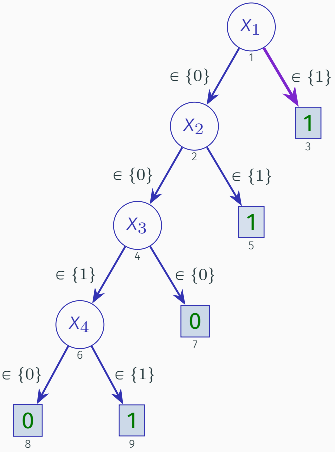
- Constrained AXps/CXps find other applications!

An example

- Instance: $((1, 1, 1, 1), 1)$
- Unconstrained AXps:



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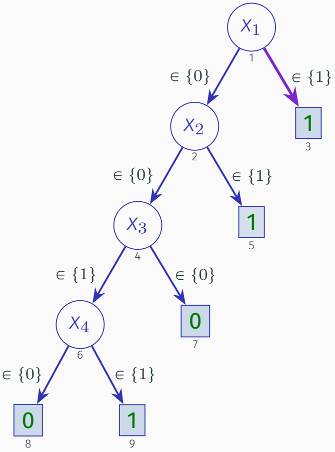

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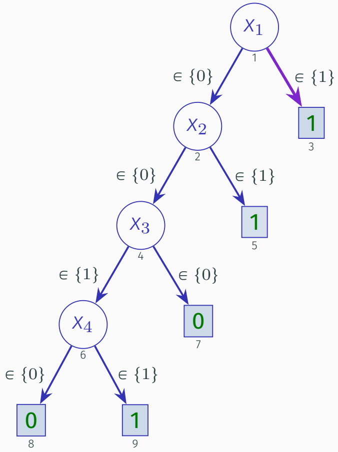

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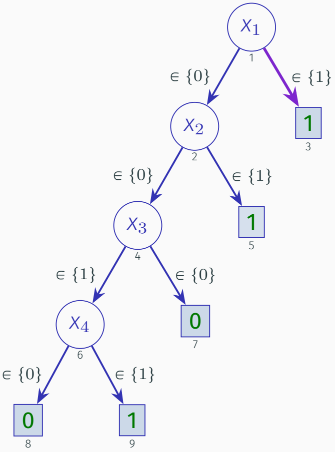

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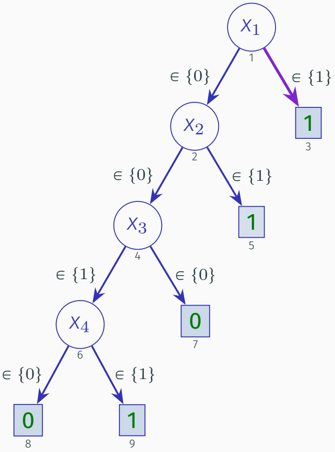

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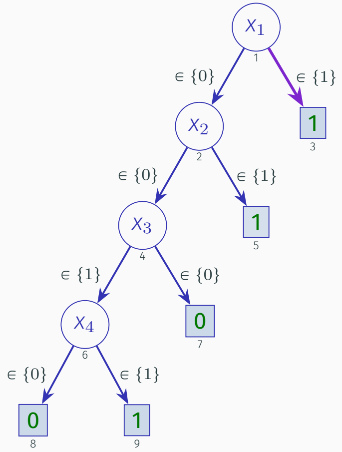

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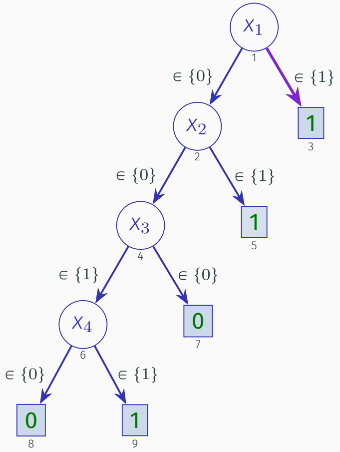

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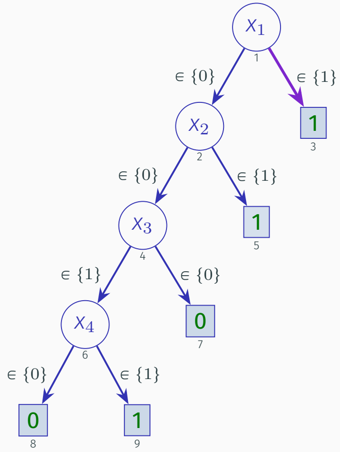

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Outline – Unit #06

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- Change definition of WAXp/WCXp to account for l_p distance to $\mathbf{v}$:

$$\forall(\mathbf{x} \in \mathbb{F}).\left[\bigwedge\nolimits_{j\in\mathcal{X}}(x_j = v_j) \wedge \left(\|\mathbf{x}-\mathbf{v}\|_{l_p} \leqslant \epsilon\right)\right] \rightarrow (\sigma(\mathbf{x}))$$

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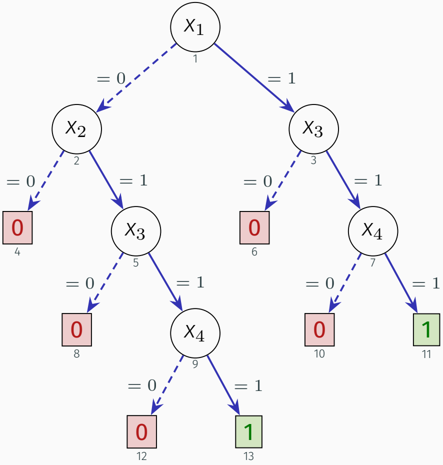
  - Norm l_p is arbitrary, e.g. Hamming, Manhattan, Euclidean, etc.
  - **Distance-restricted explanations**: $\mathfrak{d}$AXp/$\mathfrak{d}$CXp

An example – DT & instance $((1, 1, 1, 1), 1)$



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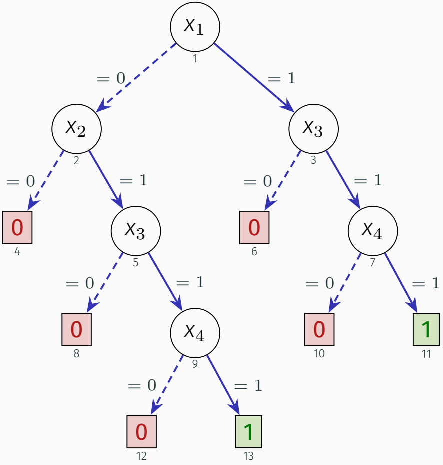

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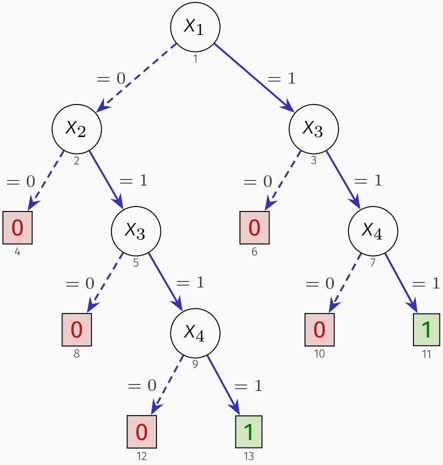

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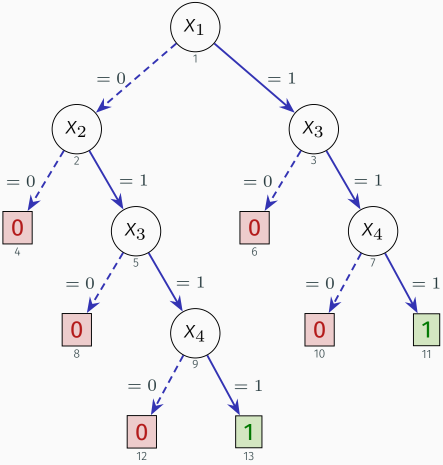

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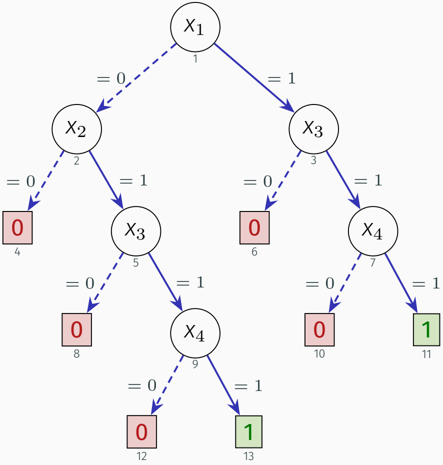

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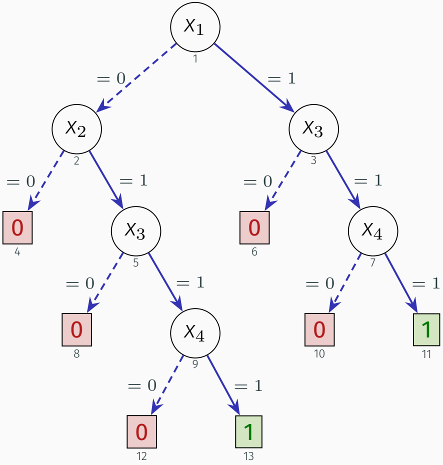
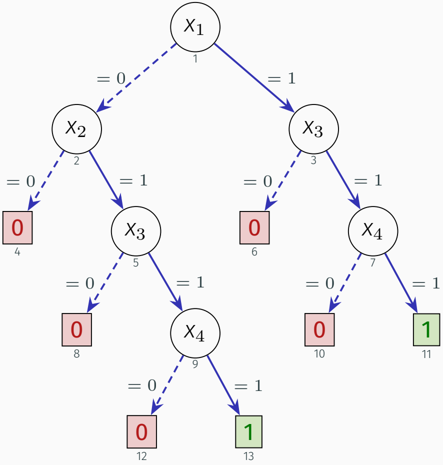
- Distance-restricted AXps/CXps, $\mathfrak{d}$AXp/$\mathfrak{d}$CXp, with Hamming distance (l_0) and $\epsilon = 1$:



An example – DT & instance $((1,1,1,1),1)$

- Plain AXps/CXps:
  - AXps? $\{\{1,3,4\},\{2,3,4\}\}$
  - CXps? $\{\{1,2\},\{3\},\{4\}\}$

- Distance-restricted AXps/CXps, $\mathfrak{d}$AXp/$\mathfrak{d}$CXp, with Hamming distance (l_0) and $\epsilon = 1$:
  - Points of interest: $\{(1,1,1,1),(0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0)\}$



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  - AXps? $\{\{1,3,4\},\{2,3,4\}\}$
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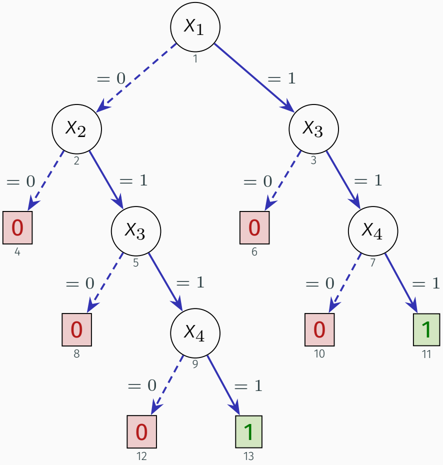
  - ∂AXps?



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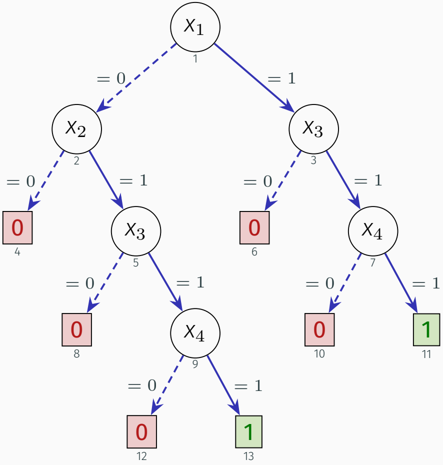
  - $\mathfrak{d}$AXps? $\{\{3,4\}\}$
  - $\mathfrak{d}$CXps?



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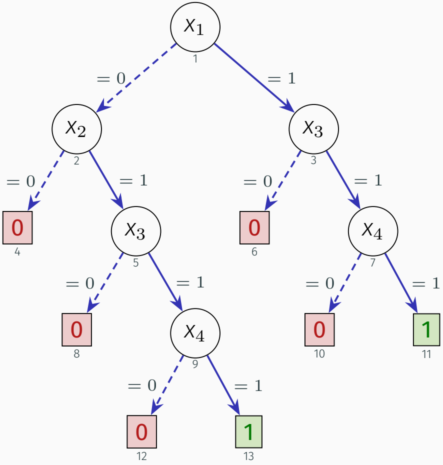
  - $\mathfrak{d}$AXps? $\{\{3,4\}\}$
  - $\mathfrak{d}$CXps? $\{\{3\},\{4\}\}$



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  - AXps? $\{\{1,3,4\},\{2,3,4\}\}$
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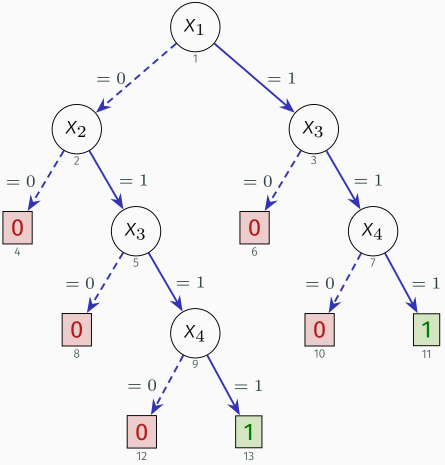
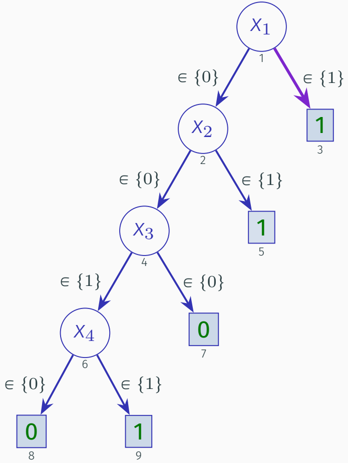
  - $\mathfrak{d}$AXps? $\{\{3,4\}\}$
  - $\mathfrak{d}$CXps? $\{\{3\},\{4\}\}$
- Given ϵ, larger adversarial examples are excluded



Another example – DT & instance $((1, 1, 1, 1), 1)$



Another example – DT & instance $((1,1,1,1),1)$

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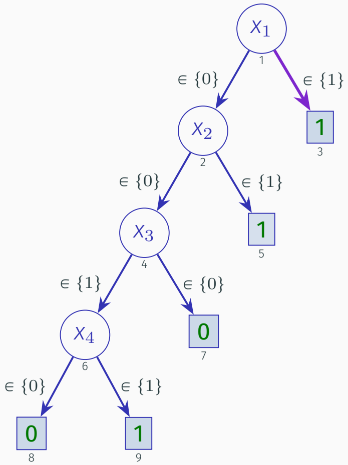
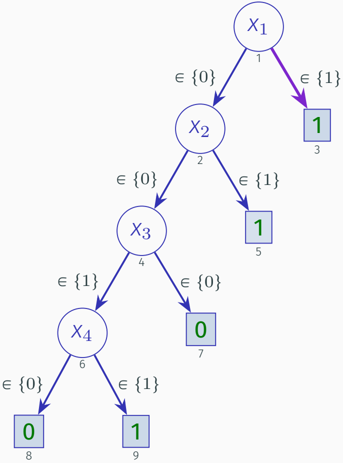


Another example – DT & instance $((1, 1, 1, 1), 1)$

- Plain AXps/CXps:
  - AXps?



Another example – DT & instance $((1,1,1,1),1)$

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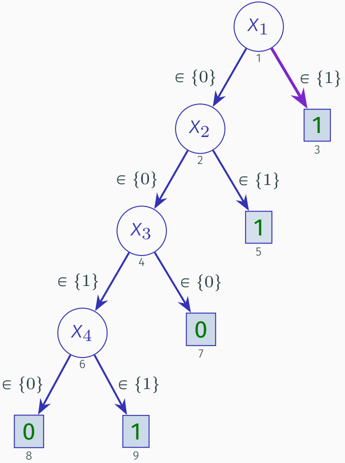
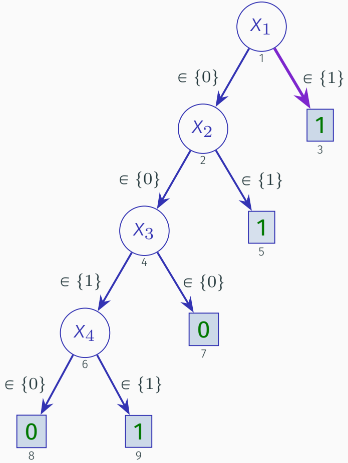


Another example – DT & instance $((1, 1, 1, 1), 1)$

- Plain AXps/CXps:
  - AXps? $\{\{1\}, \{2\}, \{3, 4\}\}$
  - CXps? $\{\{1, 2, 3\}, \{1, 2, 4\}\}$



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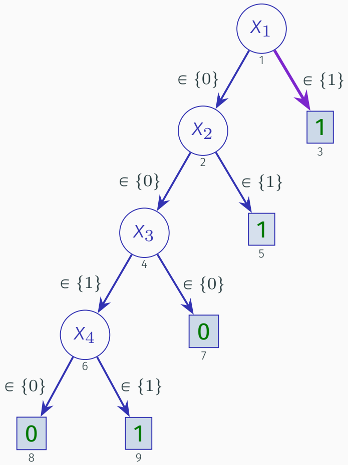

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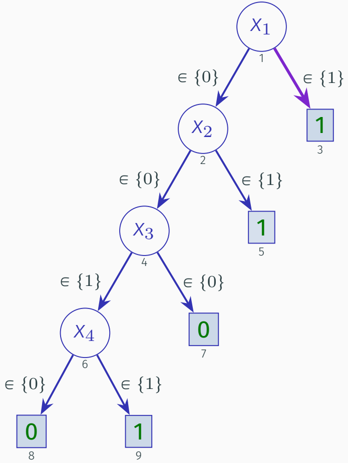
  - Constant function...



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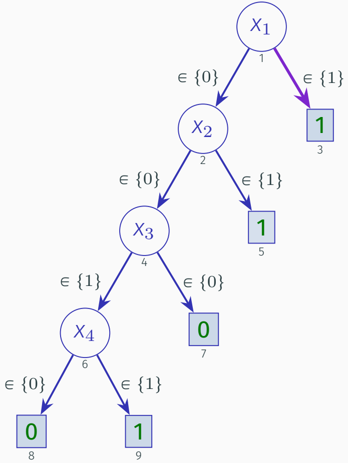
  - $\mathfrak{d}$AXps?



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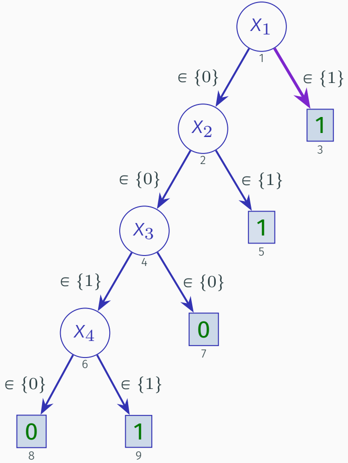
  - Constant function...
  - $\mathfrak{d}$AXps? $\{\emptyset\}$



- Distance-restricted WAXps/WCXps:

$$\forall(\mathbf{x}\in\mathbb{F}).\left[\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\land\left(\|\mathbf{x}-\mathbf{v}\|_{l_p}\leqslant\epsilon\right)\right]\rightarrow(\sigma(\mathbf{x}))$$

$$\exists(\mathbf{x}\in\mathbb{F}).\left[\bigwedge\nolimits_{j\in\mathcal{X}}(x_j=v_j)\land\left(\|\mathbf{x}-\mathbf{v}\|_{l_p}\leqslant\epsilon\right)\right]\land(\neg\sigma(\mathbf{x}))$$

Relating explanations with adversarial examples

- Distance-restricted WAXps/WCXps:

$$\forall(\mathbf{x} \in \mathbb{F}).\left[\bigwedge\nolimits_{j \in \mathcal{X}}(x_j = v_j) \land \left(\|\mathbf{x} - \mathbf{v}\|_{l_p} \leqslant \epsilon\right)\right] \rightarrow (\sigma(\mathbf{x}))$$

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- Given norm l_p and distance ϵ, there exists a (distance-restricted) WCXp iff there exists an adversarial example
  - Use robustness tool to decide existence of WCXp
  - But, WAXp decided given non existence of CXp!

Relating explanations with adversarial examples

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- Efficiency of distance-restricted explanations correlates with efficiency of finding adversarial examples
  - One can use most complete robustness tools, e.g. VNN-COMP [BMB+23]
- Clear scalability improvements for explaining NNs (see next) [HM23a, WWB23, IHM+24a, IHM+24b]

Basic algorithm

[HM23a]

Input: Arguments: ϵ; Parameters: $\mathcal{E}$, p
Output: One $\mathfrak{d}$AXp $\mathcal{S}$

function FindAXpDel(ϵ; $\mathcal{E}$, p)
  $\mathcal{S} \leftarrow \mathcal{F}$ ▷ Initially, no feature is allowed to change
  for $i \in \mathcal{F}$ **do** ▷ Invariant: $\mathfrak{d}$WAXp($\mathcal{S}$)
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{i\}$
    outc $\leftarrow$ FindAdvEx($\epsilon, \mathcal{S}; \mathcal{E}, p$)
    if outc **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$
  return $\mathcal{S}$ ▷ $\mathfrak{d}$WAXp($\mathcal{S}$) $\land$ minimal($\mathcal{S}$) $\rightarrow$ $\mathfrak{d}$AXp($\mathcal{S}$)

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- **Obs:** Efficiency of logic-based XAI tracks efficiency of robustness tools

Basic algorithm

[HM23a]

Input: Arguments: ϵ; Parameters: $\mathcal{E}$, p
Output: One ∂AXp $\mathcal{S}$

```
function FindAXpDel(ϵ; ℰ, p)
    𝒮 ← ℱ                                   ▷ Initially, no feature is allowed to change
    for i ∈ ℱ do                            ▷ Invariant: ∂WAXp(𝒮)
        𝒮 ← 𝒮\{i}
        outc ← FindAdvEx(ϵ, 𝒮; ℰ, p)
        if outc then
            𝒮 ← 𝒮 ∪ {i}
    return 𝒮                                ▷ ∂WAXp(𝒮) ∧ minimal(𝒮) → ∂AXp(𝒮)
```

- **Obs:** Efficiency of logic-based XAI tracks efficiency of robustness tools
- **Limitation:** Running time grows with number of features

Results for NNs in 2023 (using Marabou [KHI+19])

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
			$\epsilon = 0.1$				$\epsilon = 0.05$		
ACASXU_1_5	#1	3	5	185.9	0	2	5	113.8	0
	#2	2	5	273.8	0	1	5	33.2	0
	#3	0	5	714.2	0	0	5	4.3	0
ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
	#2	3	5	226.4	0	2	5	125.1	0
	#3	2	5	1740.6	0	2	5	173.6	0
ACASXU_3_5	#1	4	5	43.6	0	2	5	59.4	0
	#2	3	5	5039.4	0	2	5	4303.8	1
	#3	2	5	5574.9	1	2	5	2660.3	0
ACASXU_3_6	#1	1	5	6225.0	1	0	5	51.0	0
	#2	3	5	4957.2	1	2	5	1897.3	0
	#3	1	5	196.1	0	1	5	919.2	0
ACASXU_3_7	#1	3	5	6256.2	0	4	5	26.9	0
	#2	4	5	311.3	0	1	5	6958.6	1
	#3	2	5	7756.5	1	1	5	7807.6	1
ACASXU_4_1	#1	2	5	12413.0	2	1	5	5090.5	1
	#2	1	5	5035.1	1	0	5	2335.6	0
	#3	4	5	1237.3	0	4	5	1143.4	0
ACASXU_4_2	#1	4	5	15.9	0	4	5	12.1	0
	#2	3	5	1507.6	0	1	5	111.3	0
	#3	2	5	5641.6	2	0	5	1639.1	0

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[HM23a]

DNN	points	|AXp|	#Calls	Time	#TO	|AXp|	#Calls	Time	#TO
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	#2	2	5	273.8	0	1	5	33.2	0
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ACASXU_3_1	#1	0	5	2219.3	0	0	5	14.2	0
	#2	2	5	4263.5	1	0	5	1853.1	0
	#3	1	5	581.8	0	0	5	355.9	0
ACASXU_3_2	#1	3	5	13739.3	2	1	5	6890.1	1
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Scales to a few **hundred** neurons

Recent improvements

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    $\mathcal{S} \leftarrow \mathcal{S} \backslash \{i\}$
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    if outc **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$
  return $\mathcal{S}$ ▷ $\mathfrak{d}$WAXp($\mathcal{S}$) $\wedge$ minimal($\mathcal{S}$) $\rightarrow$ $\mathfrak{d}$AXp($\mathcal{S}$)

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- To drop features from $\mathcal{S} \subseteq \mathcal{F}$, it is open whether paralellization might be applicable
  - Algorithm FindAXpDel is mostly sequential (see above)
  - Exploit parallelization for other algorithms, e.g. dichotomic search [IHM+24b]

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  for $i \in \mathcal{F}$ **do** ▷ Invariant: $\mathfrak{d}\text{WAXp}(\mathcal{S})$
    $\mathcal{S} \leftarrow \mathcal{S} \backslash \{i\}$
    outc $\leftarrow$ FindAdvEx($\epsilon, \mathcal{S}; \mathcal{E}, p$)
    if outc **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}$
  return $\mathcal{S}$ ▷ $\mathfrak{d}\text{WAXp}(\mathcal{S}) \land \text{minimal}(\mathcal{S}) \rightarrow \mathfrak{d}\text{AXp}(\mathcal{S})$

- To drop features from $\mathcal{S} \subseteq \mathcal{F}$, it is open whether paralellization might be applicable
  - Algorithm FindAXpDel is mostly sequential (see above)
  - Exploit parallelization for other algorithms, e.g. dichotomic search [IHM+24b]
- However, to decide whether $\mathcal{S}$ is an AXp, we can exploit parallelization:
  - Recall: $\text{AXp}(\mathcal{X}) := \text{WAXp}(\mathcal{X}) \land \forall(t \in \mathcal{X}).\neg\text{WAXp}(\mathcal{X} \backslash \{t\})$
  - Each $\neg\text{WAXp}(\cdot)$ (and also $\text{WAXp}(\cdot)$) check can be run in parallel!
  - Do this opportunistically, i.e. when set $\mathcal{S}$ is expected to be AXp [IHM+24b]

More recent results (from 2024)...

[IHM+24a, IHM+24b]

Model	Deletion							SwiftXplain						
	avgC	nCalls	Len	Mn	Mx	avg	TO	avgC	nCalls	Len	FD%	Mn	Mx	avg
gtsrb-dense	0.06	1024	448	52.0	76.3	63.1	0	0.23	54	447	77.4	10.8	14.0	12.2
gtsrb-convSmall	0.06	1024	309	59.2	82.6	65.1	0	0.22	74	313	39.7	15.1	19.5	16.2
gtsrb-conv	—	—	—	—	—	—	100	96.49	45	174	33.2	3858.7	6427.7	4449.4
mnist-denseSmall	0.28	784	177	190.9	420.3	220.4	0	0.77	111	180	15.5	77.6	104.4	85.1
mnist-dense	0.19	784	231	138.1	179.9	150.6	0	0.75	183	229	11.5	130.1	145.5	136.8
mnist-convSmall	—	—	—	—	—	—	100	98.56	52	116	21.3	4115.2	6858.3	5132.8

More recent results (from 2024)...

[IHM+24a, IHM+24b]

Model	Deletion							SwiftXplain						
	avgC	nCalls	Len	Mn	Mx	avg	TO	avgC	nCalls	Len	FD%	Mn	Mx	avg
gtsrb-dense	0.06	1024	448	52.0	76.3	63.1	0	0.23	54	447	77.4	10.8	14.0	12.2
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Scales to **tens of thousands** of neurons!

Largest for MNIST: **10142** neurons
Largest for GSTRB: **94308** neurons

Outline – Unit #06

Probabilistic (formal) explanations

[WMHK21, IIN+22, IHI+22, ABOS22, IHI+23, IMM24]

- Explanation size is critical for human understanding [Mil56]
- Probabilistic explanations provide smaller explanations, by trading off rigor of explanation by explanation size

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- Definition of weak probabilistic AXp $\mathcal{X} \subseteq \mathcal{F}$:

$$\mathsf{WPAXp}(\mathcal{X}) \quad := \quad \mathbf{Pr}(\kappa(\mathbf{x}) = c) \,|\, \mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}) \geq \delta$$

  - Obs: $\mathbf{x}_{\mathcal{X}} = \mathbf{v}_{\mathcal{X}}$ requires points $\mathbf{x} \in \mathbb{F}$ to match the values of $\mathbf{v}$ for the features dictated by $\mathcal{X}$
  - Obs: for $\delta = 1$ we obtain a WAXp

Definitions

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- Weak probabilistic AXp (WPAXp):

$$\text{WeakPAXp}(\mathcal{X};\mathbb{F},\kappa,\mathbf{v},c,\delta) :=$$

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  - Recent approximate algorithms for complex ML models [IMM24]

Results for decision trees

Dataset	DT N	DT A	Path M	Path m	Path avg	δ	MinPAXp Length M	MinPAXp Length m	MinPAXp Length avg	MinPAXp Prec avg	MinPAXp Time avg	LmPAXp Length M	LmPAXp Length m	LmPAXp Length avg	LmPAXp Prec avg	LmPAXp $m_{\subseteq}$	LmPAXp Time avg	Anchor D	Anchor Length M	Anchor Length m	Anchor Length avg	Anchor Length $F_{\notin P}$	Anchor Prec avg	Anchor Time avg
						100	11	3	6.8	100	2.34	11	3	6.9	100	100	0.00	d	12	2	7.0	26.8	76.8	0.96
adult	1241	89	14	3	10.7	95	11	3	6.2	98.4	5.36	11	3	6.3	98.6	99.0	0.01	u	12	3	10.0	29.4	93.7	2.20
						90	11	2	5.6	94.6	4.64	11	2	5.8	95.2	96.4	0.01							
						100	12	1	4.4	100	0.35	12	1	4.4	100	100	0.00	d	31	1	4.8	58.1	32.9	3.10
dermatology	71	100	13	1	5.1	95	12	1	4.1	99.7	0.37	12	1	4.1	99.7	99.3	0.00	u	34	1	13.1	43.2	87.2	25.13
						90	11	1	4.0	98.8	0.35	11	1	4.0	98.8	100	0.00							
						100	12	2	4.8	100	0.93	12	2	4.9	100	100	0.00	d	36	2	7.9	44.8	69.4	1.94
kr-vs-kp	231	100	14	3	6.6	95	11	2	3.9	98.1	0.97	11	2	4.0	98.1	100	0.00	u	12	2	3.6	16.6	97.3	1.81
						90	10	2	3.2	95.4	0.92	10	2	3.3	95.4	99.0	0.00							
						100	12	4	8.2	100	16.06	11	4	8.2	100	100	0.00	d	16	3	13.2	43.1	71.3	12.22
letter	3261	93	14	4	11.8	95	12	4	8.0	99.6	18.28	11	4	8.0	99.5	100	0.00	u	16	3	13.7	47.3	66.3	10.15
						90	12	4	7.7	97.7	16.35	10	4	7.8	97.8	100	0.00							
						100	14	3	6.4	100	0.92	14	3	6.5	100	100	0.00	d	35	2	8.6	55.4	33.6	5.43
soybean	219	100	16	3	7.3	95	14	3	6.4	99.8	0.95	14	3	6.4	99.8	100	0.00	u	35	3	19.2	66.0	75.0	38.96
						90	14	3	6.1	98.1	0.94	14	3	6.1	98.2	98.5	0.00							
						0	12	3	7.4	100	1.23	12	3	7.5	100	100	0.01	d	38	2	6.3	65.3	63.3	24.12
spambase	141	99	14	3	8.5	95	9	1	3.7	96.1	2.16	9	1	3.8	96.5	100	0.01	u	57	3	28.0	86.2	65.3	834.70
						90	6	1	2.4	92.4	2.15	8	1	2.4	92.2	100	0.01							

Results for naive Bayes classifiers

Dataset	(#F #I)	NBC	AXp		$\mathrm{LmPAXp}_{\leqslant 9}$				$\mathrm{LmPAXp}_{\leqslant 7}$				$\mathrm{LmPAXp}_{\leqslant 4}$			
		A%	Length	δ	Length	Precision	W%	Time	Length	Precision	W%	Time	Length	Precision	W%	Time
adult	(13 200)	81.37	6.8± 1.2	98	6.8± 1.1	100± 0.0	100	0.003	6.3± 0.9	99.61± 0.6	96	0.023	4.8± 1.3	98.73± 0.5	48	0.059
				95	6.8± 1.1	99.99± 0.2	100	0.074	5.9± 1.0	98.87± 1.8	99	0.058	3.9± 1.0	96.93± 1.1	80	0.071
				93	6.8± 1.1	99.97± 0.4	100	0.104	5.7± 1.3	98.34± 2.6	100	0.086	3.4± 0.9	95.21± 1.6	90	0.093
				90	6.8± 1.1	99.95± 0.6	100	0.164	5.5± 1.4	97.86± 3.4	100	0.100	3.0± 0.8	93.46± 1.5	94	0.103
agaricus	(23 200)	95.41	10.3± 2.5	98	7.7± 2.7	99.12± 0.8	92	0.593	6.4± 3.0	98.75± 0.6	87	0.763	6.0± 3.1	98.67± 0.5	29	0.870
				95	6.9± 3.1	97.62± 2.1	95	0.954	5.3± 3.2	96.59± 1.6	92	1.273	4.8± 3.3	96.24± 1.2	55	1.217
				93	6.5± 3.1	96.65± 2.8	95	1.112	4.8± 3.1	95.38± 1.9	93	1.309	4.3± 3.1	94.92± 1.3	64	1.390
				90	5.9± 3.3	94.95± 4.1	96	1.332	4.0± 3.0	92.60± 2.8	95	1.598	3.6± 2.8	92.08± 1.7	76	1.830
chess	(37 200)	88.34	12.1± 3.7	98	8.1± 4.1	99.27± 0.6	64	0.383	5.9± 4.9	98.70± 0.4	64	0.454	5.7± 5.0	98.65± 0.4	46	0.457
				95	7.7± 3.8	98.51± 1.4	68	0.404	5.5± 4.4	97.90± 0.9	64	0.483	5.3± 4.5	97.85± 0.8	46	0.478
				93	7.3± 3.5	97.56± 2.4	68	0.419	5.0± 4.1	96.26± 2.2	64	0.485	4.8± 4.1	96.21± 2.1	64	0.493
				90	7.3± 3.5	97.29± 2.9	70	0.413	4.9± 4.0	95.99± 2.6	64	0.483	4.8± 4.0	95.93± 2.5	64	0.543
vote	(17 81)	89.66	5.3± 1.4	98	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.95± 0.2	100	0.007	4.6± 1.1	99.60± 0.4	64	0.014
				95	5.3± 1.4	100± 0.0	100	0.000	5.3± 1.3	99.93± 0.3	100	0.008	4.1± 1.0	98.25± 1.7	64	0.018
				93	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.1± 0.9	98.10± 1.9	64	0.018
				90	5.3± 1.4	100± 0.0	100	0.000	5.2± 1.3	99.78± 1.1	100	0.012	4.0± 1.2	97.24± 3.1	64	0.022
kr-vs-kp	(37 200)	88.07	12.2± 3.9	98	7.8± 4.2	99.19± 0.5	64	0.387	6.5± 4.7	98.99± 0.4	64	0.427	6.1± 4.9	98.88± 0.3	43	0.457
				95	7.3± 3.9	98.29± 1.4	64	0.416	6.0± 4.3	97.89± 1.1	64	0.453	5.5± 4.5	97.79± 0.9	43	0.462
				93	6.9± 3.5	97.21± 2.5	69	0.422	5.6± 3.8	96.82± 2.2	64	0.448	5.2± 4.0	96.71± 2.1	43	0.468
				90	6.8± 3.5	96.65± 3.1	69	0.418	5.4± 3.8	95.69± 3.0	64	0.468	5.0± 4.0	95.59± 2.8	61	0.487
mushroom	(23 200)	95.51	10.7± 2.3	98	7.5± 2.4	98.99± 0.7	90	0.641	6.5± 2.6	98.74± 0.5	83	0.751	6.3± 2.7	98.70± 0.4	18	0.828
				95	6.5± 2.6	97.35± 1.8	96	1.011	5.1± 2.5	96.52± 1.0	90	1.130	5.0± 2.5	96.39± 0.8	54	1.113
				93	5.8± 2.8	95.77± 2.7	96	1.257	4.4± 2.5	94.67± 1.6	94	1.297	4.2± 2.4	94.48± 1.3	65	1.324

Results for decision diagrams

Dataset	#I	#F	OMDD		δ	MinPAXp					LmPAXp					
						Length			Prec	Time	Length			Prec	$m_{\subseteq}$	Time
			#N	A%		M	m	avg	avg	avg	M	m	avg	avg		avg
lending	100	9	1103	81.7	100	9	6	8.0	100	24.24	9	6	7.9	100	100	1.57
					95	9	5	7.8	99.7	21.48	9	6	7.8	99.8	100	1.49
					90	9	4	7.2	96	24.65	9	5	7.4	97.0	100	1.48
monk2	100	6	70	79.3	100	6	4	5.1	100	0.10	6	4	5.1	100	100	0.03
					95	6	4	5.1	100	0.09	6	4	5.1	100	100	0.03
					90	6	3	4.8	98.1	0.09	6	3	4.8	98.1	100	0.03
postoperative	74	8	109	80	100	8	4	6.1	100	0.26	8	4	6.2	100	100	0.04
					95	8	2	6.0	99.3	0.25	8	2	6.0	99.3	100	0.04
					90	8	2	5.3	95.9	0.23	8	2	5.4	96.6	94.6	0.04
tic_tac_toe	100	9	424	70.3	100	9	5	7.7	100	3.60	9	5	7.8	100	100	0.38
					95	9	5	7.5	99.5	3.24	9	5	7.7	99.6	99.0	0.38
					90	9	3	7.3	98.3	4.06	9	3	7.5	98.6	98.0	0.38
xd6	100	9	76	83.1	100	9	4	4.6	100	0.10	9	4	4.6	100	100	0.03
					95	9	3	3.8	97	0.09	9	3	3.8	97.0	99.0	0.03
					90	9	3	3.3	94.8	0.10	9	3	3.4	94.6	100	0.03

Remarks on LmPAXps

[IHI+23]

- LmPAXps ignore non-monotonicity, and so overapproximate PAXps
  - Theoretical guarantees, but may be reducible
- For DTs, computation of LmPAXps is in P
- Experimental results confirm LmPAXps match PAXps in most cases
- Recent results on approximating LmPAXps for RFs [IMM24]

Outline – Unit #06

Surrogate models in logic-based XAI

[BAMT21]

- Motivation:
  - Logic-based XAI does not yet scale for highly complex ML models
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- Report computed explanation as explanation for the complex ML model

Certified explainer (for monotonic classification)

[HM23c]

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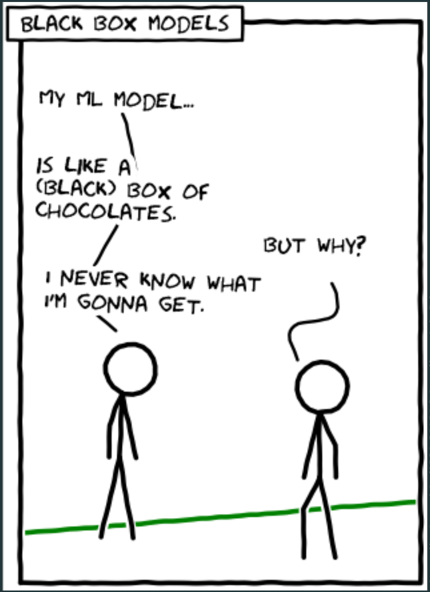
- Certification envisioned for **any** explainability algorithm

Plan for this course – light at the end of the tunnel...

- Lecture 01 – unit(s):
  - #01: Foundations
- Lecture 02 – unit(s):
  - #02: Principles of symbolic XAI – feature selection
  - #03: Tractability in symbolic XAI (& myth of interpretability)
- Lecture 03 – unit(s):
  - #04: Intractability in symbolic XAI (& myth of model-agnostic XAI)
  - #05: Explainability queries
- Lecture 04 – unit(s):
  - #06: Recent, emerging & advanced topics
- Lecture 05 – unit(s):
  - #07: Principles of symbolic XAI – feature attribution (& myth of Shapley values in XAI)
  - #08: Corrected feature attribution – nuSHAP
  - #09: Conclusions & research directions

Questions?



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