ECE 464/564 Homework-2 Solutions

1 a) $y[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] + u[n]$ ---- (1)

and $x[n] = u[n]$ [unit-step input] ---- (2)

Applying z-transform on (1) & (2) we get

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-z^{-1}}, \quad |z|>1$$

For Unit step input, $X(z) = \frac{1}{1-z^{-1}}$ $|z|>1$.

$$\therefore H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-z^{-1}}}{\frac{1}{1-z^{-1}}} \quad \text{----(3)}$$

$$= \frac{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}\right)+\left(1-\frac{1}{3}z^{-1}\right)\left(1-z^{-1}\right)+\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{3-\frac{19}{6}z^{-1}+\frac{2}{3}z^{-2}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z}, \quad |z|>1/3$$

P2

Applying Inverse Z- transform, to the eqn. below,

$$Y(z)\left[1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right]=X(z)\left[3-\frac{19}{6}z^{-1}+\frac{2}{3}z^{-2}\right]$$

We get

$$\boxed{y[n]-\frac{7}{12}y[n-1]+\frac{1}{12}y[n-2]=3x[n]-\frac{19}{6}x[n-1]+\frac{2}{3}x[n-2]}$$

(b) From eqn ③, we see that

$$H(z)=\frac{(1-z^{-1})}{(1-\frac{1}{3}z^{-1})}+\frac{(1-z^{-1})}{(1-\frac{1}{4}z^{-1})}+1.$$

We can take the Inverse Z-Transform,

$$h[n]=\left(\frac{1}{3}\right)^n u[n]-\left(\frac{1}{3}\right)^{n-1}u[n-1]+\left(\frac{1}{4}\right)^n u[n]-\left(\frac{1}{4}\right)^{n-1}u[n-1]+\delta[n]$$

Using the Property that

if $x(n)\xleftrightarrow{Z}X(z)$

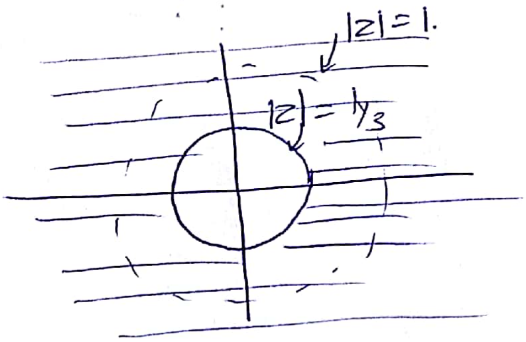
$x(n-K)\xleftrightarrow{Z}z^{-K}X(z)$.

Since it is mentioned Initial rest Condition it is a causal system.

(c) For $\frac{1-z^{-1}}{(1-\frac{1}{3}z^{-1})}$ ROC is $|z|>1/3$

& For $\frac{1-z^{-1}}{(1-\frac{1}{4}z^{-1})}$ ROC is $|z|>1/4$.

So the ROC for $H(z)$ would be $|z| > 1/3$



Since ROC Contains $|z| = 1$

The system is Stable.

2. The difference equation of an LTI system is:

$$y[n-1] - \frac{5}{2} y[n] + y[n+1] = x[n]$$

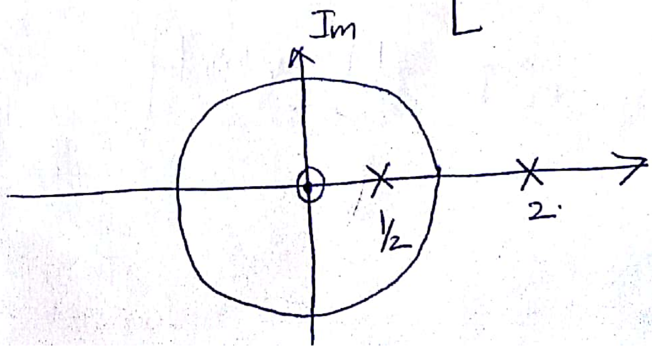
Applying Z-transform to the system:

$$z^{-1} Y(z) - \frac{5}{2} Y(z) + z Y(z) = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{\left(z^{-1} - \frac{5}{2} + z\right)} = H(z)$$

$$\Rightarrow H(z) = \frac{z^{-1}}{\left(1 - \frac{5}{2} z^{-1} + z^{-2}\right)} = \frac{z^{-1}}{\left(1 - 2z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)}$$

$$\Rightarrow H(z) = \frac{2}{3} \left[\frac{1}{1 - 2z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}} \right]$$



Pole-Zero plot

P4

3 Possible ROCs:

1. $|z| < \frac{1}{2}$

2. $\frac{1}{2} < |z| < 2$

3. $|z| > 2$.

Case 1: $|z| < \frac{1}{2}$

Clearly the system is unstable as it does not include $|z| = 1$.

Then $h[n] = \frac{2}{3}\left[-(2)^n u[n-1] + \left(\frac{1}{2}\right)^n u[-n-1]\right]$

Case 2: $\frac{1}{2} < |z| < 2$

System is stable as it contains $|z| = 1$.

Then $h[n] = \frac{2}{3}\left[-(2)^n u[-n-1] - \left(\frac{1}{2}\right)^n u[n]\right]$

Case 3: $|z| > 2$

Unstable system, but is causal (as the ROC lies outside the largest pole).

$h[n] = \frac{2}{3}\left[(2)^n u[n] - \left(\frac{1}{2}\right)^n u[n]\right]$

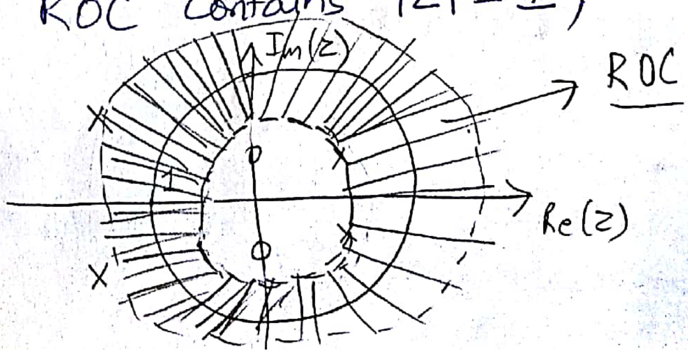
3)

(a) Since the ROC is not mentioned here, we can't determine whether the system is stable or not (as we don't know if $|z| = 1$ is inside the ROC).

(b) For a system to be causal, ROC must lie outside the largest pole. As we don't know the ROC here, we can't say whether the system is causal or not.

(c) Given the system is causal, so ROC lies outside the largest pole, which is outside $|z|=1$. So, ROC does not include $|z|=1$. Hence the system is unstable. (FALSE)

(d) TRUE; Because the system is stable, ROC contains $|z| = 1$, Since it lies outside two poles - it is Right-sided But as it also lies inside 2 other poles - it is left-sided as well.



P6

4) $h_1[n] = \delta[n] + \delta[n-4]$

$$H_1(e^{j\omega}) = e^{-j\omega \cdot 0} + e^{-j\omega 4} = 1 + e^{-j4\omega} = e^{-j2\omega}\left[e^{j2\omega} + e^{-j2\omega}\right]$$

$$= 2e^{-j2\omega} \cos 2\omega$$

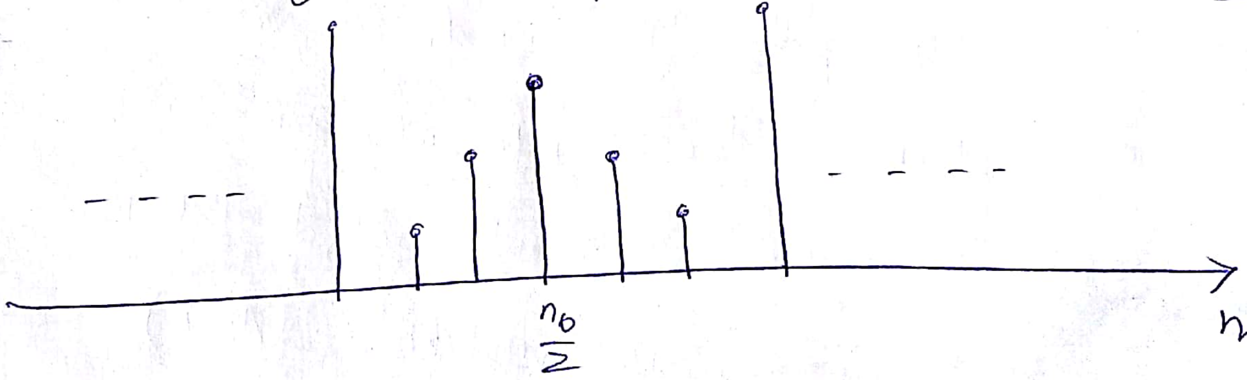
$$\measuredangle H_1(e^{j\omega}) = -2\omega.$$

$$\text{grd}\left(H_1(e^{j\omega})\right) = -\frac{d}{d\omega}(-2\omega) = 2$$

(or)

We can alternatively solve the problem using the relationship between symmetry and group delay

Let $h[n]$ be a symmetric impulse about $\frac{n_0}{2}$



We can write the Fourier Transform of $h[n]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[\underbrace{m-n}_{l}]\, e^{-j\omega n.} \quad \left(m = 2 \cdot \frac{n_0}{2} = n_0\right)$$

Let $l = m - n$, so $n = m - l$.

& $H(e^{j\omega}) = \sum_{l=-\infty}^{\infty} h[l]\, e^{-j\omega(m-l)}$

Then $H(e^{j\omega}) = e^{-j\omega m} \underbrace{\sum_{l=-\infty}^{\infty} h(l)\, e^{j\omega l}}_{H(e^{-j\omega})}$.

$$= e^{-j\omega m} H(e^{-j\omega}) \quad \text{----} \; (1).$$

Let us look at the relationship between $H(e^{j\omega})$ and $H(e^{-j\omega})$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

& $e^{-j\omega n} = \cos \omega n - j \sin \omega n.$

$$H(e^{j\omega}) = Re\left(H(e^{j\omega})\right) + j\, Im\left(H(e^{j\omega})\right)$$

$$= \sum_{n=-\infty}^{\infty} h[n] \cos(\omega n) - j \sum_{n=-\infty}^{\infty} h[n] \sin(\omega n) \text{---} \; (2)$$

In the same way,

$$H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cos \omega n + j \sum_{n=-\infty}^{\infty} h[n] \sin(\omega n) \text{---} \; (3)$$

From (2) & (3) we know

$$H^*(e^{j\omega}) = H(e^{-j\omega}) \quad \text{------} \; (4)$$

From ① and ④, we can have

$$H(e^{j\omega}) = e^{-j\omega m} H^*(e^{j\omega})$$

We express $H(e^{j\omega})$ & $H^*(e^{j\omega})$ in Amplitude, Phase notation.

$$H(e^{j\omega}) = Ae^{j\phi(\omega)} = e^{-j\omega m} A e^{-j\phi(\omega)}$$

$$\Rightarrow e^{j2\phi(\omega)} = e^{-j\omega m}.$$

$$\Rightarrow 2\phi(\omega) = -\omega m$$

$$\Rightarrow \phi(\omega) = -\frac{m\omega}{2} + 2\pi K, \quad K \in Z.$$

$$grd\left(H(e^{j\omega})\right) = -\frac{d\phi(\omega)}{d\omega} = \frac{m}{2} = \frac{n_0}{2}.$$

$\therefore$ $\boxed{\text{The Symmetry } \frac{n_0}{2} \text{ is the group delay.}}$

(1) Group Delay is 2

(2) Group Delay is 1.5

(3) Group Delay = 2

(4) $h_4[n] \rightarrow$ Group Delay $= 3$

(5) $h_5[n] \rightarrow$ Group Delay $= 3$.

(6) $h_6[n] \Rightarrow$ Group Delay $= 3.5$

(5) From Fig. 3, we see that

There are 3 narrow band signals with $\omega = 0.12\pi$, 0.3π and 0.5π.

The narrow band signal with $\omega = 0.12\pi$, has a gain $\cong 1.8$ and group delay $\cong$ 40 samples.

The narrow band signal with $\omega = 0.3\pi$, has a gain = 1.7 and group delay $\approx$ 80 samples.

The narrow band signal $\omega = 0.5\pi$ has gain ≈ 0.

This means this signal disappears after filter A. The group delay is nearly equal to 0.

$\therefore$ $y_2[n]$ in Fig. 4 can only be the solution.