P1

ECE 464/564 Homework-7
Solutions

(1) (a) $H(z) = \dfrac{2}{1-e^{-0.2}z^{-1}} - \dfrac{1}{1-e^{-0.4}z^{-1}}$

$T_d = 2$,

$h[n] = 2h_c[2n]$, $h_c(t)$ is real

Poles of the discrete time system are at $z = e^{-0.2}, e^{-0.4}$

So, $s_1 T_d = -0.2$ $\qquad$ $s_2 T_d = -0.4$

$\Rightarrow s_1 = -0.1$ $\qquad$ $\Rightarrow s_2 = -0.2$

And $T_d \cdot A_1 = 2$, $\qquad$ $T_d A_2 = -1$

$\Rightarrow A_1 = 1$ $\qquad$ $\Rightarrow A_2 = -0.5$

The discrete-time system can be expressed as

$$H(z) = \sum_{k=1}^{2} \frac{T_d A_k}{1-e^{s_k T_d} z^{-1}}$$

The corresponding s-domain poles are at $s = -0.2, -0.1$. The continuous-time system has a transfer function

$$H_c(s) = \sum_{k=1}^{2} \frac{A_k}{s-s_k} = \frac{1}{s+0.1} - \frac{0.5}{s+0.2} = \frac{0.5(s+0.3)}{(s+0.1)(s+0.2)}$$

(P2)

(b) Using Bilinear Transformation, we have

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{1-z^{-1}}{1+z^{-1}} \quad (\because T_d = 2)$$

$$\Rightarrow s = \frac{z-1}{z+1} \Rightarrow z = \frac{1+s}{1-s}$$

So, $$H_c(s) = H(z)\Big|_{z=\frac{1+s}{1-s}} = \frac{2}{1-e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1-e^{-0.4}\left(\frac{1-s}{1+s}\right)}$$

$$= \frac{2(1+s)}{(1+s)-e^{-0.2}(1-s)} - \frac{(1+s)}{(1+s)-e^{-0.4}(1-s)}$$

$$= \frac{2(1+s)}{(1-e^{-0.2})+s(1+e^{-0.2})} - \frac{(1+s)}{(1-e^{-0.4})+s(1+e^{-0.4})}$$

$$= \left(\frac{2}{1+e^{-0.2}}\right)\frac{(1+s)}{s+\left(\frac{1-e^{-0.2}}{1+e^{-0.2}}\right)} - \left(\frac{1}{1+e^{-0.4}}\right)\frac{(1+s)}{s+\left(\frac{1-e^{-0.4}}{1+e^{-0.4}}\right)}$$

$$= \frac{1.1(1+s)}{(s+0.1)} - \frac{0.598(1+s)}{(s+0.197)}$$

$$= \frac{(1+s)(0.58+0.1569)}{(s+0.1)(s+0.197)}$$

2

(a) $$H_c(s) = \frac{s+a}{(s+a)^2 + b^2} = \frac{s+a}{(s+a+bj)(s+a-bj)}$$

$$= \frac{1/2}{s+a+bj} + \frac{1/2}{s+a-bj}$$

The system has two poles at $-(a+bj)$, $-(a-bj)$

Taking inverse Laplace transform,

$$h_c(t) = \frac{1}{2} e^{-(a+bj)t} u(t) + \frac{1}{2} e^{-(a-bj)t} u(t)$$

$$\Rightarrow h_1[n] = T h_c(nT) = \frac{T}{2}\left[e^{-(a+bj)nT} + e^{-(a-bj)nT}\right] u[n]$$

Taking Z-transform, we get,

$$H_1(z) = \frac{T}{2}\left[\frac{1}{1-e^{-(a+bj)T}z^{-1}} + \frac{1}{1-e^{-(a-bj)T}z^{-1}}\right]$$

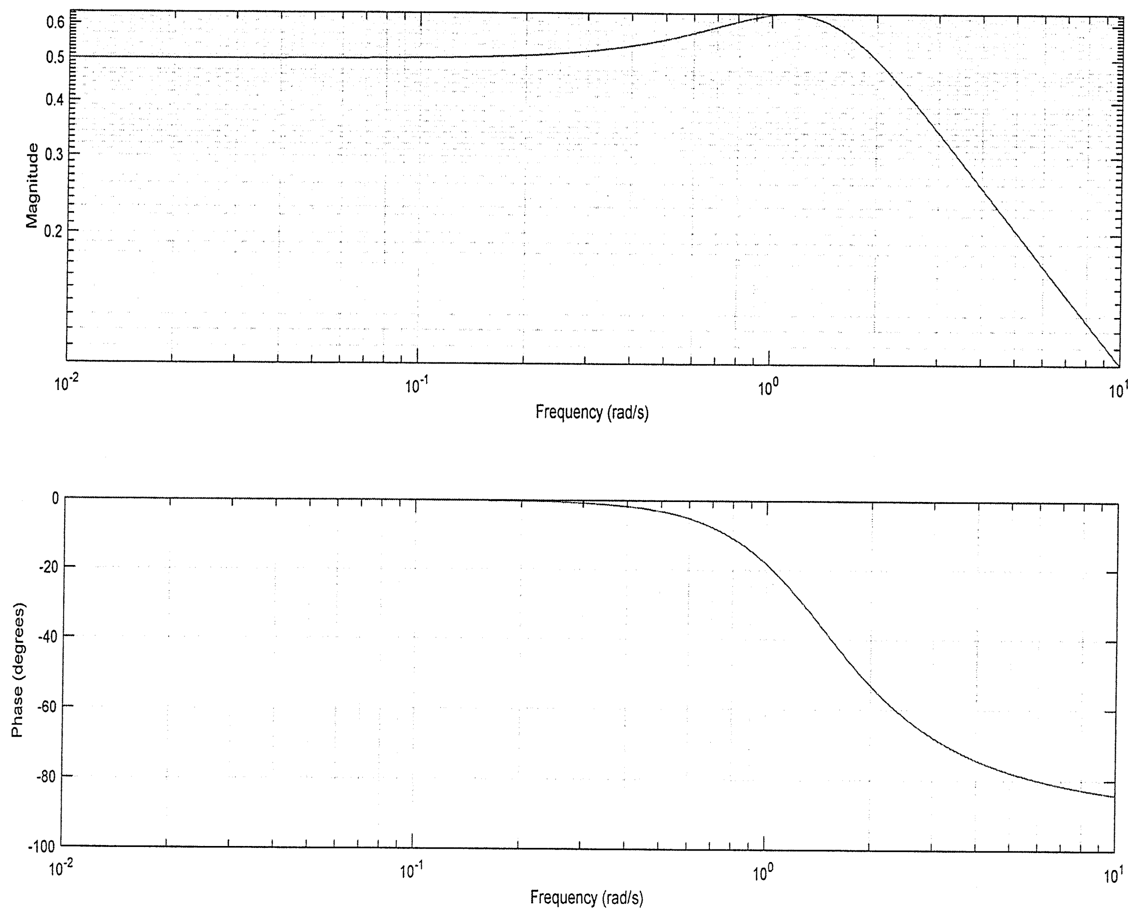
(b) $$H_1(z) = \frac{T}{2}\left[\frac{2-(e^{bjT}+e^{-bjT})e^{-aT}z^{-1}}{1-(e^{bjT}+e^{-bjT})e^{-aT}z^{-1}+e^{-2aT}z^{-2}}\right]$$

$$= T\left[\frac{1-\cos bT e^{-aT} z^{-1}}{1-2\cos bT e^{-aT} z^{-1} + e^{-2aT} z^{-2}}\right]$$

Frequency response of the Continuous Time filter:

$$H_c(s) = \frac{s+a}{(s+a)^2 + b^2}$$

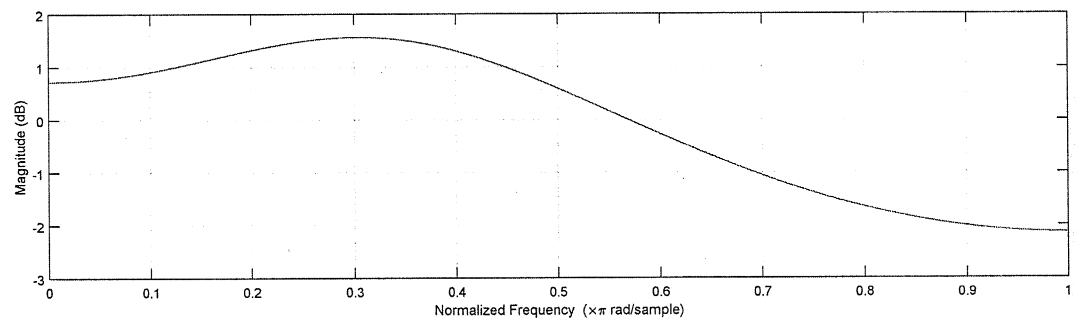
with a=1, b =1

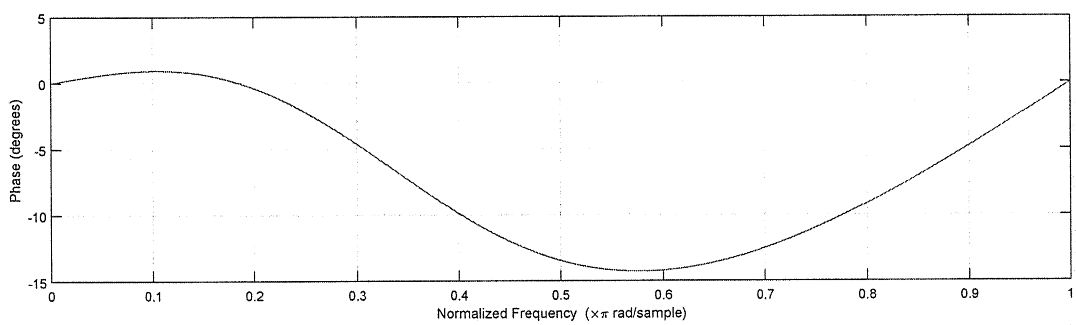


Frequency response of the Discrete Time filter:

With T=1

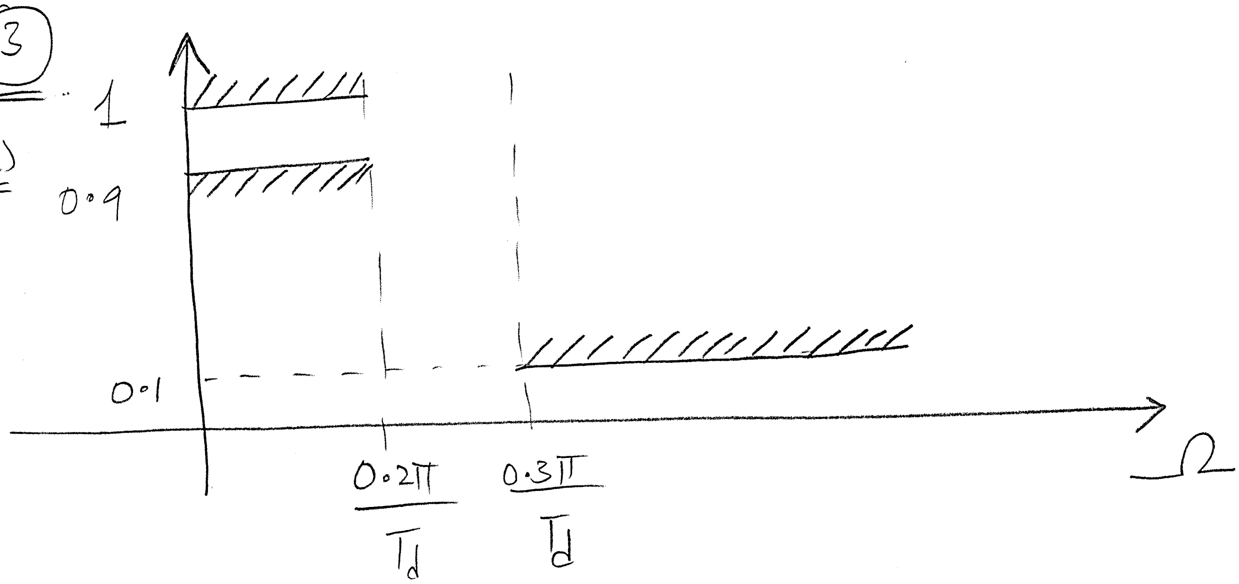
$$H(z) = \frac{1 - 0.198Z^{-1}}{1 - 0.397Z^{-1} + 0.135Z^{-2}}$$





③

(a)



(b) $\delta_p = 0.1, \quad \delta_s = 0.1, \quad \frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p} = \frac{0.3\pi}{0.2\pi} = 1.5$

$$N \geq \frac{\left[\log\left(\frac{1}{\delta_s^2} - 1\right) - \log\left(\left(\frac{1}{1-\delta_p}\right)^2 - 1\right)\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$N \geq 7.4546$, we should choose $N = 8$

$\because$ we have N, we need to find Ω_c.

$$\Omega_p\left(\frac{1}{\left(\frac{1}{1-\delta_p}\right)^2 - 1}\right)^{\frac{1}{2N}} \leq \Omega_c \leq \Omega_s\left(\frac{1}{\frac{1}{\delta_s^2} - 1}\right)^{\frac{1}{2N}}$$

$$0.2\pi\left(\frac{1}{\left(\frac{10}{9}\right)^2 - 1}\right)^{\frac{1}{16}} \leq \Omega_c \leq 0.3\pi\left(\frac{1}{100 - 1}\right)^{\frac{1}{16}}$$

$0.219\pi \le \Omega_c \le 0.225\pi$

Let us choose $\Omega_c = 0.7$

(c) So, $H_c(s)\,H_c(-s) = \left[\dfrac{1}{1+\left(\dfrac{s}{j\Omega_c}\right)^{2N}}\right]$

with $N = 8$, $\Omega_c = 0.7$

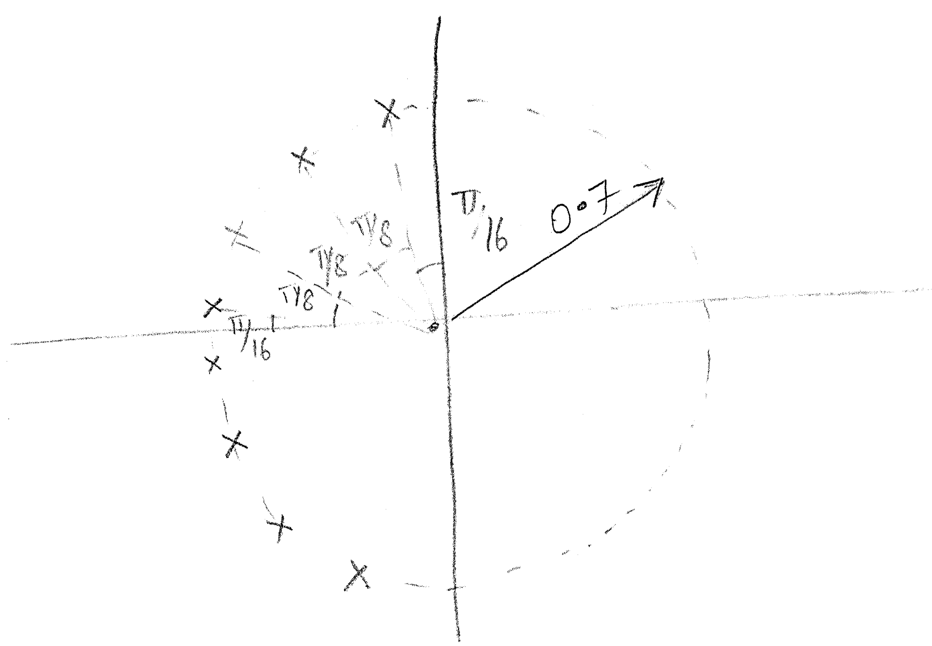
We would have 4 pole-pairs in the left half of the s-plane with the coordinates:

Pole pair 1: $0.7e^{\pm j\frac{9\pi}{16}} = -0.1366 \pm j0.6865$

Pole pair 2: $0.7e^{\pm j\frac{11\pi}{16}} = -0.3889 \pm j0.5820$

Pole pair 3: $0.7e^{\pm j\frac{15\pi}{16}} = -0.582 \pm j0.3889$

Pole pair 4: $0.7e^{\pm j\frac{15\pi}{16}} = -0.6865 \pm j0.1366$.



```
%%HW-7 Problem 3%%
clear all;
clc;

[N,Wn] = buttord(0.2*pi,0.3*pi,0.915,20,'s');
[z,p,k]=butter(N,Wn,'s');
[b,a]=zp2tf(z,p,k);
p
Hs= tf(b,a)
[numd,dend]=impinvar(b,a,1);
Hz=tf(numd,dend,1)
```

p =

-0.6936 + 0.1380i

-0.6936 - 0.1380i

-0.5880 + 0.3929i

-0.5880 - 0.3929i

-0.3929 + 0.5880i

-0.3929 - 0.5880i

-0.1380 + 0.6936i

-0.1380 - 0.6936i

Hs =

0.06257

----

s^8 + 3.625 s^7 + 6.57 s^6 + 7.727 s^5 + 6.426 s^4 + 3.865 s^3 + 1.643 s^2 + 0.4535 s + 0.06257

Continuous-time transfer function.

Hz =

1.23e-13 z^8 + 7.798e-06 z^7 + 0.0005763 z^6 + 0.003549 z^5 + 0.004546 z^4 + 0.00144 z^3

+ 9.449e-05 z^2 + 5.148e-07 z

----

z^8 - 4.479 z^7 + 9.26 z^6 - 11.39 z^5 + 9.038 z^4 - 4.718 z^3 + 1.576 z^2 - 0.307 z + 0.02665

Sample time: 1 seconds

Discrete-time transfer function.

```
%%HW-7 Problem-4 second way%%
clear all;
clc;

[N,Wn] = cheb1ord(0.2*pi,0.3*pi,1,15,'s')
[b,a]=cheby1(N,Wn,0.2*pi,'s')
[num,den]=bilinear(b,a,1)
tf(num,den,1)
```

N =

4

Wn =

0.6283

b =

0 0 0 0 0.0494

(4) n:

$$\frac{1}{1+\epsilon^2 V_n^2\left(\frac{\Omega_s}{\Omega_p}\right)} \leq \delta_s^2$$

Here $(1-\delta_p) = 10^{-1/20} = 0.8913$

$\delta_s = 10^{-\frac{15}{20}} = 0.1778$

$\Omega_c = \Omega_p = 0.2\pi$

$$1+\epsilon^2 V_n^2\left(\frac{\Omega_s}{\Omega_p}\right) \geq \left(\frac{1}{\delta_s^2}\right)$$

$$\epsilon^2 V_n^2\left(\frac{\Omega_s}{\Omega_p}\right) \geq \left(\frac{1}{\delta_s^2}-1\right)$$

$$\boxed{\epsilon = \sqrt{\frac{1}{(1-\delta_p)^2}-1}} \quad \text{--- (1)}$$

$$V_n^2\left(\frac{\Omega_s}{\Omega_p}\right) \geq \frac{1}{\epsilon^2}\left(\frac{1}{\delta_s^2}-1\right)$$

$$V_n\left(\frac{\Omega_s}{\Omega_p}\right) \geq \sqrt{\frac{\left(\frac{1}{\delta_s^2}-1\right)}{\epsilon^2}}$$

$$\cosh\left(n\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right) \geq \sqrt{\frac{\left(\frac{1}{\delta_s^2}-1\right)}{\epsilon^2}}$$

$$n\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right) \geq \cosh^{-1}\left(\sqrt{\frac{\frac{1}{\delta_s^2}-1}{\epsilon^2}}\right)$$

$$\boxed{n \geq \frac{\cosh^{-1}\left(\frac{\sqrt{\frac{1}{\delta_s^2}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}} \quad \text{--- (2)}$$

Using (1), we estimate $\epsilon = \sqrt{\frac{1}{(1-0.89)^2}-1} = 0.51$

and $\Omega_c = \Omega_p = 0{\cdot}2\pi$

$\Omega_s = 0{\cdot}3\pi$

$$\therefore \quad n \geqslant \cosh^{-1}\left(\frac{\sqrt{\left(\frac{1}{0{\cdot}1778}\right)^2 - 1}}{0{\cdot}51}\right) = 3{\cdot}2$$

so $n = 4$

```
a =

   1.0000    0.7011    0.6406    0.2294    0.0531


num =

   0.0020    0.0080    0.0120    0.0080    0.0020


den =

   1.0000   -3.0016    3.6946   -2.1670    0.5084


ans =

  0.002 z^4 + 0.008002 z^3 + 0.012 z^2 + 0.008002 z + 0.002
  ---------------------------------------------------------
      z^4 - 3.002 z^3 + 3.695 z^2 - 2.167 z + 0.5084

Sample time: 1 seconds
Discrete-time transfer function.
```