History of Combinatorial Games

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Abstract

In the first years of the 20th century the analysis of the game of NIM (by the mathematician Charles L. Bouton) triggered the outburst of a completely new mathematical subject: Combinatorial Game Theory. NIM is actually a representative of a family of games, very important in that scientific area. The aim of this paper is to give a first survey of the historical development of this mathematical field.

1 What is a combinatorial game?

Nowadays it is not usual to observe the rising of a totally new mathematical subject. In mathematics, subjects like geometry, number theory, probability, functional analysis, etc, were established a long time ago. Of course, the modern times brought advances for all these areas, but their importance and existence was very well known by the previous generations of mathematicians.

The motivation for this paper is a rare occurrence: the rising of a new mathematical subject, Combinatorial Game Theory. In 1970s, the work of Elwyn Berlekamp (b. 1940), John Conway (b. 1937), and Richard Guy (b. 1916) launched the bases and the mathematical language for a new kind of scientific research.

Actually, during the 20th century, two distinct game theories arose. One is linked to Von Neumann, John Nash, prisoner's dilemma, etc. In 1944, with the seminal work, *Theory of Games and Economic Behavior*, Von Neumann (1903-1957) and Morgenstern (1902-1977) proposed a very interesting way to analyze mathematically some decision processes [1]. Since then, some scientists have even won nobel prizes with work related to economics based in this mathematical theory.

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Von Neumann's game theory, called Classical Game Theory, is about games and outcomes determined by a payoff matrix. In typical games of Classical Game Theory players *play simultaneously*. The goal of Classical Game Theory is to determine the best possible payoff depending upon the players possible strategies. In Classical Game Theory simultaneous decisions imply that players must decide without knowing the opponent's choices: there is hidden information. Sometimes, the theory needs probabilistic tools to solve some problems.

In 1976, Conway published his *On Numbers and Games* [3] (in fact, previously, in 1974, the book *Surreal Numbers* by Donald E. Knuth (b.1938) was published based in a conversation between Knuth and Conway where the latter explained his mathematical idea [2]). Later, in 1982, Berlekamp, Conway, and Guy presented their *Winning Ways* [5]. In these two works we can see a different game theory. They propose a unified mathematical theory to analyze games without chance and without hidden information where two players take turns moving *alternately*. Before them, Charles Bouton (1869-1922), Roland Sprague (1894-1967), Patrick Grundy (1917-1959) and others presented some partial work [7, 9, 8]. However, a complete and consistent theory only appeared in [5], what we call Combinatorial Game Theory. Classical Game Theory and Combinatorial Game Theory are distinct mathematical subjects.

A game without chance and without hidden information where two players take turns moving alternately allows us to build the *complete game tree* containing all the information about the game. Consider the simple game TOADS-AND-FROGS. In this game one player moves Toads (only Eastward) and the other player moves Frogs (only to the West). Each player may move one of his animals either one square or jumping over an opposing creature. The player unable to move loses. Next picture shows the complete tree from a established starting position.

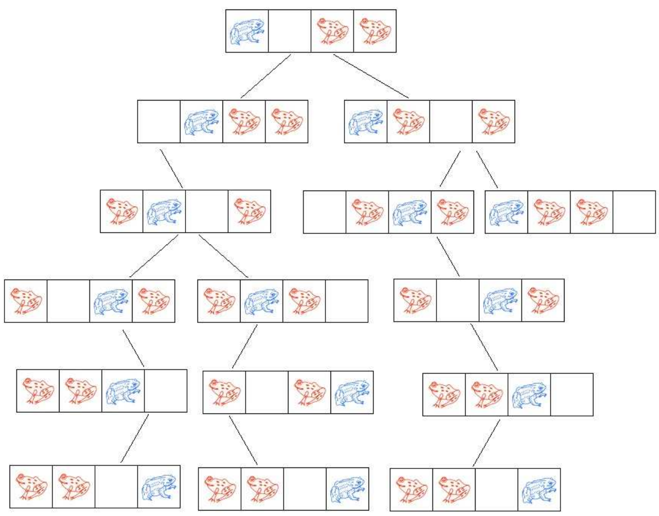


Figure 1: Complete game tree.

In this text we will adopt a restricted definition of combinatorial game which is prevalent in the specialized literature. With such restrictions, Conway developed important mathematical tools to study a large family of games.

Definition 1. *(Combinatorial game)*
A combinatorial game is a game which satisfies the following conditions:

1. *There are two players who take turns moving alternately;*
2. *No chance devices such dice, spinners, or card deals are involved, and each player is aware of all the details of the game state at all times;*
3. *The rules of a combinatorial game ensure that the depth of the game tree is finite, and the winner is often determined on the basis of who made the last move. Under normal play, the last player to move wins, while in misère play, the last player loses.*

Examples of games **not** covered by these conditions are DOTS-AND-BOXES and GO, since these are scoring games, the last person to move is not guaranteed to have either the highest or the lowest score; CHESS and CHECKERS, since the game can end in a draw and the depth of the game tree is infinite; BACKGAMMON, since there is a chance element; MASTERMIND, since there is hidden information; FOX-AND-GEESE, since when fox escapes has "eternal life" (so, the depth of the game tree is infinite).

The most famous combinatorial game is the game of NIM. This game is played with piles of stones. On his turn, each player can remove any number of stones from any pile. Under Normal Play rules, whoever takes the last counter wins while under Misère Play rules whoever takes the last counter loses. The game, in Normal version, is the first combinatorial game that was solved with a mathematical approach.

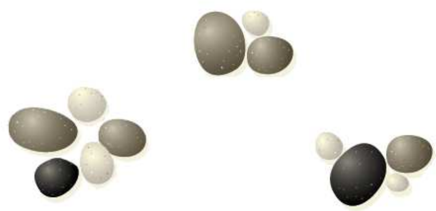


Figure 2: Game of NIM.

For mathematical aspects we recommend [3], [4], and [5]. The present paper presents historical evidence of ancient combinatorial games and shows a systematic view of the modern development of Combinatorial Game Theory.

2 The Chinese Connection

In the history of games the “Chinese origins” are omnipresent. It has been said that NIM and WYTHOFF’S NIM were born in ancient China. In 1902, C. L. Bouton claimed that NIM was widely played in America and was called FAN-TAN by the Chinese [7]. However, the German mathematician Paul Ahrens, in a 1902 article in *Naturwissenschaftliche Wochenschrift* [22], says that Bouton admitted that he had confused NIM and FAN-TAN. FAN-TAN is a Chinese game where you bet on the number of counters (remainder when dividing by 4) in someone’s hand. Parker describes a similar game, based on odd and even, popular in Ceylon and “certainly one of the earliest of all games” [23]. This kind of games seems to be very far from the nature of NIM.

The game WYTHOFF’S NIM, described by Willem Wythoff (1865-1939) in 1907 [17], has two piles and a player can take any amount from one pile or the same amount from both piles. About this game, in [15], a A. P. Domoryad’s Russian book on mathematical games, we can read “The theory of the Chinese national game TSYANSHIDZI (“picking stones”) is much more complicated (...)”. Also, in [5] it is called CHINESE NIM or TSYAN-SHIZI, but Richard Guy says he recalls this was based on Domoryad...

In [16], Martin Gardner (b. 1914) says “(...) an older counter take-away game said to have been played in China under the name TSYANSHIDZI, which means “choosing stones” (...)”. It is also made by Yaglom and Yaglom in [24]: “(...) NIM and WYTHOFF’s games are both played in China under the name TSAN-SHITSI (which means “choosing stones”) (...)”. These references look circular. Without a primordial reference it seems that the Chinese origin is not very well supported...

For now, the only fact observed is the following one: the origin of the mysterious term is the Chinese word

撿石子

The standard transcription should be *Jian Shizi*, which is *Pinyin* (the romanization system used in the People’s Republic of China). It should be written as *Jian Shizi* (and not as *Jianshizi* or *Jian Shi Zi*), because “jian” means to “pick up” and “shizi” is the word for “stones”. The correct translation should be “picking stones” or “picking up stones”. The reason it is sometimes translated as “choosing stones” is related to the fact that “jian” is occasionally confused with a homophone character that means “to choose”. However this is an empty linguistic discussion without historical relevance.

3 *Viribus Quantitatis*

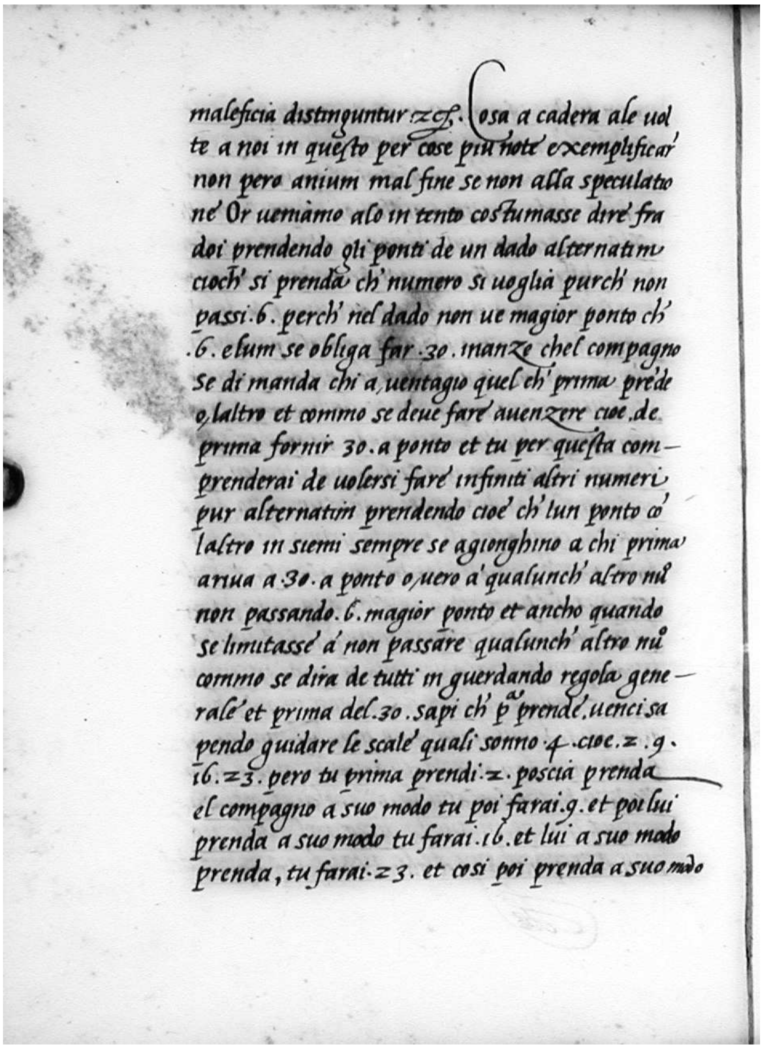
Luca Pacioli (1445-1517) was a mathematician of the Renaissance who wrote several important works on mathematics, including *Summa de arithmetica, geometria, proportioni et proportionalita* and the *De divina proportione.* Leonardo da Vinci made beautiful geometric drawings for the *De divina proportione.*

Figure 3: Luca Pacioli by Jacopo de Barbari (1495).

One of his works is the *De viribus quantitatis* [10], a treatise on mathematics and magic. Written between 1496 and 1508, the book was rediscovered after David Singmaster, a mathematician, came across a reference to it in a 19th-century manuscript. An English translation was published for the first time in 2007 [25].

The manuscript includes the first reference to a one-pile game (Uri 175-181, see Appendix 1): "effecto afiniri qualunch' numero na'ze al compagno anon prendere piu de un termi(n)ato .n." (effect to finish whatever number is before the company not taking more than a limiting number). Pacioli proposes a game where the players can add a number less than 7 to a pile, and the goal is to achieve 30. Pacioli describes how to win this case and the general game. To achieve the number 30, a player must achieve, by order, the numbers 2, 9, 16 and 23.

Nowadays, there is a class of games called *subtraction games.* We saw before that NIM is a game played with piles of stones and a move is to choose a pile and remove any number of them. Changing the rules we can define the SUBTRACTION$(s_1, \ldots, s_k)$: the game is played like NIM but the players just can remove a number of stones if it is an element of $\{s_1, \ldots, s_k\}$. The Pacioli game is an *additive* version, an ADDITION game. In fact, it is the same to play



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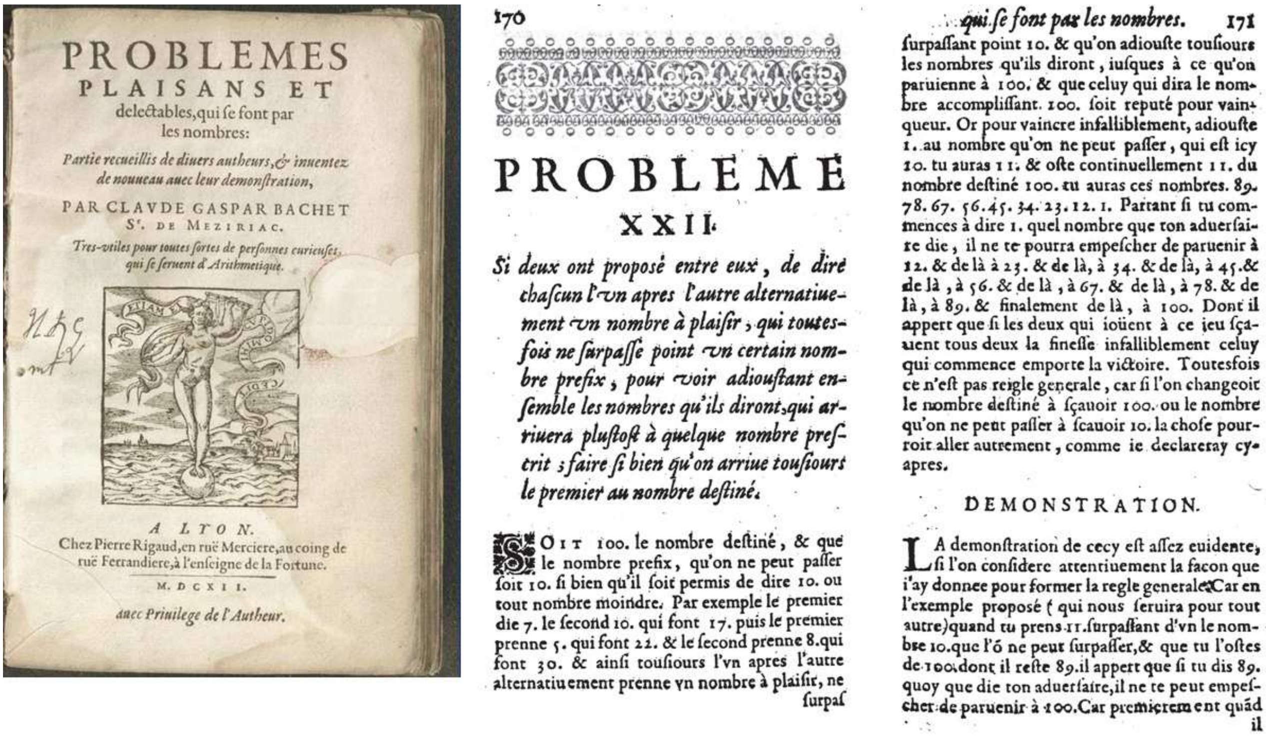
se limitasse a non passare qualunch altro nu
commo se dira de tutti in guerdando regola gene-
rale et prima del 30. sapi ch pa prende uenci sa
pendo guidare le scale quali sonno 4. cioe. 2. 9.
16. 23. pero tu prima prendi 2. poscia prenda
el compagno a suo modo tu poi farai 9. et poi lui
prenda a suo modo tu farai 16. et lui a suo modo
prenda, tu farai 23. et cosi poi prenda a suo modo

Figure 4: *Viribus*'s page about the ADDITION$(1, 2, 3, 4, 5, 6)$.

the game SUBTRACTION$(1, 2, 3, 4, 5, 6)$ from a pile with 30 stones (the goal is to take out all the stones) or to begin without stones and add numbers from the set $\{1, 2, 3, 4, 5, 6\}$ trying to achieve 30 stones. This is the first known reference to a combinatorial game.

4 *Problèmes Plaisans et Delectables*

The *Problèmes Plaisans et Delectables*, of which the first edition was issued in 1612, was written by Claude Gaspard Bachet de Méziriac (1581-1638), a French mathematician. The book has a rich set of arithmetical tricks and questions [26].



PROBLEMES
PLAISANS ET
delectables, qui ſe font par
les nombres:
Partie recueillis de diuers autheurs, & inuentez de nouueau auec leur demonſtration,
PAR CLAVDE GASPAR BACHET S^r. DE MEZIRIAC.
Tres-vtiles pour toutes ſortes de perſonnes curieuſes, qui ſe ſeruent d'Arithmetique.
A LYON.
Chez Pierre Rigaud, en ruë Merciere, au coing de ruë Ferrandiere, à l'enſeigne de la Fortune.
M. DCXII.
auec Priuilege de l'Autheur.

170

PROBLEME
XXII.

Si deux ont proposé entre eux, de dire chaſcun l'vn apres l'autre alternatiuement vn nombre à plaiſir, qui toutesfois ne ſurpaſſe point vn certain nombre prefix, pour voir adiouſtant enſemble les nombres qu'ils diront, qui arriuera pluſtoſt à quelque nombre preſcrit; faire ſi bien qu'on arriue touſiours le premier au nombre deſtiné.

SOIT 100. le nombre deſtiné, & que le nombre prefix, qu'on ne peut paſſer ſoit 10. ſi bien qu'il ſoit permis de dire 10. ou tout nombre moindre. Par exemple le premier die 7. le ſecond 10. qui font 17. puis le premier prenne 5. qui font 22. & le ſecond prenne 8. qui font 30. & ainſi touſiours l'vn apres l'autre alternatiuement prenne vn nombre à plaiſir, ne ſurpaſ-

qui ſe font par les nombres. 171

ſurpaſſant point 10. & qu'on adiouſte touſiours les nombres qu'ils diront, iuſques à ce qu'on paruienne à 100. & que celuy qui dira le nombre accompliſſant. 100. ſoit reputé pour vainqueur. Or pour vaincre infalliblement, adiouſte 1. au nombre qu'on ne peut paſſer, qui eſt icy 10. tu auras 11. & oſte continuellement 11. du nombre deſtiné 100. tu auras ces nombres. 89. 78. 67. 56. 45. 34. 23. 12. 1. Partant ſi tu commences à dire 1. quel nombre que ton aduerſaire die, il ne te pourra empeſcher de paruenir à 12. & de là à 23. & de là, à 34. & de là, à 45. & de là, à 56. & de là, à 67. & de là, à 78. & de là, à 89. & finalement de là, à 100. Dont il appert que ſi les deux qui ioüent à ce ieu ſçauent tous deux la fineſſe infalliblement celuy qui commence emporte la victoire. Toutesfois ce n'eſt pas reigle generale, car ſi l'on changeoit le nombre deſtiné à ſçauoir 100. ou le nombre qu'on ne peut paſſer à ſcauoir 10. la choſe pourroit aller autrement, comme ie declareray cy-apres.

DEMONSTRATION.

LA demonſtration de cecy eſt aſſez euidente, ſi l'on conſidere attentiuement la facon que i'ay donnee pour former la regle generale. Car en l'exemple proposé (qui nous ſeruira pour tout autre) quand tu prens 11. ſurpaſſant d'vn le nombre 10. que l'õ ne peut ſurpaſſer, & que tu l'oſtes de 100. dont il reſte 89. il appert que ſi tu dis 89. quoy que die ton aduerſaire, il ne te peut empeſcher de paruenir à 100. Car premierement quãd il

Figure 5: Cover and the two pages about Bachet's game in *Problèmes Plaisans et Delectables* (first edition, 1612).

The 22th problem is an ADDITION game similar to Pacioli's one. Instead of 30, the goal is to achieve 100. The players can add any whole number less than 11. As in *De viribus quantitatis*, we can see in *Problèmes Plaisans et Delectables* the winning strategy. The first player must achieve, by order, the numbers 1, 12, 23, 34, 45, 56, 67, 78 and 89. As in Pacioli's manuscript, this is an old occurrence of an analysis, by a mathematician, of a combinatorial game.

5 A Voyage to the Pacific Ocean

Captain James Cook (1728-1779), in 1778, during his third voyage described the native game KONANE [18]:

(...) *One of their games resembles our game of draughts; but, from the number of squares, it seems to be much more intricate. The board is of the length of about two feet, and is divided into two hundred and thirty-eight squares, fourteen in a row* [hence a 14×17 board]. *In this game they use black and white pebbles, which they move from one square to another.* (...)

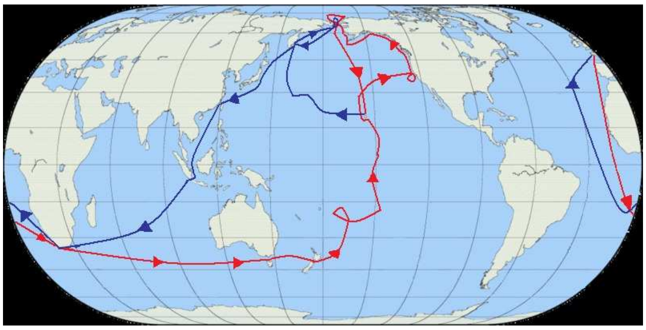


Figure 6: James Cook's third voyage.

One of the main references about this game is [19]. The rules are explained by Keneth Emmory as he learned from an old hawaiian women (see Appendix 2):

(...) *I shall explain it as I learned it from a women of nearly ninety years, Kaahaaina Naihe, from Kailua, Hawaii – the only native left who is known to be acquainted with the game.* (...)

Briefly, exemplifying with a modern chess board the rules are the following. In the starting position of a KONANE game, a rectangular checkered board is filled in such a way that no two stones of the same color occupy adjacent squares. In the opening, two adjacent pieces are removed. After this, a player moves by taking one of his stones and jumping orthogonally over an opposing stone into an empty square. The jumped stone is removed. A player can make multiple jumps on his turn but cannot change direction mid-turn. Multiple jumps are not mandatory. The winner is the player who makes the last move. We can see some examples of legal moves in the next figure:

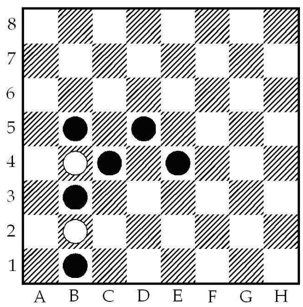


Figure 7: Example of a KONANE's position.

White has three legal moves: taking one stone with the move B4-D4, taking two stones with the move B4-F4 or taking one stone with B4-B6. Black has just one legal move: taking one stone with C4-A4.

KONANE is a combinatorial game satisfying all the criteria of the Definition 1. In [19], we can see many descriptions of native terms like *papamu* (arrangement of rows of pits), *kaka'i* (the line of positions bordering the *papamu*), *"lawe ili keokeo, paani ka eleele"* (a saying, "removing the whites is playing with the blacks").

The KONANE's boards reported in average 134 holes each. In the middle of the board was set a piece of a bone or a human tooth marking an important position, the *piko*. Hawaii provided basalt and coral pebbles for the pieces (size under an inch in diameter and slightly flattened) [20]. In [20], we can also read "(...) A game sometimes lasted an entire day; in a match, often a large number of games were played before determining the winner".

There is a large number of examples of KONANE's boards. The next example is a photo from the catalog *The Hawaiian Portion of the Polynesian Collections in the Peabody Museum of Salem.*

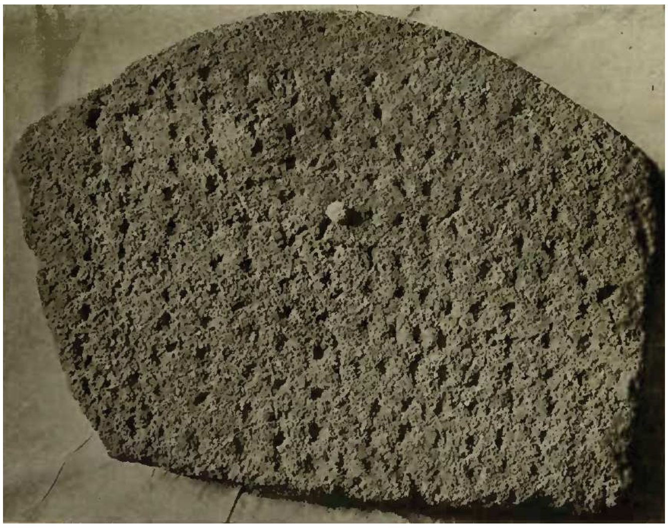


Figure 8: KONANE's board.

KONANE declined in popularity after the arrival of Westerners. However, in the last times, a volte-face occurred and nowadays students in schools learn the game as early as first or second grade [20].

6 Was NIM's first occurrence African?

In [27], a very important reference about African games, Charles Béart describes the TIOUK-TIOUK, a board game played in Ivory Coast (see Appendix 3). The game is played with small sticks and stones on a squared board drawn in sand. The board's dimension is variable. In the initial position, each player begins with a line of pieces as it is shown in the next figure.

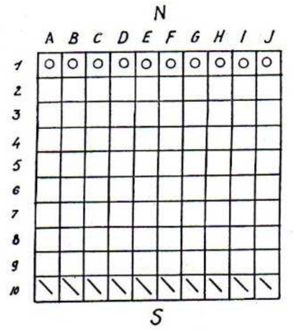


Figure 9: TIOUK TIOUK's initial position.

There is one piece for each player on each column. Each player may move any own piece to another empty cell in the same column, provided he does not jump over the opponent's piece in that column. The goal is to immobilize the opponent: the player who can not move looses.

Béart says "C'est généralement un griot qui propose une partie à un berger" (Usually a *griot* (repository of oral tradition) invites a shepherd to play). He also says that a *griot* never makes mistakes: he always wins playing second ("Le griot ne triche pas, ce n'est nécessaire, il est sûr de gagner, quand il voudra, s'il n'a pas le trait (...)"). This observation is interesting as it shows the preoccupation with the application of winning strategies. In fact, the second player can win with a symmetry strategy. Maybe the opponents made agreements to avoid symmetric moves in the first moves.

It is important to understand first the concept of *reversibility*. A reversible move is one for which the opponent can respond to in such a way that his prospects are at least as good as they were before. In [5] it is possible to find very good examples of this concept.

TIOUK-TIOUK is not a combinatorial game as in definition 1 because the retroceding moves are allowed. Rigorously, the complete game tree has infinite depth. However, retroceding moves are reversible. For example, if a player retrocedes a piece two cells in one column, the opponent may advance his piece in that column exactly two cells restoring the position. Because of this fact, a winning strategy doesn't account with retroceding moves. This justifies why the winning strategy for TIOUK-TIOUK can be achieved with the winning strategy for NIM. If retroceding moves was not allowed, two pieces in one column with k cells between them would act exactly as a pile with k stones in NIM so, in some sense, TIOUK-TIOUK *is* the game of NIM and this is the real importance of this game. These arguments can be seen in [5], but the authors call NORTHCOTT to a horizontal version TIOUK-TIOUK. Being the Béart's anthropological work rigorous, maybe this African game is the first world's occurrence of NIM.

7 *Cyclopedia of Games* and *The Canterbury Puzzles*

KAYLES (derived from the French "quilles" meaning skittles) is an ancient game where players threw sticks or cudgels at a line of pins. This inspired the recreational mathematicians Samuel Loyd (1841-1911) and Henry Dudeney (1857-1930), who published a very interesting version.

In 1914, after Loyd's death, his book *Cyclopedia of 5000 Puzzles* was published by his son [28]. In 1907, Dudeney published his *The Canterbury Puzzles* [29]. Both books are important references in the history of recreational mathematics. Dudeney and Loyd exchanged puzzles and mathematical problems for a while, but Dudeney broke off the correspondence and accused Loyd of stealing his puzzles and publishing them under his own name. So, about KAYLES, it is better to say that the game was introduced by Dudeney and also by Loyd, who called it *Rip Van Winkle's Game*.

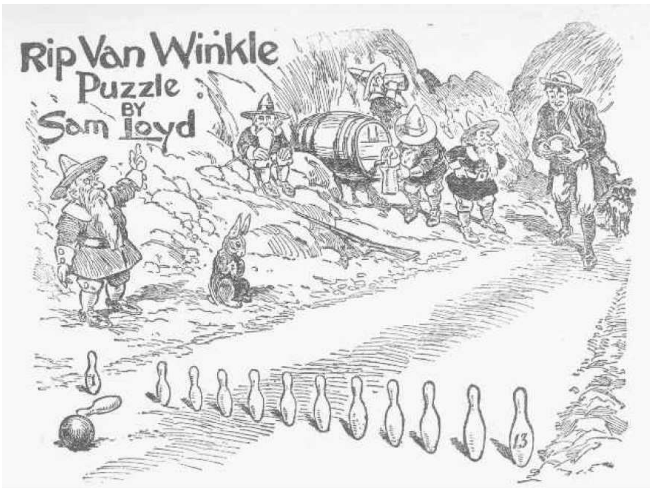


Figure 10: KAYLES in the *Cyclopedia of 5000 Puzzles*.

In their version, the players become so skilful that they can take out any desired pin or any two adjacent ones. The goal is simple: whoever is unable to knock down a pin loses.

There are many ways to generalize the game of NIM. For instance, we can impose the rule that in the game, when we remove k stones from a pile, we partition what remains of that pile into just 0 or 1 or 2 piles. For technical details, this class of combinatorial games is called the class of *octal games* [5]. KAYLES can be seen as a pile game where the players just can take 1 or 2 stones leaving 0, 1 or 2 piles. Therefore, KAYLES is one more example in literature of a combinatorial game (an octal game).

Figure 11: KAYLES in the *The Canterbury Puzzles*.

8 The rising of Combinatorial Game Theory

In the 20th century, a new mathematical subject emerged, based in combinatorial games. Richard Nowakowski, mathematician and researcher in Combinatorial Game Theory, published the principal historic reference about this period [30]. Nowakowski remarked three important historical and mathematical processes in the 20th century history of Combinatorial Game Theory:

1. Development of mathematical theory for the analysis of impartial games (a combinatorial game is *impartial* if both players always have the same moves in all positions, else it is called *partizan*). Seminal results were the Bouton's analysis (1902) and the establishment (with a proof) of the Sprague-Grundy Theorem (1935/1939).
2. Development of a general mathematical theory for the analysis of combinatorial games. Here, the principal moments were the publications of *Surreal Numbers*, in 1974, by

Donald E. Knuth (based in a conversation between Knuth and Conway where Conway explained his mathematical idea [2]), *On Numbers and Games*, in 1976, by John Conway [3], and "the Holy Bible" of Combinatorial Game Theory, *Winning Ways* , in 1982, by Elwyn Berlekamp, John Conway and Richard Guy [5].

3. The modern mathematical work, rich of open problems.

About the first two items, Nowakowski presented two schemes:

Figure 12: 20th century history of Combinatorial Game Theory [30].

We include also two cultural important moments:

- The presentation of *NimRod* in 1951's Festival of Britain (on the 5th of May 1951, the *nimrod* computer made its public bow, designed exclusively to play the game of NIM).
- The 1961's Alain Resnais movie *L'Année dernière à Marienbad* where mìsere version of NIM is a crucial part of the argument.

8.1 Bouton's work and the origin of the word "nim"

The emergence of combinatorial game theory occurred with the analysis of impartial normal play. The game of NIM is impartial since both players have the same moves in all NIM's positions. The mathematician Charles L. Bouton (1869-1922) analyzed NIM exhaustively discovering "the good" strategy based in an important mathematical operation (nim addition as it is called modernly). After, it took three decades before it was proved that each impartial combinatorial game is equivalent to a NIM position (see 8.4). This is the reason why Bouton's work was the fundamental turning point in the history of Combinatorial Game Theory: this equivalence and the Bouton's strategy allowed the understanding that impartial combinatorial games form an entire class that can be analyzed with strong mathematical tools.

Mathematically speaking, Bouton defined an operation for whole numbers: *write the numbers in binary and add without carrying* (in fact, the terms were not exactly these, but the global idea was). For example, consider the numbers 9 and 11. Their binary representations are 1001 and 1011. The next scheme shows the addition without carrying:

$$\begin{array}{ccccc} & 1 & 0 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 \end{array}$$

Using the modern symbol "$\oplus$" for nim-sum, we have $9 \oplus 11 = 2$ (10 is the binary representation of 2). Bouton proved two facts: a) Considering a NIM position, if the nim-sum of the numbers of stones in each set is null, every move provokes a new position with not null nim-sum; if the nim-sum of the numbers of stones in each set is not null, there is at least one move provoking a new position with null nim-sum. The strategy is, if it is possible, to choose moves provoking a null nim-sum.

As we can see in modern literature, the easier way to operate the nim-sum is to think about the sum of distinct powers of two of the members and canceling repetitions in pairs. Some examples:

$$\begin{aligned} 5 \oplus 3 &= (4+1) \oplus (2+1) \\ &= (4+\cancel{1}) + (2+\cancel{1}) = 6 \end{aligned}$$

$$\begin{aligned} 11 \oplus 22 \oplus 35 &= (8+2+1) \oplus (16+4+2) \oplus (32+2+1) \\ &= (8+\cancel{2}+\cancel{1}) + (16+4+\cancel{2}) + (32+2+\cancel{1}) = 62 \end{aligned}$$

There is a big speculation about the word "nim". "Nim" is an obsolete English verb meaning "take". Also, "NIM" is "WIN" by geometric transformation. There is a character pronounced "nian" which means to pick up or take (it is not pronounced "nian" in standard Mandarin Chinese - rather it is pronounced "ning"). Maybe this was one more argument for the "Chinese origin" of NIM.

拧

Figure 13: Chinese character.

The more plausible origin of the name can be found in one correspondence in *Mathematical Gazette*. Bouton did his PhD in Leipzig so it is likely that the name owes much to the German verb *nimm* meaning "take" (imperative singular of the German verb *nehmen*).

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2334. *The Name of the Game of Nim.**

The game usually called NIM is very well-known both to mathematicians and to professional gamblers and for the same reason to both : while appearing to be a game of chance, it has, in fact, a perfect theory of winning. In a standard discussion of the theory of this game—that at G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers* (2nd ed.), p. 116 ff.—the game is referred to as " The Chinese Game of Nim ". A reference such as this might well lead—and indeed has often led—people to think that *Nim* was the Chinese name for the game. A priori, this would seem improbable (for, save in the Cantonese dialect, Chinese words do not end in *-m*) and is, in fact, not the case.

Two citations are relevant here :

(1) W. Ahrens, *Mathematische unterhaltungen und spiele* (2nd ed., 1910)† i, 72 : " Der ursprung des spiels ist unbekannt. In Amerika wird es auf einigen Colleges bisweilen dort auch wohl auf Jahrmärkten gespielt ; es soll aber auch in Deutschland schon seit Jahrzehnten bekannt sein‡. Auch in China soll es gespielt werden und möglicherweise ist dies sein Ursprungsland. Jedenfalls trägt es einen chinesischen Namen ' Fan-Tan '. Da man jedoch mit demselben Namen auch ein ganz anderes, bei Chinesen beliebtes Spiel bezeichnet, das ein reines Glücksspiel ist, so haben wir hier den . . . von Chs. L. Bouton vorgeschlagenen Namen akzeptiert."

(2) C. L. Bouton, " Nim, a game with a complete mathematical theory," *Annals of Mathematics*, II. iii [1901–2], 35 : " The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American Colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it."

Professor Bouton does not tell us why he gave the name *Nim*§ to the game. He died in 1922|| and it may well be that there are people still living who actually know the answer to this question. It would be pleasant if one such should read this note. But in the absence of evidence of this kind the etymology of the name is for discussion. It may, of course, be that the name is arbitrary. But, in fact, word-coinages which are truly arbitrary, both consciously and sub-consciously, are exceedingly rare**—Dean Swift appears to constitute a notable exception here. It seems therefore more probable that *nim* is not arbitrary but that it is merely the imperative singular of the German verb *nehmen* " to take ", semantically a most suitable nomenclature. This form is today spelt *nimm*; it may be that Professor Bouton dropped the final *m*.

* I am very grateful to Professor W. Simon (London) and Professor H. H. Dubs (Oxford) for advice on the Chinese side of the matter.

† It is to this work that Hardy and Wright refer.

‡ But Ahrens does not give the German name of the game, nor have I been able to discover it.

§ The word does not appear either in the *New English Dictionary* or in W. A. Craigie, *A Dictionary of American English*.

|| He was Assistant Professor of Mathematics at Harvard from 1904 to 1914 when he became Associate Professor, a position he held until his death. (Information kindly given me by the Alumni Records Office of Harvard University.)

** Cf. my remark on Dr. Aavik's coinage *relv* " weapon " (now part of the normal Estonian vocabulary) at *Transactions of the Philological Society* 1938, p. 68, note 2.

Figure 14: Text in *Mathematical Gazzette*, 1953.

CORRESPONDENCE.

THE NAME OF THE GAME OF NIM.

To the Editor of the *Mathematical Gazette*.

SIR,—On p. 119 of the May, 1953, issue of the *Mathematical Gazette* I read the conjecture by Professor Alan S. C. Ross that Professor C. L. Bouton derived the name of the game " Nim " from the imperative singular of the German verb *nehmen*. I am writing to verify this conjecture. Professor Bouton was a teacher of mine, and later I became his colleague. I was a frequent visitor in his home. He had had numerous mathematical and personal contacts with Germany, and received his Ph.D. from Leipzig in 1898. I distinctly recall his saying to me that he had chosen the name from the German word " nimm ", as a word that might well be used frequently during the play of the game, but had dropped the final " m ".

Yours, etc.,

University of Harvard. J. L. WALSH

Figure 15: Message to the Editor of the *Mathematical Gazzete* by J. L. Walsh, Bouton's pupil and colleague.

8.2 Wythoff's work

As it was written in section 2, the origin of WYTHOFF'S NIM is also not clear. Probably Willem Wythoff himself introduced the game in 1907 [17]. Also, the game was given, independently, by Rufus Issacs [33]. WYTHOFF'S NIM can be played on a quarter-infinite chessboard, extending upwards and to the right. A chess queen is placed in some cell of the board. On each turn, a player moves the queen like in chess, except that the queen can only move left, down, or diagonally down-left. Wins the player who takes the queen to the corner.

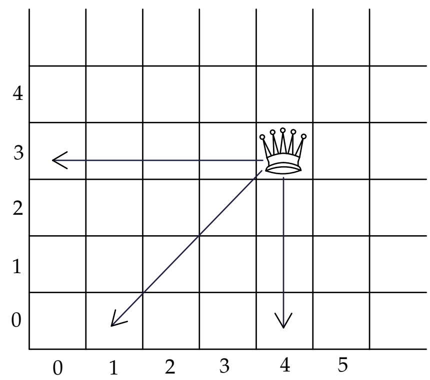

Figure 16: WYTHOFF'S NIM.

In mathematics and the arts, two quantities are in the golden ratio if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one. The golden ratio is an irrational number, $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803....$

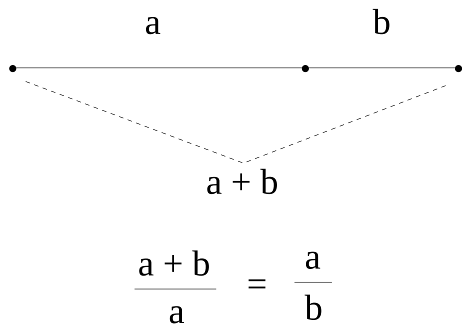

Figure 17: Golden ratio.

Amazingly, WYTHOFF'S NIM is related to the golden ratio and Fibonacci numbers. The positions where Next player loses (Previous player wins) are given by $(\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor)$ and $(\lfloor \varphi^2 n \rfloor, \lfloor \varphi n \rfloor)$ [5]. This is a classical result in Combinatorial Game Theory.

There is one property that some games have, making Combinatorial Game Theory extraordinarily useful: the board breaks up into separate components and the players have to choose a component in which to play. This aspect is so important, that the separation of the components is related to a mathematical operation with its own name: *disjunctive sum*.

WYTHOFF'S NIM provided a beautiful mathematical analysis, however, the game doesn't break up into disjoint components. The game turned to be still a wrong direction for the Sprague-Grundy Theorem exposed in subsection 8.4.

8.3 Emanuel Lasker and his variant of NIM

Emanuel Lasker (1968-1941) was a German chess player, mathematician, and philosopher who was World Chess Champion for 27 years. He is still generally regarded as one of the strongest players ever.

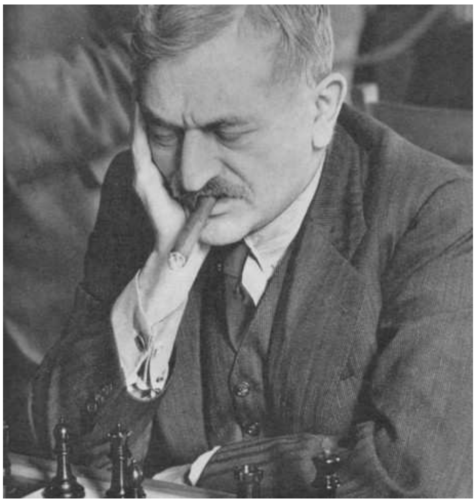

Figure 18: Emanuel Lasker.

In 1931, Emanuel Lasker introduced the game LASKER'S NIM - the rules are as in ordinary NIM with the extra option of splitting a heap into two smaller non-empty ones without removing any stone. The Lasker's game accounts with the disjunctive sum aspect. He solved the game (for mathematical aspects see [31]) and remarked the existence of positions where the Next player wins and the existence of positions where the Previous player wins [32]:

If a configuration that we are examining can be brought by a permissible move into a losing position, then the configuration under examination is a winning position. If we cannot do so, then is a losing configuration. There is no third choice.

Amazingly, Lasker worked with equivalent classes:

We see first that two groups of piles can be "equivalent" in that in every losing configuration in which one group appears, it can be replaced by the other without altering the win character of that configuration. Merging two equivalent configurations results in a losing position.

Reading this last sentence, mathematicians who know a little about Combinatorial Game Theory immediately understand how close Lasker was! However, as Jörg Bewersdorff said in [31], Lasker missed developing the whole theory: He missed the understanding of the relation to ordinary NIM. According to Richard Guy, others were close (for example, Michael Goldberg when he worked on KAYLES [30]).

8.4 The Sprague–Grundy theory

Working independently, Roland P. Sprague, in 1935, and Patrick M. Grundy, in 1939, proved the important theorem stating that every impartial game under Normal Play rules is equivalent to some pile of stones in NIM [9, 8] (see [4] for a beautiful proof). Because of this theorem, many authors now refer *nim-value* of a impartial game to the size of the equivalent nim-heap. The implications of this theorem are just fantastic (this became known as Sprague-Grundy Theory): when we analyze an impartial combinatorial game, first we try to understand to which pile it is equivalent and, after, we apply the Bouton's work. Of course, the first step can be hard, but it is the way to work the problem.

For example, Pacioli's ADDITION$(1, 2, 3, 4, 5, 6)$ (with 30 as the goal-number) is equivalent to a pile with 2 stones and Bachet's ADDITION$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ (with 100 as the goal-number) is equivalent to a pile with 1 stone (see [5] for mathematical details). Also, Loyd and Dudeney's KAYLES with 13 pins is equivalent to a pile with 1 stone: a winning position can be achieved using a symmetry strategy, on his first move, the first player should move so that the row is broken into two sections of equal length (see [5] for mathematical details).

Richard Guy worked with Cedric Smith, who had worked with Grundy. This conduced to Guy's analysis of DAWSON CHESS and hundreds of other games to witch the theory could be applied. An important seminal paper was published by Richard Guy in 1956 [34]. Richard Guy is still active in the present moment (He is 94 years old).

Figure 19: Richard Guy (June, 2005).

8.5 *NimRod*

On 6 October 1951 an electronic computer, the *NimRod*, from the English company Ferranti was the star of the Berlin Industrial Show for three weeks. The *NimRod* was designed exclusively to play the game of NIM. We can read in [12]:

The original machine stood 9 feet ny 12 feet by 5 feet, but most of this was just to house the display. Electronics took up less than two percent of that volume. It ran at a healthy *six kilowatts, though four of these were for the display lamps.(...) It housed 480 valves, all 12AT7 double-triodes. Only 350 of them actually took part in the action; the rest were spares being "burned in". 120 relays drove the displays. A few germanium diodes were used as 'OR' gates. Most of the connections were directly soldered, rather than using pluggable connectors, for reliability.*

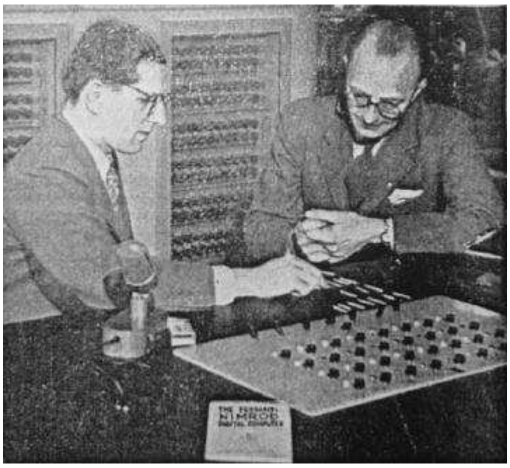

Figure 20: *NimRod.*

Also, a historical note by originator John Bennett, an Australian who went to Cambridge to do a PhD in Computer Science and joined Ferranti as a logical design engineer in 1950:

Ferranti had undertaken to display a computer at the 1951 Festival of Britain, and late in 1950 it became evident that this promise could not be fulfilled. I suggested that a machine to play the game of NIM *against all comers should be constructed with a versatile display to illustrate the algorithm and programming principles involved. The design was implemented by a Ferranti engineer, Raymond Stuart-Williams, who later joined RCA. The machine was a great success but not quite in the way intended, as I discovered during my time as spruiker on the Festival stand. Most of the public were quite happy to gawk at the flashing lights and be impressed. A few took an interest in the algorithm and even persisted to the point of beating the machine at the game. Only occasionally did we receive any evidence that our real message about the basics of programming had been understood.*

During the Festival, visitors could purchase (for one shilling and sixpence) a detailed description of the machine.

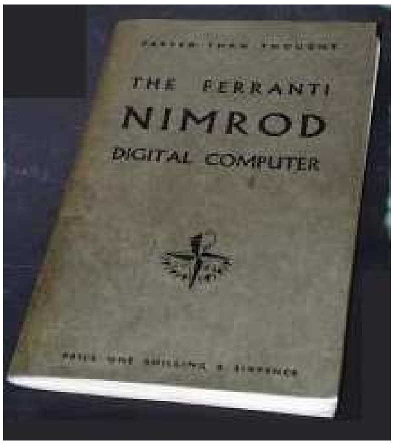

Figure 21: *NimRod* guide booklet [11].

8.6 Alain Resnais's *L'année dernière à Marienbad*

L'Année dernière à Marienbad is a 1961 French film directed by Alain Resnais, starring Delphine Seyrig, Giorgio Albertazzi, Sacha Pitoëff. The characters are unnamed in the film; in the published screenplay, the woman is referred to as "A", the first man is "X", and the man who may be her husband is "M". The mìsere version of NIM is a crucial part of the argument. It is supposed that "M" doesn't make mistakes (or not?). The movie is very enigmatic (maybe the only "linear" scenes are the NIM games!)

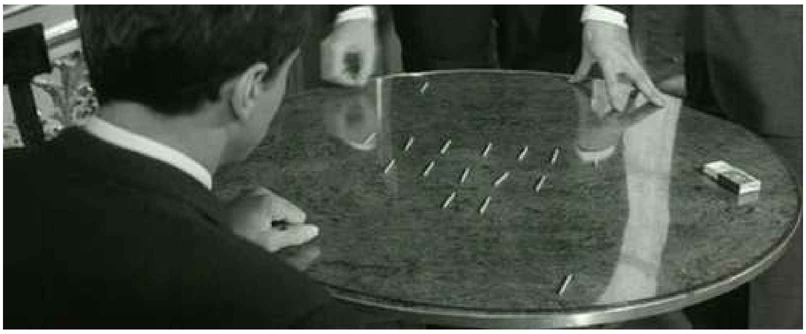

Figure 22: *L'Année dernière à Marienbad*.

In the movie, NIM mìsere games are played from the starting position $7 - 5 - 3 - 1$. It is known that, beginning with this position, First player loses. However, from 4 complete games in the movie (they can be seen in [13]), "M" is First player in two. So, the following movie's conversation is not mathematically correct:

"M": Je connais un jeu lequel je gagne toujours.
"X": Si vous ne pouvez pas perdre, ce n'est pas un jeu.
"M": Je peux perdre, mais je gagne toujours.

The fact is that "M" just make a mistake in the entire movie. Maybe an Alain Resnais's mistake... maybe not... (see Appendix 4)

8.7 GO and partizan games

After the establishment of Sprague-Grundy theory, it lacked a general mathematical theory for combinatorial games also including the partizan games. In 1953, John Milnor wrote a first paper on partizan games, based in his research in Classical Game Theory [35]. He understood that there are games were players have desire to move – *hot games* (this is an easy concept, for example, for a chess player). John Milnor gave the first step, he thought about an important idea: analyze what happens when there are many copies of a position in view to obtain a *mean-value*, useful to its evaluation.

After, in 1957, Olof Hanner advanced one more step using his interest in the ancient oriental game of GO. This game is not a combinatorial game (it is a scoring game), but when playing its endgames, the idea of disjunctive sum of independent components emerges. In a Stockholm bookstore Hanner found a copy of the English edition of the go book by Takagawa (National Go Champion of Japan), *How to Play Go*, and studied it accurately. In his own words (see [36]),

Takagawa's book gives two examples of full games. The second of these is given with a few remaining securing stones to be played. The book says that the game ended with Black winning by one point. I become interested in the way the remaining stones could be played. Being a mathematician I made a calculation and found that by best play Black should have won by 2 points. I was a little surprised that a professional player did not have a full grasp of this phase of the game. I wrote a scientific paper on the ideas behind my calculations .

In fact, GO was directly related to the rising of the general Combinatorial Game Theory. Jon Diamond formed the Go club at Cambridge University and starting the magazine of the British Go Association (British Go Journal), he was also British Champion for 12 years, starting at age 18, until he retired as Champion in 1977. So, Conway learned the game in his Cambridge times [3]:

(...) by which at that time we meant the Nim-like theory developed independently by Roland Sprague and Peter Michael Grundy for sums of impartial games–those for which the two

players have exactly the same legal moves. I had long intended to see what would become of the theory when this restriction was dropped, but only got around to doing so when the then British Go Champion became a member of the Cambridge University Pure Mathematics Department. Astonishingly, it was the resulting attempt to understand GO *that led to the discovery of the Surreal Numbers! This happened because the typical* GO *endgame was visibly a sum of games in the sense of this book, making it clear that this notion was worthy of deep study in its own right. The Surreal Numbers then emerged as the simplest domain to which it applies!*

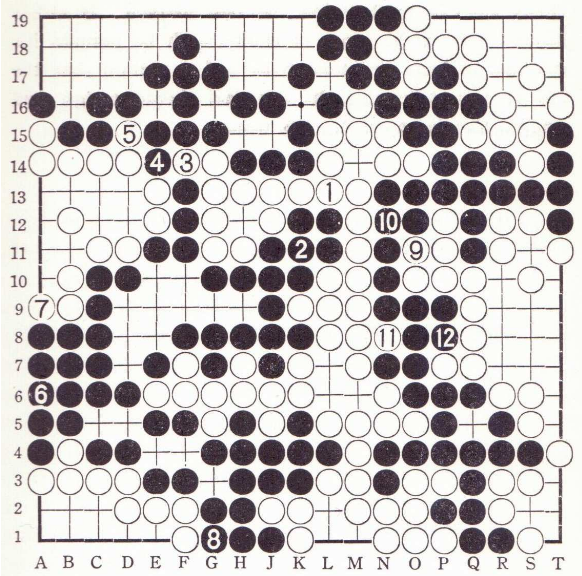

Diagram shows the neutral points filled in to facilitate counting. Note that it would make no difference which side filled in all of these points except 1, 2, 6 and 7, which were connections.

Without placing the captured stones in their respective territories, the Black territory totals 48 points and the White territory is 55 points. However, Black captured 12 White stones and White took only 4 Black stones, so the final count is 44 points for Black and 43 points for White. Black has won the game by one point!

The two sample games you have seen in this chapter were played out to the very end, that is, until there were no further plays of value on the board. This is not always

Figure 23: *The final position of the second game of the Takagawa's book.*

8.8 Conway's construction, Berlekamp, Conway, Guy, and the establishment of a wonderful mathematical theory

John Conway knew Mike Guy, Richard's son in 1960, in Cambridge. Elwyn Berlekamp met Richard Guy at a conference in 1966. Berlekamp was the world's best player of the game DOT-AND-BOXES (he still is today)–He analyzed the game with the help of a Guy's paper [6]. Berlekamp has not lost a game of DOT-AND-BOXES in over 40 years. He suggested that they write a book and Guy suggested adding Conway. This was the starting point of a great mathematical team [30].

In 1970, Conway presented some lectures at Cambridge exposing the idea behind the general Combinatorial Game Theory. After, he spoke with the mathematician Donald Knuth about his ideas about games. In 1974, Donald Knuth, during a "bad moment" with his wife, spend a week in a Norway Hotel, and wrote the *Surreal Numbers* based in Conway's ideas [2]. *Surreal Numbers* was the first book exposing Conway's construction and the term *Surreal numbers* was a Knuth's invention.

In 1976, Conway published his *On Numbers and Games* exposing mathematically the entire idea. In his own words:

(...) *for seven consecutive days I sat down and typed from 8:30 am until midnight, with just an hour for lunch, and ever since have described this book as "having been written in a week. Not entirely honest, because there were loose ends still to be tied up, and Chapter 16 was written just before the book appeared, while Chapter 13 was largely copied from a paper, "Hackenbush, Welter and Prune", that had been written a year earlier. But also not entirely dishonest.*

The most surprising immediate result was a threat of legal action from Elwyn Berlekamp. But somehow Berlekamp, Conway and Guy must have patched this up, because the two volume set *Winning Ways* appeared in 1982 [5], and they remain good friends.

Conway's inductive definition constructs the complete set of combinatorial games. Amazingly, he created a complete abstract notation that is the key of combinatorial game theory and extended the real numbers in a similar way to that Dedekind extended the rational numbers to the reals. Conway remarked that some games are numbers, other games are infinitesimals, other games are not numbers neither infinitesimals. Conway's construction gives a new very elegant construction of real numbers with the advantage that his construction doesn't need to have the rational numbers as base [3].

However, saying that Conway invented Combinatorial game Theory is not correct. One can

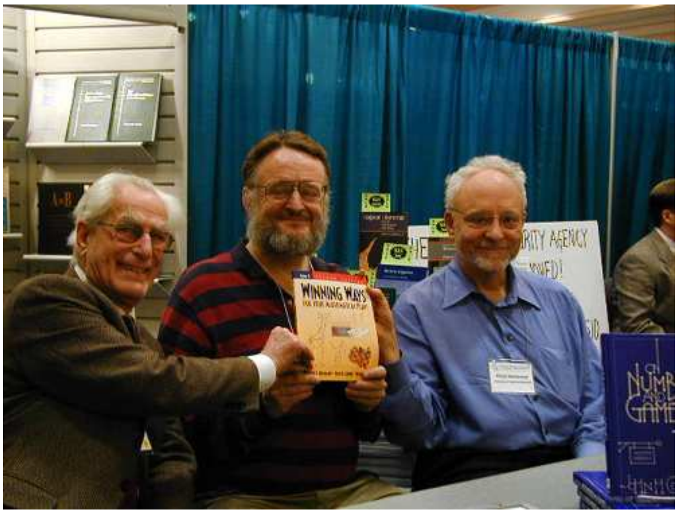

Figure 24: Guy, Conway and Berlekamp (with Winning Ways and On Numbers and Games).

say he invented the modern notation and first understood the construction, both of which are invaluable contributions to the subject. The partizan theory in general, however, is due largely to [5] and its authors (which includes Conway) even if it was published later. Of course, bits and pieces of this were developed before, and we can't forget the importance of the Sprague-Grundy theory, which was already well understood at the time.

8.9 Modern times

Nowadays, Combinatorial Game Theory is a established mathematical subject. In *Combinatorial Games: Selected Bibliography with a Succinct Gourmet Introduction*, organized by the mathematician Aviezri S. Fraenkel, we can observe already 1400 scientific papers related to Combinatorial Game Theory [37]. Many research papers on this subject were presented at conferences held in 1994 and 2000 in Berkeley, CA, at the Mathematical Sciences Research Institute. Proceedings of those conferences were later published by Cambridge University Press under the titles *Games of No Chance* and *More Games of No Chance*. We can enumerate some actual mathematical research directly related to Combinatorial Game Theory:

1. Hot Games: Games in which there is advantage in moving first. This area is really difficult for computer scientists and mathematicians. The mathematicians want to know the exact values (full understanding) and work with retrograde analysis. The computer scientist wants good heuristics.

2. All-Small Games: Games in which either both players have a move or neither does. This is a very interesting class of combinatorial games where all values are infinitesimal.

3. Loopy Games: Extending the restricted definition of combinatorial game, it is possible

to consider games in which the play is not guaranteed to end. Recent advances about loopy games were obtained by Aaron Siegel.

4. Scoring Games: Relaxing the conditions that define combinatorial game, it is possible to analyze scoring games like GO and DOTS-AND-BOXES. Great work done by Elwyn R. Berlekamp.

5. Biding Games: Games where players bid for the right to play next. Also, these games are not pure combinatorial games but some techniques can be used.

6. Three or More Players Games: In these games the players could form coalitions for all or some of the game. We remark J. Propp and A. Cincotti research.

7. Misère: In the past, many researchers believed that the strategy for a Misère game was to take the Normal play strategy and tweak it at the end of the game. This is true for NIM, it is not true in general.Misère play is a very difficult mathematical subject. We remark the advances achieved by T. Plambeck and A. Siegel.

It is also important to observe some interesting applications of Combinatorial Game Theory. The golden application is, of course, games themselves (and games have an independent position in human culture). However, we can enumerate some more applications:

1. Mathematical Importance of the Combinatorial Game Theory: Conway's construction gives a new very elegant construction of real numbers with the advantage that his construction doesn't need to have the rational numbers as base. Also, the algebraic concepts behind the theory are very interesting and can be proper analyzed with the help of Group Theory techniques.

2. Computer Systems: Combinatorial Game Theory is very useful when the board breaks up into separate components and the players have to choose a component in which to play. Decomposition issues also lie at the heart of the design problem of computer systems. The crucial design decision is usually how to partition the overall system into tractable modules.

3. Artificial Intelligence: Researchers in Combinatorial Game Theory and researchers in AI share a common interest in tree-searching and tree-pruning algorithms, and in the potential to combine such algorithms with the decomposition algorithms of Combinatorial Game Theory to provide the tools which will support computer attacks on much harder problems.

4. Complexity: branch of the theory of computation in computer science and mathematics that focuses on classifying computational problems according to their inherent difficulty. Many examples of harder mathematical problems are combinatorial games. Some consider this intimate connection with complexity theory to be important justification for studying Combinatorial Game Theory. Aviezri Fraenkel made important work about it.

5. Encrypting and Information Theory: Some error-correcting code technology is related to Combinatorial Game Theory. Elwyn R. Berlekamp did important work about it.

9 Final remarks

In this section we just list some key-ideas of this paper:

1. The rising of a new mathematical subject justifies the effort of writing a rigorous *History of Combinatorial Games.*
2. The "Chinese origin" is not clear. There is no primordial reference.
3. One game of Pacioli's *Viribus Quantitatis* (1496-1508) is an additive version of a subtraction game.
4. One game of Bachet's *Problèmes Plaisans et Delectables* (1612) is an additive version of a subtraction game.
5. KONANE described by Captain James Cook in 1778 (during his third voyage) is an ancient pure combinatorial game.
6. It is possible that TIOUK-TIOUK described by Charles Béart is the first ocurrence of NIM.
7. KAYLES, an octal game, appeared in Loyd's *Cyclopedia of 5000 Puzzles* (1914) and Dudeney's *The Canterbury Puzzles* (1907). Both books are important references in the history of recreational mathematics.
8. Bouton and Wythoff's work (first years of the 20th century), Lasker, Sprague, and Grundy's work (30's), Milnor, Hanner, Smith, and Guy's work (50's) and Conway, Berlekamp, and Guy's work (since 60's) were the fundamental base for the establishment of the Combinatorial Game Theory as a mathematical new subject.
9. NimRod (1951) and *L'Année dernière à Marienbad* (1961) were remarkable cultural events directly related to combinatorial games (NIM).

Acknowledgments

We thank Dr. Thierry de Paulis, game historian (Secretary General of "Le Vieux Papier"), for suggesting very interesting explorations.

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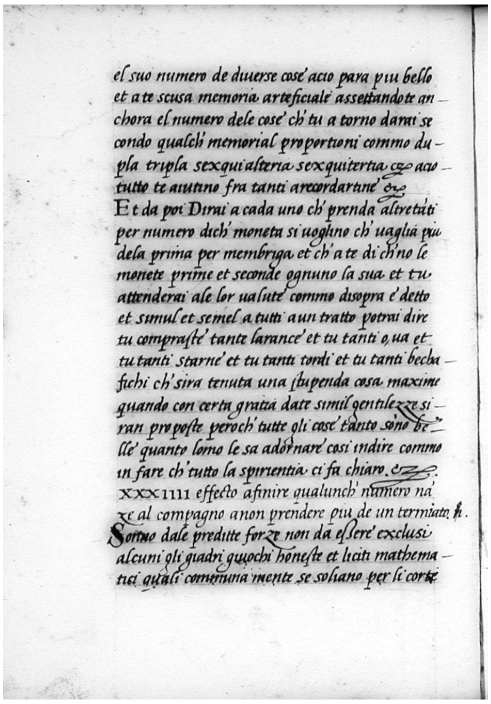
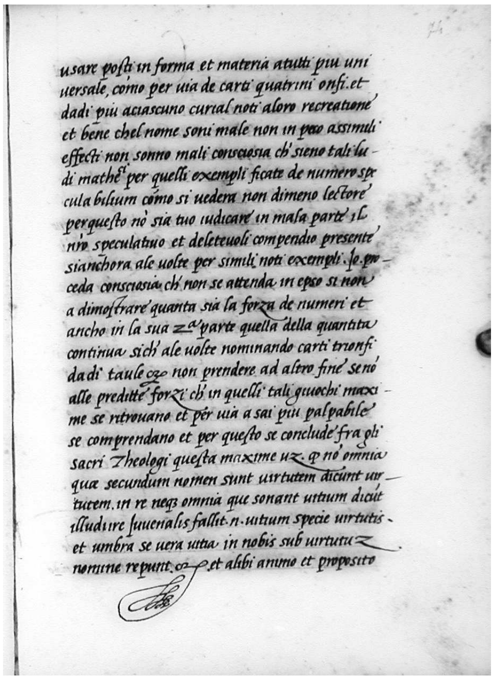
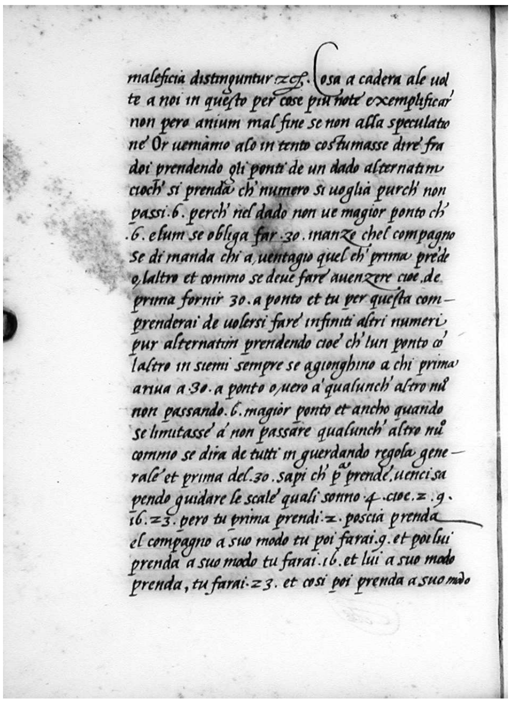
Appendix 1: *Viribus Quantitatis* (Uri p. 175-181)



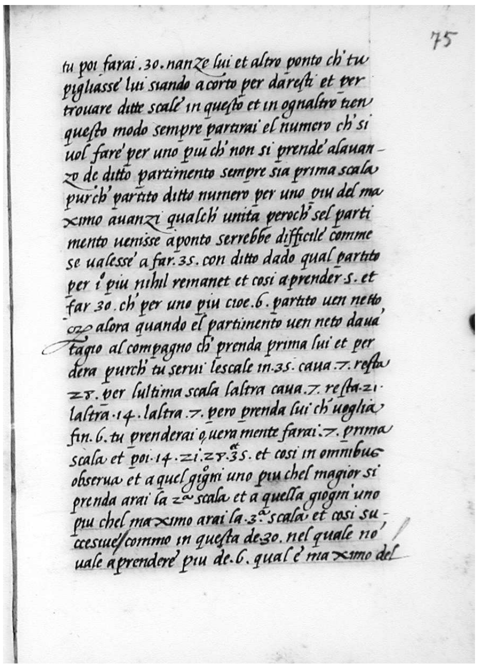
el suo numero de diuerse cose acio para piu bello
et a te scusa memoria arteficiale assettandote an-
chora el numero dele cose ch tu a torno darai se
condo qualch memorial proportioni commo du-
pla tripla sexquialteria sexquitertia &c acio-
tutto te aiutino fra tanti arecordartine &c
Et da poi Dirai a cada uno ch prenda altretāti
per numero dich moneta si uoghino ch uaglia piu
dela prima per membriga et ch a te di ch no le
monete prime et seconde ognuno la sua et tu
attenderai ale lor ualute commo disopra e detto
et simul et semel a tutti a un tratto potrai dire
tu comprasti tante larance et tu tanti oua et
tu tanti starne et tu tanti tordi et tu tanti becha-
fichi ch sira tenuta una stupenda cosa maxime
quando con certa gratia date simil gentilezze si-
ran proposte peroch tutte gli cose tanto sono be-
lle quanto lomo le sa adornare cosi indire commo
in fare ch tutto la spirientia ci fa chiaro. &c.
XXXIIII effecto afinire qualunch numero nā
ze al compagno anon prendere piu de un terminato. h.
Sonno dale preditte forze non da essere exclusi
alcuni gli giadri giuochi honeste et liciti mathema-
tici quali communa mente se soliano per li corte



usare posti in forma et materia atutti piu uni
uersale, como per uia de carti quatrini onfi. et
dadi piu aciascuno curial noti aloro recreatione
et bene chel nome soni male non in peo assimili
effecti non sonno mali consciosia ch sieno tali lu-
di mathē per quelli exempli ficate de numero spe
culabilium como si uedera non dimeno lectore
per questo nō sia tuo iudicare in mala parte il
nro speculatiuo et deletteuoli compendio presente
sianchora ale uolte per simili noti exempli. se pro-
ceda consciosia ch non se attenda in epso si non
a dimostrare quanta sia la forza de numeri et
ancho in la sua 2ª parte quella della quantita
continua sich ale uolte nominando carti trionfi
dadi taule &c non prendere ad altro fine senō
alle preditte forzi ch in quelli tali giuochi maxi-
me se ritrouano et per uia a sai piu palpabile
se comprendano et per questo se conclude fra gli
sacri Theologi questa maxime uz. q nō omnia
quæ secundum nomen sunt uirtutem dicunt uir-
tutem. in re neqz omnia que sonant uitium dicūt
illud ire Iuuenalis fallit. n. uitium specie uirtutis.
et umbra se uera uitia in nobis sub uirtutum
nomine repunt. &c et alibi animo et proposito



maleficia distinguntur &c. Cosa a cadera ale uol
te a noi in questo per cose piu note exemplificar
non pero anium mal fine se non alla speculatio
ne Or ueniamo alo in tento costumasse dire fra
doi prendendo gli ponti de un dado alternatim
cioch si prenda ch numero si uoglia purch non
passi. 6. perch nel dado non ue magior ponto ch
.6. elum se obliga far. 30. inanze chel compagno
se di manda chi a uentagio quel ch prima prēde
o laltro et commo se deue fare auenzere cioe de
prima fornir 30. a ponto et tu per questa com-
prenderai de uolersi fare infiniti altri numeri
pur alternatim prendendo cioe ch lun ponto cō
laltro in siemi sempre se agionghino a chi prima
ariua a 30. a ponto o uero a qualunch altro nū
non passando. 6. magior ponto et ancho quando
se limitasse a non passare qualunch altro nū
commo se dira de tutti in guerdando regola gene-
rale et prima del. 30. sapi ch pª prende. uenci sa
pendo guidare le scale quali sonno 4. cioe. 2. 9.
16. 23. pero tu prima prendi. 2. poscia prenda
el compagno a suo modo tu poi farai. 9. et poi lui
prenda a suo modo tu farai. 16. et lui a suo modo
prenda, tu farai. 23. et cosi poi prenda a suo modo



75

tu poi farai. 30. nanze lui et altro ponto ch tu
pigliasse lui siando a corto per daresti et per
trouare ditte scale in questo et in ognaltro tien
questo modo sempre partirai el numero ch si
uol fare per uno piu ch non si prende alauan-
zo de ditto partimento sempre sia prima scala
purch partito ditto numero per uno piu del ma
ximo auanzi qualch unita peroch sel parti
mento uenisse aponto serrebbe difficile comme
se ualesse a far. 35. con ditto dado qual partito
per 1º piu nihil remanet et cosi a prender 5. et
far 30. ch per uno piu cioe. 6. partito uen netto,
&c alora quando el partimento uen neto dauā
tagio al compagno ch prenda prima lui et per
dera purch tu serui le scale in. 35. caua. 7. resta
28. per lultima scala laltra caua. 7. resta. 21.
laltra. 14. laltra. 7. pero prenda lui ch uoglia
fin. 6. tu prenderai o uera mente farai. 7. prima
scala et poi. 14. 21. 28. et 35. et cosi in omnibus
obserua et a quel giōgni uno piu chel magior si
prenda arai la 2ª scala et a quella giogni uno
piu chel maximo arai la. 3ª. scala et cosi su-
ccesiue/commo in questa de. 30. nel quale nō
uale aprendere piu de. 6. qual e maximo del

del dado dico ch' parta .30. per uno piu di .6. cioe per
.7. neuen .4. et auanza .2. Or questo .2. dico ch' sia
prima scala del auenimento non si fa caso et a
questo giogni .7. fa .9. per la 2ª scala alla quale
a giogni .7. fa .16. per la 3ª scala et a questa giogni
.7. fa 23. per la 4ª scala et uariando tu or qua
or la per non dare auiso fa pure ch' te retroui ale
ditte scale in modo ch'l compagno non sena corga
et sia bello
Et se hauesse ualuto a far .40. pur con lo ponto de
un dado partiresti pur per .1°. piu del maximo cioe
per .7. neuen .5. et auanza .5. et di questo .5. ch'
auanza farai scala prima poi per la 2ª giogni
7 a .5. fa .12. per la 2ª giogniui .7. fa .19. per la 3ª
giogni .7. fa 26. per la 4ª giogni .7. fa 23. per la
5ª poi prenda lui ch' uoglia fin a .6. tu farai .40.
nanze de lui. E cosi si ualesse a .50. partirai .50.
in .7. neuen .7. et auanza .1°. et questo .1°. sia pª
scala a far .50. al quale giogni .7. fara .8. per la
2ª a giogni .7. fara .15. per la 3ª et cosi succe
siue arai per .50. septe scale cioe lultima .43. et
sic in alijs etc. Et quando ualesse a pigliare
altri ponti o ch' del dado, sequi el simile commo
si fossero stese tutti li carti de un sol giuocho.

cioe tutte quelle de bastoni senza le figure acio
non abagli in li ponti metendo gli ponti de un
sol giuocho esser fin a .10. cioe .10. bastoni .10. cope
.10. denari et 10 spate. etc. Or dicendo ognuno
alternatim prenda et riprenda gli ponti de una
sola carta a chi prima fa .100. or qui non si po
prendere magior ponto ch' .10. una o uero piu uol
te peroch' gli e promesse poter prender una carta
piu uolte a chi uuoli dico medesimamente ch'
parta 100. in uno piu de .10. cioe in .11. neuen .9.
et auanza .1°. per la prima scala per la 2ª
giogniui .11. fa .12. per la 3ª giogniui .11. fa 23.
per la 4ª giogniui .11. fa .34. la 5ª 45. la 6ª 56
la 7ª 67. la 8ª 78. la 9ª 89. la 10ª non ne peroch' se
rrebbe .100. donca la prima sia .1°. lultima 9ª .89. le
quali arai a mente per lo 100. et faralo. et cosi per
lo .30. con ditti carti parti .30. per .1°. piu de .10. cioe
per .11. neuen .2. et auanza .8. per la prima s
cala ala quale giogni .11. fa .19. per la 2ª et ulti
etc. con ditte carti prendendo ognuno una sia
piu netto peroch' non ognuno intende prendere un
ponto del dado ma ben ognuno intendera pren
der una carta // Ma per far .30. con le carti sira
piu bello a ponere in taula solo .6. carti cioe fin .6.

ponti cosi .1. 2 3. 4. 5. 6. quali uengano a essere gli
ponti deli .6. facce de uno dado resolute et tu prendi
la .2. etc. commo fo detto e per questo in tutti
numeri te regerai

XXXV effecto de saper trouare .3. uarie cose
diuise fra .3. persone et .4. diuise fra .4. et de quate uorrai etc.

A Nchora la medesima de .3. cose uarie dispensate per
una gentil regola. farala in questo modo. uz. darai
allamico cioe aluno .12. alaltro .24. et alaltro .36.
per numero scienter et poi dirai chi a la prima casa
getti uia la mita di quello ch' io gli ho dato et chi
a la 2ª getti uia gli $\frac{2}{3}$ di quelli gli ho dati et
chi a la 3ª getti uia gli $\frac{3}{4}$ di queli gli ho dati et
facto questo dirai ch' gionghino tutti li loro resti
o uero auanzi insiemi et dicha te la summa la
quale po hauenire in .6. modi. et non piu cioe ch'
sira 23. o 24. o uero 25. o uero 27. o uero 28.
o uero 29. perch' 26. non po mai restare fra
tutti // Se la summa de lor resti sia .23. et tu di
rai ch' colui a chi tu desti .12. habbia la prima casa
et a chi tu desti .24. la 2ª et a chi tu desti 36.
habbia la 3ª // Commo se ad Antonio dessi .12.
elui gra exempli hauesse el dº. prima casa e A
Benedetto hauesse dato .24. et hauesse la 2ª casa

desotto in mediate se
da el modo a far con
faue et monete in
questo .35. effecto

Appendix 2: *The Island of Lanai: A Survey of Native Culture* (p. 84-85)

abruptly. This shallow basin takes up the entire top of the stone except for a rim 2 inches wide.

WATER-WORN BOWLDERS ON HOUSE SITES

At many house sites along the coast are one or two flat, oblong, water-worn bowlders. These may be seats or pillows, but may as well have served other purposes.

One of these stones (B4012) from the middle of the platform at Site 42, Kaunolu, weighs 26 pounds and is 13 inches long, 6 inches wide, and 3 inches thick at the middle. Both of the flattened surfaces are concave longitudinally. This stone is unusual because of the lenticular gashes 1 to 2 inches long and about .3 inches deep cut at various angles on both of the flattened surfaces. One end of this bowlder has been polished in the center, probably by the grinding of implements upon it. The material is gray, somewhat vesicular basalt.

PITTED SLABS FOR THE GAME OF KONANE

On the platform of many houses, in the middle front or at a front corner (fig. 3, *a-b;* Pl. IX, *D*) are stone slabs pitted with rows of holes for the game of *konane*. The flat top of bowlders or the surface of a ledge near a house site may also be marked for this game. The dots average an inch apart, .2 of an inch deep and .5 of an inch in diameter. Such an arrangement of rows of pits is called a *papamu*.

The *papamu* brought from Naupaka (B 4048) is a slab of cellular basalt 3 inches thick, and 14 inches square. The rows of pits are 9 by 9, arranged, not very evenly, to occupy entire surface. The under surface of this board is fairly smooth, slightly concave, and the corners of one end rounded, strongly suggesting one-half of a broken platter.

In the *papamu* from Lanai the number of pits in a row varies from 8 to 20. Their arrangement is shown by the following examples: at Kalama, (three specimens) 9 by 13, 9 by 13, 11 by 11; at Kalamaiki, (two specimens) 9 by 10 and 10 by 10; at Keone, 8 by 13; at Kiei, 9 by 10; at Honopu, 11 by 13; at Keanapapa, (two specimens) 8 by 11 and 13 by 15; at Kuahua, 8 by 8; at Kaunolu, (three specimens) 13 by 20, 13 by 13, and 15 by 15.

As the game of *konane* has heretofore been described in little more detail than a game resembling checkers, I shall explain it as I learned it from a woman of nearly ninety years, Kaahaaina Naihe, from Kailua, Hawaii—the only native left who is known to be acquainted with the game.

The *papamu* used was a wooden board from the Kuhio collection in the Bishop Museum. It has 12 files of dots, 15 rows deep. The first board tried (B998) had evidently been cut down from a larger one, on which the positions were set quincuncially, and a modern molding added to the rim which rendered the board useless for playing as we soon discovered. But in the middle of this board is set a piece of bone or a human tooth which marked a very important position, the *piko*. The line of positions boardering the *papamu* is called the *kakai*.

The "men" we used were flat, black beach pebbles an inch in diameter and white coral pebbles of the same size. Kaahaaina thought these pebbles too large, she preferred ones of half this size and a board with positions for fifty white and fifty black pebbles. The only *konane* pebble from Lanai (B4141) from a house site near Keomuku, is a black polished basalt pebble almost perfectly round, 1.2 inches in diameter and .5 inches thick.

In the game of *konane* the two players sit opposite with the *papamu* set end on between them. Both players participate in setting (*komo*) the pebbles (*ili*) on the dots until they are all covered alternately with the black pebbles (*ka eleele* or *ele*) and the white pebbles (*ke keokeo* or *kea*). Then it is decided who shall pick up the first *ili*, which must be one at the center (*piko*) one laterally next to it, or one at the corner. If the first person to choose picks up a black next to the center *ili*, then his opponent must pick up the white center *ili*; but, if he picks up a black corner *ili*, then his opponent must pick up a white one from one side or the other of the corner. If a player removes a black at the beginning he plays with the blacks and removes the whites which he jumps. "*Lawe ili keokeo, paani ka eleele.*" (Removing the whites is playing with the blacks.)

The game now proceeds by each player jumping in turn. If a person can not jump in turn, the game is ended, and the blocked man loses. Jumping must proceed away from or towards the player, to one side or the other, but never in two directions in one move and never diagonally. One may jump over and remove a line of men of rival color, providing there is a vacant position at the end of the line and providing none of the men are separated by more than one vacant position. The term *holo* means to jump and *ku'i* (strike back) means to jump over the same course of the last move but in the opposite direction, thereby removing the man just placed by the opponent. To win is *ai;* to loose, *make.*

The betting in *konane* was sometimes very heavy and a large number of games played before determining the winner. Men and women often played together. The game was not tapu to the common people.

Kalokuokamaile of Napoopoo, Hawaii, says that the game was frequently played on the plaits of the *lauhala* mat and that the term *kaholo* meant the taking of many *iliili*, two, three, and up to five.

BOWLING STONES, ULUMAIKA

On the great flat, south of Kanepuu hill near Kapukaloa, is a level hard-packed strip of earth which seems originally to have been about 5 feet wide and more than 100 feet long. On this track the game of *maika* was played, judging from the several score of *ulumaika* stones gathered there by Mr. Munro and myself. There were also many broken *ulumaika* lying on or near the track. I am told by the natives that the game was also played below the hill Puu Nana i Hawaii.

The discoidal *ulumaika* range from 4.5 inches to 1.5 inches in diameter. More than half of the specimens are ground very smooth and some are polished. An average specimen (B4168) is 3 inches in diameter, 1.2 inches thick on the slightly convex rim, 1.7 inches thick at the center, and weighs 13 ounces. The material is basalt and entirely ground smooth.

Of the 17 fragments of *ulumaika* which I collected about the plateau (B4166 and B4167), most of them at the playing track, 14 are about half of a broken *ulumaika*, the result of a smart blow. In 5 of the stones the break is parallel to the sides, and in 9 transverse. Except for a few grass hummocks the ground was clear about

Appendix 3: *Jeux et Jouets de l'Ouest Africain* (p. 470-471)

III. *LE TIOUK-TIOUK*

Le **tiouk-tiouk,** *t'uk t'uk :* argot peul = baiser, avec toutes les acceptions de l'argot français [38]. Se joue avec des bâtonnets et des crottes, des grains ou des cailloux, sur un damier dessiné dans le sable ; ou fait de petites cuvettes, comprenant 6, 8, 10 ou 12 rangées dans chaque sens, les rangées devant être en nom-

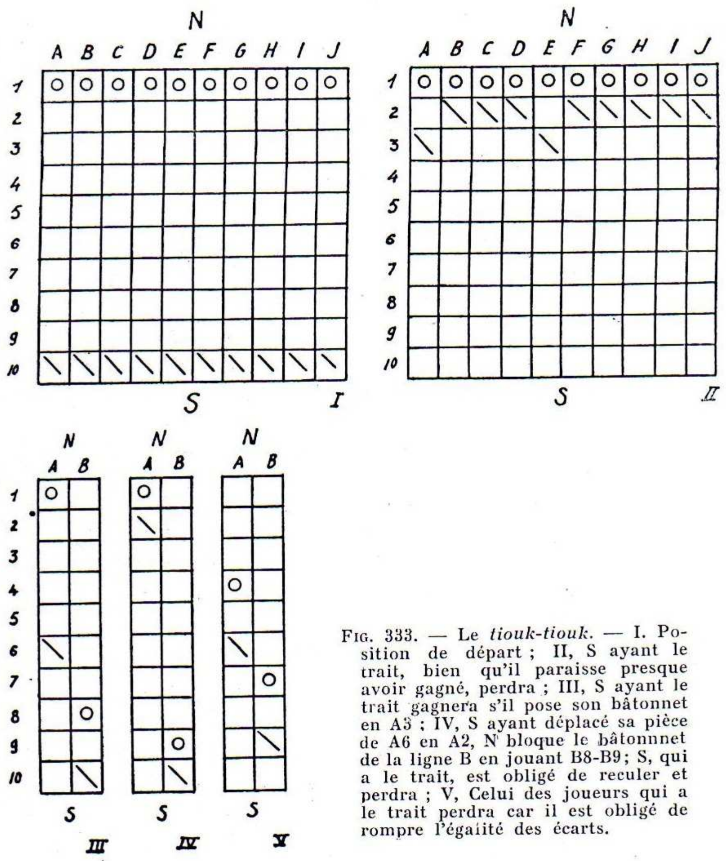


Fig. 333. — Le *tiouk-tiouk.* — I. Position de départ ; II, S ayant le trait, bien qu'il paraisse presque avoir gagné, perdra ; III, S ayant le trait gagnera s'il pose son bâtonnet en A3 ; IV, S ayant déplacé sa pièce de A6 en A2, N bloque le bâtonnnet de la ligne B en jouant B8-B9; S, qui a le trait, est obligé de reculer et perdra ; V, Celui des joueurs qui a le trait perdra car il est obligé de rompre l'égalité des écarts.

bre pair. Une seule rangée de pions pour chaque joueur, ceux-ci étant placés à l'avance sur la rangée de départ (fig. 333).

C'est généralement un griot qui propose une partie à un berger, et généreusement il lui accorde le trait. De gros enjeux sont engagés, et tous les artifices d'un bon joueur de bonneteau employés. Cependant, le griot ne triche pas, ce n'est pas

[38] Aussi de l'argot arabe, cf. *nik-nik* et, sabir algérien, « niquer », passé dans l'argot militaire français.

nécessaire, il est sûr de gagner, quand il voudra, s'il n'a pas le trait, et à peu près sûr de gagner s'il l'a.

Chaque pion peut se déplacer en avant ou en arrière à volonté, et d'autant de cases qu'il lui plaît, mais ne peut pas sauter par-dessus le pion du partenaire.

A gagné qui arrive à bloquer tous les pions du partenaire.

Entre deux joueurs non initiés, la partie durera très longtemps, et, après d'innombrables aller-retour des pièces, sera gagnée par le plus malin.

Si le joueur initié n'a pas le trait, il est sûr de gagner, tous ses pions fussent-ils bloqués, sauf deux (fig. 333, II).

Le jeu est très facile à suivre si, au lieu d'en considérer l'ensemble, on le suit seulement sur deux rangées [39] (fig. 333, III, IV, V).

Au départ, les pièces opposées sont à égale distance. Celui qui joue le premier et qui, par conséquent rompt cet équilibre, perdra. Il suffira au second de rétablir l'égalité des distances en avançant sa pièce dans la deuxième colonne de telle sorte que les intervalles soient les mêmes dans les deux colonnes. Le second joueur continuera cette tactique jusqu'à la fin de la partie en égalisant chaque fois sur le plus petit intervalle proposé par le premier joueur, qui, finalement, sera bloqué et perdra.

IV. *LES CARRÉS MAGIQUES*

Khalouma youmboué, *haluma yumbwé :* la Crête de l'Hyène [40].

On creuse neuf grottes pour les deux joueurs : l'Antilope (*héli*) et le Léopard (*barat*). Le Léopard donne trois fois 15 cailloux à l'Antilope et l'invite à les disposer par exemple de telle façon qu'en totalisant dans n'importe quel sens le contenu des grottes on obtienne 15 (fig. 334).

15		15	15	15		15
	↖	=	=	=	↗	
15	=	4	9	2	=	15
15	=	3	5	7	=	15
15	=	8	1	6	=	15
	↙	=	=	=	↘	
15		15	15	15		15

FIG. 334. — Carré magique pour **Khalouma youmbié.**

C'est le plus élémentaire des carrés magiques, le carré, ou talisman de Saturne, que l'on gravait sur plomb durant le Moyen-Age européen, et qui est, naturellement, bénéfique le samedi [41]. C'est aussi le diagramme Lo-Chou, ou Ecrit du Lac, apporté à Ta Yu, Yu le Grand, par une tortue (GUÉNON, *La Grande Triade*, p. 112 et suiv.).

Si l'Antilope ne réussit pas, le Léopard le réalise et empoche l'enjeu.

L'on prétend que certains Peuls en présentent un nouveau à chaque assemblée, il faut croire que les premiers sont rapidement oubliés, car, si le nombre des carrés magiques n'est pas limité, le nombre des pions l'est pratiquement très vite (voir, p.), d'autres carrés à vertu magique.

[39] Mon ami Henry BERGER, de Saint-Louis, m'a suggéré cette manière d'analyser la partie.

[40] M. HOUIS.

[41] En Guinée, la Crête de l'Hyène est le symbole de la vitesse et les coureurs la portent en amulette, (M. HOUIS).

Appendix 4: *L'année dernière à Marienbad* (games)

Game 1

Initial Position	● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	P
After X's move	● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	N
After M's move	● ● ● ● ● ● ● ● ● ● ● ● ●	P
After X's move	● ● ● ● ● ● ● ●	N
After M's move	● ● ● ● ● ●	P
After X's move	● ● ● ● ●	N
After M's move	● ● ●	P
After X's move	● ●	N
After M's move	●	P

Game 2

Initial Position	● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	P	
After S's move	● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	N	
After M's move	● ● ● ● ● ● ● ● ● ● ● ● ● ●	P	
After S's move	● ● ● ● ● ● ● ● ● ● ● ● ●	N	
After M's move	● ● ● ● ● ● ● ● ● ● ● ●	P	
After S's move	● ● ● ● ● ● ● ● ● ● ●	N	
After M's move	● ● ● ● ● ● ● ● ● ●	N	The only M's mistake in the entire movie
After S's move	● ● ● ● ● ● ● ● ●	N	Error
After M's move	● ● ● ● ● ●	P	
After S's move	● ● ● ● ●	N	
After M's move	● ● ● ●	P	
After S's move	● ● ●	N	
After M's move	●	P	

(P-Previous player wins; N-Next player wins)

Game 3

Initial Position	● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	P	
After M's move	● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	N	
After X's move	● ● ● ● ● ● ● ● ● ● ● ● ● ●	P	
After M's move	● ● ● ● ● ● ● ● ● ● ● ● ●	N	
After X's move	● ● ● ● ● ● ● ● ● ● ● ●	N	Error
After M's move	● ● ● ● ● ● ● ● ● ●	P	
After X's move	● ● ● ● ● ● ● ● ●	N	
After M's move	● ● ● ● ● ● ● ●	P	
After X's move	● ● ● ● ● ● ●	N	
After M's move	● ● ● ● ● ●	P	
After X's move	● ● ● ● ●	N	
After M's move	● ● ● ●	P	
After X's move	● ● ●	N	
After M's move	●	P	

(P-Previous player wins; N-Next player wins)

Game 4

Initial Position	● ● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	P	
After M's move	● ● ● ● ● ● ● ● ● ● ● ● ● ● ●	N	
After X's move	● ● ● ● ● ● ● ● ● ● ● ● ●	N	Error
After M's move	● ● ● ● ● ● ● ●	P	
After X's move	● ● ● ● ● ●	N	
After M's move	● ● ●	P	