Dark energy with a map: Importing techniques from non-linear elastic media into modified gravity

Jonathan A. Pearson*

School of Physics & Astronomy
University of Nottingham
Nottingham, NG7 2RD
and
Centre for Particle Theory,
Department of Mathematical Sciences,
Durham University, South Road,
Durham, DH1 3LE, U.K.

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Abstract

This paper is intended to be expository in nature. The idea is to create a bridge between techniques and developments in non-linear elasticity theory, and modified gravity theories.

*E-mail: j.pearson@nottingham.ac.uk

Contents

1 Introduction

Suppose one wanted to construct a description of water flowing through a pipe. Given that one knows that water is constituted from "particulate" molecules, one could construct a particle description. With the best will and all available computing power, such a description would fail to describe almost all systems of physical interest. Instead, one moves to a coarse-grained fluids description where one attempts to describe the collective behavior of the particles on a "large scale". Scalar field models of dark energy and modified gravity are prevalent in modern cosmology and it is our contention that in an important sense these are equivalent to constructing a particulate description of water.

The aims of this paper include the elucidation of the construction of non-linear material models, and showing how ideas, schematic scenarios, and model building techniques, can be imported into the language of cosmology.

It is useful for our purposes to imagine that the theory of materials comes in two branches. The first is the theory of *continuous media*: these are supposed to be space filling substances. Relativistic realizations of such media were the subject of [1–3], but under the presumption that the medium was adequently described within the framework of perturbation theory (admittedly, for the applications those studies had in mind, this was a perfectly reasonable restriction). The second is the theory of *solitons*: these are almost the polar opposite to continuous media, in that they are localised configurations and are highly non-linear deformations of the appropriate fields. In both descriptions of materials (i.e., continuous and localised) the idea of a map from the material manifold into space-time is heavily (and successfully) used. It appears that the important distinction between how the two types of theories are formulated is what information about the material manifold and its map is used to construct the theory.

In some sense the idea of describing a medium is similar to the idea of using multiple scalar fields to build dark energy models: the medium description is constructed with a set of three scalar fields. Except now one obtains a concrete interpretation of what the scalar fields *are*. Knowing what the fields are significantly enhances physical insight, and guides the choice of functions or parameters used to parameterize available freedom in the theory.

- In Bucher and Spergel, [1], the linearized theory is constructed in detail.
- See Carter and Quintana [4, 5], Karlovini [6–9] and [10–13]; [14] [15] and [16]
- Elasticity and "hyper -elasticity" have been further developed in [17], [18]

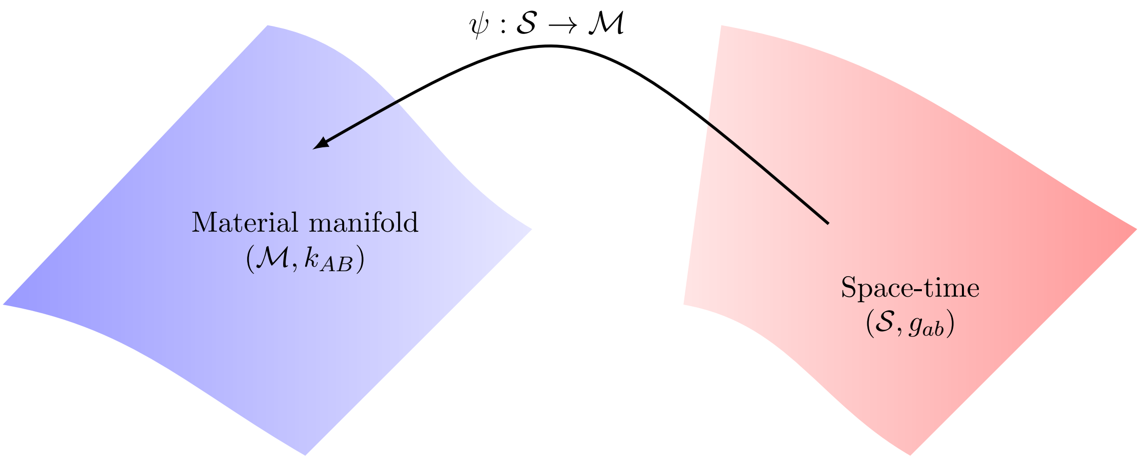

Figure 1: Schematic depiction of the map ψ that associates a point in the material space $\mathcal{M}$ with a point in space-time $\mathcal{S}$. We have also shown which metric is associated with which manifold (and the associated labelling of the indices).

- The pull-back idea is very similar to the restoration of non-linear diffeomorphism invariance utilised by massive gravity theories [19].
- See effective field theory of perfect fluids, [20]
- Note that [6] take the tensor k_{AB} to be fixed on the material space.
- see [21] [22]
- Solids in inflation context [23–25]
- see [26] for exact analytic solutions for perturbed single-component cosmology
- [11], [27]

1.1 Deformation theory and cosmology

The current state of affairs in cosmology is that the Universe is accelerating in its expansion, with many avenues being pursued in order to explain this observation. The summary is that the prediction obtained from General Relativity (GR) for how the Universe should look doesn't match up with observations of how the Universe does look (unless, for example, some form of exotic matter is included). One popular way of understanding how to tackle this mis-match is to write the gravitational field equations that actually describes the Universe as

$$G_{ab} = 8\pi G \left(T_{ab} + U_{ab}\right) . \tag{1.1}$$

The tensor U_{ab} contains all the deviations or deformations (to begin using the terminology we aim to develop) of the field equations which describe the actual Universe away from the GR (+ standard matter content) predictions.

The modern cosmology community is busy with developing candidate theories which could provide the components of the tensor U_{ab}, and with working out the observation consequences of their given form of the tensor. We would like to suggest a different approach (or at least, a different philosophy for attacking the problem). Explaining this approach, and showing how it can be used, is the subject of this paper.

In the theory of deformations (in particular, we have in mind theories of relativistic elasticity) one imagines two states of a material. The first state is a relaxed configuration, and the second is a strained configuration. The deformation which was imparted on the material to take it from being relaxed to being strained isn't necessarily small (if it was small, one would speak about linear elasticity theory). The theory of deformations prescribes a tool-kit for writing down terms in the field equations which are allowed, given classes or forms of deformation. For example, if the deformation is performed "on" some perfect fluid or perfect solid, then it is known that the quantities U_{ab} takes on the form

$$U_{ab}^{\text{fluid}} = \rho u_a u_b + P\gamma_{ab}, \qquad U_{ab}^{\text{solid}} = \rho u_a u_b + P_{ab}. \tag{1.2}$$

The energy-momentum tensors written above *become* those for a fluid or solid when some extra theoretical structure is used. Namely, an *equation of state*. For readers who are used to the literature in modern cosmology, this phrase is often used to describe the link between the dark energy pressure P and density ρ, via an equation of the form $P(t) = w(t)\rho(t)$. In the context of material models, an equation of state is the Lagrangian density.

When one constructs "conventional" models of dark energy or modified gravity, one has a some freedom to choose various types of quantities: these are, e.g., functional forms of the potential, or the kinetic terms which appear in the Lagrangian density. This may seem like an obvious point, but the choice of a restriction on a theory can have implications for (a) its applicability, and (b) its physical naturalness/interpretation.

1.2 Fluids and solids

The distinction between a *fluid* and a *solid* isn't one of the best explained concepts in the literature. Fluids are commonly used as a description for the content of the

Universe in cosmology: but they are only a sub-class of a more general description for "content"; or, to perhaps use a more physically transparent terminology: for materials. A more general description of material is that of a solid; obviously, we won't go so far as to say *the* general material description. Below we will outline some of the salient pieces to the construction of a material model: full explanations are given in the rest of this paper.

In the descriptions of both solids and fluids one has a notion of a *material metric* ${k^a}_b$ on a *material space*, whose determinant is related to the particle number density, n. A convenient decomposition of this metric is ${k^a}_b = n^{2/3}{\eta^a}_b$. With this decomposition of ${k^a}_b$, the conformal metric ${\eta^a}_b$ is uni-modular, i.e., it has unit determinant. The action for both a fluid and a solid is of the form

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L}\left(n, [\boldsymbol{\eta}], [\boldsymbol{\eta}^2]\right). \tag{1.3}$$

Since this is the "answer" from which everything else is obtained it is worth taking a little time to explain the interpretation of various terms. Firstly, n is the number density of particles and is identified with the determinant of the material metric $n = \sqrt{\det k_{AB}}$ on the material manifold (the coordinates ϕ^A specify the locations of the particles on the material manifold). The square-braces in (1.3) denote traces of the mixed components of the uni-modular tensor, ${\eta^a}_b = \gamma^{ac}\eta_{cb}$. That is, the action (1.3) is dependant upon the two independent invariants of the tensor

$$\eta_{ab} = n^{-2/3}k_{AB}\partial_a\phi^A\partial_b\phi^B. \tag{1.4}$$

It is useful to split up the Lagrangian density as $\mathcal{L} = n\epsilon$, where ϵ is the energy per particle and n retaints its interpretation as the particle number density. In the cases of fluids or solids, ϵ is a function with the following dependencies:

$$\epsilon_{\text{fluid}} = \epsilon_{\text{fluid}}(n), \qquad \epsilon_{\text{solid}} = \epsilon_{\text{solid}}(n, {\eta^a}_b). \tag{1.5}$$

This makes the distinction between solids and fluids explicit: it is the dependence of the energy per particle on the uni-modular tensor ${\eta^a}_b$ which makes the description that of a solid rather than a fluid. Later on we will see that the physical consequence of this dependence is that the substance is able to support anisotropic stress (whereas fluids can't): this manifests as *rigidity*. It is worth noting that a fluid is a highly symmetric solid, and a pressureless fluid has $\epsilon_{\text{fluid}}(n) = \bar{\epsilon}_0$, a constant.

In Figure 2 we show how the materials are related.

Another concept which is used is that of a *perfect fluid*. This is supposed to be a substance whose energy-momentum tensor can be put into the form

$$T_{ab} = \rho u_a u_b + P\gamma_{ab}, \tag{1.6}$$

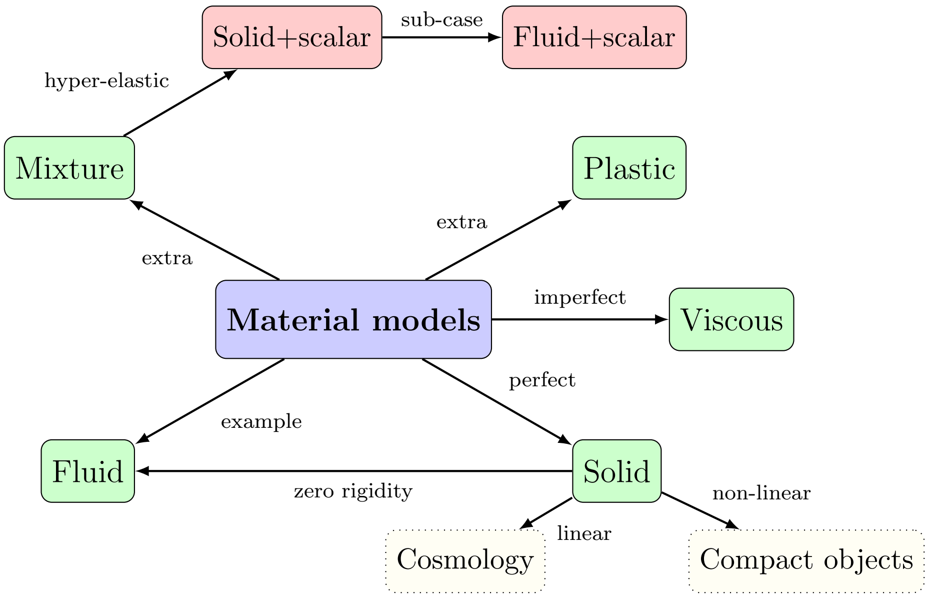

Figure 2: Road-map containing some of the simplest material models. This picture coarsely shows how some of the common classes of materials are related. For example, we see that a fluid is a perfect solid with zero rigidity. We have also shown that the linear theory of solids has been applied to cosmology, and the non-linear theory to compact objects (such as neutron stars).

in which ρ and P are the fluid's energy density and pressure respectively, u^a is the velocity of the fluid and $\gamma_{ab} = g_{ab} + u_a u_b$ is the orthogonal projection operator. If the energy-momentum tensor for a *fluid* is not of this form (for example, if there is anisotropic stress or heat flux) then the *fluid* is said to be *imperfect*. We are deliberately being careful about only using the term *fluid*: a solid can be categorised in a similar sense, but a perfect solid manifestly has an anisotropic part to the energy-momentum tensor (this is a distinguishing feature of a solid from a fluid).

In some sense the main result of this review is to obtain an understanding of the theory of a relativistic solid: useful geometric structures on the manifold of particle locations, the action, and energy-momentum tensor. It is rather involved, but is worthwhile since expressions and formulae obtain physical meaning.

1.3 Conventions

We use lower-case latin letters, $a, b, c, \ldots$ to denote space-time indices, and upper-case latin letters, $A, B, C, \ldots$ to denote indices on the material manifold. The space-time metric is decomposed as

$$g_{ab} = \gamma_{ab} - u_a u_b, \tag{1.7}$$

Symbol	Meaning
$\pounds_X$	Lie derivative operator along the vector X^μ
$(\mathcal{S}, g_{ab})$	Space-time manifold and metric
$(\mathcal{M}, k_{ab})$	Material manifold and metric
u_a	Time-like unit-vector; $u^a u_a = -1$
$\gamma_{ab} = g_{ab} + u_a u_b$	Orthogonal projector; $u^a \gamma_{ab} = 0$
n	Particle number density; $n^2 = \det k_{AB}$
$\eta^a{}_b = n^{-2/3} k^a{}_b$	Uni-modular tensor
ϵ	Equation of state

Table 1: Summary of commonly used symbols

in which u_a and γ_{ab} are the 4-velocity and spatial metric, satisfying

$$u^a u_a = -1, \qquad u^a \gamma_{ab} = 0. \tag{1.8}$$

We use the orthogonally projected derivative

$$\overline{\nabla}_a A^{b\cdots}{}_{c\cdots} = \gamma^d{}_a \gamma^b{}_e \cdots \gamma^f{}_c \cdots \nabla_d A^{c\cdots}{}_{f\cdots} \tag{1.9}$$

and the expansion (extrinsic curvature) tensor

$$\Theta_{ab} = \overline{\nabla}_{(a} u_{b)}. \tag{1.10}$$

It immediately follows that $\overline{\nabla}_a$ is the connection compatible with γ_{ab}, since

$$\overline{\nabla}_a \gamma_{cd} = 0. \tag{1.11}$$

We will use angular braces to denote the symmetric, trace-free part of a tensor:

$$A_{\langle ab\rangle} = A_{(ab)} - \tfrac{1}{3} A^c{}_c \gamma_{ab}. \tag{1.12}$$

1.3.1 First, second, and third fundamental tensors

Here we briefly review some of Carter's technology [28–30] for dealing with branes and imbeddings.

The idea is that writing $x^\mu{}_{,i} = \partial x^\mu / \partial \sigma^i$, with σ^i the worldsheet coordinates, induces a metric on the world-sheet, $\overline{g}_{ij} = g_{\mu\nu} x^\mu{}_{,i} x^\nu{}_{,j}$. Instead of working with this quantity (which is written in terms of worldsheet coordinates), it is much more convenient to work with $\overline{g}^{\mu\nu} = \overline{g}^{ij}x^{\mu}{}_{,i}x^{\nu}{}_{,j}$. This invites decomposition of the space-time metric according to

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \perp_{\mu\nu} . \tag{1.13}$$

Here $\overline{g}_{\mu\nu}$ is the world-sheet tangential metric: it is the first fundamental form. Also, $\perp_{\mu\nu}$ is orthogonal to the world-sheet. These satisfy

$$\overline{g}_{\mu\nu}\perp^{\mu}{}_{\alpha} = 0, \qquad \overline{g}_{\beta\nu}\overline{g}^{\alpha\nu} = \overline{g}^{\alpha}{}_{\beta}. \tag{1.14}$$

The space-time covariant derivative projected into the world-sheet is

$$\overline{\nabla}_{\mu} = \overline{g}^{\alpha}{}_{\mu}\nabla_{\alpha}. \tag{1.15}$$

The second fundamental tensor is defined via

$$K_{\mu\nu}{}^{\rho} = \overline{g}^{\sigma}{}_{\nu}\overline{\nabla}_{\mu}\overline{g}^{\rho}{}_{\sigma}. \tag{1.16}$$

The first fundamental tensor determines the tangential derivative of the world-sheet metric via

$$\overline{\nabla}_{\mu}\overline{g}_{\alpha\beta} = 2K_{\mu(\alpha\beta)}. \tag{1.17}$$

The second fundamental tensor satisfies

$$\perp^{\mu}{}_{\alpha}K_{\mu\nu}{}^{\lambda} = 0, \qquad \overline{g}_{\lambda}{}^{\sigma}K_{\mu\nu}{}^{\lambda} = 0. \tag{1.18}$$

It is convenient to introduce the extrinsic curvature vector as the trace of the first fundamental tensor,

$$K^{\mu} \equiv K^{\alpha}{}_{\alpha}{}^{\mu}, \tag{1.19}$$

which satisfies

$$\overline{g}^{\mu}{}_{\nu}K^{\nu} = 0. \tag{1.20}$$

The third fundamental tensor is

$$\Xi_{\lambda\mu\nu}{}^{\rho} \equiv \overline{g}^{\sigma}{}_{\mu}\overline{g}^{\tau}{}_{\nu}\perp^{\rho}{}_{\alpha}\overline{\nabla}_{\lambda}K_{\sigma\tau}{}^{\alpha}. \tag{1.21}$$

The more common decomposition of (1.13) comes in the 3+1 form, via the identifications

$$\overline{g}_{\mu\nu} = \gamma_{\mu\nu}, \qquad \perp_{\mu\nu} = u_{\mu}u_{\nu}. \tag{1.22}$$

$$\nabla_{\mu}\gamma_{\alpha\beta} = 2K_{\mu(\alpha}u_{\beta)}. \tag{1.23}$$

$$K_{\mu\alpha\beta} = K_{\mu(\alpha}u_{\beta)} \tag{1.24}$$

It follows that (1.17) evaluates to

$$\overline{\nabla}_{\mu}\gamma_{\alpha\beta}. \tag{1.25}$$

1.4 Perturbed solids

The majority of this review will be focussed on the general theory of solids: the deformations performed on the solid or medium may be arbitrarily large. Whilst this is very general, it also yields a theory which is complicated to work with. There are a substantial number of physical systems for whom the non-linear theory of elasticity is "over-kill": understanding the governing equations that describe small deformations of the solid from its equilibrium configuration is often sufficient. For this reason we shall review the theory of perturbed solids.

Comprehensive reviews, applications, and examples in the relativistic theory have already been presented [1–3, 5, 14, 31, 39], as well as the non-relativistic theory being the main subject of a classic book by Landau and Lifshitz [32].

The physical picture one should constantly keep in mind is that a continuous medium has "two states": relaxed and deformed. The former occurs when there are no forces on the medium, and the latter will induce strains and forces on other surrounding materials and fields (notably the metric). In some sense the "point" of a model is to catalogue the possible ways in which a material can influence surrounding media and fields.

The coordinates of the undeformed medium are represented by $\overline{x}^a$, and those of the deformed medium are by x^a. These are related via

$$x^a = \overline{x}^a + \xi^a(x^b). \tag{1.26}$$

The crucial piece here is the deformation vector, $\xi^a(x^b)$, which as we explicitly show, is dependent upon the space-time coordinates (different locations can deform by different amounts).

The metric of a space-time which contains a perturbed medium is given by

$$g_{ab} = \overline{g}_{ab} + h_{ab} + 2\nabla_{(a}\xi_{b)}. \tag{1.27}$$

The metric of the unperturbed space-time is $\overline{g}_{ab}$, and the metric fluctuations due to "intrinsic", or extra-material contents, is given by h_{ab}. The presence of the perturbed medium is encapsulated by the term involving the deformation field,

$$\xi^a(x^b) = x^a - \overline{x}^a. \tag{1.28}$$

One may recognise the final term is (1.27) as that which arises in standard perturbation theory after one performs the diffeomorphism

$$x^a \to x^a + \xi^a(x^b). \tag{1.29}$$

Of course, this recognition is accurate. There is an additional concept to appreciate however: interpretation. The ξ^a-field describes all the fluctuations of the medium away from its equilibrium configuration. In addition, the deformation field ξ^a is orthogonal,

$$u_a\xi^a = 0. \tag{1.30}$$

A more elegant, and geometrically intuitive way to write the corrections to the metric is by writing all of the non-background terms in (1.27) as

$$\delta_{\mathrm{L}} g_{ab} = \delta_{\mathrm{E}} g_{ab} + \pounds_{\xi} g_{ab}, \tag{1.31}$$

wherein one can hopefully recognise the usual expression for the Lie derivative of the metric along the vector ξ^a,

$$\pounds_{\xi} g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{1.32}$$

1.4.1 Non-relativistic solids

A non-relativistic solid is one for whom there are no gravitational effects. As examples, one imagines an eraser, rubber band, trampolines: these kinds of materials.

Under a deformation the coordinates of a non-relativistic solid alter according to $x^i \to x^i + \xi^i(x^j)$. If the line element in the solid before the deformation is $\mathrm{d}\ell^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, then after the deformation the line element has metric given by

$$g_{ij} = \delta_{ij} + 2\varepsilon_{ij}, \tag{1.33}$$

where we defined the strain tensor,

$$\varepsilon_{ij} \equiv \partial_{(i}\xi_{j)}. \tag{1.34}$$

The components of the strain tensor ε_{ij} contain all information about the deformation performed on the body. We now require information about the manner in which the body responds to the given deformation. What this actually entails is an understanding of the stress tensor, σ^{ij}, for a given strain tensor ε_{ij}. If one computes the divergence of the stress tensor one obtains the components of the force, which can be equated to the acceleration of the deformation vectors, to obtain the equation of motion

$$F^i = \partial_j\sigma^{ij} = \rho\ddot{\xi}^i. \tag{1.35}$$

In broad-brush-terms there are two possibilities which are useful to consider.

1. The stress tensor is proportional to the strain tensor:

$$\sigma^{ij} = E^{ijkl}\varepsilon_{kl}. \tag{1.36}$$

The components E^{ijkl} precisely prescribe the strength of certain forces for given deformations (we will have much more to say about this later); they are the components of the elasticity tensor. Materials for whom (1.36) holds are Hookean elastic solids.

2. The stress tensor is proportional to the rate-of-strain tensor:

$$\sigma^{ij} = V^{ijkl}\dot{\varepsilon}_{kl}. \tag{1.37}$$

The components V^{ijkl} play a similar role to the components of the elasticity tensor for a Hookean solid, except here we are considering viscous solids and V^{ijkl} are the components of the viscosity tensor.

Of course, the given physical material may require an amalgamation of the two cases, whereby stress is proportional to both strain, and rate-of-strain, in which case

$$\sigma^{ij} = E^{ijkl}\varepsilon_{kl} + V^{ijkl}\dot{\varepsilon}_{kl}. \tag{1.38}$$

These models describe visco-elastic solids (also known as Kelvin-Voigt solids). Since (1.38) contains both the elastic (1.36) and viscous (1.37) as sub-cases, we will proceed with the Kelvin-Voigt expression (1.38). Using (1.34) and (1.38), the equation of motion (1.35) is

$$\rho\ddot{\xi}^i = E^{ijkl}\partial_j\partial_{(k}\xi_{l)} + V^{ijkl}\partial_j\partial_{(k}\dot{\xi}_{l)}. \tag{1.39}$$

The problem of describing the solid considerably simplifies when one assumes some symmetry of the solid, for example material isotropy. In such cases (other symmetries require more freedom than we are about to introduce), the material tensors only have two independant components each, and decompose completely as

$$E^{ijkl} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij}g^{kl} + 2\mu g^{i(k}g^{l)j}, \tag{1.40a}$$
$$V^{ijkl} = \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij}g^{kl} + 2\nu g^{i(k}g^{l)j}. \tag{1.40b}$$

Using the decompositions (1.40), the stress-tensor (1.38) becomes

$$\sigma^{ij} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij}\partial_k\xi^k + \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij}\partial_k\dot{\xi}^k + 2\mu\partial^{(i}\xi^{j)} + 2\nu\partial^{(i}\dot{\xi}^{j)}, \tag{1.41}$$

and the equation of motion (1.39) becomes

$$\rho\ddot{\xi}^i = \left(\beta + \tfrac{1}{3}\mu\right)\partial^i\partial_k\xi^k + \mu\partial_k\partial^k\xi^i + \left(\lambda + \tfrac{1}{3}\nu\right)\partial^i\partial_k\dot{\xi}^k + \nu\partial_k\partial^k\dot{\xi}^i. \tag{1.42}$$

We shall provide a simple example which highlights the seperate modes of propagation inherent in a material medium. Consider the simple case where the deformation vector has only two components; we can expand ξ^i using two scalars ϕ and ψ in an orthonormal basis $(\hat{x}^i, \hat{y}^i)$ via

$$\xi^i = \phi\hat{x}^i + \psi\hat{y}^i. \tag{1.43a}$$

Now suppose that these scalars depend on time, and only one of the two available spatial directions; that is, we set

$$\partial_i\phi = \phi'\hat{x}_i, \qquad \partial_i\psi = \psi'\hat{x}_i. \tag{1.43b}$$

Putting the decomposition of the deformation vector (1.43) into the equation of motion (1.42) yields an equation with two independant projections (one along $\hat{x}^i$, and one along $\hat{y}^i$); these projections leads to the requirement that the following two equations are satisfied:

$$\ddot{\phi} - \frac{\lambda + \frac{4}{3}\nu}{\rho}\dot{\phi}'' = \frac{\beta + \frac{4}{3}\mu}{\rho}\phi'', \qquad \ddot{\psi} - \frac{\nu}{\rho}\dot{\psi}'' = \frac{\mu}{\rho}\psi''. \tag{1.44}$$

In the purely elastic case (i.e., where all components of the viscosity tensor vanish), it is with relative ease that one realises a plane wave ansatz $\phi \sim e^{\mathrm{i}(\omega t + kx)}$ solves the equations of motion, and that ϕ and ψ travel with different speeds: these are the longitudinal and transverse sound speeds

$$c_{\mathrm{L}}^2 = \frac{\beta + \frac{4}{3}\mu}{\rho}, \qquad c_{\mathrm{T}}^2 = \frac{\mu}{\rho}. \tag{1.45}$$

1.4.2 Relativistic solids

2 Describing non-linear materials

In this section we will take some time to build a description of a medium. We will introduce the notion of a material manifold, and geometric structures on the material manifold: coordinates, metric, connection, and volume form. There will be an important step where we relate structures in the material manifold to structures in space-time. We will want to obtain fields and energies in space-time due to structures in the material manifold. There will be some instances where we "mix" material space and space-time indices; this is unavoidable in the course of exposing some interesting part of the formalism. That said, all final results (equations of motion etc) will be expressed solely in terms of space-time indicies.

2.1 The material manifold, particle number density, and map

It is important to understand the minimal assumed geometric structure in building a model. We assume that there is a 3D manifold $\mathcal{M}$ which is endowed with a particle density form $n_{ABC} = n_{[ABC]}$. This is the material manifold. There will be an associated metric on the material manifold, which we call k_{AB} and will discuss further later on.

The points of $\mathcal{M}$ are particles of the medium, and they do not move: the dynamics in space-time comes from the maps from the material manifold to space-time, not the motion of the particles in material space. We shall let $\mathcal{S}'$ be the submanifold of the full space-time manifold $\mathcal{S}$ which is the subset of the spacetime that the material passes through. Then invoke a map ψ which takes a location in space-time and points at a location in the material manifold;

$$\psi : \mathcal{S}' \longrightarrow \mathcal{M}. \tag{2.1}$$

For all points p in $\mathcal{M}' = \psi(\mathcal{S}')$, the inverse map at that point, $\psi^{-1}(\mathrm{p})$, is a single time-like curve in $\mathcal{S}'$: these are the flow-lines of the particles.

Let ϕ^A be coordinates in material space. Then their gradients with respect to the space-time coordinates x^a can be computed

$$\psi^A{}_a = \frac{\partial \phi^A}{\partial x^a}. \tag{2.2}$$

The $\psi^A{}_a$ are the components of the configuration gradient. The time-like projection of these must vanish

$$u^a \psi^A{}_a = 0. \tag{2.3}$$

This is equivalent to setting the Lie derivative of the ϕ^A in the time-like direction to zero:

$$\pounds_u \phi^A = 0, \tag{2.4}$$

which has a more direct physical interpretation of saying that the material coordinates are static with respect to coordinate time. In Section 7.3 we shall explain what happens when this condition is relaxed: the upshot is that one ends up describing "scalar field theories", rather than "solid theories". In that section we also explore the technology required when the material manifolds dimension isn't just three, and where the material lives on some brane in a higher dimensional bulk.

One can conceive of scalars, vectors, forms, and tensors on the material manifold: the material metric and particle density form are examples, as are a few other tensors we introduce later on. Collectively, we call such quantities "material tensors", and they have components whose indices are denoted with captial latin letters. We relate tensors in the material and space-time manifolds using the technology of pull-backs and push-forwards:

- $\psi^\star$ is the pull-back of a covariant tensor from $\mathcal{M}'$ to $\mathcal{S}'$ and is denoted to act on a material tensor as

  $$N_{ab\cdots z} = \psi^\star N_{AB\cdots Z}. \tag{2.5}$$

  In "coordinates" notation the pull-back is

  $$N_{ab\cdots z} = \psi^A{}_a \psi^B{}_b \cdots \psi^Z{}_z N_{AB\cdots Z}, \tag{2.6}$$

  where $\psi^A{}_a$ are the components of the configuration gradient (2.2).

- $\psi_\star$ denotes the push-forward of a contravariant tensor from $\mathcal{S}'$ to $\mathcal{M}'$, and is denoted to act on a space-time tensor as

  $$M^{AB\cdots Z} = \psi_\star M^{ab\cdots z}, \tag{2.7}$$

  and in coordinates it reads

  $$M^{AB\cdots Z} = \psi^A{}_a \psi^B{}_b \cdots \psi^Z{}_z M^{ab\cdots z}. \tag{2.8}$$

The most important corollary of (2.3) is that any tensor on space-time which was constructed as the pull-back of a tensor on the material space will automatically be orthogonal. That is, for the schematic example (2.6),

$$u^a N_{ab\cdots z} = u^b N_{ab\cdots z} = \cdots = u^z N_{ab\cdots z} = 0. \tag{2.9}$$

The integral of the particle number density form n_{ABC} over some volume in the material manifold $\mathcal{M}$ is the number of particles in that volume (by definition). The pull-back of the particle volume-form to space-time is

$$n_{abc} = \psi^{\star} n_{ABC}. \tag{2.10}$$

Note that the property (2.3) makes n_{abc} an orthogonal space-time field. Using the space-time volume form ϵ_{abcd} the particle current n^a is

$$n^a = \frac{1}{3!}\epsilon^{abcd} n_{bcd}, \tag{2.11}$$

and is manifestly conserved,

$$\nabla_a n^a = 0. \tag{2.12}$$

This conservation follows since n_{abc} is a closed 3-form due to n_{ABC} being a closed 3-form on material space (an n-form in n-dimensional space is closed). What this also means is that to break (2.12) and have $\nabla_a n^a \neq 0$ one requires n^a not to be related to the volume form on material space: i.e., there is no volume form on material space. Note that the particle current n^a is the dual of the volume form n_{abc}.

It follows by orthogonality of n_{abc} that the particle current (2.11) is time-like

$$n^a = n u^a, \tag{2.13}$$

where the particle number density n is given by

$$n = \sqrt{-n^a n_a}. \tag{2.14}$$

We have

$$\epsilon_{abc} = \epsilon_{abcd} u^d, \qquad n_{abc} = n \epsilon_{abc}. \tag{2.15}$$

A useful invariant is

$$n^2 = \frac{1}{3!} n^{abc} n_{abc}. \tag{2.16}$$

Note that from the conservation equation for n^a, (2.12), and (2.13), one obtains an evolution equation for the particle number density,

$$\dot{n} = -n\Theta, \tag{2.17}$$

where $\Theta = \Theta^a{}_a$ is the trace of the extrinsic curvature tensor.

Another way of writing down (2.11) is found after combining (2.13) and (2.15) to give

$$n_{abc} = \epsilon_{abcd} n^d. \tag{2.18}$$

The expression (2.18) directly shows that the 3-form n_{abc} is the dual to n^a, and will highlight the connection to Kalb-Ramond fields. A Kalb-Ramond field is a 2-index object that transforms as a 2-form; its components satisfy

$$B_{ab} = B_{[ab]}. \tag{2.19}$$

The 3-form field strength F_{abc} corresponding to B_{ab} is an exact form constructed by taking the "derivative"

$$F = \mathrm{d}B, \tag{2.20a}$$

which works out in this case as

$$F_{abc} = 3\nabla_{[a} B_{bc]}. \tag{2.20b}$$

Since F_{abc} is an exact form, it is therefore a closed form[1]: the expression of automatic closure is given by

$$\nabla_{[a} F_{bcd]} = 0. \tag{2.21}$$

Related to the 3-form field strength F_{abc} is its dual $\widetilde{F}^a$, which is constructed via

$$F_{abc} = \epsilon_{abcd} \widetilde{F}^d. \tag{2.22}$$

By virtue of the automatic closure (2.21) it follows that $\widetilde{F}^a$ is conserved,

$$\nabla_a \widetilde{F}^a = 0. \tag{2.23}$$

It should therefore be clear that the particle number density current n^a, which is the dual of the number density form n_{abc}, is the field strength tensor of some field of Kalb-Ramond type. ***It is worth finding [33]***

[1] A *closed* form C, say, is a form for whom $\mathrm{d}C = 0$. Let A be a p-form, then $F = \mathrm{d}A$ is an *exact* $(p+1)$-form. Since $\mathrm{d}^2 = 0$, it follows that $\mathrm{d}F = 0$; in words this statement is: *an exact form is a closed form.*

2.2 Material metric

We invoke the existence of a metric k_{AB} on the material manifold $\mathcal{M}$ whose volume form is the particle density form n_{ABC} introduced in Section 2.1. This metric will enable us to introduce a Levi-Civita connection in the material manifold, which can be pulled-back to space-time to aid the evaluation of derivatives of material tensors. Before we explain this fairly complicated construction we shall elucidate some other useful structures on the material manifold.

Indices on material tensors can be contracted with the indices of other material tensors. Equivalently, indices on space-time tensors can also be contracted with those of other space-time tensors (a space-time scalar can be formed if contraction leaves no spare indices). Importantly, space-time tensors can be the pulled-back version of a material tensor, as in the discussion in the previous section. As an example, consider an arbitrary material tensor $A_{ABC\cdots}$ which is "pulled-back" to give a space-time tensor $A_{abc\cdots}$ according to the usual prescription $A_{abc\cdots} = \psi^\star A_{ABC\cdots}$. Then, after contracting some indices with the space-time metric,

$$B_{c\cdots} = g^{ab} A_{abc\cdots} = g^{ab} \psi^\star A_{ABC\cdots} \tag{2.24}$$

is a legitimate space-time tensor. One can also contract indices of material tensors on the material manifold $\mathcal{M}$, with the push-forward of space-time tensors. As an example, consider the push-forward of the inverse space-time metric tensor

$$g^{AB} = \psi_\star g^{ab} \tag{2.25a}$$

being contracted with an arbitrary material tensor,

$$g^{AB} C_{ABC\cdots} = \psi_\star g^{ab} C_{ABC\cdots}. \tag{2.25b}$$

From the orthogonality of the material mappings it follows that

$$g^{AB} = \psi_\star \gamma^{ab}, \tag{2.25c}$$

where we remind that γ^{ab} is the orthogonal part of the space-time metric as defined in (1.7).

Note that g^{AB} is the push-forward of the space-time metric to the material manifold, and does not necessarily coincide with the material metric k_{AB}. Infact, quantifying its non-coincidence is extremely important in quantifying the state of a material. With this in mind, we define a material tensor η_{AB}, which depends on the number density n, such that the push-forward of the space-time metric g^{AB} is exactly the inverse of η_{AB} when the material is in its unsheared state. That is,

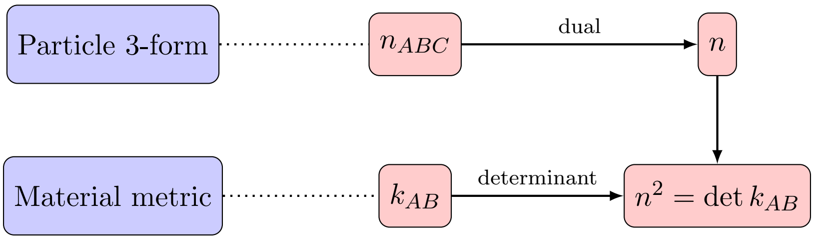

Figure 3: Explanation of the link between geometrical objects in the particle 3-form and material-metric formulations of elasticity theory.

$g^{AC}\eta_{CB} = \delta^A{}_B$ (the Kronecker-delta) when the energy is at its minimum $\epsilon = \check{\epsilon}(n)$. What this means is that $g^{AB} = \eta^{-1AB}$ in what is henceforth defined as the *unsheared state*. Consequently, the deviation of the actual value of g^{AB} from η^{-1AB}, which we write as

$$s^{AB} = \tfrac{1}{2}\left(g^{AB} - \eta^{-1AB}\right), \tag{2.26}$$

quantifies the shear of the system.

Writing the volume form of η_{AB} as ϵ_{ABC} it follows that

$$n_{ABC} = n\epsilon_{ABC}. \tag{2.27}$$

Note that $\epsilon_{abc} = \psi^\star\epsilon_{ABC}$. The particle density form n_{ABC} is a fixed material space tensor, and is independent of n.

It is now useful and helps physical insight, to define the material tensor k_{AB} as the metric on the material manifold $\mathcal{M}$. k_{AB} is conformal to η_{AB}, and has the particle density form n_{ABC} as its volume form. Therefore

$$k_{AB} = n^{2/3}\eta_{AB}. \tag{2.28}$$

This tells us that the (square-root of the) determinant of k_{AB} is the particle number density, n:

$$n = \sqrt{\det k_{AB}}. \tag{2.29}$$

See Figure 3 for a cartoon of the relationship between the material metric k_{AB} and particle form n_{ABC}.

The pull-back of k_{AB} gives a space-time tensor,

$$k_{ab} = \psi^\star k_{AB}, \tag{2.30}$$

and will play an important role in what follows. Specifically, using (2.6) the pull-back (2.30) reads

$$k_{ab} = \psi^A{}_a\psi^B{}_b k_{AB}. \tag{2.31}$$

An application of (2.3) is that since k_{ab} is a space-time field, it is orthogonal

$$u^a k_{ab} = 0. \tag{2.32}$$

We will frequently use the mixed version of the tensor, wherein indices are raised with the space-time metric,

$$k^a{}_b = g^{ac} k_{bc}, \tag{2.33}$$

which is also orthogonal,

$$u^a k^b{}_a = 0. \tag{2.34}$$

As a consequence of (2.34), (2.33) can be re-expressed as

$$k^a{}_b = \gamma^{ac} k_{bc}. \tag{2.35}$$

From (2.35) it follows that

$$\frac{\partial k^a{}_b}{\partial g^{cd}} = \delta^a{}_{(c} k_{d)b}. \tag{2.36}$$

Similarly, the pull-back of η_{AB} gives an orthogonal space-time tensor

$$\eta_{ab} = \psi^\star \eta_{AB}, \tag{2.37}$$

and we also use the mixed version of the tensor,

$$\eta^a{}_b = \gamma^{ac} \eta_{cb}. \tag{2.38}$$

From the pull-back of the relationship (2.28) we obtain

$$k^a{}_b = n^{2/3} \eta^a{}_b. \tag{2.39}$$

Since we have set everything up so that n^2 is the determinant of k_{ab}, it follows from (2.39) that $\eta^a{}_b$ is a uni-modular tensor:

$$\det(\eta^a{}_b) = 1. \tag{2.40}$$

This property will be useful later on.

We now elucidate some consequences of the n-dependence of k_{AB}. In what follows it will be convenient to denote differentiation with respect to n with a prime. Using (2.28) to compute k'_{AB} yields

$$n\eta'_{AB} = -\tfrac{2}{3}\eta_{AB} + \tau_{AB}, \tag{2.41}$$

in which

$$\tau_{AB} \equiv n^{1/3} k'_{AB}. \tag{2.42}$$

Since $n'_{ABC} = 0$ (by definition) it follows that $(\det k_{AB})' = 0$, and therefore $k^{-1AB} k'_{AB} = 0$, and hence

$$\eta^{-1AB} \tau_{AB} = 0. \tag{2.43}$$

Thus, we see that τ_{AB} is traceless; it is called the *compressional distortion tensor*, and measures deformations of the medium that *aren't* due to conformal rescalings of the material metric upon varying the particle density. Hence, computing the trace of (2.41) with respect to η^{-1AB} yields

$$n\eta^{-1AB} \eta'_{AB} = -2. \tag{2.44}$$

Note that from (2.42) it follows trivially, but more usefully, $k'_{AB} = n^{-1/3}\tau_{AB}$, and so if the material varies only conformally (i.e. is uniformly compressed) k_{AB} is independent of n since $\tau_{AB} = 0$ for these types of deformations.

The push-forward of (2.41) reads

$$n\eta'_{ab} = -\tfrac{2}{3}\eta_{ab} + \tau_{ab}. \tag{2.45}$$

And so, in the case where $\eta_{ab} = \eta_{ab}(n)$, it is simple to see that

$$[\eta_{ab}]^{\cdot} = \eta'_{ab}[n]^{\cdot}, \tag{2.46}$$

where $[X]^{\cdot}$ denotes the material derivative of X. After using (2.17) to replace $[n]^{\cdot} = \dot{n}$ we obtain the evolution equation:

$$[\eta_{ab}]^{\cdot} = \left(\tfrac{2}{3}\eta_{ab} - \tau_{ab}\right)\Theta. \tag{2.47}$$

2.3 Material covariant derivative

It is convenient at this point to introduce the covariant derivative on the material manifold which is compatible with the material metric. Let $\overline{\widetilde{\nabla}}_A$ be the Levi-Civita connection for k_{AB}; i.e.,

$$\overline{\widetilde{\nabla}}_C k_{AB} = 0. \tag{2.48}$$

There is a reason for our including two different "accents" above the del-symbol. The pushed-forward version of $\overline{\widetilde{\nabla}}_A$, denoted as $\overline{\widetilde{\nabla}}_a$, is allowed to act on space-time tensors; note that it will be orthogonal, and so is taken to be the orthogonal projection of some space-time derivative $\widetilde{\nabla}_a$ according to

$$\overline{\widetilde{\nabla}}_a A^{b\cdots}{}_{c\cdots} = \gamma^d{}_a \gamma^b{}_e \cdots \gamma^f{}_c \cdots \widetilde{\nabla}_d A^{e\cdots}{}_{f\cdots}. \tag{2.49}$$

For any space-time vector Y^a the difference between any two connections can be written as

$$\left(\tilde{\bar{\nabla}}_a - \bar{\nabla}_a\right) Y^c = \mathfrak{D}^c{}_{ab} Y^b \tag{2.50}$$

in which $\mathfrak{D}^c{}_{ab}$ is the (symmetric) relativistic difference tensor [2] defined as

$$\mathfrak{D}^c{}_{ab} = \tfrac{1}{2} k^{-1cd} \left(\bar{\nabla}_a k_{bd} + \bar{\nabla}_b k_{ad} - \bar{\nabla}_d k_{ab}\right), \tag{2.51}$$

where k^{-1cd} is defined via

$$k^{-1cd} k_{ca} = \gamma^d{}_a, \tag{2.52}$$

and is orthogonal $k^{-1cd} u_c = 0$. Due to the applications in mind, we actually call $\mathfrak{D}^c{}_{ab}$ the relativistic elasticity difference tensor.

Using this construction, one finds that $\tilde{\bar{\nabla}}_a$ is the connection which is compatible with k_{ab},

$$\tilde{\bar{\nabla}}_a k_{cd} = 0. \tag{2.53}$$

As an example of using this technology, suppose that $B^{a\cdots}{}_{b\dots}$ is a tensor function of g^{ab} and k_{ab}. Then taking its derivative with $\tilde{\bar{\nabla}}_a$ yields

$$\tilde{\bar{\nabla}}_a B^{b\cdots}{}_{c\cdots} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}} \tilde{\bar{\nabla}}_a g^{ef} + \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial k_{ef}} \tilde{\bar{\nabla}}_a k_{ef} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}} \tilde{\bar{\nabla}}_a g^{ef}, \tag{2.54}$$

where the second equality holds via (2.53). We can go one step further and realise that

$$\tilde{\bar{\nabla}}_a B^{b\cdots}{}_{c\cdots} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}} \left(\tilde{\bar{\nabla}}_a g^{ef} - \bar{\nabla}_a g^{ef}\right) = 2 \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}} \mathfrak{D}^{ef}{}_a. \tag{2.55}$$

The second term in braces, $\bar{\nabla}_a g^{ef}$, vanishes by (1.11), and the final equality holds by (2.50). Finally, since

$$\bar{\nabla}_a B^{b\cdots}{}_{c\cdots} = \tilde{\bar{\nabla}}_a B^{b\cdots}{}_{c\cdots} - \left(\tilde{\bar{\nabla}}_a - \bar{\nabla}_a\right) B^{b\cdots}{}_{c\cdots}, \tag{2.56}$$

then it follows by repeated application of (2.50) on the last term, that orthogonally projected derivative is

$$\bar{\nabla}_a B^{b\cdots}{}_{c\cdots} \;=\; 2 \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{de}} \mathfrak{D}^{de}{}_a - B^{d\cdots}{}_{c\dots} \mathfrak{D}^b{}_{ad} - \cdots + B^{b\cdots}{}_{d\dots} \mathfrak{D}^d{}_{ac} + \cdots. \tag{2.57}$$

[2] The conditions required for this definition to hold are

2.4 Constructing the set of scalar invariants

We will formally introduce it later, but we are interested in constructing the equation of state ρ which will be a scalar function of the state of the system which is integrated to give the action. There are a few different sets of scalar invariants one could use: formally they are identical, but different choices will enhance, or hide, insight into the physical behavior. And so, we are interested in finding the complete list of scalar invariants which will specify the state of the system. The invariants are constructed from the pull-back of the material metric, $k^a{}_b$.

As a candidate set of invariants, note that *the* three independent scalar invariants of the mixed components of the pulled-back material metric $k^a{}_b$ are

$$I_1 = [\mathbf{k}], \qquad I_2 = [\mathbf{k}^2], \qquad I_3 = [\mathbf{k}^3], \tag{2.58}$$

in which we denoted traces with square braces,

$$I_n = \mathrm{Tr}(\mathbf{k}^n) = [\mathbf{k}^n] = k^a{}_b k^b{}_c \cdots k^f{}_a, \tag{2.59}$$

with $k^a{}_b$ defined from k_{ab} via (2.35). This is a complete list of independent invariants (any other invariants can be computed from these) due to the orthogonality of $k^a{}_b$ (2.34). Since n_{ABC} is the volume form of k_{AB}, the particle number density n is also a scalar invariant of $k^a{}_b$; by the Cayley-Hamilton theorem, the determinant is related to the other invariants via

$$n^2 = \det(k^a{}_b) = \tfrac{1}{3!}\left([\mathbf{k}]^3 - 3[\mathbf{k}][\mathbf{k}^2] + 2[\mathbf{k}^3]\right). \tag{2.60}$$

We could use $\{I_1, I_2, I_3\}$ as the list of invariants which could be the arguments of the equation of state, but we shall instead choose n and the independent scalar invariants of the uni-modular tensor $\eta^a{}_b$, defined in (2.38) since this will help the comparison between solid and fluid descriptions. The important consequence of uni-modularity is that $\eta^a{}_b$ only has two independent invariants (rather than 3 which could be expected from a symmetric rank-2 tensor in 3D). The invariants are linked via the Cayley-Hamilton theorem as

$$3! = [\boldsymbol{\eta}]^3 - 3[\boldsymbol{\eta}][\boldsymbol{\eta}^2] + 2[\boldsymbol{\eta}^3]. \tag{2.61}$$

Notice that (2.61) can be rewritten as

$$2\left([\boldsymbol{\eta}^3] - 3\right) = 3[\boldsymbol{\eta}]\left([\boldsymbol{\eta}^2] - \tfrac{1}{3}[\boldsymbol{\eta}]^2\right). \tag{2.62}$$

To summarise, we have shown that there are two equivalent ways to write the most general equation of state for a solid: both have a maximum of three arguments.

They are

$$\rho = \rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right) \tag{2.63a}$$

and

$$\rho = \rho\left(n, [\boldsymbol{\eta}], \left[\boldsymbol{\eta}^2\right]\right). \tag{2.63b}$$

We remind that $k^a{}_b$ is the pull-back of a tensor whose volume form is n_{ABC} and (squared) determinant is the particle number density, n. Secondly, $\eta^a{}_b$ is a uni-modular tensor whose inverse η^{-1AB} co-incides with the push-forward of the space-time metric when the material is in the unsheared state. The latter formulation is somewhat favorable, since it becomes easy to connect to a scenario in which the solid "becomes" like a fluid, since ρ becomes independent of $[\boldsymbol{\eta}^n]$.

Before we continue it is worth noting some useful ways to compute derivatives of functions which depend on quantities which regularly appear in the construction, most notably functions which depend on n or $\eta^a{}_b$. First of all, the derivative of the number density n with respect to the space-time metric is given by

$$\frac{\partial n}{\partial g^{ab}} = \frac{1}{2} n \gamma_{ab}. \tag{2.64}$$

When $Y = Y(k^a{}_b)$ is any quantity that depends only on the $k^a{}_b$, then its derivative with respect to the space-time metric is

$$\frac{\partial Y}{\partial g^{ab}} = k_{c(a} \frac{\partial Y}{\partial k^{b)}{}_c}. \tag{2.65}$$

For any quantity $Z = Z(n, \eta^a{}_b)$, and using (2.39) as a decomposition of the degrees of freedom in $k^a{}_b$, we obtain

$$\frac{\partial Z}{\partial g^{ab}} = \frac{1}{2} n \gamma_{ab} \frac{\partial Z}{\partial n} + \eta_{c\langle a} \frac{\partial Z}{\partial \eta^{b\rangle}{}_c}, \tag{2.66}$$

where the angular brackets denote the symmetric trace-free part of the tensor, as defined in (1.12). For each quantity n, Y, and Z as defined here,

$$u^a \frac{\partial n}{\partial g^{ab}} = 0, \qquad u^a \frac{\partial Y}{\partial g^{ab}} = 0, \qquad u^a \frac{\partial Z}{\partial g^{ab}} = 0. \tag{2.67}$$

2.5 Deformations about a relaxed state

It is important to understand how to deal with a deformed medium. Before we give some explicit expressions for deformations of the solid, we shall illustrate the philosophy via "non-linear sigma models" from field theory.

2.5.1 Example from non-linear sigma models

One of the important ideas in continuous mechanics is that of the assumed existence of a relaxed state: this is supposed to be some configuration that minimizes some measure of "energy". This concept is absolutely vital in the study of solitons. As the simplest example, consider the Lagrangian density for a real scalar field ϕ living in a Higgs potential,

$$\mathcal{L} = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2. \tag{2.68}$$

The relaxed configuration of this scalar is when $\phi = \pm\eta$ (commonly known as the vacuum manifold). It is simple to find the Lagrangian density for fluctuations about the relaxed state; substituting $\phi = \eta + \delta\phi$ into (2.68) and expanding to quadratic order in $\delta\phi$ yields

$$\mathcal{L} = -\frac{1}{2}\partial_a\delta\phi\partial^a\delta\phi - \frac{1}{2}\lambda\eta^2(\delta\phi)^2. \tag{2.69}$$

The Lagrangian that results is that for a massive scalar field. This example is depressingly simple since there isn't a non-trivial Lagrangian that describes the field *in* the relaxed state. For that we shall move to a more complicated example and think about a multi-scalar field model whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathfrak{k}_{IJ}\partial_a\Phi^I\partial^a\Phi^J - V(\Phi^I). \tag{2.70}$$

There are supposed to be n fields here, and so $I = 1,\dots,n$, and the set of symmetric quantities $\mathfrak{k}_{IJ}$ are supposed to play the role of a metric in field space. Before we continue we want to make it plainly clear that this isn't the most general Lagrangian density that can be constructed out of single derivatives.

Suppose that the energy gets minimized when the fields Φ^I are consigned to live on a sub-manifold, $\mathcal{V}$ say, of dimension $q \le n$. The "vacuum manifold" $\mathcal{V}$ can be coordinatized by q scalars ϕ^A, say, with $A = 1,\dots,q$. Hence, the original set of fields Φ^I are some function of the fields ϕ^A,

$$\Phi^I = \Phi^I(\phi^A), \tag{2.71}$$

when the configuration is in its relaxed state. From (2.71) it is clear that

$$\partial_a\Phi^I = \frac{\partial\Phi^I}{\partial\phi^A}\partial_a\phi^A. \tag{2.72}$$

Putting (2.72) into (2.70) gives

$$\mathcal{L} = -\frac{1}{2}\mathfrak{g}_{AB}(\phi)\partial_a\phi^A\partial^a\phi^B, \tag{2.73}$$

in which we defined

$$\mathfrak{g}_{AB}(\phi) \equiv \mathfrak{k}_{IJ}\frac{\partial \Phi^I}{\partial \phi^A}\frac{\partial \Phi^J}{\partial \phi^B}, \tag{2.74}$$

which is interpreted as the metric on the field submanifold $\mathcal{V}$. The field equations for the ϕ^A derived from (2.73) are given by

$$g^{ab}\nabla_a\nabla_b\phi^A + \Gamma^A{}_{BC}\nabla_a\phi^B\nabla^a\phi^C = 0, \tag{2.75}$$

where

$$\Gamma^A{}_{BC} = \tfrac{1}{2}\mathfrak{g}^{AD}\left(\partial_B\mathfrak{g}_{CD} + \partial_C\mathfrak{g}_{BD} - \partial_D\mathfrak{g}_{BC}\right) \tag{2.76}$$

are the Christoffel symbols for the metric in the field submanifold.

One of the simplest ways (we can think of, at least) to see how study fluctuations or deformations away from the relaxed state is to first imagine that the relaxed state is specified by the condition

$$\frac{\partial \Phi^I_0}{\partial \phi^A_0} = \mathfrak{J}^I{}_A, \tag{2.77}$$

where the "0" subscripts are used to specify that the configuration is relaxed, and the gothic-J is used to denote the relaxed Jacobian. Using (2.77) to compute (2.74) gives a simple expression for the submanifolds metric in the relaxed state,

$$\bar{\mathfrak{g}}_{AB} = \mathfrak{k}_{IJ}\mathfrak{J}^I{}_A\mathfrak{J}^J{}_B. \tag{2.78}$$

It should be evident that the Christoffel symbols in the field submanifold (2.76) are zero for this relaxed state if $\mathfrak{k}_{IJ}$ is flat and the Jacobians $\mathfrak{J}^I{}_A = \delta^I{}_A$. We have denoted $\bar{\mathfrak{g}}_{AB}$ as the field submanifolds metric in the relaxed state. In a deformed state the derivatives of Φ^I with respect to the ϕ^A must differ from their values in the relaxed state by some amount which can be packaged into a rank-2 tensor $\mathfrak{d}^I{}_A$ (this is a gothic-d, for "deformation") via

$$\frac{\partial \Phi^I}{\partial \phi^A} = \mathfrak{J}^I{}_A + \mathfrak{d}^I{}_A, \tag{2.79}$$

where we will not make any assumptions about the size of the $\mathfrak{d}^I{}_A$. Putting (2.79) into (2.74) gives

$$\mathfrak{g}_{AB} = \bar{\mathfrak{g}}_{AB} + 2\mathfrak{d}_{AB} + \mathfrak{d}^I{}_A\mathfrak{d}_{IB}. \tag{2.80}$$

This expression is very similar to what is used in the non-linear Stuckelberg trick in the massive gravity literature (see, e.g., [19, 34]). It is therefore apparent that the deviation of $\mathfrak{g}_{AB}$ from $\bar{\mathfrak{g}}_{AB}$ is contained within the tensor

$$s_{AB} = \mathfrak{d}_{AB} + \tfrac{1}{2}\mathfrak{d}^I{}_A\mathfrak{d}_{IB}, \tag{2.81}$$

so that

$$s_{AB} = \tfrac{1}{2}\left(\mathfrak{g}_{AB} - \overline{\mathfrak{g}}_{AB}\right). \tag{2.82}$$

Hence, we now have a measure on how deformed the material is: when $s_{AB} = 0$ one has $\mathfrak{g}_{AB} = \overline{\mathfrak{g}}_{AB}$ which is the relaxed metric, and any $s_{AB} \neq 0$ means that the material is deformed in some way. If the deformations are small then one can safely assume that $\mathfrak{d}_{AB}$ is a small quantity and so $S_{AB} = \mathfrak{d}_{AB}$.

2.5.2 Deformations of the material

To make more explicit contact to the construction we gave in section 2.1, suppose that the ϕ^A are given by an "expansion" (which isn't necessarily small) about some fiducial state,

$$\phi^A = \overline{\phi}^A + \pi^A. \tag{2.83}$$

Then the configuration gradient (2.2) can be evaluated

$$\psi^A{}_a = \overline{J}^A{}_a + \partial_a \pi^A, \tag{2.84}$$

in which the configuration gradient computed in the fiducial state is

$$\overline{J}^A{}_a = \frac{\partial \overline{\phi}^A}{\partial x^a}. \tag{2.85}$$

Using (2.84) to provide an expression for the configuration gradient to compute the pull-back k_{ab} of the material metric k_{AB} via (2.31) yields

$$k_{ab} = \overline{k}_{ab} + 2\partial_{(a}\xi_{b)} + \Pi_{ab}, \tag{2.86}$$

in which we defined

$$\begin{aligned} \overline{k}_{ab} &\equiv k_{AB}\overline{J}^A{}_a\overline{J}^B{}_b, && (2.87a)\\ \partial_a\xi_b &\equiv k_{AB}\overline{J}^A{}_a\partial_b\pi^B, && (2.87b)\\ \Pi_{ab} &\equiv k_{AB}\partial_a\pi^A\partial_b\pi^B. && (2.87c) \end{aligned}$$

The Π_{ab}-term is neglected if the deformations are small. The quantity $\overline{k}_{ab}$ is the pull-back of the material metric when the material is in its unstrained state (i.e. when the $\pi^A = 0$ identically).

The state of the system is the contained within the tensor

$$S_{ab} \equiv \frac{1}{2}\left(k_{ab} - \overline{k}_{ab}\right), \tag{2.88}$$

quantifying the difference between the actual value of k_{ab} and its value in the fiducial state.

3 Quantifying the state of the material

Armed with the map, material metric, and set of scalar invariants, it remains to understand how to quantify the state of the material in terms of its effects on space-time. This quantification is acheived by constructing a material action which can be appended to the Einstein-Hilbert action, and from which one can derive the energy-momentum tensor which sources the gravitational field equations.

Along the way there are various useful auxiliary quantities, and useful pieces of technology that can be used to help understand what is going on.

3.1 Constant volume shear tensor

We define the *constant volume shear tensor*

$$s^a{}_b = \tfrac{1}{2}\left(\gamma^a{}_b - \eta^a{}_b\right). \tag{3.1}$$

This is a space-time tensor which quantifies the difference between the actual value of $\gamma^a{}_b$ and the unsheared value $\eta^a{}_b$ as described in Section 2.2. The definition (3.1) follows from the pull-back of (2.26), which was defined in the material manifold.

3.2 The equation of state and material action

The idea is to compute everything from a "master function" (to use Carter's terminology). This master function will be the piece of freedom which corresponds to the specification of the type or class of materials under consideration (much like a potential function $V(\phi)$ controls what types of canonical scalar field theories one is studying).

It is the energy density ρ which plays the role of the master function; in what follows we will refer to ρ as the *equation of state*. On a first pass we write down a material action given by the integral of the equation of state which has, as its sole arguments, the components of the pulled-back material metric:

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\rho\left(k^a{}_b\right). \tag{3.2}$$

However, our discussion in Section 2.4 has shown that we can go further, and we can realise that ρ is a function of any possible scalar invariants discussed in Section 2.4, and so the material action is given by the general expression

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{3.3}$$

We should note that the assumption of ρ being a function of the invariants of $k^a{}_b$ is akin to asking that the material is isotropic. The energy-momentum tensor is derived from varying S_{M} using the usual expression,

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{M}}}{\delta g^{ab}}, \tag{3.4}$$

which gives

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}}. \tag{3.5}$$

It is convenient to re-express the equation of state in terms of the particle number density n and the energy per particle, ϵ, via

$$\rho = n\epsilon. \tag{3.6}$$

And so, rather than ask for the form of ρ, we ask for the form of ϵ, and then write the matter action (3.3) as

$$S_{\text{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\, n\epsilon\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{3.7}$$

3.3 Variation of the material action and measure-weighted variation

Varying the action (3.3) yields

$$\delta S = \int \mathrm{d}^4x\, \sqrt{-g}\, \lozenge\rho. \tag{3.8}$$

We have used the "diamond derivative" notation to denote measure-weighted variations, defined to act on a quantity Q via

$$\lozenge^n Q \equiv \frac{1}{\sqrt{-g}}\delta_{\text{L}}{}^n\left(\sqrt{-g}\,Q\right), \tag{3.9}$$

in which δ_{L} is the *Lagrangian variation* operator. The role of δ_{L} is to encorporate both intrinsic variations of a field, and variations due to some other process (such as symmetry transformations). Before we evaluate (3.8) we want to explain some interesting properties and uses for the first measure-weighted variation $\lozenge Q$.

The first measure-weighted variation of this quantity Q is

$$\lozenge Q = \delta_{\text{L}} Q - \tfrac{1}{2} Q g_{ab}\delta_{\text{L}} g^{ab}. \tag{3.10}$$

When Q is a function of a set of scalars χ^A and their derivatives $\partial_a\chi^A$, say, and the metric g_{ab}, then it is a simple exercise to observe that

$$\lozenge Q = \frac{\partial Q}{\partial \chi^A}\delta\chi^A + \frac{\partial Q}{\partial\partial_a\chi^A}\partial_a\delta\chi^A + \left(\frac{\partial Q}{\partial g^{ab}} - \frac{1}{2}Q g_{ab}\right)\delta g^{ab}. \tag{3.11}$$

The second term can be rearranged by integrating by parts (without neglecting any total derivatives) to give

$$\Diamond Q = \mathcal{E}_A \delta_{\mathrm{L}} \chi^A + \tfrac{1}{2} T_{ab} \delta_{\mathrm{L}} g^{ab} + \nabla_a \vartheta^a, \tag{3.12}$$

where we defined

$$\mathcal{E}_A \equiv \frac{\partial Q}{\partial \chi^A} - \nabla_a \frac{\partial Q}{\partial \partial_a \chi^A}, \tag{3.13a}$$

$$T_{ab} \equiv 2 \frac{\partial Q}{\partial g^{ab}} - Q g_{ab}, \tag{3.13b}$$

$$\vartheta^a \equiv \frac{\partial Q}{\partial \partial_a \chi^A} \delta_{\mathrm{L}} \chi^A. \tag{3.13c}$$

The ϑ^a-term in (3.12) only contributes to the boundary and can be made to vanish by choice of boundary conditions: it won't play a role in what follows.

Suppose the variations δ_{L} are due to diffeomorphisms generated by the vector ξ^a and intrinsic arbitrary variations (of the type usually considered when using variational principles), then the variations δ_{L} in (3.12) should be replaced with

$$\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \mathcal{L}_\xi, \tag{3.14}$$

in which the Lie derivatives are

$$\mathcal{L}_\xi \chi^A = \xi^a \nabla_a \chi^A, \qquad \mathcal{L}_\xi g^{ab} = -2 \nabla^{(a} \xi^{b)}. \tag{3.15}$$

so that

$$\Diamond Q = \mathcal{E}_A \delta_{\mathrm{E}} \chi^A + \tfrac{1}{2} T_{ab} \delta_{\mathrm{E}} g^{ab} + \xi^a \left(\mathcal{E}_A \nabla_a \chi^A + \nabla^b T_{ab} \right) - \nabla^{(a} \left(\xi^{b)} T_{ab} \right). \tag{3.16}$$

Note that the final term only contributes to the boundary. And so, we can read off from (3.16) that diffeomorphism invariance is ensured when the coefficient of the diffeomorphism generating field ξ^a vanishes, namely

$$\mathcal{E}_A \nabla_a \chi^A + \nabla^b T_{ab} = 0. \tag{3.17}$$

We can also read off from (3.16) that the condition for the theory is stationary under arbitrary variations in the scalars χ^A (this is the usual statement of the variational principle) is that the coefficient $\mathcal{E}_A$ of the arbitrary variations $\delta_{\mathrm{E}} \chi^A$ should vanish:

$$\mathcal{E}_A = 0. \tag{3.18}$$

It is immediately clear from its definition (3.13a) that the conditions (3.18) are just the equations of motion of the scalars χ^A. By inspecting (3.17) it is manifest that when the equations of motion (3.18) are satisified, the energy-momentum tensor is conserved

Let us now return to the problem at hand: evaluation of (3.8) for the material medium. At the top of Section 3.2 we stated that the equation of state ρ (i.e. the integrand of the material action) is a function of the pulled-back metric $k^a{}_b$ alone, (3.2). This means that $\delta_\mathrm{L}\rho$ can be written as

$$\delta_\mathrm{L}\rho = \frac{\partial \rho}{\partial g^{ab}}\delta_\mathrm{L} g^{ab}, \tag{3.19}$$

which can be used to obtain

$$\lozenge\rho = \frac{1}{2}\left(-\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}}\right)\delta_\mathrm{L} g^{ab}, \tag{3.20}$$

which we remind is the integrand of the first variation of the action.

3.4 The energy-momentum tensor

The quantity in braces in (3.20) is precisely the definition of the energy-momentum tensor

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}}. \tag{3.21}$$

We are able to further evaluate this expression, and in particular we can deduce the "types" of contributions to T_{ab} from knowledge of what ρ is a function of. Since $\rho = \rho\left(k^a{}_b\right)$, using (2.65) gives

$$\frac{\partial \rho}{\partial g^{ab}} = k_{c(a}\frac{\partial \rho}{\partial k^{b)}{}_c}, \tag{3.22}$$

which, by virtue of (2.34), means that

$$u^a\frac{\partial \rho}{\partial g^{ab}} = 0. \tag{3.23}$$

And so, assuming an equation of state ρ has been given as a function of the invariants of the pulled-back material metric $k^a{}_b$, the energy-momentum tensor of the solid (3.21) is given by

$$T_{ab} = \rho u_a u_b + P_{ab}, \tag{3.24}$$

in which the pressure tensor P_{ab} is given by

$$P_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho\gamma_{ab}. \tag{3.25}$$

By virtue of (3.23) the pressure tensor (3.25) is orthogonal,

$$u^a P_{ab} = 0. \tag{3.26}$$

The important thing to note is that there is no heat flux term in T_{ab}: this a consequence of the orthogonality of the mapping between the material manifold and spacetime.

After using the solid form of the energy-momentum tensor (3.24), the variation of the energy density (3.19) can be written as

$$\delta_{\mathrm{L}}\rho = \tfrac{1}{2}\left(\rho\gamma_{ab} + P_{ab}\right)\delta_{\mathrm{L}}g^{ab}. \tag{3.27}$$

After rewriting the equation of state ρ in terms of an energy per particle, ϵ, via (3.6), the pressure tensor (3.25) takes on the more compact form

$$P_{ab} = 2n\frac{\partial\epsilon}{\partial g^{ab}}. \tag{3.28}$$

When the energy per particle ϵ is written in a (still general) way to only depend on the number density n and uni-modular tensor $\eta^a{}_b$, i.e., $\epsilon = \epsilon(n, \eta^a{}_b)$, we can use (2.66) to further evaluate the pressure tensor (3.28), yielding the rather attractive expression

$$P_{ab} = p\gamma_{ab} + \pi_{ab}, \tag{3.29}$$

in which we have identified the pressure scalar p,

$$p = n^2\frac{\partial\epsilon}{\partial n}, \tag{3.30a}$$

and the (traceless) anisotropic stress tensor

$$\pi_{ab} = 2n\eta_{c\langle a}\frac{\partial\epsilon}{\partial\eta^{b\rangle}{}_c}. \tag{3.30b}$$

This highlights that dependence of ϵ on the number density n is linked to isotropic pressure p, and dependence of ϵ on the uni-modular tensor $\eta^a{}_b$ is linked to anistropic stress π_{ab}.

There is nothing "imperfect" about the construction of the substance so far: there is no dissipation, everything is conserved, and is constructed from a very geometrical point of view. However, the pressure tensor (3.29) has anisotropic stress (3.30b). For a *fluid* this would signal an imperfection, but it is exactly this anisotropic stress which makes the theory that of a *solid*.

It has recently become popular to suggest that an observation of anisotropic stress would point towards modified gravity rather than dark energy [35–38]. What we are about to state is not a comment on a claim made by any of these articles, but it is worth pointing out. Whilst it is true that modified gravity models have anisotropic stress, it is also true that material models can contribute towards anisotropic stress. Infact, material models constitute the simplest and physically "most intuitive" additions to the Einstin-Hilbert and standard matter content gravitational model.

3.5 Example equation of state

It is instructive to specify an example equation of state and obtain the energy-momentum tensor. We will make the same choice as described in [6]. To begin with it is useful to recall the covariant form of the constant volume shear tensor (3.1), which we repeat here for completeness:

$$s_{ab} = \tfrac{1}{2}(\gamma_{ab} - \eta_{ab}). \tag{3.31}$$

There are two methods to raise indices (and thus construct traces). These methods are

$$s^a{}_b = \gamma^{ac} s_{cb}, \qquad \hat{s}^a{}_b = \eta^{-1ac} s_{cb}. \tag{3.32}$$

In matrix form these respectively read

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{1} - \boldsymbol{\eta}), \qquad \hat{\mathbf{s}} = \tfrac{1}{2}(\boldsymbol{\eta}^{-1} - \mathbf{1}). \tag{3.33}$$

In [6] the equation of state ϵ is picked to be a function of the particle number density n and only one invariant of $\eta^a{}_b$. The explicit form of ϵ is

$$\epsilon = \check{\epsilon}_0(n) + \frac{\check{\mu}(n)}{n}\overline{s}^2, \tag{3.34}$$

and where $\overline{s}^2$ is the shear scalar,

$$\overline{s}^2 \equiv \tfrac{1}{36}\left([\boldsymbol{\eta}]^3 - [\boldsymbol{\eta}^3] - 24\right). \tag{3.35}$$

Notice that by using (2.61), the choice of shear scalar (3.35) is equivalent to

$$\overline{s}^2 = \tfrac{1}{24}\left([\boldsymbol{\eta}]^2 - [\boldsymbol{\eta}^2]\right)[\boldsymbol{\eta}] - \tfrac{3}{4}. \tag{3.36}$$

Using (3.34) matter action is therefore given by

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\left\{n\check{\epsilon}_0 + \tfrac{1}{36}\check{\mu}\left([\boldsymbol{\eta}]^3 - [\boldsymbol{\eta}^3] - 24\right)\right\}. \tag{3.37}$$

Using (3.34) the pressure tensor is given by (3.29) where the isotropic pressure (3.30a) is

$$p = \check{p} + (\check{\Omega} - 1)\sigma, \tag{3.38a}$$

and the anisotropic stress (3.30b) is given by

$$\pi_{ab} = \tfrac{1}{6}\check{\mu}\left([\boldsymbol{\eta}]^2 \eta_{\langle ab\rangle} - \eta^{cd}\eta_{c\langle a}\eta_{b\rangle d}\right), \tag{3.38b}$$

and where the three quantities appearing in the pressure (3.38a) are

$$\check{p} = n^2 \frac{\mathrm{d}\check{\epsilon}}{\mathrm{d}n}, \qquad \check{\Omega} = \frac{n}{\check{\mu}}\frac{\mathrm{d}\check{\mu}}{\mathrm{d}n}, \qquad \sigma = \check{\mu}s^2. \tag{3.39}$$

4 Equation of motion

Obtaining the equation of state and energy-momentum tensor is clearly only part of the story. One must also obtain equations of motion: these come from the conservation equation

$$\nabla_a T^{ab} = 0. \tag{4.1}$$

If the material is the only source to the gravitational field equations, then (4.1) follows by diffeomorphism invariance, and also by the Bianchi identity. Using the solid form (3.24) for T_{ab} the two independent (i.e., time-like and orthogonal) projections of (4.1) are

$$\dot{\rho} + (\rho\gamma^{ab} + p^{ab})\Theta_{ab} = 0, \tag{4.2a}$$

$$(\rho\gamma^{ab} + p^{ab})\dot{u}_b + \overline{\nabla}_b p^{ab} = 0. \tag{4.2b}$$

We used the orthogonally projected derivative $\overline{\nabla}_b$, as defined in (1.9)

If the energy per particle ϵ is a function only of the scalar invariants of $k^a{}_b$ (regardless of what the invariants are), then using (3.28) in conjunction with (2.57), the orthogonally projected derivative of the pressure tensor is

$$\overline{\nabla}_b p^{ab} = \left(E^{ab}{}_{cd} - \gamma^a{}_c p^b{}_d\right) \mathfrak{D}^{cd}{}_b, \tag{4.3}$$

in which we used the elasticity difference tensor $\mathfrak{D}^{cd}{}_b$ as defined in (2.51), and introduced the relativistic *elasticity tensor*, $E^{ab}{}_{cd}$, defined via

$$E^{ab}{}_{cd} \equiv 2\frac{\partial p^{ab}}{\partial \gamma^{cd}} - p^{ab}\gamma_{cd}. \tag{4.4}$$

Using (2.64) and (3.28), the elasticity tensor can be written as the second derivative of the energy per particle ϵ via

$$E^{abcd} = 4n\frac{\partial^2 \epsilon}{\partial\gamma_{ab}\partial\gamma_{cd}}. \tag{4.5}$$

At a later stage it will be convenient to use the relativistic *Hadamard elasticity tensor*, $A^{ab}{}_{cd}$, defined via

$$A^{ab}{}_{cd} \equiv E^{ab}{}_{cd} - \gamma^a{}_c p^b{}_d, \tag{4.6}$$

in which case (4.3) becomes

$$\overline{\nabla}_b p^{ab} = A^{ab}{}_{cd}\mathfrak{D}^{cd}{}_b. \tag{4.7}$$

Using (4.3) and the Hadamard tensor (4.6), the orthogonal projection (4.2b) can be written as

$$(\rho\gamma^{ab} + p^{ab})\dot{u}_b + A^{abcd}\mathfrak{D}_{cbd} = 0. \tag{4.8}$$

A rather convenient form of the equations of motion is given in [4, 39]; in terms of the material derivative the pressure tensor satisfies

$$\left[p^{ab}\right]^{\cdot} = -p^{ab}\theta - E^{abcd}\theta_{cd}. \tag{4.9}$$

In the more usual notation of "time-like" derivatives, the equations of motion for the materials energy density and pressure tensor are given by

$$u^a\rho_{;a} = -\rho u^a{}_{;a} - p^{ab}u_{a;b}, \tag{4.10a}$$

$$u^c p^{ab}{}_{;c} = 2p^{c(a}u^{b)}{}_{;c} + 2p^{c(a}u^{b)}\dot{u}_c - p^{ab}u^c{}_{;c} - E^{abcd}u_{c;d}, \tag{4.10b}$$

where we defined the acceleration vector $\dot{u}_a = u^b\theta_{ab}$.

4.1 Speed of sound

Here we review how to compute the speed of sound of the medium [39].

Sound wavefronts are characteristic hypersurfaces across which the acceleration vector $\dot{u}^a$ has a jump discontinuity (the velocity u^a and the metric remain continuous). Following Carter, we denote discontinuities across the wavefront with square braces; and so we set

$$[\dot{u}^a] = \alpha\iota^a, \tag{4.11}$$

in which α is the amplitude of the wavefront and ι^a is the polarization vector satisfying the space-like normalization condition

$$\iota^a\iota_a = 1. \tag{4.12}$$

Since the acceleration and velocity vectors are mutually orthogonal,

$$u_a\dot{u}^a = 0 \tag{4.13}$$

it follows that the polarization vector and the velocity vector are orthogonal

$$u_a\iota^a = 0. \tag{4.14}$$

The *propagation direction vector* ν^a is specified with the same orthonormality conditions as the polarization vector, namely

$$\nu^a \nu_a = 1, \qquad \nu^a u_a = 0. \tag{4.15}$$

The normal to the characteristic hypersurface is in the direction of the vector λ_a, defined via

$$\lambda_a = \nu_a - v u_a. \tag{4.16}$$

The scalar

$$v = \lambda^a u_a \tag{4.17}$$

is the speed of propagation.

The derivatives of the density, pressure tensor, and velocity fields on the characteristic hypersurface are given in terms of quantities $\sigma, \kappa^a, \tau^{ab}$ via

$$[\rho_{;a}] = \sigma \lambda_a, \tag{4.18a}$$

$$[u^a{}_{;b}] = \kappa^a \lambda_b, \tag{4.18b}$$

$$\left[p^{ab}{}_{;c}\right] = \tau^{ab} \lambda_c. \tag{4.18c}$$

We now show how to determine the values of $\sigma, \kappa^a, \tau^{ab}$ in terms of v, α, and ι^a. First, contracting (4.18b) with u^b gives (4.11) on the left-hand-side, and $v\kappa^a$ on the right-hand-side, and thus one obtains

$$v\kappa^a = \alpha \iota^a. \tag{4.19a}$$

Taking the discontinuity of the projections of the conservation equation (4.10a) and (4.10b), and then multiplying by v respectively yields

$$v^2 \sigma = -\alpha \left(\rho \iota^a \lambda_a + p^{ab} \iota_a \lambda_b\right), \tag{4.19b}$$

$$v^2 \tau^{ab} = \alpha \left(2 v u^{(a} p^{b)c} \iota_c + 2 p^{c(a} \iota^{b)} \lambda_c - p^{ab} \iota^c \lambda_c - E^{abcd} \iota_c \lambda_d\right). \tag{4.19c}$$

Putting the general form of the energy-momentum tensor (3.24) into the conservation equation (4.1)

$$\left(u^b \rho_{;b} + \rho u^b{}_{;b}\right) u^a + \rho \dot{u}^a + p^{ab}{}_{;b} = 0. \tag{4.20}$$

Taking the discontinuity of the general formula (4.20) and using (4.18) yields

$$\left(\upsilon\sigma + \rho\kappa^b\lambda_b\right) u^a + \rho\alpha\iota^a + \tau^{ab}\lambda_b = 0. \tag{4.21}$$

Now using (4.19) for κ^a, σ, and τ^{ab} yields

$$\upsilon^2\left(\rho\gamma^{ab} + p^{ab}\right)\iota_b + p^{bc}\lambda_b\lambda_c\iota^a - E^{abcd}\lambda_b\iota_c\lambda_d = 0. \tag{4.22}$$

By using the relativistic Hadamard tensor A^{abcd}, defined in (4.6), the equation (4.22) becomes

$$\left[\upsilon^2\left(\rho\gamma^{ab} + p^{ab}\right) - Q^{ab}\right]\iota_b = 0. \tag{4.23}$$

where we have introduced the Fresnel tensor Q^{ab} which is defined in terms of the Hadamard tensor and the propagation vector ν_a via

$$Q^{ac} \equiv A^{abcd}\nu_b\nu_d, \tag{4.24}$$

after noting that the Hadamard tensor is orthogonal on all indices. Orthogonality of the Hadamard tensor carries over to give orthogonality of the Fresnel tensor,

$$u_a Q^{ab} = 0. \tag{4.25}$$

Since every term in the characterstic equation (4.23) is orthogonal, it is essentially a 3-dimensional equation. The eigenvalues υ^2 are the squared sound speed (in general there will be three values).

Although we will show where this comes from later on, it is worth our providing an example of the explicit computation of the sound speed. In the case of an isoptropic elastic solid close to a ground state, the pressure tensor is specified in terms of the isotropic pressure scalar as $p^{ab} = p\gamma^{ab}$, and the elasticity tensor is given by

$$E^{abcd} = \left(\beta - \tfrac{1}{3}p\right)\gamma^{ab}\gamma^{cd} + 2\left(\mu + p\right)\left(\gamma^{a(c}\gamma^{d)b} - \tfrac{1}{3}\gamma^{ab}\gamma^{cd}\right); \tag{4.26}$$

the coefficients p, β, and μ, are repectively the isotopic pressure, bulk modulus, and modulus of rigidity. The Hadamard tensor in this case is given by

$$A^{abcd} = \beta\gamma^{ab}\gamma^{cd} + 2p\gamma^{a[d}\gamma^{b]c} + 2\mu\left(\gamma^{a(c}\gamma^{d)b} - \tfrac{1}{3}\gamma^{ab}\gamma^{cd}\right), \tag{4.27}$$

and the Fresnel tensor works out as

$$Q^{ab} = \left(\beta + \tfrac{1}{3}\mu\right)\nu^a\nu^b + \mu\gamma^{ab}. \tag{4.28}$$

Hence, the characteristic equation (4.23) becomes

$$\left[v^2\left(\rho+p\right)\gamma^{ab}-\mu\gamma^{ab}-\left(\beta+\tfrac{1}{3}\mu\right)\nu^a\nu^b\right]\iota_b=0. \tag{4.29}$$

There are two solutions: the first is where the polarization and propagation vectors are aligned, $\nu_a=\iota_a$ in which case the eigenvalue is

$$v^2=\frac{\beta+\frac{4}{3}\mu}{\rho+p}\equiv c_{\mathrm{L}}^2. \tag{4.30}$$

Secondly, where the polarization and propagation vectors are orthogonal: $\nu_a\iota^a=0$, in which case the eigenvalue is

$$v^2=\frac{\mu}{\rho+p}\equiv c_{\mathrm{T}}^2. \tag{4.31}$$

We therefore have two sound speeds; c_{L}^2 which is the speed of propagation of longitudinal modes, and c_{T}^2 which is the speed of propagation of transverse modes.

4.2 Equations of motion from the action

The action will be a function of the metric g^{ab}, and a set of scalars ϕ^A and their derivatives $\partial_a\phi^A$ (and possibly other material space tensors; we leave that out for now). Thus,

$$S_{\mathrm{M}}=\int \mathrm{d}^4x\,\sqrt{-g}\,\rho\left(g^{ab},\phi^A,\partial_a\phi^A\right). \tag{4.32}$$

Under Lagrangian variations δ_{L} in g^{ab} and ϕ^A, the variation in the action is

$$\delta_{\mathrm{L}}S_{\mathrm{M}}=\int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{1}{2}T_{ab}\delta_{\mathrm{L}}g^{ab}-\mathcal{E}_A\delta_{\mathrm{L}}\phi^A\right], \tag{4.33}$$

where T_{ab} is the energy-momentum tensor defined in the usual manner,

$$T_{ab}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{ab}}=2\frac{\partial\rho}{\partial g^{ab}}-\rho g_{ab}, \tag{4.34}$$

and

$$\mathcal{E}_A=\nabla_a\left(\frac{\partial\rho}{\partial\partial_a\phi^A}\right)-\frac{\partial\rho}{\partial\phi^A}. \tag{4.35}$$

To make it clear, δ_{L} simply stands for "some" variation: we have not yet specified what generates it; however, this has given us a clear notational method for imposing general covariance. We will, without loss of generality, assume that δ_{L} has two parts:

$$\delta_{\mathrm{L}}=\delta_{\mathrm{E}}+\delta_{\xi}. \tag{4.36}$$

The first part, δ_{E}, will be due to intrinsic variations, and δ_ξ will be the variation induced by changes in the coordinates $x^a \to x^a + \xi^a(x^b)$. For the metric g^{ab} and the set of scalars ϕ^A,

$$\delta_\xi g^{ab} = -2\nabla^{(a}\xi^{b)}, \qquad \delta_\xi \phi^A = \xi^a \partial_a \phi^A. \tag{4.37}$$

After integrating by parts, the resulting variation in the action is

$$\delta_\xi S_{\mathrm{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\left[\xi^a \left(\nabla_b T^{ab} - \mathcal{E}_A \partial_a \phi^A\right)\right]. \tag{4.38}$$

The variation in the action due to the intrinsic variations of the fields is

$$\delta_{\mathrm{E}} S_{\mathrm{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\frac{1}{2} T_{ab}\delta_{\mathrm{E}} g^{ab} - \mathcal{E}_A \delta_{\mathrm{E}} \phi^A\right]. \tag{4.39}$$

In particular,

$$\frac{\delta_{\mathrm{E}} S_{\mathrm{M}}}{\delta_{\mathrm{E}} \phi^A} = -\mathcal{E}_A, \tag{4.40}$$

which must vanish via the variational principle. General covariance requires the action to be invariant under changes in the coordinates, and so $\delta_\xi S_{\mathrm{M}} = 0$ when

$$\nabla^b T_{ab} = \mathcal{E}_A \partial_a \phi^A. \tag{4.41}$$

Therefore, energy-momentum conservation only holds if the equations of motion of the ϕ^A are satisfied. Put another way: conservation of energy-momentum implies the equations of motion of the elastic medium are satisfied. Note from the orthogonality of the mapping (2.3) that the time-like projection of (4.41) is automatically satisfied:

$$u^a \nabla^b T_{ab} = 0. \tag{4.42}$$

It then follows that the orthogonal projection of (4.41) implies the vanishing of (4.40).

5 The Carter-Quintana perfect solid

Carter and Quintana conclude their paper with an exposition of the equations for a *perfect elastic solid*. Before we give their equation of state and energy-momentum tensor, we shall discuss physical issues regarding the existence (or otherwise) of locally relaxed states of the material.

5.1 Strain and shear tensors

The strain tensor is linked to the assumption about the existence of a locally relaxed state of a material – this is the unstrained state. In the unstrained state the energy per particle ϵ is supposed to be minimum when γ_{ab} takes on a particular value, k_{ab} say. This invites a quantification of the state of strain of the material by measuring the difference between the actual value of γ_{ab} and its unstrained value k_{ab} via the *strain tensor*, e_{ab}, defined as

$$e_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - k_{ab}\right). \tag{5.1}$$

Recalling that the energy per particle is denoted as ϵ, we define ϵ_0 to be the energy per particle in the unstrained state. The Hookean idealization takes the energy per particle to be of quadratic form in the strain tensor

$$\epsilon = \epsilon_0 + \tfrac{1}{2}K^{abcd}e_{ab}e_{cd}. \tag{5.2}$$

The elasticity tensor E^{abcd} relates to K^{abcd} via

$$E^{abcd} = nK^{abcd}. \tag{5.3}$$

Hence the energy density can be written as

$$\rho = \frac{n}{n_0}\rho_0 + \tfrac{1}{2}E^{abcd}e_{ab}e_{cd}, \tag{5.4}$$

and the pressure tensor is related to the strain tensor via

$$p^{ab} = -E^{abcd}e_{cd}. \tag{5.5}$$

The tensor k_{ab} can be thought of as a Riemannian metric on material space; the pull-back formalism means that $u^a k_{ab} = 0$. Associated with the value ϵ_0 of ϵ in the unstrained state are the values ρ_0 of the energy density ρ, and n_0 of the particle number density n.

The complication which Carter invites is that not all physical systems of interest will have a state which is locally relaxed, thus negating the existence of k_{ab} and rendering this construction impotent. This leads to the introduction of the shear tensor.

Rather than ask for the relaxed state to be a state where the energy per particle is minimum, we ask for a state in which ϵ is minimized subject to the restriction of constant particle number density. This is the unsheared state, and motivates the introduction of $\eta_{ab}(n)$ which is the value of γ_{ab} in the unsheared state with particle number density n. Again, to quantify the state of shear we define the *constant volume shear tensor* via

$$s_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - \eta_{ab}\right), \tag{5.6}$$

which (to reinforce the point) is the difference between the actual value of γ_{ab} and its value in the unsheared state.

We define $\check{\rho}(n)$ to be the energy density in the unsheared state, and hence

$$\check{\rho} = n\check{\epsilon}. \tag{5.7}$$

When ϵ does have an absolute minimum, at some particle number density n_0, one can keep the previous notions of the strain tensor; indeed

$$\eta_{ab}(n_0) = k_{ab}, \tag{5.8a}$$

$$\check{\rho}(n_0) = \rho_0, \tag{5.8b}$$

$$\check{\epsilon}(n_0) = \epsilon_0. \tag{5.8c}$$

5.2 The equation of state

The compressional distortion tensor is supposed to vanish, $\tau_{ab} = 0$, the reference tensors satisfy

$$[\eta_{ab}]^{\cdot} = \tfrac{2}{3}\eta_{ab}\Theta, \tag{5.9a}$$

$$[\eta^{-1ab}]^{\cdot} = -\tfrac{2}{3}\eta^{-1ab}\Theta, \tag{5.9b}$$

and the strain tensor satisfies

$$[s_{ab}]^{\cdot} = \tfrac{2}{3}s_{ab}\Theta + \sigma_{ab}. \tag{5.9c}$$

Here, $[X]^{\cdot}$ denotes the *material derivative* of X (this is explained the first half of Carter and Quintana, and we will do so later on). One can obtain

$$\eta_{ab} = (n/n_0)^{-2/3}k_{ab}, \tag{5.10}$$

The solid is supposed to be isotropic with respect to its unsheared states. Hence, the energy per particle (recall, $\rho = \epsilon n$, and ϵ is the energy per particle) is a function only of invariants. There are a maximum of three invariants: they are taken to be the particle number density n and the two independent invariants of the shear tensor $s^a{}_b$. The particular combination of these are taken to be

$$s^2 \equiv \left(\eta^{-1ad}\eta^{-1bc} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right) s_{ab}s_{cd} = \lceil \mathbf{s}^2 \rceil - \tfrac{1}{3}\lceil \mathbf{s} \rceil^2, \tag{5.11a}$$

$$l \equiv \eta^{-1ab}\eta^{-1cd}\eta^{-1ef} s_{bc}s_{de}s_{fa} = \lceil \mathbf{s}^3 \rceil. \tag{5.11b}$$

We used the notation $\lceil \mathbf{X} \rceil$ for traces which are taken with η^{-1ab} (as opposed to $[\mathbf{X}]$ which was used to denote traces with g^{ab}): this choice is for simplicity of the resulting formulae and does not lose generality. Hence, the most general form of the equation of state is a function with three arguments:

$$\epsilon = F(n, s^2, l). \tag{5.12}$$

The action for Einsteinian gravity with the CQ solid is thus

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G} - nF(n, s^2, l) \right]. \tag{5.13}$$

5.3 The energy-momentum tensor

The energy density ρ, and pressure tensor p^{ab} are given by

$$\rho = nF, \tag{5.14a}$$

$$\begin{aligned} p^{ab} = &\left\{ n^2 F_{,n} + ns^2\left(\frac{4}{3}F_{,s^2} + F_{,l}\right) - n\left(2l + \tfrac{1}{3}\lceil \mathbf{s} \rceil^2\right) F_{,l} \right\} \gamma^{ab} \\ &-2nF_{,s^2}\left(\eta^{-1a(c}\eta^{-1d)b} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right) s_{cd} \\ &-3nF_{,l}\eta^{-1a(c}\eta^{-2d)b}\eta^{-1ef}s_{ce}s_{df}. \end{aligned} \tag{5.14b}$$

These expressions contain the corrections to the typos which were present in [4], and which were pointed out (by the same authors) in [5]. The isotropic pressure is found from the trace of (5.14b), and is given by

$$p = n^2 F_{,n} + 2n\left(\frac{2}{3}s^2 F_{,s^2} - lF_{,l}\right). \tag{5.15}$$

5.4 The quasi-Hookean solid

We will make use of the notation

$$\begin{aligned}
\check{\epsilon}(n) &= F(n,0,0), && (5.16a)\\
\check{\rho}(n) &= nF(n,0,0), && (5.16b)\\
\check{p}(n) &= n^2\frac{\partial F}{\partial n}(n,0,0), && (5.16c)\\
\beta(n) &= n^3\frac{\partial^2 F}{\partial n^2}(n,0,0) + 2n^2\frac{\partial F}{\partial n}(n,0,0), && (5.16d)\\
\mu(n) &= n\frac{\partial F}{\partial s^2}(n,0,0). && (5.16e)
\end{aligned}$$

The quantities $\check{\rho}, \check{p}, \beta, \mu$ are the unsheared energy density, bulk and rigidity moduli. The value of the elasticity tensor in the state of zero shear strain is

$$\check{E}^{abcd}(n) = (\beta - \tfrac{1}{3}\check{p})\eta^{-1ab}\eta^{-1cd} + 2(\mu + \check{p})(\eta^{-1a(c}\eta^{-1b)d} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}). \qquad (5.17)$$

Note that this is the elasticity tensor we computed the sound speeds for just after equation (4.26).

The Lagrangian for the Carter-Quintana solid in the quasi-Hookean limit (which we shall refer to as a "quasi-Hookean solid") is linear in s^2, and independent of l:

$$F_{\text{qHs}} = \check{\epsilon} + \tfrac{\mu(n)}{n}s^2. \qquad (5.18)$$

For this quasi-Hookean solid the energy density and pressure tensor are respectively given by

$$\rho = \check{\rho} + \mu s^2, \qquad (5.19a)$$

$$p^{ab} = \left\{\check{p} + \left(n\mu' + \tfrac{1}{3}\mu\right)s^2\right\}\gamma^{ab} - 2\mu\left\{\eta^{-1a(c}\eta^{-1b)d} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right\}s_{cd}. \qquad (5.19b)$$

5.5 Exact non-linear equations of motion

This is based on the discussion in [5], and the aim is to compute the equations of motion for General Relativity sourced by a non-linear elastic solid.

Start off with the matter flow velocity

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \qquad (5.20)$$

normalised via

$$u^a u_a = -c^2. \qquad (5.21)$$

Now introduce the flow gradient tensor v_{ab} via

$$u_{a;b} = v_{ab} - \tfrac{1}{c^2}\dot{u}_a u_b, \tag{5.22}$$

where

$$\dot{u}_a = u^b u_{a;b} \tag{5.23}$$

is the acceleration vector. The flow gradient tensor is orthogonal to the flow,

$$u^a v_{ab} = 0. \tag{5.24}$$

Let f_{ab} be an orthogonal covariant tensor. Then the Lie derivative of f_{ab} along the direction of the flow velocity is

$$\pounds_u f_{ab} = \gamma^c{}_a \gamma^d{}_b u^e f_{cd;e} + f_{cb} v^c{}_a + f_{ac} v^c{}_b. \tag{5.25}$$

One has

$$\pounds_u \gamma_{ab} = 2\theta_{ab}, \tag{5.26}$$

where θ_{ab} is related to the flow gradient v_{ab} via

$$v_{ab} = \theta_{ab} + \omega_{ab}, \tag{5.27}$$

with

$$\theta_{[ab]} = 0, \qquad \omega_{(ab)} = 0. \tag{5.28}$$

From the Ricci identity

$$u_{a;[b;c]} = \tfrac{1}{2} u_d R^d{}_{abc} \tag{5.29}$$

one obtains the Lie derivative of the flow gradient in the direction of u^a,

$$\pounds_u v_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{2}\dot{u}_a \dot{u}_b - u^c u^d R_{acbd}. \tag{5.30}$$

The anti-symmetric portion of (5.30) gives

$$\pounds_u \omega_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{[c;d]}, \tag{5.31}$$

and the symmetric portion of (5.30) gives

$$\tfrac{1}{2}\pounds_u \pounds_u \gamma_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{(c;d)} + v^c{}_a v_{cb} + \tfrac{1}{c^2}\dot{u}_a \dot{u}_b - u^c u^d R_{acbd}. \tag{5.32}$$

We also use the Weyl tensor

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}\left(R^{b]}{}_{d]} - \tfrac{1}{6}Rg^{b]}{}_{d]}\right). \tag{5.33}$$

The *relative strain tensor* is

$$e_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - \kappa_{ab}\right), \tag{5.34}$$

where the *strain reference tensor* is κ_{ab} and satisfies

$$\pounds_u \kappa_{ab} = 0. \tag{5.35}$$

Hence, from (5.26),

$$\theta_{ab} = \pounds_u e_{ab}, \tag{5.36}$$

meaning that the expansion tensor θ_{ab} quantifies the rate of relative strain. Carter sets $\kappa_{ab} = 0$. Hence

$$\begin{aligned} \pounds_u \pounds_u e_{ab} &= \gamma^c{}_a \gamma^d{}_b \dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{c^2}\dot{u}_a \dot{u}_b \\ &\quad - u^c u^d C_{acbd} - \tfrac{1}{2}\gamma_{ab}\left(u^c u^d R_{cd} + \tfrac{1}{3}Rc^2\right) + \tfrac{1}{2}\gamma^c{}_a \gamma^d{}_b R_{cd} c^2. \end{aligned} \tag{5.37}$$

Now we set the energy-momentum tensor to be that for a perfect elastic solid,

$$T^{ab} = \rho u^a u^b + p^{ab}. \tag{5.38}$$

One can compute p^{AB} from the energy as a function of strain, $\epsilon(\gamma_{AB})$ via

$$p^{AB} = -\epsilon\gamma^{AB} - 2\frac{\partial \epsilon}{\partial \gamma_{AB}}. \tag{5.39}$$

This is valid for any linear or non-linear function of strain, for which the conservation law

$$T^{ab}{}_{;b} = 0 \tag{5.40}$$

holds.

We now specify the gravitational theory, which we take to be General Relativity for whom the Ricci tensor is given by

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \tfrac{1}{2}Tg_{ab}\right), \tag{5.41}$$

where

$$T = T^a{}_a = p^a{}_a - \rho c^2 \tag{5.42}$$

is the trace of the energy-momentum tensor. In General Relativity the Weyl tensor satisifies

$$C_{abcd} = C_{cdab} = C_{[ab][cd]}, \qquad C^{a}{}_{[bcd]} = 0, \qquad C^{ab}{}_{ac} = 0, \tag{5.43}$$

and the Bianchi identities impose

$$C^{abcd}{}_{;d} = \frac{8\pi G}{c^4}\left(T^{c[a;b]} - \tfrac{1}{3}g^{a[a}T^{;b]}\right). \tag{5.44}$$

We denote

$$C_{ab} = u^c u^d C_{acbd} \tag{5.45}$$

for the electric part of the the Weyl tensor.

From the conservation equation one obtains

$$\rho\dot{u}^a = -p^{ab}{}_{;b} + \tfrac{1}{c^2}u^a p^{bc}\theta_{bc}. \tag{5.46}$$

Hence (5.31) and (5.37) become

$$\pounds_u\omega_{ab} = \frac{1}{\rho^2}\gamma_{c[a}\gamma^d{}_{b]}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) + \frac{1}{\rho c^2}\omega_{ab}p^{cd}\pounds_u e_{cd}. \tag{5.47a}$$

$$\begin{aligned}\pounds_u\pounds_u e_{ab} \;=\;& \frac{1}{\rho^2}\gamma_{c(a}\gamma^d{}_{b)}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) - C_{ab} - \frac{4\pi G}{3}\rho\gamma_{ab}\\ &+\omega_{ca}\omega^c{}_b + 2\omega^c{}_{(a}\pounds_u e_{b)c} + g^{cd}\left(\pounds_u e_{ac}\right)\left(\pounds_u e_{bd}\right) + \frac{4\pi G}{c^2}\left(P_{ab} - \tfrac{2}{3}p^c{}_c\gamma_{ab}\right)\\ &+\frac{1}{c^2\rho^2}\left[\gamma_{ac}\gamma_{bd}p^{ce}{}_{;e}p^{df}{}_{;f} + \rho p^{cd}\left(\pounds_u e_{cd}\right)\left(\pounds_u e_{ab}\right)\right].\end{aligned} \tag{5.47b}$$

5.6 Slow roll parameter

We take this opportunity to recall that for inflating an FLRW Universe one requires smallness of the slow-roll parameter ϵ_{slow}, defined as

$$\epsilon_{\text{slow}} \equiv -\frac{\dot{H}}{H^2} = \frac{3(\rho+P)}{2\rho}. \tag{5.48}$$

Using (5.14a) and (5.15) the slow-roll parameter (5.48) evaluates for the CQ perfect solid to give

$$\epsilon_{\text{slow}} = \frac{3}{2}\left[1 + \frac{\partial\log F}{\partial\log n} + \frac{4}{3}\frac{\partial\log F}{\partial\log s^2} - 2\frac{\partial\log F}{\partial\log l}\right]. \tag{5.49}$$

The structure of (5.49) suggests a separable ansatz for the functional form of F:

$$F(n, s^2, l) = x(n)y(s^2)z(l), \tag{5.50}$$

since (5.49) becomes

$$\epsilon_{\text{slow}} = \frac{3}{2}\left[1 + nx' + \frac{4}{3}s^2 y' - 2lz'\right], \tag{5.51}$$

in which a prime is used to denote derivative with respect to the sole argument of the given function.

6 General isotropic configurations of elastic solids

The aim of this section is to understand the physics of elastic solids in isotropic configurations. We will study (a) elastic stars, (b) stars immersed in an elastic solid. The former problem has been studied before, but under the guise of "anisotropic stars", without mention of the anisotropy coming from elasticity. The latter problem is relevant for understanding how an elastic "bath" could affect the properties of compact objects (specifically, we will be looking out for some analogue of a "screen").

See [11]. There are some axially symmetric solutions in [40–43].

6.1 Eigenvalue decomposition

Here we review the technology laid out in [6, 12, 44] for dealing with isotropic elastic solids. The main point is to identify the maximum number of eigenvalues and eigenvectors, and to use a common eigenvector basis with which space-time and material quantities can be expanded. In this section we make no assumption about symmetries like spherical, axial, or static; in the next section we will, and consequently a lot of the expressions simplify from the general case.

The eigenvalues of the pulled-back material metric $k^a{}_b$ are written as $n^2_{\mathbb{A}}$, where $\mathbb{A} = 1, 2, 3$ label each of the "principle" directions. The particle density n is given in terms of the eigenvalues as

$$n = n_1 n_2 n_3, \tag{6.1}$$

which are interpretable as the principle linear particle densities. In an orthonormal basis $\{e^a{}_{\mathbb{A}}\}$ it follows that the space-time metric decomposes as

$$g_{ab} = -u_a u_b + \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}}, \tag{6.2}$$

and the pulled-back material metric decomposes as

$$k_{ab} = \sum_{\mathbb{A}=1}^{3} n^2_{\mathbb{A}} e_{a\mathbb{A}} e_{\mathbb{A}}. \tag{6.3}$$

Hence, $\frac{\partial}{\partial g^{ab}}$ acting on a quantity X which is a function of scalar invariants of $k^a{}_b$, is

$$\frac{\partial X}{\partial g^{ab}} = \frac{1}{2} \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}} n_{\mathbb{A}} \frac{\partial X}{\partial n_{\mathbb{A}}}. \tag{6.4}$$

Using this, the pressure tensor is given by

$$P_{ab} = \sum_{\mathbb{A}=1}^{3} P_{\mathbb{A}} e_{a\mathbb{A}} e_{b\mathbb{A}}, \tag{6.5}$$

in which the principle values of the pressure tensor are given by

$$P_{\mathbb{A}} = n n_{\mathbb{A}} \frac{\partial \epsilon}{\partial n_{\mathbb{A}}}. \tag{6.6}$$

Using $k^a{}_b = n^{2/3}\eta^a{}_b$ naturally splits the pressure tensor into the pressure scalar and anisotropic pressure as we now show. Denoting the eigenvalues of $\eta^a{}_b$ as α_μ^2, and are related to the n_μ via

$$\alpha_{\mathbb{A}} = \frac{n_{\mathbb{A}}}{n^{1/3}} = \left(\frac{z_{\mathbb{A}+2}}{z_{\mathbb{A}+1}}\right)^{1/3}, \tag{6.7}$$

in which

$$z_{\mathbb{A}} = \frac{n_{\mathbb{A}+1}}{n_{\mathbb{A}+2}} = \frac{\alpha_{\mathbb{A}+1}}{\alpha_{\mathbb{A}+2}}. \tag{6.8}$$

It follows that

$$n_{\mathbb{A}} \frac{\partial}{\partial n_{\mathbb{A}}} = n \frac{\partial}{\partial n} + z_{\mathbb{A}+2} \frac{\partial}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1} \frac{\partial}{\partial z_{\mathbb{A}+1}}, \tag{6.9a}$$

and furthermore that

$$\eta_{c\langle a} \frac{\partial}{\partial \eta^{b\rangle}{}_c} = \frac{1}{2} \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}} \left(z_{\mathbb{A}+2} \frac{\partial}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1} \frac{\partial}{\partial z_{\mathbb{A}+1}} \right). \tag{6.9b}$$

The principle pressures are given by the sum

$$p_{\mathbb{A}} = p + \pi_{\mathbb{A}}, \tag{6.10a}$$

where the eigenvalues $\pi_{\mathbb{A}}$ are

$$\pi_{\mathbb{A}} = n \left(z_{\mathbb{A}+2} \frac{\partial \epsilon}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1} \frac{\partial \epsilon}{\partial z_{\mathbb{A}+1}} \right), \tag{6.10b}$$

which satisfy

$$\sum_{\mathbb{A}=1}^{3} \pi_{\mathbb{A}} = 0. \tag{6.11}$$

In terms of the $\pi_{\mathbb{A}}$, the anisotropic pressure tensor is

$$\pi_{ab} = \sum_{\mathbb{A}=1}^{3} \pi_{\mathbb{A}} e_{a\mathbb{A}} e_{b\mathbb{A}}. \tag{6.12}$$

6.2 Static spherically symmetric configurations

We continue to review [6, 12, 44], and use the technology outlined in the previous section to construct the relevant equations to describe static spherically symmetric configurations.

The metric for static spherically symmetric space-time decomposes as

$$g_{ab} = -u_a u_b + \gamma_{ab}, \tag{6.13a}$$

where the orthogonal metric γ_{ab} splits up into a "radial" vector and "angular" metric via

$$\gamma_{ab} = r_a r_b + t_{ab}. \tag{6.13b}$$

The velocity vector u_a, radial vector r_a, and totally orthogonal metric t_{ab} are given by

$$u_a = -e^{\nu(r)} \left(\mathrm{d}t\right)_a, \qquad r_a = e^{\lambda(r)} \left(\mathrm{d}r\right)_a, \qquad t_{ab} = r^2 \left(\mathrm{d}\Omega^2\right)_{ab} \tag{6.13c}$$

and $\lambda(r)$ is specified by the "Schwarzschild mass" function $m(r)$ via

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r}. \tag{6.13d}$$

The gravitational field equations set the Einstein tensor equal to the usual form of the solid energy-momentum tensor

$$T_{ab} = \rho u_a u_b + P_{ab}, \tag{6.14a}$$

in which the only pressure tensor compatible with the spherical symmetry decomposes as

$$P_{ab} = p_{\mathrm{r}} r_a r_b + p_{\mathrm{t}} t_{ab}. \tag{6.14b}$$

One should interpret p_{r} as the radial pressure, and p_{t} as the tangential pressure.

In all generality (i.e., for any static spherically symmetric configuration) the Einstein equations $G^a{}_b = \kappa T^a{}_b$ for the metric (6.13) and energy-momentum tensor (6.14) are given by

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\left(r - 2m\right)}, \tag{6.15a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2}\kappa r^2 \rho, \tag{6.15b}$$

$$\frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}r} = -\left(\rho + p_{\mathrm{r}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\left(r - 2m\right)} + 6\frac{q}{r}, \tag{6.15c}$$

where the difference between the radial and tangential pressures is quantified via

$$q \equiv \tfrac{1}{3}\left(p_{\mathrm{t}} - p_{\mathrm{r}}\right). \tag{6.16}$$

Since the metric variable ν does not appear in (6.15b) or (6.15c), we do not need to consider it in what follows, if all we are interested in is the profiles of the elastic matter fields.

The crucial part of making the source to the field equations that due to an elastic solid, is to compute ρ, p_{r}, and p_{t} from the equation of state. For that we must decompose the material metric, and pull it back to space-time. Symmetry dictates that the equation of state ϵ has only two arguments,

$$\epsilon = \epsilon(n_{\mathrm{r}}, n_{\mathrm{t}}), \tag{6.17}$$

and the particle number density n is given by

$$n = n_{\mathrm{r}} n_{\mathrm{t}}^2. \tag{6.18}$$

Note that n_{r} and n_{t} are interpretable as the radial and tangential linear number densities. The material metric must have symmetries similar to the space-time metric to preserve isotropy; in the material space the material metric takes on the form

$$k_{AB} = \tilde{r}_A \tilde{r}_B + \tilde{t}_{AB}. \tag{6.19}$$

The basis vector and tangential tensor in the material manifold are given by

$$\tilde{r}_A = e^{\tilde{\lambda}}\left(\mathrm{d}\tilde{r}\right)_A, \qquad \tilde{t}_{AB} = \tilde{r}^2\left(\mathrm{d}\tilde{\Omega}^2\right)_{AB}. \tag{6.20}$$

The pulled back material metric is given by

$$k_{ab} = n_{\mathrm{r}}^2 r_a r_b + n_{\mathrm{t}}^2 t_{ab}, \tag{6.21}$$

where the two space-time components of the pulled-back material metric (6.21) are given by

$$n_{\mathrm{r}} = e^{\tilde{\lambda}-\lambda}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}, \qquad n_{\mathrm{t}} = \frac{\tilde{r}}{r}. \tag{6.22}$$

The mapping between the material and space-time manifolds is thus defined through the relationship between the radial coordinate in the material manifold, and that in space-time $\tilde{r} = \tilde{r}(r)$: this is entirely encapsulated by the two space-time functions $n_{\rm r}$ and $n_{\rm t}$. The constant volume shear tensor, defined in (3.1) is given by

$$s_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - n^{-2/3}k_{ab}\right), \tag{6.23}$$

where we used (2.39) to replace the uni-modular tensor η_{ab} with n and k_{ab}. Using (6.13b), (6.18), and (6.21) we thus obtain

$$s_{ab} = \tfrac{1}{2}\left(\left[1 - \left(\frac{n_{\rm r}}{n_{\rm t}}\right)^{4/3}\right] r_a r_b + \left[1 - \left(\frac{n_{\rm t}}{n_{\rm r}}\right)^{4/3}\right] t_{ab}\right). \tag{6.24}$$

Hence, we observe that $s_{ab} = 0$ when, and only when, the radial and tangential number densities are identical:

$$s_{ab} = 0 \qquad \Longleftrightarrow \qquad n_{\rm r} = n_{\rm t}. \tag{6.25}$$

Thus, $s_{ab} = 0$ when the solid "becomes" a fluid.

Rather than work with n_r and n_t, it is useful to work with the combinations

$$n \equiv n_{\rm r} n_{\rm t}^2 = \left(\frac{\tilde{r}}{r}\right)^3 z, \tag{6.26a}$$

$$z \equiv \frac{n_{\rm r}}{n_{\rm t}} = e^{\tilde{\lambda}-\lambda}\frac{r}{\tilde{r}}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}. \tag{6.26b}$$

One should keep in mind that n is the particle number density. Also, note that $z = 1$ when $n_{\rm r} = n_{\rm t}$: this is the fluid limit. In terms of (n, z) as defined in (6.26) the field equations (6.15b, 6.15c) can be written as

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2}\kappa r^2 \rho, \tag{6.27a}$$

$$\frac{\mathrm{d}n}{\mathrm{d}r} = \frac{n}{r\beta_{\rm r}}\left[-(\rho + p_{\rm r})\frac{m + \frac{1}{2}\kappa r^3 p_{\rm r}}{r - 2m} + 6q + 3z\frac{\partial p_{\rm r}}{\partial z}\left(z e^{\lambda-\tilde{\lambda}} - 1\right)\right], \tag{6.27b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = z\left[\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r} - \frac{3}{r}\left(z e^{\lambda-\tilde{\lambda}} - 1\right)\right], \tag{6.27c}$$

in which $\tilde{\lambda}$ parameterizes the curvature of the material manifold (which can be set to zero), and where $\{\rho, p_{\rm r}, q, \beta_{\rm r}\}$ are given in terms of the two-parameter equation of state $\epsilon = \epsilon(n, z)$ via

$$\rho = n\epsilon, \qquad p_{\rm r} = n^2\frac{\partial \epsilon}{\partial n} - 2q, \tag{6.28a}$$

$$q = -\frac{1}{2} n z \frac{\partial \epsilon}{\partial z}, \qquad \beta_{\mathrm{r}} = n \frac{\partial p_{\mathrm{r}}}{\partial n} + z \frac{\partial p_{\mathrm{r}}}{\partial z}. \tag{6.28b}$$

Explicitly,

$$\beta_{\mathrm{r}} = n \left(2n \frac{\partial \epsilon}{\partial n} + n^2 \frac{\partial^2 \epsilon}{\partial n^2} + 2z \frac{\partial \epsilon}{\partial z} + 2nz \frac{\partial^2 \epsilon}{\partial n \partial z} + z^2 \frac{\partial^2 \epsilon}{\partial z^2} \right). \tag{6.29}$$

By way of a "fiducial" example, consider the quasi-Hookean ansatz for the equation of state,

$$\epsilon = \check{\epsilon} + \frac{1}{n} \check{\mu} s^2, \tag{6.30}$$

where a quantity with an overhead "check" symbol denotes that the quantity only depends on n, and

$$s^2 = \tfrac{1}{6} \left(z^{-1} - z \right)^2 \tag{6.31}$$

is the shear scalar we defined in (3.35), although there we used the symbol $\overline{s}^2$, and we have now expressed it in terms of z as defined in (6.26b). Note that in the fluid limit $z = 1$, the shear scalar $s^2 = 0$. The Einstein equations (6.27) can be formulated in the independant variables $(m, \check{p}, z)$, and become

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2} \kappa r^2 \rho, \tag{6.32a}$$

$$\frac{\mathrm{d}\check{p}}{\mathrm{d}r} = \frac{\check{\beta}}{r \beta_{\mathrm{r}}} \left[-\left(\rho + p_{\mathrm{r}} \right) \frac{m + \frac{1}{2} \kappa r^3 p_{\mathrm{r}}}{r - 2m} + 6q + 4 \left(z e^{\lambda - \tilde{\lambda}} - 1 \right) \left(\check{\mu} + 3\sigma + \tfrac{3}{2} \left(1 - \check{\Omega} \right) q \right) \right], \tag{6.32b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{z}{r} \left[\frac{r}{\check{\beta}} \frac{\mathrm{d}\check{p}}{\mathrm{d}r} - 3 \left(z e^{\lambda - \tilde{\lambda}} - 1 \right) \right], \tag{6.32c}$$

in which

$$\sigma = \check{\mu} s^2, \qquad q = \check{\mu} \chi, \qquad \rho = \check{\rho} + \sigma, \tag{6.33a}$$

$$p_{\mathrm{r}} = p - 2q, \qquad p = \check{p} + \left(\check{\Omega} - 1 \right) \sigma, \tag{6.33b}$$

$$\chi = \tfrac{1}{6} \left(z^{-2} - z^2 \right), \qquad \check{\beta} = \left(\check{\rho} + \check{p} \right) \frac{\mathrm{d}\check{p}}{\mathrm{d}\check{\rho}}, \qquad \check{\Omega} = \frac{\check{\beta}}{\check{\mu}} \frac{\mathrm{d}\check{\mu}}{\mathrm{d}\check{p}}, \tag{6.33c}$$

$$\beta_{\mathrm{r}} = \beta + 4 \left[\sigma + \left(\check{\Omega} - \frac{1}{2} \right) \right], \qquad \beta = \check{\beta} + \frac{4}{3} \check{\mu} + \left[\check{\Omega} \left(\check{\Omega} - 1 \right) + \check{\beta} \frac{\mathrm{d}\check{\Omega}}{\mathrm{d}\check{p}} \right]. \tag{6.33d}$$

One now requires two functions of state to complete the system of equations: $\check{\rho}\,(\check{p})$ and $\check{\mu}\,(\check{p})$. One may be slightly uncomfortable with this idea: if that is the case, then one should recall that in the more familiar "scalar field models" one needs to pick parameterizations of the Lagrangian density (even the canonical theory has one free function, the potential $V(\phi)$). An explicit relationship between $\check{\rho}$, $\check{\mu}$ and $\check{p}$ is provided by

$$\check{\rho} = \frac{p_{\mathrm{c}}}{\check{\Gamma}-1}\left[w\frac{\check{p}}{p_{\mathrm{c}}} + \left(\frac{\check{p}}{p_{\mathrm{c}}}\right)^{1/\check{\Gamma}}\right], \qquad \check{\mu} = k\check{p}. \tag{6.34}$$

This is a modified polytropic equation of state. The parameter p_{c} indicates the pressure scale at which the transition between the linear and polytropic behavior occurs. The parameter k quantifies the rigidity, and vanishes in the fluid limit.

6.2.1 My spherical symmetry calculations

Here we use the metric

$$\mathrm{d}s^2 = -e^{2\nu(r)}\mathrm{d}t^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{6.35}$$

and the total energy-momentum tensor

$$T^a{}_b = \mathrm{diag}\left(-\rho(r), p_{\mathrm{r}}(r), p_{\mathrm{t}}(r), p_{\mathrm{t}}(r)\right). \tag{6.36}$$

The components are: energy density ρ, radial pressure p_{r}, and tangential pressure p_{t}. One should bear in mind that these components could be formed from multiple constituents (although we will come back to that case later on).

The non-zero components of the Einstein tensor computed from the metric (6.35) are given by

$$G^0{}_0 = -\frac{2m'}{r^2}, \tag{6.37a}$$

$$G^r{}_r = -\frac{2}{r^3}\left(m + r\left(2m - r\right)\nu'\right) \tag{6.37b}$$

$$\begin{aligned} G^\theta{}_\theta = G^\phi{}_\phi &= \frac{1}{r^3}\Bigg(- m\left(-1 + r\nu' + 2r^2\nu'^2 + 2r^2\nu''\right) \\ &\quad + r\left(-m'\left(1 + r\nu'\right) + r\left(\nu' + r\nu'^2 + r\nu''\right)\right)\Bigg). \end{aligned} \tag{6.37c}$$

The 00- and rr-Einstein field equations $G^a{}_b = \kappa T^a{}_b$ thus read

$$m' = \tfrac{1}{2}\kappa r^2\rho, \tag{6.38a}$$

$$\nu' = \frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\,(r-2m)}. \tag{6.38b}$$

The only non-zero entry of the conservation equation $\nabla_a T^a{}_b = 0$ is

$$p_{\mathrm{r}}' = -\left(\rho + p_{\mathrm{r}}\right) \frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\,(r-2m)} + \frac{2}{r}\left(p_{\mathrm{t}} - p_{\mathrm{r}}\right). \tag{6.38c}$$

The system of three equations (6.38) constitute the *generalized Tolman-Oppenheimer-Volkov* equations, where a star is constructed from an anisotropic fluid. Notice that there are three equations for five unknowns (i.e., $m(r), \nu(r), \rho(r), p_{\mathrm{r}}(t)$, and $p_{\mathrm{t}}(r)$). And so, two "equations of state" must be given: these are typically taken to relate the pressures to the density. Note that the metric variable ν does not appear in (6.38a) or (6.38c): it does not need to be solved for at the same time as m and p_{r} (this is statement which helps computations).

There are some simple circumstances under which the generalized TOV equations become analytically soluble [45]. When the density is taken to be constant throughout the star, $\rho = \rho_0$, it is simple to integrate (6.38a) to obtain

$$m(r) = \frac{4\pi G}{3}\rho_0 r^3. \tag{6.39}$$

We now assume an equation of state (which is the final piece of information required to close the system of equations) in the form

$$p_{\mathrm{t}} = p_{\mathrm{r}} + A f(p_{\mathrm{r}}, r)\left(\rho + p_{\mathrm{r}}\right) r^n, \tag{6.40a}$$

with the function f given by

$$f(p_{\mathrm{r}}, r) = \frac{\rho + 3p_{\mathrm{r}}}{1 - 2m/r}, \tag{6.40b}$$

and where there are two free parameters, A and n. Using this, (6.38c) can be written as

$$p_{\mathrm{r}}' = -\left(\rho_0 + p_{\mathrm{r}}\right)\left(\rho_0 + 3p_{\mathrm{r}}\right) \frac{\frac{4\pi G}{3} - 2Ar^{n-2}}{1 - \frac{8\pi G\rho_0}{3}r^2} r. \tag{6.41}$$

When $n = 2$, this can be integrated to give

$$p_{\mathrm{r}}(r) = \rho_0 \left[\frac{(1-2m/r)^Q - (1-2M/R)^Q}{3\,(1-2M/R)^Q - (1-2m/r)^Q}\right] \tag{6.42}$$

with

$$p_{\mathrm{r}}(r = R) = 0, \qquad m(R) = M, \qquad Q = \frac{1}{2} - \frac{3A}{4\pi G}. \tag{6.43}$$

The pressure at the centre of the configuration is given by

$$p_{\rm c} = \rho_0 \frac{1 - (1 - 2M/R)^Q}{3\,(1 - 2M/R)^Q - 1}. \tag{6.44}$$

The critical combination of M and R for which the central pressure becomes infinite is found by noting when the denominator of $p_{\rm c}$ becomes zero; i.e., when

$$(2M/R)_{\rm crit} = 1 - \left(\frac{1}{3}\right)^{1/Q}. \tag{6.45}$$

One requires $(2M/R)_{\rm crit} \geq 0$ for physical systems. This is known as *Buchdahl's inequality* [46, 47] (although it also exists for more general setups than that which we are considering here). In the perfect-fluid case there is no anisotropy, $A = 0$, and so $Q = 1/2$, and so

$$(2M/R)_{\rm crit} = 8/9. \tag{6.46}$$

When $A = 2\pi G/3$, one has $Q = 0$ and thus

$$(2M/R)_{\rm crit} = 1, \tag{6.47}$$

which is the Schwarzschild value.

Note that

$$\sqrt{-g} = \left(1 - \frac{2m}{r}\right)^{-1/2} e^{\nu} r^2 \sin\theta. \tag{6.48}$$

The total mass in a volume is given by

$$M \equiv \int \mathrm{d}^3x\, \sqrt{-g}\, m(r) \tag{6.49}$$

6.3 Stars immersed in an elastic solid

It should be clear from our presentation so far that an elastic solid is an anisotropic fluid. In [45, 48–51] anisotropic stars are studied. See also [52, 53], which study stars (i.e., solutions to the TOV equations) immersed in Chaplygin-gas; also, [54] study stars in a Chameleon scalar-field background. Also, [55] look at stars in anisotropic fluids, but in the context of wormholes. In non of these papers is any mention made of the link between anisotropy and elasticity.

We would like to understand

1. Stars in vacuum

2. Stars in fluids

3. Stars in solids

Point 1. is about solving the Tolman-Oppenheimer-Volkov equations (usually studied with a model neutron-star equation of state). The second is about solving the TOV equations, but in a non-empty space-time.

6.3.1 Example model

A simple example of a model of a star immersed in a "cosmological" background is

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_{\text{star}} + \mathcal{L}_{\text{cosm}} \right], \tag{6.50}$$

wherein R is the Ricci scalar, $\mathcal{L}_{\text{star}}$ is the Lagrangian density of the matter which gives rise to the "star", and $\mathcal{L}_{\text{cosm}}$ is the Lagrangian density of the "cosmological" dark energy. The field equations are given by

$$G^a{}_b = \kappa \left(T^a{}_b + U^a{}_b \right), \tag{6.51}$$

in which $T^a{}_b$ is the energy-momentum tensor for the "star Lagrangian", and $U^a{}_b$ is the energy-momentum tensor for the cosmological dark energy (to be in-keeping with some of our other work, we call $U^a{}_b$ the dark energy-momentum tensor). If the model is minimally coupled then the two energy-momentum tensors are separately conserved; else, they have a coupling current which transfers energy between the cosmological and stellar matter fields.

Typical examples of the dark energy Lagrangians are

$$\mathcal{L}_{\text{cosm}} \in \begin{cases} \text{scalar field}, \\ \text{solid}, \\ \text{perfect fluid}. \end{cases} \tag{6.52}$$

It is to be remarked that the perfect fluid is a particular limit of the solid (a solid with vanishing rigidity is a perfect fluid). In each case, the components of the dark energy-momentum tensor are given respectively by

$$U^a{}_b \in \begin{cases} \partial^a\phi\partial_b\phi + g^a{}_b \left(\frac{1}{2}\partial_c\phi\partial^c\phi - V(\phi) \right), \\ \rho u^a u_b + p\gamma^a{}_b + \pi^a{}_b, \\ \rho u^a u_b + p\gamma^a{}_b. \end{cases} \tag{6.53}$$

The total source to the gravitional field equations can be written as

$$T^a_{\text{tot}\,b} = T^a{}_b + U^a{}_b. \tag{6.54}$$

This will engender the following decomposition for the total energy density ρ^{tot}, the radial pressure $p_{\text{r}}^{\text{tot}}$, and tangential pressure $p_{\text{t}}^{\text{tot}}$ components,

$$\rho^{\text{tot}} = \rho^{\text{m}} + \rho^{\text{d}}, \tag{6.55a}$$

$$p_{\text{r}}^{\text{tot}} = p_{\text{r}}^{\text{m}} + p_{\text{r}}^{\text{d}} \tag{6.55b}$$

$$p_{\text{t}}^{\text{tot}} = p_{\text{t}}^{\text{m}} + p_{\text{t}}^{\text{d}} \tag{6.55c}$$

Components with an "m" super-script label correspond to the component of the stellar energy-momentum tensor, and those with a "d" super-script to the components of the dark energy-momentum tensor. This will mean that the field equations (in the minimally coupled case) are given by an appropriately modified version of (6.38). To be specific,

$$\nu' = \frac{m + \frac{1}{2}\kappa r^3 p_{\text{r}}^{\text{tot}}}{r\,(r - 2m)}, \tag{6.56a}$$

$$m' = \frac{1}{2}\kappa r^2 \rho^{\text{tot}}, \tag{6.56b}$$

$$p_{\text{r}}^{\text{i}\,\prime} = -\left(\rho^{\text{i}} + p_{\text{r}}^{\text{i}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\text{r}}^{\text{tot}}}{r\,(r - 2m)} + \frac{2}{r}\left(p_{\text{t}}^{\text{i}} - p_{\text{r}}^{\text{i}}\right), \tag{6.56c}$$

where there is a copy of (6.56c) for each $\text{i} \in \{\text{m}, \text{d}\}$. For a perfect fluid model of the star, the matter components satisfy

$$p_{\text{r}}^{\text{m}} = p_{\text{t}}^{\text{m}}, \qquad p_{\text{r}}^{\text{m}}(R_0) = 0, \qquad \rho^{\text{m}}(r > R_0) = 0. \tag{6.57}$$

That is, the radial and tangential pressures are identical, the pressure vanishes at the radius R_0, which is supposed to be the surface of the star, and the stellar density vanishes outside of the star.

Integrating (6.56b) from $r = 0$ to $r = R_\infty$ yields a total mass M_∞, which can be broken up into three contributions as

$$M_\infty = M_\star^{\text{m}} + M_\star^{\text{d}} + M^{\text{d}}, \tag{6.58}$$

where

$$M_\star^{\text{m}} \equiv \frac{1}{2}\kappa \int_0^{R_0} \mathrm{d}r\, r^2\, \rho^{\text{m}}, \tag{6.59a}$$

$$M_\star^{\mathrm{d}} \equiv \frac{1}{2}\kappa \int_0^{R_0} \mathrm{d}r\, r^2\, \rho^{\mathrm{d}}, \tag{6.59b}$$

$$M^{\mathrm{d}} \equiv \frac{1}{2}\kappa \int_{R_0}^{R_\infty} \mathrm{d}r\, r^2\, \rho^{\mathrm{d}} \tag{6.59c}$$

These can be interpreted as the mass of the star, the mass of the dark substance inside the star, and the mass of the dark substance outside of the star (i.e., in the cosmology).

7 Mixing solids, fluids, and scalar fields

We may also be interested in composite descriptions where there are solids interacting with scalar fields or fluids.

7.1 Solids and fluids

We are interested in constructing the action for a fluid coupled to a solid. It is useful to note that in Alkistis et al [13] consider a fluid coupled to a scalar field.

See [56], [57], [58], [59].

Suppose one had two fluids; each fluid have currents a^μ and b^μ. The allowed invariants are

$$a \equiv (-a^\mu a_\mu)^{1/2}, \qquad x \equiv (-a^\mu b_\mu)^{1/2}, \qquad b = (-b^\mu b_\mu)^{1/2}. \tag{7.1}$$

7.2 Solids and scalar fields

The theory of a solid was constructed from all invariants of the elements of the set $\{n, u^a, \eta^a{}_b\}$. Note that in [13] a similar question was asked, but for a fluid; thus, they formed all invariants out of $\{n, u^a, \phi, \nabla_a\phi\}$. Thus, the Lagrangian for a scalar+fluid mixture has at most four scalar arguments, and can be written as

$$\mathcal{L}_{\rm s+f} = \mathcal{L}_{\rm s+f}\left(n, \phi, u^a\nabla_a\phi, \nabla_a\phi\nabla^a\phi\right). \tag{7.2}$$

Suppose we now had a scalar field in the matter action: we now want to form all invariants from the set $\{n, u^a, \eta^a{}_b, \phi, \nabla_a\phi\}$. See Section 7.3 for a discussion on hyper-elasticity which may be more helpful.

It seems that for the solid+scalar mixture, the invariants are

$$\left\{n, \quad [\boldsymbol{\eta}], \quad [\boldsymbol{\eta}^2], \quad u^a\nabla_a\phi, \quad \nabla^a\phi\nabla_a\phi, \quad \eta^a{}_b\nabla_a\phi\nabla^b\phi, \quad \eta^a{}_b\eta^b{}_c\nabla_a\phi\nabla^c\phi, \qquad \dots\right\} \tag{7.3}$$

Define

$$C^a{}_b{}^c \equiv \eta^a{}_b\nabla^c\phi. \tag{7.4}$$

Then we can form

$$\mathcal{I} = \left\{C^a{}_a{}^c\nabla_c\phi, \qquad \dots\right\}. \tag{7.5}$$

We cannot (or at least, we are not aware of how to) construct all invariants out of these fields. Infact, it does not seem that this construction is the most elegant one can envisage – for that one should turn to our review of hyper-elasticity, which we save for the next subsection. However, we can still obtain field equations. Let us begin from a statement about the field content of the scalar+solid theory

$$\mathcal{L} = \mathcal{L}(g_{ab}, n^a, \eta^a{}_b, \phi, \partial_a\phi). \tag{7.6}$$

In a cosmological context, one can isolate the projections of the relevant quantities, at the level of the background:

$$g_{ab} = \gamma_{ab} - u_a u_b, \qquad \partial_a\phi = -\dot{\phi}u_a, \qquad n^a = nu^a, \qquad \eta^a{}_b = \gamma^a{}_c\gamma_b{}^d\eta^c{}_d. \tag{7.7a}$$

Note that by orthogonality

$$\eta^a{}_b\partial_a\phi = n^a\eta^b{}_a = 0. \tag{7.7b}$$

Also, on the background, the uni-modular tensor $\eta^a{}_b$ is strictly isotropic, and so completely decomposes as

$$\eta^a{}_b = \omega\gamma^a{}_b. \tag{7.7c}$$

Thus,

$$[\boldsymbol{\eta}] = 3\omega, \qquad [\boldsymbol{\eta}^2] = 3\omega^2. \tag{7.7d}$$

By combining (7.7) we find that on the cosmological background there are only 5 scalar invariants, and so the Lagrangian for the background is a function with 5 arguments:

$$\mathcal{L} = \mathcal{L}\left(\phi, n, \omega, u^a\partial_a\phi, \partial^a\phi\partial_a\phi\right). \tag{7.8}$$

The variation in the function $\mathcal{L}$ is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial g_{ab}}\delta_{\mathrm{L}}g_{ab} + \frac{\partial\mathcal{L}}{\partial n^a}\delta_{\mathrm{L}}n^a + \frac{\partial\mathcal{L}}{\partial\eta^a{}_b}\delta_{\mathrm{L}}\eta^a{}_b + \frac{\partial\mathcal{L}}{\partial\phi}\delta_{\mathrm{L}}\phi + \frac{\partial\mathcal{L}}{\partial\partial_a\phi}\partial_a\delta_{\mathrm{L}}\phi. \tag{7.9}$$

Since

$$\delta_{\mathrm{L}}n^a = -\tfrac{1}{2}n^a g^{cd}\delta_{\mathrm{L}}g_{cd}, \tag{7.10}$$

one can find

$$\delta_{\mathrm{E}}n^a = -\tfrac{1}{2}n^a g^{cd}\delta_{\mathrm{E}}g_{cd} + n^b\nabla_b\xi^a - \nabla_b\left(n^a\xi^b\right) \tag{7.11}$$

Note that

$$\mathcal{L} = \mathcal{L}(\phi, \partial_a\phi, k^a{}_b) \tag{7.12}$$

7.3 Hyper-elasticity as a road to mixing scalars and solids

The idea of hyper-elasticity [18] may give an elegant insight as to how to incorporate general coupling between a solid and a scalar field.

7.3.1 Hyper-elasticity

Suppose the theory can be constructed from a Lagrangian density $\mathcal{L}$ formed as a function of $(p+1)$ scalar fields and their gradients,

$$\mathcal{L} = \mathcal{L}\left(\phi^0, \phi^1, \ldots, \phi^p, \partial_a \phi^0, \partial_a \phi^1, \ldots, \partial_a \phi^p\right) \tag{7.13}$$

where the derivatives are with respect to worldsheet coordinates $\overline{x}^a$, with $a = 0, \ldots, p$. The background has a metric $g_{\mu\nu}$ with $\mu = 0, \ldots, p+q$. Note that the worldsheet has co-dimension q. The background induces a metric on the worldsheet

$$\overline{g}_{ab} = g_{\mu\nu} x^\mu{}_{,a} x^\nu{}_{,b}. \tag{7.14}$$

The determinant $\overline{g}$ of the worldsheet metric is used as the measure with which to integrate the Lagrangian density to give the action:

$$S = \int \mathrm{d}^{p+1}\overline{x} \sqrt{|\overline{g}|}\, \mathcal{L}. \tag{7.15}$$

The worldsheet metric must have Lorentzian signature, and the field gradients must be independant. This means that the fields can be used as a set of coordinates on the worldsheet,

$$\overline{x}^a = \phi^a, \tag{7.16}$$

and thus the Lagrangian density $\mathcal{L}$ will depend on just the undifferentiated values ϕ^a and on the set of induced metric components $\overline{g}_{ab}$ (there will be $\frac{1}{2}p(p+1)$ of these).

The worldsheet energy-momentum tensor will always be given by

$$T^{ab} = \frac{2}{\sqrt{-\overline{g}}} \frac{\partial \sqrt{-\overline{g}}\, \mathcal{L}}{\partial \overline{g}_{ab}}, \tag{7.17}$$

and the corresponding hyper-elasticity tensor on the world-sheet is

$$\mathfrak{C}^{abcd} = \frac{1}{\sqrt{-\overline{g}}} \frac{\partial \sqrt{-\overline{g}}\, T^{ab}}{\partial \overline{g}_{cd}}. \tag{7.18}$$

We should note that the hyper-elasticity tensor is related to the elasticity tensor E^{abcd}, but it is not exactly the same. There is a minor, and one major difference: first, $\mathfrak{C}^{abcd} \sim -2E^{abcd}$. Secondly, the elasticity tensor is purely orthogonal (i.e., purely spatial in the sense that $u_a E^{abcd} = 0$), whereas the hyper-elasticity tensor is not. These worldsheet tensors can be pulled back to give background space-time tensors via

$$T^{\mu\nu} = T^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b} = 2\frac{\partial \mathcal{L}}{\partial \overline{g}_{\mu\nu}} + \mathcal{L}\overline{g}^{\mu\nu}, \tag{7.19a}$$

$$\mathfrak{C}^{\mu\nu\alpha\beta} = \mathfrak{C}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\alpha}{}_{,c} x^{\beta}{}_{,d} = \frac{\partial T^{\mu\nu}}{\partial \overline{g}_{\alpha\beta}} + \tfrac{1}{2} T^{\mu\nu}\overline{g}^{\alpha\beta}. \tag{7.19b}$$

The first fundamental tensor of the worldsheet, $\overline{g}^{\mu\nu}$, is constructed from the induced metric $\overline{g}^{ab}$ via

$$\overline{g}^{\mu\nu} = \overline{g}^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b}. \tag{7.20}$$

When the codimension $q = 0$, $\overline{g}^{\mu\nu} = g^{\mu\nu}$, and $\overline{g}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$. In general, $\overline{g}^{\mu}{}_{\nu}$ is a projector, giving rise to a worldsheet derivative operator

$$\overline{\nabla}_{\nu} = \overline{g}^{\mu}{}_{\nu} \nabla_{\mu}. \tag{7.21}$$

In addition to the energy-momentum, first fundamental, and hyper-elasticity tensors, there is another tensor which plays an important role in governing the dynamics: the hyper-Hadamard tensor, given by

$$\mathfrak{H}^{\mu\nu\rho\sigma} = g^{\mu\rho} T^{\nu\sigma} + 2\mathfrak{C}^{\mu\nu\rho\sigma}. \tag{7.22}$$

The standard decomposition of the metric into tangential and orthogonal parts

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \perp_{\mu\nu} \tag{7.23}$$

implies an associated decomposition of the hyper-Hadamard tensor into worldsheet tangential and orthogonal parts,

$$\mathfrak{H}^{\mu\nu\rho\sigma} = \overline{\mathfrak{H}}^{\mu\nu\rho\sigma} + {}^{\perp}\mathfrak{H}^{\mu\nu\rho\sigma}. \tag{7.24}$$

The worldsheet tangential part is

$$\overline{\mathfrak{H}}^{\mu\nu\rho\sigma} = \overline{\mathfrak{H}}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\rho}{}_{,c} x^{\sigma}{}_{,d}, \tag{7.25}$$

with the components of the worldsheet hyper-Hadamard tensor given by

$$\overline{\mathfrak{H}}^{abcd} = \overline{g}^{ac} T^{bd} + 2\mathfrak{C}^{abcd}. \tag{7.26}$$

The worldsheet orthogonal part is

$${}^{\perp}\mathfrak{H}^{\mu\nu\rho\sigma} = \perp^{\mu\rho} T^{\nu\sigma}. \tag{7.27}$$

The full space-time metric decomposes first into a piece which is confined to the world sheet, and one which is perpendicular; the confined piece then decomposes into the spatial part and time-like part in the usual fashion:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \perp_{\mu\nu}, \tag{7.28}$$

$$= \gamma_{\mu\nu} - u_\mu u_\nu + \perp_{\mu\nu} . \tag{7.29}$$

For the system to "be" classified as hyper-elastic, the system should include a sub-system of ordinary elastic type. That is, all scalar fields except one (ϕ^0, say) only have space-like gradients; i.e., their configuration gradients satisfy the orthogonality condition (2.3). What this means is that

$$u^a \phi^A{}_{,a} = 0, \qquad \text{with} \qquad A = 1, \ldots, p, \tag{7.30a}$$

with $\overline{g}_{ab} u^a u^b = -1$, but

$$u^a \phi^0{}_{,a} \neq 0. \tag{7.30b}$$

We will use this notation: the small letters "$a, b, \ldots$" denote arbitrary worldsheet coordinates, but the larger letters "$A, B, \ldots$" denote the coordinate system specified in terms of (7.16) by

$$\overline{x}^0 = \phi^0, \qquad \overline{x}^A = \phi^A. \tag{7.31}$$

Using (7.31) it is apparent that there are three "types" of components of the induced metric (7.14):

$$\overline{g}^{00} = \overline{g}^{ab} \phi^0{}_{,a} \phi^0{}_{,b}, \tag{7.32a}$$

$$\overline{g}^{A0} = \overline{g}^{ab} \phi^A{}_{,a} \phi^0{}_{,b}, \tag{7.32b}$$

$$\overline{g}^{AB} = \overline{g}^{ab} \phi^A{}_{,a} \phi^B{}_{,b}. \tag{7.32c}$$

Setting

$$\mu_a = \phi^0{}_{,a}, \qquad \psi^A{}_a = \phi^A{}_{,a}, \tag{7.33}$$

the components (7.32) become

$$\overline{g}^{00} = \overline{g}^{ab} \mu_a \mu_b, \tag{7.34a}$$

$$\overline{g}^{A0} = \overline{g}^{ab} \psi^A{}_a \mu_b, \tag{7.34b}$$

$$\overline{g}^{AB} = \overline{g}^{ab} \psi^A{}_a \psi^B{}_b. \tag{7.34c}$$

These are the components of the induced metric on which the Lagrangian $\mathcal{L}$ can depend. The Lagrangian $\mathcal{L}$ will be a function of

$$\mathcal{L} = \mathcal{L}\left(\overline{g}^{00}, \overline{g}^{A0}, \overline{g}^{AB}\right). \tag{7.35}$$

Since $\overline{g}^{AB}$ are the components of a $p \times p$ matrix there are p invariants that can be formed from $\overline{g}^{AB}$ alone. There is one invariant that can be formed from $\overline{g}^{A0}$ alone, and $\overline{g}^{00}$ is already an invariant. This is just something to bear in mind before we explicitly give the catalogue of invariants.

By (7.30a), it follows that

$$\overline{g}^{AB} = \gamma^{ab}\psi^{A}{}_{a}\psi^{B}{}_{b} = \gamma^{AB}, \tag{7.36}$$

after writing the worldsheet metric in the usual way,

$$\overline{g}^{ab} = \gamma^{ab} - u^a u^b. \tag{7.37}$$

Since the Lagrangian only depends on the field gradients $\phi^{a}{}_{,b}$ via the induced metric components, the generic variation in the Lagrangian density (7.13) will be given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial\overline{g}^{ab}}\delta\overline{g}^{ab}. \tag{7.38}$$

After separating out the coordinates (7.31), this variation becomes

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^0}\delta\phi^0 + \frac{\partial\mathcal{L}}{\partial\phi^A}\delta\phi^A + L_{00}\delta\overline{g}^{00} + 2L_{A0}\delta\overline{g}^{A0} + L_{AB}\delta\overline{g}^{AB}, \tag{7.39}$$

in which we defined the partial derivatives of $\mathcal{L}$ with respect to $\overline{g}^{ab}$ as

$$L_{ab} = \frac{\partial\mathcal{L}}{\partial\overline{g}^{ab}}. \tag{7.40a}$$

The contravariant components of L^{ab} are given by

$$L^{ab} = -\frac{\partial\mathcal{L}}{\partial\overline{g}_{ab}}. \tag{7.40b}$$

The tensor L_{ab} is a worldsheet tensor.

These are the required ingredients for computing the energy-momentum tensor (7.17). One finds

$$T^{ab} = -2L^{ab} + \overline{g}^{ab}\mathcal{L}. \tag{7.41}$$

One should note that the energy-momentum tensor (7.41) is not of the form of a perfect solid which we gave in (3.24). Writing the energy-momentum tensor in terms of the energy density ρ, heat flux q^a, and pressure tensor p^{ab} without loss of generality,

$$T^{ab} = \rho u^a u^b + 2q^{(a}u^{b)} + p^{ab} \tag{7.42}$$

one obtains

$$\rho = -\left(\mathcal{L} + 2u_a u_b L^{ab}\right), \tag{7.43a}$$

$$q^a = \gamma^a{}_b u_c L^{bc}, \tag{7.43b}$$

$$p^{ab} = \gamma^{ab}\mathcal{L} - 2\gamma^a{}_c \gamma^b{}_d L^{cd}. \tag{7.43c}$$

The existence of the L_{A0} term in (7.39) gives rise to the heat flux contribution (7.43b). Later on we will give an example where the Lagrangian density separates into "scalar" and "solid" terms with no interaction between the two: this will switch off the heat flux. The L_{00} term also gives rise to the extra contribution to the energy density (7.43a) compared to the equivalent expression for a perfect solid which we gave in (5.14a).

To be able to compute the hyper-elasticity tensor (7.18) one also needs

$$L_{abcd} = \frac{\partial L_{ab}}{\partial \overline{g}^{cd}} = \frac{\partial^2 \mathcal{L}}{\partial \overline{g}^{ab} \partial \overline{g}^{cd}}, \tag{7.44}$$

which gives

$$\frac{\partial^2 \mathcal{L}}{\partial \overline{g}_{ab} \partial \overline{g}_{cd}} = L^{abcd} + L^{a(c}\overline{g}^{d)b} + \overline{g}^{a(c}L^{d)b}. \tag{7.45}$$

Using these ingredients, the hyper-elasticity tensor (7.18) is given by

$$\begin{aligned} \mathfrak{C}^{abcd} &= 2\left(L^{abcd} + L^{a(c}\overline{g}^{d)b} + \overline{g}^{a(c}L^{d)b}\right) \\ &\quad - \left(L^{ab}\overline{g}^{cd} + \overline{g}^{ab}L^{cd}\right) + \tfrac{1}{2}\mathcal{L}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right). \end{aligned} \tag{7.46}$$

Using (7.46) the hyper-Hadamard tensor (7.26) is given by

$$\overline{\mathfrak{H}}^{abcd} = 4L^{abcd} + 2L^{ac}\overline{g}^{bd} + 2L^{a[d}\overline{g}^{b]c} + 2\overline{g}^{a[d}L^{b]c} + L\overline{g}^{a[d}\overline{g}^{b]c}. \tag{7.47}$$

When we computed the generic variation of the Lagrangian density (7.38) we already used the fact that Lagrangian density only depends on the field gradients via the induced metric components. Reversing this gives the Eulerian variation of the Lagrangian, which is the variation with respect to fixed values of the background coordinates x^μ and metric $g_{\mu\nu}$. The variation is

$$\delta_{\mathrm{E}}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + J^a{}_b \delta\phi^b{}_{,a}, \tag{7.48}$$

where the currents are given in terms of the L_{ab}, defined in (7.40a), via

$$J^a{}_b = 2\overline{g}^{ac} L_{bd}\phi^b{}_{,c}. \tag{7.49}$$

The variational field equations are therefore

$$\overline{\nabla}_b J^b{}_a = \frac{\partial \mathcal{L}}{\partial \phi^a}. \tag{7.50}$$

If we again separate out the coordinates as in (7.31) then the currents (7.49) read

$$J^a{}_0 = 2\overline{g}^{ab}\left(L_{00}\phi^0{}_{,b} + L_{A0}\phi^A{}_{,b}\right), \tag{7.51a}$$

$$J^a{}_A = 2\overline{g}^{ab}\left(L_{A0}\phi^0{}_{,b} + L_{AB}\phi^B{}_{,b}\right). \tag{7.51b}$$

7.3.2 The symplectic current, and evaluation of sound speeds

Let $\delta x^\mu = \xi^\mu$ specify the background coordinate displacements, for whom conservation of the energy-momentum tensor is a consequence. Now suppose that $\delta x^\mu = \eta^\mu$ is another solution, for example that which results from a symmetry. Then the equation of motion of the generic perturbation vector ξ^μ is given by an expression

$$\overline{\nabla}_\mu \Omega^\mu = 0. \tag{7.52}$$

The symplectic current Ω^μ is given by

$$\Omega^\mu\left\{\xi,\eta\right\} = \eta^\mu \mathfrak{O}^\nu{}_\mu\left\{\eta\right\} - \xi^\mu \mathfrak{O}^\nu{}_\mu\left\{\eta\right\} \tag{7.53}$$

in which the "hyper-Hadamard operator" is defined via

$$\mathfrak{O}^\nu{}_\mu\left\{\xi\right\} = \mathfrak{H}_\mu{}^\nu{}_\rho{}^\sigma \overline{\nabla}_\sigma \xi^\rho, \tag{7.54}$$

and the hyper-Hadamard tensor is given in terms of the hyper-elasticity tensor via

$$\mathfrak{H}_\mu{}^\nu{}_\rho{}^\sigma = g_{\mu\rho}T^{\nu\sigma} + 2\mathfrak{C}_\mu{}^\nu{}_\rho{}^\sigma \tag{7.55}$$

Consider the worldsheet hypersurface which has normal vector

$$\lambda_a = \lambda_\mu x^\mu{}_{,a}, \qquad \lambda^\mu \perp^\mu{}_\nu = 0. \tag{7.56}$$

Then the characteristic equation governing the second derivative of the perturbation vector ξ^μ is given by

$$\left[\nabla_\mu \nabla_\nu \xi^\rho\right]_-^+ = \lambda_\mu \lambda_\nu \zeta^\rho, \tag{7.57}$$

in which ζ^μ is a measure of the discontinuity whose speed we are about to measure: these are the speeds of the wavefronts. The discontinuity of the divergence of the symplectic current is

$$\left[\nabla_\mu \Omega^\mu \left\{\xi, \eta\right\}\right]_-^+ = \eta^\mu \mathfrak{H}_\mu{}^\nu{}_\rho{}^\sigma \lambda_\nu \lambda_\sigma \zeta^\rho. \tag{7.58}$$

When $\overline{g}_{\mu\nu}\zeta^\nu = 0$, i.e., ζ^μ is worldsheet orthogonal, then the characteristic vector λ_μ must be a null eigenvector of $T^{\mu\nu}$:

$$\lambda_\mu \lambda_\nu T^{\mu\nu} = 0. \tag{7.59}$$

When ζ^μ is tangential to the worldsheet, the characteristic equation is expressible in terms of worldsheet tensors via

$$\zeta^\mu = \zeta^a x^\mu{}_{,a}, \tag{7.60}$$

with

$$\mathcal{Q}_{ab}\zeta^b = 0 \tag{7.61}$$

and where $\mathcal{Q}_{ab}$ is called the characteristic matrix, defined via

$$\mathcal{Q}_{ac} = \overline{\mathfrak{H}}_a{}^b{}_c{}^d \lambda_b \lambda_d. \tag{7.62}$$

The condition for ζ^b to be an intrinsic characteristic vector is therefore that $\det(\mathcal{Q}_{ab}) = 0$. Using (7.47) for the hyper-Hadamard tensor, the characteristic matrix is given by

$$\mathcal{Q}_{ac} = 2\left(L_{ac}\overline{g}^{bd} + 2L_a{}^b{}_c{}^d\right)\lambda_b \lambda_d. \tag{7.63}$$

7.3.3 Separable case

A simplified example is where the Lagrangian splits as

$$\mathcal{L}(\phi^0, \ldots, \phi^p, \phi^0{}_{,a}, \ldots, \phi^p{}_{,a}) = \mathcal{L}_{\mathrm{s}}(\phi^0, \phi^0{}_{,a}) + \mathcal{L}_{\mathrm{e}}(\phi^1, \ldots, \phi^p{}_{,a}, \phi^1{}_{,a}, \ldots, \phi^p{}_{,a}). \tag{7.64}$$

That is, $\mathcal{L}_{\mathrm{s}}$ is only dependent on ϕ^0 and its space-time gradient (which includes its time-like gradient), and $\mathcal{L}_{\mathrm{e}}$ is only dependent of the scalars which don't have a time-like gradient (that makes $\mathcal{L}_{\mathrm{e}}$ the action for an elastic solid). With this setup,

$\mathcal{L}_{\rm s}$ is the Lagrangian for a "normal" scalar field (generically of k-essence type), since it depends only on ϕ^0 and

$$\mu_a = \phi^0{}_{,a}, \tag{7.65}$$

the gradient 1-form. Hence, (7.32a) becomes

$$\overline{g}^{00} = \overline{g}^{ab}\mu_a\mu_b = -\mu^2. \tag{7.66}$$

Since we are working in the separable case, neither $\mathcal{L}_{\rm s}$ or $\mathcal{L}_{\rm e}$ will depend on $\overline{g}^{0A}$; this means that the "entrainment" effect vanishes, and

$$L_{A0} = 0 \tag{7.67}$$

in (7.39). For the other terms in the variation of the Lagrangian (7.39) one obtains

$$L_{00} = \frac{\partial \mathcal{L}}{\partial \overline{g}^{00}} = -\frac{\partial \mathcal{L}_{\rm s}}{\partial \mu^2}, \tag{7.68a}$$

and

$$L_{AB} = \frac{\partial \mathcal{L}_{\rm e}}{\partial \overline{g}^{AB}} = \tfrac{1}{2}\left(\mathcal{L}_{\rm e}\gamma_{AB} - P_{AB}\right), \tag{7.68b}$$

where P_{AB} is the pressure tensor of the medium, definable as

$$P^{AB} = \frac{2}{\sqrt{|\gamma|}}\frac{\partial \sqrt{|\gamma|}\mathcal{L}_{\rm e}}{\partial \gamma_{AB}}, \tag{7.69}$$

in clear analogy with the worldsheet energy-momentum tensor (7.17). Note that we have used (7.36) to set $\overline{g}_{AB} = \gamma_{AB}$. One then obtains the elasticity tensor

$$E^{AB}{}_{CD} = 2\frac{\partial P^{AB}}{\partial \overline{g}^{CD}} - P^{AB}\gamma_{CD}. \tag{7.70}$$

Hence, since $P_{ab} = P_{AB}\phi^A{}_{,a}\phi^B{}_{,b}$, one obtains from (7.68b) the contravariant components of the pressure tensor on the worldsheet in arbitrary coordinates,

$$P^{ab} = \mathcal{L}_{\rm e}\gamma^{ab} - 2\gamma^{ac}\gamma^{bd}L_{cd}, \tag{7.71}$$

and the elasticity tensor

$$E^{abcd} = \left(\gamma^{ab}\gamma^{cd} - 2\gamma^{a(c}\gamma^{d)b}\right)\mathcal{L}_{\rm e} + P^{a(c}\gamma^{d)b} + \gamma^{a(c}P^{d)b} - 4\gamma^{ae}\gamma^{bf}\gamma^{cg}\gamma^{dh}L_{efgh}. \tag{7.72}$$

The total worldsheet energy-momentum tensor can be written in separated form as

$$T^{ab} = T^{ab}_{\rm s} + T^{ab}_{\rm e}, \tag{7.73}$$

in which the scalar field contribution is

$$T^{ab}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}}\overline{g}^{ab} - 2L^{ab}_{\mathrm{s}}, \tag{7.74a}$$

and the contribution due to the elastic medium is

$$T^{ab}_{\mathrm{e}} = \rho_{\mathrm{e}} u^a u^b + P^{ab}, \tag{7.74b}$$

with $\rho_{\mathrm{e}} = -\mathcal{L}_{\mathrm{e}}$, P^{ab} as given by (7.71) and

$$L^{ab}_{\mathrm{s}} = -L'_{\mathrm{s}}\mu^a\mu^b, \qquad L'_{\mathrm{s}} = -L_{00}. \tag{7.75}$$

The energy density ρ_{s} and isotropic pressure P_{s} of the scalar contribution can be read off from (7.74a) as

$$\rho_{\mathrm{s}} = 2\mu^2 L'_{\mathrm{s}} - \mathcal{L}_{\mathrm{s}}, \qquad P_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}}. \tag{7.76}$$

In analogue with the separated energy-momentum tensor (7.73), the hyper-elasticity tensor can also be expressed in separated form,

$$\mathfrak{C}^{abcd} = \mathfrak{C}^{abcd}_{\mathrm{s}} + \mathfrak{C}^{abcd}_{\mathrm{e}}. \tag{7.77}$$

The scalar and elastic contributions are

$$\begin{aligned} \mathfrak{C}^{abcd}_{\mathrm{s}} &= 2\left(L^{abcd}_{\mathrm{s}} + L^{a(c}_{\mathrm{s}}\overline{g}^{d)b} + \overline{g}^{a(c}L^{d)b}_{\mathrm{s}}\right) \\ &\quad - \left(L^{ab}_{\mathrm{s}}\overline{g}^{cd} + \overline{g}^{ab}L^{cd}_{\mathrm{s}}\right) + \tfrac{1}{2}\mathcal{L}_{\mathrm{s}}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right), \end{aligned} \tag{7.78a}$$

$$\begin{aligned} \mathfrak{C}^{abcd}_{\mathrm{e}} &= -\tfrac{1}{2}E^{abcd} + 2u^{(a}P^{b)(c}u^{d)} - \tfrac{1}{2}\left(P^{ab}u^c u^d + P^{cd}u^a u^b\right) \\ &\quad + \tfrac{1}{2}\rho_{\mathrm{e}}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right), \end{aligned} \tag{7.78b}$$

in which

$$L^{abcd}_{\mathrm{s}} = L''_{\mathrm{s}}\mu^a\mu^b\mu^c\mu^d, \qquad L''_{\mathrm{s}} = \frac{\partial L'_{\mathrm{s}}}{\partial \mu^2}. \tag{7.79}$$

In this separated case the cross-component of the characteristic matrix (7.63) vanishes

$$\mathcal{Q}_{0A} = 0. \tag{7.80}$$

This gives a decoupling of the characteric modes ζ^a for the scalar and elastic parts.

7.3.4 Remark on fluids

We would like to remark about perfect fluids. A perfect fluid is characterised by the Lagrangian only being a function of the determinant of the material metric: in the language of this section, that is $\mathcal{L}_{\mathrm{e}} = \mathcal{L}_{\mathrm{e}}(|\gamma_{AB}|)$, where $|\gamma_{AB}|$ is supposed to be the determinant of γ_{AB}. It is useful to recall the identity

$$\frac{\partial|\gamma|}{\partial\gamma^{AB}} = -|\gamma|\gamma_{AB}. \tag{7.81}$$

In this case the spatial parts of (7.40a) evaluate to

$$L_{AB} = -L_{\mathrm{F}}\gamma_{AB}, \qquad L_{\mathrm{F}} \equiv |\gamma|\frac{\partial\mathcal{L}_{\mathrm{e}}}{\partial|\gamma|}, \tag{7.82}$$

and the pressure tensor becomes

$$P^{ab} = P_{\mathrm{F}}\gamma^{ab}, \tag{7.83}$$

with

$$P_{\mathrm{F}} = \mathcal{L}_{\mathrm{e}} + 2L_{\mathrm{F}}. \tag{7.84}$$

We would like to offer a generalization, and consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}\left(\overline{g}^{00}, \overline{g}^{0A}, |\overline{g}^{AB}|\right). \tag{7.85}$$

We remember that we can still write $\overline{g}^{00} = -\mu^2$. One then obtains for the components (7.40a)

$$\begin{aligned}
L_{00} &= -\frac{\partial\mathcal{L}}{\partial\mu^2}, && (7.86a)\\
L_{0A} &= \frac{\partial\mathcal{L}}{\partial\overline{g}^{0A}}, && (7.86b)\\
L_{AB} &= -L_{\mathrm{F}}\overline{g}_{AB}, && (7.86c)
\end{aligned}$$

where we set

$$\mu^2 = -\overline{g}^{00}, \qquad L_{\mathrm{F}} \equiv |\overline{g}^{CD}|\frac{\partial\mathcal{L}}{\partial|\overline{g}^{CD}|}. \tag{7.87}$$

7.3.5 Non-separable case

In the general case, $\mathcal{L}$ will be a function of all invariants formed out of the components (7.32) of the induced metric (7.14). And so, regarding $\overline{g}^{ab}$ as a rank-2 tensor in $p+1$ dimensions, there are $p+1$ invariants that can be constructed. The first few such invariants are

$$I_0 = \det \overline{\mathbf{g}}, \tag{7.88a}$$

$$I_1 = [\overline{\mathbf{g}}], \tag{7.88b}$$

$$I_2 = \frac{1}{2!}\left([\overline{\mathbf{g}}]^2 - [\overline{\mathbf{g}}^2]\right), \tag{7.88c}$$

$$I_3 = \frac{1}{3!}\left([\overline{\mathbf{g}}]^3 - 3[\overline{\mathbf{g}}^2][\overline{\mathbf{g}}] + 2[\overline{\mathbf{g}}^3]\right), \tag{7.88d}$$

$$I_4 = \frac{1}{4!}\Big([\overline{\mathbf{g}}]^4 + 8[\overline{\mathbf{g}}][\overline{\mathbf{g}}^3] + 3[\overline{\mathbf{g}}^2]^2 - 6[\overline{\mathbf{g}}]^2[\overline{\mathbf{g}}^2] - 6[\overline{\mathbf{g}}^4]\Big). \tag{7.88e}$$

These are elementary symmetric polynomials. The Cayley-Hamilton theorem imposes

$$I_{p+1} = I_0, \tag{7.89}$$

and sets all higher I_n to zero. Note that

$$I_1 = \overline{g}^0{}_0 + \overline{g}^A{}_A, \tag{7.90a}$$

$$I_2 = \overline{g}^0{}_0\overline{g}^A{}_A - \overline{g}^A{}_0\overline{g}^0{}_A + \frac{1}{2}\left[\overline{g}^A{}_A\overline{g}^B{}_B - \overline{g}^A{}_B\overline{g}^B{}_A\right] \tag{7.90b}$$

8 Generalization away from orthogonal mappings

Our aim here is to use the previous sections results and ideas to construct a theory for whom the orthogonality of the mappings (2.3) doesn't hold. We will keep to the notion of a material metric, k_{AB}, and coordinates ϕ^A on the material space; we carry on using (2.2) to define the configuration gradients:

$$\phi^A{}_{,a} = \psi^A{}_a. \tag{8.1}$$

The pull-back of the material metric to space-time is

$$k_{ab} = k_{AB}\phi^A{}_{,a}\phi^B{}_{,b}. \tag{8.2}$$

The energy-momentum tensor is

$$T_{ab} = \mathcal{L}g_{ab} - 2L_{ab}, \tag{8.3}$$

where

$$L_{ab} \equiv \frac{\partial \mathcal{L}}{\partial g^{ab}}. \tag{8.4}$$

It follows that L_{ab} can be thought of as the pull-back of some material-manifold tensor L_{AB}

$$L_{ab} = L_{AB}\phi^A{}_{,a}\phi^B{}_{,b}, \qquad L_{AB} \equiv \frac{\partial \mathcal{L}}{\partial k^{AB}}. \tag{8.5}$$

Hence, the energy-momentum tensor is

$$T_{ab} = \mathcal{L}g_{ab} - 2L_{AB}\phi^A{}_{,a}\phi^B{}_{,b}. \tag{8.6}$$

Suppose we wanted to compute $u^a u^b L_{ab}$. Then

$$u^a u^b L_{ab} = L_{AB} u^a \phi^A{}_{,a} u^b \phi^B{}_{,b}, \tag{8.7}$$

but

$$u^A = u^a \phi^A{}_{,a} \tag{8.8}$$

is the push-forward of u^a. In coordinate free notation the push-forward takes contravariant tensors in space-time and returns a contravariant tensor on the material space via a schematic expression of the form $B^{AB\cdots} = \psi_\star B^{ab\cdots}$, as we explained in the discussion leading up to (2.8). Hence

$$u^a u^b L_{ab} = u^A u^B L_{AB}. \tag{8.9}$$

Similarly, if we errected a vector l^a in space-time that is orthonormal to u^a, i.e., $l^a u_a = 0$ and $l^a l_a = 1$, then

$$l^a L_{ab} = l^A L_{AB} \phi^B{}_{,b} \tag{8.10}$$

where

$$l^A = l^a \phi^A{}_{,a} \tag{8.11}$$

is the push-forward of l^a. We will also introduece a tensor γ^{AB} on the material manifold, to be the tensor for whom l^A is an eigenvector,

$$l^A \gamma^B{}_A = l^B. \tag{8.12}$$

We can now use this to obtain a complete decomposition of the allowed freedom in L_{AB}:

$$L_{AB} = \left(u_C u_D L^{CD}\right) u_A u_B + 2\left(u_C l_D L^{CD}\right) u_{(A} l_{B)} + \gamma_{A(C}\gamma_{D)B} L^{CD}. \tag{8.13}$$

In the hyper-elastic category one has the splitting

$$\left(\phi^A{}_{,a}\right) \longrightarrow \left(\phi^0{}_{,a}, \phi^{\overline{A}}{}_{,a}\right) \tag{8.14}$$

in which

$$u^a \phi^0{}_{,a} \neq 0, \qquad u^a \phi^{\overline{A}}{}_{,a} = 0. \tag{8.15}$$

This means that the pull-back of the material metric splits up as

$$k_{ab} = k_{00} \phi^0{}_{,a} \phi^0{}_{,b} + 2 k_{\overline{A}0} \phi^0{}_{,a} \phi^{\overline{A}}{}_{,b} + k_{\overline{AB}} \phi^{\overline{A}}{}_{,a} \phi^{\overline{B}}{}_{,b}. \tag{8.16}$$

There are three projections of the pulled-back material metric:

$$u^a u^b k_{ab} = u^a u^b k_{00} \phi^0{}_{,a} \phi^0{}_{,b}, \tag{8.17a}$$

$$u^a \gamma^b{}_c k_{ab} = 2 u^a \gamma^b{}_c k_{\overline{A}0} \phi^0{}_{,a} \phi^{\overline{A}}{}_{,b}, \tag{8.17b}$$

$$\gamma^a{}_c \gamma^b{}_d k_{ab} = \gamma^a{}_c \gamma^d{}_b k_{\overline{AB}} \phi^{\overline{A}}{}_{,a} \phi^{\overline{B}}{}_{,b}. \tag{8.17c}$$

It is convenient, and does not lose generality, to set

$$\mu_a \equiv \phi^0{}_{,a}, \qquad l_a \equiv k_{\overline{A}0} \phi^{\overline{A}}{}_{,a}, \qquad \overline{k}_{ab} \equiv k_{\overline{AB}} \phi^{\overline{A}}{}_{,a} \phi^{\overline{B}}{}_{,b}, \tag{8.18}$$

in which

$$u^a l_a = 0, \qquad u^a \overline{k}_{ab} = 0. \tag{8.19}$$

The pull-back of the material metric now naturally splits up as

$$k_{ab} = \mu_a \mu_b + 2\mu_{(a} l_{b)} + \overline{k}_{ab}. \tag{8.20}$$

Suppose we wrote $\overline{k}_{ab}$ as the pull-back with respect to some new map $\widehat{\psi}$ of some new tensor, $\overline{k}_{ab} = \widehat{\psi}^\star \overline{k}_{AB}$, where in coordinates

$$\overline{k}_{ab} = \overline{k}_{\widehat{A}\widehat{B}} \widehat{\phi}^{\widehat{A}}{}_{,a} \widehat{\phi}^{\widehat{B}}{}_{,b}, \tag{8.21}$$

and the components of this new configuration gradient satisfy

$$u^a \widehat{\phi}^{\widehat{A}}{}_{,a} = 0. \tag{8.22}$$

9 Skyrmion and ferromagnetic dark energy

In this section we want to review some "material models" which have been extensively studied in the literature, albeit within the context of field theory and particularly within the soliton community. When a soliton is constructed in a theory mildly more complicated than a (real) scalar field theory it is usual for the language of "mappings" and an "extra metric" to be used. The difference in topologies of the space-time and field manifolds ends up introducing interesting topological soliton solutions.

As a first non-trivial example, in the Skyrmion model it is natural to speak of a map $\phi : M \to N$, where M is the 4D space-time manifold whose (pseudo-Riemannian) metric is $g_{\mu\nu}$ and N is the 3D manifold whose Riemannian metric is γ_{IJ}. The Lagrangian density for the Skyrmion model is given by

$$\mathcal{L}_{\text{Sky}} = -\frac{1}{2}[\mathbf{S}] + \frac{1}{4}\alpha^2\left([\mathbf{S}^2] - [\mathbf{S}]^2\right), \tag{9.1a}$$

where we have used the bracket-notation for the trace, for example $[\mathbf{S}] = S^{\alpha}{}_{\alpha}$, and where the rank-2 tensor in question is

$$S_{\alpha\beta} = \gamma_{IJ}\partial_\alpha\phi^I\partial_\beta\phi^J. \tag{9.1b}$$

See [60, 61] for a presentation of the Skyrmion model written in this way. This is the pull-back of the metric γ_{IJ}. Note that we are using the mostly-positive signature. One is able to form Skyrmion configurations in space-time: these are topological solitons [62] which have been extensively studied in the field theory literature, mostly for the purposes of modelling atomic nuclei. It is therefore rather clear that this setup produces extended objects in space-time: these are *materials*.

There exists multi-scalar generalizations of the galileon and Horndeski theories, for example [63–66]. There one of the main building blocks is $X_{IJ} = \frac{1}{2}\nabla_\alpha\phi_I\nabla^\alpha\phi_J$. **Is this equivalent to (9.1b) or not? I think this is the push-forward of g, rather than (9.1b) which is the pull-back of h** . There is some related work in [67]. Gibbons made a note about energy conditions in the Skyrme model [68], and concluded that the Skyrme term probably isn't a good dark energy candidate.

A simple sub-model is Lagrangian with $\alpha = 0$ and metric $\gamma_{IJ} = \frac{1}{2}\delta_{IJ}$, where δ_{IJ} is the Kronecker-delta. The Lagrangian density is thus

$$\mathcal{L} = -\tfrac{1}{4}\delta_{IJ}\partial^\mu\phi^I\partial_\mu\phi^J = -\frac{1}{4}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}. \tag{9.2}$$

In the field theory literature this is referred to as an $O(3)$-sigma model. When $\boldsymbol{\phi}$ is a unit vector the theory describes the dynamics of a (non-relativistic) ferromagnet;

this unit constraint is usually enforced via a Lagrange multiplier:

$$\mathcal{L} = -\frac{1}{4}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} + \lambda \left(\eta^2 - \boldsymbol{\phi} \cdot \boldsymbol{\phi}\right). \tag{9.3}$$

The ferromagnets described by this theory are known to posess exact analytic solutions which are rational functions in the complex number $z = x + \mathrm{i}y$, Hopf knots, isolated and leapfrogging vortex rings (in a modified "anisotropic" version of the theory). The equations of motion that follow from the Lagrangian density (9.3) are

$$\Box \boldsymbol{\phi} + \eta^{-2} \left(\nabla^\mu \boldsymbol{\phi} \cdot \nabla_\mu \boldsymbol{\phi}\right) \boldsymbol{\phi} = \mathbf{0}. \tag{9.4}$$

We enforce the unit constraint, thus removing λ from the equations of motion, by computing the second derivative of $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \eta^2$, and using the resulting formula in the equation of motion.

In [69] the cosmology of sigma-models are computed.

The solutions of field theories such as the Skyrmion model (9.1) which are generally searched for are localised objects (they are the topological solitons). These solutions can be thought about as providing some energy-momentum tensor, $T_{\mu\nu}$, which sits on the right-hand-side of the Einstein field equations.

For the perturbations of the two relevant invariants which appear in the Lagrangian we obtain

$$\delta \left(S^\alpha{}_\alpha\right) = g^{\mu\nu} \delta S_{\mu\nu} - S^{\mu\nu} \delta g_{\mu\nu}, \tag{9.5a}$$

$$\delta \left(S^{\alpha\beta} S_{\alpha\beta}\right) = 2 S^{\mu\nu} \delta S_{\mu\nu} - 2 S^{(\mu}{}_\alpha S^{\nu)\alpha} \delta g_{\mu\nu}. \tag{9.5b}$$

Also, the perturbation of the pulled-back metric (9.1b) is

$$\delta S_{\mu\nu} = \partial_\mu \phi^I \partial_\nu \phi^J \delta \gamma_{IJ} + 2 \gamma_{IJ} \partial_{(\mu} \phi^I \partial_{\nu)} \delta \phi^J. \tag{9.5c}$$

Using (9.5) the variation of the Skyrmion Lagrangian density (9.1a) can be written as

$$\delta \mathcal{L} = \frac{1}{2} \mathcal{E}^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J \delta \gamma_{IJ} - \nabla_{(\mu} \left(\mathcal{E}^{\mu\nu} \gamma_{IJ} \partial_{\nu)} \phi^I\right) \delta \phi^J - \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}, \tag{9.6}$$

after integrating by parts, and by setting

$$\mathcal{E}^{\mu\nu} \equiv \alpha^2 S^{\mu\nu} - \left(1 + \alpha^2 S^\alpha{}_\alpha\right) g^{\mu\nu}, \tag{9.7a}$$

$$T^{\mu\nu} \equiv \alpha^2 S^{(\mu}{}_\alpha S^{\nu)\alpha} - \left(1 + \alpha^2 S^\alpha{}_\alpha\right) S^{\mu\nu}. \tag{9.7b}$$

The equations of motion of the ϕ^I can be read off from (9.6) as

$$\nabla_{(\mu} \left(\mathcal{E}^{\mu\nu} \gamma_{IJ} \partial_{\nu)} \phi^I\right) = 0. \tag{9.8}$$

10 Higher order perturbations

This is a quick review of language obtained from some of Carter's notes [70] on brane dynamics. We note that a very recent article [71] obtained some similar expressions, with applications to dRGT massive gravity in mind. There are also papers with higher-order cosmological perturbation theory in mind (see e.g., [72]).

Let ξ^μ be the components of a vector at a position with coordinates x^μ. Then let there be an affinely parameterized geodesic (with affine parameter ϵ) beginning at x^μ and for whom the initial normalised tangent vector is the ξ^μ. What this means is that we shall have initial conditions $x^\mu_{\{0\}} = x^\mu, \dot{x}^\mu_{\{0\}} = \xi^\mu$ corresponding to the geodesic equation

$$\ddot{x}^\mu_{\{\epsilon\}} + \Gamma^\mu{}_{\alpha\beta\{\epsilon\}}\dot{x}^\alpha_{\{\epsilon\}}\dot{x}^\beta_{\{\epsilon\}} = 0. \tag{10.1}$$

The solution will be given by an expression of the form

$$x^\mu_{\{\epsilon\}} = x^\mu + \epsilon \$_\xi x^\mu + \epsilon^2 \tfrac{1}{2!}\$^2_\xi x^\mu + \epsilon^3 \tfrac{1}{3!}\$^3_\xi x^\mu + \mathcal{O}(\epsilon^4), \tag{10.2}$$

with $\$_\xi$ denoting differentiation with respect to the affine parameter given that the initial tangent is ξ^μ. For the first, second, and third orders one has

$$\$_\xi x^\mu = \xi^\mu, \tag{10.3a}$$

$$\$^2_\xi x^\mu = -\Gamma^\mu{}_{\lambda\nu}\xi^\lambda\xi^\nu, \tag{10.3b}$$

$$\$^3_\xi x^\mu = \left(\Gamma^\mu{}_{\lambda\alpha}\Gamma^\lambda{}_{\rho\sigma} - \partial_\sigma\Gamma^\mu{}_{\alpha\rho}\right)\xi^\alpha\xi^\rho\xi^\sigma. \tag{10.3c}$$

Carter's plan was to use this to create a systematic calculus which can be used to generalize perturbation theory to higher orders. To explain this further, we are well aware that to first order in the displacement vector, the infinitesimal Lagrangian $\delta_{\rm L}$ and Eulerian $\delta_{\rm E}$ perturbations are related via the Lie derivative,

$$\delta_{\rm L} - \delta_{\rm E} = \pounds_\xi, \tag{10.4}$$

in which $\pounds_\xi$ denotes the Lie derivative operator along the vector ξ^μ. One then wants to construct finite Lagrangian $\Delta_{\rm L}$ and Eulerian $\Delta_{\rm E}$ perturbations in some expansion

$$\Delta_{\rm L} = \delta_{\rm L} + \tfrac{1}{2}\delta^2_{\rm L} + \ldots, \tag{10.5a}$$

$$\Delta_{\rm E} = \delta_{\rm E} + \tfrac{1}{2}\delta^2_{\rm E} + \ldots. \tag{10.5b}$$

The required expansion would be of the form

$$\Delta_{\text{L}} - \Delta_{\text{E}} = \$_\xi + \tfrac{1}{2}\$_\xi^2 + \tfrac{1}{3!}\$_\xi^3 + \dots \tag{10.6}$$

Notice that to first order in the displacement vectors, $\Delta_{\text{L}} - \Delta_{\text{E}} = \pounds_\xi$.

The form of the "action" of the derivatives $\$_\xi^n$ on space-time field will be different depending on the types of fields (e.g., scalar, vector, and tensor). The displacement $x^\mu \mapsto x^\mu_{\{\epsilon\}}$ will determine a pullback mapping of a space-time field. For example, for a scalar, vector, and rank-2 tensor fields, the mapping engenders

$$\begin{aligned} \phi &\longmapsto \phi_{\{\epsilon\}}\{x\}, && (10.7a)\\ A_\mu &\longmapsto A_{\{\epsilon\}\mu}\{x\}, && (10.7b)\\ B_{\mu\nu} &\longmapsto B_{\{\epsilon\}\mu\nu}\{x\}, && (10.7c)\end{aligned}$$

with

$$\begin{aligned} \phi_{\{\epsilon\}}\{x\} &= \phi\{x_{\{\epsilon\}}\}, && (10.8a)\\ A_{\{\epsilon\}\mu}\{x\} &= A_{\{\epsilon\}\nu}\{x_{\{\epsilon\}}\}x^\nu_{\{\epsilon\},\mu}, && (10.8b)\\ B_{\{\epsilon\}\mu\nu}\{x\} &= B_{\{\epsilon\}\alpha\beta}\{x_{\{\epsilon\}}\}x^\alpha_{\{\epsilon\},\mu}x^\beta_{\{\epsilon\},\nu}. && (10.8c)\end{aligned}$$

It is evident that we will need the components of the Jacobians, $x^\nu_{\{\epsilon\},\mu}$, which via (10.3) work out as

$$x^\mu_{\{\epsilon\},\lambda} = \delta^\mu{}_\lambda + \epsilon\xi^\mu{}_{,\lambda} - \tfrac{1}{2}\epsilon^2\left(\Gamma^\mu{}_{\nu\rho,\lambda}\xi^\nu\xi^\rho + 2\Gamma^\mu{}_{\nu\rho}\xi^{(\nu}\xi^{\rho)}{}_{,\sigma}\right) + \mathcal{O}(\epsilon^3) \tag{10.9}$$

In the case of a scalar field ϕ the expansion will be of the form

$$\phi_{\{\epsilon\}} = \phi + \phi_{,\mu}\left(x^\mu_{\{\epsilon\}} - x^\mu\right) + \tfrac{1}{2}\phi_{,\mu\nu}\left(x^\mu_{\{\epsilon\}} - x^\mu\right)\left(x^\nu_{\{\epsilon\}} - x^\nu\right) + \mathcal{O}(\epsilon^2). \tag{10.10}$$

The expansion to higher order in the displacement vectors for a scalar ϕ, vector A_μ, and rank-2 tensor $B_{\mu\nu}$, are

$$\phi_{\{\epsilon\}} = \phi + \epsilon\$_\xi\phi + \tfrac{1}{2}\epsilon^2\$_\xi^2\phi + \mathcal{O}(\epsilon^3), \tag{10.11a}$$

$$A_{\{\epsilon\}\mu} = A_\mu + \epsilon\$_\xi A_\mu + \tfrac{1}{2}\epsilon^2\$_\xi^2 A_\mu + \mathcal{O}(\epsilon^3), \tag{10.11b}$$

$$B_{\{\epsilon\}\mu\nu} = B_{\mu\nu} + \epsilon\$_\xi B_{\mu\nu} + \tfrac{1}{2}\epsilon^2\$_\xi^2 B_{\mu\nu} + \mathcal{O}(\epsilon^3). \tag{10.11c}$$

The contributions to first order in ξ^μ are

$$\$_\xi\phi = \xi^\mu\nabla_\mu\phi, \tag{10.12a}$$

$$\$_\xi A_\mu = \xi^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \xi^\nu, \tag{10.12b}$$

$$\$_\xi B_{\mu\nu} = \xi^\rho \nabla_\rho B_{\mu\nu} + B_{\rho\nu} \nabla_\mu \xi^\rho + B_{\mu\rho} \nabla_\nu \xi^\rho; \tag{10.12c}$$

and to second order in ξ^μ are

$$\$^2_\xi \phi = \xi^\mu \xi^\nu \nabla_\mu \nabla_\nu \phi, \tag{10.13a}$$

$$\$^2_\xi A_\mu = \xi^\nu \xi^\rho \left(\nabla_\rho \nabla_\nu A_\mu - R_{\mu\nu}{}^\lambda{}_\rho A_\lambda\right) + 2\xi^\nu \left(\nabla_\mu \xi^\rho\right) \nabla_\nu A_\rho, \tag{10.13b}$$

$$\begin{aligned} \$^2_\xi B_{\mu\nu} &= \xi^\rho \xi^\sigma \left(\nabla_\rho \nabla_\sigma B_{\mu\nu} - R_{\mu\nu}{}^\lambda{}_\sigma B_{\lambda\nu} - R_{\nu\rho}{}^\lambda{}_\sigma B_{\mu\lambda}\right) \\ &\quad +2\xi^\rho \left(\nabla_\mu \xi^\lambda \nabla_\rho B_{\lambda\nu} + \nabla_\nu \xi^\lambda \nabla_\rho B_{\mu\lambda}\right) + 2B_{\rho\sigma} \nabla_\mu \xi^\rho \nabla_\nu \xi^\sigma. \end{aligned} \tag{10.13c}$$

The Riemann tensor is defined via

$$\left(\nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\nu\right) A_\rho = -R_{\mu\nu}{}^\lambda{}_\rho A_\lambda. \tag{10.14}$$

We remark that the first order contributions (10.11) are just the Lie derivatives. However, for example, the second Lie derivative of a scalar does not coincide with the second dollar-derivative of a scalar:

$$\pounds^2_\xi \phi = \$^2_\xi \phi + \left(\xi^\mu \nabla_\mu \xi^\nu\right) \nabla_\nu \phi. \tag{10.15}$$

As an important example of a symmetric rank-2 tensor, the case of the space-time metric works out as

$$g_{\{\epsilon\}\mu\nu} = g_{\mu\nu} + \epsilon \$ g_{\mu\nu} + \tfrac{1}{2}\epsilon^2 \$^2 g_{\mu\nu} + \mathcal{O}(\epsilon^2), \tag{10.16}$$

with

$$\$ g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)}, \tag{10.17a}$$

$$\$^2 g_{\mu\nu} = 2\left(\nabla_{(\mu} \xi^\rho\right) \nabla_{\nu)} \xi_\rho - 2\xi^\rho \xi^\sigma R_{\mu\rho\nu\sigma}. \tag{10.17b}$$

11 Final discussion

- Whats the point?
- what have we learnt?
- where next?

A Tensor invariants

For a symmetric rank-2 tensor in 3D, $A_{ab} = A_{(ab)}$, there are three invariants:

$$\begin{aligned} I_0 &= \det \mathbf{A}, & \text{(A.1a)} \\ I_1 &= [\mathbf{A}] = A^a{}_a, & \text{(A.1b)} \\ I_2 &= \frac{1}{2}\left([\mathbf{A}]^2 - [\mathbf{A}^2]\right) = \frac{1}{2}\left((A^a{}_a)^2 - A^a{}_b A^b{}_a\right). & \text{(A.1c)} \end{aligned}$$

These invariants are the unique combinations that represent the total-derivative contractions of the matrix $A^a{}_b = \partial^a \partial_b \pi$. Since we are in 3D, the following holds via the Cayley-Hamilton theorem

$$\det \mathbf{A} = \frac{1}{3!}\left([\mathbf{A}]^3 - 3[\mathbf{A}^2][\mathbf{A}] + 2[\mathbf{A}^3]\right). \tag{A.2}$$

In nD there are a maximum of n invariants.

We may also be interested in the invariants of rank-3 tensors in 3D; these are substantially more complicated to compute. An interesting thing to note is that tensors which are symmetric in the last two indices, $B_{abc} = B_{a(bc)}$, are sometimes called "peizo-electric tensors" [73]. Then the quadratic invariants are

$$B_{aab}B_{ccb}, \qquad B_{abb}B_{acc}, \qquad B_{abc}B_{abc}, \qquad B_{abc}B_{bac}, \qquad B_{aab}B_{bcc} \tag{A.3}$$

B Relation to statistical cumulants

Note that the tensor invariants seem related to the mean, variance, skewness, etc, in statistics; the culumants are

$$\begin{aligned} \kappa_1 &= \mu_1, & \text{(B.1a)} \\ \kappa_2 &= \mu_2 - \mu_1^2, & \text{(B.1b)} \\ \kappa_3 &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3, & \text{(B.1c)} \\ \kappa_4 &= \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4 & \text{(B.1d)} \end{aligned}$$

And so, for the spatial 2x2 matrix $k^a{}_b$ only κ_1, κ_2, and κ_3 are required to specify the invariants.

C Symplectic current

C.0.6 Symplectic structure

Suppose an action is given by the integral over a Lagrangian density which is a function of fields q^A and their derivatives $q^A{}_{,i}$,

$$\mathcal{L} = \mathcal{L}(q^A, q^A{}_{,i}). \tag{C.1}$$

A generic variation is thus given by

$$\delta\mathcal{L} = \mathcal{L}_{,A}\delta q^A + p^i{}_A \delta q^A{}_{,i}, \tag{C.2}$$

and is easily re-writable as

$$\delta\mathcal{L} = \mathcal{E}_A \delta q^A + \vartheta^i{}_{,i} \tag{C.3}$$

in which we defined

$$\mathcal{E}_A \equiv \mathcal{L}_{,A} - p^i{}_{A,i}, \tag{C.4a}$$

$$\vartheta^i \equiv p^i{}_A \delta q^A. \tag{C.4b}$$

The "on-shell" dynamically permissible variations require the vanishing of (C.4a), $\mathcal{E}_A = 0$. The vector (C.4b) is the Liouville 1-form. The pseudo-Hamiltonian density is

$$\mathcal{H} = p^i{}_A q^A{}_{,i} - \mathcal{L}, \tag{C.5}$$

and whose generic variation can be expressed as

$$\delta\mathcal{H} = q^A{}_{,i}\delta p^i{}_A - \mathcal{L}_{,A}\delta q^A. \tag{C.6}$$

For the on-shell variations (i.e., those with $\mathcal{E}_A = 0$) the variation in the pseudo-Hamiltonian is

$$\delta\mathcal{H} = q^A{}_{,i}\delta p^i{}_A - p^i{}_{A,i}\delta q^A, \tag{C.7}$$

and

$$\vartheta^i{}_{,i} = 0. \tag{C.8}$$

Now suppose that there are two variations, $\grave{\delta}$ and $\acute{\delta}$. Then acting these onto the Lagrangian (C.1) yields

$$\grave{\delta}\acute{\delta}\mathcal{L} = \mathcal{E}_A \grave{\delta}\acute{\delta} q^A + \left(\grave{\delta}\mathcal{E}_A\right)\acute{\delta} q^A + \left(\grave{\delta} p^i{}_A \acute{\delta} q^A + p^i{}_A \grave{\delta}\acute{\delta} q^A\right)_{,i}. \tag{C.9}$$

Since the two variations commute

$$\grave{\delta}\acute{\delta}\mathcal{L} = \acute{\delta}\grave{\delta}\mathcal{L} \tag{C.10}$$

it follows from (C.9) that

$$\left(\grave{\delta}\mathcal{E}_A\right)\acute{\delta}q^A - \left(\acute{\delta}\mathcal{E}_A\right)\grave{\delta}q^A = \widehat{\varpi}^i{}_{,i} \tag{C.11}$$

with

$$\widehat{\varpi}^i \equiv \acute{\delta}p^i{}_A\grave{\delta}q^A - \grave{\delta}p^i{}_A\acute{\delta}q^A, \tag{C.12}$$

which is the symplectic 2-form. For the pair of variations which are on-shell, i.e., $\mathcal{E}_A = 0$, it follows from (C.11) that

$$\widehat{\varpi}^i{}_{,i} = 0. \tag{C.13}$$

Rather than working with internal coordinates i etc, we construct tensorial versions of the surface currents,

$$\Theta^\mu = \frac{1}{\sqrt{-\overline{g}}}x^\mu{}_{,i}\vartheta^i, \qquad \Omega^\mu = \frac{1}{\sqrt{-\overline{g}}}x^\mu{}_{,i}\widehat{\varpi}^i. \tag{C.14}$$

Then (C.8) and (C.13) are expressible as

$$\overline{\nabla}_\mu\Theta^\mu = 0, \qquad \overline{\nabla}_\mu\Omega^\mu = 0. \tag{C.15}$$

Take

$$\delta\mathcal{L} = \mathcal{L}_{,a}\delta_{\mathrm{L}}\phi^a + p^\mu{}_a\partial_\mu\delta_{\mathrm{L}}\phi^a \tag{C.16}$$

which rearranges to

$$\delta\mathcal{L} = \left(\mathcal{L}_{,a} - \partial_\mu p^\mu{}_a\right)\delta_{\mathrm{L}}\phi^a + \partial_\mu\vartheta^\mu, \tag{C.17}$$

with

$$\vartheta^\mu = p^\mu{}_a\delta_{\mathrm{L}}\phi^a. \tag{C.18}$$

Replacing the Lagrangian variation $\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \pounds_\xi$ yields

$$\vartheta^\mu = p^\mu{}_a \tag{C.19}$$

D Equations of motion from a partial theory

It is instructive to look at the variational principle as a method of constructing equations of motion. Suppose that one has a Lagrangian which contains a scalar field ϕ, the metric $g_{\mu\nu}$, and some "other" field, s^{A}, say (here "A" is a label, not an index). The variation of the action will yield

$$\delta S = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\mathcal{E}\delta\phi + \frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} - \mathcal{F}_{\text{A}}\delta s^{\text{A}} \right]. \tag{D.1}$$

It is important to note that this is the matter action and may append another action (which may also contain more terms for the scalar field, for example). If the variation δ was generated by coordinate shifts, then the variation in the action is found from (D.1) by replacing all instances of "δ" by the Lie derivative $\pounds_\xi$ yielding

$$\delta S = \int \mathrm{d}^4x\, \sqrt{-g}\, \left[\mathcal{E}\xi^\mu \nabla_\mu\phi + T_{\mu\nu}\nabla^{(\mu}\xi^{\nu)} - \mathcal{F}_{\text{A}}\pounds_\xi s^{\text{A}} \right]. \tag{D.2}$$

This must vanish if the theory is to be diffeomorphism invariant, resulting in

$$\nabla_\mu T^{\mu\nu} = f^\nu_{[\phi]} + f^\nu_{[s]}. \tag{D.3}$$

The terms on the RHS are force densities. For this action we always have

$$f^\nu_{[\phi]} = \mathcal{E}\nabla^\nu\phi, \tag{D.4}$$

and the precise form of $f^\nu_{[s]}$ will depend on whether s^{A} is a scalar, vector, tensor, etc, field.

If $T^{\mu\nu}$ is supposed to be the energy-momentum tensor (i.e. it sits on the RHS of the Einstein equations), it must be conserved and therefore the RHS of (D.3) must vanish. This yields the force balance condition

$$f^\nu_{[\phi]} = -f^\nu_{[s]}. \tag{D.5}$$

As an explicit example, if $s_{\text{A}} \to s$, a scalar, then

$$f^\mu_{[s]} = -\mathcal{F}\nabla^\mu s. \tag{D.6}$$

The force balance condition (D.5) explicitly reads

$$\mathcal{E}\nabla_\mu\phi = \mathcal{F}\nabla_\mu s. \tag{D.7}$$

This explicitly sources the "equation of motion" for the scalar ϕ with the field s. We can rewrite this as

$$\mathcal{E} = \mathcal{F}\frac{\nabla^\mu\phi\nabla_\mu s}{\nabla_\nu\phi\nabla^\nu\phi}. \tag{D.8}$$

What we would like is for the equation of motion of the scalar to be of the form

$$\mathcal{E} = \alpha\dot{\phi}, \tag{D.9}$$

that is,

$$\alpha\dot{\phi} = \mathcal{F}\frac{\nabla^{\mu}\phi\nabla_{\mu}s}{\nabla_{\nu}\phi\nabla^{\nu}\phi}. \tag{D.10}$$

When the fields ϕ and s only have time-like derivatives,

$$\mathcal{F} = \frac{\dot{\phi}^2}{\dot{s}}\alpha. \tag{D.11}$$

E Pull-back formalism *à la* Arkani-Hamed

The pullback formalism we are about to discuss was developed by Arkani-Hamed et al [74, 75] motivated by a graph theory model for "theory space". They developed a method to construct an effective theory of gravity to restore coordinate invariance of linearized gravity theories. Interestingly, the pullback formalism was also laid out by Carter et al [4, 76], more than twenty years before Arkani, during the development of relativistic elasticity theory: the mathematical language between the two motivations is identical. The Arkani-motivation was developed by the analogy with gauge field theory: there it is natural to talk of a pullback in terms of a gauge field. Infact, when numerical codes are developed, the equations of motion are discretized using a pullback and gauge field to link between lattice sites. The construction of the pullback formulation is very mathematical, but should be familiar to those who have studied differential geometry in some detail.

E.1 Setup

Consider a manifold, $\mathcal{U}$, with a point on $\mathcal{U}$ being denoted by $x_{\mathcal{U}}$. Now let us have a map from the manifold to the real line:

$$\phi : \mathcal{U} \to \mathbb{R}, \tag{E.1}$$

which, in a more usual notation, reads $\phi(x) \in \mathbb{R}$. Let us also have an "internal map"

$$f : \mathcal{U} \to \mathcal{U} \tag{E.2}$$

which is more obviously written as $x \to f(x)$. The map ϕ is a scalar field on $\mathcal{U}$ if it transforms under f according to

$$\phi \to \phi \circ f. \tag{E.3}$$

In coordinates this reads

$$\phi(x) \to \phi(f(x)). \tag{E.4}$$

One could think of f as being a change of coordinates; later on we will give a concrete example of the map. A vector field $\mathbf{A}$ on $\mathcal{U}$ transforms under f according to

$$\mathbf{A} \to \mathbf{A} \circ f, \tag{E.5}$$

the components of $\mathbf{A}$ transform as

$$A_\mu(x) \to \frac{\partial f^\alpha}{\partial x^\mu}(x) A_\alpha(f(x)). \tag{E.6}$$

Similarly, a tensor field

$$\mathbf{T} \to \mathbf{T} \circ f, \tag{E.7}$$

and its components

$$T_{\mu\nu}(x) \to \frac{\partial f^\alpha}{\partial x^\mu}(x) \frac{\partial f^\beta}{\partial x^\nu}(x) T_{\alpha\beta}(f(x)) \tag{E.8}$$

This holds, for example, for the metric $\mathbf{g} = g_{\mu\nu}(x) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.

Let us now introduce a second manifold, $\mathcal{V}$. This manifold is endowed with an equivalent set of maps and fields as $\mathcal{U}$. The internal map in manifold $\mathcal{V}$ is denoted as $f_\mathcal{V}$, so that $x_\mathcal{V} \to f_\mathcal{V}(x_\mathcal{V})$. In the language of Arkani, the manifolds $\mathcal{U}, \mathcal{V}$ are two sites of a graph connected by a link, and in the language of Carter, the manifolds contain the relaxed and strained positions of a material body. Note that we now have two manifolds and two internal transformations: $f_\mathcal{V}, f_\mathcal{U}$. We can operate each independently, but any link between the manifolds will induce some interesting structure between the transformations. This is exactly the structure we impose by including a pullback map.

Let us have a map Y between these two manifolds

$$Y : \mathcal{U} \to \mathcal{V}. \tag{E.9}$$

The map between the manifolds is called a *link field* in the language of Arkani. The link field $Y(x_\mathcal{U})$ will associate a point $x_\mathcal{U}$ in $\mathcal{U}$ (the point $x_\mathcal{U}$ has coordinates $x^\mu_\mathcal{U}$) with a point in $\mathcal{V}$ whose coordinates are $x^\mu_\mathcal{V}$:

$$Y(x_\mathcal{U}) = x_\mathcal{V}. \tag{E.10}$$

The link field $Y_{\mathcal{V}\mathcal{U}}$ is a pullback map[3] from $\mathcal{V}$ to $\mathcal{U}$. Under a composition of maps, the link field transforms as

$$Y \to f_{\mathcal{V}}^{-1} \circ Y \circ f_{\mathcal{U}}, \tag{E.11}$$

or, in coordinates,

$$x_{\mathcal{V}}^{\mu}(x_{\mathcal{U}}) \to (f_{\mathcal{V}}^{-1})^{\mu}(Y(f_{\mathcal{U}}(x_{\mathcal{U}}))). \tag{E.12}$$

So, consider the object

$$\Psi = \psi_{\mathcal{V}} \circ Y, \tag{E.13}$$

where $\psi_{\mathcal{V}}$ is a scalar field in $\mathcal{V}$, then

$$\Psi \to (\psi_{\mathcal{V}} \circ f_{\mathcal{V}}) \circ (f_{\mathcal{V}}^{-1} \circ Y \circ f_{\mathcal{U}}) = \psi_{\mathcal{V}} \circ Y \circ f_{\mathcal{U}} = \Psi \circ f_{\mathcal{U}}, \tag{E.14}$$

i.e. Ψ transforms as a scalar under the mapping in $\mathcal{U}$:

$$\Psi \to \Psi \circ f_{\mathcal{U}}. \tag{E.15}$$

For example, the scalar field $\phi_{\mathcal{V}}(x_{\mathcal{V}})$ in $\mathcal{V}$, then

$$\Phi = \phi_{\mathcal{V}} \circ Y \quad \Rightarrow \quad \Phi(x_{\mathcal{U}}) = \phi_{\mathcal{V}}(Y(x_{\mathcal{U}})) \tag{E.16}$$

is a scalar under $f_{\mathcal{U}}$. Similarly,

$$A_{\mu}(x_{\mathcal{U}}) = \frac{\partial x_{\mathcal{V}}^{\alpha}}{\partial x_{\mathcal{U}}^{\mu}}(x_{\mathcal{U}}) a_{\mathcal{V}\alpha}(Y(x_{\mathcal{U}})), \tag{E.17}$$

$$T_{\mu\nu}(x_{\mathcal{U}}) = \frac{\partial x_{\mathcal{V}}^{\alpha}}{\partial x_{\mathcal{U}}^{\mu}}(x_{\mathcal{U}}) \frac{\partial x_{\mathcal{V}}^{\beta}}{\partial x_{\mathcal{U}}^{\nu}}(x_{\mathcal{U}}) t_{\mathcal{V}\alpha\beta}(Y(x_{\mathcal{U}})) \tag{E.18}$$

are vectors and tensors respectively under $f_{\mathcal{U}}$. The point is, the pullback field (link field) enables the evaluation of scalar, vector, tensor etc fields in one manifold to be determined in terms of coordinates on another manifold.

[3]**Pullback** Consider a map F between two manifolds M, W; the map is $F : M \to W$. The manifold M has coordinates m and W has coordinates w. Thus, $F(m) = w$. Now let us have a real valued function f on W, written as $f : W \to \mathbb{R}$. We define the pullback of f (which is on W) to M, written as $F^* f$, to be the composition $f \circ F : M \to \mathbb{R}$. So, let us pick a point m in M, and act the pullback on m. This is written as $F^* f(m) = (f \circ F)(m) = f(F(m))$.

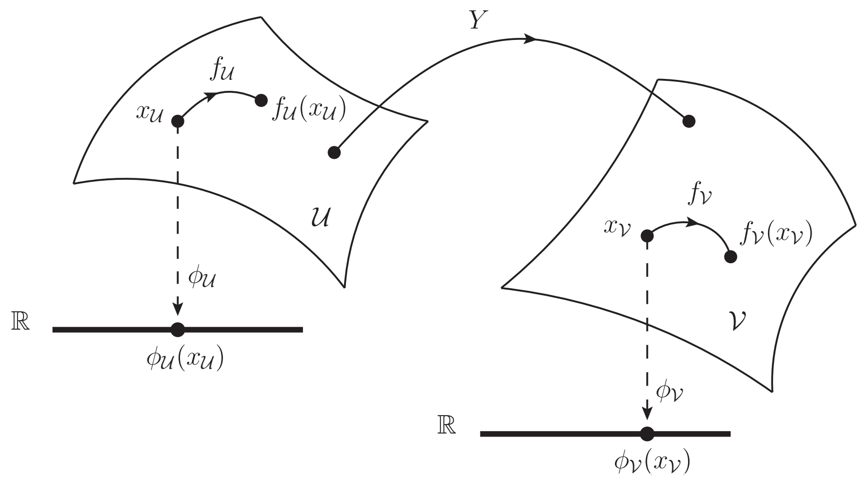

Figure 4: The manifolds $\mathcal{U}, \mathcal{V}$, the internal mappings $f_\mathcal{U}, f_\mathcal{V}$, the scalar field maps $\phi_\mathcal{U}, \phi_\mathcal{V}$ and the link field Y.

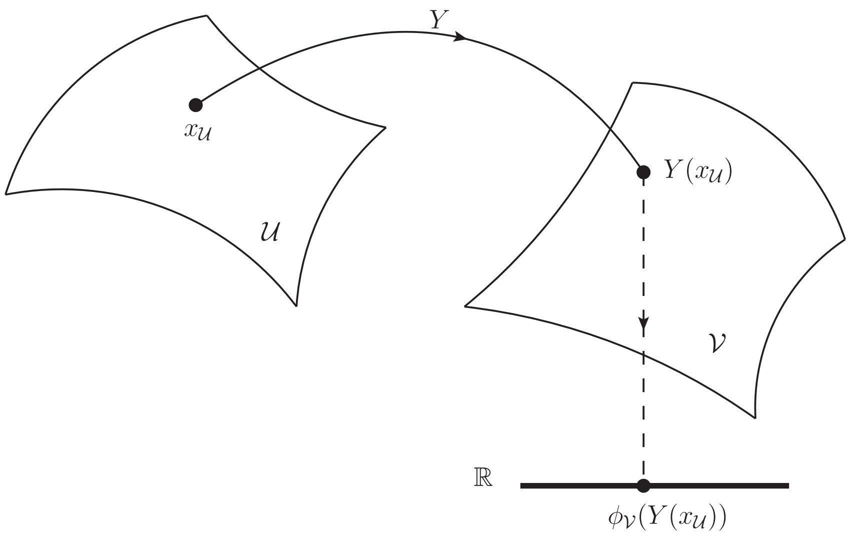

Figure 5: The link field Y maps a point $x_\mathcal{U}$ in $\mathcal{U}$ to a point $x_\mathcal{V} = Y(x_\mathcal{U})$ in $\mathcal{V}$. The scalar field map $\phi_\mathcal{V}$ in $\mathcal{V}$ then computes the value of the scalar field at that point.

E.2 Studying the link field

We will now study the link field, first in the simplest possible case, where it is the identity (also called the unitary gauge), then in the case where it is perturbatively close to the identity. We will use a deformation vector ξ^μ to describe the deviation of the link field from the identity. This will also describe the deviations from the unitary gauge.

In the simplest case, the link field is the identity:

$$Y = \mathrm{id} \qquad \Rightarrow \qquad x^\mu_\mathcal{V} = x^\mu_\mathcal{U}. \tag{E.19}$$

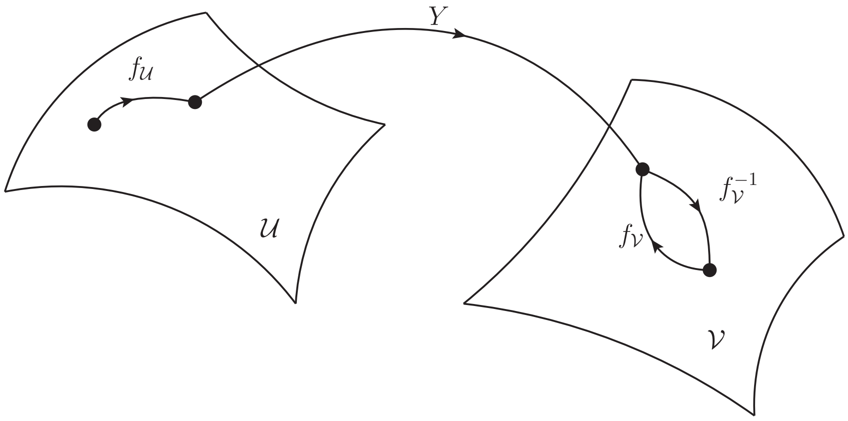

Figure 6: The internal mapping $f_{\mathcal{U}}$ transforms between points in $\mathcal{U}$. The link field Y then takes the transformed point from $\mathcal{U}$ into the transformed point in $\mathcal{V}$. The inverse mapping $f_{\mathcal{V}}^{-1}$ then finds the "original" point in $\mathcal{V}$ (the inverse mapping is assumed to exist).

and the mapping between the manifolds is "trivial".

In the "next simplest" case the link field is perturbed away from the identity with a deformation vector, so that

$$Y(x_{\mathcal{U}}) = x_{\mathcal{V}} = x_{\mathcal{U}} + \xi(x_{\mathcal{U}}) \qquad \Rightarrow \qquad x^{\mu}_{\mathcal{V}} = x^{\mu}_{\mathcal{U}} + \xi^{\mu}(x_{\mathcal{U}}). \tag{E.20}$$

Notice that in the unitary gauge (E.19) the deformation vector is identically zero: $\xi^{\mu} = 0$. We can now insert (E.20) into (E.18), which we write for clarity as

$$T_{\mu\nu}(x_{\mathcal{U}}) \;=\; \left[\frac{\partial x^{\alpha}_{\mathcal{V}}(x_{\mathcal{U}})}{\partial x^{\mu}_{\mathcal{U}}}\right]\left[\frac{\partial x^{\beta}_{\mathcal{V}}(x_{\mathcal{U}})}{\partial x^{\nu}_{\mathcal{U}}}\right]\Big[t_{\mathcal{V}\alpha\beta}(Y_{\mathcal{V}\mathcal{U}}(x_{\mathcal{U}}))\Big]. \tag{E.21}$$

Inserting (E.20) into these terms yields

$$\frac{\partial x^{\alpha}_{\mathcal{V}}(x_{\mathcal{U}})}{\partial x^{\mu}_{\mathcal{U}}} = \frac{\partial (x^{\alpha}_{\mathcal{U}} + \xi^{\alpha})}{\partial x^{\mu}_{\mathcal{U}}} = \delta^{\alpha}_{\mu} + \nabla_{\mu}\xi^{\alpha}, \tag{E.22}$$

$$t_{\mathcal{V}\alpha\beta}(Y_{\mathcal{V}\mathcal{U}}(x_{\mathcal{U}})) = t_{\mathcal{V}\alpha\beta} + \xi^{\rho}\nabla_{\rho}t_{\mathcal{V}\alpha\beta} + \frac{1}{2}\xi^{\rho}\xi^{\sigma}\nabla_{\rho}\nabla_{\sigma}t_{\mathcal{V}\alpha\beta} + \ldots \tag{E.23}$$

Thus,

$$T_{\mu\nu}(x_{\mathcal{U}}) = t_{\mathcal{V}\mu\nu} + \xi^{\rho}\nabla_{\rho}t_{\mathcal{V}\mu\nu} + 2t_{\mathcal{V}\alpha(\mu}\nabla_{\nu)}\xi^{\alpha} + \ldots. \tag{E.24}$$

All terms on the right-hand-side are functions of $x_{\mathcal{U}}$. The series continues in quadratic and higher powers of the vector ξ^{μ}. By our initial construction, $T_{\mu\nu}(x_{\mathcal{U}})$ transforms as a tensor in $\mathcal{U}$ under $f_{\mathcal{U}}$. The second term on the RHS will vanish when

$t_{\mathcal{V}\mu\nu}$ is taken to be the metric. We will call ξ^μ the deformation vector. In Arkani-Hamed et al [75] the ξ^μ are called Goldstone fields, and they find it instructive to write the Goldstone field in "pion" notation $\xi \equiv \pi$, and decompose the pion field

$$\pi_\mu = A_\mu + \nabla_\mu \sigma, \tag{E.25}$$

where $\nabla_\mu A^\mu = 0$. The Goldstone pions enable an interesting discussion of massive gravity theories to take place, where, for example, the σ-field has a ghost, unless the Pauli-Feirz mass term is used. The fields $\{\pi, A, \sigma\}$ are called Stuckleberg fields [**?**].

E.3 The internal transformation is a coordinate shift

Now suppose that the internal transformation f is a shift in position:

$$x_{\mathcal{U}} \to f_{\mathcal{U}}(x_{\mathcal{U}}), \qquad f_{\mathcal{U}}(x_{\mathcal{U}}) = x_{\mathcal{U}} + \epsilon_{\mathcal{U}}(x_{\mathcal{U}}). \tag{E.26}$$

Similarly,

$$f_{\mathcal{V}}(x_{\mathcal{V}}) = x_{\mathcal{V}} + \epsilon_{\mathcal{V}}(x_{\mathcal{V}}) \tag{E.27}$$

Under this coordinate transformation,

$$T_{\mu\nu}(x_{\mathcal{U}}) \to T_{\mu\nu}(x_{\mathcal{U}} + \epsilon_{\mathcal{U}}) = T_{\mu\nu}(x_{\mathcal{U}}) + \pounds_{\epsilon_{\mathcal{U}}} T_{\mu\nu}, \tag{E.28}$$

where the Lie derivative is

$$\pounds_{\epsilon_{\mathcal{U}}} T_{\mu\nu} = \epsilon^\alpha_{\mathcal{U}} \nabla_\alpha T_{\mu\nu} + 2T_{\alpha(\mu} \nabla_{\nu)} \epsilon^\alpha_{\mathcal{U}}. \tag{E.29}$$

Thus, we say that under the transformation $f_{\mathcal{U}}$ we have, $T_{\mu\nu}(x_{\mathcal{U}}) \to T_{\mu\nu}(x_{\mathcal{U}}) + \delta T_{\mu\nu}(x_{\mathcal{U}})$ where

$$\delta T_{\mu\nu}(x_{\mathcal{U}}) = \epsilon^\alpha_{\mathcal{U}} \nabla_\alpha T_{\mu\nu} + 2T_{\alpha(\mu} \nabla_{\nu)} \epsilon^\alpha_{\mathcal{U}}. \tag{E.30}$$

Similarly, under $f_{\mathcal{V}}$,

$$\delta t_{\mu\nu}(x_{\mathcal{V}}) = \epsilon^\alpha_{\mathcal{V}} \nabla_\alpha t_{\mu\nu} + 2t_{\alpha(\mu} \nabla_{\nu)} \epsilon^\alpha_{\mathcal{V}}. \tag{E.31}$$

By using (E.12) we see that under the transformation $f_{\mathcal{U}}$ the link field transforms into

$$Y(x_{\mathcal{U}}) \to Y(f_{\mathcal{U}}(x_{\mathcal{U}})) \tag{E.32}$$

where

$$\begin{aligned} Y(f_{\mathcal{U}}(x_{\mathcal{U}})) &= Y(x_{\mathcal{U}} + \epsilon_{\mathcal{U}}(x_{\mathcal{U}})) \\ &= x_{\mathcal{U}} + \epsilon_{\mathcal{U}}(x_{\mathcal{U}}) + \xi(x_{\mathcal{U}} + \epsilon_{\mathcal{U}}(x_{\mathcal{U}})) \\ &= x_{\mathcal{U}} + \epsilon_{\mathcal{U}}(x_{\mathcal{U}}) + \xi(x_{\mathcal{U}}) + \epsilon^{\mu}_{\mathcal{U}}\nabla_{\mu}\xi \\ &= x_{\mathcal{U}} + \xi + \delta\xi, \end{aligned} \tag{E.33}$$

where

$$\delta\xi^{\mu} = \epsilon^{\mu}_{\mathcal{U}} + \epsilon^{\alpha}_{\mathcal{U}}\nabla_{\alpha}\xi^{\mu}. \tag{E.34}$$

Thus, we have established how the deformation vector (Goldstone boson) shifts under an infinitesimal coordinate transformation $f_{\mathcal{U}}$.

Now, because $f_{\mathcal{V}}(x) = x + \epsilon_{\mathcal{V}}(x)$, and $f^{-1}_{\mathcal{V}} \circ f_{\mathcal{V}} = \mathrm{id}$ by definition, we can obtain

$$f^{-1}_{\mathcal{V}}(x) = x - \epsilon_{\mathcal{V}}(x). \tag{E.35}$$

We now check that this inverse function satisfies the required property:

$$f^{-1}_{\mathcal{V}} \circ f_{\mathcal{V}}(x) = f^{-1}_{\mathcal{V}}(x + \epsilon_{\mathcal{V}}(x)) = (x + \epsilon_{\mathcal{V}}(x)) - \epsilon_{\mathcal{V}}(x) = x = \mathrm{id}. \tag{E.36}$$

Now, from (E.12)

$$Y(x_{\mathcal{U}}) \to f^{-1}_{\mathcal{V}}(Y_{\mathcal{V}\mathcal{U}}(f_{\mathcal{U}}(x_{\mathcal{U}}))) = f^{-1}_{\mathcal{V}}(Y) = Y - \epsilon_{\mathcal{V}}(Y). \tag{E.37}$$

So, under $f_{\mathcal{V}}$,

$$\begin{aligned} Y \to Y - \epsilon_{\mathcal{V}}(Y) &= x + \xi - \epsilon_{\mathcal{V}}(x + \xi) \\ &= x + \xi - \epsilon_{\mathcal{V}} - \xi^{\alpha}\nabla_{\alpha}\epsilon_{\mathcal{V}} - \frac{1}{2}\xi^{\alpha}\xi^{\beta}\nabla_{\alpha}\nabla_{\beta}\epsilon_{\mathcal{V}} + \dots \\ &= x + \xi + \delta\xi, \end{aligned} \tag{E.38}$$

where

$$\delta\xi^{\mu} = -\epsilon^{\mu}_{\mathcal{V}}(x + \xi) = -\epsilon^{\mu}_{\mathcal{V}} - \xi^{\alpha}\nabla_{\alpha}\epsilon^{\mu}_{\mathcal{V}} - \frac{1}{2}\xi^{\alpha}\xi^{\beta}\nabla_{\alpha}\nabla_{\beta}\epsilon^{\mu}_{\mathcal{V}} + \dots \tag{E.39}$$

Now we have an expression for how the deformation vector transforms under the internal transformations $f_{\mathcal{U}}$ and $f_{\mathcal{V}}$:

$$\delta\xi^{\mu} = \epsilon^{\mu}_{\mathcal{U}} - \epsilon^{\mu}_{\mathcal{V}} + \epsilon^{\alpha}_{\mathcal{U}}\nabla_{\alpha}\xi^{\mu} - \xi^{\alpha}\nabla_{\alpha}\epsilon^{\mu}_{\mathcal{V}} - \frac{1}{2}\xi^{\alpha}\xi^{\beta}\nabla_{\alpha}\nabla_{\beta}\epsilon^{\mu}_{\mathcal{V}} + \dots \tag{E.40}$$

Under $f_{\mathcal{V}}$,

$$\delta T_{\mu\nu} = \delta t_{\mu\nu} + \xi^{\rho}\nabla_{\rho}\delta t_{\mu\nu} + 2\delta t_{\alpha(\mu}\nabla_{\nu)}\xi^{\alpha} + \delta\xi^{\rho}\nabla_{\rho}t_{\mu\nu} + 2t_{\alpha(\mu}\nabla_{\nu)}\delta\xi^{\alpha} \tag{E.41}$$

E.4 Example: massive gravity

As an example, we could take the tensor field to be the metric, in which case

$$h_{\mu\nu} = g_{\mu\nu} - \frac{\partial x^{\alpha}_{\mathcal{V}}}{\partial x^{\mu}_{\mathcal{U}}}\frac{\partial x^{\beta}_{\mathcal{V}}}{\partial x^{\nu}_{\mathcal{U}}}\bar{g}_{\alpha\beta}(Y_{\mathcal{V}\mathcal{U}}(x_{\mathcal{U}})). \tag{E.42}$$

After using the link field $Y(x) = x - \xi$, and writing $\mathtt{h} = g - \bar{g}$,

$$h_{\mu\nu} = \mathtt{h}_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \tag{E.43}$$

The deformation vector can be decomposed into a scalar piece and a divergence-less vector:

$$h_{\mu\nu} = \mathtt{h}_{\mu\nu} + 2\nabla_{(\mu}A_{\nu)} + 2\nabla_{\mu}\nabla_{\nu}\sigma. \tag{E.44}$$

A generic mass term in the action will be

$$S \supset S_{\text{mass}} = \int_{M} \mathcal{W}^{\mu\nu\alpha\beta} h_{\mu\nu} h_{\alpha\beta}, \tag{E.45}$$

and so will contain terms such as

$$\mathtt{h}^2, \quad \mathtt{h}\nabla A, \quad \nabla A\nabla A, \quad \mathtt{h}\nabla\nabla\sigma, \quad \nabla A\nabla\nabla\sigma, \quad \nabla\nabla\sigma\nabla\nabla\sigma. \tag{E.46}$$

These second derivative terms of the σ-field in the Lagrangian will generically produce ghosts, unless the way in which they are contracted made them vanish. Arkani observes that the combination of the coefficients that is required to make these terms vanish gives rise to the Pauli-Feirz term, $h^{\mu\nu}h_{\mu\nu} - h^2$. The Pauli-Feirz mass-tensor is

$$\mathcal{W}^{\mu\nu\alpha\beta}_{(\text{PF})} = g^{\mu(\alpha}g^{\beta)\nu} - g^{\mu\nu}g^{\alpha\beta}. \tag{E.47}$$

F Multi-constituent fluids

Here we review Carter's construction [77]. See also [78], [79], [80]

Suppose there are a set of currents, n^{μ}_{X}; here, the subscript "X" is a constituent index, and when they are repeated they are assumed to be summed over (unless explicitly stated). In the material space, the variation of the master function Λ is

$$\mathrm{d}\Lambda = \frac{\partial\Lambda}{\partial\gamma_{IJ}}\mathrm{d}\gamma_{IJ} + \frac{\partial\Lambda}{\partial^{\perp}n^{I}_{\text{X}}}\mathrm{d}^{\perp}n^{I}_{\text{X}} + \frac{\partial\Lambda}{\partial n^{//}_{\text{X}}}\mathrm{d}n^{//}_{\text{X}}. \tag{F.1}$$

The convective derivative in terms of space-time fields is thus

$$\mathrm{d}_{\text{C}}\Lambda \;=\; \frac{\partial\Lambda}{\partial\gamma_{\mu\nu}}\mathrm{d}_{\text{C}}\gamma_{\mu\nu} + \frac{\partial\Lambda}{\partial^{\perp}n^{\mu}_{\text{X}}}\mathrm{d}^{\perp}_{\text{C}}n^{\mu}_{\text{X}} + \frac{\partial\Lambda}{\partial n^{//}_{\text{X}}}\mathrm{d}_{\text{C}}n^{//}_{\text{X}}. \tag{F.2}$$

We can bring the longitudinal and orthogonal projective pieces together and write

$$\mathrm{d}_{\mathrm{C}}\Lambda = \frac{\partial\Lambda}{\partial\gamma_{\mu\nu}}\mathrm{d}_{\mathrm{C}}\gamma_{\mu\nu} + \frac{\partial\Lambda}{\partial n^{\mu}_{\mathrm{X}}}\mathrm{d}_{\mathrm{C}}n^{\mu}_{\mathrm{X}} \tag{F.3}$$

in which

$$\frac{\partial\Lambda}{\partial n^{\mu}_{\mathrm{X}}} = \frac{\partial\Lambda}{\partial^{\perp}n^{\mu}_{\mathrm{X}}} - u_{\mu}\frac{\partial\Lambda}{\partial n^{//}_{\mathrm{X}}}. \tag{F.4}$$

with

$${}^{\perp}n^{\mu}_{\mathrm{X}} = \gamma^{\mu}{}_{\nu}n^{\nu}_{\mathrm{X}}, \qquad n^{//}_{\mathrm{X}} = -u_{\mu}n^{\mu}_{\mathrm{X}}. \tag{F.5}$$

The Lagrangian variation is then given by

$$\mathrm{d}_{\mathrm{L}}\Lambda = \chi^{\mathrm{X}}_{\mu}\mathrm{d}_{\mathrm{L}}n^{\mu}_{\mathrm{X}} + \frac{\partial\Lambda}{\partial g_{\mu\nu}}\mathrm{d}_{\mathrm{L}}g_{\mu\nu}, \tag{F.6}$$

in which the momentum covectors are

$$\chi^{\mathrm{X}}_{\mu} = \frac{\partial\Lambda}{\partial n^{\mu}_{\mathrm{X}}}, \tag{F.7a}$$

and

$$\frac{\partial\Lambda}{\partial g_{\mu\nu}} = \frac{\partial\Lambda}{\partial\gamma_{\mu\nu}} - u^{\rho}\chi^{\mathrm{X}}_{\rho}n^{\sigma}_{\mathrm{X}}\left(\gamma^{(\mu}{}_{\sigma}u^{\nu)} - \tfrac{1}{2}u_{\sigma}u^{\mu}u^{\nu}\right). \tag{F.7b}$$

The momentum covectors χ^{X}_{μ} can be decomposed in a physically intuitive manner as

$$\chi^{\mathrm{X}}_{\mu} = \mu^{\mathrm{X}}u_{\mu} + p^{\mathrm{X}}_{\mu}, \tag{F.8}$$

where the effective inertia μ^{X} and 3-momentum p^{X}_{μ} are given by

$$\mu^{\mathrm{X}} = -\frac{\partial\Lambda}{\partial n^{//}_{\mathrm{X}}}, \qquad p^{\mathrm{X}}_{\mu} = \frac{\partial\Lambda}{\partial^{\perp}n^{\mu}_{\mathrm{X}}}. \tag{F.9}$$

The variations of all quantities have two origins. The first is Eulerian variation, d_{E}, which is variation with respect to the gravitational background. Secondly there are variations due to displacements of the world-lines of the flow. These are produced by ξ^{μ} which is displacement of the flow, and ξ^{μ}_{X} for independent displacements of the world-lines of each constituent of the substance. This all translates into the following expressions for the Lagrangian variation of the metric and currents,

$$\mathrm{d}_{\mathrm{L}}g_{\mu\nu} = \mathrm{d}_{\mathrm{E}}g_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}, \tag{F.10a}$$

$$\mathrm{d}_{\mathrm{L}}n^{\mu}_{\mathrm{X}} = -n^{\mu}_{\mathrm{X}}\left(\nabla_{\nu}\eta^{\nu}_{\{\mathrm{X}\}} + \tfrac{1}{2}g^{\alpha\beta}\mathrm{d}_{\mathrm{L}}g_{\alpha\beta}\right) + n^{\nu}_{\mathrm{X}}\nabla_{\nu}\eta^{\mu}_{\{\mathrm{X}\}} - \eta^{\nu}_{\{\mathrm{X}\}}\nabla_{\nu}n^{\mu}_{\mathrm{X}}, \tag{F.10b}$$

in which η^μ_{X} represents the difference between the wordline displacement of each constituent

$$\eta^\mu_{\mathrm{X}} = \xi^\mu_{\mathrm{X}} - \xi^\mu, \tag{F.11}$$

and where the curly-braces are used to exclude the use of summations over the constituent indices. The Eulerian variation of the currents can be written as

$$\mathrm{d}_{\mathrm{E}} n^\mu_{\mathrm{X}} = -\frac{1}{2} n^\mu_{\mathrm{X}} g^{\alpha\beta} \mathrm{d}_{\mathrm{E}} g_{\alpha\beta} + n^\nu_{\mathrm{X}} \nabla_\nu \xi^\mu_{\{\mathrm{X}\}} - \nabla_\nu \left(n^\mu_{\mathrm{X}} \xi^\nu_{\{\mathrm{X}\}} \right). \tag{F.12}$$

Putting this together, we arrive at the measure-weighted Eulerian variation of the master function,

$$\frac{1}{\sqrt{-g}} \mathrm{d}_{\mathrm{E}} \left(\sqrt{-g}\, \Lambda \right) = \frac{1}{2} T^{\mu\nu} \mathrm{d}_{\mathrm{E}} g_{\mu\nu} + f_\mu \xi^\mu + f^{\mathrm{X}}_\mu \xi^\mu_{\mathrm{X}} + \nabla_\mu S^\mu, \tag{F.13}$$

in which the energy-momentum tensor is given by

$$T^{\mu\nu} = 2 \frac{\partial \Lambda}{\partial g_{\mu\nu}} + \left(\Lambda - \chi^{\mathrm{X}}_\sigma n^\sigma_{\mathrm{X}} \right) g^{\mu\nu}, \tag{F.14}$$

the individual forces are

$$f^{\mathrm{X}}_\mu = \chi^{\mathrm{X}}_\mu \nabla_\nu n^\nu_{\{\mathrm{X}\}} + 2 n^\nu_{\{\mathrm{X}\}} \nabla_{[\nu} \chi^{\mathrm{X}}_{\mu]}, \tag{F.15}$$

the self-force is

$$f_\mu = \nabla_\nu T^\nu{}_\mu - \sum_{\mathrm{X}} f^{\mathrm{X}}_\mu, \tag{F.16}$$

and the total derivative current is given by

$$S^\mu \equiv \left(T^{\mu\nu} - \Lambda g^{\mu\nu} \right) \xi_\nu + 2 n^{[\mu}_{\{\mathrm{X}\}} \eta^{\nu]}_{\mathrm{X}} \pi^{\mathrm{X}}_\nu. \tag{F.17}$$

By recalling (F.7b), the energy-momentum tensor (F.14) can be written in a more conventional manner, where we isolate the energy density, flux, and pressure tensor as

$$T^{\mu\nu} = \rho u^\mu u^\nu + 2 Q^{(\mu} u^{\nu)} + P^{\mu\nu}, \tag{F.18}$$

with

$$\rho = {}^{\perp} n_{\mathrm{X}} p^{\mathrm{X}}_\mu - \Lambda, \tag{F.19a}$$

$$Q^\mu = \mu^{\mathrm{X}\perp} n^\mu_{\mathrm{X}}, \tag{F.19b}$$

$$P^{\mu\nu} = 2 \frac{\partial \Lambda}{\partial \gamma_{\mu\nu}} + \left(\Lambda - \chi^{\mathrm{X}}_\sigma n^\sigma_{\mathrm{X}} \right) \gamma^{\mu\nu}. \tag{F.19c}$$

F.1 Two constituent model

- Two constituent fluid was studied in [79], and the cosmology in [81] (although that was just at the background)

Let us expose all formulae in the two-constituent model. The particle currents are

$$n^{\mu}_{\scriptscriptstyle 0} = s^{\mu}, \qquad n^{\mu}_{\scriptscriptstyle 1} = n^{\mu}, \tag{F.20}$$

and the momentum covectors are

$$\chi^{\scriptscriptstyle 0}_{\mu} = \theta_{\mu}, \qquad \chi^{\scriptscriptstyle 1}_{\mu} = \chi_{\mu}. \tag{F.21}$$

We also assume covariance, which means that the master function is a function of scalars formed out of all particle currents:

$$\Lambda = \Lambda(n, s, j), \tag{F.22}$$

where

$$n = (-n_{\mu} n^{\mu})^{1/2}, \qquad s = (-s_{\mu} s^{\mu})^{1/2}, \qquad j = (-n_{\mu} s^{\mu})^{1/2}. \tag{F.23}$$

Notice that j quantifies the leaking between the currents (also known as entrainment). The convective derivative of the master function (F.3) now reads

$$\mathrm{d}_{\scriptscriptstyle \mathrm{C}}\Lambda = \frac{\partial \Lambda}{\partial \gamma_{\mu\nu}} \mathrm{d}_{\scriptscriptstyle \mathrm{C}}\gamma_{\mu\nu} + \frac{\partial \Lambda}{\partial n^{\mu}} \mathrm{d}_{\scriptscriptstyle \mathrm{C}} n^{\mu} + \frac{\partial \Lambda}{\partial s^{\mu}} \mathrm{d}_{\scriptscriptstyle \mathrm{C}} s^{\mu}, \tag{F.24}$$

in which

$$\frac{\partial \Lambda}{\partial \gamma_{\mu\nu}} = -\frac{1}{2n}\frac{\partial \Lambda}{\partial n} n^{\mu} n^{\nu} - \frac{1}{2s}\frac{\partial \Lambda}{\partial s} s^{\mu} s^{\nu} - \frac{1}{2j}\frac{\partial \Lambda}{\partial j} n^{\mu} s^{\nu}, \tag{F.25a}$$

$$\frac{\partial \Lambda}{\partial n^{\mu}} = -\frac{1}{n}\frac{\partial \Lambda}{\partial n} n_{\mu} - \frac{1}{2j}\frac{\partial \Lambda}{\partial j} s_{\mu}, \tag{F.25b}$$

$$\frac{\partial \Lambda}{\partial s^{\mu}} = -\frac{1}{s}\frac{\partial \Lambda}{\partial s} s_{\mu} - \frac{1}{2j}\frac{\partial \Lambda}{\partial j} n_{\mu}. \tag{F.25c}$$

Hence, the Lagrangian variation of the master function is

$$\mathrm{d}_{\scriptscriptstyle \mathrm{L}}\Lambda = \frac{\partial \Lambda}{\partial g_{\mu\nu}} \mathrm{d}_{\scriptscriptstyle \mathrm{L}} g_{\mu\nu} + \chi_{\mu} \mathrm{d}_{\scriptscriptstyle \mathrm{L}} n^{\mu} + \theta_{\mu} \mathrm{d}_{\scriptscriptstyle \mathrm{L}} s^{\mu}, \tag{F.26}$$

where we read off the expressions for the momentum covectors,

$$\chi_{\mu} \equiv \frac{\partial \Lambda}{\partial n^{\mu}} = g_{\mu\nu} \left(B^{\mathrm{n}} n^{\nu} + C s^{\nu} \right), \tag{F.27a}$$

$$\theta_\mu = \frac{\partial \Lambda}{\partial s^\mu} = g_{\mu\nu}\left(B^{\mathrm{s}} s^\nu + C n^\nu\right), \tag{F.27b}$$

where the cofficients are

$$B^{\mathrm{n}} \equiv -\frac{1}{n}\frac{\partial \Lambda}{\partial n}, \qquad B^{\mathrm{s}} \equiv -\frac{1}{s}\frac{\partial \Lambda}{\partial s}, \qquad C \equiv -\frac{1}{2j}\frac{\partial \Lambda}{\partial j}. \tag{F.28}$$

We now work in the matter frame and pick n^μ to be parallel to u^μ,

$$n^\mu = n u^\mu, \tag{F.29}$$

and set

$$s^\mu = s^\star (u^\mu + w^\mu), \tag{F.30}$$

with $u^\mu w_\mu = 0$. Hence, the momenta are

$$\chi_\mu = \left(B^{\mathrm{n}} n + C s^\star\right) u_\mu + C s^\star w_\mu, \tag{F.31a}$$

$$\theta_\mu = \left(B^{\mathrm{s}} s^\star + C n\right) u_\mu + B^{\mathrm{s}} s^\star w_\mu \tag{F.31b}$$

We then define

$$\theta^\star \equiv -u^\mu \theta_\mu = B^{\mathrm{s}} s^\star + C n, \qquad \theta^\sharp \equiv B^{\mathrm{s}} s^\star, \tag{F.32}$$

so that

$$\theta_\mu = -\theta^\star u_\mu + \theta^\sharp w_\mu. \tag{F.33}$$

For the heat flux current in the energy-momentum tensor (F.19b) we obtain

$$Q^\mu = \theta^\star s^\star w^\mu. \tag{F.34}$$

F.2 Including dissipative effects

The idea is to also include the effects of dissipation, which have so far neglected. Carters construction for such effects takes as a starting point his variational formalism and deforms it in some way. Fundamentally, dissipative effects need to be included “by hand”: at what level the “by hand” bit is done will have an effect on how physically intuitive the resulting effect is. And so, Carter’s argument is that is is better to introduce the dissipation by hand to a set of equations which were constructed from a well-posed variational principle, than to construct dissipation by hand from the outset.

We set

$$f_\mu^{\text{ext}} = f_\mu + \sum_{\text{X}} f_\mu^{\text{X}}, \tag{F.35}$$

so that

$$\nabla_\nu T^\nu{}_\mu = f_\mu^{\text{ext}}. \tag{F.36}$$

This allows us to begin to think about how to include dissipative effects.

If we now separate out the zeroth constituent from the force balance,

$$f_\mu^{\text{ext}} = f_\mu + f_\mu^{0} + \sum_{\text{A}} f_\mu^{\text{A}}, \tag{F.37}$$

and set

$$n_0^\mu = s^\mu, \qquad \chi_\mu^0 = \Theta_\mu \tag{F.38}$$

to be the entropy and temperature. Now define

$$\beta_0^\mu = \frac{s^\mu}{-s^\nu \Theta_\nu} \tag{F.39}$$

And so it follows from the definition of f_μ^0 (F.15) that

$$\nabla_\mu s^\mu = -\beta_0^\mu f_\mu^0. \tag{F.40}$$

And thus, the "second law of thermodynamics" requirement $\nabla_\mu s^\mu \geq 0$ translates into

$$-\beta_0^\mu f_\mu^0 \geq 0. \tag{F.41}$$

F.3 Example of the equation of state

Here we are interested in thinking about the possible invariants that the master function can be a function of. Since we have a set of currents n_{X}^μ it seems natural to write

$$\Lambda = \Lambda(\sigma), \tag{F.42}$$

with

$$\sigma = l^{\text{XY}} g_{\mu\nu} n_{\text{X}}^\mu n_{\text{Y}}^\nu, \tag{F.43}$$

where the symmetric quantities l^{XY} determine the mixing (or, entrainment) between the constituents. From (F.5) it follows that

$$n_{\text{X}}^\mu = -n_{\text{X}}^{/\!/} u^\mu + {}^\perp n_{\text{X}}^\mu, \tag{F.44}$$

and thus

$$\frac{\partial \sigma}{\partial n^{//}_{\mathrm{X}}} = -2u_\nu n^\nu_{\mathrm{Y}} l^{\mathrm{XY}}, \qquad \frac{\partial \sigma}{\partial^\perp n^\mu_{\mathrm{X}}} = 2\gamma_{\nu\mu} n^\nu_{\mathrm{Y}} l^{\mathrm{XY}} \tag{F.45}$$

and therefore

$$\mu^{\mathrm{X}} = -2\frac{\partial \Lambda}{\partial \sigma} n^{//}_{\mathrm{Y}} l^{\mathrm{XY}}, \qquad p^{\mathrm{X}}_\mu = 2\frac{\partial \Lambda}{\partial \sigma}{}^\perp n_{\mathrm{Y}\mu} l^{\mathrm{XY}} \tag{F.46}$$

We write

$$l^{\mathrm{XY}} = \delta^{\mathrm{XY}} + e^{\mathrm{XY}}, \tag{F.47}$$

where δ^{XY} is the Kronecker-delta and e^{XY} controls the entrainment (i.e., it is traceless).

START AGAIN!!

Let the master function be dependent upon

$$\Lambda = \Lambda(\sigma_{\mathrm{XY}}) \tag{F.48}$$

where

$$\sigma_{\mathrm{XY}} = g_{\mu\nu} n^\mu_{\mathrm{X}} n^\nu_{\mathrm{Y}}. \tag{F.49}$$

G Saint-Venant compatibility conditions

See [82, 83]

Begin with

$$2\nabla_{[\sigma}\nabla_{\rho]} A_\nu = R^\beta{}_{\nu\rho\sigma} A_\beta, \tag{G.1}$$

$$\nabla_{[\mu}\nabla_{\nu]} B_{\alpha\beta} = B_{\lambda(\alpha} R^\lambda{}_{\beta)\nu\mu}, \tag{G.2}$$

Now,

$$\delta R^a{}_{bcd} = \nabla_c \delta\Gamma^a{}_{bd} - \nabla_d \delta\Gamma^a{}_{bc} \tag{G.3}$$

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}\overline{g}^{ad}\left(\nabla_b \delta g_{cd} + \nabla_c \delta g_{bd} - \nabla_d \delta g_{bc}\right) \tag{G.4}$$

And so

$$\delta R^a{}_{bec} = \overline{g}^{ad}\left(\delta g_{f(b} R^f{}_{d)ce} + \nabla_c \nabla_{[d}\delta g_{b]e} + \nabla_e \nabla_{[b}\delta g_{d]c}\right) \tag{G.5}$$

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