

# Anisotropic Stars II: Stability

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## Abstract

We investigate the stability of self-gravitating spherically symmetric anisotropic spheres under radial perturbations. We consider both the Newtonian and the full general-relativistic perturbation treatment. In the general-relativistic case, we extend the variational formalism for spheres with isotropic pressure developed by Chandrasekhar. We find that, in general, when the tangential pressure is greater than the radial pressure, the stability of the anisotropic sphere is enhanced when compared to isotropic configurations. In particular, anisotropic spheres are found to be stable for smaller values of the adiabatic index  $\gamma$ .

**KEYWORDS:** Radial Perturbations, Stars, Anisotropic Pressure

## 1 Introduction

In a recent paper [1], we presented a series of new exact solutions of the Einstein field equations for self-gravitating, general-relativistic spheres with anisotropic pressure. We have found that the presence of pressure anisotropy, for a large variety of *ansatze* for its functional form, has important implications for the physical properties of self-gravitating objects. Namely, both the object's critical mass and surface redshift are modified, and may violate well-known bounds for isotropic objects ( $2M/R < 8/9$  and  $z_s \leq 2$ ). We have shown that this is true not only for stars of constant energy density, but also for objects with  $\rho \propto 1/r^2$ , often used to model neutron star interiors. Given the fact that pressure isotropy is an assumption not required by imposing

spherical symmetry, it is clearly of great relevance to investigate if, indeed, these anisotropic configurations are stable against radial perturbation and, thus, better candidates to exist in Nature.

The aim of the present paper is, then, to develop a formalism which can be used to test the stability of anisotropic spheres against small radial perturbations. Our formalism is a generalization of the variational principle used for investigating the stability properties of isotropic objects. We reduce the stability analysis to an eigenvalue problem, where the eigenvalues are the frequencies of the radial modes.

The dynamical stability of isotropic spheres has been extensively studied by various authors [2, 3, 4, 5, 6]. A calculation based on the concept of extremal energy was presented by Fowler [3]. Cocke [4] performed a calculation based on the method of extremal entropy. Gleiser, and Gleiser and Watkins applied Chandrasekhar's variational method to investigate the stability of boson stars, self-gravitating spheres of complex scalar fields, which are naturally anisotropic [6].

Chandrasekhar considered the dynamical stability of isotropic spheres as an eigenvalue problem [2]. He used an analytical approach to compute the eigenfrequencies of radial oscillations for isotropic spherical stars. The study of the stability of a star thus becomes a Sturm-Liouville problem.

The main result of these studies is that, for dynamical stability in general relativity, isotropic spheres must have an adiabatic index (or exponent)

$$\gamma \geq \frac{4}{3} + \kappa \frac{M}{R}, \quad (1)$$

where  $\kappa$  is a number of order unity, that depends on the structure of the star, and  $M$  and  $R$  are the star's mass and radius, respectively. For white dwarfs,  $\kappa = 2.25$ .

The stability of anisotropic spheres in general relativity was studied numerically by Hillebrandt and Steinmetz [7]. An analytical approach in the spirit of Chandrasekhar's work for isotropic spheres, however, does not seem to exist for anisotropic spheres. Our goal is to obtain this approach.

This paper is organized as follows. In the next section, we obtain exact solutions for several examples of Newtonian anisotropic spheres, and study their stability properties. This will give us some insight into the effects of anisotropy on the stability of self-gravitating objects. We then proceed, in section 3, to derive the full general-relativistic perturbation formalism for

anisotropic spheres. In section 4, we apply the formalism to anisotropic spheres of constant energy density. In section 5, we apply it to anisotropic spheres with  $\rho \propto 1/r^2$ . In both sections, we follow the exact solutions derived in [1]. In section 6 we summarize our results, and discuss possible avenues for future work.

## 2 Newtonian Anisotropic Spheres

### 2.1 Exact solutions for Newtonian Anisotropic Spheres

We consider the dynamics of anisotropic spheres under the influence of Newtonian gravity. The equation of hydrostatic equilibrium with anisotropic pressure in Newtonian gravity is

$$p'_r = -\frac{m(r)\rho(r)}{r^2} + \frac{2}{r}(p_t - p_r) \quad (2)$$

where  $p_r$  is the radial pressure,  $p_t$  is the tangential pressure,  $\rho$  is the energy density and

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr', \quad (3)$$

is the mass contained in a sphere of radius  $r$ . This equation may be obtained as the Newtonian limit of the generalized Tolman-Oppenheimer-Volkov equation for general relativistic hydrodynamical equilibrium, or it may be derived using the principles of Newtonian fluid mechanics.

The pressure in isotropic spheres with constant density,  $\rho_0$ , is given by

$$p_r = \frac{2\pi}{3}\rho_0^2(R^2 - r^2). \quad (4)$$

We note that, in Newtonian gravity, the pressure at the center of a sphere with constant density and isotropic pressure can only become infinite if the radius of the sphere is infinite.

We will now solve eq. 2 for various ansatze connecting  $p_r$  and  $p_t$  at constant density  $\rho_0$ . These ansatze are chosen so as to correspond to the choices we will make for the full general relativistic cases.

**Case I:**  $p_t - p_r = C \rho_0^2 r^2$

This ansatz assumes that the anisotropy term in eq. (2) is proportional to

the first term on the right hand side of the equation, *i.e.*, the anisotropy is chosen to mimic the behavior of the purely gravitational term. This ansatz can be interpreted as the Newtonian limit of the ansatz that Bowers and Liang used to solve the full general relativistic TOV equation [12]. With this ansatz eq. (2) becomes

$$p'_r = -\frac{4\pi}{3}\rho_0^2 r + 2C\rho_0^2 r \quad (5)$$

and the solution is

$$p_r = \rho_0^2 \left( \frac{2\pi}{3} - C \right) (R^2 - r^2) \quad (6)$$

Since we are considering spheres with constant energy density  $\rho_0$ , from eq. (3),

$$m(r) = \frac{4}{3}\pi\rho_0 r^3 . \quad (7)$$

Therefore, here we can also write

$$p_r = \rho_0 \left( \frac{1}{4} - \frac{3C}{8\pi} \right) \left( \frac{2M}{R} - \frac{2m}{r} \right) . \quad (8)$$

Comparing this solution with the isotropic case ( $C = 0$ ), we see that the radial pressure has the same spatial behavior in both cases, and they differ only by a multiplicative factor that depends on the amount of anisotropy  $C$ . If we define an effective density:

$$\bar{\rho} = \left( 1 - \frac{3}{2\pi}C \right)^{1/2} \rho_0 , \quad (9)$$

then we can write

$$p_r = \frac{2\pi}{3}\bar{\rho}^2 (R^2 - r^2) . \quad (10)$$

Thus, in this model the effect of the anisotropy can be considered as a scaling of the density of the sphere. We can take this scaling interpretation a step further by reintroducing into the expression for  $p_r$  the gravitational constant  $G$ :

$$p_r = \rho_0^2 G \left( \frac{2\pi}{3} - C \right) (R^2 - r^2) . \quad (11)$$

We now define

$$\bar{G} = \left( 1 - \frac{3C}{2\pi} \right) G \quad (12)$$

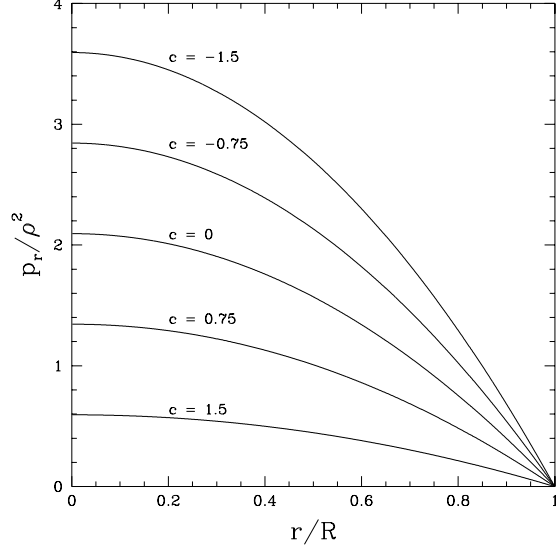


Figure 1: Radial pressure  $p_r$  as a function of  $r$  for the ansatz  $p_t - p_r = C\rho^2 r^2$ , parameterized by values of  $C$ .

and find that

$$p_r = \frac{2\pi}{3}\rho_0^2\bar{G}\left(R^2 - r^2\right). \quad (13)$$

Thus, we can also interpret anisotropy as a variation of the gravitational constant. This explains why positive values of  $C$  and hence smaller values of  $\bar{G}$  lead to smaller values of the radial pressure. For smaller  $\bar{G}$  the gravitational force between the particles in the sphere is decreased and this leads to a decrease in the radial pressure needed to stabilize the sphere. Negative values of  $C$  have the opposite effect, i.e.,  $\bar{G}$  is increased and correspondingly  $p_r$  is also increased. We plot the radial pressure  $p_r$  as a function of the radius  $r$  for several values of the anisotropy  $C$  in figure 1.

We note that, for  $C = 2\pi/3$  ( $\bar{G} = 0$ ), the radial pressure vanishes and becomes negative if  $C > 2\pi/3$ ; in this case, no bound solutions are possible. It is interesting to note that the solution with  $C = 2\pi/3$  has the following form

$$p_r = 0 \quad p_t = \frac{2\pi}{3}\rho_0^2 r^2. \quad (14)$$

Hence, for this particular solution, the sphere is held together by purely tan-

gential stresses. The particles that constitute the sphere are considered to be in random circular orbits [11].

**Case II:**  $p_t - p_r = C \rho_0 p_r r^2$

The solution to eq. (2) with this ansatz is

$$p_r = \frac{2\pi}{3C} \rho_0 \left[ 1 - e^{-C\rho_0(R^2-r^2)} \right]. \quad (15)$$

For this solution, the pressure at the center is

$$p_c = \frac{2\pi}{3C} \rho_0 \left[ 1 - e^{-C\rho_0 R^2} \right], \quad (16)$$

and all values of  $C$  are allowed. For  $C$  small,

$$p_r = \frac{2\pi}{3} \rho_0^2 (R^2 - r^2), \quad (17)$$

and

$$p_t = \frac{2\pi}{3} \rho_0^2 (R^2 - r^2) (1 + C\rho_0 r^2). \quad (18)$$

**Case III:**  $p_t - p_r = C p_r^2 r^2$

Here the solution has two distinct forms depending on whether  $C$  is greater than or less than zero. For  $C < 0$ , the solution is

$$p_r = \rho_0 \left( \frac{2\pi}{3|C|} \right)^{1/2} \tan \left[ \rho_0 \left( \frac{2\pi|C|}{3} \right)^{1/2} (R^2 - r^2) \right], \quad (19)$$

with

$$p_c = \rho_0 \left( \frac{2\pi}{3|C|} \right)^{1/2} \tan \left[ \rho_0 \left( \frac{2\pi|C|}{3} \right)^{1/2} R^2 \right]. \quad (20)$$

Thus  $p_c$  becomes infinite if  $R^2 = \frac{1}{\rho_0} \left( \frac{3\pi}{8|C|} \right)^{1/2}$ , a result that is not possible for Newtonian isotropic spheres with constant density. However, we note that the values of  $C$  for which  $p_c$  becomes infinite in Newtonian gravity are quite large, on the order of  $10^4$  for  $2M/R = 0.05$ . We will see that for the general relativistic case when  $2M/R \sim 1$ , the values of  $C$  for which  $p_c$  becomes

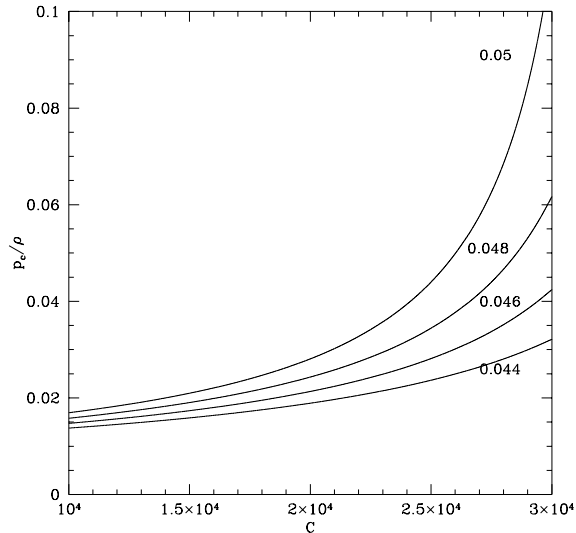


Figure 2: Core pressure  $p_c$  as a function of  $C$  for the ansatz  $p_t - p_r = Cp_r^2 r^2$ , parameterized by values of  $2M/R$ .

infinite are of order 1. We plot the core pressure  $p_c$  as a function of  $C$  for various values of  $2M/R$  in figure 2.

When  $C > 0$  the solution is

$$p_r = \left(\frac{2\pi}{3C}\right)^{1/2} \rho_0 \left[ \frac{\exp \left[ \left(\frac{2\pi C}{3}\right)^{1/2} \rho_0 (r^2 - R^2) \right] - 1}{\exp \left[ \left(\frac{2\pi C}{3}\right)^{1/2} \rho_0 (r^2 - R^2) \right] + 1} \right]. \quad (21)$$

For this solution  $p_c$  is always positive and finite.

## 2.2 Stability of Newtonian Anisotropic Spheres

We now proceed to investigate the effects of small perturbations on the solutions obtained above. This study is important since it allows us to compute the frequencies and normal modes of oscillations, enabling us to establish the dynamical stability of our solutions. We will follow closely the formalism outlined in Shapiro and Teukolsky [8] for isotropic spheres, modifying it

where necessary to the anisotropic case.

It is useful to distinguish between Eulerian and Lagrangian perturbations of the fluid variables. If  $Q(\vec{x}, t)$  is any fluid variable, the Eulerian change relative to the unperturbed value  $Q_0(\vec{x}, t)$  is defined as

$$\delta Q(\vec{x}, t) = Q(\vec{x}, t) - Q_0(\vec{x}, t). \quad (22)$$

The Lagrangian change is defined as

$$\Delta Q(\vec{x}, t) = Q[\vec{x} + \vec{\xi}(\vec{x}, t), t] - Q_0(\vec{x}, t) \quad (23)$$

where  $\vec{\xi}(\vec{x}, t)$  is an infinitesimal displacement of the fluid element.

The Eulerian approach considers changes in the fluid variables at a particular point in space, whereas the Lagrangian approach considers changes in a particular fluid element. The relationship between the two is

$$\Delta = \delta + \vec{\xi} \cdot \vec{\nabla}. \quad (24)$$

The following equations govern the dynamics of the unperturbed system:

1. The continuity equation that connects the density  $\rho$  and velocity  $v$ ,

$$\frac{\partial \rho}{\partial t} + (\rho v)' = 0 ; \quad (25)$$

2. The momentum equation

$$\frac{dv}{dt} = -\frac{1}{\rho} p_r' - \Phi' + \frac{2}{r} (p_t - p_r) , \quad (26)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{d}{dr} ; \quad (27)$$

3. Poisson's equation, the equation that determines the gravitational potential  $\Phi$ ,

$$\frac{1}{r^2} (r^2 \Phi')' = 4\pi \rho . \quad (28)$$



We have adopted spherical symmetry since we are considering radial perturbations.

A Lagrangian perturbation of the momentum equation gives,

$$\Delta \left( \frac{dv}{dt} + \frac{1}{\rho} p'_r + \Phi' - \frac{2}{r} (p_t - p_r) \right) = 0. \quad (29)$$

We note the following:

- From the continuity equation it follows that

$$\Delta \rho = -\rho \frac{1}{r^2} (r^2 \xi)'. \quad (30)$$

- A perturbation of Poisson's equation gives

$$(\delta \Phi)' = -4\pi \rho \xi. \quad (31)$$

- The adiabatic exponent  $\gamma$  is defined using the following expression,

$$\Delta p_r \equiv p_r \gamma \frac{\Delta \rho}{\rho}. \quad (32)$$

- We also find it convenient to introduce the following symbol,

$$\Pi \equiv p_t - p_r. \quad (33)$$

Using eqs. (27)-(30) we can evaluate each term of the perturbed momentum equation:

$$\Delta \frac{dv}{dt} = \frac{d^2 \xi}{dt^2}, \quad (34)$$

$$\Delta \left( \frac{1}{\rho} p'_r \right) = -\frac{\Delta \rho}{\rho^2} p'_r + \frac{1}{\rho} \Delta p'_r = \frac{2}{\rho r} \xi p'_r + \frac{1}{\rho} \left[ -\gamma p_r \frac{1}{r^2} (r^2 \xi)' \right]', \quad (35)$$

$$\Delta(\Phi)' = \frac{d}{dr} + \xi \nabla^2 \Phi - \frac{2}{r} \xi \frac{d\Phi}{dr} = \frac{2\xi}{\rho r} p'_r - \frac{2}{\rho r^2} \Pi \xi, \quad (36)$$

and

$$\Delta \left( \frac{2}{\rho r} \Pi \right) = \frac{2}{\rho r} \frac{1}{r^2} (\xi r^2)' \Pi - \frac{2}{\rho r^2} \xi \Pi + \frac{2}{\rho r} \delta \Pi. \quad (37)$$

Combining all terms in eqs. (34)- (37), we find that radial perturbations are governed by the following equation:

$$\ddot{\xi} - \frac{1}{\rho} \left[ \gamma p_r \frac{1}{r^2} (r^2 \xi)' \right]' + \frac{4}{\rho r} \xi p_r' - \frac{6}{\rho r^2} \xi \Pi - \frac{2}{\rho r} (\xi)' \Pi - \frac{2}{\rho r} \xi \Delta \Pi = 0. \quad (38)$$

We now assume that all variables have a time dependence of the form  $e^{i\omega t}$ . Substituting this form of the time dependence in the above equation, we arrive at an eigenvalue equation for radial oscillations of a Newtonian spherical star;

$$\left[ \gamma p_r \frac{1}{r^2} (r^2 \xi)' \right]' - \frac{4}{r} \xi p_r' + \frac{6}{\rho r^2} \xi \Pi + \frac{2}{r} (\xi)' \Pi + \frac{2}{r} \xi \Delta \Pi + \rho \omega^2 \xi = 0. \quad (39)$$

The boundary conditions for this equation are

$$\xi = 0 \quad \text{at} \quad r = 0, \quad (40)$$

$$\Delta p_r = 0 \quad \text{at} \quad r = R. \quad (41)$$

Equation (39) subject to the boundary conditions (40) and (41) is a Sturm-Liouville eigenvalue problem for  $\omega^2$ . The general theory of these equations gives the following results [10]:

1. The eigenvalues are real and form an infinite discrete sequence,

$$\omega_0^2 \leq \omega_1^2 \leq \omega_2^2 \dots\dots\dots$$

2. The  $\xi_n$  are orthogonal with a weight function  $\rho r^2$ :

$$\int_0^R \xi_n \xi_m \rho r^2 dr = 0, \quad m \neq n.$$

3. The  $\xi_n$  form a complete basis for any function satisfying the boundary conditions 40 and 41.

An important consequence of these results is that, if the fundamental mode of the star is stable ( $\omega_0^2 \geq 0$ ), then all radial modes are stable. Conversely, if the star is radially unstable, the fastest growing instability will be via the fundamental mode ( $\omega_0^2$  more negative than all other  $\omega_n^2$ ).

Equation (39) can be solved for  $\omega^2$ . Multiplying by  $\xi r^2$  and integrating from 0 to  $R$  we find

$$\omega^2 = \frac{\int_0^R \left\{ \gamma p_r \frac{1}{r^2} (r^2 \xi)^{r^2} + 4r \xi^2 p'_r - 6 \xi^2 \Pi - 2r \xi \xi' \Pi - 2r \xi \Delta \Pi \right\} dr}{\int_0^R \rho \xi^2 r^2 dr} . \quad (42)$$

We will now compute the frequency of oscillation for some anisotropic spheres. We will only consider models with  $\gamma = \text{const}$ . First, we note that for stars with isotropic pressure and constant density, under a self-similar deformation  $\xi = \text{const} \times r$ ,

$$\omega_0^2 = 4\pi\rho_0\left(\gamma - \frac{4}{3}\right) . \quad (43)$$

This result says that isotropic stars with  $\gamma = 4/3$  are marginally stable. If  $\gamma$  is less than  $4/3$  then dynamical instability will occur, while if  $\gamma$  is greater than  $4/3$  the star is stable relative to the deformation  $\xi = r$ .

Computing  $\omega$  from eq. (42) for the anisotropic model Case I, with the deformation  $\xi = r$ , we find

$$\omega^2 = 6\rho_0 \left[ \left( \frac{2\pi}{3} - C \right) \left( \gamma - \frac{4}{3} \right) \right] \quad (44)$$

Thus, not surprisingly for this model, anisotropy results in a scaling of the frequency of oscillation. We had already seen earlier that the effect of the anisotropy was equivalent to a scaling of the density, and since the frequency of oscillation is proportional to the density, this result is expected. Thus, for this case, positive anisotropy may slow down the growth of instabilities, but will not reverse their trend (recall that we must have  $C \leq 2\pi/3$ .)

We next consider the model Case II. For this model,

$$\omega^2 = 4\pi\rho_0 \left[ \gamma - \frac{4}{3} + \frac{16}{21} C \rho_0 R^2 \right] \quad (45)$$

The fundamental frequency occurs for  $\omega^2 = 0$ , and this corresponds here to

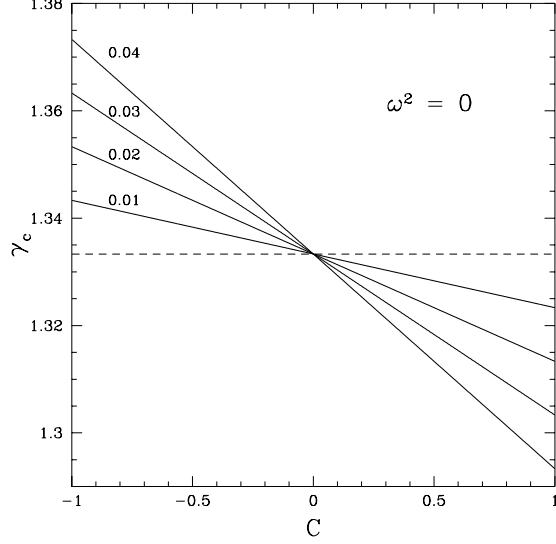


Figure 3: Adiabatic index  $\gamma_c$ , for which  $\omega^2 = 0$ , as a function of  $C$  for the ansatz  $p_t - p_r = C\rho_0 p_r r^2$  (Case II) parameterized by values of  $\rho_0^2 R$ .

$$\gamma_c = \frac{4}{3} - \frac{16}{21}C\rho_0 R^2 . \quad (46)$$

We observe that, since  $\rho_0 R^2$  is always a positive quantity, depending on the sign of  $C$  the value of  $\gamma$  for which  $\omega^2 = 0$  can be less than or greater than  $4/3$ , indicating that positive (negative) values of anisotropy can increase (decrease) the stability of the star.

We plot  $\gamma_c$  as a function of  $C$  for  $\omega^2 = 0$  in figure 3 for the ansatz  $p_t - p_r = C\rho p_r r^2$ .

This concludes our stability study of Newtonian anisotropic spheres. We have seen that the presence of anisotropic pressure in a self-gravitating system can have dramatic effects on the dynamics and stability of the system. In particular, there are some novel features that are present only if the pressure is anisotropic, e.g., infinite core pressure, zero radial pressure and stable objects with  $\gamma < 4/3$ . We will now proceed to study the perturbation problem for relativistic anisotropic compact spheres.

### 3 Stability of General Relativistic Anisotropic Spheres: General Formalism

In this section we will study perturbations of exact solutions of the general relativistic field equations for anisotropic spheres. In particular, we will be concerned with perturbations that preserve spherical symmetry. Under these perturbations, radial motions will ensue. We will develop an analytical approach generalizing work by Chandrasekhar for isotropic spheres [2].

In this section  $\rho_o, \nu_o, \lambda_o, p_{t_o}$  and  $p_{r_o}$  are values of the dynamical variables that satisfy the equations for static equilibrium. The perturbed variables will be written as  $\rho, \nu, \lambda, p_t$  and  $p_r$ , respectively.

#### 3.1 The Perturbed Energy-Momentum Tensor

The energy-momentum tensor for a spherically symmetric spacetime is [9]

$$T_\mu^\nu = (\rho + p_r)u_\mu u^\nu - g_\mu^\nu p_r - l_\mu l^\nu (p_t - p_r) - k_\mu k^\nu (p_t - p_r). \quad (47)$$

Here,

$$u^\mu = \frac{dx^\mu}{ds}, \quad (48)$$

and

$$l_\mu = \delta_\mu^\theta, \quad l^\nu = \delta_\theta^\nu, \quad k_\mu = \delta_\mu^\phi, \quad k^\nu = \delta_\nu^\phi. \quad (49)$$

Since we are considering only radial motions, we will take

$$u^t = e^{-\frac{\nu}{2}}, \quad u_t = e^{\frac{\nu}{2}}, \quad (50)$$

and

$$u^r = v e^{-\frac{\nu_o}{2}}, \quad u_r = v e^{\lambda_o - \frac{\nu_o}{2}}, \quad (51)$$

with

$$v = \frac{dr}{dt}. \quad (52)$$

It should be quite clear when a subscript refers to a time-like coordinate, such as  $u^t$ , or tangential pressure, such as  $p_t$ . We reserve the index  $o$  to unperturbed metric and physical quantities.

Writing  $\rho = \rho_o + \delta\rho$ ,  $p_r = p_{r_o} + \delta p_r$ ,  $p_t = p_{t_o} + \delta p_t$ ,  $\lambda = \lambda_o + \delta\lambda$ , and  $\nu = \nu_o + \delta\nu$  we find that, to first order in  $v$

$$T_t^t = \rho, \quad T_r^r = -p_r, \quad T_\theta^\theta = T_\phi^\phi = -p_t \quad (53)$$

and

$$T_t^r = (\rho_o + p_{r_o})v, \quad T_r^t = (\rho_o + p_{r_o})ve^{(\lambda_o - \nu_o)}. \quad (54)$$

### 3.2 Perturbations of the Dynamical Variables

The set of equations governing radial motions can be written as [2]:

$$(re^{-\lambda})' = 1 - 8\pi\rho r^2, \quad (55)$$

$$\nu' = r(e^\lambda - 1) + 8\pi p_r r e^\lambda, \quad (56)$$

$$\frac{\dot{\lambda}}{r} e^{-\lambda} = 8\pi T_t^r, \quad (57)$$

$$\dot{T}_r^t + \frac{1}{2}T_r^t(\dot{\lambda} + \dot{\nu}) = -p_r' - \frac{1}{2}(\rho + p_r)\nu' + \frac{2}{r}\Pi \quad (58)$$

with

$$\Pi \equiv (p_t - p_r). \quad (59)$$

The zeroth order (or static equilibrium) equations are:

$$(re^{-\lambda_o})' = 1 - 8\pi\rho_o r^2, \quad (60)$$

$$\nu_o' = r(e^{\lambda_o} - 1) + 8\pi p_{r_o} r e^{\lambda_o}, \quad (61)$$

and

$$p_{r_o}' = -(\rho_o + p_{r_o})\frac{\nu_o'}{2} + \frac{2}{r}\Pi_o. \quad (62)$$

We also have the identity

$$\frac{e^{-\lambda_o}}{r}(\lambda_o' + \nu_o') = 8\pi(p_{r_o} + \rho_o). \quad (63)$$

We now linearize eqs. (55)- (58), taking into consideration eqs. (60) - (62). Since we consider all perturbations to be of order  $v$ , we find that to first order in  $v$ , eqs. (55)- (58) imply

$$\left(r e^{-\lambda_o} \delta \lambda\right)' = 8 \pi r^2 \delta \rho, \quad (64)$$

$$\frac{e^{-\lambda_o}}{r} (\delta \nu' - \nu_o' \delta \lambda) = \frac{e^{\lambda_o}}{r^2} \delta \lambda + 8 \pi \delta p_r, \quad (65)$$

$$\delta \dot{\lambda} \frac{e_o^{-\lambda}}{r} = -8 \pi (\rho_o + p_{r_o}) v, \quad (66)$$

$$e^{\lambda_o - \nu_o} (p_{r_o} + \rho_o) \dot{v} + (\delta p_r)' + \frac{1}{2} (p_{r_o} + \rho_o) (\delta \nu)' - \frac{1}{2} (\delta \rho + \delta p_r) \nu_o' + \frac{2}{r} \delta \Pi = 0. \quad (67)$$

We now introduce a “Lagrangian displacement”  $\xi$  defined by

$$v = \frac{\partial \xi}{\partial t}. \quad (68)$$

Integrating eq. (66), we find that

$$\delta \lambda \frac{e^{-\lambda_o}}{r} = -8 \pi (\rho_o + p_{r_o}) \xi. \quad (69)$$

Taking into consideration eq. (63), the above equation becomes

$$\delta \lambda = -\xi (\lambda_o' + \nu_o'). \quad (70)$$

We can now combine eqs. (64) and (70) to get

$$\delta \rho = -\frac{1}{r^2} \left[ r^2 (\rho_o + p_{r_o}) \xi \right]'. \quad (71)$$

Substituting for the expression for  $p_{r_o}'$  from eq. (62) into the above equation, we find that

$$\delta \rho = -\xi \rho_o' - (\rho_o + p_{r_o}) \frac{1}{r^2} e^{\nu_o/2} \left( r^2 e^{-\nu_o/2} \xi \right) - \frac{2\xi}{r} \Pi_o. \quad (72)$$

We now consider eq. (65). Using eqs. (62) and (63) we find

$$(p_{r_o} + \rho_o) (\delta \nu)' = \left[ \delta p_r - (p_{r_o} + \rho_o) \left( \nu_o' + \frac{1}{r} \right) \xi \right] (\lambda_o + \nu_o). \quad (73)$$

We note that eqs. (70), (72) and (73) allows us to express  $\delta \lambda$ ,  $\delta \rho$ , and  $\delta \nu$  in terms of  $\delta p_r$ ,  $v$ , and the unperturbed variables. We need to impose an

extra condition on the system in order to obtain an expression for  $\delta p_r$ . The condition we shall impose is the conservation of baryon number. Further, we note that  $\delta\Pi$  can always be expressed in terms of the unperturbed variables once  $\delta p_r$  is given in terms of these variables.

Chandrasekhar [2] derived an expression for  $\delta p_r$  from the law of conservation of baryon number in general relativity. Since we are not making any new assumptions with respect to Chandrasekhar (except, of course, we are considering anisotropic pressure), we will only outline the basic steps of his derivation here.

The law of conservation of baryon number density in general relativity can be written as

$$(nu^\alpha)_{;\alpha} = 0 , \quad (74)$$

where  $n$  is the number density for baryons and  $u^\alpha$  is the four-velocity of the fluid. Taking

$$n = n_o(r) + \delta n(r, t) , \quad (75)$$

and recalling that, to first order in  $v$ ,  $u^\alpha$  is given by eqs. (50) and (51), eq. (74) becomes

$$e^{-\nu_o/2} \delta \dot{n} + \frac{1}{r^2} (n_o r^2 v e^{-\nu_o/2}) + \frac{1}{2} n_o e^{-\nu_o/2} \delta \dot{\lambda} + \frac{1}{2} e^{-\nu_o/2} v (\lambda_o + \nu_o)' = 0. \quad (76)$$

Since  $v = \dot{\xi}$ , eq. (76) integrates to give

$$\delta n + \frac{e^{\nu_o/2}}{r^2} (n_o r^2 v e^{-\nu_o/2})' + \frac{1}{2} n_o [\delta \lambda + \xi (\lambda_o + \nu_o)'] = 0. \quad (77)$$

The last term on the left-hand side of eq. (77) vanishes on account of eq. (70), and we obtain

$$\delta n = -\frac{e^{\nu_o/2}}{r^2} (n_o r^2 v e^{-\nu_o/2})' . \quad (78)$$

The first law of thermodynamics in general relativity is obtained by combining

$$u_\nu T^{\mu\nu}{}_{;\mu} = 0, \quad (79)$$

with the law of conservation of baryon number given by eq. (74). Using the expressions for  $T^{\mu\nu}$  and  $u_\mu$  from eqs. (47), (50) and (51), we find that,

$$p_r d\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right) + \frac{2}{r} (p_t - p_r) \frac{v}{n} dr = 0. \quad (80)$$



Thus, in general, the equation of state is given by

$$n \equiv n(\rho, p_r, \Pi) . \quad (81)$$

However, since we are considering systems where the tangential pressure is given in terms of the radial pressure and the density (recall that in generating exact solutions in the previous chapter we assumed various ansatze for  $p_t - p_r$ ), we will take

$$n \equiv n(\rho, p_r) . \quad (82)$$

For this  $n$ , we have

$$\delta n = \frac{\partial n_o}{\partial \rho} \delta \rho + \frac{\partial n_o}{\partial p_r} \delta p_r + \frac{\partial n_o}{\partial r} dr. \quad (83)$$

Substituting for  $\delta n$  from eq. (78) and  $\delta \rho$  from eq. (72), we find that

$$\begin{aligned} -\xi \frac{dn_0}{dr} - n_o \frac{e^{\nu_o}}{r^2} (r^2 e^{\nu_o/2} \xi)' &= - (p_{r_o} + \rho_o) \frac{e^{\nu_o/2}}{r^2} (r^2 e^{\nu_o/2} \xi)' \frac{\partial n_0}{\partial \rho} \\ &- \xi \frac{d\rho}{dr} \frac{\partial n_0}{\partial \rho} - \frac{2\xi}{r} \Pi_o \frac{\partial n_0}{\partial \rho} + \frac{\partial n_o}{\partial p_r} \delta p_r + \frac{\partial n_o}{\partial r} dr. \end{aligned} \quad (84)$$

Dividing through out by  $(\partial n_o / \partial p_r)$  gives, to first order,

$$\begin{aligned} -\xi \frac{dp_{r_o}}{dr} - \frac{1}{\frac{\partial n_o}{\partial p_r}} n_o \frac{e^{\nu_o}}{r^2} (r^2 e^{\nu_o/2} \xi)' &= - (p_{r_o} + \rho_o) \frac{e^{\nu_o/2}}{r^2} (r^2 e^{\nu_o/2} \xi)' \frac{\frac{\partial n_o}{\partial \rho}}{\frac{\partial n_o}{\partial p_r}} \\ &- \xi \frac{d\rho}{dr} \frac{\frac{\partial n_o}{\partial \rho}}{\frac{\partial n_o}{\partial p_r}} - \frac{2\xi}{r} \Pi_o \frac{\frac{\partial n_o}{\partial \rho}}{\frac{\partial n_o}{\partial p_r}} + \delta p_r + \frac{\frac{dn_o}{dr}}{\frac{\partial n_o}{\partial p_r}} dr. \end{aligned} \quad (85)$$

Solving for  $\delta p_r$  we find

$$\delta p_r = -p_{r_o}' \xi - \frac{1}{p_{r_o} \frac{\partial n_o}{\partial p_r}} p_{r_o} \left[ n_o - \frac{\partial n_o}{\partial \rho} (\rho_o + p_{r_o}) \right] \frac{e^{\nu_o/2}}{r} (r^2 e^{\nu_o/2} \xi)' + \frac{2\xi}{r} \Pi_o \frac{\partial p_{r_o}}{\partial \rho_o}. \quad (86)$$

We can rewrite this as

$$\delta p_r = -p_{r_o}' - \gamma p_{r_o} \frac{e^{\nu_o/2}}{r^2} (r^2 e^{\nu_o/2} \xi)' + \frac{2\xi}{r} \Pi_o \frac{\partial p_{r_o}}{\partial \rho_o}, \quad (87)$$

with  $\gamma$  being the adiabatic exponent defined as

$$\gamma \equiv \frac{1}{p_r (\partial n / \partial p_r)} \left[ n - (\rho + p_r) \frac{\partial n}{\partial p_r} \right]. \quad (88)$$

### 3.3 The Pulsation Equation

We now assume that all perturbations have a time dependence of the form  $e^{i\omega t}$ . Further, considering  $\delta\lambda$ ,  $\delta\nu$ ,  $\delta\rho$ ,  $\delta p_r$  and  $\delta\Pi$  to now represent the amplitude of the various perturbations with the same time dependence we find, from eq. (67), that

$$\begin{aligned} \omega^2 (\rho_o + p_{ro}) \xi e^{\lambda_o - \nu_o} &= (\delta p_r)' + \delta p_r \left[ \frac{1}{2} \lambda_o' + \nu_o' \right] + \frac{1}{2} \delta \rho \nu_o' \\ &\quad - \frac{1}{2} (\rho_o + p_{ro}) \left( \nu_o' + \frac{1}{r} \right) \xi (\lambda_o' + \nu_o') - \frac{2}{r} \delta \Pi . \end{aligned} \quad (89)$$

Substituting the expressions for the various amplitudes in the above equation we find

$$\begin{aligned} \omega^2 (\rho_o + p_{ro}) \xi e^{\lambda_o - \nu_o} &= -(p_{ro}' \xi)' - \frac{1}{2} (\rho_o + p_{ro}) \left( \nu_o' + \frac{1}{r} \right) \xi (\lambda_o' + \nu_o') \\ &\quad - e^{-(\lambda_o + 2\nu_o)/2} \left[ e^{(\lambda_o + 2\nu_o)/2} \gamma p_{ro} \frac{e^{\nu_o/2}}{r^2} (r^2 e^{-\nu_o/2} \xi)' \right]' + \left[ \frac{1}{2} \lambda_o' + \nu_o' \right] p_{ro}' \xi \\ &\quad - e^{-(\lambda_o + 2\nu_o)/2} \left[ e^{(\lambda_o + 2\nu_o)/2} \frac{2}{r} \xi \Pi_o \frac{\partial p_r}{\partial \rho} \right]' - \frac{2}{r} \delta \Pi . \end{aligned} \quad (90)$$

From eq. (62) it follows that

$$p_{ro}'' = -(\rho_o' + p_{ro}') \frac{\nu_o'}{2} + (\rho_o + p_{ro}) \frac{\nu_o''}{2} + \left( \frac{2}{r} \Pi_o \right)' , \quad (91)$$

and

$$\nu_o' = \frac{-2r p_{ro}' + 4\Pi_o}{r(\rho_o + p_{ro})} . \quad (92)$$

Also, we have

$$\nu_o'' - \nu_o' \lambda_o' - \frac{1}{r} \lambda_o' = 16\pi (\Pi_o + p_{ro}) e^{\lambda_o} - \frac{1}{2} \nu_o'^2 - \frac{1}{r} \nu_o' . \quad (93)$$

Using eqs. (91), (92) and (93) in (90) we arrive at the pulsation equation i.e., the equation that governs radial oscillations:

$$\begin{aligned}
\omega^2 (\rho_o + p_{r_o}) \xi e^{\lambda_o - \nu_o} &= \frac{4}{r} p_{r_o}' \xi - e^{-(\lambda_o + 2\nu_o)/2} \left[ e^{(\lambda_o + 3\nu_o)/2} \gamma \frac{p_{r_o}}{r^2} (r^2 e^{-\nu_o/2} \xi)' \right]' \quad (94) \\
&= 8\pi e^{\lambda_o} (\Pi_o + p_{r_o}) (\rho_o + p_{r_o}) \xi - \frac{1}{(\rho_o + p_{r_o})} (p_{r_o}')^2 \xi + \frac{4 p_{r_o}' \Pi_o \xi}{r(\rho_o + p_{r_o})} - \frac{4 \Pi_o^2 \xi}{r^2 (\rho_o + p_{r_o})} \\
&\quad - e^{-(\lambda_o + 2\nu_o)/2} \left[ e^{(\lambda_o + 2\nu_o)/2} \frac{2}{r} \xi \Pi_o \left( \frac{\partial p_r}{\partial \rho} + 1 \right) \right]' - \frac{8}{r^2} \Pi_o \xi - \frac{2}{r} \delta \Pi .
\end{aligned}$$

The boundary conditions imposed on this equation are

$$\xi = 0 \text{ at } r = 0 \quad \text{and} \quad \delta p_r = 0 \text{ at } r = R. \quad (95)$$

The pulsation eq. (94), together with the boundary conditions eq. (95), reduce to an eigenvalue problem for the frequency  $\omega$  and amplitude  $\xi$ . This is equivalent to the Sturm-Liouville problem we encountered while studying Newtonian gravity in chapter 2. Multiplying eq. (94) by  $r^2 \xi e^{(\lambda+\nu)/2}$  and integrating over the entire range of  $r$  we find that

$$\begin{aligned}
\omega^2 \int_0^R e^{(3\lambda-\nu)/2} (\rho + p_r) r^2 \xi^2 dr &= 4 \int_0^R e^{(\lambda+\nu)/2} p_r' r \xi^2 dr \quad (96) \\
&+ \int_0^R e^{(\lambda+3\nu)/2} \gamma \frac{p_r}{r^2} \left[ \left( r^2 e^{-\nu/2} \xi \right)' \right]^2 - \int_0^R e^{(\lambda+\nu)/2} \frac{r^2 \xi^2}{(\rho + p_r)} (p_r')^2 dr \\
&8\pi \int_0^R e^{(3\lambda+\nu)/2} (\Pi + p_r) (\rho + p_r) r^2 \xi^2 dr + 4 \int_0^R e^{(\lambda+\nu)/2} \frac{p_r' \Pi}{(\rho + p_r)} r \xi^2 dr \\
&- 4 \int_0^R e^{(\lambda+\nu)/2} \frac{\Pi^2}{(\rho + p_r)} \xi^2 dr - 8 \int_0^R e^{(\lambda+\nu)/2} \Pi \xi^2 dr - 2 \int_0^R e^{(\lambda+\nu)/2} \delta \Pi r \xi^2 dr \\
&- \int_0^R e^{-\nu/2} \left[ e^{(\lambda+2\nu)/2} \frac{2}{r} \xi \Pi \left( \frac{\partial p_r}{\partial \rho} + 1 \right) \right]' r^2 \xi^2 dr ,
\end{aligned}$$

where we have dropped the subscripts as no longer necessary. The orthogonality condition is now

$$\int_0^R e^{(3\lambda-\nu)/2} (\rho + p_r) r^2 \xi^i \xi^j dr = 0 \quad (i \neq j) , \quad (97)$$

where  $\xi^i$  and  $\xi^j$  are the proper solutions belonging to different eigenvalues  $\omega^2$ .

## 4 Stability of Anisotropic Spheres: $\rho = \text{const.}$

Chandrasekhar studied the dynamical stability of isotropic spheres with constant density  $\rho$  and constant adiabatic index  $\gamma$  using the above formalism. The full solution of Einstein's field equations for constant-density isotropic spheres is well known [9]. Thus, writing

$$y = 1 - \frac{r^2}{\alpha^2} \equiv 1 - \eta^2 \quad \text{and} \quad y_1 = 1 - \frac{R^2}{\alpha^2} \equiv 1 - \eta_1^2 \quad (98)$$

with

$$\alpha^2 = \frac{3}{8\pi\rho}, \quad (99)$$

the complete interior static isotropic solution for  $\rho = \text{const}$  is

$$p = \rho \left[ \frac{y - y_1}{3y_1 - y} \right], \quad e^\lambda = \frac{1}{y^2} \quad \text{and} \quad e^\nu = \frac{1}{4}(3y_1 - y)^2. \quad (100)$$

Here, we will apply the formalism just developed to the Bowers-Liang solution for anisotropic spheres [12],

$$p_r = \rho \left[ \frac{y^{2Q} - y_1^{2Q}}{3y_1^{2Q} - y^{2Q}} \right], \quad e^\lambda = \frac{1}{y^2} \quad \text{and} \quad e^\nu = \frac{1}{4}(3y_1^{2Q} - y^{2Q})^{1/Q} \quad (101)$$

and

$$p_t - p_r = C \frac{\rho^2 r^2}{y^2} \frac{4y^{2Q} y_1^{2Q}}{(3y_1^{2Q} - y^{2Q})^2}, \quad (102)$$

with

$$Q = \frac{1}{2} - \frac{3C}{4\pi} \equiv \frac{1}{2} - \frac{k}{2}. \quad (103)$$

In his stability analysis for isotropic stars, Chandrasekhar used the following trial function

$$\xi = \eta e^{\nu/2} = \frac{1}{2} \eta (3y_1 - y), \quad (104)$$

and found that, to first order in  $2M/R$ , the frequency of oscillation is given by

$$\omega^2 = \frac{1}{2\alpha^2} [(3\gamma - 4) - \frac{1}{14} (\frac{2M}{R}) (54\gamma - 53)]. \quad (105)$$

The first term reproduces the results for Newtonian isotropic spheres [ cf. eq. (43)] and the second term represents a correction due to general relativity.

We now turn our attention to the anisotropic case. A trial function that generalizes (104) to include the effects of anisotropy is

$$\xi_c = \eta e^{Q\nu} , \quad (106)$$

However, we found that for the anisotropic case the corresponding integrals in the expression for  $\omega^2$  cannot be computed analytically if this substitution is made. In order to compute the integrals analytically, we used trial functions of the following form

$$\xi_1 = \eta^{\frac{1}{2}} e^{Q\nu} , \quad (107)$$

and

$$\xi_2 = \eta^{\frac{3}{2}} e^{Q\nu} . \quad (108)$$

For the trial function  $\xi_1$ , we found, after integrating all terms in eq. (96), [for small  $2M/R$ ]

$$\omega^2 = \frac{25}{16\alpha^2} \left[ \gamma - \frac{32}{25} - k\left(\gamma - \frac{52}{25}\right) \right] - \frac{25}{12\alpha^2} \left[ \gamma - \frac{23}{25} - k\left(\frac{15}{8}\gamma - \frac{209}{100}\right) \right] \left( \frac{2M}{R} \right). \quad (109)$$

Thus, for this model, stable oscillations will occur if

$$\gamma \geq \frac{32}{25} - \frac{4}{5}k + \left[ \frac{36}{75} - k \right] \left( \frac{2M}{R} \right) . \quad (110)$$

For the trial function  $\xi_2$  integrating eq. (96) gives,

$$\omega^2 = \frac{49}{32\alpha^2} \left[ \gamma - \frac{64}{49} - k\left(\gamma - \frac{141}{49}\right) \right] - \frac{245}{128\alpha^2} \left[ \gamma - \frac{48}{49} - k\left(\frac{931}{490}\gamma - \frac{586}{245}\right) \right] \left( \frac{2M}{R} \right) \quad (111)$$

and the condition for stable oscillations becomes

$$\gamma \geq \frac{64}{49} - \frac{11}{7}k + \left[ \frac{20}{49} - \frac{327}{196}k \right] \left( \frac{2M}{R} \right) . \quad (112)$$

An examination of the two expressions for  $\gamma$  above shows that for positive anisotropy ( $k > 0$ )  $\gamma$  is smaller than the corresponding isotropic value, implying that positive  $k$  leads to more stable configurations, while negative

values of  $k$  will have a destabilizing effect. Further, we note that since for small values of  $k$  the various new analytical solutions we found previously for constant-density anisotropic spheres [1] have a similar form to the Bowers-Liang solution, we expect the relationship found here between the sign of  $k$  and the stability of the sphere to hold also for those new solutions.

It is useful to compare the results Chandrasekhar found using the trial function  $\xi$  [eq. (104)] (denoted by  $\gamma_{Ch}$ ), with the results obtained with our trial functions  $\xi_1$  and  $\xi_2$  (denoted by  $\gamma_1$  and  $\gamma_2$ ) in the isotropic limit ( $k = 0$ ). For stable oscillations we must have

$$\begin{aligned}\gamma_{Ch} &\geq \frac{4}{3} + \frac{19}{42}\left(\frac{2M}{R}\right) = 1.333 + 0.4523\left(\frac{2M}{R}\right), \\ \gamma_1 &\geq \frac{32}{25} + \frac{36}{75}\left(\frac{2M}{R}\right) = 1.280 + 0.480\left(\frac{2M}{R}\right), \\ \gamma_2 &\geq \frac{64}{49} + \frac{20}{49}\left(\frac{2M}{R}\right) = 1.306 + 0.4081\left(\frac{2M}{R}\right).\end{aligned}\tag{113}$$

It is known that Chandrasekhar's trial function becomes exact in the limit  $(2M/R) \rightarrow 0$ . Comparing the results for the three trial functions above, we see that in the exact limit our results differ from the exact value by  $\approx 5\%$ . This leads us to believe that our trial functions generate results that are qualitatively correct.

## 5 Stability of Anisotropic Spheres: $\rho \propto 1/r^2$

In [1] we found several exact solutions for anisotropic stellar configurations with the following expression for the energy density

$$\rho = \frac{1}{8\pi} \left( \frac{a}{r^2} + 3b \right),\tag{114}$$

where both  $a$  and  $b$  are constants. The choice of the values for  $a$  and  $b$  is dictated by the physical configuration under consideration. For example,  $a = 3/7$  corresponds to the Misner-Zapolsky solution for ultra high-density neutron star cores [13].

If we model the pressure anisotropy as

$$p_t - p_r = \frac{1}{8\pi} \left( \frac{c}{r^2} + d \right),\tag{115}$$

then an exact solution of the field equations with  $b = d = 0$  is

$$e^{-\lambda} = 1 - a = I_0 \quad (116)$$

$$e^{\frac{\mu}{2}} = A_+ \left(\frac{r}{R}\right)^{1+q} + A_- \left(\frac{r}{R}\right)^{1-q}, \quad (117)$$

with

$$q \equiv \frac{(1 + c - 2a)^{\frac{1}{2}}}{(1 - a)^{\frac{1}{2}}}, \quad (118)$$

and the constants  $A_+$  and  $A_-$  fixed by boundary conditions. For the case under consideration here ( $b = d = 0$ ), the boundary conditions are

$$e^{-\lambda(R)} = e^{\nu(R)} = I_0^2, \quad \text{and} \quad e^{\nu(R)} \frac{d\nu}{dr} \Big|_R = \frac{a}{R}. \quad (119)$$

Applying the boundary conditions we find

$$A_+ = \frac{I_0}{2} + \frac{1 - 3I_0^2}{4qI_0} \quad \text{and} \quad A_- = A_+(q \rightarrow -q). \quad (120)$$

The radial pressure for  $q$  real, after substituting the expressions for  $A_+$  and  $A_-$ , is

$$8\pi p_r = \frac{(3I_0^2 - 1)^2 - 4q^2 I_0^4}{r^2} \left[ \frac{R^{2q} - r^{2q}}{(3I_0^2 - 1 + 2qI_0^2)R^{2q} + (1 - 3I_0^2 + 2qI_0^2)r^{2q}} \right]. \quad (121)$$

For this case we found that using the following trial function

$$\xi = r^2(\rho + p_r)e^\nu \quad (122)$$

all the integrals were, after some tedious work, exactly integrable. In table I, we present results for the frequencies of radial oscillations  $\omega^2$  as a function of the anisotropy parameter,  $c$ , for given values of the density parameter. We also give, in table II, the values of  $\gamma_c$  above which stable oscillations are possible. Here we see that the effect of a positive anisotropy is to reduce the value of  $\gamma$ , thus giving rise to a more stable configuration when compared with the corresponding isotropic model. In particular, for the Misner-Zapolsky solution ( $a = 3/7$ ), we find that a small positive pressure anisotropy in the equation of state improves the neutron star's core stability.

$a = 2/9$	$\omega^2 R^2 = 0.95(\gamma - 1.79) + (101.1 - 52.6 \gamma)c$
$a = 2/7$	$\omega^2 R^2 = 2.3(\gamma - 1.83) + (122.3 - 59.3 \gamma)c$
$a = 3/7$	$\omega^2 R^2 = 0.57(\gamma - 1.93) + (15.2 - 5.1 \gamma)c$
$a = 3.4/7$	$\omega^2 R^2 = 0.4(\gamma - 2.6) + (8.9 - 2.3 \gamma)c$
$a = 3.49/7$	$\omega^2 R^2 = 0.36(\gamma - 2.76) + (8.0 - 1.97 \gamma)c$

Table 1:  $\omega^2$  *vs.*  $c$  for given values of  $a$ .

$a = 2/9$	$c_{max} = 0.0016$	$\gamma_c = 1.79 - 6.87 c$
$a = 2/7$	$c_{max} = 0.0028$	$\gamma_c = 1.83 - 13.39 c$
$a = 3/7$	$c_{max} = 0.083$	$\gamma_c = 1.93 - 5.55 c$
$a = 3.4/7$	$c_{max} = 0.11$	$\gamma_c = 2.6 - 2.84 c$
$a = 3.47/7$	$c_{max} = 0.12$	$\gamma_c = 2.75 - 7.29 c$

Table 2:  $\gamma_c$  *vs*  $c$  for given values of  $a$ .

## 6 Conclusion

We have studied the stability of anisotropic compact spheres against radial perturbations in the framework of Newtonian gravity and general relativity. In both cases we have seen that the presence of anisotropic pressure can have significant effects.

We have found that there are Newtonian anisotropic spheres with constant energy density whose core pressure can go to infinity, without requiring the radius of the sphere to be infinite. A result of this nature is not possible for Newtonian isotropic spheres with constant energy density. Furthermore, a stability analysis of some of these models shows that there can exist stable anisotropic spheres with an adiabatic exponent  $\gamma < 4/3$ . In the corresponding isotropic case, instability immediately sets in if  $\gamma < 4/3$ .

We have extended the formalism developed by Chandrasekhar to study the stability of general relativistic isotropic spheres against radial perturbations to anisotropic spheres. In particular, we have applied this formalism to study anisotropic spheres with constant energy density and with energy



densities with an  $1/r^2$  profile, used to model ultra-dense neutron star interiors, for example. We have found that in both cases there can exist stable relativistic anisotropic spheres with values of the adiabatic exponent that would necessarily imply instability in isotropic spheres. In particular, this is true whenever the tangential pressure is larger than the radial pressure for all models we investigated. These results may explain the higher stability of certain neutron stars with anisotropic deviations near their core, and other gravitationally-bound compact objects such as boson stars, which are naturally anisotropic. Work along these lines is currently in progress.

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