Relativistic solids and their applications 1: reviewing the construction

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Abstract

The non-relativistic theory of solids is highly developed, has extensive applications, and is very intuitive. On the other hand, the relativistic theory of solids is relatively under-developed and has not been applied to many situations. The purpose of this article is to act as a dictionary to translate ideas and mathematics between the communities of neutron star and cosmologists. [`Purpose`], [`Highlights`],

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1 Introduction

The concept of material models is rather simple: in what manner does, or can, a substance respond under a given stimulus. What factors about the substance are important in order for useful dynamical information to be extracted? Especially information about how the substance imparts energy and momentum into surrounding materials. These concepts and associated problems are best explained by analogy.

First, suppose one wanted to construct a description of water flowing through a pipe. Given that one knows that water is constituted from "particulate" molecules, one could construct a particle description. This would be built from knowledge – or a guessed understanding – of how water molecules interact with their neighbours and surfaces inside the pipe. With the best will and all available computing power, such a description would fail to describe almost all systems of physical interest. Instead, one moves to a coarse-grained fluids description where one attempts to describe the collective behavior of the particles on a "large scale".

Secondly, suppose one wanted to construct a description of how an object, such as a table, responds to being kicked. The impact of the externally applied kick is transmitted via inter-molecular bonds within the object to release some kind of energy in the form of motion, sound, or heat. Ones intuition has been built up to such an extenet that the precise details of the inter-molecular bonds are irrelevant if one wanted to understand the large-scale response of the object to the impact. However, ones intuition well aware of the fact that if the object were made of different materials (which, on a fundamental level, means that the objects constitutive inter-molecular bonds are different in nature), then the object could respond very differently. The amount of kicking required to move the table depends on what the table is made of (bendy, versus stiff matierials). And so, one builds a working picture of the object: it is vital to have some understanding of some of the underlying micro-physical make-up of the object when building up an understanding of the macro-physical response of the object to macro-physical impacts.

Scalar field models of dark energy and modified gravity are prevalent in modern cosmology and it is our contention that in an important sense these are equivalent to constructing a particulate description of water, or a molecular picture of a table. We would like to offer a change in philosophy in building models.

The aims of this paper include the elucidation of the construction of non-linear material models, and showing how ideas, schematic scenarios, and model building techniques, can be imported into the language of cosmology.

It is useful for our purposes to imagine that the theory of materials comes in

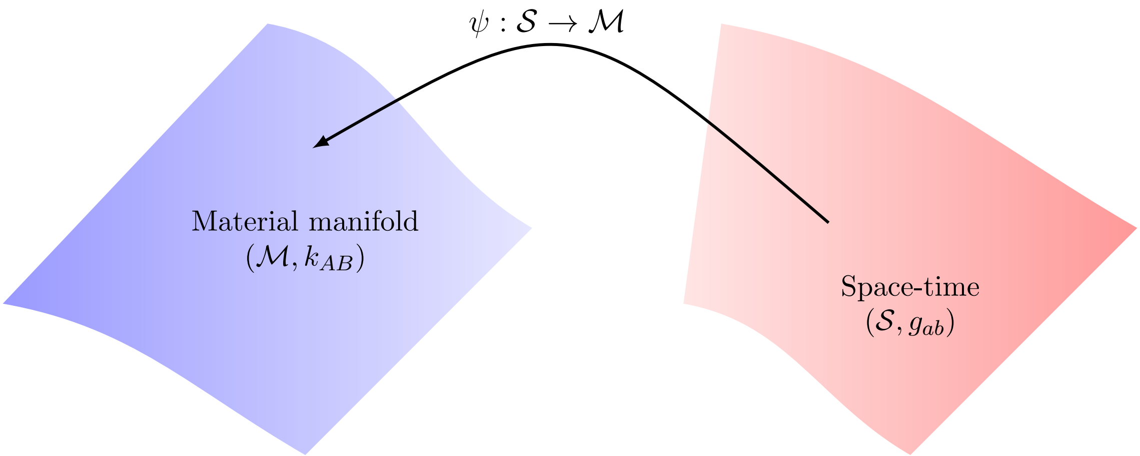

Figure 1: Schematic depiction of the map ψ that associates a point in the material space $\mathcal{M}$ with a point in space-time $\mathcal{S}$. We have also shown which metric is associated with which manifold (and the associated labelling of the indices).

two branches. The first is the theory of *continuous media*: these are supposed to be space filling substances. Relativistic realizations of such media were the subject of [1–3], but under the presumption that the medium was adequently described within the framework of perturbation theory (admittedly, for the applications those studies had in mind, this was a perfectly reasonable restriction). The second is the theory of *solitons*: these are almost completely opposite to continuous media, in the sense that they are localised configurations and are highly non-linear deformations of the appropriate fields. In both descriptions of materials (i.e., continuous and localised) the idea of a map from the material manifold into space-time is heavily (and successfully) used. It appears that the important distinction between how the two types of theories are formulated is what information about the material manifold and its map is used to construct the theory.

In some sense the idea of describing a medium is similar to the idea of using multiple scalar fields to build dark energy models: the medium description is constructed with a set of three scalar fields. Except now one obtains a concrete interpretation of what the scalar fields *are*. Knowing what the fields are significantly enhances physical insight, and guides the choice of functions or parameters used to parameterize available freedom in the theory.

- In Bucher and Spergel, [1], the linearized theory is constructed in detail.
- See Carter and Quintana [4, 5], Karlovini [6–9] and [10–13]; [14] [15] and [16]
- Elasticity and "hyper -elasticity" have been further developed in [17], [18]

- The pull-back idea is very similar to the restoration of non-linear diffeomorphism invariance utilised by massive gravity theories [19].
- See effective field theory of perfect fluids, [20]
- Note that [6] take the tensor k_{AB} to be fixed on the material space.
- see [21] [22]
- Solids in inflation context [23–25]
- see [26] for exact analytic solutions for perturbed single-component cosmology
- [11], [27]

1.1 Deformation theory and cosmology

The current state of affairs in cosmology is that the Universe is accelerating in its expansion, with many avenues being pursued in order to explain this observation. The summary is that the prediction obtained from General Relativity (GR) for how the Universe should look doesn't match up with observations of how the Universe does look (unless, for example, some form of exotic matter is included). One popular way of understanding how to tackle this mis-match is to write the gravitational field equations that actually describes the Universe as

$$G_{ab} = 8\pi G\left(T_{ab} + U_{ab}\right). \tag{1.1}$$

The tensor U_{ab} contains all the deviations or deformations (to begin using the terminology we aim to develop) of the field equations which describe the actual Universe away from the GR (+ standard matter content) predictions.

The modern cosmology community is busy with developing candidate theories which could provide the components of the tensor U_{ab}, and with working out the observational consequences of their given form of the tensor by using observational probes such as the distances to supernovae, the Cosmic Microwave Background radiation, and the effects of the evolution of gravitational perturbations on the propagation of photons. We would like to suggest a different approach (or at least, a different philosophy for attacking the problem). Explaining this approach, and showing how it can be used, is the subject of this paper.

In the theory of deformations (in particular, we have in mind theories of relativistic elasticity) one imagines two states of a material. The first state is a relaxed configuration, and the second is a strained configuration. The deformation which was imparted on the material to take it from being relaxed to being strained isn't necessarily small (if it was small, one would speak about linear elasticity theory). The theory of deformations prescribes a tool-kit for writing down terms in the field equations which are allowed, given classes or forms of deformation. For example, if the deformation is performed "on" some perfect fluid or perfect solid, then it is known that the quantities U_{ab} takes on the form

$$U_{ab}^{\text{fluid}} = \rho u_a u_b + P\gamma_{ab}, \qquad U_{ab}^{\text{solid}} = \rho u_a u_b + P_{ab}. \tag{1.2}$$

The energy-momentum tensors written above *become* those for a fluid or solid when some extra theoretical structure is used. Namely, an *equation of state*. For readers who are used to the literature in modern cosmology, this phrase is often used to describe the link between the dark energy pressure P and density ρ, via an equation of the form $P(t) = w(t)\rho(t)$. In the context of material models, an equation of state is the Lagrangian density.

When one constructs "conventional" models of dark energy or modified gravity, one has a some freedom to choose various types of quantities: these are, e.g., functional forms of the potential, or the kinetic terms which appear in the Lagrangian density. This may seem like an obvious point, but the choice of a restriction on a theory can have implications for (a) its applicability, and (b) its physical naturalness/interpretation. This is a particularly pertintant point, and so therefore we want to take inspiration from the extremely well developed field of the *mechanics of solids*.

1.2 Fluids and solids

The distinction between a *fluid* and a *solid* isn't one of the best explained concepts in the literature. Fluids are commonly used as a description for the "source term" in a gravitational theory, since they are both mathematically simple and physically intuitive. But fluids are only a sub-class of a more general description for "content". To perhaps use a more physically transparent terminology: for materials. A more general description of material is that of a solid; obviously, we won't go so far as to say *the* general material description. Below we will outline some of the salient pieces to the construction of a material model: full explanations are given in the rest of this paper.

In the descriptions of both solids and fluids one has a notion of a *material metric* k_{AB} on a *material space*, whose determinant is related to the particle number density, $n = \sqrt{\det k_{AB}}$. A convenient decomposition of this metric is $k_{AB} = n^{2/3}\eta_{AB}$. With this decomposition of k_{AB}, the conformal metric η_{AB} is uni-modular, i.e., it has unit determinant. Note that all indices are of "capital latin" type: this indicates that they correspond to quantities defined on the material manifold. Such quantities can be "pulled-back" to the space-time manifold. For example, the components of the uni-modular tensor in the space-time manifold are constructed from the set of three scalars ϕ^A and k_{AB} via

$$\eta_{ab} = n^{-2/3} k_{AB} \partial_a \phi^A \partial_b \phi^B. \tag{1.3}$$

The ϕ^A are the coordinates on the material manifold: physically they specify the locations of the particles.

The action for the material is constructed by integrating the Lagrangian density whose arguments are all possible scalar quantities formed from the available structures in space-time which are the pulled-back counterparts of structures on the material manifold. Schematically put, the material Lagrangian can be written as $\mathcal{L} = \mathcal{L}(k^a{}_b)$, although this has not yet made manifest the scalar arguments. The action for both a fluid and a solid is of the general form

$$S = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}\left(n, [\boldsymbol{\eta}], [\boldsymbol{\eta}^2]\right). \tag{1.4}$$

The square-braces in (1.4) denote traces of the mixed components of the uni-modular tensor, $\eta^a{}_b = \gamma^{ac}\eta_{cb}$.

It is useful to split up the Lagrangian density as $\mathcal{L} = n\epsilon$, where ϵ is the energy per particle and n retains its interpretation as the particle number density. In the cases of fluids or solids, ϵ is a function with the following dependencies:

$$\epsilon_{\text{fluid}} = \epsilon_{\text{fluid}}(n), \qquad \epsilon_{\text{solid}} = \epsilon_{\text{solid}}(n, \eta^a{}_b). \tag{1.5}$$

This makes the distinction between solids and fluids explicit: it is the dependence of the energy per particle on the uni-modular tensor $\eta^a{}_b$ which makes the description that of a solid rather than of a fluid. Later on we will see that the physical consequence of this dependence is that the substance is able to support anisotropic stress (whereas fluids can't): this manifests as *rigidity*. It is worth noting that a fluid is a highly symmetric solid, and a pressureless fluid has $\epsilon_{\text{fluid}}(n) = \bar{\epsilon}_0$, a constant.

In Figure 2 we show how the materials are related.

Another concept which is used is that of a *perfect fluid*. This is supposed to be a substance whose energy-momentum tensor can be put into the form

$$T_{ab} = \rho u_a u_b + P\gamma_{ab}, \tag{1.6}$$

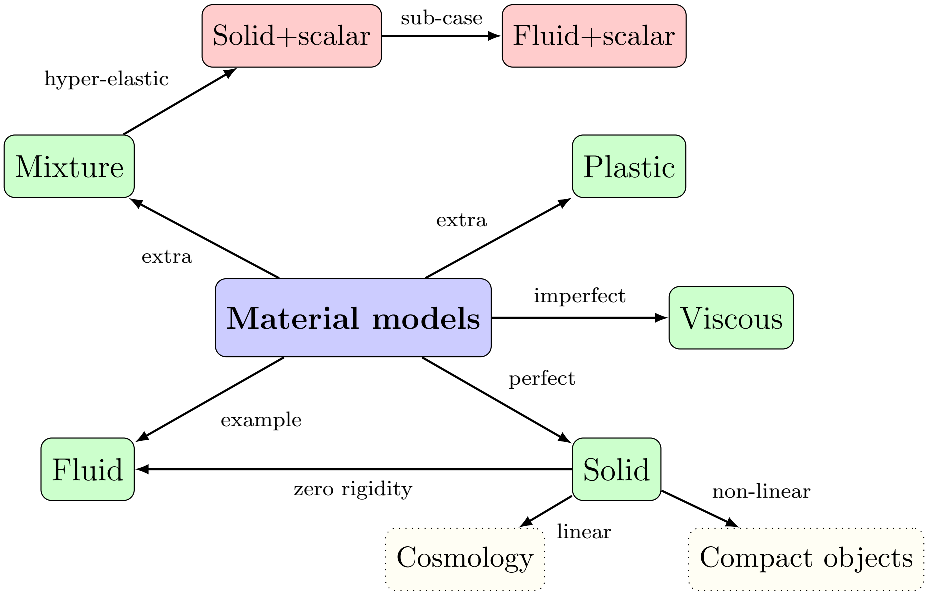

Figure 2: Road-map containing some of the simplest material models. This picture coarsely shows how some of the common classes of materials are related. For example, we see that a fluid is a perfect solid with zero rigidity. We have also shown that the linear theory of solids has been applied to cosmology, and the non-linear theory to compact objects (such as neutron stars).

in which ρ and P are the fluid's energy density and pressure respectively, u^a is the velocity of the fluid and $\gamma_{ab} = g_{ab} + u_a u_b$ is the orthogonal projection operator. If the energy-momentum tensor for a *fluid* is not of this form (for example, if there is anisotropic stress or heat flux) then the *fluid* is said to be *imperfect*. We are deliberately being careful about only using the term *fluid*: a solid can be categorised in a similar sense, but a perfect solid manifestly has an anisotropic part to the energy-momentum tensor (this distinguishes a solid from a fluid).

In some sense the main result of this review is to obtain an understanding of the theory of a relativistic solid: useful geometric structures on the manifold of particle locations, the action, and energy-momentum tensor. It is rather involved, but is worthwhile since expressions and formulae obtain physical meaning.

1.3 Conventions

We use lower-case latin letters, $a, b, c, \ldots$ to denote space-time indices, and upper-case latin letters, $A, B, C, \ldots$ to denote indices on the material manifold. The space-time metric is decomposed as

$$g_{ab} = \gamma_{ab} - u_a u_b, \tag{1.7}$$

Symbol	Meaning
$\pounds_X$	Lie derivative operator along the vector X^μ
$(\mathcal{S}, g_{ab})$	Space-time manifold and metric
$(\mathcal{M}, k_{ab})$	Material manifold and metric
u_a	Time-like unit-vector; $u^a u_a = -1$
$\gamma_{ab} = g_{ab} + u_a u_b$	Orthogonal projector; $u^a \gamma_{ab} = 0$
n	Particle number density; $n^2 = \det k_{AB}$
$\eta^a{}_b = n^{-2/3} k^a{}_b$	Uni-modular tensor
ϵ	Equation of state

Table 1: Summary of commonly used symbols

in which u_a and γ_{ab} are the 4-velocity and spatial metric, satisfying

$$u^a u_a = -1, \qquad u^a \gamma_{ab} = 0. \tag{1.8}$$

We use the orthogonally projected derivative

$$\overline{\nabla}_a A^{b\cdots}{}_{c\cdots} = \gamma^d{}_a \gamma^b{}_e \cdots \gamma^f{}_c \cdots \nabla_d A^{e\cdots}{}_{f\cdots} \tag{1.9}$$

and the expansion (extrinsic curvature) tensor

$$\Theta_{ab} = \overline{\nabla}_{(a} u_{b)}. \tag{1.10}$$

It immediately follows that $\overline{\nabla}_a$ is the connection compatible with γ_{ab}, since

$$\overline{\nabla}_a \gamma_{cd} = 0. \tag{1.11}$$

We will use angular braces to denote the symmetric, trace-free part of a tensor:

$$A_{\langle ab\rangle} = A_{(ab)} - \tfrac{1}{3} A^c{}_c \gamma_{ab}. \tag{1.12}$$

1.3.1 First, second, and third fundamental tensors

Here we briefly review some of Carter's technology [28–30] for dealing with branes and imbeddings.

The idea is that writing $x^\mu{}_{,i} = \partial x^\mu / \partial \sigma^i$, with σ^i the worldsheet coordinates, induces a metric on the world-sheet, $\overline{g}_{ij} = g_{\mu\nu} x^\mu{}_{,i} x^\nu{}_{,j}$. Instead of working with this quantity (which is written in terms of worldsheet coordinates), it is much more convenient to work with $\overline{g}^{\mu\nu} = \overline{g}^{ij} x^{\mu}{}_{,i} x^{\nu}{}_{,j}$. This invites decomposition of the space-time metric according to

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \perp_{\mu\nu} . \tag{1.13}$$

Here $\overline{g}_{\mu\nu}$ is the world-sheet tangential metric: it is the first fundamental form. Also, $\perp_{\mu\nu}$ is orthogonal to the world-sheet. These satisfy

$$\overline{g}_{\mu\nu} \perp^{\mu}{}_{\alpha} = 0, \qquad \overline{g}_{\beta\nu} \overline{g}^{\alpha\nu} = \overline{g}^{\alpha}{}_{\beta}. \tag{1.14}$$

The space-time covariant derivative projected into the world-sheet is

$$\overline{\nabla}_{\mu} = \overline{g}^{\alpha}{}_{\mu} \nabla_{\alpha}. \tag{1.15}$$

The second fundamental tensor is defined via

$$K_{\mu\nu}{}^{\rho} = \overline{g}^{\sigma}{}_{\nu} \overline{\nabla}_{\mu} \overline{g}^{\rho}{}_{\sigma}. \tag{1.16}$$

The first fundamental tensor determines the tangential derivative of the world-sheet metric via

$$\overline{\nabla}_{\mu} \overline{g}_{\alpha\beta} = 2K_{\mu(\alpha\beta)}. \tag{1.17}$$

The second fundamental tensor satisfies

$$\perp^{\mu}{}_{\alpha} K_{\mu\nu}{}^{\lambda} = 0, \qquad \overline{g}_{\lambda}{}^{\sigma} K_{\mu\nu}{}^{\lambda} = 0. \tag{1.18}$$

It is convenient to introduce the extrinsic curvature vector as the trace of the first fundamental tensor,

$$K^{\mu} \equiv K^{\alpha}{}_{\alpha}{}^{\mu}, \tag{1.19}$$

which satisfies

$$\overline{g}^{\mu}{}_{\nu} K^{\nu} = 0. \tag{1.20}$$

The third fundamental tensor is

$$\Xi_{\lambda\mu\nu}{}^{\rho} \equiv \overline{g}^{\sigma}{}_{\mu} \overline{g}^{\tau}{}_{\nu} \perp^{\rho}{}_{\alpha} \overline{\nabla}_{\lambda} K_{\sigma\tau}{}^{\alpha}. \tag{1.21}$$

The more common decomposition of (1.13) comes in the 3+1 form, via the identifications

$$\overline{g}_{\mu\nu} = \gamma_{\mu\nu}, \qquad \perp_{\mu\nu} = u_{\mu} u_{\nu}. \tag{1.22}$$

$$\nabla_{\mu} \gamma_{\alpha\beta} = 2K_{\mu(\alpha} u_{\beta)}. \tag{1.23}$$

$$K_{\mu\alpha\beta} = K_{\mu(\alpha} u_{\beta)} \tag{1.24}$$

It follows that (1.17) evaluates to

$$\overline{\nabla}_{\mu} \gamma_{\alpha\beta}. \tag{1.25}$$

1.4 Perturbed solids

The majority of this review will be focussed on the general theory of solids: the deformations performed on the solid or medium may be arbitrarily large. Whilst this is very general, it also yields a theory which is complicated to work with. There are a substantial number of physical systems for whom the non-linear theory of elasticity is "over-kill": understanding the governing equations that describe small deformations of the solid from its equilibrium configuration is often sufficient. For this reason we shall review the theory of perturbed solids.

Comprehensive reviews, applications, and examples in the relativistic theory have already been presented [1–3, 5, 14, 31, 32], as well as the non-relativistic theory being the main subject of a classic book by Landau and Lifshitz [33].

The physical picture one should constantly keep in mind is that a continuous medium has "two states": relaxed and deformed. The former occurs when there are no forces on the medium, and the latter will induce strains and forces on other surrounding materials and fields (notably the metric). In some sense the "point" of a model is to catalogue the possible ways in which a material can influence surrounding media and fields.

1.4.1 Non-relativistic solids

A non-relativistic solid is one for whom there are no gravitational effects. As examples, one imagines an eraser, rubber band, trampolines: these kinds of materials. Another distinguishing feature of non-relativistic solids from relativistic ones is that the pressure of the solid is negligible.

The locations within a relaxed non-relativistic solid are denoted by x^i. Under a deformation the coordinates alter according to

$$x^i \longrightarrow x^i + \xi^i(x^j). \tag{1.26}$$

If the line element in the solid before the deformation is $\mathrm{d}\ell^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, then after the deformation the line element has metric given by

$$g_{ij} = \delta_{ij} + 2\varepsilon_{ij}, \tag{1.27}$$

in which we defined the strain tensor,

$$\varepsilon_{ij} \equiv \partial_{(i}\xi_{j)}. \tag{1.28}$$

The components of the strain tensor ε_{ij} contain all information about the deformation performed on the body. We now require information about the manner in which the body responds to the given deformation. This entails an understanding of the stress tensor, σ^{ij}, for a given strain tensor ε_{ij}. This is where the "physics" comes in.

Although this seems like a tangential calculation, consider that if one computes the divergence of the stress tensor one obtains the components of the force. These can be equated to the acceleration of the deformation vectors to obtain the equation of motion

$$F^i = \partial_j \sigma^{ij} = \rho \ddot{\xi}^i. \tag{1.29}$$

And so, if the stress tensor can be related to the strain tensor (which, we remind is constructed from the derivatives of the deformation vector), then the equation of motion (1.29) becomes a closed set of equations.

In broad-brush-terms there are two cases which are useful to consider and describe a huge class of physically useful solids.

1. The stress tensor is proportional to the strain tensor:

$$\sigma^{ij} = E^{ijkl} \varepsilon_{kl}. \tag{1.30}$$

   The components, E^{ijkl}, precisely prescribe the strength of certain forces for given deformations (we will have much more to say about this later); they are the components of the elasticity tensor, and there are a fixed number of them for any given material in a space-time with given dimension. This "given number" is rather large for a material with arbitrary symmetry, but dramatically reduces once the material is imposed to have certain symmetries. Materials for whom (1.30) holds are Hookean elastic solids.

2. The stress tensor is proportional to the rate-of-strain tensor:

$$\sigma^{ij} = V^{ijkl} \dot{\varepsilon}_{kl}. \tag{1.31}$$

   The components V^{ijkl} play a similar role to the components of the elasticity tensor for a Hookean solid, except here we are considering viscous solids and V^{ijkl} are the components of the viscosity tensor.

Of course, the given physical material may require an amalgamation of the two cases, whereby stress is proportional to both strain, and rate-of-strain, in which case

$$\sigma^{ij} = E^{ijkl} \varepsilon_{kl} + V^{ijkl} \dot{\varepsilon}_{kl}. \tag{1.32}$$

The expression (1.32) describes visco-elastic solids (also known as Kelvin-Voigt solids). Since (1.32) contains both the elastic (1.30) and viscous (1.31) models as sub-cases, we will proceed with the Kelvin-Voigt expression (1.32). Using (1.28) and (1.32), the equation of motion (1.29) is

$$\rho \ddot{\xi}^i = E^{ijkl} \partial_j \partial_{(k} \xi_{l)} + V^{ijkl} \partial_j \partial_{(k} \dot{\xi}_{l)}. \tag{1.33}$$

The problem of describing the solid considerably simplifies when one assumes some symmetry of the solid, for example material isotropy. In such cases (other symmetries require more freedom than we are about to introduce), the material tensors only have two independant components each, and decompose completely as

$$E^{ijkl} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij} g^{kl} + 2\mu g^{i(k} g^{l)j}, \tag{1.34a}$$

$$V^{ijkl} = \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij} g^{kl} + 2\nu g^{i(k} g^{l)j}. \tag{1.34b}$$

Using the decompositions (1.34), the stress-tensor (1.32) becomes

$$\sigma^{ij} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij} \partial_k \xi^k + \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij} \partial_k \dot{\xi}^k + 2\mu \partial^{(i} \xi^{j)} + 2\nu \partial^{(i} \dot{\xi}^{j)}, \tag{1.35}$$

and the equation of motion (1.33) becomes

$$\rho \ddot{\xi}^i = \left(\beta + \tfrac{1}{3}\mu\right) \partial^i \partial_k \xi^k + \mu \partial_k \partial^k \xi^i + \left(\lambda + \tfrac{1}{3}\nu\right) \partial^i \partial_k \dot{\xi}^k + \nu \partial_k \partial^k \dot{\xi}^i. \tag{1.36}$$

We shall provide a simple example which highlights the seperate modes of propagation inherent in a material medium. Consider the simple case where the deformation vector has only two components; we can expand ξ^i using two scalars ϕ and ψ in an orthonormal basis $(\hat{x}^i, \hat{y}^i)$ via

$$\xi^i = \phi \hat{x}^i + \psi \hat{y}^i. \tag{1.37a}$$

Now suppose that these scalars depend on time, and only one of the two available spatial directions; that is, we set

$$\partial_i \phi = \phi' \hat{x}_i, \qquad \partial_i \psi = \psi' \hat{x}_i. \tag{1.37b}$$

Putting the decomposition of the deformation vector (1.37) into the equation of motion (1.36) yields an equation with two independant projections (one along $\hat{x}^i$, and one along $\hat{y}^i$); these projections leads to the requirement that the following two equations are satisfied:

$$\ddot{\phi} - \frac{\lambda + \frac{4}{3}\nu}{\rho} \dot{\phi}'' = \frac{\beta + \frac{4}{3}\mu}{\rho} \phi'', \qquad \ddot{\psi} - \frac{\nu}{\rho} \dot{\psi}'' = \frac{\mu}{\rho} \psi''. \tag{1.38}$$

In the purely elastic case (i.e., where all components of the viscosity tensor vanish), it is with relative ease that one realises a plane wave ansatz $\phi \sim e^{\mathrm{i}(\omega t+kx)}$ solves the equations of motion, and that ϕ and ψ travel with different speeds: these are the longitudinal and transverse sound speeds

$$c_{\mathrm{L}}^2 = \frac{\beta + \frac{4}{3}\mu}{\rho}, \qquad c_{\mathrm{T}}^2 = \frac{\mu}{\rho}. \tag{1.39}$$

1.4.2 Relativistic solids

In the relativistic theory one needs to carefully describe the perturbations; there are intrinsic variations in the metric, and pre-existing matter fields, as well as perturbations in the continuous medium.

The coordinates of the undeformed medium are represented by $\overline{x}^a$, and those of the deformed medium are by x^a. These are related via

$$x^a = \overline{x}^a + \xi^a(x^b). \tag{1.40}$$

The crucial piece here is the deformation vector, $\xi^a(x^b)$, which as have we explicitly shown via our notation, is dependent upon the space-time coordinates (different locations can deform by different amounts).

The metric of a space-time which contains a perturbed medium is given by expanding the metric to linear order in intrinsic metric perturbations, h_{ab}, and in the deformation vector,

$$g_{ab} = \overline{g}_{ab} + h_{ab} + 2\nabla_{(a}\xi_{b)}. \tag{1.41}$$

The metric of the unperturbed space-time is $\overline{g}_{ab}$, and the metric fluctuations due to "intrinsic", or extra-material contents, is given by h_{ab}. The presence of the perturbed medium is encapsulated by the term involving the deformation field,

$$\xi^a = x^a - \overline{x}^a. \tag{1.42}$$

One may recognise the final term in (1.41) as that which arises in standard perturbation theory after one performs the diffeomorphism

$$x^a \to x^a + \xi^a(x^b). \tag{1.43}$$

Of course, this recognition is accurate. There is an additional concept to appreciate however: interpretation. The ξ^a-field describes all the fluctuations of the medium away from its equilibrium configuration. In addition, the deformation field ξ^a is orthogonal,

$$u_a\xi^a = 0. \tag{1.44}$$

A more elegant, and geometrically intuitive way to write the corrections to the metric is by writing all of the non-background terms in (1.41) as

$$\delta_{\mathrm{L}} g_{ab} = \delta_{\mathrm{E}} g_{ab} + \pounds_{\xi} g_{ab}, \tag{1.45}$$

wherein one can hopefully recognise the usual expression for the Lie derivative of the metric along the vector ξ^a,

$$\pounds_{\xi} g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{1.46}$$

The field equations for the perturbations (in the conventional sense) of a gravitating system which only contains an elastic medium are given by

$$\delta_{\mathrm{E}} G^{\mu\nu} = 8\pi G \, \delta_{\mathrm{E}} T^{\mu\nu}, \tag{1.47}$$

where we used the symbol "δ_{E}" to denote intrinsic variations. The source term, $\delta_{\mathrm{E}} T^{\mu\nu}$, is constructed from a term which contains the variations in the energy-momentum tensor

$$\delta_{\mathrm{E}} T^{\mu\nu} = \delta_{\mathrm{L}} T^{\mu\nu} - \pounds_{\xi} T^{\mu\nu}. \tag{1.48}$$

$$\pounds_{\xi} T^{\mu\nu} = \xi^{\alpha}\nabla_{\alpha} T^{\mu\nu} - 2T^{\alpha(\mu}\nabla_{\alpha}\xi^{\nu)} \tag{1.49}$$

In the visco-elastic case,

$$\delta_{\mathrm{L}} T^{\mu\nu} = -\frac{1}{2}\left(W^{\mu\nu\alpha\beta} + T^{\mu\nu} g^{\alpha\beta}\right)\delta_{\mathrm{L}} g_{\alpha\beta} - V^{\mu\nu\alpha\beta}\delta_{\mathrm{L}} K_{\alpha\beta}; \tag{1.50}$$

$$W^{\mu\nu\alpha\beta} = E^{\mu\nu\alpha\beta} + P^{\mu\nu}u^{\alpha}u^{\beta} + P^{\alpha\beta}u^{\mu}u^{\nu} - 4u^{(\alpha}P^{\beta)(\mu}u^{\nu)} - \rho u^{\mu}u^{\nu}u^{\alpha}u^{\beta} \tag{1.51}$$

The elasticity and viscosity tensors have the symmetries

$$E^{\mu\nu\alpha\beta} = E^{(\mu\nu)(\alpha\beta)} = E^{\alpha\beta\mu\nu}, \qquad V^{\mu\nu\alpha\beta} = V^{(\mu\nu)(\alpha\beta)}, \tag{1.52}$$

and are orthogonal on all indices,

$$u_{\mu}E^{\mu\nu\alpha\beta} = 0, \qquad u_{\mu}V^{\mu\nu\alpha\beta} = u_{\alpha}V^{\mu\nu\alpha\beta} = 0. \tag{1.53}$$

2 Describing non-linear materials

In this section we will take some time to build a description of a medium. We will introduce the notion of a material manifold, and geometric structures on the material manifold: coordinates, metric, connection, and volume form. There will be an important step where we relate structures in the material manifold to structures in space-time. We will want to obtain fields and energies in space-time due to structures in the material manifold. There will be some instances where we "mix" material space and space-time indices; this is unavoidable in the course of exposing some interesting part of the formalism. That said, all final results (equations of motion etc) will be expressed solely in terms of space-time indicies.

2.1 The material manifold, particle number density, and map

One imagines that there is a continuous distribution of particles in the space-time manifold, $\mathcal{S}$. These particles carve out world-lines. Attached to a given world-line, one can associate three coordinates, specifying the location of the particle. The set of these three-coordinates forms the *material manifold*, which we denote by $\mathcal{M}$. This is a useful starting point in our description of the material. We assume that the 3D material manifold $\mathcal{M}$ is endowed with a particle density form, denoted by $n_{ABC} = n_{[ABC]}$. The integral of n_{ABC} over some region in $\mathcal{M}$ tells us about the number of particles of the medium that reside in that region. We will also assume that there is an associated metric on the material manifold, which we call k_{AB}, but we will discuss it later on.

The points of $\mathcal{M}$ are particles of the medium, and they do not move: the dynamics in space-time comes from the maps from the material manifold to space-time, not the motion of the particles in material space. Let $\mathcal{S}'$ be the submanifold of the full space-time manifold $\mathcal{S}$ which is the subset of the spacetime that the material passes through. Then invoke a map ψ which takes a location in space-time and points at a location in the material manifold;

$$\psi : \mathcal{S}' \longrightarrow \mathcal{M}. \tag{2.1}$$

For all points p in $\mathcal{M}' = \psi(\mathcal{S}')$, the inverse map at that point, $\psi^{-1}(\mathrm{p})$, is a single time-like curve in $\mathcal{S}'$: these are the flow-lines of the particles. This construction is the analogue of allowing a scalar field, ϕ say, to pervade the Universe: rather than (2.1), for a real scalar field ϕ then one has $\phi : \mathcal{S} \to \mathbb{R}$.

Let ϕ^A be coordinates in material space. Then their gradients with respect to the space-time coordinates x^a can be computed

$$\psi^A{}_a \equiv \frac{\partial \phi^A}{\partial x^a} = \phi^A{}_{,a} = \partial_a \phi^A; \tag{2.2}$$

in which we have given the definition of the $\psi^A{}_a$ and a list of useful notational alternatives. The $\psi^A{}_a$ are the components of the configuration gradient. These would be the components of the Jacobian associated with a coordinate transformation if the dimension of the material manifold were to be the same as the dimension of the space-time manifold. The time-like projection of these must vanish

$$u^a \psi^A{}_a = 0. \tag{2.3}$$

This is equivalent to setting the Lie derivative of the ϕ^A in the time-like direction to zero:

$$\pounds_u \phi^A = 0. \tag{2.4}$$

This has a direct physical interpretation of saying that the material coordinates are static with respect to coordinate time, or that a given world-line corresponds to a given particle. If this condition is relaxed one ends up describing "hyper-elastic", rather than "elastic" theories, and bears resemblance to a theory which mixes a scalar with a solid.

One can conceive of scalars, vectors, forms, and tensors on the material manifold. The material metric and particle density form are examples, and there will be a few others which we will introduce later on. Collectively, we call such quantities "material tensors", and they have components whose indices are denoted with captial latin letters. We relate tensors in the material and space-time manifolds using technology from differential geometry of pull-backs and push-forwards, as summarised below.

- $\psi^\star$ is the pull-back of a covariant tensor from $\mathcal{M}'$ to $\mathcal{S}'$ and is denoted to act on a material tensor as

  $$N_{ab\cdots z} = \psi^\star N_{AB\cdots Z}. \tag{2.5a}$$

  In "coordinates" notation the pull-back is

  $$N_{ab\cdots z} = \psi^A{}_a \psi^B{}_b \cdots \psi^Z{}_z N_{AB\cdots Z}, \tag{2.5b}$$

  where $\psi^A{}_a$ are the components of the configuration gradient (2.2).

- $\psi_\star$ denotes the push-forward of a contravariant tensor from $\mathcal{S}'$ to $\mathcal{M}'$, and is denoted to act on a space-time tensor as

  $$M^{AB\cdots Z} = \psi_\star M^{ab\cdots z}, \tag{2.6a}$$

  and in coordinates it reads

  $$M^{AB\cdots Z} = \psi^A{}_a \psi^B{}_b \cdots \psi^Z{}_z M^{ab\cdots z}. \tag{2.6b}$$

Since we are assuming the existence of the pull-back we only need ever work with space-time tensors. This brings with it a conceptual simplicity: we only work with space-time indices, and a special subset of the space-time tensors will correspond to material tensors, and will also obtain a corresponding interpretation. That said, it is sometimes helpful to perform intermediate calculations entirely within the material manifold.

The most important corollary of (2.3) is that any tensor on space-time which corresponds to the pull-back of a tensor on the material space will automatically be orthogonal. That is, for the schematic example (2.5b),

$$u^a N_{ab\cdots z} = u^b N_{ab\cdots z} = \cdots = u^z N_{ab\cdots z} = 0. \tag{2.7}$$

This property can be extremely useful. For example, consider the expression $u^a \nabla_c N_{ab}$, where $N_{ab} = \psi^\star N_{AB}$ is the pull-back of a material tensor. After performing a simple manipulation one finds

$$u^a \nabla_c N_{ab} = -K^a{}_c N_{ab}. \tag{2.8}$$

That is, this particular contraction of the time-like unit vector u^a with the space-time covariant derivative of N_{ab} is given by the un-differentiated values of N_{ab} and the extrinsic curvature tensor K_{ab}. Furthermore, it follows by the orthogonality of K_{ab} that $u^a u^c \nabla_c N_{ab} = 0$.

Another, related, corollary is that the orthogonal part of the metric, $\gamma_{ab} = g_{ab} + u_a u_b$, can be used to raise and lower indices of the space-time tensor counterpart of a pulled-back material tensor, e.g., $g^{ac} N_{ab\cdots z} = \gamma^{ac} N_{ab\cdots z} = N^c{}_{b\cdots z}$.

The integral of the particle number density form n_{ABC} over some volume in the material manifold $\mathcal{M}$ is the number of particles in that volume (by definition). The pull-back of the particle volume-form to space-time is

$$n_{abc} = \psi^\star n_{ABC}. \tag{2.9}$$

Note that (2.3) means that n_{abc} is an orthogonal space-time field. Using the space-time volume form ϵ_{abcd}, the dual in space-time of n_{abc} yields the vector

$$n^a = \frac{1}{3!} \epsilon^{abcd} n_{bcd}. \tag{2.10}$$

This vector n^a carries the interpretation of being the particle current, and is manifestly conserved,

$$\nabla_a n^a = 0. \tag{2.11}$$

This conservation follows since n_{abc} is a closed 3-form due to n_{ABC} being a closed 3-form on material space (an n-form in n-dimensional space is closed). What this also means is that to break (2.11) and have $\nabla_a n^a \neq 0$ one requires n^a not to be related to the volume form on material space.

It follows by orthogonality of n_{abc} that the particle current (2.10) is time-like

$$n^a = nu^a, \tag{2.12}$$

where the particle number density n is given by

$$n = \sqrt{-n^a n_a}. \tag{2.13}$$

Putting together some of the above relations, one can obtain the useful expressions,

$$\epsilon_{abc} = \epsilon_{abcd} u^d, \qquad n_{abc} = n\epsilon_{abc}, \tag{2.14a}$$

as well as realising that the number density n can be constructed from the particle 3-form via

$$n^2 = \frac{1}{3!} n^{abc} n_{abc}. \tag{2.14b}$$

Note that from the conservation equation for n^a, (2.11), and (2.12), one obtains an evolution equation for the particle number density,

$$\dot{n} = -n\Theta, \tag{2.15}$$

where $\Theta = \Theta^a{}_a$ is the trace of the extrinsic curvature tensor.

Another way of expressing the duality relation (2.10) is found after combining (2.12) and (2.14a) to give

$$n_{abc} = \epsilon_{abcd} n^d. \tag{2.16}$$

The expression (2.16) directly shows that the 3-form n_{abc} is the dual to n^a, and will highlight the connection to Kalb-Ramond fields. A Kalb-Ramond field is a 2-index object that transforms as a 2-form; its components satisfy

$$B_{ab} = B_{[ab]}. \tag{2.17}$$

The 3-form field strength F_{abc} corresponding to B_{ab} is an exact form constructed by taking the "derivative"

$$F = \mathrm{d}B, \tag{2.18a}$$

which works out in this case as

$$F_{abc} = 3\nabla_{[a}B_{bc]}. \tag{2.18b}$$

Since F_{abc} is an exact form, it is therefore a closed form[1]: the expression of automatic closure is given by

$$\nabla_{[a}F_{bcd]} = 0. \tag{2.19}$$

Related to the 3-form field strength F_{abc} is its dual $\widetilde{F}^a$, which is constructed via

$$F_{abc} = \epsilon_{abcd}\widetilde{F}^d. \tag{2.20}$$

By virtue of the automatic closure (2.19) it follows that $\widetilde{F}^a$ is conserved,

$$\nabla_a\widetilde{F}^a = 0. \tag{2.21}$$

It should therefore be clear that the particle number density current n^a, which is the dual of the number density form n_{abc}, is the field strength tensor of some field of Kalb-Ramond type. ***It is worth finding [34]***

2.2 Material metric

We invoke the existence of a metric k_{AB} on the material manifold $\mathcal{M}$ whose volume form is the particle density form n_{ABC} introduced in Section 2.1. This metric will enable us to introduce a Levi-Civita connection in the material manifold, which can be pulled-back to space-time to aid the evaluation of derivatives of material tensors. Before we explain this fairly complicated construction we shall elucidate some other useful structures on the material manifold.

Indices on material tensors can be contracted with the indices of other material tensors. Equivalently, indices on space-time tensors can also be contracted with those of other space-time tensors (a space-time scalar can be formed if contraction leaves no spare indices). Importantly, space-time tensors can be the pulled-back version of a material tensor, as in the discussion in the previous section. As an example, consider an arbitrary material tensor $A_{ABC\cdots}$ which is "pulled-back" to give a space-time tensor $A_{abc\cdots}$ according to the usual prescription $A_{abc\cdots} = \psi^\star A_{ABC\cdots}$. Then, after contracting some indices with the space-time metric,

$$B_{c\cdots} = g^{ab}A_{abc\cdots} = g^{ab}\psi^\star A_{ABC\cdots} \tag{2.22}$$

[1] A *closed* form C, say, is a form for whom $\mathrm{d}C = 0$. Let A be a p-form, then $F = \mathrm{d}A$ is an *exact* $(p+1)$-form. Since $\mathrm{d}^2 = 0$, it follows that $\mathrm{d}F = 0$; in words this statement is: *an exact form is a closed form.*

is a legitimate space-time tensor. One can also contract indices of material tensors on the material manifold $\mathcal{M}$, with the push-forward of space-time tensors. As an example, consider the push-forward of the inverse space-time metric tensor

$$g^{AB} = \psi_{\star} g^{ab} \tag{2.23a}$$

being contracted with an arbitrary material tensor,

$$g^{AB} C_{ABC\cdots} = \psi_{\star} g^{ab} C_{ABC\cdots}. \tag{2.23b}$$

From the orthogonality of the material mappings it follows that

$$g^{AB} = \psi_{\star} \gamma^{ab}, \tag{2.23c}$$

where we remind that γ^{ab} is the orthogonal part of the space-time metric as defined in (1.7).

Note that g^{AB} is the push-forward of the space-time metric to the material manifold, and does not necessarily coincide with the material metric k_{AB}. Infact, quantifying its non-coincidence is extremely important in quantifying the state of a material. With this in mind, we define a material tensor η_{AB}, which depends on the number density n, such that the push-forward of the space-time metric g^{AB} is exactly the inverse of η_{AB} when the material is in its unsheared state. That is, $g^{AC}\eta_{CB} = \delta^{A}{}_{B}$ (the Kronecker-delta) when the energy is at its minimum $\epsilon = \check{\epsilon}(n)$. What this means is that $g^{AB} = \eta^{-1AB}$ in what is henceforth defined as the *unsheared state*. Consequently, the deviation of the actual value of g^{AB} from η^{-1AB}, which we write as

$$s^{AB} = \tfrac{1}{2}\left(g^{AB} - \eta^{-1AB}\right), \tag{2.24}$$

quantifies the shear of the system.

Writing the volume form of η_{AB} as ϵ_{ABC} it follows that

$$n_{ABC} = n\epsilon_{ABC}. \tag{2.25}$$

Note that $\epsilon_{abc} = \psi^{\star}\epsilon_{ABC}$. The particle density form n_{ABC} is a fixed material space tensor, and is independent of n.

It is now useful and helps physical insight, to define the material tensor k_{AB} as the metric on the material manifold $\mathcal{M}$. k_{AB} is conformal to η_{AB}, and has the particle density form n_{ABC} as its volume form. Therefore

$$k_{AB} = n^{2/3}\eta_{AB}. \tag{2.26}$$

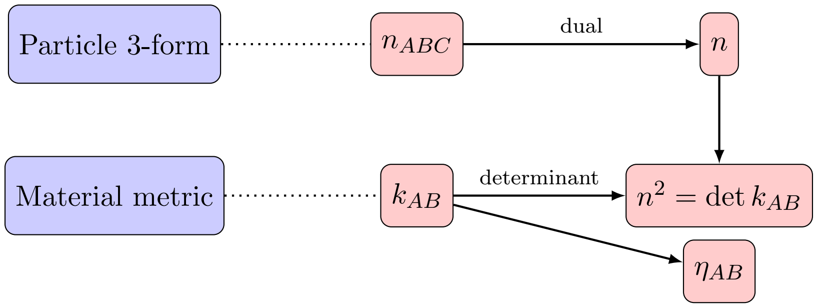

Figure 3: Explanation of the link between geometrical objects in the particle 3-form and material-metric formulations of elasticity theory. In the "particle 3-form" formulation, the only peice of information about the geometrical structure of the material manifold that is actually used is n, the dual of the particle 3-form. In the "material metric" formulation, one posits a metric on the material manifold which has more pieces of information which are used: its determinant, n, and quantites η_{AB} which keep track of the shear-like parts of k_{AB}. The point is that the material metric construction keeps track of more information about the material manifold than the particle 3-form construction. In this way the former is more general than the latter.

This tells us that the (square-root of the) determinant of k_{AB} is the particle number density, n:

$$n = \sqrt{\det k_{AB}}. \tag{2.27}$$

See Figure 3 for a cartoon of the relationship between the material metric k_{AB} and particle form n_{ABC}.

The pull-back of the material metric k_{AB} gives a space-time tensor,

$$k_{ab} = \psi^{\star} k_{AB}, \tag{2.28}$$

and will play an important role in what follows. Specifically, using (2.5b) the pull-back (2.28) reads

$$k_{ab} = \psi^{A}{}_{a} \psi^{B}{}_{b} k_{AB}. \tag{2.29}$$

The corrorlary of (2.3) which we keep coming back to is that k_{ab} is an orthogonal space-time field

$$u^a k_{ab} = 0. \tag{2.30}$$

We will frequently use the (space-time) tensor with mixed indices; to concrete our notation, space-time indices are raised with the space-time metric,

$$k^{a}{}_{b} = g^{ac} k_{bc}. \tag{2.31}$$

This mixed space-time tensor is also orthogonal,

$$u^a k^b{}_a = 0. \tag{2.32}$$

A consequence of (2.32) is that the indices on k_{ab} can be raised and lowered using the orthogonal space-time metric,

$$k^a{}_b = \gamma^{ac} k_{bc}. \tag{2.33}$$

From (2.33) it follows that

$$\frac{\partial k^a{}_b}{\partial g^{cd}} = \delta^a{}_{(c} k_{d)b}. \tag{2.34}$$

In a similar fashion, the pull-back of η_{AB} gives an orthogonal space-time tensor

$$\eta_{ab} = \psi^\star \eta_{AB}, \tag{2.35}$$

and we also use the mixed version of the tensor,

$$\eta^a{}_b = \gamma^{ac} \eta_{cb}. \tag{2.36}$$

From the pull-back of the relationship (2.26) we obtain

$$k^a{}_b = n^{2/3} \eta^a{}_b. \tag{2.37}$$

Since we have set everything up so that n^2 is the determinant of k_{ab}, it follows from (2.37) that $\eta^a{}_b$ is a uni-modular tensor:

$$\det(\eta^a{}_b) = 1. \tag{2.38}$$

This property will be useful later on.

We now elucidate some consequences of the n-dependence of k_{AB}. In what follows it will be convenient to denote differentiation with respect to n with a prime. Using (2.26) to compute k'_{AB} yields

$$n\eta'_{AB} = -\tfrac{2}{3}\eta_{AB} + \tau_{AB}, \tag{2.39}$$

in which

$$\tau_{AB} \equiv n^{1/3} k'_{AB}. \tag{2.40}$$

Since $n'_{ABC} = 0$ (by definition) it follows that $(\det k_{AB})' = 0$, and therefore $k^{-1AB} k'_{AB} = 0$, and hence

$$\eta^{-1AB} \tau_{AB} = 0. \tag{2.41}$$

Thus, we see that τ_{AB} is traceless; it is called the *compressional distortion tensor*, and measures deformations of the medium that *aren't* due to conformal rescalings of the material metric upon varying the particle density. Hence, computing the trace of (2.39) with respect to η^{-1AB} yields

$$n\eta^{-1AB}\eta'_{AB} = -2. \tag{2.42}$$

Note that from (2.40) it follows trivially, but more usefully, $k'_{AB} = n^{-1/3}\tau_{AB}$, and so if the material varies only conformally (i.e. is uniformly compressed) k_{AB} is independent of n since $\tau_{AB} = 0$ for these types of deformations.

The push-forward of (2.39) reads

$$n\eta'_{ab} = -\tfrac{2}{3}\eta_{ab} + \tau_{ab}. \tag{2.43}$$

And so, in the case where $\eta_{ab} = \eta_{ab}(n)$, it is simple to see that

$$[\eta_{ab}]^{\cdot} = \eta'_{ab}[n]^{\cdot}, \tag{2.44}$$

where $[X]^{\cdot}$ denotes the material derivative of X. After using (2.15) to replace $[n]^{\cdot} = \dot{n}$ we obtain the evolution equation:

$$[\eta_{ab}]^{\cdot} = \left(\tfrac{2}{3}\eta_{ab} - \tau_{ab}\right)\Theta. \tag{2.45}$$

2.3 Material covariant derivative

It is convenient at this point to introduce the covariant derivative on the material manifold which is compatible with the material metric. Let $\overline{\widetilde{\nabla}}_A$ be the Levi-Civita connection for k_{AB}; i.e.,

$$\overline{\widetilde{\nabla}}_C k_{AB} = 0. \tag{2.46}$$

There is a reason for our including two different "accents" above the del-symbol. The pushed-forward version of $\overline{\widetilde{\nabla}}_A$, denoted as $\overline{\widetilde{\nabla}}_a$, is allowed to act on space-time tensors; note that it will be orthogonal, and so is taken to be the orthogonal projection of some space-time derivative $\widetilde{\nabla}_a$ according to

$$\overline{\widetilde{\nabla}}_a A^{b\cdots}{}_{c\cdots} = \gamma^d{}_a\gamma^b{}_e\cdots\gamma^f{}_c\cdots\widetilde{\nabla}_d A^{c\cdots}{}_{f\cdots}. \tag{2.47}$$

For any space-time vector Y^a the difference between any two connections can be written as

$$\left(\overline{\widetilde{\nabla}}_a - \overline{\nabla}_a\right)Y^c = \mathfrak{D}^c{}_{ab}Y^b \tag{2.48}$$

in which $\mathfrak{D}^c{}_{ab}$ is the (symmetric) relativistic difference tensor [2] defined as

$$\mathfrak{D}^c{}_{ab} = \tfrac{1}{2}k^{-1cd}\left(\overline{\nabla}_a k_{bd} + \overline{\nabla}_b k_{ad} - \overline{\nabla}_d k_{ab}\right), \tag{2.49}$$

where k^{-1cd} is defined via

$$k^{-1cd}k_{ca} = \gamma^d{}_a, \tag{2.50}$$

and is orthogonal $k^{-1cd}u_c = 0$. Due to the applications in mind, we actually call $\mathfrak{D}^c{}_{ab}$ the relativistic elasticity difference tensor.

Using this construction, one finds that $\overline{\overline{\nabla}}_a$ is the connection which is compatible with k_{ab},

$$\overline{\overline{\nabla}}_a k_{cd} = 0. \tag{2.51}$$

As an example of using this technology, suppose that $B^{a\cdots}{}_{b\cdots}$ is a tensor function of g^{ab} and k_{ab}. Then taking its derivative with $\overline{\overline{\nabla}}_a$ yields

$$\overline{\overline{\nabla}}_a B^{b\cdots}{}_{c\cdots} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}}\overline{\overline{\nabla}}_a g^{ef} + \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial k_{ef}}\overline{\overline{\nabla}}_a k_{ef} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}}\overline{\overline{\nabla}}_a g^{ef}, \tag{2.52}$$

where the second equality holds via (2.51). We can go one step further and realise that

$$\overline{\overline{\nabla}}_a B^{b\cdots}{}_{c\cdots} = \frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}}\left(\overline{\overline{\nabla}}_a g^{ef} - \overline{\nabla}_a g^{ef}\right) = 2\frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{ef}}\mathfrak{D}^{ef}{}_a. \tag{2.53}$$

The second term in braces, $\overline{\nabla}_a g^{ef}$, vanishes by (1.11), and the final equality holds by (2.48). Finally, since

$$\overline{\nabla}_a B^{b\cdots}{}_{c\cdots} = \overline{\overline{\nabla}}_a B^{b\cdots}{}_{c\cdots} - \left(\overline{\overline{\nabla}}_a - \overline{\nabla}_a\right) B^{b\cdots}{}_{c\cdots}, \tag{2.54}$$

then it follows by repeated application of (2.48) on the last term, that orthogonally projected derivative is

$$\overline{\nabla}_a B^{b\cdots}{}_{c\cdots} \quad = \quad 2\frac{\partial B^{b\cdots}{}_{c\cdots}}{\partial g^{de}}\mathfrak{D}^{de}{}_a - B^{d\cdots}{}_{c\cdots}\mathfrak{D}^b{}_{ad} - \cdots + B^{b\cdots}{}_{d\cdots}\mathfrak{D}^d{}_{ac} + \cdots . \tag{2.55}$$

2.4 Constructing scalar invariants

We will formally introduce it later, but we are interested in constructing the equation of state ρ which will be a scalar function of the state of the system, and is integrated to give the action. What this requires, from the theoretical construction which concerns us at the moment, is an understanding of the allowed scalar quantities one

[2] The conditions required for this definition to hold are

can form from objects which specify the state of the system. The scalar invariants are constructed from the pull-back of the material metric, $k^a{}_b$.

There are a few different sets of scalar invariants one could use: formally they will be identical, but different choices will help or hide insight into the physical behavior. And so, we are interested in finding the complete list of scalar invariants which will specify the state of the system.

As a candidate set of invariants, *the* three independent scalar invariants of the mixed components of the pulled-back material metric $k^a{}_b$ are

$$I_1 = [\mathbf{k}], \qquad I_2 = [\mathbf{k}^2], \qquad I_3 = [\mathbf{k}^3], \tag{2.56}$$

in which we denoted traces with square braces,

$$I_n = \mathrm{Tr}(\mathbf{k}^n) = [\mathbf{k}^n] = k^a{}_b k^b{}_c \cdots k^f{}_a, \tag{2.57}$$

with $k^a{}_b$ defined from k_{ab} via (2.33). This is a complete list of independent invariants (any other invariants can be computed from these) due to the orthogonality of $k^a{}_b$ (2.32).

Since n_{ABC} is the volume form of k_{AB}, the particle number density n is also a scalar invariant of $k^a{}_b$; by the Cayley-Hamilton theorem, the determinant is related to the other invariants via

$$n^2 = \det(k^a{}_b) = \tfrac{1}{3!}\left([\mathbf{k}]^3 - 3[\mathbf{k}][\mathbf{k}^2] + 2[\mathbf{k}^3]\right). \tag{2.58}$$

We could use $\{I_1, I_2, I_3\}$ as defined in (2.56) as the list of invariants which could be the arguments of the equation of state, but we shall also consider the particle number density n and the independent scalar invariants of the uni-modular tensor $\eta^a{}_b$, defined in (2.36) since this will help the comparison between solid and fluid descriptions. The important consequence of uni-modularity is that $\eta^a{}_b$ only has two independent invariants (rather than 3 which could be expected from a symmetric rank-2 tensor in 3D). The invariants are linked via the Cayley-Hamilton theorem as

$$3! = [\boldsymbol{\eta}]^3 - 3[\boldsymbol{\eta}][\boldsymbol{\eta}^2] + 2[\boldsymbol{\eta}^3]. \tag{2.59}$$

Notice that (2.59) can be rewritten as

$$2\left([\boldsymbol{\eta}^3] - 3\right) = 3[\boldsymbol{\eta}]\left([\boldsymbol{\eta}^2] - \tfrac{1}{3}[\boldsymbol{\eta}]^2\right). \tag{2.60}$$

To summarise, we have shown that there are two equivalent ways to write the most general equation of state for a solid: both have a maximum of three arguments. They are

$$\rho = \rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right) \tag{2.61a}$$

and

$$\rho = \rho\left(n, [\boldsymbol{\eta}], [\boldsymbol{\eta}^2]\right). \tag{2.61b}$$

We remind that $k^a{}_b$ is the pull-back of a tensor whose volume form is n_{ABC} and (squared) determinant is the particle number density, n. Secondly, $\eta^a{}_b$ is a unimodular tensor whose inverse η^{-1AB} co-incides with the push-forward of the space-time metric when the material is in the unsheared state. The latter formulation is somewhat favorable, since it becomes easy to connect to a scenario in which the solid "becomes" like a fluid, since ρ becomes independent of $[\boldsymbol{\eta}^n]$.

Before we continue it is worth noting some useful ways to compute derivatives of functions which depend on quantities which regularly appear in the construction, most notably functions which depend on n or $\eta^a{}_b$. First of all, the derivative of the number density n with respect to the space-time metric is given by

$$\frac{\partial n}{\partial g^{ab}} = \frac{1}{2} n \gamma_{ab}. \tag{2.62}$$

When $Y = Y(k^a{}_b)$ is any quantity that depends only on the $k^a{}_b$, then its derivative with respect to the space-time metric is

$$\frac{\partial Y}{\partial g^{ab}} = k_{c(a} \frac{\partial Y}{\partial k^{b)}{}_c}. \tag{2.63}$$

For any quantity $Z = Z(n, \eta^a{}_b)$, and using (2.37) as a decomposition of the degrees of freedom in $k^a{}_b$, we obtain

$$\frac{\partial Z}{\partial g^{ab}} = \frac{1}{2} n \gamma_{ab} \frac{\partial Z}{\partial n} + \eta_{c\langle a} \frac{\partial Z}{\partial \eta^{b\rangle}{}_c}, \tag{2.64}$$

where the angular brackets denote the symmetric trace-free part of the tensor, as defined in (1.12). For each quantity n, Y, and Z as defined here,

$$u^a \frac{\partial n}{\partial g^{ab}} = 0, \qquad u^a \frac{\partial Y}{\partial g^{ab}} = 0, \qquad u^a \frac{\partial Z}{\partial g^{ab}} = 0. \tag{2.65}$$

[Fill this in...]

$$\frac{\partial [\mathbf{k}]}{\partial g^{ab}} = \tag{2.66a}$$

$$\frac{\partial [\mathbf{k}^2]}{\partial g^{ab}} = \tag{2.66b}$$

$$\frac{\partial [\mathbf{k}^3]}{\partial g^{ab}} = \tag{2.66c}$$

2.5 Deformations about a relaxed state

It is important to understand how to deal with a deformed medium. Before we give some explicit expressions for deformations of the solid, we shall illustrate the philosophy via "non-linear sigma models" from field theory.

2.5.1 Example from non-linear sigma models

One of the important ideas in continuous mechanics is that of the assumed existence of a relaxed state: this is supposed to be some configuration that minimizes some measure of "energy". This concept is absolutely vital in the study of solitons. As the simplest example, consider the Lagrangian density for a real scalar field ϕ living in a Higgs potential,

$$\mathcal{L} = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2. \tag{2.67}$$

The relaxed configuration of this scalar is when $\phi = \pm\eta$ (commonly known as the vacuum manifold). It is simple to find the Lagrangian density for fluctuations about the relaxed state; substituting $\phi = \eta + \delta\phi$ into (2.67) and expanding to quadratic order in $\delta\phi$ yields

$$\mathcal{L} = -\frac{1}{2}\partial_a\delta\phi\partial^a\delta\phi - \frac{1}{2}\lambda\eta^2(\delta\phi)^2. \tag{2.68}$$

The Lagrangian that results is that for a massive scalar field, and it describes the perturbations about the relaxed state.

This example was simple enough to demonstrate the idea of a "relaxed state" in a non-linear field theory, but it was in some sense "too" simple since there isn't a non-trivial Lagrangian that describes the field *in* the relaxed state. For that we shall move to a more complicated example and think about a multi-scalar field model whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathfrak{k}_{IJ}\partial_a\Phi^I\partial^a\Phi^J - V(\Phi^I). \tag{2.69}$$

There are supposed to be n fields here, and so $I = 1, \ldots, n$, and the set of symmetric quantities $\mathfrak{k}_{IJ}$ are supposed to play the role of a metric in field space. Before we continue we want to make it plainly clear that this isn't the most general Lagrangian density that can be constructed out of single derivatives.

Suppose that the energy gets minimized when the fields Φ^I are consigned to live on a sub-manifold, $\mathcal{V}$ say, of dimension $q \leq n$; in this state the potential energy $V(\Phi^I) = 0$. The "vacuum manifold" $\mathcal{V}$ can be coordinatized by q scalars ϕ^A, say, with $A = 1, \ldots, q$. Hence, when the configuration is in its relaxed state the original set of fields Φ^I are expressible as a function of the fields ϕ^A,

$$\Phi^I = \Phi^I(\phi^A). \tag{2.70}$$

By simple application of the chain rule, (2.70) provides

$$\partial_a \Phi^I = \frac{\partial \Phi^I}{\partial \phi^A} \partial_a \phi^A. \tag{2.71}$$

Putting (2.71) into (2.69) gives

$$\mathcal{L} = -\frac{1}{2} \mathfrak{g}_{AB}(\phi) \partial_a \phi^A \partial^a \phi^B, \tag{2.72}$$

in which we defined

$$\mathfrak{g}_{AB}(\phi) \equiv \mathfrak{k}_{IJ} \frac{\partial \Phi^I}{\partial \phi^A} \frac{\partial \Phi^J}{\partial \phi^B}. \tag{2.73}$$

The $\mathfrak{g}_{AB}$ are interpreted as the components of the metric on the field submanifold $\mathcal{V}$. The field equations for the ϕ^A derived from (2.72) are given by

$$g^{ab} \nabla_a \nabla_b \phi^A + \Gamma^A{}_{BC} \nabla_a \phi^B \nabla^a \phi^C = 0, \tag{2.74}$$

where

$$\Gamma^A{}_{BC} = \tfrac{1}{2} \mathfrak{g}^{AD} \left(\partial_B \mathfrak{g}_{CD} + \partial_C \mathfrak{g}_{BD} - \partial_D \mathfrak{g}_{BC} \right) \tag{2.75}$$

are the Christoffel symbols for the metric in the field submanifold.

One of the simplest ways (we can think of, at least) to see how study fluctuations or deformations away from the relaxed state is to first imagine that the relaxed state is specified by the condition

$$\frac{\partial \Phi_0^I}{\partial \phi_0^A} = \mathfrak{J}^I{}_A, \tag{2.76}$$

where the "0" subscripts are used to specify that the configuration is relaxed, and the gothic-J is used to denote the relaxed Jacobian. Using (2.76) to compute (2.73) gives a simple expression for the submanifolds metric in the relaxed state,

$$\bar{\mathfrak{g}}_{AB} = \mathfrak{k}_{IJ} \mathfrak{J}^I{}_A \mathfrak{J}^J{}_B. \tag{2.77}$$

It should be evident that the Christoffel symbols in the field submanifold (2.75) are zero for this relaxed state if $\mathfrak{k}_{IJ}$ is flat and the Jacobians $\mathfrak{J}^I{}_A = \delta^I{}_A$. We have denoted $\bar{\mathfrak{g}}_{AB}$ as the field submanifolds metric in the relaxed state. In a deformed state the derivatives of Φ^I with respect to the ϕ^A must differ from their values in the relaxed state by some amount which can be packaged into a rank-2 tensor $\mathfrak{d}^I{}_A$ (this is a gothic-d, for "deformation") via

$$\frac{\partial \Phi^I}{\partial \phi^A} = \mathfrak{J}^I{}_A + \mathfrak{d}^I{}_A, \tag{2.78}$$

where we will not make any assumptions about the size of the $\mathfrak{d}^I{}_A$. Putting (2.78) into (2.73) gives

$$\mathfrak{g}_{AB} = \bar{\mathfrak{g}}_{AB} + 2\mathfrak{d}_{AB} + \mathfrak{d}^I{}_A \mathfrak{d}_{IB}. \tag{2.79}$$

This expression is very similar to what is used in the non-linear Stuckelberg trick in the massive gravity literature (see, e.g., [19, 35]). It is therefore apparent that the deviation of $\mathfrak{g}_{AB}$ from $\bar{\mathfrak{g}}_{AB}$ is contained within the tensor

$$s_{AB} = \mathfrak{d}_{AB} + \tfrac{1}{2}\mathfrak{d}^I{}_A \mathfrak{d}_{IB}, \tag{2.80}$$

so that

$$s_{AB} = \tfrac{1}{2}\left(\mathfrak{g}_{AB} - \bar{\mathfrak{g}}_{AB}\right). \tag{2.81}$$

Hence, we now have a measure on how deformed the material is: when $s_{AB} = 0$ one has $\mathfrak{g}_{AB} = \bar{\mathfrak{g}}_{AB}$ which is the relaxed metric, and any $s_{AB} \neq 0$ means that the material is deformed in some way. If the deformations are small then one can safely assume that $\mathfrak{d}_{AB}$ is a small quantity and so $S_{AB} = \mathfrak{d}_{AB}$.

2.5.2 Deformations of the material

To make more explicit contact to the construction we gave in section 2.1, suppose that the actual values of the material coordinates ϕ^A are related by way of an "expansion" (which is not necessarily small) about some fiducial state whose coordinates were $\bar{\phi}^A$. That is,

$$\phi^A = \bar{\phi}^A + \pi^A. \tag{2.82}$$

Then the configuration gradient (2.2) can be evaluated

$$\psi^A{}_a = \bar{J}^A{}_a + \partial_a \pi^A, \tag{2.83}$$

in which the configuration gradient computed in the fiducial state is

$$\bar{J}^A{}_a = \frac{\partial \bar{\phi}^A}{\partial x^a}. \tag{2.84}$$

Using (2.83) to provide an expression for the configuration gradient to compute the pull-back k_{ab} of the material metric k_{AB} via (2.29) yields

$$k_{ab} = \overline{k}_{ab} + 2\partial_{(a}\xi_{b)} + \Pi_{ab}, \tag{2.85}$$

in which we defined

$$\begin{aligned} \overline{k}_{ab} &\equiv k_{AB}\overline{J}^{A}{}_{a}\overline{J}^{B}{}_{b}, && (2.86a)\\ \partial_a\xi_b &\equiv k_{AB}\overline{J}^{A}{}_{a}\partial_b\pi^B, && (2.86b)\\ \Pi_{ab} &\equiv k_{AB}\partial_a\pi^A\partial_b\pi^B. && (2.86c) \end{aligned}$$

The Π_{ab}-term is neglected if the deformations are small. The quantity $\overline{k}_{ab}$ is the pull-back of the material metric when the material is in its unstrained state (i.e. when the $\pi^A = 0$ identically).

It should thus be clear that important information about the state of the system is the contained within the tensor

$$S_{ab} \equiv \frac{1}{2}\left(k_{ab} - \overline{k}_{ab}\right), \tag{2.87}$$

since it quantifies the difference between the actual value of k_{ab} and its value in the fiducial state.

3 Quantifying the state of the material

Armed with the map, material metric, and set of scalar invariants, it remains to understand how to quantify the state of the material. This will guided by understanding the effects of the material on space-time. This quantification is acheived by constructing a material action which can be appended to the Einstein-Hilbert action, and from which one can derive the energy-momentum tensor which sources the gravitational field equations.

Along the way there are various useful auxiliary quantities, and useful pieces of technology that can be used to help understand what is going on.

3.1 Constant volume shear tensor

We define the *constant volume shear tensor*

$$s^a{}_b = \tfrac{1}{2}\left(\gamma^a{}_b - \eta^a{}_b\right). \tag{3.1}$$

This is a space-time tensor which quantifies the difference between the actual value of $\gamma^a{}_b$ and the unsheared value $\eta^a{}_b$ as described in Section 2.2. The definition (3.1) follows from the pull-back of (2.24), which was defined in the material manifold.

[Do something to build intuition of s_{ab}]

3.2 The equation of state and material action

The idea is to quantify the state of the material from a "master function" (to use Carter's terminology). This master function will be the piece of freedom which corresponds to the specification of the type or class of materials under consideration. This is much like the specification of the potential function $V(\phi)$ which controls what types of canonical scalar field theories one is studying.

For a material, the energy density, ρ, plays the role of the master function; in what follows we will refer to ρ as the *equation of state*. On a first pass we write down a material action given by the integral of the equation of state which has, as its sole arguments, the mixed components of the pulled-back material metric:

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \rho\left(k^a{}_b\right). \tag{3.2}$$

This is as far as one can go in generality without asking anything further of the material. However, if one demands that the material is isotropic (i.e., impose $SO(3)$ symmetry upon the material) then this constitutes a constraint on ρ as being a function of any possible scalar invariants discussed in Section 2.4. With this constraint imposed, the material action is given the integral of a tri-variate scalar function,

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\, \rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{3.3}$$

It is convenient to re-express the equation of state in terms of the particle number density n and the energy per particle, ϵ, via

$$\rho = n\epsilon. \tag{3.4}$$

And so, rather than ask for the form of ρ, we ask for the form of ϵ, and then write the matter action (3.3) as

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\, \sqrt{-g}\, n\epsilon\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{3.5}$$

3.3 Variation of the material action and measure-weighted variation

Varying the action (3.3) yields

$$\delta S = \int \mathrm{d}^4x\, \sqrt{-g}\, \Diamond\rho. \tag{3.6}$$

We have used the "diamond derivative" notation to denote measure-weighted variations, defined to act on a quantity Q via

$$\lozenge^n Q \equiv \frac{1}{\sqrt{-g}} {\delta_{\mathrm{L}}}^n \left(\sqrt{-g}\, Q\right), \tag{3.7}$$

in which δ_{L} is the *Lagrangian variation* operator. The role of δ_{L} is to encorporate both intrinsic variations of a field, and variations due to some other process (such as symmetry transformations). Before we evaluate (3.6) we want to explain some interesting properties and uses for the first measure-weighted variation $\lozenge Q$.

The first measure-weighted variation of this quantity Q is

$$\lozenge Q = \delta_{\mathrm{L}} Q - \tfrac{1}{2} Q g_{ab} \delta_{\mathrm{L}} g^{ab}. \tag{3.8}$$

When Q is a function of a set of scalars χ^A and their derivatives $\partial_a \chi^A$, say, and the metric g_{ab}, then it is a simple exercise to observe that

$$\lozenge Q = \frac{\partial Q}{\partial \chi^A} \delta \chi^A + \frac{\partial Q}{\partial \partial_a \chi^A} \partial_a \delta \chi^A + \left(\frac{\partial Q}{\partial g^{ab}} - \frac{1}{2} Q g_{ab} \right) \delta g^{ab}. \tag{3.9}$$

The second term can be rearranged by integrating by parts (without neglecting any total derivatives) to give

$$\lozenge Q = \mathcal{E}_A \delta_{\mathrm{L}} \chi^A + \tfrac{1}{2} T_{ab} \delta_{\mathrm{L}} g^{ab} + \nabla_a \vartheta^a, \tag{3.10}$$

where we defined

$$\mathcal{E}_A \equiv \frac{\partial Q}{\partial \chi^A} - \nabla_a \frac{\partial Q}{\partial \partial_a \chi^A}, \tag{3.11a}$$

$$T_{ab} \equiv 2 \frac{\partial Q}{\partial g^{ab}} - Q g_{ab}, \tag{3.11b}$$

$$\vartheta^a \equiv \frac{\partial Q}{\partial \partial_a \chi^A} \delta_{\mathrm{L}} \chi^A. \tag{3.11c}$$

The ϑ^a-term in (3.10) only contributes to the boundary and can be made to vanish by choice of boundary conditions: it won't play a role in what follows.

Suppose the variations δ_{L} are due to diffeomorphisms generated by the vector ξ^a and intrinsic arbitrary variations (of the type usually considered when using variational principles), then the variations δ_{L} in (3.10) should be replaced with

$$\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \pounds_\xi, \tag{3.12}$$

in which the Lie derivatives are

$$\pounds_\xi \chi^A = \xi^a \nabla_a \chi^A, \qquad \pounds_\xi g^{ab} = -2 \nabla^{(a} \xi^{b)}. \tag{3.13}$$

so that

$$\Diamond Q = \mathcal{E}_A \delta_{\mathrm{E}} \chi^A + \tfrac{1}{2} T_{ab} \delta_{\mathrm{E}} g^{ab} + \xi^a \left(\mathcal{E}_A \nabla_a \chi^A + \nabla^b T_{ab} \right) - \nabla^{(a} \left(\xi^{b)} T_{ab} \right). \tag{3.14}$$

Note that the final term only contributes to the boundary. And so, we can read off from (3.14) that diffeomorphism invariance is ensured when the coefficient of the diffeomorphism generating field ξ^a vanishes, namely

$$\mathcal{E}_A \nabla_a \chi^A + \nabla^b T_{ab} = 0. \tag{3.15}$$

We can also read off from (3.14) that the condition for the theory is stationary under arbitrary variations in the scalars χ^A (this is the usual statement of the variational principle) is that the coefficient $\mathcal{E}_A$ of the arbitrary variations $\delta_{\mathrm{E}} \chi^A$ should vanish:

$$\mathcal{E}_A = 0. \tag{3.16}$$

It is immediately clear from its definition (3.11a) that the conditions (3.16) are just the equations of motion of the scalars χ^A. By inspecting (3.15) it is manifest that when the equations of motion (3.16) are satisified, the energy-momentum tensor is conserved

Let us now return to the problem at hand: evaluation of (3.6) for the material medium. At the top of Section 3.2 we stated that the equation of state ρ (i.e. the integrand of the material action) is a function of the pulled-back metric $k^a{}_b$ alone, (3.2). This means that $\delta_{\mathrm{L}} \rho$ can be written as

$$\delta_{\mathrm{L}} \rho = \frac{\partial \rho}{\partial g^{ab}} \delta_{\mathrm{L}} g^{ab}, \tag{3.17}$$

which can be used to obtain

$$\Diamond \rho = \frac{1}{2} \left(-\rho g_{ab} + 2 \frac{\partial \rho}{\partial g^{ab}} \right) \delta_{\mathrm{L}} g^{ab}. \tag{3.18}$$

We remind that (3.18) is the integrand of the first variation of the action.

3.4 The energy-momentum tensor

We have in mind that the material constitutes only part of the dynamics of the entire "universe": there is also the possibility of gravitational dynamics (not nessecarily limited to those prescribed by General Relativity), and also other matter, fluid, or scalar-field sources. In the case that General Relativity provides the gravitational dynamics, the gravitational field equations are given by

$$G_{ab} = 8\pi G \sum_{\mathrm{i}} T^{\mathrm{i}}_{ab}, \tag{3.19}$$

where T^{i}_{ab} is the energy-momentum tensor for the i^{th} source. Below we will be concerned with computing the energy-momentum tensor for the material. We will only use the symbol T_{ab} for the material's energy-momentum tensor, but one should keep in mind that it can be added to any additional energy-momentum tensors.

The energy-momentum tensor is derived from varying the material action S_{M} using the usual expression,

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{ab}}. \tag{3.20}$$

The quantity in braces in (3.18) is precisely the variation requires to work out the right-hand-side of (3.20); and so, the energy-momentum tensor is given by

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}}. \tag{3.21}$$

We are able to further evaluate this expression, and in particular we can deduce the "types" of contributions to T_{ab} from knowledge of what ρ is a function of. Since the equation of state only depends on the mixed components of the pulled-back material metric,

$$\rho = \rho\left(k^a{}_b\right), \tag{3.22}$$

using (2.63) gives an expression for the final term in (3.20):

$$\frac{\partial \rho}{\partial g^{ab}} = k_{c(a}\frac{\partial \rho}{\partial k^{b)}{}_c}. \tag{3.23}$$

Since the right-hand-side of this expression is orthogonal by virtue of (2.32), so is the left-hand-side,

$$u^a\frac{\partial \rho}{\partial g^{ab}} = 0. \tag{3.24}$$

And so, assuming an equation of state ρ has been given in the form of (3.22), the energy-momentum tensor of the solid (3.21) can be written as

$$T_{ab} = \rho u_a u_b + P_{ab}, \tag{3.25a}$$

in which the pressure tensor P_{ab} is given by

$$P_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho\gamma_{ab}. \tag{3.25b}$$

By virtue of (3.24) the pressure tensor (3.25b) is orthogonal,

$$u^a P_{ab} = 0. \tag{3.26}$$

The important thing to note is that there is no heat flux term in T_{ab}:

$$u^a \gamma^b{}_c T_{ab} = 0. \tag{3.27}$$

This a consequence of the orthogonality of the mapping between the material manifold and spacetime.

After using the solid form of the energy-momentum tensor (3.25a), the variation of the energy density (3.17) can be written as

$$\delta_{\mathrm{L}} \rho = \tfrac{1}{2} \left(\rho \gamma_{ab} + P_{ab} \right) \delta_{\mathrm{L}} g^{ab}. \tag{3.28}$$

After rewriting the equation of state ρ in terms of an energy per particle, ϵ, via (3.4), the pressure tensor (3.25b) takes on the more compact form

$$P_{ab} = 2n \frac{\partial \epsilon}{\partial g^{ab}}. \tag{3.29}$$

When the energy per particle ϵ is written in a (still general) way to only depend on the number density n and the components of the uni-modular tensor $\eta^a{}_b$, i.e.,

$$\epsilon = \epsilon(n, \eta^a{}_b), \tag{3.30}$$

we can use (2.64) to further evaluate the pressure tensor (3.29), yielding the rather attractive expression

$$P_{ab} = p\gamma_{ab} + \pi_{ab}, \tag{3.31}$$

in which we have identified the pressure scalar p,

$$p = n^2 \frac{\partial \epsilon}{\partial n}, \tag{3.32a}$$

and the (traceless) anisotropic stress tensor

$$\pi_{ab} = 2n \eta_{c\langle a} \frac{\partial \epsilon}{\partial \eta^{b\rangle}{}_c}. \tag{3.32b}$$

We remind that the energy density is given by

$$\rho = n\epsilon. \tag{3.32c}$$

This highlights that dependence of ϵ on the number density n is linked to isotropic pressure p, and dependence of ϵ on the uni-modular tensor $\eta^a{}_b$ is linked to anistropic stress π_{ab}.

Note that we have not assumed the material to be isotropic: at no point did we ask for the equation of state to be constructed out of the scalar invariants of $\eta^a{}_b$ or $k^a{}_b$. When we do so we will find some simplifications.

There is nothing "imperfect" about the construction of the substance so far: there is no dissipation, everything is conserved, and is constructed from a very geometrical point of view. However, the pressure tensor (3.31) has anisotropic stress (3.32b). For a *fluid* this would signal an imperfection, but it is exactly this anisotropic stress which makes the theory that of a *solid*.

It has recently become popular to suggest that an observation of anisotropic stress would point towards modified gravity rather than dark energy [36–39]. What we are about to state is not a comment on a claim made by any of these articles, but it is worth pointing out. Whilst it is true that modified gravity models have anisotropic stress, it is also true that material models can contribute towards anisotropic stress. Infact, material models constitute the simplest and physically "most intuitive" additions to the Einstin-Hilbert and standard matter content gravitational model.

3.5 Example equation of state

It is instructive to specify an example equation of state and obtain the energy-momentum tensor. We will make the same choice as described in [6], and write down a particular equation of state (which is only a subset of all the possible models) for isotropic materials.

To begin with it is useful to recall the covariant form of the constant volume shear tensor (3.1), which we repeat here for completeness:

$$s_{ab} = \tfrac{1}{2}(\gamma_{ab} - \eta_{ab}). \tag{3.33}$$

There are two methods to raise indices (and thus construct traces). These methods are

$$s^a{}_b = \gamma^{ac}s_{cb}, \qquad \hat{s}^a{}_b = \eta^{-1ac}s_{cb}. \tag{3.34}$$

In matrix form these respectively read

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{1} - \boldsymbol{\eta}), \qquad \hat{\mathbf{s}} = \tfrac{1}{2}(\boldsymbol{\eta}^{-1} - \mathbf{1}). \tag{3.35}$$

The equation of state ϵ is chosen to be a function of the particle number density n and a particular combination of the invariant of $\eta^a{}_b$. The explicit form of ϵ is

$$\epsilon = \check{\epsilon}_0(n) + \frac{\check{\mu}(n)}{n}\overline{s}^2, \tag{3.36}$$

in which $\overline{s}^2$ is the shear scalar defined from the invariants of $\eta^a{}_b$ via

$$\overline{s}^2 \equiv \tfrac{1}{36}\left([\boldsymbol{\eta}]^3 - [\boldsymbol{\eta}^3] - 24\right). \tag{3.37}$$

Notice that by using the Cayley-Hamilton relation (2.59), the choice of shear scalar (3.37) is equivalent to

$$\bar{s}^2 = \tfrac{1}{24}\left([\boldsymbol{\eta}]^2 - [\boldsymbol{\eta}^2]\right)[\boldsymbol{\eta}] - \tfrac{3}{4}. \tag{3.38}$$

Using (3.36) the material action is therefore given by

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\left\{n\check{\epsilon}_0 + \tfrac{1}{36}\check{\mu}\left([\boldsymbol{\eta}]^3 - [\boldsymbol{\eta}^3] - 24\right)\right\}. \tag{3.39}$$

The pressure tensor is given by (3.31) where the isotropic pressure scalar (3.32a) is

$$p = \check{p} + (\check{\Omega} - 1)\sigma, \tag{3.40a}$$

and the anisotropic stress (3.32b) is given by

$$\pi_{ab} = \tfrac{1}{6}\check{\mu}\left([\boldsymbol{\eta}]^2\eta_{\langle ab\rangle} - \eta^{cd}\eta_{c\langle a}\eta_{b\rangle d}\right), \tag{3.40b}$$

and where the three quantities appearing in the pressure (3.40a) are

$$\check{p} = n^2\frac{\mathrm{d}\check{\epsilon}_0}{\mathrm{d}n}, \qquad \check{\Omega} = \frac{n}{\check{\mu}}\frac{\mathrm{d}\check{\mu}}{\mathrm{d}n}, \qquad \sigma = \check{\mu}s^2. \tag{3.41}$$

4 Equation of motion and propagation of sound

Obtaining the equation of state and energy-momentum tensor is clearly only part of the story. One must also obtain equations of motion: these come from the conservation equation

$$\nabla_a T^{ab} = 0. \tag{4.1}$$

If the material is the only source to the gravitational field equations, then (4.1) follows by diffeomorphism invariance, and also by the Bianchi identity. If there are multiple sources to the gravitational field equations, then (4.1) holds if and only if T^{ab} is interpreted as the sum of the individual energy-momentum tensors (of which the materials EMT can be an additive contribution). Only in the case when these sources are "decoupled", their individual energy-momentum tensors are independently conserved.

We will proceed by first proving this statement, then continue by providing a useful way to write down (4.1) in a rather physically intuitive manner.

4.1 Equations of motion from the action

Without assuming the existence of the pulled-back material metric, one should be convinced that the action will be a function of the metric g^{ab}, a set of scalars ϕ^A representing the particle positions in the material manifold, and their derivatives $\partial_a\phi^A$. There may be other material space tensors, but we shall leave that complicating possibility out for now. With these considerations, the action can be written as

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\rho\left(g^{ab},\phi^A,\partial_a\phi^A\right). \tag{4.2}$$

Under Lagrangian variations δ_{L} in the available fields, g^{ab} and ϕ^A, the corresponding variation in the action is

$$\delta_{\mathrm{L}}S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\left[\frac{1}{2}T_{ab}\delta_{\mathrm{L}}g^{ab} - \mathcal{E}_A\delta_{\mathrm{L}}\phi^A\right]. \tag{4.3}$$

We set T_{ab} to be the energy-momentum tensor, as defined in the usual manner,

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{ab}} = 2\frac{\partial\rho}{\partial g^{ab}} - \rho g_{ab}, \tag{4.4}$$

and the contribution to $\delta_{\mathrm{L}}S_{\mathrm{M}}$ multiplying the perturbed scalars is

$$\mathcal{E}_A = \nabla_a\left(\frac{\partial\rho}{\partial\partial_a\phi^A}\right) - \frac{\partial\rho}{\partial\phi^A}. \tag{4.5}$$

We used the symbol δ_{L} to stand for "some" arbitrary variation operator which acts on the fields: we have not yet specified what generates the variation. As we now show, this has given us a clear notational method for imposing general covariance. We will, without loss of generality, assume that δ_{L} decomposes into two parts:

$$\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \delta_{\xi}. \tag{4.6}$$

The first part, δ_{E}, will be due to intrinsic variations, and δ_ξ will be the variation induced by changes in the coordinates, $x^a \to x^a + \xi^a(x^b)$. For the metric g^{ab} and the set of scalars ϕ^A one find that the relevant expressions for δ_ξ acting of each of the fields is given by the Lie derivative,

$$\delta_\xi g^{ab} = -2\nabla^{(a}\xi^{b)}, \qquad \delta_\xi\phi^A = \xi^a\partial_a\phi^A. \tag{4.7}$$

After using these expressions and integrating by parts, the variation in the action generated by space-time coordinate transformation is given by

$$\delta_\xi S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\left[\xi^a\left(\nabla_b T^{ab} - \mathcal{E}_A\,\partial_a\phi^A\right)\right]. \tag{4.8}$$

In a similar, and simpler fashion, the variation in the action due to the intrinsic variations of the fields is

$$\delta_{\mathrm{E}} S_{\mathrm{M}} = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{2} T_{ab} \delta_{\mathrm{E}} g^{ab} - \mathcal{E}_A \, \delta_{\mathrm{E}} \phi^A \right]. \tag{4.9}$$

We can use this to obtain the functional derivative of the action with respect to the intrinsic variations in the scalars,

$$\frac{\delta_{\mathrm{E}} S_{\mathrm{M}}}{\delta_{\mathrm{E}} \phi^A} = -\mathcal{E}_A. \tag{4.10}$$

The variational principle demands that this expression must vanish, and yields the equation of motion satisfied by the scalars ϕ^A. General covariance requires the action to be invariant under changes in the coordinates. We read off from (4.8) that $\delta_\xi S_{\mathrm{M}} = 0$ for arbitrary ξ^a when

$$\nabla^b T_{ab} = \mathcal{E}_A \, \partial_a \phi^A. \tag{4.11}$$

This links conservation of energy-momentum and the equations of motion of the scalars. Put another way: energy-momentum conservation implies the satisfaction of the equations of motion of the elastic medium. Due to the orthogonality of the mapping (2.3)

$$u^a \partial_a \phi^A = 0, \tag{4.12}$$

and so therefore the time-like projection of (4.11) is automatically satisfied:

$$u^a \nabla^b T_{ab} = 0. \tag{4.13}$$

It then follows that the orthogonal projection of (4.11) implies the vanishing of (4.10):

$$\gamma^a{}_c \nabla^b T_{ab} = 0 \qquad \Longleftrightarrow \qquad \mathcal{E}_A = 0. \tag{4.14}$$

Therefore, conservation of energy-momentum guarantees that the scalars ϕ^A satisfy their equation of motion.

4.2 The fluid equations

Inspired by the result of the previous section, we now present the conservation equations in a physically intuitive manner.

Using the solid form (3.25a) for T_{ab} the two independent (i.e., time-like and orthogonal) projections of the conservation equation (4.1) are

$$\dot{\rho} + (\rho \gamma^{ab} + p^{ab}) \Theta_{ab} = 0, \tag{4.15a}$$

$$(\rho\gamma^{ab} + p^{ab})\dot{u}_b + \overline{\nabla}_b p^{ab} = 0. \tag{4.15b}$$

We used the orthogonally projected derivative $\overline{\nabla}_b$, as defined in (1.9)

If the energy per particle ϵ is a function only of the components $k^a{}_b$ (regardless of whether the material is isotropic), then using (3.29) in conjunction with (2.55), the orthogonally projected derivative of the pressure tensor can be written as

$$\overline{\nabla}_b p^{ab} = \left(E^{ab}{}_{cd} - \gamma^a{}_c p^b{}_d\right) \mathfrak{D}^{cd}{}_b, \tag{4.16}$$

in which we used the elasticity difference tensor $\mathfrak{D}^{cd}{}_b$ as defined in (2.49), and introduced the relativistic *elasticity tensor*, $E^{ab}{}_{cd}$, defined via

$$E^{ab}{}_{cd} \equiv 2\frac{\partial p^{ab}}{\partial \gamma^{cd}} - p^{ab}\gamma_{cd}. \tag{4.17}$$

It is manifest from this definition that the elasticity tensor has the following symmetries in its indices:

$$E^{abcd} = E^{(ab)(cd)}. \tag{4.18}$$

Using (2.62) and (3.29), the elasticity tensor can be written as the second derivative of the energy per particle ϵ via

$$E^{abcd} = 4n\frac{\partial^2 \epsilon}{\partial \gamma_{ab} \partial \gamma_{cd}}. \tag{4.19}$$

This final relationship informs us that the elasticity tensor is also symmetric under interchange of the first and last pair of indices,

$$E^{abcd} = E^{cdab} \tag{4.20}$$

in addition to the simple symmetries (4.18).

It will be convenient to use the relativistic *Hadamard elasticity tensor*, $A^{ab}{}_{cd}$, defined via the elasticity tensor via

$$A^{ab}{}_{cd} \equiv E^{ab}{}_{cd} - \gamma^a{}_c p^b{}_d. \tag{4.21}$$

Using the Hadamard elasticity tensor, the orthogonally projected derivative of the pressure tensor (4.16) takes on a particularly simple form,

$$\overline{\nabla}_b p^{ab} = A^{ab}{}_{cd} \mathfrak{D}^{cd}{}_b, \tag{4.22}$$

and the orthogonal projection (4.15b) of the conservation equations can be written as

$$(\rho\gamma^{ab} + p^{ab})\dot{u}_b + A^{abcd}\mathfrak{D}_{cbd} = 0. \tag{4.23}$$

A rather convenient form of the equations of motion is given in [4, 31]; in terms of the material derivative [need to explain the material derivative somewhere] the pressure tensor satisfies

$$\left[p^{ab}\right]^{\cdot} = -p^{ab}\theta - E^{abcd}\theta_{cd}. \tag{4.24}$$

In the more usual notation of "time-like" derivatives, the equations of motion for the materials energy density and pressure tensor are given by

$$u^a\rho_{;a} = -\rho u^a{}_{;a} - p^{ab}u_{a;b}, \tag{4.25a}$$

$$u^c p^{ab}{}_{;c} = 2p^{c(a}u^{b)}{}_{;c} + 2p^{c(a}u^{b)}\dot{u}_c - p^{ab}u^c{}_{;c} - E^{abcd}u_{c;d}, \tag{4.25b}$$

where we defined the acceleration vector $\dot{u}_a = u^b\theta_{ab}$.

4.3 Speed of sound

Here we review how to compute the speed of sound of the medium [31].

Sound wavefronts are characteristic hypersurfaces across which the acceleration vector $\dot{u}^a$ has a jump discontinuity (the velocity u^a and the metric remain continuous). Following Carter, we denote discontinuities across the wavefront with square braces; and so we set

$$[\dot{u}^a] = \alpha\iota^a, \tag{4.26}$$

in which α is the amplitude of the wavefront and ι^a is the polarization vector satisfying the space-like normalization condition

$$\iota^a\iota_a = 1. \tag{4.27}$$

Since the acceleration and velocity vectors are mutually orthogonal,

$$u_a\dot{u}^a = 0 \tag{4.28}$$

it follows that the polarization vector and the velocity vector are orthogonal

$$u_a\iota^a = 0. \tag{4.29}$$

The *propagation direction vector* ν^a is specified with the same orthonormality conditions as the polarization vector, namely

$$\nu^a\nu_a = 1, \qquad \nu^a u_a = 0. \tag{4.30}$$

The normal to the characteristic hypersurface is in the direction of the vector λ_a, defined via

$$\lambda_a = \nu_a - \upsilon u_a. \tag{4.31}$$

The scalar

$$\upsilon = \lambda^a u_a \tag{4.32}$$

is the speed of propagation.

The derivatives of the density, velocity, and pressure tensor fields on the characteristic hypersurface are given in terms of quantities $\sigma, \kappa^a, \tau^{ab}$ via

$$[\rho_{;a}] = \sigma \lambda_a, \tag{4.33a}$$

$$[u^a{}_{;b}] = \kappa^a \lambda_b, \tag{4.33b}$$

$$\left[p^{ab}{}_{;c}\right] = \tau^{ab} \lambda_c. \tag{4.33c}$$

We now show how to determine the values of $\sigma, \kappa^a, \tau^{ab}$ in terms of υ, α, and ι^a. First, contracting (4.33b) with u^b gives (4.26) on the left-hand-side, and $\upsilon\kappa^a$ on the right-hand-side, and thus one obtains

$$\upsilon \kappa^a = \alpha \iota^a. \tag{4.34a}$$

Taking the discontinuity of the projections of the conservation equation (4.25a) and (4.25b), and then multiplying by υ respectively yields

$$\upsilon^2 \sigma = -\alpha \left(\rho \iota^a \lambda_a + p^{ab} \iota_a \lambda_b\right), \tag{4.34b}$$

$$\upsilon^2 \tau^{ab} = \alpha \left(2\upsilon u^{(a} p^{b)c} \iota_c + 2p^{c(a} \iota^{b)} \lambda_c - p^{ab} \iota^c \lambda_c - E^{abcd} \iota_c \lambda_d\right). \tag{4.34c}$$

Putting the general form of the energy-momentum tensor (3.25a) into the conservation equation (4.1)

$$\left(u^b \rho_{;b} + \rho u^b{}_{;b}\right) u^a + \rho \dot{u}^a + p^{ab}{}_{;b} = 0. \tag{4.35}$$

Taking the discontinuity of the general formula (4.35) and using (4.33) yields

$$\left(\upsilon\sigma + \rho \kappa^b \lambda_b\right) u^a + \rho \alpha \iota^a + \tau^{ab} \lambda_b = 0. \tag{4.36}$$

Now using (4.34) for κ^a, σ, and τ^{ab} yields

$$v^2 \left(\rho\gamma^{ab} + p^{ab}\right) \iota_b + p^{bc}\lambda_b\lambda_c\iota^a - E^{abcd}\lambda_b\iota_c\lambda_d = 0. \tag{4.37}$$

By using the relativistic Hadamard tensor A^{abcd}, defined in (4.21), the equation (4.37) becomes

$$\left[v^2 \left(\rho\gamma^{ab} + p^{ab}\right) - Q^{ab}\right] \iota_b = 0. \tag{4.38}$$

where we have introduced the Fresnel tensor Q^{ab} which is defined in terms of the Hadamard tensor and the propagation vector ν_a via

$$Q^{ac} \equiv A^{abcd}\nu_b\nu_d, \tag{4.39}$$

after noting that the Hadamard tensor is orthogonal on all indices. Orthogonality of the Hadamard tensor carries over to give orthogonality of the Fresnel tensor,

$$u_a Q^{ab} = 0. \tag{4.40}$$

Since every term in the characterstic equation (4.38) is orthogonal, it is essentially a 3-dimensional equation. The eigenvalues v^2 are the squared sound speed (in general there will be three values).

Although we will show where this comes from later on, it is worth our providing an example of the explicit computation of the sound speed. In the case of an isoptropic elastic solid close to a ground state, the pressure tensor is specified in terms of the isotropic pressure scalar as $p^{ab} = p\gamma^{ab}$, and the elasticity tensor is given by

$$E^{abcd} = \left(\beta - \tfrac{1}{3}p\right)\gamma^{ab}\gamma^{cd} + 2\left(\mu + p\right)\left(\gamma^{a(c}\gamma^{d)b} - \tfrac{1}{3}\gamma^{ab}\gamma^{cd}\right); \tag{4.41}$$

the coefficients p, β, and μ, are repectively the isotopic pressure, bulk modulus, and modulus of rigidity. The Hadamard tensor in this case is given by

$$A^{abcd} = \beta\gamma^{ab}\gamma^{cd} + 2p\gamma^{a[d}\gamma^{b]c} + 2\mu\left(\gamma^{a(c}\gamma^{d)b} - \tfrac{1}{3}\gamma^{ab}\gamma^{cd}\right), \tag{4.42}$$

and the Fresnel tensor works out as

$$Q^{ab} = \left(\beta + \tfrac{1}{3}\mu\right)\nu^a\nu^b + \mu\gamma^{ab}. \tag{4.43}$$

Hence, the characteristic equation (4.38) becomes

$$\left[v^2\left(\rho + p\right)\gamma^{ab} - \mu\gamma^{ab} - \left(\beta + \tfrac{1}{3}\mu\right)\nu^a\nu^b\right]\iota_b = 0. \tag{4.44}$$

There are two solutions: the first is where the polarization and propagation vectors are aligned, $\nu_a = \iota_a$ in which case the eigenvalue is

$$v^2 = \frac{\beta + \frac{4}{3}\mu}{\rho + p} \equiv c_{\mathrm{L}}^2. \tag{4.45}$$

Secondly, where the polarization and propagation vectors are orthogonal: $\nu_a \iota^a = 0$, in which case the eigenvalue is

$$v^2 = \frac{\mu}{\rho + p} \equiv c_{\mathrm{T}}^2. \tag{4.46}$$

We therefore have two sound speeds; c_{L}^2 which is the speed of propagation of longitudinal modes, and c_{T}^2 which is the speed of propagation of transverse modes.

5 The Carter-Quintana perfect solid

Carter and Quintana conclude their paper with an exposition of the equations for a *perfect elastic solid*. Before we give their equation of state and energy-momentum tensor, we shall discuss physical issues regarding the existence (or otherwise) of locally relaxed states of the material. See also [40] for a presentation of FRW solutions to the Carter-Quintana elastic solid system.

5.1 Strain and shear tensors

The strain tensor is linked to the assumption about the existence of a locally relaxed state of a material – this is the unstrained state. In the unstrained state the energy per particle ϵ is supposed to be minimum when γ_{ab} takes on a particular value, k_{ab} say. This invites a quantification of the state of strain of the material by measuring the difference between the actual value of γ_{ab} and its unstrained value k_{ab} via the *strain tensor*, e_{ab}, defined as

$$e_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - k_{ab}\right). \tag{5.1}$$

Recalling that the energy per particle is denoted as ϵ, we define ϵ_0 to be the energy per particle in the unstrained state. The Hookean idealization takes the energy per particle to be of quadratic form in the strain tensor

$$\epsilon = \epsilon_0 + \tfrac{1}{2}K^{abcd}e_{ab}e_{cd}. \tag{5.2}$$

The elasticity tensor E^{abcd} relates to K^{abcd} via

$$E^{abcd} = nK^{abcd}. \tag{5.3}$$

Hence, since $\rho = n\epsilon$, the energy density can be written as

$$\rho = \frac{n}{n_0}\rho_0 + \tfrac{1}{2}E^{abcd}e_{ab}e_{cd}, \tag{5.4}$$

and the pressure tensor is related to the strain tensor via

$$p^{ab} = -E^{abcd}e_{cd}. \tag{5.5}$$

The tensor k_{ab} can be thought of as a Riemannian metric on material space; the pull-back formalism means that $u^a k_{ab} = 0$. Associated with the value ϵ_0 of ϵ in the unstrained state are the values ρ_0 of the energy density ρ, and n_0 of the particle number density n.

The complication which Carter invites is that not all physical systems of interest will have a state which is locally relaxed, thus negating the existence of k_{ab} and rendering this construction impotent. This leads to the introduction of the shear tensor.

Rather than ask for the relaxed state to be a state where the energy per particle is minimum, we ask for a state in which ϵ is minimized subject to the restriction of constant particle number density. This is the unsheared state, and motivates the introduction of $\eta_{ab}(n)$ which is the value of γ_{ab} in the unsheared state with particle number density n. Again, to quantify the state of shear we define the *constant volume shear tensor* via

$$s_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - \eta_{ab}\right), \tag{5.6}$$

which (to reinforce the point) is the difference between the actual value of γ_{ab} and its value in the unsheared state.

We define $\check{\rho}(n)$ to be the energy density in the unsheared state, and hence

$$\check{\rho} = n\check{\epsilon}. \tag{5.7}$$

When ϵ does have an absolute minimum, at some particle number density n_0, one can keep the previous notions of the strain tensor; indeed

$$\eta_{ab}(n_0) = k_{ab}, \tag{5.8a}$$

$$\check{\rho}(n_0) = \rho_0, \tag{5.8b}$$

$$\check{\epsilon}(n_0) = \epsilon_0. \tag{5.8c}$$

5.2 The equation of state

The "physics" of the solid that Carter and Quintana had in mind was that it was supposed to have vanishing compressional distortion. That is, the compressional distortion tensor is supposed to vanish, $\tau_{ab} = 0$. This means that the reference tensors satisfy

$$[\eta_{ab}]^{\cdot} = \tfrac{2}{3}\eta_{ab}\Theta, \tag{5.9a}$$

$$[\eta^{-1ab}]^{\cdot} = -\tfrac{2}{3}\eta^{-1ab}\Theta, \tag{5.9b}$$

and the strain tensor satisfies

$$[s_{ab}]^{\cdot} = \tfrac{2}{3}s_{ab}\Theta + \sigma_{ab}. \tag{5.9c}$$

Here, $[X]^{\cdot}$ denotes the *material derivative* of X (this is explained the first half of Carter and Quintana, and we will do so later on). One can obtain

$$\eta_{ab} = (n/n_0)^{-2/3}k_{ab}, \tag{5.10}$$

The solid is supposed to be isotropic with respect to its unsheared states. Hence, the energy per particle (recall, $\rho = \epsilon n$, and ϵ is the energy per particle) is a function only of invariants. There are a maximum of three invariants: they are taken to be the particle number density n and the two independent invariants of the shear tensor $s^a{}_b$. The particular combination of these are taken to be

$$s^2 \equiv \left(\eta^{-1ad}\eta^{-1bc} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right) s_{ab}s_{cd} = \lceil \mathbf{s}^2 \rceil - \tfrac{1}{3}\lceil \mathbf{s} \rceil^2, \tag{5.11a}$$

$$l \equiv \eta^{-1ab}\eta^{-1cd}\eta^{-1ef}s_{bc}s_{de}s_{fa} = \lceil \mathbf{s}^3 \rceil. \tag{5.11b}$$

We used the notation $\lceil \mathbf{X} \rceil$ for traces which are taken with η^{-1ab} (as opposed to $[\mathbf{X}]$ which was used to denote traces with g^{ab}): this choice is for simplicity of the resulting formulae and does not lose generality. Recall that the shear tensor s_{ab} is related to the material metric k_{ab} and particle number density via

$$k_{ab} = n^{2/3}\left(\gamma_{ab} - 2s_{ab}\right). \tag{5.12}$$

Hence, the most general form of the equation of state is a function with three arguments:

$$\epsilon = F(n, s^2, l). \tag{5.13}$$

The action for Einsteinian gravity with the CQ solid is thus

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{R}{16\pi G} - nF(n, s^2, l)\right]. \tag{5.14}$$

5.3 The energy-momentum tensor

The energy density ρ, and pressure tensor p^{ab} are given by

$$\rho = nF, \tag{5.15a}$$

$$\begin{aligned} p^{ab} \;=\; & \left\{n^2 F_{,n} + ns^2\left(\frac{4}{3}F_{,s^2} + F_{,l}\right) - n\left(2l + \tfrac{1}{3}\lceil \mathbf{s}\rceil^2\right)F_{,l}\right\}\gamma^{ab} \\ & -2nF_{,s^2}\left(\eta^{-1a(c}\eta^{-1d)b} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right)s_{cd} \\ & -3nF_{,l}\eta^{-1a(c}\eta^{-2d)b}\eta^{-1ef}s_{ce}s_{df}. \end{aligned} \tag{5.15b}$$

These expressions contain the corrections to the typos which were present in [4], and which were pointed out (by the same authors) in [5]. The isotropic pressure is found from the trace of (5.15b), and is given by

$$p = n^2 F_{,n} + 2n\left(\frac{2}{3}s^2 F_{,s^2} - lF_{,l}\right). \tag{5.16}$$

5.4 Exact non-linear equations of motion

This is based on the discussion in [5], and the aim is to compute the equations of motion for General Relativity sourced by a non-linear elastic solid.

Start off with the matter flow velocity

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \tag{5.17}$$

normalised via

$$u^a u_a = -c^2. \tag{5.18}$$

Now introduce the flow gradient tensor v_{ab} via

$$u_{a;b} = v_{ab} - \tfrac{1}{c^2}\dot{u}_a u_b, \tag{5.19}$$

where

$$\dot{u}_a = u^b u_{a;b} \tag{5.20}$$

is the acceleration vector. The flow gradient tensor is orthogonal to the flow,

$$u^a v_{ab} = 0. \tag{5.21}$$

Let f_{ab} be an orthogonal covariant tensor. Then the Lie derivative of f_{ab} along the direction of the flow velocity is

$$\pounds_u f_{ab} = \gamma^c{}_a \gamma^d{}_b u^e f_{cd;e} + f_{cb}v^c{}_a + f_{ac}v^c{}_b. \tag{5.22}$$

One has

$$\pounds_u \gamma_{ab} = 2\theta_{ab}, \tag{5.23}$$

where θ_{ab} is related to the flow gradient v_{ab} via

$$v_{ab} = \theta_{ab} + \omega_{ab}, \tag{5.24}$$

with

$$\theta_{[ab]} = 0, \qquad \omega_{(ab)} = 0. \tag{5.25}$$

From the Ricci identity

$$u_{a;[b;c]} = \tfrac{1}{2} u_d R^d{}_{abc} \tag{5.26}$$

one obtains the Lie derivative of the flow gradient in the direction of u^a,

$$\pounds_u v_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{2} \dot{u}_a \dot{u}_b - u^c u^d R_{acbd}. \tag{5.27}$$

The anti-symmetric portion of (5.27) gives

$$\pounds_u \omega_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{[c;d]}, \tag{5.28}$$

and the symmetric portion of (5.27) gives

$$\tfrac{1}{2} \pounds_u \pounds_u \gamma_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{(c;d)} + v^c{}_a v_{cb} + \tfrac{1}{c^2} \dot{u}_a \dot{u}_b - u^c u^d R_{acbd}. \tag{5.29}$$

We also use the Weyl tensor

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2 g^{[a}{}_{[c} \left(R^{b]}{}_{d]} - \tfrac{1}{6} R g^{b]}{}_{d]} \right). \tag{5.30}$$

The *relative strain tensor* is

$$e_{ab} = \tfrac{1}{2} \left(\gamma_{ab} - \kappa_{ab} \right), \tag{5.31}$$

where the *strain reference tensor* is κ_{ab} and satisfies

$$\pounds_u \kappa_{ab} = 0. \tag{5.32}$$

Hence, from (5.23),

$$\theta_{ab} = \pounds_u e_{ab}, \tag{5.33}$$

meaning that the expansion tensor θ_{ab} quantifies the rate of relative strain. Carter sets $\kappa_{ab} = 0$. Hence

$$\begin{aligned} \pounds_u \pounds_u e_{ab} \;=\;& \gamma^c{}_a \gamma^d{}_b \dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{c^2} \dot{u}_a \dot{u}_b \\ & - u^c u^d C_{acbd} - \tfrac{1}{2} \gamma_{ab} \left(u^c u^d R_{cd} + \tfrac{1}{3} R c^2 \right) + \tfrac{1}{2} \gamma^c{}_a \gamma^d{}_b R_{cd} c^2. \end{aligned} \tag{5.34}$$

Now we set the energy-momentum tensor to be that for a perfect elastic solid,

$$T^{ab} = \rho u^a u^b + p^{ab}. \tag{5.35}$$

One can compute p^{AB} from the energy as a function of strain, $\epsilon(\gamma_{AB})$ via

$$p^{AB} = -\epsilon\gamma^{AB} - 2\frac{\partial\epsilon}{\partial\gamma_{AB}}. \tag{5.36}$$

This is valid for any linear or non-linear function of strain, for which the conservation law

$$T^{ab}{}_{;b} = 0 \tag{5.37}$$

holds.

We now specify the gravitational theory, which we take to be General Relativity for whom the Ricci tensor is given by

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \tfrac{1}{2}T g_{ab}\right), \tag{5.38}$$

where

$$T = T^a{}_a = p^a{}_a - \rho c^2 \tag{5.39}$$

is the trace of the energy-momentum tensor. In General Relativity the Weyl tensor satisifies

$$C_{abcd} = C_{cdab} = C_{[ab][cd]}, \qquad C^a{}_{[bcd]} = 0, \qquad C^{ab}{}_{ac} = 0, \tag{5.40}$$

and the Bianchi identities impose

$$C^{abcd}{}_{;d} = \frac{8\pi G}{c^4}\left(T^{c[a;b]} - \tfrac{1}{3}g^{a[a}T^{;b]}\right). \tag{5.41}$$

We denote

$$C_{ab} = u^c u^d C_{acbd} \tag{5.42}$$

for the electric part of the the Weyl tensor.

From the conservation equation one obtains

$$\rho\dot{u}^a = -p^{ab}{}_{;b} + \tfrac{1}{c^2}u^a p^{bc}\theta_{bc}. \tag{5.43}$$

Hence (5.28) and (5.34) become

$$\pounds_u\omega_{ab} = \frac{1}{\rho^2}\gamma_{c[a}\gamma^d{}_{b]}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) + \frac{1}{\rho c^2}\omega_{ab}p^{cd}\pounds_u e_{cd}. \tag{5.44a}$$

$$\begin{aligned}
\pounds_u\pounds_u e_{ab} \;=\;& \frac{1}{\rho^2}\gamma_{c(a}\gamma^d{}_{b)}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) - C_{ab} - \frac{4\pi G}{3}\rho\gamma_{ab} \\
&+\omega_{ca}\omega^c{}_b + 2\omega^c{}_{(a}\pounds_u e_{b)c} + g^{cd}\left(\pounds_u e_{ac}\right)\left(\pounds_u e_{bd}\right) + \frac{4\pi G}{c^2}\left(P_{ab} - \tfrac{2}{3}p^c{}_c\gamma_{ab}\right) \\
&+\frac{1}{c^2\rho^2}\left[\gamma_{ac}\gamma_{bd}p^{ce}{}_{;e}p^{df}{}_{;f} + \rho p^{cd}\left(\pounds_u e_{cd}\right)\left(\pounds_u e_{ab}\right)\right].
\end{aligned} \tag{5.44b}$$

5.5 Slow roll parameter

We take this opportunity to recall that for inflating an FLRW Universe one requires smallness of the slow-roll parameter $\epsilon_{\rm slow}$, defined as

$$\epsilon_{\rm slow} \equiv -\frac{\dot{H}}{H^2} = \frac{3(\rho + P)}{2\rho}. \tag{5.45}$$

Using (5.15a) and (5.16) the slow-roll parameter (5.45) evaluates for the CQ perfect solid to give

$$\epsilon_{\rm slow} = \frac{3}{2}\left[1 + \frac{\partial \log F}{\partial \log n} + \frac{4}{3}\frac{\partial \log F}{\partial \log s^2} - 2\frac{\partial \log F}{\partial \log l}\right]. \tag{5.46}$$

The structure of (5.46) suggests a separable ansatz for the functional form of F:

$$F(n, s^2, l) = x(n)y(s^2)z(l), \tag{5.47}$$

since (5.46) becomes

$$\epsilon_{\rm slow} = \frac{3}{2}\left[1 + nx' + \frac{4}{3}s^2y' - 2lz'\right], \tag{5.48}$$

in which a prime is used to denote derivative with respect to the sole argument of the given function.

6 The solid action

Recall from (3.3) that the action for the solid can be written as a tri-variant function whose arguments are the three independent scalar invariants formed from the mixed-components of the pulled-back material metric:

$$S = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}\left([\mathbf{k}], [\mathbf{k}^2], [\mathbf{k}^3]\right). \tag{6.1}$$

Our focus in this section is on providing useful ways to construct workable sub-theories from this action.

6.1 Derivative counting

Suppose we counted the terms by keeping track of the number of derivatives. The pulled-back material metric $k^a{}_b$ has two space-time derivatives, and so the $n_{\rm th}$-trace has $2n$-inverse-powers of a mass-scale M,

$$[\mathbf{k}^n] \sim \frac{1}{M^{2n}}. \tag{6.2}$$

We can use this to write the complete list of terms in the solid-action which come with a given power of M. Up to sixth-order the Lagrangian density is

$$\mathcal{L} = \frac{1}{M^2} b\,[\mathbf{k}] + \frac{1}{M^4}\left(c_1\,[\mathbf{k}^2] + c_2\,[\mathbf{k}]^2\right) + \frac{1}{M^6}\left(d_1\,[\mathbf{k}^3] + d_2\,[\mathbf{k}][\mathbf{k}^2] + d_3\,[\mathbf{k}]^3\right) + \ldots \tag{6.3}$$

We have introduced the dimensionless coefficients, $\{b, c_i, d_i\}$, to control the strength which which a given term influences the action.

6.2 The quasi-Hookean solid

We will make use of the notation

$$\check{\epsilon}(n) = F(n,0,0), \tag{6.4a}$$

$$\check{\rho}(n) = nF(n,0,0), \tag{6.4b}$$

$$\check{p}(n) = n^2 \frac{\partial F}{\partial n}(n,0,0), \tag{6.4c}$$

$$\beta(n) = n^3 \frac{\partial^2 F}{\partial n^2}(n,0,0) + 2n^2 \frac{\partial F}{\partial n}(n,0,0), \tag{6.4d}$$

$$\mu(n) = n \frac{\partial F}{\partial s^2}(n,0,0). \tag{6.4e}$$

The quantities $\check{\rho}, \check{p}, \beta, \mu$ are the unsheared energy density, bulk and rigidity moduli. The value of the elasticity tensor in the state of zero shear strain is

$$\check{E}^{abcd}(n) = (\beta - \tfrac{1}{3}\check{p})\eta^{-1ab}\eta^{-1cd} + 2(\mu + \check{p})(\eta^{-1a(c}\eta^{-1b)d} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}). \tag{6.5}$$

Note that this is the elasticity tensor we computed the sound speeds for just after equation (4.41).

The Lagrangian for the Carter-Quintana solid in the quasi-Hookean limit (which we shall refer to as a "quasi-Hookean solid") is linear in s^2, and independent of l:

$$F_{\text{qHs}} = \check{\epsilon} + \tfrac{\mu(n)}{n} s^2. \tag{6.6}$$

For this quasi-Hookean solid the energy density and pressure tensor are respectively given by

$$\rho = \check{\rho} + \mu s^2, \tag{6.7a}$$

$$p^{ab} = \left\{\check{p} + \left(n\mu' + \tfrac{1}{3}\mu\right) s^2\right\}\gamma^{ab} - 2\mu\left\{\eta^{-1a(c}\eta^{-1b)d} - \tfrac{1}{3}\eta^{-1ab}\eta^{-1cd}\right\} s_{cd}. \tag{6.7b}$$

6.3 Solid $\rightarrow$ fluid $\rightarrow \Lambda$

The theory, as constructed, is the general description for a perfect solid. The particular solid under examination is determined by the dependance of ϵ on its arguments. This is simple to impose, has interesting consequences, and the physics alters in a meaningful way. For example, when

$$\epsilon\left(n, \eta^{a}{}_{b}\right) \longrightarrow \epsilon(n), \tag{6.8}$$

the anisotropic stress tensor (3.32b) vanishes, and one is left with a theory which describes a perfect fluid. Furthermore, when

$$\epsilon(n) = \frac{\epsilon_0}{n}, \tag{6.9}$$

where ϵ_0 is a constant, one finds that $\rho = -p$. That is, the theory is that of a cosmological constant (the Lagrangian is just $\mathcal{L} = \epsilon_0$, a constant).

7 Final remarks

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