The theory of relativistic solids and their applications 1: the theory

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Abstract

This paper is intended to be expository in nature. The idea is to create a bridge between techniques and developments in non-linear elasticity theory, and modified gravity theories.

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Contents

Part I
Review of the theory

1 Introduction

Materials are everywhere. They have been used for thousands of years, modelled for hundreds, and manipulated for tens (making some allowances for inaccuracies in favour of flamboyant language). It is therefore rather surprising that they are rather under-studied, as a class of theories, in particular contexts which have become rather fashionable in recent times.

Modelling a material is a game of asking rather physically simple questions. For example, in what manner does a substance respond under a given stimulus? What factors about the substance are important in order for useful dynamical information to be extracted? Especially information about how the substance imparts energy and momentum into surrounding materials. These concepts and associated problems are best explained by analogy.

First, suppose one wanted to construct a description of water flowing through a pipe. Given that one knows that water is constituted from "particulate" molecules, one could construct a particle description. This would be built from knowledge – or a guessed understanding – of how water molecules interact with their neighbours and surfaces inside the pipe. With the best will and all available computing power, such a description would fail to describe almost all systems of physical interest. Instead, one moves to a coarse-grained fluids description where one attempts to describe the collective behavior of the particles on a "large scale".

Secondly, suppose one wanted to construct a description of how an object, such as a table, responds to being kicked. The impact of the externally applied kick is transmitted via inter-molecular bonds within the object to release some kind of energy in the form of motion, sound, or heat. Ones intuition has been built up to such an extent that the precise details of the inter-molecular bonds are irrelevant if one wanted to understand the large-scale response of the object to the impact. However, ones intuition is well aware of the fact that if the object were made of different materials (which, on a fundamental level, means that the bonds between the molecules which constitute the object are different in nature), then the object could respond very differently. The amount of kicking required to move the table depends on what the table is made of (bendy, versus stiff matierials). And so, one builds a working picture of the object: it is vital to have some understanding of some of the underlying micro-physical make-up of the object when building up an understanding of the macro-physical response of the object to macro-physical impacts.

It is useful for our purposes to imagine that the theory of materials comes in two branches. The first is the theory of *continuous media*: these are supposed to be space filling substances and one should imagine being immersed within the medium. Relativistic realizations of such media were the subject of [1–3], but under the presumption that the medium was adequently described within the framework of perturbation theory (admittedly, this was a perfectly reasonable restriction for the applications those studies had in mind). The second is the theory of *solitons*: these are almost completely opposite to continuous media, in the sense that they are localised configurations and are highly non-linear deformations of the appropriate fields. In both descriptions of materials (i.e., continuous and localised) the idea of a map from the material manifold into space-time is relied upon. It appears that the important distinction between how the two types of theories are formulated is what information about the material manifold and its map is used to construct the theory. This qualifies as an important piece of intuition which we advocate using.

The development of the theory of solids has a long history, dating back to Cauchy and Poisson's works in the 1820's. As with a lot of theoretical physics, Landau and Lifshitz have a classic text [33] on the subject. These early works, however, focussed on the linear non-relativistic category of theories in which gravitational perturbations are neglected, and the material under consideration has negligible pressure and is only minutely deformed from its relaxed state. Rather a lot of effort was put into obtaining the relativistic non-linear (and linear) theory in the 1960's, with a majority of problems being solved in Carter and Quintana's formulation from the 1970's [4, 5]. The motivation for most of these works from the 60's and 70's on relativistic elasticity came from a desire to understand the emission and interaction of gravitational waves from neutron stars [17]. There has also been, and indeed is ongoing, focus on using elastic solids as a generalisation of a perfect fluid, in regards to the modelling on the internal constituents of relativistic stars [6–12], being able to provide a generalisation of the Tolman-Oppenheimer-Volkov equations.

Scalar field models of dark energy and modified gravity are prevalent in modern cosmology and it is our contention that in an important sense these are equivalent to constructing a particulate description of water, or a molecular picture of a table. One of the aims of this paper is to offer a change in philosophy in building models.

The change in philosophy of the construction of dark sector theories which we are advocating is outlined as follows, and is illustrated in Figure 1. As a first pass at allowing gravitational dynamics to be sourced by a non-constant contribution, most models are constructed using perfect fluid or scalar-field-theory modifications. Perfect fluids are physically natural models since they actually describe a material which is supposed to have some effects. This is in contrast to scalar-field models which are a simple mathematical tool for providing non-trivial dynamics. In this sense, a perfect solid is a physically natural generalisation of a perfect fluid. However, the required mathematical tools are significantly more complicated than those for scalar-field theories. Fortunately, these tools have been developed for almost as long a time as tensor calculus has existed: these tools have been applied

In some sense the idea of describing a medium is similar to the idea of using multiple scalar fields to build dark energy models: the medium description is constructed with a set of three scalar fields. Except now one obtains a concrete interpretation of what the scalar fields *are*. Knowing what the fields are significantly enhances physical insight, and guides the choice of functions or parameters used to parameterize available freedom in the theory.

The aims of this paper include the elucidation of the construction of non-linear material models, and showing how ideas, schematic scenarios, and model building techniques, can be imported into the language of cosmology.

- In Bucher and Spergel, [1], the linearized theory is constructed in detail.

- See Karlovini and [13] [14] [15] and [16]

- The pull-back idea is very similar to the restoration of non-linear diffeomorphism invariance utilised by massive gravity theories [19].

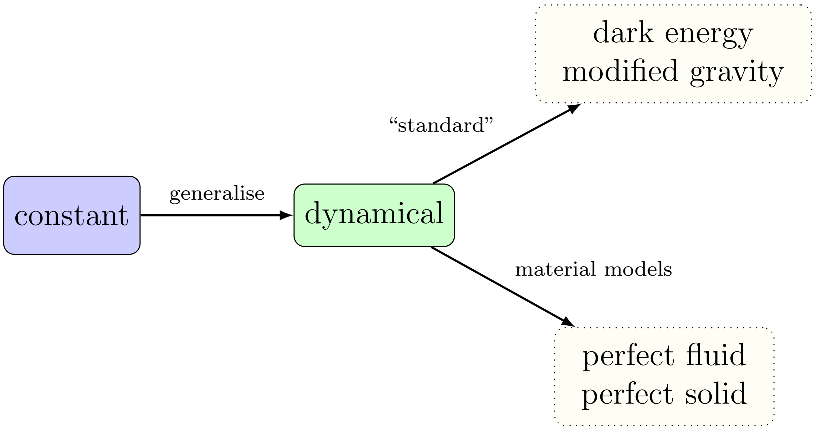

Figure 1: Schematic illustration of the philosophy behind model-class choices. The simplest generalisation of a constant modification to the gravitational field equations is to include a dynamical component. At that point one can either go down the route labelled here as "standard", in which one can include dark energy and/or modified gravity contributions into the field equations. Alternatively, one can include material models, where in this illustration we have shown perfect fluid and solids as examples. These routes are not mutually exclusive, since, e.g., simple scalar field models of dark energy can be thought of as perfect fluids.

- See effective field theory of perfect fluids, [20]
- see [21] [22]
- Solids in inflation context [23–25]
- see [26] for exact analytic solutions for perturbed single-component cosmology
- [11], [27]
- Carter and elastic theory for neutron stars [**?**]
- solids in coset construction [**? ?**]
- Relativistic hydrodynamics lecture notes [**?**]
- EFT of broken spatial diffs [**?**]; massive gravity and elasticity [32]

1.1 Deformation theory and cosmology

The current state of affairs in cosmology is that the Universe is accelerating in its expansion. A huge business has boomed with the expressed intention of explaining this observation; this has yielded a huge literature of both phenomenologically and theoretically motivated modifications to gravity. A quick and highly inadequate summary is that the prediction obtained from General Relativity (GR) for how the Universe should look doesn't match up with observations of how the Universe does look (unless, for example, some form of exotic matter is included).

One popular way of understanding how to tackle this mis-match is to write the gravitational field equations that actually describes the Universe as

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + U_{\mu\nu} \right) . \tag{1.1}$$

Here, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is Einstein's tensor, and $T_{\mu\nu}$ is the energy-momentum tensor of all *known* matter sources (such as radiation, baryonic, and dark matter). The tensor $U_{\mu\nu}$ is the dark energy-momentum tensor, and contains all the deviations or deformations (to begin using the terminology we aim to develop) of the field equations which describe the actual Universe away from the GR (+ standard matter content) predictions.

The modern cosmology community is busy with developing candidate theories which could provide the components of the tensor $U_{\mu\nu}$. The focus is also on working out the observational consequences of their given form of the tensor by using observational probes such as the distances to supernovae, the Cosmic Microwave Background radiation, and the effects of the evolution of gravitational perturbations on the propagation of photons.

These candidate theories usually fall into the class of "dark energy" or "modified gravity", and are generally constructed in order to satisfy mathematical principles. Whilst this is an entirely sensible philosophy (the principle of building models whilst asking for symmetries to hold has served theoretical physics very well), we would like to suggest a different approach, or at least, a different philosophy for attacking the problem. Explaining this approach, and showing how it can be used, is the subject of this paper.

In the theory of deformations (in particular, we have in mind theories of relativistic elasticity) one imagines two states of a material. The first state is a relaxed configuration, and the second is a strained configuration. The deformation which was imparted on the material to take it from being relaxed to being strained isn't necessarily small (if it was small, one would speak about linear elasticity theory). The theory of deformations prescribes a tool-kit for writing down terms in the field equations which are allowed, given classes or categories of deformation. For example, if the deformation is performed "on" some perfect fluid or perfect solid, then it is known that the quantities $U_{\mu\nu}$ takes on the form

$$U^{\{\text{fluid}\}}_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}, \qquad U^{\{\text{solid}\}}_{\mu\nu} = \rho u_\mu u_\nu + P_{\mu\nu}. \tag{1.2}$$

The energy-momentum tensors written above *become* those for a fluid or solid when some extra theoretical structure is used. Namely, an *equation of state*. For readers who are used to the literature in modern cosmology, this phrase is often used to describe the link between the dark energy pressure P and density ρ, via an equation of the form $P(t) = w(t)\rho(t)$. In the context of material models, an equation of state is the Lagrangian density.

When one constructs "conventional" models of dark energy or modified gravity, one has a some freedom to choose various types of quantities: these are, e.g., functional forms of the potential, or the kinetic terms which appear in the Lagrangian density. This may seem like an obvious point, but the choice of a restriction on a theory can have implications for (a) its applicability, and (b) its physical naturalness/interpretation. This is a particularly pertinent point, and so therefore we want to take inspiration from the extremely well developed field of the *mechanics of solids* and use that as a model building guide.

1.2 Fluids and solids

The distinction between a *fluid* and a *solid* isn't one of the best explained concepts in the literature. Fluids are commonly used as a description for the "source term" in a gravitational theory, since they are both mathematically simple and physically intuitive. But fluids are only a sub-class of a more general description for "material content". A more general description of material is that of a solid; obviously, we won't go so far as to say *the* general material description. Below we will outline some of the salient pieces to the construction of a material model: these are the take-home points of the construction. More complete motivations, explanations, and proofs are given in the subsequent sections of this paper.

In the descriptions of both solids and fluids one has a notion of a *material manifold* endowed with a *material metric* k_{AB}. This is a symmetric rank-2 tensor living on a 3D manifold, whose determinant is related to the particle number density, $n = \sqrt{\det k_{AB}}$. A convenient decomposition of this metric is $k_{AB} = n^{2/3}\mathfrak{U}_{AB}$. With this decomposition of k_{AB}, the conformal metric $\mathfrak{U}_{AB}$ is uni-modular, i.e., it has unit determinant. Note that all indices are of "capital latin" type: this indicates that they correspond to quantities defined on the material manifold. Such quantities can be "pulled-back" to the space-time manifold by means of a map. For example, the components of the uni-modular tensor in the space-time manifold are constructed from the set of three scalars ϕ^A and k_{AB} via

$$\mathfrak{U}_{\mu\nu} = n^{-2/3} k_{AB} \partial_\mu \phi^A \partial_\nu \phi^B. \tag{1.3}$$

The ϕ^A are the coordinates on the material manifold: physically they specify the locations of the particles.

Schematically put, the Lagrangian density for a general material can be written as a general function of the pulled-back components of the material metric, $\mathcal{L} = \mathcal{L}(k^\mu{}_\nu)$. The action for an isotropic material is constructed by integrating the Lagrangian density whose arguments are all possible scalar quantities formed from the available structures in space-time which are the pulled-back counterparts of structures on the material manifold. The action for both a fluid and a solid is of the general form

$$S = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}\left(n, [\mathfrak{U}], [\mathfrak{U}^2]\right). \tag{1.4}$$

The square-braces in (1.4) denote traces of the mixed components of the uni-modular tensor, $\mathfrak{U}^\mu{}_\nu = \gamma^{\mu\alpha}\mathfrak{U}_{\alpha\nu}$.

It is useful to split up the Lagrangian density as $\mathcal{L} = n\epsilon$, where ϵ is the energy per particle and n retains its interpretation as the particle number density. In the cases of fluids or solids, ϵ is a function with the following dependencies:

$$\epsilon_{\text{fluid}} = \epsilon_{\text{fluid}}(n), \qquad \epsilon_{\text{solid}} = \epsilon_{\text{solid}}(n, \mathfrak{U}^\mu{}_\nu). \tag{1.5}$$

A given expression of the energy-per-particle constitutes an equation of state for the material.

This makes the distinction between solids and fluids explicit: it is the dependence of the equation of state on the uni-modular tensor $\mathfrak{U}^\mu{}_\nu$ which makes the description that of a solid rather than of a fluid. Later on we will see that the physical

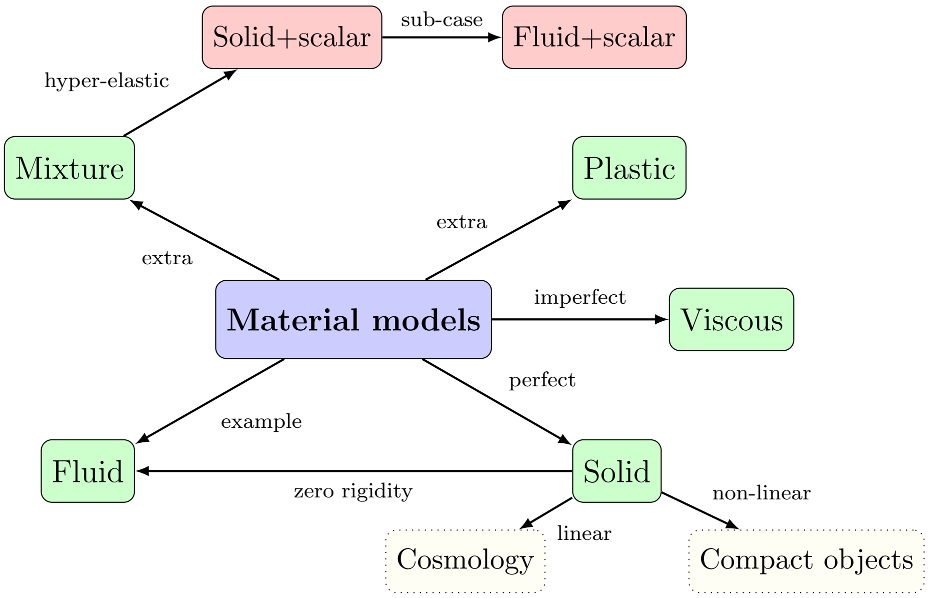

Figure 2: Illustration containing some of the simplest material models. This picture coarsely shows how some of the common classes of materials are related. For example, we see that a fluid is a perfect solid with zero rigidity. We have also shown that the linear theory of solids has been applied to cosmology, and the non-linear theory to compact objects (such as neutron stars). There are also imperfect materials, such as viscous solids and plastics. In addition, there are models which dynamically mix the degrees of freedom of a solid with those of a scalar; they can be categorised as "hyper-elastic", in the sense of Carter.

consequence of this dependence is that the substance is able to support anisotropic stress, whereas fluids can't. This anisotropic stress is the manifestation of *rigidity*. It is worth noting that a fluid is a highly symmetric solid, and a pressureless fluid has $\epsilon_{\text{fluid}}(n) = \bar{\epsilon}_0$, a constant.

Another concept which is used is that of a *perfect fluid*. This is supposed to be a substance whose energy-momentum tensor can be put into the form $T_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}$, in which ρ and P are the fluid's energy density and pressure respectively, u^μ is the velocity of the fluid and $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the orthogonal projection operator. If the energy-momentum tensor for a *fluid* is not of this form (for example, if there is anisotropic stress or heat flux) then the *fluid* is said to be *imperfect*. We are deliberately being careful about only using the term *fluid*: a solid can be categorised in a similar sense, but a perfect solid manifestly has an anisotropic part to the energy-momentum tensor (this distinguishes a solid from a fluid).

Solids and fluids are only two examples of a more general category of "material models". In Figure 2 we name a few other classes of materials: such as viscous and plastic ones. We have also pointed out how some of the examples are related, and the current applications of some of the types.

In some sense the goal of this review is to obtain an understanding of the theory of a relativistic solid: useful geometric structures on the manifold of particle locations, the action, and energy-momentum tensor. It is rather involved, but is worthwhile since expressions and formulae obtain physical meaning.

Symbol	Meaning
$\pounds_X$	Lie derivative operator along the vector X^μ
$(\mathcal{S}, g_{\mu\nu})$	Space-time manifold and metric
$(\mathcal{M}, k_{\mu\nu})$	Material manifold and metric
u_μ	Time-like unit-vector; $u^\mu u_\mu = -1$
$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$	Orthogonal projector; $u^\mu \gamma_{\mu\nu} = 0$
n	Particle number density; $n^2 = \det k_{AB}$
$\mathfrak{U}^\mu{}_\nu = n^{-2/3} k^\mu{}_\nu$	Uni-modular tensor; $\det \mathfrak{U} = 1$
ϵ	Equation of state (energy-per-particle)

Table 1: Summary of commonly used symbols, and some of their useful properties.

1.3 Conventions and outline

We use this subsection to collect our notation, and provide a layout of this paper. In Table 1 we present and summarise a set of commonly occuring quantities, tensors, and operators.

We use greek letters, $\alpha, \beta, \mu, \ldots$ to denote space-time indices, and upper-case latin letters, $A, B, C, \ldots$ to denote indices on the material manifold. The space-time metric is decomposed as

$$g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu, \tag{1.6}$$

in which u_μ and $\gamma_{\mu\nu}$ are the 4-velocity and spatial metric, satisfying

$$u^\mu u_\mu = -1, \qquad u^\mu \gamma_{\mu\nu} = 0. \tag{1.7}$$

We use the orthogonally projected derivative

$$\overline{\nabla}_\mu A^{\alpha\cdots}{}_{\lambda\cdots} = \gamma^\nu{}_\mu \gamma^\alpha{}_\beta \cdots \gamma^\kappa{}_\lambda \cdots \nabla_\nu A^{\beta\cdots}{}_{\kappa\cdots} \tag{1.8}$$

and the expansion (extrinsic curvature) tensor

$$\Theta_{\mu\nu} = \overline{\nabla}_{(\mu} u_{\nu)}. \tag{1.9}$$

It immediately follows that $\overline{\nabla}_\mu$ is the connection compatible with $\gamma_{\mu\nu}$, since

$$\overline{\nabla}_\mu \gamma_{\alpha\beta} = 0. \tag{1.10}$$

We will use angular braces to denote the symmetric, trace-free part of a tensor:

$$A_{\langle\mu\nu\rangle} = A_{(\mu\nu)} - \tfrac{1}{3} A^\alpha{}_\alpha \gamma_{\mu\nu}. \tag{1.11}$$

In section 2 we describe how to model materials using a pull-back formalism. This section introduces a substantial portion of the notation and termninology used in the subbsequent parts of the paper. In section 3 we outline how to quantify the state of a material, in terms of its action and energy-momentum tensor.

2 Describing non-linear materials

In this section we will take some time to build a description of a medium. We will introduce the notion of a material manifold, and geometric structures on the material manifold: coordinates, metric, connection, and volume form. There will be an important step where we relate structures in the material manifold to structures in space-time. We will want to obtain fields and energies in space-time due to structures in the material manifold. There will be some instances where we "mix" material space and space-time indices; this is unavoidable in the course of exposing some interesting part of the formalism. That said, all final results (equations of motion etc) will be expressed solely in terms of space-time indicies.

2.1 The material manifold, particle number density, and map

One imagines that there is a continuous distribution of particles in the space-time manifold, $\mathcal{S}$. These particles carve out world-lines. In order to specify the location of the particle, one can attach three coordinates to a given world-line. The set of these three-coordinates forms the *material manifold*, which we denote by $\mathcal{M}$. We assume that the 3D material manifold $\mathcal{M}$ is endowed with a particle density form, denoted by $n_{ABC} = n_{[ABC]}$. The integral of n_{ABC} over some region in $\mathcal{M}$ tells us about the number of particles of the medium that reside in that region. We will also assume that there is an associated metric on the material manifold, which we call k_{AB}, but we will discuss it later on.

The points of $\mathcal{M}$ are particles of the medium, and they do not move: the dynamics in space-time comes from the maps from the material manifold to space-time, not the motion of the particles in material space. This is enforced by a condition we explain below.

Now suppose that $\mathcal{S}'$ is the submanifold of the full space-time manifold $\mathcal{S}$ that the material passes through. Then invoke a map ψ which takes a location in space-time and points at a location in the material manifold;

$$\psi : \mathcal{S}' \longrightarrow \mathcal{M}. \tag{2.1}$$

This is illustrated in Figure 3. For all points p in $\mathcal{M}' = \psi(\mathcal{S}')$, the inverse map at that point, $\psi^{-1}(\mathrm{p})$, is a single time-like curve in $\mathcal{S}'$: these are the flow-lines of the particles. This construction is the analogue of allowing a scalar field, ϕ say, to pervade the Universe: for a real scalar field ϕ, rather than (2.1) one has $\phi : \mathcal{S} \to \mathbb{R}$.

Let ϕ^A be coordinates in material space. Then their gradients with respect to the space-time coordinates x^a can be computed

$$\psi^A{}_\mu \equiv \frac{\partial \phi^A}{\partial x^\mu} = \phi^A{}_{,\mu} = \partial_\mu \phi^A; \tag{2.2}$$

in which we have given the definition of the $\psi^A{}_\mu$ and a list of useful notational alternatives. The $\psi^A{}_\mu$ are the components of the configuration gradient. These would be the components of the Jacobian associated with a coordinate transformation if the dimension of the material manifold were to be the same as the dimension of the space-time manifold.

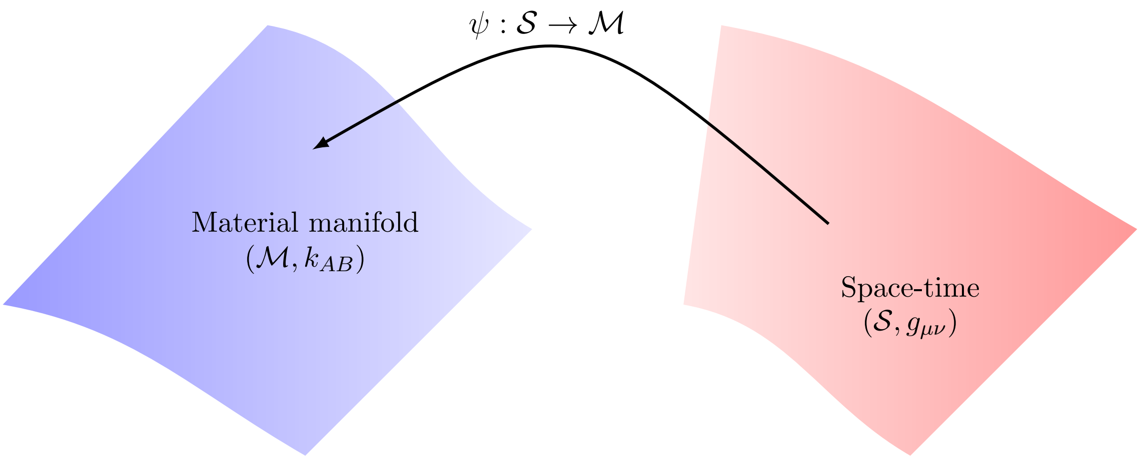

Figure 3: Schematic depiction of the map ψ that associates a point in the material space $\mathcal{M}$ with a point in space-time $\mathcal{S}$. We have also shown which metric is associated with which manifold (and the associated labelling of the indices).

The enforcement of the material coordinates being static with respect to changes in coordinate time, or in orther words that a given world-line corresponds to a given particle, is that the time-like projection of the configuration gradients must vanish

$$u^\mu \psi^A{}_\mu = 0. \tag{2.3}$$

This is equivalent to setting the Lie derivative of the ϕ^A in the time-like direction to zero:

$$\pounds_u \phi^A = 0. \tag{2.4}$$

It can be useful to imagine that the whole world-line of a particle in space-time is associated with one point on $\mathcal{M}$, in which case the value of the coordinate on $\mathcal{M}$ cannot be dependent upon the "time" coordinate. Thus, (2.3) must hold. If the condition (2.3) is relaxed one ends up describing "hyper-elastic", rather than "elastic" theories, and bears resemblance to a theory which mixes a scalar with a solid. That case has been explained in [18], and will not be further explored in this review.

One can conceive of scalars, vectors, forms, and tensors on the material manifold. The material metric and particle density form are examples, and there will be others which we will introduce later on. Collectively, we call such quantities "material tensors", and they have components whose indices are denoted with captial latin letters. Tensors in the material and space-time manifolds are related to each other using technology from differential geometry of pull-backs and push-forwards, as summarised below.

- $\psi^\star$ is the pull-back of a covariant tensor from $\mathcal{M}'$ to $\mathcal{S}'$ and is denoted to act on a material tensor as

  $$N_{\mu\nu\cdots\lambda} = \psi^\star N_{AB\cdots Z}. \tag{2.5a}$$

  In "coordinates" notation the pull-back is

  $$N_{\mu\nu\cdots\lambda} = \psi^A{}_\mu \psi^B{}_\nu \cdots \psi^Z{}_\lambda N_{AB\cdots Z}, \tag{2.5b}$$

  where $\psi^A{}_\mu$ are the components of the configuration gradient (2.2).

- $\psi_\star$ denotes the push-forward of a contravariant tensor from $\mathcal{S}'$ to $\mathcal{M}'$, and is denoted to act on a space-time tensor as

$$M^{AB\cdots Z} = \psi_\star M^{\mu\nu\cdots\lambda}, \tag{2.6a}$$

  and in coordinates it reads

$$M^{AB\cdots Z} = \psi^A{}_\mu \psi^B{}_\nu \cdots \psi^Z{}_\lambda M^{\mu\nu\cdots\lambda}. \tag{2.6b}$$

Since we are assuming the existence of the pull-back we only need ever work with space-time tensors. This brings with it a conceptual simplicity: we only work with space-time indices, and a special subset of the space-time tensors will correspond to material tensors, and will also obtain a corresponding interpretation. That said, it is sometimes helpful to perform intermediate calculations entirely within the material manifold.

The most important corollary of (2.3) is that any tensor on space-time which corresponds to the pull-back of a tensor on the material space will automatically be orthogonal. That is, for the schematic example (2.5b),

$$u^\mu N_{\mu\nu\cdots\lambda} = u^\nu N_{\mu\nu\cdots\lambda} = \cdots = u^\lambda N_{\mu\nu\cdots\lambda} = 0. \tag{2.7}$$

This property can be extremely useful. Another related corollary is that the orthogonal part of the metric, $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, can be used to raise and lower indices of the space-time tensor counterpart of a pulled-back material tensor, e.g.,

$$g^{\mu\alpha} N_{\mu\nu\cdots\lambda} = g^{\mu\alpha} N_{\mu\nu\cdots\lambda} = N^\alpha{}_{\nu\cdots\lambda}. \tag{2.8}$$

Now consider the expression $u^\mu \nabla_\alpha N_{\mu\nu}$, where $N_{\mu\nu} = \psi^\star N_{AB}$ is the pull-back of a material tensor. After performing a simple manipulation one finds

$$u^\mu \nabla_\alpha N_{\mu\nu} = -K^\mu{}_\alpha N_{\mu\nu}. \tag{2.9}$$

That is, this particular contraction of the time-like unit vector u^μ with the space-time covariant derivative of N_{ab} is given by the un-differentiated values of $N_{\mu\nu}$ and the extrinsic curvature tensor $K_{\mu\nu}$. Furthermore, it follows by the orthogonality of $k_{\mu\nu}$ that $u^\mu u^\alpha \nabla_\alpha N_{\mu\nu} = 0$.

The integral of the particle number density form n_{ABC} over some volume in the material manifold $\mathcal{M}$ is the number of particles in that volume (by definition). The pull-back of the particle volume-form to space-time is

$$n_{\mu\nu\lambda} = \psi^\star n_{ABC}. \tag{2.10}$$

Note that (2.3) means that $n_{\mu\nu\alpha}$ is an orthogonal space-time field. Using the space-time volume form $\epsilon_{\mu\nu\alpha\beta}$, the dual in space-time of $n_{\mu\nu\alpha}$ yields the vector

$$n^\mu = \frac{1}{3!}\epsilon^{\mu\nu\alpha\beta} n_{\nu\alpha\beta}. \tag{2.11}$$

This vector n^μ carries the interpretation of being the particle current, and is manifestly conserved,

$$\nabla_\mu n^\mu = 0. \tag{2.12}$$

This conservation follows since $n_{\mu\nu\alpha}$ is a closed 3-form due to n_{ABC} being a closed 3-form on material space (an n-form in n-dimensional space is closed). What this also means is that to break (2.12) and have $\nabla_\mu n^\mu \neq 0$ one requires n^μ not to be related to the volume form on material space.

It follows by orthogonality of $n_{\mu\nu\alpha}$ that the particle current (2.11) is time-like

$$n^\mu = n u^\mu, \tag{2.13}$$

where the particle number density n is given by

$$n = \sqrt{-n^\mu n_\mu}. \tag{2.14}$$

Putting together some of the above relations, one can obtain the useful expressions,

$$\epsilon_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\beta} u^\beta, \qquad n_{\mu\nu\alpha} = n\epsilon_{\mu\nu\alpha}, \tag{2.15a}$$

as well as realising that the number density n can be constructed from the particle 3-form via

$$n^2 = \frac{1}{3!} n^{\mu\nu\alpha} n_{\mu\nu\alpha}. \tag{2.15b}$$

Note that from the conservation equation for n^μ, (2.12), and (2.13), one obtains an evolution equation for the particle number density,

$$\dot{n} = -n\Theta, \tag{2.16}$$

where $\Theta = \Theta^\mu{}_\mu$ is the trace of the extrinsic curvature tensor.

Another way of expressing the duality relation (2.11) is found after combining (2.13) and (2.15a) to give

$$n_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\beta} n^\beta. \tag{2.17}$$

This expression helps to highlight the connection to Kalb-Ramond fields. A Kalb-Ramond field is a 2-index object that transforms as a 2-form; its components satisfy $B_{\mu\nu} = B_{[\mu\nu]}$. The 3-form field strength $F_{\mu\nu\alpha}$ corresponding to $B_{\mu\nu}$ is an exact form constructed by taking the "derivative" $F = \mathrm{d}B$, which works out in component form in this case as $F_{\mu\nu\alpha} = 3\nabla_{[\mu} B_{\nu\alpha]}$. Since $F_{\mu\nu\alpha}$ is an exact form, it is therefore a closed form[1]: the expression of automatic closure is given by $\nabla_{[\mu} F_{\nu\alpha\beta]} = 0$. Related to the 3-form field strength $F_{\mu\nu\alpha}$ is its dual F^μ, which is constructed via $F_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\beta} F^\beta$. By virtue of the automatic closure it follows that F^μ is conserved, $\nabla_\mu F^\mu = 0$.

It should therefore be clear that the particle number density current n^μ, which is the dual of the number density form $n_{\mu\nu\alpha}$, can be interpreted as the field strength tensor of some field of Kalb-Ramond type. ***It is worth finding [34]***

[1]A *closed* form C, say, is a form for whom $\mathrm{d}C = 0$. Let A be a p-form, then $F = \mathrm{d}A$ is an *exact* $(p+1)$-form. Since $\mathrm{d}^2 = 0$, it follows that $\mathrm{d}F = 0$; in words this statement is: *an exact form is a closed form.*

2.2 Material metric

We invoke the existence of a metric k_{AB} on the material manifold $\mathcal{M}$ whose volume form is the particle density form n_{ABC} introduced in Section 2.1. This metric will enable us to introduce a Levi-Civita connection in the material manifold, which can be pulled-back to space-time to aid the evaluation of derivatives of material tensors. Before we explain this fairly complicated construction we shall elucidate some other useful structures on the material manifold.

Indices on material tensors can be contracted with the indices of other material tensors. Equivalently, indices on space-time tensors can also be contracted with those of other space-time tensors (a space-time scalar can be formed if contraction leaves no spare indices). Importantly, space-time tensors can be the pulled-back conterpart of a material tensor, as in the discussion in the previous section. As an example, consider an arbitrary material tensor $A_{ABC\cdots}$ which is "pulled-back" to give a space-time tensor $A_{\mu\nu\alpha\cdots}$ according to the usual prescription $A_{\mu\nu\alpha\cdots} = \psi^{\star} A_{ABC\cdots}$. Then, after contracting some indices with the space-time metric,

$$B_{\alpha\cdots} = g^{\mu\nu} A_{\mu\nu\alpha\cdots} = g^{\mu\nu} \psi^{\star} A_{ABC\cdots} \tag{2.18}$$

is a legitimate space-time tensor. One can also contract indices of material tensors on the material manifold $\mathcal{M}$, with the push-forward of space-time tensors. As an example, consider the push-forward of the inverse space-time metric tensor

$$g^{AB} = \psi_{\star} g^{\mu\nu} \tag{2.19a}$$

being contracted with an arbitrary material tensor,

$$g^{\mu\nu} C_{ABC\cdots} = \psi_{\star} g^{\mu\nu} C_{ABC\cdots}. \tag{2.19b}$$

From the orthogonality of the material mappings it follows that

$$g^{AB} = \psi_{\star} \gamma^{\mu\nu}, \tag{2.19c}$$

where we remind that $\gamma^{\mu\nu}$ is the orthogonal part of the space-time metric as defined in (1.6).

Note that g^{AB} is the push-forward of the space-time metric to the material manifold, but it does not necessarily coincide with the material metric k_{AB}. Infact, quantifying its non-coincidence is extremely important in quantifying the state of a material. With this in mind, we define a material tensor $\mathfrak{U}_{AB}$, which depends on the number density n, such that the push-forward of the space-time metric g^{AB} is exactly the inverse of $\mathfrak{U}_{AB}$ when the material is in its unsheared state. That is, $g^{AC}\mathfrak{U}_{CB} = \delta^{A}{}_{B}$ (the Kronecker-delta) when the energy is at its minimum $\epsilon = \check{\epsilon}(n)$. What this means is that $g^{AB} = \mathfrak{U}^{-1AB}$ in what is henceforth defined as the *unsheared state*. Consequently, the deviation of the actual value of g^{AB} from $\mathfrak{U}^{-1AB}$, which we write as

$$s^{AB} = \tfrac{1}{2}\left(g^{AB} - \mathfrak{U}^{-1AB}\right), \tag{2.20}$$

quantifies the shear of the system.

Writing the volume form of $\mathfrak{U}_{AB}$ as ϵ_{ABC} it follows that

$$n_{ABC} = n\epsilon_{ABC}. \tag{2.21}$$

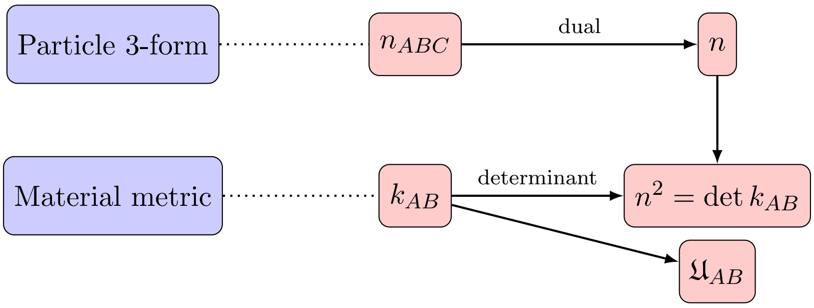

Figure 4: Explanation of the link between geometrical objects in the particle 3-form and material-metric formulations of elasticity theory. In the "particle 3-form" formulation, the only peice of information about the geometrical structure of the material manifold that is actually used is n, the dual of the particle 3-form. In the "material metric" formulation, one posits a metric on the material manifold which has more pieces of information which are used: its determinant, n, and quantites η_{AB} which keep track of the shear-like parts of k_{AB}. The point is that the material metric construction keeps track of more information about the material manifold than the particle 3-form construction. In this way the former is more general than the latter.

Note that $\epsilon_{\mu\nu\alpha} = \psi^\star \epsilon_{ABC}$. The particle density form n_{ABC} is a fixed material space tensor, and is independent of n.

It is now useful and helps physical insight, to define the material tensor k_{AB} as the metric on the material manifold $\mathcal{M}$. k_{AB} is conformal to $\mathfrak{U}_{AB}$, and has the particle density form n_{ABC} as its volume form. Therefore

$$k_{AB} = n^{2/3}\mathfrak{U}_{AB}. \tag{2.22}$$

This tells us that the (square-root of the) determinant of k_{AB} is the particle number density, n:

$$n = \sqrt{\det k_{AB}}\,. \tag{2.23}$$

See Figure 4 for a cartoon of the relationship between the material metric k_{AB} and particle form n_{ABC}.

The pull-back of the material metric $k_{\mu\nu}$ gives a space-time tensor,

$$k_{\mu\nu} = \psi^\star k_{AB}, \tag{2.24}$$

and will play an important role in what follows. Specifically, using (2.5b) the pull-back (2.24) reads

$$k_{\mu\nu} = \psi^A{}_\mu \psi^B{}_\nu k_{AB}. \tag{2.25}$$

The corollary of (2.3) which we keep coming back to is that $k_{\mu\nu}$ is an orthogonal space-time field

$$u^\mu k_{\mu\nu} = 0. \tag{2.26}$$

We will frequently use the (space-time) tensor with mixed indices; to concrete our notation, space-time indices are raised with the space-time metric,

$$k^\mu{}_\nu = g^{\mu\alpha} k_{\alpha\nu}. \tag{2.27}$$

This mixed space-time tensor is also orthogonal,

$$u^\nu k^\mu{}_\nu = 0. \tag{2.28}$$

A consequence of (2.28) is that the indices on $k_{\mu\nu}$ can be raised and lowered using the orthogonal space-time metric,

$$k^\mu{}_\nu = \gamma^{\mu\alpha} k_{\alpha\nu}. \tag{2.29}$$

From (2.29) it follows that

$$\frac{\partial k^\mu{}_\nu}{\partial g^{\alpha\beta}} = \delta^\mu{}_{(\alpha} k_{\beta)\nu}. \tag{2.30}$$

In a similar fashion, the pull-back of $\mathfrak{U}_{AB}$ gives an orthogonal space-time tensor

$$\mathfrak{U}_{\mu\nu} = \psi^\star \mathfrak{U}_{AB}, \tag{2.31}$$

and we also use the mixed version of the tensor,

$$\mathfrak{U}^\mu{}_\nu = \gamma^{\mu\alpha} \mathfrak{U}_{\alpha\nu}. \tag{2.32}$$

From the pull-back of the relationship (2.22) we obtain

$$k^\mu{}_\nu = n^{2/3} \mathfrak{U}^\mu{}_\nu. \tag{2.33}$$

Since we have set everything up so that n^2 is the determinant of $k_{\mu\nu}$, it follows from (2.33) that $\mathfrak{U}^\mu{}_\nu$ is a uni-modular tensor:

$$\det(\mathfrak{U}^\mu{}_\nu) = 1. \tag{2.34}$$

This property will be useful later on.

We now elucidate some consequences of the n-dependence of k_{AB}. In what follows it will be convenient to denote differentiation with respect to n with a prime. Using (2.22) to compute k'_{AB} yields

$$n\mathfrak{U}'_{AB} = -\tfrac{2}{3}\mathfrak{U}_{AB} + \tau_{AB}, \tag{2.35}$$

in which

$$\tau_{AB} \equiv n^{1/3} k'_{AB}. \tag{2.36}$$

Since $n'_{ABC} = 0$ (by definition) it follows that $(\det k_{AB})' = 0$, and therefore $k^{-1AB} k'_{AB} = 0$, and hence

$$\mathfrak{U}^{-1AB} \tau_{AB} = 0. \tag{2.37}$$

Thus, we see that τ_{AB} is traceless; it is called the *compressional distortion tensor*, and measures deformations of the medium that *aren't* due to conformal rescalings of the material metric upon varying the particle density. Hence, computing the trace of (2.35) with respect to $\mathfrak{U}^{-1AB}$ yields

$$n\mathfrak{U}^{-1AB} \mathfrak{U}'_{AB} = -2. \tag{2.38}$$

Note that from (2.36) it follows trivially, but more usefully, $k'_{AB} = n^{-1/3}\tau_{AB}$, and so if the material varies only conformally (i.e. is uniformly compressed) k_{AB} is independent of n since $\tau_{AB} = 0$ for these types of deformations.

The push-forward of (2.35) reads

$$n\mathfrak{U}'_{\mu\nu} = -\tfrac{2}{3}\mathfrak{U}_{\mu\nu} + \tau_{ab}. \tag{2.39}$$

And so, in the case where $\mathfrak{U}_{\mu\nu} = \mathfrak{U}_{\mu\nu}(n)$, it is simple to see that

$$[\mathfrak{U}_{\mu\nu}]^{\cdot} = \mathfrak{U}'_{\mu\nu}[n]^{\cdot}, \tag{2.40}$$

where $[X]^{\cdot}$ denotes the material derivative of X. After using (2.16) to replace $[n]^{\cdot} = \dot{n}$ we obtain the evolution equation:

$$[\mathfrak{U}_{\mu\nu}]^{\cdot} = \left(\tfrac{2}{3}\mathfrak{U}_{\mu\nu} - \tau_{\mu\nu}\right)\Theta. \tag{2.41}$$

2.3 Material covariant derivative

It is convenient at this point to introduce the covariant derivative on the material manifold which is compatible with the material metric. Let $\overline{\overline{\nabla}}_A$ be the Levi-Civita connection for k_{AB}; i.e.,

$$\overline{\overline{\nabla}}_C k_{AB} = 0. \tag{2.42}$$

There is a reason for our including two different "accents" above the del-symbol. The pushed-forward version of $\overline{\overline{\nabla}}_A$, denoted as $\overline{\widetilde{\nabla}}_\mu$, is allowed to act on space-time tensors; note that it will be orthogonal, and so is taken to be the orthogonal projection of some space-time derivative $\widetilde{\nabla}_\mu$ according to

$$\overline{\widetilde{\nabla}}_\mu A^{\alpha\cdots}{}_{\beta\ldots} = \gamma^\nu{}_\mu \gamma^\alpha{}_\lambda \cdots \gamma^\kappa{}_\beta \cdots \widetilde{\nabla}_\nu A^{\lambda\cdots}{}_{\kappa\ldots}. \tag{2.43}$$

For any space-time vector Y^μ the difference between any two connections can be written as

$$\left(\overline{\widetilde{\nabla}}_\mu - \overline{\nabla}_\mu\right) Y^\alpha = \mathfrak{D}^\alpha{}_{\mu\nu} Y^\nu \tag{2.44}$$

in which $\mathfrak{D}^\alpha{}_{\mu\nu}$ is the (symmetric) relativistic difference tensor [2] defined as

$$\mathfrak{D}^\alpha{}_{\mu\nu} = \tfrac{1}{2} k^{-1\alpha\beta}\left(2\overline{\nabla}_{(\mu} k_{\nu)\beta} - \overline{\nabla}_\beta k_{\mu\nu}\right), \tag{2.45}$$

where $k^{-1\alpha\beta}$ is defined via

$$k^{-1\mu\beta} k_{\beta\nu} = \gamma^\mu{}_\nu, \tag{2.46}$$

and is orthogonal $u_\mu k^{-1\mu\nu} = 0$. Due to the applications in mind, we actually call $\mathfrak{D}^\alpha{}_{\mu\nu}$ the relativistic elasticity difference tensor.

Using this construction, one finds that $\overline{\widetilde{\nabla}}_\mu$ is the connection which is compatible with $k_{\mu\nu}$,

$$\overline{\widetilde{\nabla}}_\mu k_{\alpha\beta} = 0. \tag{2.47}$$

[2] The conditions required for this definition to hold are

As an example of using this technology, suppose that $B^{\mu\cdots}{}_{\nu\ldots}$ is a tensor function of $g^{\mu\nu}$ and $k_{\mu\nu}$. Then taking its derivative with $\overline{\overline{\nabla}}_\mu$ yields

$$\overline{\overline{\nabla}}_\mu B^{\alpha\cdots}{}_{\beta\ldots} = \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial g^{\lambda\kappa}} \overline{\overline{\nabla}}_\mu g^{\lambda\kappa} + \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial k_{\lambda\kappa}} \overline{\overline{\nabla}}_\mu k_{\lambda\kappa} = \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial g^{\lambda\kappa}} \overline{\overline{\nabla}}_\mu g^{\lambda\kappa}, \tag{2.48}$$

where the second equality holds via (2.47). We can go one step further and realise that

$$\overline{\overline{\nabla}}_\mu B^{\alpha\cdots}{}_{\beta\ldots} = \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial g^{\lambda\kappa}} \left(\overline{\overline{\nabla}}_\mu g^{\lambda\kappa} - \overline{\nabla}_\mu g^{\lambda\kappa} \right) = 2 \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial g^{\lambda\kappa}} \mathfrak{D}^{\lambda\kappa}{}_\mu. \tag{2.49}$$

The second term in braces, $\overline{\nabla}_\mu g^{\lambda\kappa}$, vanishes by (1.10), and the final equality holds by (2.44). Finally, since

$$\overline{\nabla}_\mu B^{\alpha\cdots}{}_{\beta\ldots} = \overline{\overline{\nabla}}_\mu B^{\alpha\cdots}{}_{\beta\ldots} - \left(\overline{\overline{\nabla}}_\mu - \overline{\nabla}_\mu \right) B^{\alpha\cdots}{}_{\beta\ldots}, \tag{2.50}$$

then it follows by repeated application of (2.44) on the last term, that orthogonally projected derivative is

$$\overline{\nabla}_\mu B^{\alpha\cdots}{}_{\beta\ldots} \;=\; 2 \frac{\partial B^{\alpha\cdots}{}_{\beta\ldots}}{\partial g^{\lambda\kappa}} \mathfrak{D}^{\lambda\kappa}{}_\mu - B^{\lambda\cdots}{}_{\beta\ldots} \mathfrak{D}^{\alpha}{}_{\mu\lambda} - \cdots + B^{\alpha\cdots}{}_{\lambda\ldots} \mathfrak{D}^{\lambda}{}_{\mu\beta} + \cdots . \tag{2.51}$$

3 Quantifying the state of the material

Armed with the map and material metric it remains to understand how to quantify the state of the material. This will guided by understanding the effects of the material on space-time, and is acheived by constructing a material action which can be appended to the Einstein-Hilbert action, from which one can derive the energy-momentum tensor which sources the gravitational field equations.

Along the way there are various useful auxiliary quantities, and useful pieces of technology that can be used to help understand what is going on.

Before we continue it is worth noting some useful ways to compute derivatives of functions which depend on quantities which regularly appear in the construction, most notably functions which depend on n or $\mathfrak{U}^\mu{}_\nu$. First of all, the derivative of the number density n with respect to the space-time metric is given by

$$\frac{\partial n}{\partial g^{\mu\nu}} = \frac{1}{2} n \gamma_{\mu\nu}. \tag{3.1}$$

When $Y = Y(k^\mu{}_\nu)$ is any quantity that depends only on the $k^\mu{}_\nu$, then its derivative with respect to the space-time metric is

$$\frac{\partial Y}{\partial g^{\mu\nu}} = k_{\alpha(\mu} \frac{\partial Y}{\partial k^{\nu)}{}_\alpha}. \tag{3.2}$$

For any quantity $Z = Z(n, \mathfrak{U}^\mu{}_\nu)$, and using (2.33) as a decomposition of the degrees of freedom in $k^\mu{}_\nu$, we obtain

$$\frac{\partial Z}{\partial g^{\mu\nu}} = \frac{1}{2} n \gamma_{\mu\nu} \frac{\partial Z}{\partial n} + \mathfrak{U}_{\alpha\langle\mu} \frac{\partial Z}{\partial \mathfrak{U}^{\nu\rangle}{}_\alpha}, \tag{3.3}$$

where the angular brackets denote the symmetric trace-free part of the tensor, as defined in (1.11). For each quantity n, Y, and Z as defined here,

$$u^{\mu}\frac{\partial n}{\partial g^{\mu\nu}} = 0, \qquad u^{\mu}\frac{\partial Y}{\partial g^{\mu\nu}} = 0, \qquad u^{\mu}\frac{\partial Z}{\partial g^{\mu\nu}} = 0. \tag{3.4}$$

3.1 The equation of state and material action

The idea is to quantify the state of the material from a "master function" (to use Carter's terminology). This master function will be the piece of freedom which corresponds to the specification of the type or class of materials under consideration. This is much like the specification of the potential function $V(\phi)$ which controls what types of canonical scalar field theories one is studying.

For a material, the energy density, ρ, plays the role of the master function; in what follows we will refer to ρ as the *equation of state*. On a first pass we write down a material action given by the integral of the equation of state which has, as its sole arguments, the mixed components of the pulled-back material metric:

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\,\rho\left(k^{\mu}{}_{\nu}\right). \tag{3.5}$$

It is convenient to re-express the equation of state in terms of the particle number density n and the energy per particle, ϵ, via

$$\rho = n\epsilon. \tag{3.6}$$

And so, rather than ask for the form of ρ, we ask for the form of ϵ.

This is as far as one can go in generality without asking anything further of the material.

3.2 Variation of the material action and measure-weighted variation

The material action gives everything we need. But to continue we need to understand how it behaves under application of the variational principle; to do this, we will explain some useful technology.

Varying the action (3.5) yields

$$\delta S = \int \mathrm{d}^4x\,\sqrt{-g}\,\Diamond\rho. \tag{3.7}$$

We have used the "diamond derivative" notation to denote measure-weighted variations, defined to act on a quantity Q via

$$\Diamond Q \equiv \frac{1}{\sqrt{-g}}\delta_{\mathrm{L}}\left(\sqrt{-g}\,Q\right), \tag{3.8}$$

in which δ_{L} is the *Lagrangian variation* operator. The role of δ_{L} is to encorporate both intrinsic variations of a field, and variations due to some other process (such as symmetry transformations); we will return to the explicit description of this operator later on. The first measure-weighted variation of this quantity Q is

$$\Diamond Q = \delta_{\mathrm{L}} Q - \tfrac{1}{2} Q g_{\mu\nu} \delta_{\mathrm{L}} g^{\mu\nu}. \tag{3.9}$$

Before we evaluate (3.7) for the material, we want to explain some interesting properties and uses for the first measure-weighted variation.

Let us suppose that Q is a function which depends on a set of scalars χ^A, their space-time derivatives $\partial_\mu \chi^A$, and the metric $g_{\mu\nu}$,

$$Q = Q(\chi^A, \partial_\mu \chi^A, g_{\mu\nu}). \tag{3.10}$$

Then, using (3.9), we find a rather compact form of the measure-weighted variation of the quantity Q:

$$\Diamond Q = \tfrac{1}{2} T_{\mu\nu} \delta_{\mathrm{L}} g^{\mu\nu} - \mathcal{E}_A \delta_{\mathrm{L}} \chi^A + \nabla_\mu \left(\vartheta^\mu{}_A \delta_{\mathrm{L}} \chi^A \right). \tag{3.11}$$

We have not removed any total derivative terms, and we defined

$$\mathcal{E}_A \equiv \nabla_\mu \frac{\partial Q}{\partial \partial_\mu \chi^A} - \frac{\partial Q}{\partial \chi^A}, \tag{3.12a}$$

$$T_{\mu\nu} \equiv 2 \frac{\partial Q}{\partial g^{\mu\nu}} - Q g_{\mu\nu}, \tag{3.12b}$$

$$\vartheta^\mu{}_A \equiv \frac{\partial Q}{\partial \partial_\mu \chi^A}. \tag{3.12c}$$

The ϑ^μ-term in (3.11) only contributes to the boundary and can be made to vanish by choice of boundary conditions: it won't play a role in what follows. Interpretations of the quantities $\mathcal{E}_A$ and $T_{\mu\nu}$ are probably rather obvious, but we will wait for a moment before making concrete statements about what each means.

Now, suppose the variations δ_{L} have two origins: the first is due to diffeomorphisms generated by the vector ξ^a, and the second is intrinsic arbitrary variations (of the type usually considered when using variational principles). Then the variations δ_{L} in (3.11) should be replaced with

$$\delta_{\mathrm{L}} = \delta_{\mathrm{E}} + \pounds_\xi, \tag{3.13a}$$

in which the Lie derivatives of the scalars and metric are

$$\pounds_\xi \chi^A = \xi^\mu \nabla_\mu \chi^A, \qquad \pounds_\xi g^{\mu\nu} = -2 \nabla^{(\mu} \xi^{\nu)}. \tag{3.13b}$$

Hence, putting (3.13) into (3.11) yields

$$\Diamond Q = \tfrac{1}{2} T_{\mu\nu} \delta_{\mathrm{E}} g^{\mu\nu} - \mathcal{E}_A \delta_{\mathrm{E}} \chi^A + \xi^\mu \left(\nabla^\nu T_{\mu\nu} - \mathcal{E}_A \nabla_\mu \chi^A \right) - \nabla_\mu S^\mu, \tag{3.14}$$

in which

$$S^\mu \equiv \xi^\nu \left(T^\mu{}_\nu - \vartheta^\mu{}_A \nabla_\nu \chi^A \right) - \vartheta^\mu{}_A \delta_{\mathrm{E}} \chi^A. \tag{3.15}$$

The final term only contributes to the boundary, and vanishes identically in the absence of the scalars in the scenario where $\xi^\mu T_{\mu\nu} = 0$. This could be the case, for example, if $\xi^\mu \propto u^\mu$ and $T_{\mu\nu} \propto \gamma_{\mu\nu}$ only. We can read off from (3.14) that diffeomorphism invariance is ensured when the coefficient of the diffeomorphism generating field ξ^μ vanishes, namely when

$$\nabla^\nu T_{\mu\nu} = \mathcal{E}_A \nabla_\mu \chi^A. \tag{3.16}$$

We can also read off from (3.14) that the condition for the theory to be stationary under arbitrary variations in the scalars χ^A is that the coefficient of the arbitrary variations $\delta_{\mathrm{E}}\chi^A$ should vanish, i.e.,

$$\mathcal{E}_A = 0. \tag{3.17}$$

It is immediately clear from its definition (3.12a) that the conditions (3.17) are just the Euler-Lagrange equations of motion of the scalars χ^A. By inspecting (3.16) it is manifest that the satisfaction of theequations of motion (3.17) implies conservation of the energy-momentum tensor.

Let us now return to the problem at hand: evaluation of (3.7) for the material medium. At the top of Section 3.1 we stated that the equation of state ρ (i.e. the integrand of the material action) is a function of the pulled-back metric $k^\mu{}_\nu$ alone, (3.5). This means that $\delta_{\mathrm{L}}\rho$ can be written as

$$\delta_{\mathrm{L}}\rho = \frac{\partial \rho}{\partial g^{\mu\nu}} \delta_{\mathrm{L}} g^{\mu\nu}, \tag{3.18}$$

which can be used to obtain

$$\Diamond \rho = \frac{1}{2}\left(-\rho g_{\mu\nu} + 2\frac{\partial \rho}{\partial g^{\mu\nu}}\right) \delta_{\mathrm{L}} g^{\mu\nu}. \tag{3.19}$$

We remind that (3.19) is the integrand of the first variation of the action.

3.3 The energy-momentum tensor

We have in mind that the material constitutes only part of the content of the entire "universe": there is also the possibility of gravitational dynamics (not necessarily limited to those prescribed by General Relativity), and also other matter, fluid, or scalar-field sources. In the case that General Relativity provides the gravitational dynamics, the gravitational field equations are given by

$$g_{\mu\nu} = 8\pi G \sum_{\mathrm{i}} T^{\mathrm{i}}_{\mu\nu}, \tag{3.20}$$

where $T^{\mathrm{i}}_{\mu\nu}$ is the energy-momentum tensor for the i^{th} source. Below we will be concerned with computing the energy-momentum tensor for the material. We will use the symbol $T_{\mu\nu}$ for the material's energy-momentum tensor, but one should keep in mind that it can be added to any additional energy-momentum tensors.

The energy-momentum tensor is derived from varying the material action S_{M} using the usual expression,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}. \tag{3.21}$$

The quantity in braces in (3.19) is precisely the variation required to work out the right-hand-side of (3.21); and so, the energy-momentum tensor is given by

$$T_{\mu\nu} = -\rho g_{\mu\nu} + 2\frac{\partial\rho}{\partial g^{\mu\nu}}. \tag{3.22}$$

We are able to further evaluate this expression, and in particular we can deduce the "types" of contributions to $T_{\mu\nu}$ from knowledge of what ρ is a function of. Since the equation of state only depends on the mixed components of the pulled-back material metric,

$$\rho = \rho\left(k^{\mu}{}_{\nu}\right), \tag{3.23}$$

using (3.2) gives an expression for the final term in (3.22):

$$\frac{\partial\rho}{\partial g^{\mu\nu}} = k_{\alpha(\mu}\frac{\partial\rho}{\partial k^{\nu)}{}_{\alpha}}. \tag{3.24}$$

Since the right-hand-side of this expression is orthogonal by virtue of (2.28), so is the left-hand-side,

$$u^{\mu}\frac{\partial\rho}{\partial g^{\mu\nu}} = 0. \tag{3.25}$$

And so, assuming an equation of state ρ has been given in the form of (3.23), the energy-momentum tensor of the solid (3.22) can be written as

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu} + P_{\mu\nu}, \tag{3.26a}$$

in which the pressure tensor $P_{\mu\nu}$ is given by

$$P_{\mu\nu} = 2\frac{\partial\rho}{\partial g^{\mu\nu}} - \rho\gamma_{\mu\nu}. \tag{3.26b}$$

By virtue of (3.25) the pressure tensor (3.26b) is orthogonal,

$$u^{\mu}P_{\mu\nu} = 0. \tag{3.27}$$

The important thing to note is that there is no heat flux term in $T_{\mu\nu}$:

$$u^{\mu}\gamma^{\nu}{}_{\alpha}T_{\mu\nu} = 0. \tag{3.28}$$

This a consequence of the orthogonality of the mapping between the material manifold and spacetime.

After rewriting the equation of state ρ in terms of an energy per particle, ϵ, via (3.6), the pressure tensor (3.26b) takes on the more compact form

$$P_{\mu\nu} = 2n\frac{\partial\epsilon}{\partial g^{\mu\nu}}. \tag{3.29}$$

When the energy per particle ϵ is written in a (still general) way to only depend on the number density n and the components of the uni-modular tensor $\mathfrak{U}^{\mu}{}_{\nu}$, i.e.,

$$\epsilon = \epsilon(n, \mathfrak{U}^{\mu}{}_{\nu}), \tag{3.30}$$

we can use (3.3) to further evaluate the pressure tensor (3.29), yielding the rather attractive expression

$$P_{\mu\nu} = p\gamma_{\mu\nu} + \pi_{\mu\nu}, \tag{3.31}$$

in which we have identified the pressure scalar p,

$$p = n^2 \frac{\partial \epsilon}{\partial n}, \tag{3.32a}$$

and the (traceless) anisotropic stress tensor

$$\pi_{\mu\nu} = 2n\mathfrak{U}_{\alpha\langle\mu} \frac{\partial \epsilon}{\partial \mathfrak{U}^{\nu\rangle}{}_{\alpha}}. \tag{3.32b}$$

We remind that the energy density is given by

$$\rho = n\epsilon. \tag{3.32c}$$

This highlights that dependence of ϵ on the number density n is linked to isotropic pressure p, and dependence of ϵ on the uni-modular tensor $\mathfrak{U}^{\mu}{}_{\nu}$ is linked to anisotropic stress $\pi_{\mu\nu}$. Note that n is the determinant of the material metric, and $\mathfrak{U}^{\mu}{}_{\nu}$ encodes the "other" invariants.

There is nothing "imperfect" about the construction of the substance so far: there is no dissipation, everything is conserved, and is constructed from a very geometrical point of view. However, the pressure tensor (3.31) has anisotropic stress (3.32b). For a *fluid* this would signal an imperfection, but it is exactly this anisotropic stress which makes the theory that of a *solid*.

4 Equation of motion and propagation of sound

Obtaining the equation of state and energy-momentum tensor is only part of the story. One must also obtain equations of motion: these come from the conservation equation

$$\nabla_{\mu} T^{\mu\nu} = 0. \tag{4.1}$$

If the material is the only source to the gravitational field equations, then (4.1) follows by diffeomorphism invariance, and also by the Bianchi identity. If there are multiple sources to the gravitational field equations, then (4.1) holds if and only if $T^{\mu\nu}$ is interpreted as the sum of the individual energy-momentum tensors (of which the material's EMT can be an additive contribution). Only in the case when these sources are "decoupled" are their individual energy-momentum tensors independently conserved.

We will proceed by first proving this statement, then continue by providing a useful way to write down (4.1) in a rather physically intuitive manner.

4.1 Equations of motion from the action

Without assuming the existence of the pulled-back material metric, one should be convinced that the action will be a function of the metric $g^{\mu\nu}$, a set of scalars ϕ^A, and their derivatives $\partial_\mu\phi^A$ (the scalars can be interpested as representing the particle positions in the material manifold). There may be other material space tensors, but we shall leave that complicating possibility out for now. With these considerations, the action can be written as

$$S_{\mathrm{M}} = \int \mathrm{d}^4x \sqrt{-g}\, \rho\left(g^{\mu\nu}, \phi^A, \partial_\mu\phi^A\right). \tag{4.2}$$

We now use the technology and notation introduced in section 3.2 to apply the variational principle to this action. Under Lagrangian variations δ_{L} in the available fields, $g^{\mu\nu}$ and ϕ^A, the corresponding measure-weighted variation of the action density is

$$\Diamond\rho = \tfrac{1}{2}T_{\mu\nu}\delta_{\mathrm{L}}g^{\mu\nu} - \mathcal{E}_A\delta_{\mathrm{L}}\phi^A, \tag{4.3}$$

after neglecting unimportant total derivatives. We set $T_{\mu\nu}$ to be the energy-momentum tensor, as defined in the usual manner,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}} = 2\frac{\partial\rho}{\partial g^{\mu\nu}} - \rho g_{\mu\nu}, \tag{4.4}$$

and the contribution multiplying the perturbed scalars is

$$\mathcal{E}_A = \nabla_\mu\left(\frac{\partial\rho}{\partial\partial_\mu\phi^A}\right) - \frac{\partial\rho}{\partial\phi^A}. \tag{4.5}$$

The symbol δ_{L} stands for the arbitrary variation operator which acts on the fields in a manner which we defined in (3.13). For the metric $g^{\mu\nu}$ and the set of scalars ϕ^A the relevant expressions for the Lie derivative is given by (3.13b). After integrating by parts and neglecting the total derivative, the measure-weighted variation in the action density (4.3) becomes

$$\Diamond\rho = \tfrac{1}{2}T_{\mu\nu}\delta_{\mathrm{E}}g^{\mu\nu} - \mathcal{E}_A\delta_{\mathrm{E}}\phi^A + \xi^\mu\left(\nabla^\nu T_{\mu\nu} - \mathcal{E}_A\nabla_\mu\phi^A\right). \tag{4.6}$$

This can be used to obtain the functional derivative of the action with respect to the intrinsic variations in the scalars,

$$\frac{\Diamond\rho}{\delta_{\mathrm{E}}\phi^A} = -\mathcal{E}_A. \tag{4.7}$$

The variational principle demands that this expression must vanish, and yields the equation of motion satisfied by the scalars ϕ^A. General covariance requires the action to be invariant under changes in the coordinates. We read off from (4.6) that an arbitrary coordinate transformation can be performed and not affect the material action when the coefficient of ξ^a vanishes:

$$\nabla^\nu T_{\mu\nu} = \mathcal{E}_A\,\partial_\mu\phi^A. \tag{4.8}$$

This links conservation of energy-momentum and the equations of motion of the scalars. Put another way: energy-momentum conservation implies the satisfaction of the equations of motion of the elastic medium. At first glance it seems that the system is overdetermined: there are three scalars ϕ^A which are supposed to have equations of motion, but there are four components to (4.8). This apparent over-determination is resolved by the orthogonality of the map.

The orthogonality of the mapping (2.3) can be written as $u^\mu \partial_\mu \phi^A = 0$, and so the time-like projection of (4.8) is automatically satisfied:

$$u^\mu \nabla^\nu T_{\mu\nu} = 0. \tag{4.9}$$

It then follows that the orthogonal projection of (4.8) implies the vanishing of (4.7):

$$\gamma^\mu{}_\alpha \nabla^\nu T_{\mu\nu} = 0 \qquad \Longleftrightarrow \qquad \mathcal{E}_A = 0. \tag{4.10}$$

Therefore, conservation of energy-momentum guarantees that the scalars ϕ^A satisfy their equation of motion.

4.2 The solid equations

Inspired by the result of the previous section, we now present the conservation equations in a physically intuitive manner. One must constantly keep in mind that satisfaction of the equations of motion is equivalent to energy-momentum conservation.

Using the solid form (3.26a) for $T_{\mu\nu}$ the two independent (i.e., time-like and orthogonal) projections of the conservation equation (4.1) are

$$\dot{\rho} + (\rho\gamma^{\mu\nu} + P^{\mu\nu})\Theta_{\mu\nu} = 0, \tag{4.11a}$$

$$(\rho\gamma^{\mu\nu} + P^{\mu\nu})\dot{u}_\nu + \overline{\nabla}_\nu P^{\mu\nu} = 0. \tag{4.11b}$$

We used the orthogonally projected derivative $\overline{\nabla}_\mu$, as defined in (1.8).

If the energy per particle ϵ is a function only of the components $k^\mu{}_\nu$, then using (3.29) in conjunction with (2.51), the orthogonally projected derivative of the pressure tensor can be written as

$$\overline{\nabla}_\nu P^{\mu\nu} = \left(E^{\mu\nu}{}_{\alpha\beta} - \gamma^\mu{}_\alpha P^\nu{}_\beta\right) \mathfrak{D}^{\alpha\beta}{}_\nu, \tag{4.12}$$

in which we used the elasticity difference tensor $\mathfrak{D}^{\alpha\beta}{}_\mu$ as defined in (2.45), and introduced the relativistic *elasticity tensor*, $E^{\mu\nu}{}_{\alpha\beta}$, defined via

$$E^{\mu\nu}{}_{\alpha\beta} \equiv 2\frac{\partial P^{\mu\nu}}{\partial \gamma^{\alpha\beta}} - P^{\mu\nu}\gamma_{\alpha\beta}. \tag{4.13}$$

It is manifest from this definition that the elasticity tensor has the following symmetries in its indices:

$$E^{\mu\nu\alpha\beta} = E^{(\mu\nu)(\alpha\beta)}. \tag{4.14}$$

Using (3.1) and the assumption of the existence of an equation of state, (3.29), the elasticity tensor can be written as the second derivative of the energy per particle ϵ via

$$E^{\mu\nu\alpha\beta} = 4n\frac{\partial^2\epsilon}{\partial\gamma_{\mu\nu}\partial\gamma_{\alpha\beta}}. \tag{4.15}$$

This final relationship informs us that the elasticity tensor is, in addition to the simple symmetries (4.14), also symmetric under interchange of the first and last pair of indices,

$$E^{\mu\nu\alpha\beta} = E^{\alpha\beta\mu\nu}. \tag{4.16}$$

To re-iterate: the symmetries (4.14) are inherited by the fact that $\gamma_{\mu\nu} = \gamma_{(\mu\nu)}$ and $P_{\mu\nu} = P_{(\mu\nu)}$. The rather more restrictive symmetry (4.16) came from the assumption of the existence of an equation of state.

It will be convenient to use the relativistic *Hadamard elasticity tensor*, $A^{\mu\nu}{}_{\alpha\beta}$, defined via the elasticity tensor via

$$A^{\mu\nu}{}_{\alpha\beta} \equiv E^{\mu\nu}{}_{\alpha\beta} - \gamma^{\mu}{}_{\alpha}p^{\nu}{}_{\beta}. \tag{4.17}$$

Using the Hadamard elasticity tensor, the orthogonally projected derivative of the pressure tensor (4.12) takes on a particularly simple form,

$$\overline{\nabla}_\nu P^{\mu\nu} = A^{\mu\nu}{}_{\alpha\beta}\mathfrak{D}^{\alpha\beta}{}_{\nu}, \tag{4.18}$$

and the orthogonal projection (4.11b) of the conservation equations can be written as

$$(\rho\gamma^{\mu\nu} + P^{\mu\nu})\dot{u}_\nu + A^{\mu\nu\alpha\beta}\mathfrak{D}_{\alpha\nu\beta} = 0. \tag{4.19}$$

A rather convenient form of the equations of motion is given in [4, 31]; in terms of the material derivative the pressure tensor satisfies

$$[P^{\mu\nu}]^{\cdot} = -P^{\mu\nu}\theta - E^{\mu\nu\alpha\beta}\theta_{\alpha\beta}. \tag{4.20}$$

`[need to explain the material derivative somewhere]` In the more usual notation of "time-like" derivatives, the equations of motion for the materials energy density and pressure tensor are given by

$$u^\mu\rho_{;\mu} = -\rho u^{\mu}{}_{;\mu} - P^{\mu\nu}u_{\mu;\nu}, \tag{4.21a}$$

$$u^\alpha P^{\mu\nu}{}_{;\alpha} = 2p^{\alpha(\mu}u^{\nu)}{}_{;\alpha} + 2p^{\alpha(\mu}u^{\nu)}\dot{u}_\alpha - P^{\mu\nu}u^{\alpha}{}_{;\alpha} - E^{\mu\nu\alpha\beta}u_{\alpha;\beta}, \tag{4.21b}$$

where we defined the acceleration vector $\dot{u}_\mu = u^\nu\theta_{\mu\nu}$. `[Are these equivalent to the fluid equations?]`

Specification of the components of the elasticity tensor, which must be subject to the symmetries (4.14) and (4.16), closes the solid equations (4.21).

Symbol	Meaning
$[X]$	Discontinuity of X across the sound wave-front
α	Amplitude of acceleration discontinuity
ι^μ	Space-like polarization vector
ν^μ	Propagation direction vector
$\lambda_\mu = \nu_\mu - v u_\mu$	Normal of the wave-front
$v = u^\mu \lambda_\mu$	Sound speed
$\{\sigma, \kappa^\mu, \tau^{\mu\nu}\}$	Amplitude of discontinuity of the derivative of {density, velocity, pressure tensor} on wave-front

Table 2: Summary of the symbols used to compute the sound speed.

4.3 Speed of sound

The story so far has led to us being able to write down the equations of motion, with all freedom being contained within the components of the elasticity tensor. We will now perform a calculation with these equations, and setup the equations into a particular physical configuration which will enable calculation of the speed of sound of the medium [31]. This calculation requires the introduction of many quantities: they are collected with brief definitions in Table 2.

Sound wavefronts are characteristic hypersurfaces across which the acceleration vector $\dot{u}^\mu$ has a jump discontinuity (the velocity u^μ and the metric remain continuous). Following Carter, we denote discontinuities across the wavefront with square braces; and so we set

$$[\dot{u}^\mu] = \alpha \iota^\mu, \tag{4.22}$$

in which α is the amplitude of the wavefront and ι^μ is the polarization vector satisfying the space-like normalization condition, $\iota^\mu \iota_\mu = 1$. Since the acceleration and velocity vectors are mutually orthogonal, $u_\mu \dot{u}^\mu = 0$, it follows that the polarization vector and the velocity vector are orthogonal $u_\mu \iota^\mu = 0$. The *propagation direction vector* ν^μ is specified with the same orthonormality conditions as the polarization vector, namely $\nu^\mu \nu_\mu = 1$ and $\nu^\mu u_\mu = 0$.

The normal to the characteristic hypersurface is in the direction of the vector λ_μ, defined via

$$\lambda_\mu = \nu_\mu - v u_\mu. \tag{4.23}$$

The scalar

$$v = \lambda^\mu u_\mu \tag{4.24}$$

is the speed of propagation.

The derivatives of the density, velocity, and pressure tensor fields on the characteristic hypersurface are given in terms of quantities $\sigma, \kappa^\mu, \tau^{\mu\nu}$ via

$$[\rho_{;\mu}] = \sigma \lambda_\mu, \tag{4.25a}$$

$$[u^\mu{}_{;\nu}] = \kappa^\mu \lambda_\nu, \tag{4.25b}$$

$$[P^{\mu\nu}{}_{;\alpha}] = \tau^{\mu\nu} \lambda_\alpha. \tag{4.25c}$$

We now show how to determine the values of $\sigma, \kappa^\mu, \tau^{\mu\nu}$ in terms of v, α, and ι^μ. First, contracting (4.25b) with u^ν gives (4.22) on the left-hand-side, and $v\kappa^\mu$ on the right-hand-side, and thus one obtains

$$v\kappa^\mu = \alpha\iota^\mu. \tag{4.26a}$$

Taking the discontinuity of the projections of the conservation equation (4.21a) and (4.21b), and then multiplying by v respectively yields

$$v^2\sigma = -\alpha \left(\rho\iota^\mu\lambda_\mu + P^{\mu\nu}\iota_\mu\lambda_\nu\right), \tag{4.26b}$$

$$v^2\tau^{\mu\nu} = \alpha \left(2vu^{(\mu}p^{\nu)\alpha}\iota_\alpha + 2p^{\alpha(\mu}\iota^{\nu)}\lambda_\alpha - P^{\mu\nu}\iota^\alpha\lambda_\alpha - E^{\mu\nu\alpha\beta}\iota_\alpha\lambda_\beta\right). \tag{4.26c}$$

Putting the general form of the energy-momentum tensor (3.26a) into the conservation equation (4.1)

$$\left(u^\nu\rho_{;\nu} + \rho u^\nu{}_{;\nu}\right) u^\mu + \rho\dot{u}^\mu + P^{\mu\nu}{}_{;\nu} = 0. \tag{4.27}$$

Taking the discontinuity of the general formula (4.27) and using (4.25) yields

$$\left(v\sigma + \rho\kappa^\nu\lambda_\nu\right) u^\mu + \rho\alpha\iota^\mu + \tau^{\mu\nu}\lambda_\nu = 0. \tag{4.28}$$

Now using (4.26) for κ^μ, σ, and $\tau^{\mu\nu}$ yields

$$v^2 \left(\rho\gamma^{\mu\nu} + P^{\mu\nu}\right)\iota_\nu + P^{\nu\alpha}\lambda_\nu\lambda_\alpha\iota^\mu - E^{\mu\nu\alpha\beta}\lambda_\nu\iota_\alpha\lambda_\beta = 0. \tag{4.29}$$

By using the relativistic Hadamard tensor $A^{\mu\nu\alpha\beta}$, defined in (4.17), the equation (4.29) becomes

$$\left[v^2 \left(\rho\gamma^{\mu\nu} + P^{\mu\nu}\right) - Q^{\mu\nu}\right]\iota_\nu = 0. \tag{4.30}$$

where we have introduced the Fresnel tensor $Q^{\mu\nu}$ which is defined in terms of the Hadamard tensor and the propagation vector ν_μ via

$$Q^{\mu\alpha} \equiv A^{\mu\nu\alpha\beta}\nu_\nu\nu_\beta, \tag{4.31}$$

after noting that the Hadamard tensor is orthogonal on all indices. Orthogonality of the Hadamard tensor carries over to give orthogonality of the Fresnel tensor,

$$u_\mu Q^{\mu\nu} = 0. \tag{4.32}$$

Since every term in the characterstic equation (4.30) is orthogonal, it is essentially a 3-dimensional equation. The eigenvalues v^2 are the squared sound speed (in general there will be three values).

Although we will show where this comes from later on, it is worth our providing an example of the explicit computation of the sound speed. In the case of an isoptropic elastic solid close to a ground state, the pressure tensor is specified in terms of the isotropic pressure scalar as $P^{\mu\nu} = p\gamma^{\mu\nu}$, and the elasticity tensor is given by

$$E^{\mu\nu\alpha\beta} = \left(\beta - \tfrac{1}{3}p\right)\gamma^{\mu\nu}\gamma^{\alpha\beta} + 2\left(\mu + p\right)\left(\gamma^{\mu(\alpha}\gamma^{\beta)\nu} - \tfrac{1}{3}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right); \tag{4.33}$$

the coefficients p, β, and μ, are repectively the isotopic pressure, bulk modulus, and modulus of rigidity. The Hadamard tensor in this case is given by

$$A^{\mu\nu\alpha\beta} = \beta\gamma^{\mu\nu}\gamma^{\alpha\beta} + 2p\gamma^{\mu[\beta}\gamma^{\nu]\alpha} + 2\mu\left(\gamma^{\mu(\alpha}\gamma^{\beta)\nu} - \tfrac{1}{3}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right), \tag{4.34}$$

and the Fresnel tensor works out as

$$Q^{\mu\nu} = \left(\beta + \tfrac{1}{3}\mu\right)\nu^{\mu}\nu^{\nu} + \mu\gamma^{\mu\nu}. \tag{4.35}$$

Hence, the characteristic equation (4.30) becomes

$$\left[v^2\left(\rho + p\right)\gamma^{\mu\nu} - \mu\gamma^{\mu\nu} - \left(\beta + \tfrac{1}{3}\mu\right)\nu^{\mu}\nu^{\nu}\right]\iota_{\nu} = 0. \tag{4.36}$$

There are two solutions: the first is where the polarization and propagation vectors are aligned, $\nu_{\mu} = \iota_{\mu}$ in which case the eigenvalue is

$$v^2 = \frac{\beta + \frac{4}{3}\mu}{\rho + p} \equiv c_{\mathrm{L}}^2. \tag{4.37}$$

Secondly, where the polarization and propagation vectors are orthogonal: $\nu_{\mu}\iota^{\mu} = 0$, in which case the eigenvalue is

$$v^2 = \frac{\mu}{\rho + p} \equiv c_{\mathrm{T}}^2. \tag{4.38}$$

We therefore have two sound speeds; c_{L}^2 which is the speed of propagation of longitudinal modes, and c_{T}^2 which is the speed of propagation of transverse modes.

5 The isotropic solid

So far we have not asked anything of the solid. We can however ask (or, demand, depending on your point of view) that the solid is isotropic. That is, invariant under $SO(3)$ transformations of the material coordinates. What this ends up imposing is that the action is dependent only upon scalar invariants formed from the pulled-back material metric. In this section we show what these invariants are, the resulting action, and give an example equation of state.

5.1 Constructing scalar invariants

We are interested in constructing the allowed arguments of the equation of state ρ. When the material is constrained to be isotropic, the arguments of this equation of state are formed from scalar invariants of the available material tensors. From the point of view of the theoretical construction which concerns us at the moment, this requires an understanding of the allowed scalar quantities one can form from objects which specify the state of the system. The scalar invariants are constructed from the mixed components of the pull-back of the material metric, $k^{\mu}{}_{\nu}$.

There are a few different sets of scalar invariants one could use: formally they are identical, but different choices will help or hide insight into the physical behavior. There are two sets of scalar invariants which we now describe.

As a candidate set of invariants, *the* three independent scalar invariants of the mixed components of the pulled-back material metric ${k^\mu}_\nu$ are

$$I_1 = [\mathbf{k}], \qquad I_2 = [\mathbf{k}^2], \qquad I_3 = [\mathbf{k}^3], \tag{5.1}$$

in which we denoted traces with square braces,

$$I_n = \mathrm{Tr}(\mathbf{k}^n) = [\mathbf{k}^n] = {k^\mu}_\nu {k^\nu}_\alpha \cdots {k^\lambda}_\mu, \tag{5.2}$$

with ${k^\mu}_\nu$ defined from $k_{\mu\nu}$ via (2.29). To reiterate, (5.1) is a complete list of independent invariants due to the orthogonality of ${k^\mu}_\nu$ (2.28), and any other invariants can be computed from these via the Cayley-Hamilton theorem. For example, since n_{ABC} is the volume form of $k_{\mu\nu}$, the particle number density n is also a scalar invariant of ${k^\mu}_\nu$; by the Cayley-Hamilton theorem, the determinant is related to the other invariants via

$$n^2 = \det({k^\mu}_\nu) = \tfrac{1}{3!}\left([\mathbf{k}]^3 - 3[\mathbf{k}][\mathbf{k}^2] + 2[\mathbf{k}^3]\right). \tag{5.3}$$

We could use $\{I_1, I_2, I_3\}$ as defined in (5.1) as the list of invariants for the arguments of the equation of state, but we shall also consider the particle number density n and the independent scalar invariants of the uni-modular tensor ${\mathfrak{U}^\mu}_\nu$, defined in (2.32) since this will help the comparison between solid and fluid descriptions. The important consequence of uni-modularity is that ${\mathfrak{U}^\mu}_\nu$ only has two independent invariants (rather than 3 which could be expected from a symmetric rank-2 tensor in 3D). The invariants are linked via the Cayley-Hamilton theorem as

$$3! = [\boldsymbol{\mathfrak{U}}]^3 - 3[\boldsymbol{\mathfrak{U}}][\boldsymbol{\mathfrak{U}}^2] + 2[\boldsymbol{\mathfrak{U}}^3]. \tag{5.4}$$

And so, in brief summary, we have shown that there are two equivalent ways to write the most general function of state for a solid with scalar arguments: both have a maximum of three arguments. They are

$$\rho = \rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right) \tag{5.5a}$$

and

$$\rho = \rho\left(n, [\boldsymbol{\mathfrak{U}}], \left[\boldsymbol{\mathfrak{U}}^2\right]\right). \tag{5.5b}$$

We remind that ${k^\mu}_\nu$ is the pull-back of a tensor whose volume form is n_{ABC} and (squared) determinant is the particle number density, n, and that ${\mathfrak{U}^\mu}_\nu$ is a uni-modular tensor whose inverse $\mathfrak{U}^{-1AB}$ co-incides with the push-forward of the space-time metric when the material is in the unsheared state. The latter formulation, (5.5b), is somewhat favorable, since it becomes easy to connect to a scenario in which the solid "becomes" like a fluid, since ρ becomes independent of the $[\boldsymbol{\mathfrak{U}}^n]$.

5.2 The action of an isotropic solid

When one demands that the material is isotropic then this constitutes a constraint on ρ as being a function of any possible scalar invariants discussed in Section 5.1. With this constraint imposed, the material action is given the integral of a tri-variate scalar function,

$$S_{\mathrm{M}} = \int \mathrm{d}^4x \sqrt{-g}\, \rho\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{5.6}$$

Rather than asking for the form of ρ, it is often more convenient to ask for the form of the energy-per-particle, ϵ, defined from ρ and n via (3.6), and then write the matter action (5.6) as

$$S_{\mathrm{M}} = \int \mathrm{d}^4x \sqrt{-g}\, n\epsilon\left([\mathbf{k}], \left[\mathbf{k}^2\right], \left[\mathbf{k}^3\right]\right). \tag{5.7}$$

5.3 Example equations of state

It is instructive to specify an example equation of state and obtain the energy-momentum tensor.

5.3.1 The Carter-Quintana perfect solid

Carter and Quintana conclude their paper with an exposition of the equations for a *perfect elastic solid*. Before we give their equation of state and energy-momentum tensor, we shall discuss physical issues regarding the existence (or otherwise) of locally relaxed states of the material. See also [40] for a presentation of FRW solutions to the Carter-Quintana elastic solid system.

Strain and shear tensors The strain tensor is linked to the assumption about the existence of a locally relaxed state of a material – this is the unstrained state. In the unstrained state the energy per particle ϵ is supposed to be minimum when $\gamma_{\mu\nu}$ takes on a particular value, $k_{\mu\nu}$ say. This invites a quantification of the state of strain of the material by measuring the difference between the actual value of $\gamma_{\mu\nu}$ and its unstrained value $k_{\mu\nu}$ via the *strain tensor*, $e_{\mu\nu}$, defined as

$$e_{\mu\nu} = \tfrac{1}{2}\left(\gamma_{\mu\nu} - k_{\mu\nu}\right). \tag{5.8}$$

Recalling that the energy per particle is denoted as ϵ, we define ϵ_0 to be the energy per particle in the unstrained state. The Hookean idealization takes the energy per particle to be of quadratic form in the strain tensor

$$\epsilon = \epsilon_0 + \tfrac{1}{2}K^{\mu\nu\alpha\beta}e_{\mu\nu}e_{\alpha\beta}. \tag{5.9}$$

The elasticity tensor $E^{\mu\nu\alpha\beta}$ relates to $K^{\mu\nu\alpha\beta}$ via

$$E^{\mu\nu\alpha\beta} = nK^{\mu\nu\alpha\beta}. \tag{5.10}$$

Hence, since $\rho = n\epsilon$, the energy density can be written as

$$\rho = \frac{n}{n_0}\rho_0 + \tfrac{1}{2}E^{\mu\nu\alpha\beta}e_{\mu\nu}e_{\alpha\beta}, \tag{5.11}$$

and the pressure tensor is related to the strain tensor via

$$P^{\mu\nu} = -E^{\mu\nu\alpha\beta} e_{\alpha\beta}. \tag{5.12}$$

The tensor $k_{\mu\nu}$ can be thought of as a Riemannian metric on material space; the pull-back formalism means that $u^\mu k_{\mu\nu} = 0$. Associated with the value ϵ_0 of ϵ in the unstrained state are the values ρ_0 of the energy density ρ, and n_0 of the particle number density n.

The complication which Carter invites is that not all physical systems of interest will have a state which is locally relaxed, thus negating the existence of $k_{\mu\nu}$ and rendering this construction impotent. This leads to the introduction of the shear tensor.

Rather than ask for the relaxed state to be a state where the energy per particle is minimum, we ask for a state in which ϵ is minimized subject to the restriction of constant particle number density. This is the unsheared state, and motivates the introduction of $\mathfrak{U}_{\mu\nu}(n)$ which is the value of $\gamma_{\mu\nu}$ in the unsheared state with particle number density n. Again, to quantify the state of shear we define the *constant volume shear tensor* via

$$s_{\mu\nu} = \tfrac{1}{2}\left(\gamma_{\mu\nu} - \mathfrak{U}_{\mu\nu}\right), \tag{5.13}$$

which (to reinforce the point) is the difference between the actual value of $\gamma_{\mu\nu}$ and its value in the unsheared state.

We define $\check{\rho}(n)$ to be the energy density in the unsheared state, and hence

$$\check{\rho} = n\check{\epsilon}. \tag{5.14}$$

When ϵ does have an absolute minimum, at some particle number density n_0, one can keep the previous notions of the strain tensor; indeed

$$\mathfrak{U}_{\mu\nu}(n_0) = k_{\mu\nu}, \tag{5.15a}$$
$$\check{\rho}(n_0) = \rho_0, \tag{5.15b}$$
$$\check{\epsilon}(n_0) = \epsilon_0. \tag{5.15c}$$

The equation of state The "physics" of the solid that Carter and Quintana had in mind was that it was supposed to have vanishing compressional distortion. That is, the compressional distortion tensor is supposed to vanish, $\tau_{\mu\nu} = 0$. One can obtain

$$\mathfrak{U}_{\mu\nu} = (n/n_0)^{-2/3} k_{\mu\nu}, \tag{5.16}$$

The solid is supposed to be isotropic with respect to its unsheared states. Hence, the energy per particle (recall, $\rho = \epsilon n$, and ϵ is the energy per particle) is a function only of invariants. There are a maximum of three invariants: they are taken to be the particle number density n and the two independent invariants of the shear tensor $s^a{}_b$. The particular combination of these are taken to be

$$s^2 \equiv \left(\mathfrak{U}^{-1\mu\beta}\mathfrak{U}^{-1\nu\alpha} - \tfrac{1}{3}\mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta}\right) s_{\mu\nu}s_{\alpha\beta} = \lceil \mathbf{s}^2 \rceil - \tfrac{1}{3}\lceil \mathbf{s} \rceil^2, \tag{5.17a}$$

$$l \equiv \mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta}\mathfrak{U}^{-1\lambda\kappa} s_{\nu\alpha}s_{\beta\lambda}s_{\mu\kappa} = \lceil \mathbf{s}^3 \rceil. \tag{5.17b}$$

We used the notation $\lceil \mathbf{X} \rceil$ for traces which are taken with $\mathfrak{U}^{-1\mu\nu}$ (as opposed to $[\mathbf{X}]$ which was used to denote traces with $g^{\mu\nu}$): this choice is for simplicity of the resulting formulae and does not lose generality. Recall that the shear tensor s_{ab} is related to the material metric $k_{\mu\nu}$ and particle number density via

$$k_{\mu\nu} = n^{2/3}\left(\gamma_{\mu\nu} - 2s_{\mu\nu}\right). \tag{5.18}$$

Hence, the most general form of the equation of state is a function with three arguments:

$$\epsilon = F(n, s^2, l). \tag{5.19}$$

The action for Einsteinian gravity with the CQ solid is thus

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{R}{16\pi G} - nF(n, s^2, l)\right]. \tag{5.20}$$

The energy-momentum tensor The energy density ρ, and pressure tensor $P^{\mu\nu}$ are given by

$$\rho = nF, \tag{5.21a}$$

$$\begin{aligned} P^{\mu\nu} \;=\; & \left\{n^2F_{,n} + ns^2\left(\frac{4}{3}F_{,s^2} + F_{,l}\right) - n\left(2l + \tfrac{1}{3}\lceil\mathbf{s}\rceil^2\right)F_{,l}\right\}\gamma^{\mu\nu} \\ & -2nF_{,s^2}\left(\mathfrak{U}^{-1\mu(\alpha}\mathfrak{U}^{-1\beta)\nu} - \tfrac{1}{3}\mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta}\right)s_{\alpha\beta} \\ & -3nF_{,l}\mathfrak{U}^{-1\mu(\alpha}\mathfrak{U}^{-1\beta)\nu}\mathfrak{U}^{-1\lambda\kappa}s_{\alpha\lambda}s_{\beta\kappa}. \end{aligned} \tag{5.21b}$$

These expressions contain the corrections to the typos which were present in [4], and which were pointed out (by the same authors) in [5]. The isotropic pressure is found from the trace of (5.21b), and is given by

$$p = n^2F_{,n} + 2n\left(\frac{2}{3}s^2F_{,s^2} - lF_{,l}\right). \tag{5.22}$$

The quasi-Hookean solid We will make use of the notation

$$\check{\epsilon}(n) = F(n,0,0), \tag{5.23a}$$
$$\check{\rho}(n) = nF(n,0,0), \tag{5.23b}$$
$$\check{p}(n) = n^2\frac{\partial F}{\partial n}(n,0,0), \tag{5.23c}$$
$$\beta(n) = n^3\frac{\partial^2 F}{\partial n^2}(n,0,0) + 2n^2\frac{\partial F}{\partial n}(n,0,0), \tag{5.23d}$$
$$\mu(n) = n\frac{\partial F}{\partial s^2}(n,0,0). \tag{5.23e}$$

The quantities $\check{\rho}, \check{p}, \beta, \mu$ are the unsheared energy density, bulk and rigidity moduli. The value of the elasticity tensor in the state of zero shear strain is

$$\check{E}^{\mu\nu\alpha\beta}(n) = (\beta - \tfrac{1}{3}\check{p})\mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta} + 2(\mu + \check{p})(\mathfrak{U}^{-1\mu(\alpha}\mathfrak{U}^{-1\nu)\beta} - \tfrac{1}{3}\mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta}). \tag{5.24}$$

Note that this is the elasticity tensor we computed the sound speeds for just after equation (4.33).

The Lagrangian for the Carter-Quintana solid in the quasi-Hookean limit (which we shall refer to as a "quasi-Hookean solid") is linear in s^2, and independent of l:

$$F_{\text{qHs}} = \check{\epsilon} + \tfrac{\mu(n)}{n} s^2. \tag{5.25}$$

For this quasi-Hookean solid the energy density and pressure tensor are respectively given by

$$\rho = \check{\rho} + \mu s^2, \tag{5.26a}$$

$$P^{\mu\nu} = \left\{\check{p} + \left(n\mu' + \tfrac{1}{3}\mu\right) s^2\right\} \gamma^{\mu\nu} - 2\mu \left\{\mathfrak{U}^{-1\mu(\alpha}\mathfrak{U}^{-1\beta)\nu} - \tfrac{1}{3}\mathfrak{U}^{-1\mu\nu}\mathfrak{U}^{-1\alpha\beta}\right\} s_{\alpha\beta}. \tag{5.26b}$$

Slow roll parameter We take this opportunity to recall that for inflating an FLRW Universe one requires smallness of the slow-roll parameter ϵ_{slow}, defined as

$$\epsilon_{\text{slow}} \equiv -\frac{\dot{H}}{H^2} = \frac{3(\rho + P)}{2\rho}. \tag{5.27}$$

Using (5.21a) and (5.22) the slow-roll parameter (5.27) evaluates for the CQ perfect solid to give

$$\epsilon_{\text{slow}} = \frac{3}{2}\left[1 + \frac{\partial \log F}{\partial \log n} + \frac{4}{3}\frac{\partial \log F}{\partial \log s^2} - 2\frac{\partial \log F}{\partial \log l}\right]. \tag{5.28}$$

The structure of (5.28) suggests a separable ansatz for the functional form of F:

$$F(n, s^2, l) = x(n)y(s^2)z(l), \tag{5.29}$$

since (5.28) becomes

$$\epsilon_{\text{slow}} = \frac{3}{2}\left[1 + nx' + \frac{4}{3}s^2 y' - 2lz'\right], \tag{5.30}$$

in which a prime is used to denote derivative with respect to the sole argument of the given function.

5.3.2 Karlovini and Samuelsson's solid

We will make the same choice as described in [6], and write down a particular equation of state (which is only a subset of all the possible models) for isotropic materials.

To begin with it is useful to pull-back the constant volume shear tensor (2.20), which is given by

$$s_{\mu\nu} = \tfrac{1}{2}(\gamma_{\mu\nu} - \mathfrak{U}_{\mu\nu}). \tag{5.31}$$

There are two methods to raise indices (and thus construct traces). These methods are

$$s^{\mu}{}_{\nu} = \gamma^{\mu\alpha} s_{\alpha\nu}, \qquad \hat{s}^{\mu}{}_{\nu} = \mathfrak{U}^{-1\mu\alpha} s_{\alpha\nu}. \tag{5.32}$$

In matrix form these respectively read

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{1} - \mathfrak{U}), \qquad \hat{\mathbf{s}} = \tfrac{1}{2}(\mathfrak{U}^{-1} - \mathbf{1}). \tag{5.33}$$

The equation of state ϵ is chosen to be a function of the particle number density n and a particular combination of the invariant of $\mathfrak{U}^{\mu}{}_{\nu}$. The explicit form of ϵ is

$$\epsilon = \check{\epsilon}_0(n) + \frac{\check{\mu}(n)}{n}\overline{s}^2, \tag{5.34}$$

in which $\overline{s}^2$ is the shear scalar defined from the invariants of $\mathfrak{U}^{\mu}{}_{\nu}$ via

$$\overline{s}^2 \equiv \tfrac{1}{36}\left([\mathfrak{U}]^3 - [\mathfrak{U}^3] - 24\right). \tag{5.35}$$

Notice that by using the Cayley-Hamilton relation (5.4), the choice of shear scalar (5.35) is equivalent to

$$\overline{s}^2 = \tfrac{1}{24}\left([\mathfrak{U}]^2 - [\mathfrak{U}^2]\right)[\mathfrak{U}] - \tfrac{3}{4}. \tag{5.36}$$

Using (5.34) the material action is therefore given by

$$S_{\mathrm{M}} = \int \mathrm{d}^4x\,\sqrt{-g}\left\{n\check{\epsilon}_0 + \tfrac{1}{36}\check{\mu}\left([\mathfrak{U}]^3 - [\mathfrak{U}^3] - 24\right)\right\}. \tag{5.37}$$

The pressure tensor is given by (3.31) where the isotropic pressure scalar (3.32a) is

$$p = \check{p} + (\check{\Omega} - 1)\sigma, \tag{5.38a}$$

and the anisotropic stress (3.32b) is given by

$$\pi_{\mu\nu} = \tfrac{1}{6}\check{\mu}\left([\mathfrak{U}]^2\mathfrak{U}_{\langle\mu\nu\rangle} - \mathfrak{U}^{\alpha\beta}\mathfrak{U}_{\alpha\langle\mu}\mathfrak{U}_{\nu\rangle\beta}\right), \tag{5.38b}$$

and where the three quantities appearing in the pressure (5.38a) are

$$\check{p} = n^2\frac{\mathrm{d}\check{\epsilon}_0}{\mathrm{d}n}, \qquad \check{\Omega} = \frac{n}{\check{\mu}}\frac{\mathrm{d}\check{\mu}}{\mathrm{d}n}, \qquad \sigma = \check{\mu}s^2. \tag{5.39}$$

6 Perturbed solids

The majority of this review has been focussed on the general theory of solids: the deformations performed on the solid or medium may be arbitrarily large. Whilst this is very general, it also yields a theory which is complicated to work with. There are a substantial number of physical systems for whom the non-linear theory of elasticity is "over-kill": understanding the governing equations that describe small deformations of the solid from its equilibrium configuration is often sufficient. For this reason we shall review the theory of perturbed solids.

Comprehensive reviews, applications, and examples in the relativistic theory have already been presented [1–3, 5, 14, 31, 32], as well as the non-relativistic theory being the main subject of a classic book by Landau and Lifshitz [33].

The physical picture one should constantly keep in mind is that a continuous medium has "two states": relaxed and deformed. The former occurs when there are no forces on the medium, and the latter will induce strains and forces on other surrounding materials and fields (notably the metric). In some sense the "point" of a model is to catalogue the possible ways in which a material can influence surrounding media and fields.

6.1 Non-relativistic solids

A non-relativistic solid is one for whom there are no gravitational effects. Another distinguishing feature of non-relativistic solids from relativistic ones is that the pressure of the solid is negligible. As examples, one imagines an eraser, rubber band, trampolines: these kinds of materials.

The locations within a relaxed non-relativistic solid are denoted by x^i. Under a deformation the coordinates alter according to

$$x^i \longrightarrow x^i + \xi^i(x^j), \tag{6.1}$$

where the ξ^i are supposed to be small. If the line element in the solid before the deformation is $\mathrm{d}\ell^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, then after the deformation the line element has metric given by

$$g_{ij} = \delta_{ij} + 2\varepsilon_{ij}, \tag{6.2}$$

in which we defined the strain tensor,

$$\varepsilon_{ij} \equiv \partial_{(i}\xi_{j)}. \tag{6.3}$$

The components of the strain tensor ε_{ij} contain all information about the deformation performed on the body. We now require information about the manner in which the body responds to the given deformation. This entails an understanding of the stress tensor, σ^{ij}, for a given strain tensor ε_{ij}. This is where the "physics" comes in.

Although this seems like a tangential calculation, consider that if one computes the divergence of the stress tensor one obtains the components of the force, $F^i = \partial_j\sigma^{ij}$. These can be equated to the acceleration of the deformation vectors to obtain the equation of motion

$$\rho\ddot{\xi}^i = \partial_j\sigma^{ij}. \tag{6.4}$$

And so, if the stress tensor can be related to the strain tensor (which, we remind is constructed from the derivatives of the deformation vector according to (6.3)), then the equation of motion (6.4) becomes a closed set of equations.

In broad-brush-terms there are two cases which are useful to consider and describe a huge class of physically useful solids.

1. The stress tensor is proportional to the strain tensor:

$$\sigma^{ij} = E^{ijkl}\varepsilon_{kl}. \tag{6.5}$$

The components, E^{ijkl}, precisely prescribe the strength of certain forces for given deformations (we will have much more to say about this later); they are the components of the elasticity tensor, and there are a fixed number of them for any given material in a space-time with given dimension. This "given number" is rather large for a material with arbitrary symmetry, but dramatically reduces once the material is imposed to have certain symmetries. Materials for whom (6.5) holds are Hookean elastic solids.

2. The stress tensor is proportional to the rate-of-strain tensor:

$$\sigma^{ij} = V^{ijkl}\dot{\varepsilon}_{kl}. \tag{6.6}$$

The components V^{ijkl} play a similar role to the components of the elasticity tensor for a Hookean solid, except here we are considering viscous solids and V^{ijkl} are the components of the viscosity tensor.

Of course, the given physical material may require an amalgamation of the two cases, whereby stress is proportional to both strain, and rate-of-strain, in which case

$$\sigma^{ij} = E^{ijkl}\varepsilon_{kl} + V^{ijkl}\dot{\varepsilon}_{kl}. \tag{6.7}$$

The expression (6.7) describes visco-elastic solids (also known as Kelvin-Voigt solids). Since (6.7) contains both the elastic (6.5) and viscous (6.6) models as sub-cases, we will proceed with the Kelvin-Voigt expression (6.7). Using (6.3) and (6.7), the equation of motion (6.4) is

$$\rho\ddot{\xi}^i = E^{ijkl}\partial_j\partial_{(k}\xi_{l)} + V^{ijkl}\partial_j\partial_{(k}\dot{\xi}_{l)}. \tag{6.8}$$

The elastic part only cares about the gradients of the deformation, but the viscous part also cares about the rate at which the deformation is applied.

The problem of describing the solid considerably simplifies when one assumes some symmetry of the solid, for example material isotropy. In such cases (other symmetries require more freedom than we are about to introduce), the material tensors each only have two independent components, and decompose completely as

$$E^{ijkl} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij}g^{kl} + 2\mu g^{i(k}g^{l)j}, \tag{6.9a}$$
$$V^{ijkl} = \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij}g^{kl} + 2\nu g^{i(k}g^{l)j}. \tag{6.9b}$$

We introduced $\{\beta, \mu\}$ for the components of the elasticity tensor, and $\{\lambda, \nu\}$ for the components of the viscosity tensor. Using the decompositions (6.9), the stress-tensor (6.7) becomes

$$\sigma^{ij} = \left(\beta - \tfrac{2}{3}\mu\right) g^{ij}\partial_k\xi^k + \left(\lambda - \tfrac{2}{3}\nu\right) g^{ij}\partial_k\dot{\xi}^k + 2\mu\partial^{(i}\xi^{j)} + 2\nu\partial^{(i}\dot{\xi}^{j)}, \tag{6.10}$$

and the equation of motion (6.8) becomes

$$\rho\ddot{\xi}^i = \left(\beta + \tfrac{1}{3}\mu\right)\partial^i\partial_k\xi^k + \mu\partial_k\partial^k\xi^i + \left(\lambda + \tfrac{1}{3}\nu\right)\partial^i\partial_k\dot{\xi}^k + \nu\partial_k\partial^k\dot{\xi}^i. \tag{6.11}$$

We shall provide a simple example which highlights the seperate modes of propagation inherent in a material medium. Consider the simple case where the deformation vector has only two components; we can expand ξ^i using two scalars ϕ and ψ in an orthonormal basis $(\hat{x}^i, \hat{y}^i)$ via

$$\xi^i = \phi\hat{x}^i + \psi\hat{y}^i. \tag{6.12a}$$

Now suppose that these scalars depend on time, and only one of the two available spatial directions; that is, we set

$$\partial_i\phi = \phi'\hat{x}_i, \qquad \partial_i\psi = \psi'\hat{x}_i. \tag{6.12b}$$

Putting the decomposition of the deformation vector (6.12) into the equation of motion (6.11) yields an equation with two independant projections (one along $\hat{x}^i$, and one along $\hat{y}^i$); these projections leads to the requirement that the following equations are satisfied:

$$\ddot{\phi} - \frac{\lambda + \frac{4}{3}\nu}{\rho}\dot{\phi}'' = \frac{\beta + \frac{4}{3}\mu}{\rho}\phi'', \tag{6.13a}$$

$$\ddot{\psi} - \frac{\nu}{\rho}\dot{\psi}'' = \frac{\mu}{\rho}\psi''. \tag{6.13b}$$

In the purely elastic case (i.e., where all components of the viscosity tensor vanish), it is with relative ease that one realises a plane wave ansatz $\phi \sim e^{\mathrm{i}(\omega t + kx)}$ solves the equations of motion, and that ϕ and ψ travel with different speeds: these are the longitudinal and transverse sound speeds

$$c_{\mathrm{L}}^2 = \frac{\beta + \frac{4}{3}\mu}{\rho}, \qquad c_{\mathrm{T}}^2 = \frac{\mu}{\rho}. \tag{6.14}$$

6.2 Relativistic solids

In the relativistic theory one needs to carefully describe the perturbations; there are intrinsic variations in the metric, and pre-existing matter fields, as well as perturbations in the continuous medium.

The coordinates of the undeformed medium are represented by $\overline{x}^\mu$, and those of the deformed medium are by x^μ. These are related via

$$x^\mu = \overline{x}^\mu + \xi^\mu(x^\nu). \tag{6.15}$$

The crucial piece here is the deformation vector, $\xi^\mu(x^\nu)$, which as have we explicitly shown via our notation, is dependent upon the space-time coordinates (different locations can deform by different amounts).

The metric of a space-time which contains a perturbed medium is given by expanding the metric to linear order in intrinsic metric perturbations, h_{ab}, and in the deformation vector,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \tag{6.16}$$

The metric of the unperturbed space-time is $\overline{g}_{\mu\nu}$, and the metric fluctuations due to "intrinsic", or extra-material contents, is given by $h_{\mu\nu}$. The presence of the perturbed medium is encapsulated by the term involving the deformation field,

$$\xi^\mu = x^\mu - \overline{x}^\mu. \tag{6.17}$$

One may recognise the final term in (6.16) as that which arises in standard perturbation theory after one performs the diffeomorphism

$$x^\mu \longrightarrow x^\mu + \xi^\mu(x^\nu). \tag{6.18}$$

Of course, this recognition is accurate. There is an additional concept to appreciate however: interpretation. The ξ^μ-field describes all the fluctuations of the medium away from its equilibrium configuration. In addition, the deformation field ξ^μ is orthogonal,

$$u_\mu \xi^\mu = 0. \tag{6.19}$$

The key to obtaining a picture of whats going on here is to go back to the construction we laid out in section 2.1, and imagine that the solid in space-time is a collection of particles, each of which traces out its own world-line. The deformation represented by (6.18), and constrained by (6.19), should be interpreted as a given world-line being moved from its original trajectory. Or, to use a more cohesive language: the world-lines are being deformed.

6.2.1 Lagrangian and Eulerian variations

A more elegant, and geometrically intuitive way to write the corrections to the metric is by writing all of the non-background terms in (6.16) as

$$\delta_{\mathrm{L}} g_{\mu\nu} = \delta_{\mathrm{E}} g_{\mu\nu} + \pounds_\xi g_{\mu\nu}, \tag{6.20}$$

in which we identified the intrinsic metric perturbation as

$$\delta_{\mathrm{E}} g_{\mu\nu} \equiv h_{\mu\nu}, \tag{6.21}$$

and the usual expression for the Lie derivative of the metric along the vector ξ^a,

$$\pounds_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}. \tag{6.22}$$

The expression (6.20) encapsulates a more general framework of Lagrangian and Eulerian variations, and how they are related in a system which is deformed. This distinction is only valid or useful in systems where the metric has intrinsic perturbations, possibly of the type usually considered in, say cosmological perturbatino theory.

6.2.2 The perturbed energy-momentum tensor

The field equations for the perturbations (in the conventional sense) of a gravitating system which only contains an elastic medium are given by

$$\delta_{\mathrm{E}} G^{\mu\nu} = 8\pi G\, \delta_{\mathrm{E}} T^{\mu\nu}, \tag{6.23}$$

where we used the symbol "δ_{E}" to denote intrinsic variations. The source term, $\delta_{\mathrm{E}} T^{\mu\nu}$, is constructed from a term which contains the variations in the energy-momentum tensor

$$\delta_{\mathrm{E}} T^{\mu\nu} = \delta_{\mathrm{L}} T^{\mu\nu} - \pounds_\xi T^{\mu\nu}. \tag{6.24}$$

$$\pounds_\xi T^{\mu\nu} = \xi^\alpha \nabla_\alpha T^{\mu\nu} - 2T^{\alpha(\mu}\nabla_\alpha \xi^{\nu)} \tag{6.25}$$

In the visco-elastic case,

$$\delta_{\mathrm{L}} T^{\mu\nu} = -\frac{1}{2}\left(W^{\mu\nu\alpha\beta} + T^{\mu\nu} g^{\alpha\beta}\right)\delta_{\mathrm{L}} g_{\alpha\beta} - V^{\mu\nu\alpha\beta}\delta_{\mathrm{L}} K_{\alpha\beta}; \tag{6.26}$$

$$W^{\mu\nu\alpha\beta} = E^{\mu\nu\alpha\beta} + P^{\mu\nu}u^\alpha u^\beta + P^{\alpha\beta}u^\mu u^\nu - 4u^{(\alpha}P^{\beta)(\mu}u^{\nu)} - \rho u^\mu u^\nu u^\alpha u^\beta \tag{6.27}$$

The elasticity and viscosity tensors have the symmetries

$$E^{\mu\nu\alpha\beta} = E^{(\mu\nu)(\alpha\beta)} = E^{\alpha\beta\mu\nu}, \qquad V^{\mu\nu\alpha\beta} = V^{(\mu\nu)(\alpha\beta)}, \tag{6.28}$$

and are orthogonal on all indices,

$$u_\mu E^{\mu\nu\alpha\beta} = 0, \qquad u_\mu V^{\mu\nu\alpha\beta} = u_\alpha V^{\mu\nu\alpha\beta} = 0. \tag{6.29}$$

6.3 Deformations about a relaxed state

It is important to understand how to deal with a deformed medium. Before we give some explicit expressions for deformations of the solid, we shall illustrate the philosophy via "non-linear sigma models" from field theory.

6.3.1 Example from non-linear sigma models

One of the important ideas in continuous mechanics is that of the assumed existence of a relaxed state: this is supposed to be some configuration that minimizes some measure of "energy". This concept is absolutely vital in the study of solitons. As the simplest example, consider the Lagrangian density for a real scalar field ϕ living in a Higgs potential,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2. \tag{6.30}$$

The relaxed configuration of this scalar is when $\phi = \pm\eta$ (commonly known as the vacuum manifold). It is simple to find the Lagrangian density for fluctuations about the relaxed state; substituting $\phi = \eta + \delta\phi$ into (6.30) and expanding to quadratic order in $\delta\phi$ yields

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\delta\phi\partial^\mu\delta\phi - \frac{1}{2}\lambda\eta^2(\delta\phi)^2. \tag{6.31}$$

The Lagrangian that results is that for a massive scalar field, and it describes the perturbations about the relaxed state.

This example was simple enough to demonstrate the idea of a "relaxed state" in a non-linear field theory, but it was in some sense "too" simple since there isn't a non-trivial Lagrangian that describes the field *in* the relaxed state. For that we shall move to a more complicated example and think about a multi-scalar field model whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\mathfrak{k}_{IJ}\partial_\mu\Phi^I\partial^\mu\Phi^J - V(\Phi^I). \tag{6.32}$$

There are supposed to be n fields here, and so $I = 1, \ldots, n$, and the set of symmetric quantities $\mathfrak{k}_{IJ}$ are supposed to play the role of a metric in field space. Before we continue we want to make it plainly clear that this isn't the most general Lagrangian density that can be constructed out of single derivatives.

Suppose that the energy gets minimized when the fields Φ^I are consigned to live on a sub-manifold, $\mathcal{V}$ say, of dimension $q \leq n$; in this state the potential energy

$V(\Phi^I) = 0$. The "vacuum manifold" $\mathcal{V}$ can be coordinatized by q scalars ϕ^A, say, with $A = 1, \ldots, q$. Hence, when the configuration is in its relaxed state the original set of fields Φ^I are expressible as a function of the fields ϕ^A,

$$\Phi^I = \Phi^I(\phi^A). \tag{6.33}$$

By simple application of the chain rule, (6.33) provides

$$\partial_\mu \Phi^I = \frac{\partial \Phi^I}{\partial \phi^A} \partial_\mu \phi^A. \tag{6.34}$$

Putting (6.34) into (6.32) gives

$$\mathcal{L} = -\frac{1}{2}\mathfrak{g}_{AB}(\phi)\partial_\mu \phi^A \partial^\mu \phi^B, \tag{6.35}$$

in which we defined

$$\mathfrak{g}_{AB}(\phi) \equiv \mathfrak{k}_{IJ} \frac{\partial \Phi^I}{\partial \phi^A} \frac{\partial \Phi^J}{\partial \phi^B}. \tag{6.36}$$

The $\mathfrak{g}_{AB}$ are interpreted as the components of the metric on the field submanifold $\mathcal{V}$. The field equations for the ϕ^A derived from (6.35) are given by

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi^A + \Gamma^A{}_{BC} \nabla_\mu \phi^B \nabla^\mu \phi^C = 0, \tag{6.37}$$

where

$$\Gamma^A{}_{BC} = \tfrac{1}{2}\mathfrak{g}^{AD}\left(\partial_\nu \mathfrak{g}_{CD} + \partial_C \mathfrak{g}_{BD} - \partial_D \mathfrak{g}_{BC}\right) \tag{6.38}$$

are the Christoffel symbols for the metric in the field submanifold.

One of the simplest ways (we can think of, at least) to see how study fluctuations or deformations away from the relaxed state is to first imagine that the relaxed state is specified by the condition

$$\frac{\partial \Phi_0^I}{\partial \phi_0^A} = \mathfrak{J}^I{}_A, \tag{6.39}$$

where the "0" subscripts are used to specify that the configuration is relaxed, and the gothic-J is used to denote the relaxed Jacobian. Using (6.39) to compute (6.36) gives a simple expression for the submanifolds metric in the relaxed state,

$$\bar{\mathfrak{g}}_{AB} = \mathfrak{k}_{IJ} \mathfrak{J}^I{}_A \mathfrak{J}^J{}_B. \tag{6.40}$$

It should be evident that the Christoffel symbols in the field submanifold (6.38) are zero for this relaxed state if $\mathfrak{k}_{IJ}$ is flat and the Jacobians $\mathfrak{J}^I{}_A = \delta^I{}_A$. We have denoted $\bar{\mathfrak{g}}_{AB}$ as the field submanifolds metric in the relaxed state. In a deformed state the derivatives of Φ^I with respect to the ϕ^A must differ from their values in the relaxed state by some amount which can be packaged into a rank-2 tensor $\mathfrak{d}^I{}_A$ (this is a gothic-d, for "deformation") via

$$\frac{\partial \Phi^I}{\partial \phi^A} = \mathfrak{J}^I{}_A + \mathfrak{d}^I{}_A, \tag{6.41}$$

where we will not make any assumptions about the size of the $\mathfrak{d}^I{}_A$. Putting (6.41) into (6.36) gives

$$\mathfrak{g}_{AB} = \bar{\mathfrak{g}}_{AB} + 2\mathfrak{d}_{AB} + \mathfrak{d}^I{}_A \mathfrak{d}_{IB}. \tag{6.42}$$

This expression is very similar to what is used in the non-linear Stuckelberg trick in the massive gravity literature (see, e.g., [19, 35]). It is therefore apparent that the deviation of $\mathfrak{g}_{AB}$ from $\bar{\mathfrak{g}}_{AB}$ is contained within the tensor

$$s_{AB} = \mathfrak{d}_{AB} + \tfrac{1}{2}\mathfrak{d}^I{}_A \mathfrak{d}_{IB}, \tag{6.43}$$

so that

$$s_{AB} = \tfrac{1}{2}\left(\mathfrak{g}_{AB} - \bar{\mathfrak{g}}_{AB}\right). \tag{6.44}$$

Hence, we now have a measure on how deformed the material is: when $s_{AB} = 0$ one has $\mathfrak{g}_{AB} = \bar{\mathfrak{g}}_{AB}$ which is the relaxed metric, and any $s_{AB} \neq 0$ means that the material is deformed in some way. If the deformations are small then one can safely assume that $\mathfrak{d}_{AB}$ is a small quantity and so $S_{AB} = \mathfrak{d}_{AB}$.

6.3.2 Deformations of the material

To make more explicit contact to the construction we gave in section 2.1, suppose that the actual values of the material coordinates ϕ^A are related by way of an "expansion" (which is not necessarily small) about some fiducial state whose coordinates were $\bar{\phi}^A$. That is,

$$\phi^A = \bar{\phi}^A + \pi^A. \tag{6.45}$$

Then the configuration gradient (2.2) can be evaluated

$$\psi^A{}_\mu = \bar{J}^A{}_\mu + \partial_\mu \pi^A, \tag{6.46}$$

in which the configuration gradient computed in the fiducial state is

$$\bar{J}^A{}_\mu = \frac{\partial \bar{\phi}^A}{\partial x^\mu}. \tag{6.47}$$

Using (6.46) to provide an expression for the configuration gradient to compute the pull-back $k_{\mu\nu}$ of the material metric $k_{\mu\nu}$ via (2.25) yields

$$k_{\mu\nu} = \bar{k}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} + \Pi_{\mu\nu}, \tag{6.48}$$

in which we defined

$$\begin{aligned}
\bar{k}_{\mu\nu} &\equiv k_{AB}\bar{J}^A{}_\mu \bar{J}^B{}_\nu, && (6.49\text{a})\\
\partial_\mu \xi_b &\equiv k_{AB}\bar{J}^A{}_a \partial_\nu \pi^B, && (6.49\text{b})\\
\Pi_{\mu\nu} &\equiv k_{AB}\partial_\mu \pi^A \partial_\nu \pi^B. && (6.49\text{c})
\end{aligned}$$

The $\Pi_{\mu\nu}$-term is neglected if the deformations are small. The quantity $\bar{k}_{\mu\nu}$ is the pull-back of the material metric when the material is in its unstrained state (i.e. when the $\pi^A = 0$ identically).

It should thus be clear that important information about the state of the system is the contained within the tensor

$$S_{\mu\nu} \equiv \frac{1}{2}\left(k_{\mu\nu} - \bar{k}_{\mu\nu}\right), \tag{6.50}$$

since it quantifies the difference between the actual value of $k_{\mu\nu}$ and its value in the fiducial state.

7 Final remarks

Recall from (5.6) that the action for the solid can be written as a tri-variant function whose arguments are the three independent scalar invariants formed from the mixed-components of the pulled-back material metric:

$$S = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{L}\left([\mathbf{k}], [\mathbf{k}^2], [\mathbf{k}^3]\right). \tag{7.1}$$

It has recently become popular to suggest that an observation of anisotropic stress would point towards modified gravity rather than dark energy [36–39]. What we are about to state is not a comment on a claim made by any of these articles, but it is worth pointing out. Whilst it is true that modified gravity models have anisotropic stress, it is also true that material models can contribute towards anisotropic stress. Infact, material models constitute the simplest and physically "most intuitive" additions to the Einstin-Hilbert and standard matter content gravitational model.

Derivative counting The manner in which we have presented the construction of the equation of state (i.e., the action) of a material somewhat appeared to rely on some intuition. This can be a help or hinderance, depending on ones perspective. Another way, which is rather popular, to construct Lagrangians is by counting the number of space-time derivatives.

Suppose we performed book-keeping of terms in the Lagrangian by keeping track of the number of derivatives. The pulled-back material metric $k^{\mu}{}_{\nu}$ has two space-time derivatives, and so the n^{th}-trace has $2n$-inverse-powers of a mass-scale M,

$$[\mathbf{k}^n] \sim \frac{1}{M^{2n}}. \tag{7.2}$$

We can use this to write the complete list of terms in the solid-action which come with a given power of M. Up to sixth-order the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \frac{1}{M^2} c_{00}\,[\mathbf{k}] + \frac{1}{M^4}\left(c_{11}\,[\mathbf{k}^2] + c_{12}\,[\mathbf{k}]^2\right) \\ &\quad + \frac{1}{M^6}\left(c_{31}\,[\mathbf{k}^3] + c_{32}\,[\mathbf{k}][\mathbf{k}^2] + c_{33}\,[\mathbf{k}]^3\right) + \ldots. \end{aligned} \tag{7.3}$$

We have introduced the dimensionless coefficients, $\{c_{ij}\}$, to control the strength which which a given term influences the action.

Solid $\to$ fluid $\to$ Λ The theory, as constructed, is the general description for a perfect solid. The particular solid under examination is determined by the dependance of ϵ on its arguments. This is simple to impose, has interesting consequences, and the physics alters in a meaningful way. For example, when

$$\epsilon\left(n, \mathfrak{U}^{\mu}{}_{\nu}\right) \longrightarrow \epsilon(n), \tag{7.4}$$

the anisotropic stress tensor (3.32b) vanishes, and one is left with a theory which describes a perfect fluid. Furthermore, when

$$\epsilon(n) = \frac{\epsilon_0}{n}, \tag{7.5}$$

where ϵ_0 is a constant, one finds that $\rho = -p$. That is, the theory is that of a cosmological constant (the Lagrangian is just $\mathcal{L} = \epsilon_0$, a constant).

Part II
Applications, examples, and extensions

7.1 Exact non-linear equations of motion

This is based on the discussion in [5], and the aim is to compute the equations of motion for General Relativity sourced by a non-linear elastic solid.

Start off with the matter flow velocity

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \tag{7.6}$$

normalised via

$$u^a u_a = -c^2. \tag{7.7}$$

Now introduce the flow gradient tensor v_{ab} via

$$u_{a;b} = v_{ab} - \tfrac{1}{c^2}\dot{u}_a u_b, \tag{7.8}$$

where

$$\dot{u}_a = u^b u_{a;b} \tag{7.9}$$

is the acceleration vector. The flow gradient tensor is orthogonal to the flow,

$$u^a v_{ab} = 0. \tag{7.10}$$

Let f_{ab} be an orthogonal covariant tensor. Then the Lie derivative of f_{ab} along the direction of the flow velocity is

$$\pounds_u f_{ab} = \gamma^c{}_a \gamma^d{}_b u^e f_{cd;e} + f_{cb} v^c{}_a + f_{ac} v^c{}_b. \tag{7.11}$$

One has

$$\pounds_u \gamma_{ab} = 2\theta_{ab}, \tag{7.12}$$

where θ_{ab} is related to the flow gradient v_{ab} via

$$v_{ab} = \theta_{ab} + \omega_{ab}, \tag{7.13}$$

with

$$\theta_{[ab]} = 0, \qquad \omega_{(ab)} = 0. \tag{7.14}$$

From the Ricci identity

$$u_{a;[b;c]} = \tfrac{1}{2} u_d R^d{}_{abc} \tag{7.15}$$

one obtains the Lie derivative of the flow gradient in the direction of u^a,

$$\pounds_u v_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{2}\dot{u}_a \dot{u}_b - u^c u^d R_{acbd}. \tag{7.16}$$

The anti-symmetric portion of (7.16) gives

$$\pounds_u \omega_{ab} = \gamma^c{}_a \gamma^d{}_b \dot{u}_{[c;d]}, \tag{7.17}$$

and the symmetric portion of (7.16) gives

$$\tfrac{1}{2}\pounds_u\pounds_u\gamma_{ab} = \gamma^c{}_a\gamma^d{}_b\dot{u}_{(c;d)} + v^c{}_a v_{cb} + \tfrac{1}{c^2}\dot{u}_a\dot{u}_b - u^c u^d R_{acbd}. \tag{7.18}$$

We also use the Weyl tensor

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}\left(R^{b]}{}_{d]} - \tfrac{1}{6}Rg^{b]}{}_{d]}\right). \tag{7.19}$$

The *relative strain tensor* is

$$e_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - \kappa_{ab}\right), \tag{7.20}$$

where the *strain reference tensor* is κ_{ab} and satisfies

$$\pounds_u\kappa_{ab} = 0. \tag{7.21}$$

Hence, from (7.12),

$$\theta_{ab} = \pounds_u e_{ab}, \tag{7.22}$$

meaning that the expansion tensor θ_{ab} quantifies the rate of relative strain. Carter sets $\kappa_{ab} = 0$. Hence

$$\begin{aligned}\pounds_u\pounds_u e_{ab} \quad = \quad & \gamma^c{}_a\gamma^d{}_b\dot{u}_{c;d} + v^c{}_a v_{cb} + \tfrac{1}{c^2}\dot{u}_a\dot{u}_b \\ & -u^c u^d C_{acbd} - \tfrac{1}{2}\gamma_{ab}\left(u^c u^d R_{cd} + \tfrac{1}{3}Rc^2\right) + \tfrac{1}{2}\gamma^c{}_a\gamma^d{}_b R_{cd}c^2.\end{aligned} \tag{7.23}$$

Now we set the energy-momentum tensor to be that for a perfect elastic solid,

$$T^{ab} = \rho u^a u^b + p^{ab}. \tag{7.24}$$

One can compute p^{AB} from the energy as a function of strain, $\epsilon(\gamma_{AB})$ via

$$p^{AB} = -\epsilon\gamma^{AB} - 2\frac{\partial\epsilon}{\partial\gamma_{AB}}. \tag{7.25}$$

This is valid for any linear or non-linear function of strain, for which the conservation law

$$T^{ab}{}_{;b} = 0 \tag{7.26}$$

holds.

We now specify the gravitational theory, which we take to be General Relativity for whom the Ricci tensor is given by

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \tfrac{1}{2}Tg_{ab}\right), \tag{7.27}$$

where

$$T = T^a{}_a = p^a{}_a - \rho c^2 \tag{7.28}$$

is the trace of the energy-momentum tensor. In General Relativity the Weyl tensor satisifies

$$C_{abcd} = C_{cdab} = C_{[ab][cd]}, \qquad C^a{}_{[bcd]} = 0, \qquad C^{ab}{}_{ac} = 0, \tag{7.29}$$

and the Bianchi identities impose

$$C^{abcd}{}_{;d} = \frac{8\pi G}{c^4}\left(T^{c[a;b]} - \tfrac{1}{3}g^{a[a}T^{;b]}\right). \tag{7.30}$$

We denote

$$C_{ab} = u^c u^d C_{acbd} \tag{7.31}$$

for the electric part of the the Weyl tensor.

From the conservation equation one obtains

$$\rho\dot{u}^a = -p^{ab}{}_{;b} + \tfrac{1}{c^2}u^a p^{bc}\theta_{bc}. \tag{7.32}$$

Hence (7.17) and (7.23) become

$$\pounds_u\omega_{ab} = \frac{1}{\rho^2}\gamma_{c[a}\gamma^d{}_{b]}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) + \frac{1}{\rho c^2}\omega_{ab}p^{cd}\pounds_u e_{cd}. \tag{7.33a}$$

$$\begin{aligned}\pounds_u\pounds_u e_{ab} \;=\; & \frac{1}{\rho^2}\gamma_{c(a}\gamma^d{}_{b)}\left(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}\right) - C_{ab} - \frac{4\pi G}{3}\rho\gamma_{ab} \\ & +\omega_{ca}\omega^c{}_b + 2\omega^c{}_{(a}\pounds_u e_{b)c} + g^{cd}\left(\pounds_u e_{ac}\right)\left(\pounds_u e_{bd}\right) + \frac{4\pi G}{c^2}\left(P_{ab} - \tfrac{2}{3}p^c{}_c\gamma_{ab}\right) \\ & +\frac{1}{c^2\rho^2}\left[\gamma_{ac}\gamma_{bd}p^{ce}{}_{;e}p^{df}{}_{;f} + \rho p^{cd}\left(\pounds_u e_{cd}\right)\left(\pounds_u e_{ab}\right)\right].\end{aligned} \tag{7.33b}$$

8 Application to general isotropic configurations of elastic solids

The aim of this section is to understand the physics of elastic solids in isotropic configurations. We will study (a) elastic stars, (b) stars immersed in an elastic solid. The former problem has been studied before, but under the guise of "anisotropic stars", without mention of the anisotropy coming from elasticity. The latter problem is relevant for understanding how an elastic "bath" could affect the properties of compact objects (specifically, we will be looking out for some analogue of a "screen").

See [11]. There are some axially symmetric solutions in [41–44].

8.1 Eigenvalue decomposition

Here we review the technology laid out in [6, 12, 45] for dealing with isotropic elastic solids. The main point is to identify the maximum number of eigenvalues and eigenvectors, and to use a common eigenvector basis with which space-time and material quantities can be expanded. In this section we make no assumption about symmetries like spherical, axial, or static; in the next section we will, and consequently a lot of the expressions simplify from the general case.

The eigenvalues of the pulled-back material metric $k^a{}_b$ are written as $n^2_{\mathbb{A}}$, where $\mathbb{A} = 1, 2, 3$ label each of the "principle" directions. The particle density n is given in terms of the eigenvalues as

$$n = n_1 n_2 n_3, \tag{8.1}$$

which are interpretable as the principle linear particle densities. In an orthonormal basis $\{e^a{}_{\mathbb{A}}\}$ it follows that the space-time metric decomposes as

$$g_{ab} = -u_a u_b + \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}}, \tag{8.2}$$

and the pulled-back material metric decomposes as

$$k_{ab} = \sum_{\mathbb{A}=1}^{3} n_{\mathbb{A}}^2 e_{a\mathbb{A}} e_{\mathbb{A}}. \tag{8.3}$$

Hence, $\frac{\partial}{\partial g^{ab}}$ acting on a quantity X which is a function of scalar invariants of $k^a{}_b$, is

$$\frac{\partial X}{\partial g^{ab}} = \frac{1}{2} \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}} n_{\mathbb{A}} \frac{\partial X}{\partial n_{\mathbb{A}}}. \tag{8.4}$$

Using this, the pressure tensor is given by

$$P_{ab} = \sum_{\mathbb{A}=1}^{3} P_{\mathbb{A}} e_{a\mathbb{A}} e_{b\mathbb{A}}, \tag{8.5}$$

in which the principle values of the pressure tensor are given by

$$P_{\mathbb{A}} = n n_{\mathbb{A}} \frac{\partial \epsilon}{\partial n_{\mathbb{A}}}. \tag{8.6}$$

Using $k^a{}_b = n^{2/3} \eta^a{}_b$ naturally splits the pressure tensor into the pressure scalar and anisotropic pressure as we now show. Denoting the eigenvalues of $\eta^a{}_b$ as α_μ^2, and are related to the n_μ via

$$\alpha_{\mathbb{A}} = \frac{n_{\mathbb{A}}}{n^{1/3}} = \left(\frac{z_{\mathbb{A}+2}}{z_{\mathbb{A}+1}}\right)^{1/3}, \tag{8.7}$$

in which

$$z_{\mathbb{A}} = \frac{n_{\mathbb{A}+1}}{n_{\mathbb{A}+2}} = \frac{\alpha_{\mathbb{A}+1}}{\alpha_{\mathbb{A}+2}}. \tag{8.8}$$

It follows that

$$n_{\mathbb{A}} \frac{\partial}{\partial n_{\mathbb{A}}} = n \frac{\partial}{\partial n} + z_{\mathbb{A}+2} \frac{\partial}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1} \frac{\partial}{\partial z_{\mathbb{A}+1}}, \tag{8.9a}$$

and furthermore that

$$\eta_{c\langle a} \frac{\partial}{\partial \eta^{b\rangle}{}_c} = \frac{1}{2} \sum_{\mathbb{A}=1}^{3} e_{a\mathbb{A}} e_{b\mathbb{A}} \left(z_{\mathbb{A}+2} \frac{\partial}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1} \frac{\partial}{\partial z_{\mathbb{A}+1}} \right). \tag{8.9b}$$

The principle pressures are given by the sum

$$p_{\mathbb{A}} = p + \pi_{\mathbb{A}}, \tag{8.10a}$$

where the eigenvalues $\pi_{\mathbb{A}}$ are

$$\pi_{\mathbb{A}} = n\left(z_{\mathbb{A}+2}\frac{\partial\epsilon}{\partial z_{\mathbb{A}+2}} - z_{\mathbb{A}+1}\frac{\partial\epsilon}{\partial z_{\mathbb{A}+1}}\right), \tag{8.10b}$$

which satisfy

$$\sum_{\mathbb{A}=1}^{3}\pi_{\mathbb{A}} = 0. \tag{8.11}$$

In terms of the $\pi_{\mathbb{A}}$, the anisotropic pressure tensor is

$$\pi_{ab} = \sum_{\mathbb{A}=1}^{3}\pi_{\mathbb{A}}e_{a\mathbb{A}}e_{b\mathbb{A}}. \tag{8.12}$$

8.2 Static spherically symmetric configurations

We continue to review [6, 12, 45], and use the technology outlined in the previous section to construct the relevant equations to describe static spherically symmetric configurations.

The metric for static spherically symmetric space-time decomposes as

$$g_{ab} = -u_a u_b + \gamma_{ab}, \tag{8.13a}$$

where the orthogonal metric γ_{ab} splits up into a "radial" vector and "angular" metric via

$$\gamma_{ab} = r_a r_b + t_{ab}. \tag{8.13b}$$

The velocity vector u_a, radial vector r_a, and totally orthogonal metric t_{ab} are given by

$$u_a = -e^{\nu(r)}\left(\mathrm{d}t\right)_a, \qquad r_a = e^{\lambda(r)}\left(\mathrm{d}r\right)_a, \qquad t_{ab} = r^2\left(\mathrm{d}\Omega^2\right)_{ab} \tag{8.13c}$$

and $\lambda(r)$ is specified by the "Schwarzschild mass" function $m(r)$ via

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r}. \tag{8.13d}$$

The gravitational field equations set the Einstein tensor equal to the usual form of the solid energy-momentum tensor

$$T_{ab} = \rho u_a u_b + P_{ab}, \tag{8.14a}$$

in which the only pressure tensor compatible with the spherical symmetry decomposes as

$$P_{ab} = p_{\mathrm{r}} r_a r_b + p_{\mathrm{t}} t_{ab}. \tag{8.14b}$$

One should interpret p_{r} as the radial pressure, and p_{t} as the tangential pressure.

In all generality (i.e., for any static spherically symmetric configuration) the Einstein equations $G^a{}_b = \kappa T^a{}_b$ for the metric (8.13) and energy-momentum tensor (8.14) are given by

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\,(r-2m)}, \tag{8.15a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2}\kappa r^2 \rho, \tag{8.15b}$$

$$\frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}r} = -\left(\rho + p_{\mathrm{r}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\,(r-2m)} + 6\frac{q}{r}, \tag{8.15c}$$

where the difference between the radial and tangential pressures is quantified via

$$q \equiv \tfrac{1}{3}\left(p_{\mathrm{t}} - p_{\mathrm{r}}\right). \tag{8.16}$$

Since the metric variable ν does not appear in (8.15b) or (8.15c), we do not need to consider it in what follows, if all we are interested in is the profiles of the elastic matter fields.

The crucial part of making the source to the field equations that due to an elastic solid, is to compute ρ, p_{r}, and p_{t} from the equation of state. For that we must decompose the material metric, and pull it back to space-time. Symmetry dictates that the equation of state ϵ has only two arguments,

$$\epsilon = \epsilon(n_{\mathrm{r}}, n_{\mathrm{t}}), \tag{8.17}$$

and the particle number density n is given by

$$n = n_{\mathrm{r}} n_{\mathrm{t}}^2. \tag{8.18}$$

Note that n_{r} and n_{t} are interpretable as the radial and tangential linear number densities. The material metric must have symmetries similar to the space-time metric to preserve isotropy; in the material space the material metric takes on the form

$$k_{AB} = \tilde{r}_A \tilde{r}_B + \tilde{t}_{AB}. \tag{8.19}$$

The basis vector and tangential tensor in the material manifold are given by

$$\tilde{r}_A = e^{\tilde{\lambda}}\left(\mathrm{d}\tilde{r}\right)_A, \qquad \tilde{t}_{AB} = \tilde{r}^2\left(\mathrm{d}\tilde{\Omega}^2\right)_{AB}. \tag{8.20}$$

The pulled back material metric is given by

$$k_{ab} = n_{\mathrm{r}}^2 r_a r_b + n_{\mathrm{t}}^2 t_{ab}, \tag{8.21}$$

where the two space-time components of the pulled-back material metric (8.21) are given by

$$n_{\mathrm{r}} = e^{\tilde{\lambda}-\lambda}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}, \qquad n_{\mathrm{t}} = \frac{\tilde{r}}{r}. \tag{8.22}$$

The mapping between the material and space-time manifolds is thus defined through the relationship between the radial coordinate in the material manifold, and that in space-time $\tilde{r} = \tilde{r}(r)$: this is entirely encapsulated by the two space-time functions n_{r} and n_{t}. The constant volume shear tensor, defined in (**??**) is given by

$$s_{ab} = \tfrac{1}{2}\left(\gamma_{ab} - n^{-2/3}k_{ab}\right), \tag{8.23}$$

where we used (2.33) to replace the uni-modular tensor η_{ab} with n and k_{ab}. Using (8.13b), (8.18), and (8.21) we thus obtain

$$s_{ab} = \tfrac{1}{2}\left(\left[1 - \left(\frac{n_{\mathrm{r}}}{n_{\mathrm{t}}}\right)^{4/3}\right] r_a r_b + \left[1 - \left(\frac{n_{\mathrm{t}}}{n_{\mathrm{r}}}\right)^{4/3}\right] t_{ab}\right). \tag{8.24}$$

Hence, we observe that $s_{ab} = 0$ when, and only when, the radial and tangential number densities are identical:

$$s_{ab} = 0 \qquad \Longleftrightarrow \qquad n_{\mathrm{r}} = n_{\mathrm{t}}. \tag{8.25}$$

Thus, $s_{ab} = 0$ when the solid "becomes" a fluid.

Rather than work with n_r and n_t, it is useful to work with the combinations

$$n \equiv n_{\mathrm{r}} n_{\mathrm{t}}^2 = \left(\frac{\tilde{r}}{r}\right)^3 z, \tag{8.26a}$$

$$z \equiv \frac{n_{\mathrm{r}}}{n_{\mathrm{t}}} = e^{\tilde{\lambda}-\lambda}\frac{r}{\tilde{r}}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}. \tag{8.26b}$$

One should keep in mind that n is the particle number density. Also, note that $z = 1$ when $n_{\mathrm{r}} = n_{\mathrm{t}}$: this is the fluid limit. In terms of (n, z) as defined in (8.26) the field equations (8.15b, 8.15c) can be written as

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2}\kappa r^2 \rho, \tag{8.27a}$$

$$\frac{\mathrm{d}n}{\mathrm{d}r} = \frac{n}{r\beta_{\mathrm{r}}}\left[-\left(\rho + p_{\mathrm{r}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r - 2m} + 6q + 3z\frac{\partial p_{\mathrm{r}}}{\partial z}\left(z e^{\lambda-\tilde{\lambda}} - 1\right)\right], \tag{8.27b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = z\left[\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r} - \frac{3}{r}\left(z e^{\lambda-\tilde{\lambda}} - 1\right)\right], \tag{8.27c}$$

in which $\tilde{\lambda}$ parameterizes the curvature of the material manifold (which can be set to zero), and where $\{\rho, p_{\mathrm{r}}, q, \beta_{\mathrm{r}}\}$ are given in terms of the two-parameter equation of state $\epsilon = \epsilon(n, z)$ via

$$\rho = n\epsilon, \qquad p_{\mathrm{r}} = n^2\frac{\partial \epsilon}{\partial n} - 2q, \tag{8.28a}$$

$$q = -\frac{1}{2}nz\frac{\partial \epsilon}{\partial z}, \qquad \beta_{\mathrm{r}} = n\frac{\partial p_{\mathrm{r}}}{\partial n} + z\frac{\partial p_{\mathrm{r}}}{\partial z}. \tag{8.28b}$$

Explicitly,

$$\beta_{\text{r}} = n\left(2n\frac{\partial\epsilon}{\partial n} + n^2\frac{\partial^2\epsilon}{\partial n^2} + 2z\frac{\partial\epsilon}{\partial z} + 2nz\frac{\partial^2\epsilon}{\partial n\partial z} + z^2\frac{\partial^2\epsilon}{\partial z^2}\right). \tag{8.29}$$

By way of a "fiducial" example, consider the quasi-Hookean ansatz for the equation of state,

$$\epsilon = \check{\epsilon} + \frac{1}{n}\check{\mu}s^2, \tag{8.30}$$

where a quantity with an overhead "check" symbol denotes that the quantity only depends on n, and

$$s^2 = \tfrac{1}{6}\left(z^{-1} - z\right)^2 \tag{8.31}$$

is the shear scalar we defined in (5.35), although there we used the symbol $\overline{s}^2$, and we have now expressed it in terms of z as defined in (8.26b). Note that in the fluid limit $z = 1$, the shear scalar $s^2 = 0$. The Einstein equations (8.27) can be formulated in the independant variables $(m, \check{p}, z)$, and become

$$\frac{\mathrm{d}m}{\mathrm{d}r} = \frac{1}{2}\kappa r^2\rho, \tag{8.32a}$$

$$\frac{\mathrm{d}\check{p}}{\mathrm{d}r} = \frac{\check{\beta}}{r\beta_{\text{r}}}\left[-\left(\rho + p_{\text{r}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\text{r}}}{r - 2m} + 6q + 4\left(ze^{\lambda-\tilde{\lambda}} - 1\right)\left(\check{\mu} + 3\sigma + \tfrac{3}{2}\left(1 - \check{\Omega}\right)q\right)\right], \tag{8.32b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{z}{r}\left[\frac{r}{\check{\beta}}\frac{\mathrm{d}\check{p}}{\mathrm{d}r} - 3\left(ze^{\lambda-\tilde{\lambda}} - 1\right)\right], \tag{8.32c}$$

in which

$$\sigma = \check{\mu}s^2, \qquad q = \check{\mu}\chi, \qquad \rho = \check{\rho} + \sigma, \tag{8.33a}$$

$$p_{\text{r}} = p - 2q, \qquad p = \check{p} + \left(\check{\Omega} - 1\right)\sigma, \tag{8.33b}$$

$$\chi = \tfrac{1}{6}\left(z^{-2} - z^2\right), \qquad \check{\beta} = \left(\check{\rho} + \check{p}\right)\frac{\mathrm{d}\check{p}}{\mathrm{d}\check{\rho}}, \qquad \check{\Omega} = \frac{\check{\beta}}{\check{\mu}}\frac{\mathrm{d}\check{\mu}}{\mathrm{d}\check{p}}, \tag{8.33c}$$

$$\beta_{\text{r}} = \beta + 4\left[\sigma + \left(\check{\Omega} - \frac{1}{2}\right)\right], \qquad \beta = \check{\beta} + \frac{4}{3}\check{\mu} + \left[\check{\Omega}\left(\check{\Omega} - 1\right) + \check{\beta}\frac{\mathrm{d}\check{\Omega}}{\mathrm{d}\check{p}}\right]. \tag{8.33d}$$

One now requires two functions of state to complete the system of equations: $\check{\rho}(\check{p})$ and $\check{\mu}(\check{p})$. One may be slightly uncomfortable with this idea: if that is the case, then one should recall that in the more familiar "scalar field models" one needs to pick parameterizations of the Lagrangian density (even the canonical theory has one free function, the potential $V(\phi)$). An explicit relationship between $\check{\rho}$, $\check{\mu}$ and $\check{p}$ is provided by

$$\check{\rho} = \frac{p_{\text{c}}}{\check{\Gamma} - 1}\left[w\frac{\check{p}}{p_{\text{c}}} + \left(\frac{\check{p}}{p_{\text{c}}}\right)^{1/\check{\Gamma}}\right], \qquad \check{\mu} = k\check{p}. \tag{8.34}$$

This is a modified polytropic equation of state. The parameter p_{c} indicates the pressure scale at which the transition between the linear and polytropic behavior occurs. The parameter k quantifies the rigidity, and vanishes in the fluid limit.

8.2.1 My spherical symmetry calculations

Here we use the metric

$$\mathrm{d}s^2 = -e^{2\nu(r)}\mathrm{d}t^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{8.35}$$

and the total energy-momentum tensor

$$T^a{}_b = \mathrm{diag}\left(-\rho(r), p_{\mathrm{r}}(r), p_{\mathrm{t}}(r), p_{\mathrm{t}}(r)\right). \tag{8.36}$$

The components are: energy density ρ, radial pressure p_{r}, and tangential pressure p_{t}. One should bear in mind that these components could be formed from multiple constituents (although we will come back to that case later on).

The non-zero components of the Einstein tensor computed from the metric (8.35) are given by

$$G^0{}_0 = -\frac{2m'}{r^2}, \tag{8.37a}$$

$$G^r{}_r = -\frac{2}{r^3}\left(m + r\left(2m - r\right)\nu'\right) \tag{8.37b}$$

$$\begin{aligned} G^\theta{}_\theta = G^\phi{}_\phi \quad &= \quad \frac{1}{r^3}\bigg(- m\left(-1 + r\nu' + 2r^2\nu'^2 + 2r^2\nu''\right) \\ &\quad +r\left(-m'\left(1 + r\nu'\right) + r\left(\nu' + r\nu'^2 + r\nu''\right)\right)\bigg). \end{aligned} \tag{8.37c}$$

The 00- and rr-Einstein field equations $G^a{}_b = \kappa T^a{}_b$ thus read

$$m' = \tfrac{1}{2}\kappa r^2\rho, \tag{8.38a}$$

$$\nu' = \frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\left(r - 2m\right)}. \tag{8.38b}$$

The only non-zero entry of the conservation equation $\nabla_a T^a{}_b = 0$ is

$$p_{\mathrm{r}}' = -\left(\rho + p_{\mathrm{r}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\mathrm{r}}}{r\left(r - 2m\right)} + \frac{2}{r}\left(p_{\mathrm{t}} - p_{\mathrm{r}}\right). \tag{8.38c}$$

The system of three equations (8.38) constitute the *generalized Tolman-Oppenheimer-Volkov* equations, where a star is constructed from an anisotropic fluid. Notice that there are three equations for five unknowns (i.e., $m(r), \nu(r), \rho(r), p_{\mathrm{r}}(t)$, and $p_{\mathrm{t}}(r)$). And so, two "equations of state" must be given: these are typically taken to relate the pressures to the density. Note that the metric variable ν does not appear in (8.38a) or (8.38c): it does not need to be solved for at the same time as m and p_{r} (this is statement which helps computations).

There are some simple circumstances under which the generalized TOV equations become analytically soluble [46]. When the density is taken to be constant throughout the star, $\rho = \rho_0$, it is simple to integrate (8.38a) to obtain

$$m(r) = \frac{4\pi G}{3}\rho_0 r^3. \tag{8.39}$$

We now assume an equation of state (which is the final piece of information required to close the system of equations) in the form

$$p_{\mathrm{t}} = p_{\mathrm{r}} + Af(p_{\mathrm{r}}, r)\,(\rho + p_{\mathrm{r}})\,r^n, \tag{8.40a}$$

with the function f given by

$$f(p_{\mathrm{r}}, r) = \frac{\rho + 3p_{\mathrm{r}}}{1 - 2m/r}, \tag{8.40b}$$

and where there are two free parameters, A and n. Using this, (8.38c) can be written as

$$p_{\mathrm{r}}' = -\left(\rho_0 + p_{\mathrm{r}}\right)\left(\rho_0 + 3p_{\mathrm{r}}\right)\frac{\frac{4\pi G}{3} - 2Ar^{n-2}}{1 - \frac{8\pi G\rho_0}{3}r^2}r. \tag{8.41}$$

When $n = 2$, this can be integrated to give

$$p_{\mathrm{r}}(r) = \rho_0\left[\frac{(1-2m/r)^Q - (1-2M/R)^Q}{3\,(1-2M/R)^Q - (1-2m/r)^Q}\right] \tag{8.42}$$

with

$$p_{\mathrm{r}}(r=R) = 0, \qquad m(R) = M, \qquad Q = \frac{1}{2} - \frac{3A}{4\pi G}. \tag{8.43}$$

The pressure at the centre of the configuration is given by

$$p_{\mathrm{c}} = \rho_0\frac{1-(1-2M/R)^Q}{3\,(1-2M/R)^Q - 1}. \tag{8.44}$$

The critical combination of M and R for which the central pressure becomes infinite is found by noting when the denominator of p_{c} becomes zero; i.e., when

$$(2M/R)_{\mathrm{crit}} = 1 - \left(\frac{1}{3}\right)^{1/Q}. \tag{8.45}$$

One requires $(2M/R)_{\mathrm{crit}} \geq 0$ for physical systems. This is known as *Buchdahl's inequality* [47, 48] (although it also exists for more general setups than that which we are considering here). In the perfect-fluid case there is no anisotropy, $A = 0$, and so $Q = 1/2$, and so

$$(2M/R)_{\mathrm{crit}} = 8/9. \tag{8.46}$$

When $A = 2\pi G/3$, one has $Q = 0$ and thus

$$(2M/R)_{\mathrm{crit}} = 1, \tag{8.47}$$

which is the Schwarzschild value.

Note that

$$\sqrt{-g} = \left(1 - \frac{2m}{r}\right)^{-1/2} e^{\nu} r^2 \sin\theta. \tag{8.48}$$

The total mass in a volume is given by

$$M \equiv \int \mathrm{d}^3x \sqrt{-g}\, m(r) \tag{8.49}$$

8.3 Stars immersed in an elastic solid

It should be clear from our presentation so far that an elastic solid is an anisotropic fluid. In [46, 49–52] anisotropic stars are studied. See also [53, 54], which study stars (i.e., solutions to the TOV equations) immersed in Chaplygin-gas; also, [55] study stars in a Chameleon scalar-field background. Also, [56] look at stars in anisotropic fluids, but in the context of wormholes. In non of these papers is any mention made of the link between anisotropy and elasticity.

We would like to understand

1. Stars in vacuum
2. Stars in fluids
3. Stars in solids

Point 1. is about solving the Tolman-Oppenheimer-Volkov equations (usually studied with a model neutron-star equation of state). The second is about solving the TOV equations, but in a non-empty space-time.

8.3.1 Example model

A simple example of a model of a star immersed in a "cosmological" background is

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_{\text{star}} + \mathcal{L}_{\text{cosm}}\right], \tag{8.50}$$

wherein R is the Ricci scalar, $\mathcal{L}_{\text{star}}$ is the Lagrangian density of the matter which gives rise to the "star", and $\mathcal{L}_{\text{cosm}}$ is the Lagrangian density of the "cosmological" dark energy. The field equations are given by

$$G^a{}_b = \kappa \left(T^a{}_b + U^a{}_b\right), \tag{8.51}$$

in which $T^a{}_b$ is the energy-momentum tensor for the "star Lagrangian", and $U^a{}_b$ is the energy-momentum tensor for the cosmological dark energy (to be in-keeping with some of our other work, we call $U^a{}_b$ the dark energy-momentum tensor). If the model is minimally coupled then the two energy-momentum tensors are separately conserved; else, they have a coupling current which transfers energy between the cosmological and stellar matter fields.

Typical examples of the dark energy Lagrangians are

$$\mathcal{L}_{\text{cosm}} \in \begin{cases} \text{scalar field}, \\ \text{solid}, \\ \text{perfect fluid}. \end{cases} \tag{8.52}$$

It is to be remarked that the perfect fluid is a particular limit of the solid (a solid with vanishing rigidity is a perfect fluid). In each case, the components of the dark energy-momentum tensor are given respectively by

$$U^a{}_b \in \begin{cases} \partial^a\phi\partial_b\phi + g^a{}_b\left(\frac{1}{2}\partial_c\phi\partial^c\phi - V(\phi)\right), \\ \rho u^a u_b + p\gamma^a{}_b + \pi^a{}_b, \\ \rho u^a u_b + p\gamma^a{}_b. \end{cases} \tag{8.53}$$

The total source to the gravitional field equations can be written as

$$T^a_{\text{tot}\,b} = T^a{}_b + U^a{}_b. \tag{8.54}$$

This will engender the following decomposition for the total energy density ρ^{tot}, the radial pressure $p_{\text{r}}^{\text{tot}}$, and tangential pressure $p_{\text{t}}^{\text{tot}}$ components,

$$\rho^{\text{tot}} = \rho^{\text{m}} + \rho^{\text{d}}, \tag{8.55a}$$

$$p_{\text{r}}^{\text{tot}} = p_{\text{r}}^{\text{m}} + p_{\text{r}}^{\text{d}} \tag{8.55b}$$

$$p_{\text{t}}^{\text{tot}} = p_{\text{t}}^{\text{m}} + p_{\text{t}}^{\text{d}} \tag{8.55c}$$

Components with an "m" super-script label correspond to the component of the stellar energy-momentum tensor, and those with a "d" super-script to the components of the dark energy-momentum tensor. This will mean that the field equations (in the minimally coupled case) are given by an appropriately modified version of (8.38). To be specific,

$$\nu' = \frac{m + \frac{1}{2}\kappa r^3 p_{\text{r}}^{\text{tot}}}{r\left(r - 2m\right)}, \tag{8.56a}$$

$$m' = \frac{1}{2}\kappa r^2 \rho^{\text{tot}}, \tag{8.56b}$$

$$p_{\text{r}}^{\text{i}\,\prime} = -\left(\rho^{\text{i}} + p_{\text{r}}^{\text{i}}\right)\frac{m + \frac{1}{2}\kappa r^3 p_{\text{r}}^{\text{tot}}}{r\left(r - 2m\right)} + \frac{2}{r}\left(p_{\text{t}}^{\text{i}} - p_{\text{r}}^{\text{i}}\right), \tag{8.56c}$$

where there is a copy of (8.56c) for each $\text{i} \in \{\text{m}, \text{d}\}$. For a perfect fluid model of the star, the matter components satisfy

$$p_{\text{r}}^{\text{m}} = p_{\text{t}}^{\text{m}}, \qquad p_{\text{r}}^{\text{m}}(R_0) = 0, \qquad \rho^{\text{m}}(r > R_0) = 0. \tag{8.57}$$

That is, the radial and tangential pressures are identical, the pressure vanishes at the radius R_0, which is supposed to be the surface of the star, and the stellar density vanishes outside of the star.

Integrating (8.56b) from $r = 0$ to $r = R_\infty$ yields a total mass M_∞, which can be broken up into three contributions as

$$M_\infty = M_\star^{\mathrm{m}} + M_\star^{\mathrm{d}} + M^{\mathrm{d}}, \tag{8.58}$$

where

$$M_\star^{\mathrm{m}} \equiv \frac{1}{2}\kappa \int_0^{R_0} \mathrm{d}r\, r^2\, \rho^{\mathrm{m}}, \tag{8.59a}$$

$$M_\star^{\mathrm{d}} \equiv \frac{1}{2}\kappa \int_0^{R_0} \mathrm{d}r\, r^2\, \rho^{\mathrm{d}}, \tag{8.59b}$$

$$M^{\mathrm{d}} \equiv \frac{1}{2}\kappa \int_{R_0}^{R_\infty} \mathrm{d}r\, r^2\, \rho^{\mathrm{d}} \tag{8.59c}$$

These can be interpreted as the mass of the star, the mass of the dark substance inside the star, and the mass of the dark substance outside of the star (i.e., in the cosmology).

9 Mixing solids, fluids, and scalar fields

We may also be interested in composite descriptions where there are solids interacting with scalar fields or fluids.

9.1 Solids and fluids

We are interested in constructing the action for a fluid coupled to a solid. It is useful to note that in Alkistis et al [13] consider a fluid coupled to a scalar field.

See [57], [58], [59], [60].

Suppose one had two fluids; each fluid have currents a^μ and b^μ. The allowed invariants are

$$a \equiv (-a^\mu a_\mu)^{1/2}, \qquad x \equiv (-a^\mu b_\mu)^{1/2}, \qquad b = (-b^\mu b_\mu)^{1/2}. \tag{9.1}$$

9.2 Solids and scalar fields

The theory of a solid which we have considered in this review was constructed from all invariants of the elements of the set $\{n, u^a, \eta^a{}_b\}$. In [13] a similar question was asked, but for a fluid: they asked for the general theory of a scalar+fluid mixture; thus, they found all invariants formed out of the relevant fields, which are $\{n, u^a, \phi, \nabla_a\phi\}$. There are at most four scalar invariants, and so the Lagrangian for a scalar+fluid mixture has at most four scalar arguments, and can be written as

$$\mathcal{L}_{\text{s+f}} = \mathcal{L}_{\text{s+f}}\left(n, \phi, u^a\nabla_a\phi, \nabla_a\phi\nabla^a\phi\right). \tag{9.2}$$

Suppose we now had a scalar field in the matter action: we now want to form all invariants from the set $\{n, u^a, \eta^a{}_b, \phi, \nabla_a\phi\}$. See Section 9.3 for a discussion on hyper-elasticity which may be more helpful.

For the solid+scalar mixture, there is a tower of invariants. For example, the first few are

$$\left\{\phi, \quad n, \quad u^a\nabla_a\phi, \quad \nabla^a\phi\nabla_a\phi, \quad [\boldsymbol{\eta}], \quad [\boldsymbol{\eta}^2], \quad \eta^a{}_b\nabla_a\phi\nabla^b\phi, \quad \eta^a{}_b\eta^b{}_c\nabla_a\phi\nabla^c\phi, \quad \dots\right\} \tag{9.3}$$

The first four above are present in the scalar+fluid mixture: the rest are only present in the scalar+solid mixture. To aid the construction of allowed invariants, it is useful to define the quantity

$$C^a{}_b{}^c \equiv \eta^a{}_b\nabla^c\phi. \tag{9.4}$$

Note that $u^bC^a{}_b{}^c = 0$; i.e., the tensor $C^a{}_b{}^c$ is orthogonal on its last two indices. The second useful step in cataloguing the terms is by keeping track of the number of space-time derivatives. This is done by replacing $x^a \to Mx^a$, and keeping track of the powers of M. First of all, note that the uni-modular tensor has 2-powers of M,

$$\eta_{ab} = n^{-2/3}k_{AB}\partial_a\phi^A\partial_b\phi^B \sim \frac{1}{M^2} \tag{9.5a}$$

The rank-3 tensor defined in (9.4) carries 3 powers of M,

$$C^{a}{}_{b}{}^{c} \sim \frac{1}{M^3}, \tag{9.5b}$$

the n^{th}-trace of $\boldsymbol{\eta}$ carries $2n$-powers of M,

$$[\boldsymbol{\eta}^n] \sim \frac{1}{M^{2n}}, \tag{9.5c}$$

and the canonical kinetic term for the scalar carries 2 powers of M,

$$2\mathcal{X} \equiv \partial_a\phi\partial^a\phi \sim \frac{1}{M^2}. \tag{9.5d}$$

In addition, the contraction of the scalar field derivative with the time-like unit vector carries a single power of M,

$$\mathcal{Z} \equiv u^a\partial_a\phi \sim \frac{1}{M}. \tag{9.5e}$$

Using these countings, some of the first few invariants that can be formed from the solid+scalar mixture are

$$C^{ab}{}_{b}\partial_a\phi = 2\mathcal{X}[\boldsymbol{\eta}] \sim \frac{1}{M^4}, \qquad C^{ba}{}_{b}\partial_a\phi = \eta^{ab}\partial_a\phi\partial_b\phi \sim \frac{1}{M^4}, \tag{9.6a}$$

$$C^{abc}C_{abc} = 2\mathcal{X}[\boldsymbol{\eta}^2] \sim \frac{1}{M^6}, \qquad C^{abc}C_{bac} = \eta^{ac}\eta^{b}{}_{c}\partial_a\phi\partial_b\phi \sim \frac{1}{M^6}. \tag{9.6b}$$

Of course, this list is not exhaustive, but it nicely illustrate how some of the invariants constructed using C could be related to each other. The Lagrangian, built out of such considerations, will be of the form

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{M}c_{1,1}\mathcal{Z} \\
& + \frac{1}{M^2}\left(c_{2,1}\mathcal{Z}^2 + c_{2,2}\mathcal{X} + c_{2,3}[\boldsymbol{\eta}]\right) \\
& + \frac{1}{M^3}\left(c_{3,1}\mathcal{Z}^3 + c_{3,2}\mathcal{Z}\mathcal{X} + c_{3,3}\mathcal{Z}[\boldsymbol{\eta}]\right) \\
& + \frac{1}{M^4}\left(c_{4,1}\mathcal{Z}^4 + c_{4,2}\mathcal{Z}^2\mathcal{X} + c_{4,3}\mathcal{Z}^2[\boldsymbol{\eta}] + c_{4,4}\mathcal{X}^2\right. \\
& \qquad \left. + c_{4,5}\mathcal{X}[\boldsymbol{\eta}] + c_{4,6}[\boldsymbol{\eta}]^2 + c_{4,7}[\boldsymbol{\eta}^2] + c_{4,8}\eta^{ab}\partial_a\phi\partial_b\phi\right) + \ldots
\end{aligned} \tag{9.7}$$

where the coefficients $c_{I,J} = c_{I,J}\left(n,\phi\right)$ are explicitly constructed without any derivatives, and the ellipses stand for terms in higher powers of M.

Then we can form

$$\mathcal{I} = \left\{C^{a}{}_{a}{}^{c}\nabla_c\phi, \qquad \ldots\right\}. \tag{9.8}$$

We cannot (or at least, we are not aware of how to) construct all invariants out of these fields. Infact, it does not seem that this construction is the most elegant one can envisage – for that one should turn to our review of hyper-elasticity, which we save for the next subsection. However, we can still obtain field equations. Let us begin from a statement about the field content of the scalar+solid theory

$$\mathcal{L} = \mathcal{L}(g_{ab}, n^a, \eta^a{}_b, \phi, \partial_a \phi). \tag{9.9}$$

In a cosmological context, one can isolate the projections of the relevant quantities, at the level of the background:

$$g_{ab} = \gamma_{ab} - u_a u_b, \qquad \partial_a \phi = -\dot{\phi} u_a, \qquad n^a = n u^a, \qquad \eta^a{}_b = \gamma^a{}_c \gamma_b{}^d \eta^c{}_d. \tag{9.10a}$$

Note that by orthogonality

$$\eta^a{}_b \partial_a \phi = n^a \eta^b{}_a = 0. \tag{9.10b}$$

Also, on the background, the uni-modular tensor $\eta^a{}_b$ is strictly isotropic, and so completely decomposes as

$$\eta^a{}_b = \omega \gamma^a{}_b. \tag{9.10c}$$

Thus,

$$[\boldsymbol{\eta}] = 3\omega, \qquad [\boldsymbol{\eta}^2] = 3\omega^2. \tag{9.10d}$$

By combining (9.10) we find that on the cosmological background there are only 5 scalar invariants, and so the Lagrangian for the background is a function with 5 arguments:

$$\mathcal{L} = \mathcal{L}\left(\phi, n, \omega, u^a \partial_a \phi, \partial^a \phi \partial_a \phi\right). \tag{9.11}$$

The variation in the function $\mathcal{L}$ is given by

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial g_{ab}} \delta_{\mathrm{L}} g_{ab} + \frac{\partial \mathcal{L}}{\partial n^a} \delta_{\mathrm{L}} n^a + \frac{\partial \mathcal{L}}{\partial \eta^a{}_b} \delta_{\mathrm{L}} \eta^a{}_b + \frac{\partial \mathcal{L}}{\partial \phi} \delta_{\mathrm{L}} \phi + \frac{\partial \mathcal{L}}{\partial \partial_a \phi} \partial_a \delta_{\mathrm{L}} \phi. \tag{9.12}$$

Since

$$\delta_{\mathrm{L}} n^a = -\tfrac{1}{2} n^a g^{cd} \delta_{\mathrm{L}} g_{cd}, \tag{9.13}$$

one can find

$$\delta_{\mathrm{E}} n^a = -\tfrac{1}{2} n^a g^{cd} \delta_{\mathrm{E}} g_{cd} + n^b \nabla_b \xi^a - \nabla_b \left(n^a \xi^b\right) \tag{9.14}$$

Note that

$$\mathcal{L} = \mathcal{L}(\phi, \partial_a \phi, k^a{}_b) \tag{9.15}$$

9.3 Hyper-elasticity as a road to mixing scalars and solids

The idea of hyper-elasticity [18] may give an elegant insight as to how to incorporate general coupling between a solid and a scalar field.

9.3.1 Hyper-elasticity

Suppose the theory is constructed from a Lagrangian density $\mathcal{L}$ formed as a general function of (p + 1)-scalar fields and their space-time gradients,

$$\mathcal{L} = \mathcal{L}\left(\phi^0, \phi^1, \ldots, \phi^{\mathrm{p}}, \partial_a\phi^0, \partial_a\phi^1, \ldots, \partial_a\phi^{\mathrm{p}}\right). \tag{9.16}$$

The derivatives are with respect to worldsheet coordinates $\overline{x}^a$, with $a = 0, \ldots, \mathrm{p}$. The background has a metric $g_{\mu\nu}$ with $\mu = 0, \ldots, \mathrm{p} + q$. Note that the worldsheet has co-dimension q. The background induces a metric on the worldsheet

$$\overline{g}_{ab} = g_{\mu\nu} x^{\mu}{}_{,a} x^{\nu}{}_{,b}. \tag{9.17}$$

The determinant $\overline{g}$ of the worldsheet metric is used as the measure with which to integrate the Lagrangian density to give the action:

$$S = \int \mathrm{d}^{\mathrm{p}+1}\overline{x}\sqrt{|\overline{g}|}\,\mathcal{L}. \tag{9.18}$$

The worldsheet metric must have Lorentzian signature, and the field gradients must be independant. This means that the fields can be used as a set of coordinates on the worldsheet,

$$\overline{x}^a = \phi^a, \tag{9.19}$$

and thus the Lagrangian density $\mathcal{L}$ will depend on just the undifferentiated values ϕ^a and on the set of induced metric components $\overline{g}_{ab}$ (there will be $\frac{1}{2}\mathrm{p}(\mathrm{p}+1)$ of these).

The worldsheet energy-momentum tensor will always be given by

$$T^{ab} = \frac{2}{\sqrt{-\overline{g}}}\frac{\partial\sqrt{-\overline{g}}\,\mathcal{L}}{\partial\overline{g}_{ab}}, \tag{9.20}$$

and the corresponding hyper-elasticity tensor on the world-sheet is

$$\mathfrak{C}^{abcd} = \frac{1}{\sqrt{-\overline{g}}}\frac{\partial\sqrt{-\overline{g}}\,T^{ab}}{\partial\overline{g}_{cd}}. \tag{9.21}$$

We should note that the hyper-elasticity tensor is related to the elasticity tensor E^{abcd}, but it is not exactly the same. There is a minor, and one major difference. First, there is a multiplicative factor of "-2" that distinguishes them, $\mathfrak{C}^{abcd} \sim -2E^{abcd}$. Secondly, the elasticity tensor is purely orthogonal (i.e., purely spatial in the sense that $u_a E^{abcd} = 0$), whereas the hyper-elasticity tensor is not. These worldsheet tensors can be pulled-back to give background space-time tensors via

$$T^{\mu\nu} = T^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b} = 2\frac{\partial\mathcal{L}}{\partial\overline{g}_{\mu\nu}} + \mathcal{L}\overline{g}^{\mu\nu}, \tag{9.22a}$$

$$\mathfrak{C}^{\mu\nu\alpha\beta} = \mathfrak{C}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\alpha}{}_{,c} x^{\beta}{}_{,d} = \frac{\partial T^{\mu\nu}}{\partial\overline{g}_{\alpha\beta}} + \tfrac{1}{2}T^{\mu\nu}\overline{g}^{\alpha\beta}. \tag{9.22b}$$

The first fundamental tensor of the worldsheet, $\overline{g}^{\mu\nu}$, is constructed from the induced metric $\overline{g}^{ab}$ via

$$\overline{g}^{\mu\nu} = \overline{g}^{ab} x^{\mu}{}_{,a} x^{\nu}{}_{,b}. \tag{9.23}$$

When the codimension $q = 0$, $\overline{g}^{\mu\nu} = g^{\mu\nu}$, and $\overline{g}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$. In general, $\overline{g}^{\mu}{}_{\nu}$ is a projector, giving rise to a worldsheet derivative operator

$$\overline{\nabla}_{\nu} = \overline{g}^{\mu}{}_{\nu} \nabla_{\mu}. \tag{9.24}$$

In addition to the energy-momentum, first fundamental, and hyper-elasticity tensors, there is another tensor which plays an important role in governing the dynamics: the hyper-Hadamard tensor, given by

$$\mathfrak{H}^{\mu\nu\rho\sigma} = g^{\mu\rho} T^{\nu\sigma} + 2\mathfrak{C}^{\mu\nu\rho\sigma}. \tag{9.25}$$

The standard decomposition of the metric into tangential and orthogonal parts

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \perp_{\mu\nu} \tag{9.26}$$

implies an associated decomposition of the hyper-Hadamard tensor into worldsheet tangential and orthogonal parts,

$$\mathfrak{H}^{\mu\nu\rho\sigma} = \overline{\mathfrak{H}}^{\mu\nu\rho\sigma} + {}^{\perp}\mathfrak{H}^{\mu\nu\rho\sigma}. \tag{9.27}$$

The worldsheet tangential part is

$$\overline{\mathfrak{H}}^{\mu\nu\rho\sigma} = \overline{\mathfrak{H}}^{abcd} x^{\mu}{}_{,a} x^{\nu}{}_{,b} x^{\rho}{}_{,c} x^{\sigma}{}_{,d}, \tag{9.28}$$

with the components of the worldsheet hyper-Hadamard tensor given by

$$\overline{\mathfrak{H}}^{abcd} = \overline{g}^{ac} T^{bd} + 2\mathfrak{C}^{abcd}. \tag{9.29}$$

The worldsheet orthogonal part is

$${}^{\perp}\mathfrak{H}^{\mu\nu\rho\sigma} = \perp^{\mu\rho} T^{\nu\sigma}. \tag{9.30}$$

The full space-time metric decomposes first into a piece which is confined to the world sheet, and one which is perpendicular; the confined piece then decomposes into the spatial part and time-like part in the usual fashion:

$$\begin{aligned} g_{\mu\nu} &= \overline{g}_{\mu\nu} + \perp_{\mu\nu}, &(9.31)\\ &= \gamma_{\mu\nu} - u_{\mu} u_{\nu} + \perp_{\mu\nu}. &(9.32) \end{aligned}$$

Now that we have established a technology for dealing with these hyper-elastic materials, we can move back to the main goal of describing what we actually mean by hyper-elastic. For the system to "be" classified as hyper-elastic, the system should include a sub-system of ordinary elastic type. That is, all scalar fields except one (ϕ^0, say) only have space-like gradients; i.e., their configuration gradients satisfy the orthogonality condition (2.3). What this means is that

$$u^a \phi^A{}_{,a} = 0, \qquad \text{with} \qquad A = 1, \ldots, \mathrm{p}, \tag{9.33a}$$

with $\overline{g}_{ab}u^a u^b = -1$, but

$$u^a \phi^0{}_{,a} \neq 0. \tag{9.33b}$$

We will use this notation: the small letters "$a, b, \ldots$" denote arbitrary worldsheet coordinates, but the larger letters "$A, B, \ldots$" denote the coordinate system specified in terms of (9.19) by

$$\overline{x}^0 = \phi^0, \qquad \overline{x}^A = \phi^A. \tag{9.34}$$

Using (9.34) it is apparent that there are three "types" of components of the induced metric (9.17):

$$\overline{g}^{00} = \overline{g}^{ab}\mu_a\mu_b, \tag{9.35a}$$

$$\overline{g}^{A0} = \overline{g}^{ab}\psi^A{}_a\mu_b, \tag{9.35b}$$

$$\overline{g}^{AB} = \overline{g}^{ab}\psi^A{}_a\psi^B{}_b. \tag{9.35c}$$

we set

$$\mu_a = \phi^0{}_{,a}, \tag{9.36a}$$

$$\psi^A{}_a = \phi^A{}_{,a}. \tag{9.36b}$$

These are the components of the induced metric on which the Lagrangian $\mathcal{L}$ can depend. The Lagrangian $\mathcal{L}$ will be a function of

$$\mathcal{L} = \mathcal{L}\left(\overline{g}^{00}, \overline{g}^{A0}, \overline{g}^{AB}\right). \tag{9.37}$$

Note that (9.33a) implies that

$$u^a\psi^A{}_a = 0. \tag{9.38}$$

One can perhaps recognise μ_a as defined in (9.36a) as a space-time covector whose components are the space-time derivatives of a scalar, $\mu_a = \partial_a\phi^0$. We will explicitly refer to this interpretation later on, but also note that $\overline{g}^{00}$ as defined in (9.35a) corresponds to the "kinetic term" which would appear in the canonical action for the scalar field ϕ^0; $\overline{g}^{00} = \mu^a\mu_a = \partial^a\phi^0\partial_a\phi^0$.

Since $\overline{g}^{AB}$ are the components of a p × p matrix there are p scalar invariants that can be formed from $\overline{g}^{AB}$ alone. There is one invariant that can be formed from $\overline{g}^{A0}$ alone, and $\overline{g}^{00}$ is already an invariant. This is just something to bear in mind before we explicitly give the catalogue of invariants.

By (9.33a), it follows that

$$\overline{g}^{AB} = \gamma^{ab}\psi^A{}_a\psi^B{}_b = \gamma^{AB}, \tag{9.39}$$

after writing the worldsheet metric in the usual way,

$$\overline{g}^{ab} = \gamma^{ab} - u^a u^b. \tag{9.40}$$

Since the Lagrangian only depends on the field gradients $\phi^a{}_{,b}$ via the induced metric components, the generic variation in the Lagrangian density (9.16) will be given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + L_{ab}\delta\overline{g}^{ab}. \tag{9.41}$$

in which we defined the partial derivatives of $\mathcal{L}$ with respect to $\overline{g}^{ab}$ as

$$L_{ab} = \frac{\partial\mathcal{L}}{\partial\overline{g}^{ab}}. \tag{9.42a}$$

The contravariant components of L^{ab} are given by

$$L^{ab} = -\frac{\partial\mathcal{L}}{\partial\overline{g}_{ab}}. \tag{9.42b}$$

The tensor L_{ab} is a worldsheet tensor. After separating out the coordinates (9.34), the variation (9.41) becomes

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^0}\delta\phi^0 + \frac{\partial\mathcal{L}}{\partial\phi^A}\delta\phi^A + L_{00}\delta\overline{g}^{00} + 2L_{A0}\delta\overline{g}^{A0} + L_{AB}\delta\overline{g}^{AB}. \tag{9.43}$$

These are the required ingredients for computing the energy-momentum tensor (9.20). One finds

$$T^{ab} = -2L^{ab} + \overline{g}^{ab}\mathcal{L}. \tag{9.44}$$

One should note that the energy-momentum tensor (9.44) is not of the form of a perfect solid which we gave in (3.26a). Writing the energy-momentum tensor in terms of the energy density ρ, heat flux q^a, and pressure tensor p^{ab} without loss of generality,

$$T^{ab} = \rho u^a u^b + 2q^{(a}u^{b)} + p^{ab} \tag{9.45}$$

one obtains

$$\rho = -\left(\mathcal{L} + 2u_a u_b L^{ab}\right), \tag{9.46a}$$

$$q^a = \gamma^a{}_b u_c L^{bc}, \tag{9.46b}$$

$$p^{ab} = \gamma^{ab}\mathcal{L} - 2\gamma^a{}_c\gamma^b{}_d L^{cd}. \tag{9.46c}$$

The existence of the L_{A0} term in (9.43) gives rise to the heat flux contribution (9.46b). Later on we will give an example where the Lagrangian density separates into "scalar" and "solid" terms with no interaction between the two: this will switch off the heat flux. The L_{00} term also gives rise to the extra contribution to the energy density (9.46a) compared to the equivalent expression for a perfect solid which we gave in (5.21a).

We can use this to compute the class of forces to the conservation equations due to the explicit mixing term, L_{A0}. Suppose that we write the energy-momentum tensor (9.44) as

$$T^{ab} = \breve{T}^{ab} + \widehat{T}^{ab}, \tag{9.47}$$

in which the first term includes the pure time-like and pure space-like contributions present from the solid and scalar without any mixing, and the second term is the term explicitly present due to their mixing:

$$\breve{T}^{ab} = \rho u^a u^b + p^{ab}, \tag{9.48a}$$

$$\widehat{T}^{ab} = 2q^{(a}u^{b)}. \tag{9.48b}$$

One of the important points we want to emphasise about this construction is that $\widehat{T}^{ab}$ contains the deviations of the full energy-momentum tensor T^{ab} from its perfect solid form (note that the perfect solid, fluid, and scalar field cases are contained within an expression of the form $\breve{T}^{ab}$). Then, the conservation of T^{ab} implies that the divergence of (9.48a) satisfies a forced conservation equation given by

$$\nabla_a \breve{T}^{ab} = f^b, \tag{9.49}$$

in which we defined the force to be proportional to the divergence of the mixing contribution to the overall energy-momentum tensor,

$$f^b \equiv -\nabla_a \widehat{T}^{ab}. \tag{9.50}$$

By computing the divergence of (9.48b) one finds

$$-f^b = u^b \overline{\nabla}_a q^a + \gamma^b{}_a \dot{q}^a + 2K^c{}_a \gamma^{(a}{}_c q^{b)}, \tag{9.51}$$

from which one obtains the two independent projections of the force:

$$u_a f^a = \overline{\nabla}_a q^a, \tag{9.52a}$$

$$\gamma^c{}_b f^b = -\gamma^c{}_a \dot{q}^a - 2K^d{}_a \gamma^{(c}{}_d q^{a)}. \tag{9.52b}$$

Notice that (9.52a) informs us that q^a is required to be space-varying for the time-like projection of the conservation of $\breve{T}^{ab}$ to be broken. We remind that q^a is defined from the L_{ab} via (9.46b).

To be able to compute the hyper-elasticity tensor (9.21) one also needs

$$L_{abcd} = \frac{\partial L_{ab}}{\partial \overline{g}^{cd}} = \frac{\partial^2 \mathcal{L}}{\partial \overline{g}^{ab} \partial \overline{g}^{cd}}, \tag{9.53}$$

which gives

$$\frac{\partial^2 \mathcal{L}}{\partial \overline{g}_{ab} \partial \overline{g}_{cd}} = L^{abcd} + L^{a(c} \overline{g}^{d)b} + \overline{g}^{a(c} L^{d)b}. \tag{9.54}$$

Using these ingredients, the hyper-elasticity tensor (9.21) is given by

$$\begin{aligned}\mathfrak{C}^{abcd} &= 2\left(L^{abcd} + L^{a(c}\overline{g}^{d)b} + \overline{g}^{a(c}L^{d)b}\right) \\ &\quad - \left(L^{ab}\overline{g}^{cd} + \overline{g}^{ab}L^{cd}\right) + \tfrac{1}{2}\mathcal{L}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right). \end{aligned} \tag{9.55}$$

Using (9.55) the hyper-Hadamard tensor (9.29) is given by

$$\overline{\mathfrak{H}}^{abcd} = 4L^{abcd} + 2L^{ac}\overline{g}^{bd} + 2L^{a[d}\overline{g}^{b]c} + 2\overline{g}^{a[d}L^{b]c} + L\overline{g}^{a[d}\overline{g}^{b]c}. \tag{9.56}$$

When we computed the generic variation of the Lagrangian density (9.41) we already used the fact that Lagrangian density only depends on the field gradients via the induced metric components. Reversing this gives the Eulerian variation of the Lagrangian, which is the variation with respect to fixed values of the background coordinates x^μ and metric $g_{\mu\nu}$. The variation is

$$\delta_{\mathrm{E}}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + J^a{}_b\delta\phi^b{}_{,a}, \tag{9.57}$$

where the currents are given in terms of the L_{ab}, defined in (9.42a), via

$$J^a{}_b = 2\overline{g}^{ac}L_{bd}\phi^b{}_{,c}. \tag{9.58}$$

The variational field equations are therefore

$$\overline{\nabla}_b J^b{}_a = \frac{\partial\mathcal{L}}{\partial\phi^a}. \tag{9.59}$$

If we again separate out the coordinates as in (9.34) then the currents (9.58) read

$$J^a{}_0 = 2\overline{g}^{ab}\left(L_{00}\phi^0{}_{,b} + L_{A0}\phi^A{}_{,b}\right), \tag{9.60a}$$

$$J^a{}_A = 2\overline{g}^{ab}\left(L_{A0}\phi^0{}_{,b} + L_{AB}\phi^B{}_{,b}\right). \tag{9.60b}$$

9.3.2 The symplectic current, and evaluation of sound speeds

Let $\delta x^\mu = \xi^\mu$ specify the background coordinate displacements, for whom conservation of the energy-momentum tensor is a consequence. Now suppose that $\delta x^\mu = \eta^\mu$ is another solution, for example that which results from a symmetry. Then the equation of motion of the generic perturbation vector ξ^μ is given by an expression

$$\overline{\nabla}_\mu\Omega^\mu = 0. \tag{9.61}$$

The symplectic current Ω^μ is given by

$$\Omega^\mu\{\xi,\eta\} = \eta^\mu\mathfrak{O}^\nu{}_\mu\{\eta\} - \xi^\mu\mathfrak{O}^\nu{}_\mu\{\eta\} \tag{9.62}$$

in which the "hyper-Hadamard operator" is defined via

$$\mathfrak{O}^\nu{}_\mu\{\xi\} = \mathfrak{H}_\mu{}^\nu{}_\rho{}^\sigma\overline{\nabla}_\sigma\xi^\rho, \tag{9.63}$$

and the hyper-Hadamard tensor is given in terms of the hyper-elasticity tensor via

$$\mathfrak{H}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} = g_{\mu\rho}T^{\nu\sigma} + 2\mathfrak{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} \tag{9.64}$$

Consider the worldsheet hypersurface which has normal vector

$$\lambda_a = \lambda_\mu x^\mu{}_{,a}, \qquad \lambda^\mu \perp^\mu{}_\nu = 0. \tag{9.65}$$

Then the characteristic equation governing the second derivative of the perturbation vector ξ^μ is given by

$$\left[\overline{\nabla}_\mu \overline{\nabla}_\nu \xi^\rho\right]_-^+ = \lambda_\mu \lambda_\nu \zeta^\rho, \tag{9.66}$$

in which ζ^μ is a measure of the discontinuity whose speed we are about to measure: these are the speeds of the wavefronts. The discontinuity of the divergence of the symplectic current is

$$\left[\overline{\nabla}_\mu \Omega^\mu \left\{\xi, \eta\right\}\right]_-^+ = \eta^\mu \mathfrak{H}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} \lambda_\nu \lambda_\sigma \zeta^\rho. \tag{9.67}$$

When $\overline{g}_{\mu\nu}\zeta^\nu = 0$, i.e., ζ^μ is worldsheet orthogonal, then the characteristic vector λ_μ must be a null eigenvector of $T^{\mu\nu}$:

$$\lambda_\mu \lambda_\nu T^{\mu\nu} = 0. \tag{9.68}$$

When ζ^μ is tangential to the worldsheet, the characteristic equation is expressible in terms of worldsheet tensors via

$$\zeta^\mu = \zeta^a x^\mu{}_{,a}, \tag{9.69}$$

with

$$\mathcal{Q}_{ab}\zeta^b = 0 \tag{9.70}$$

and where $\mathcal{Q}_{ab}$ is called the characteristic matrix, defined via

$$\mathcal{Q}_{ac} = \overline{\mathfrak{H}}_a{}^b{}_c{}^d \lambda_b \lambda_d. \tag{9.71}$$

The condition for ζ^b to be an intrinsic characteristic vector is therefore that $\det(\mathcal{Q}_{ab}) = 0$. Using (9.56) for the hyper-Hadamard tensor, the characteristic matrix is given by

$$\mathcal{Q}_{ac} = 2\left(L_{ac}\overline{g}^{bd} + 2L_a{}^b{}_c{}^d\right)\lambda_b \lambda_d. \tag{9.72}$$

9.3.3 Separable case

A simplified example is where the Lagrangian splits as

$$\mathcal{L}(\phi^0, \dots, \phi^p, \phi^0{}_{,a}, \dots, \phi^p{}_{,a}) = \mathcal{L}_{\mathrm{s}}(\phi^0, \phi^0{}_{,a}) + \mathcal{L}_{\mathrm{e}}(\phi^1, \dots, \phi^p, \phi^1{}_{,a}, \dots, \phi^p{}_{,a}). \tag{9.73}$$

That is, $\mathcal{L}_{\mathrm{s}}$ is only dependent on ϕ^0 and its space-time gradient (which includes its time-like gradient), and $\mathcal{L}_{\mathrm{e}}$ is only dependent of the scalars which don't have a time-like gradient. This restruction makes $\mathcal{L}_{\mathrm{e}}$ the Lagrangian density for an elastic solid. With this setup, $\mathcal{L}_{s}$ is the Lagrangian for a "normal" scalar field (generically of k-essence type), since it depends only on ϕ^0 and

$$\mu_a = \phi^0{}_{,a}, \tag{9.74}$$

the gradient 1-form. Hence, (9.35a) becomes

$$\overline{g}^{00} = \overline{g}^{ab}\mu_a\mu_b = -\mu^2. \tag{9.75}$$

This is just the "kinetic scalar" corresponding to the scalar field ϕ^0. Thus, in a more familiar language, we are asking for the Lagrangian density (9.73) to split as

$$\mathcal{L} = \mathcal{L}_{s}\left(\phi^0, -\mu^2\right) + \mathcal{L}_{e}\left(\phi^A, \overline{g}^{AB}\right). \tag{9.76}$$

The first term is the general Lagrangian for a k-essence scalar field theory, and the second is a general elastic solid Lagrangian of the type we have discussed throughout the entire review.

Since we are working in the separable case, neither $\mathcal{L}_{s}$ or $\mathcal{L}_{e}$ will depend on $\overline{g}^{0A}$; this means that the "entrainment" effect vanishes, and

$$L_{A0} = 0 \tag{9.77}$$

in (9.43). For the other terms in the variation of the Lagrangian (9.43) one obtains

$$L_{00} = \frac{\partial \mathcal{L}}{\partial \overline{g}^{00}} = -\frac{\partial \mathcal{L}_{s}}{\partial \mu^2}, \tag{9.78a}$$

and

$$L_{AB} = \frac{\partial \mathcal{L}}{\partial \overline{g}^{AB}} = \tfrac{1}{2}\left(\mathcal{L}_{e}\gamma_{AB} - P_{AB}\right), \tag{9.78b}$$

where P_{AB} is the pressure tensor of the medium, definable as

$$P^{AB} = \frac{2}{\sqrt{|\gamma|}}\frac{\partial\sqrt{|\gamma|}\mathcal{L}_{e}}{\partial\gamma_{AB}}, \tag{9.79}$$

in clear analogy with the worldsheet energy-momentum tensor (9.20). Note that we have used (9.39) to set $\overline{g}_{AB} = \gamma_{AB}$. One then obtains the elasticity tensor

$$E^{AB}{}_{CD} = 2\frac{\partial P^{AB}}{\partial \overline{g}^{CD}} - P^{AB}\gamma_{CD}. \tag{9.80}$$

Hence, since $P_{ab} = P_{AB}\phi^A{}_{,a}\phi^B{}_{,b}$, one obtains from (9.78b) the contravariant components of the pressure tensor on the worldsheet in arbitrary coordinates,

$$P^{ab} = \mathcal{L}_{e}\gamma^{ab} - 2\gamma^{ac}\gamma^{bd}L_{cd}, \tag{9.81}$$

and the elasticity tensor

$$E^{abcd} = \left(\gamma^{ab}\gamma^{cd} - 2\gamma^{a(c}\gamma^{d)b}\right)\mathcal{L}_{e} + P^{a(c}\gamma^{d)b} + \gamma^{a(c}P^{d)b} - 4\gamma^{ae}\gamma^{bf}\gamma^{cg}\gamma^{dh}L_{efgh}. \tag{9.82}$$

The total worldsheet energy-momentum tensor can be written in separated form as

$$T^{ab} = T_{\rm s}^{ab} + T_{\rm e}^{ab}, \tag{9.83}$$

in which the scalar field contribution is

$$T_{\rm s}^{ab} = \mathcal{L}_{\rm s}\overline{g}^{ab} - 2L_{\rm s}^{ab}, \tag{9.84a}$$

and the contribution due to the elastic medium is

$$T_{\rm e}^{ab} = \rho_{\rm e}u^a u^b + P^{ab}, \tag{9.84b}$$

with $\rho_{\rm e} = -\mathcal{L}_{\rm e}$, P^{ab} as given by (9.81) and

$$L_{\rm s}^{ab} = -L_{\rm s}'\mu^a\mu^b, \qquad L_{\rm s}' = -L_{00}. \tag{9.85}$$

The energy density $\rho_{\rm s}$ and isotropic pressure $P_{\rm s}$ of the scalar contribution can be read off from (9.84a) as

$$\rho_{\rm s} = 2\mu^2 L_{\rm s}' - \mathcal{L}_{\rm s}, \qquad P_{\rm s} = \mathcal{L}_{\rm s}. \tag{9.86}$$

In analogue with the separated energy-momentum tensor (9.83), the hyper-elasticity tensor can also be expressed in separated form,

$$\mathfrak{C}^{abcd} = \mathfrak{C}_{\rm s}^{abcd} + \mathfrak{C}_{\rm e}^{abcd}. \tag{9.87}$$

The scalar and elastic contributions are

$$\begin{aligned}\mathfrak{C}_{\rm s}^{abcd} \;=\;& 2\left(L_{\rm s}^{abcd} + L_{\rm s}^{a(c}\overline{g}^{d)b} + \overline{g}^{a(c}L_{\rm s}^{d)b}\right) \\ & - \left(L_{\rm s}^{ab}\overline{g}^{cd} + \overline{g}^{ab}L_{\rm s}^{cd}\right) + \tfrac{1}{2}\mathcal{L}_{\rm s}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right),\end{aligned} \tag{9.88a}$$

$$\begin{aligned}\mathfrak{C}_{\rm e}^{abcd} \;=\;& -\tfrac{1}{2}E^{abcd} + 2u^{(a}P^{b)(c}u^{d)} - \tfrac{1}{2}\left(P^{ab}u^c u^d + P^{cd}u^a u^b\right) \\ & + \tfrac{1}{2}\rho_{\rm e}\left(\overline{g}^{ab}\overline{g}^{cd} - 2\overline{g}^{a(c}\overline{g}^{d)b}\right),\end{aligned} \tag{9.88b}$$

in which

$$L_{\rm s}^{abcd} = L_{\rm s}''\mu^a\mu^b\mu^c\mu^d, \qquad L_{\rm s}'' = \frac{\partial L_{\rm s}'}{\partial\mu^2}. \tag{9.89}$$

In this separated case the cross-component of the characteristic matrix (9.72) vanishes

$$\mathcal{Q}_{0A} = 0. \tag{9.90}$$

This gives a decoupling of the characteric modes ζ^a for the scalar and elastic parts.

9.3.4 Reduction to the fluid case

We would like to remark about the reduction of the theory to include only perfect fluids. A perfect fluid is characterised by the Lagrangian only being a function of the determinant of the material metric: in the language of this section, that is

$$\mathcal{L}_{\rm e} = \mathcal{L}_{\rm e}(|\gamma_{AB}|), \tag{9.91}$$

where $|\gamma_{AB}|$ denotes the determinant of γ_{AB}. It is useful to recall the identity

$$\frac{\partial|\gamma|}{\partial\gamma^{AB}} = -|\gamma|\gamma_{AB}. \tag{9.92}$$

In this case the spatial parts of (9.42a) evaluate to

$$L_{AB} = -L_{\rm F}\gamma_{AB}, \qquad L_{\rm F} \equiv |\gamma|\frac{\partial\mathcal{L}_{\rm e}}{\partial|\gamma|}, \tag{9.93}$$

and the equivalent of (9.79) is

$$P_{AB} = (\mathcal{L}_{\rm e} + 2L_{\rm F})\,\gamma_{AB}. \tag{9.94}$$

The pressure tensor becomes

$$P^{ab} = P_{\rm F}\gamma^{ab}, \tag{9.95}$$

with the pressure scalar being given by

$$P_{\rm F} = \mathcal{L}_{\rm e} + 2L_{\rm F}. \tag{9.96}$$

We would like to offer a generalization, and consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}\left(\overline{g}^{00}, \overline{g}^{0A}, |\overline{g}^{AB}|\right). \tag{9.97}$$

We remember that we can still write $\overline{g}^{00} = -\mu^2$. One then obtains for the components (9.42a)

$$L_{00} = -\frac{\partial\mathcal{L}}{\partial\mu^2}, \tag{9.98a}$$

$$L_{0A} = \frac{\partial\mathcal{L}}{\partial\overline{g}^{0A}}, \tag{9.98b}$$

$$L_{AB} = -L_{\rm F}\overline{g}_{AB}, \tag{9.98c}$$

where we set

$$\mu^2 = -\overline{g}^{00}, \qquad L_{\rm F} \equiv |\overline{g}^{CD}|\frac{\partial\mathcal{L}}{\partial|\overline{g}^{CD}|}. \tag{9.99}$$

9.3.5 Non-separable case

In the general case, $\mathcal{L}$ will be a function of all invariants formed out of the components (9.35) of the induced metric (9.17). And so, regarding $\overline{g}^{ab}$ as a rank-2 tensor in $p+1$ dimensions, there are $p+1$ invariants that can be constructed. The first few such invariants are

$$
\begin{aligned}
I_0 &= \det \overline{\mathbf{g}}, && (9.100a)\\
I_1 &= [\overline{\mathbf{g}}], && (9.100b)\\
I_2 &= \frac{1}{2!}\left([\overline{\mathbf{g}}]^2 - [\overline{\mathbf{g}}^2]\right), && (9.100c)\\
I_3 &= \frac{1}{3!}\left([\overline{\mathbf{g}}]^3 - 3[\overline{\mathbf{g}}^2][\overline{\mathbf{g}}] + 2[\overline{\mathbf{g}}^3]\right), && (9.100d)\\
I_4 &= \frac{1}{4!}\Big([\overline{\mathbf{g}}]^4 + 8[\overline{\mathbf{g}}][\overline{\mathbf{g}}^3] + 3[\overline{\mathbf{g}}^2]^2 - 6[\overline{\mathbf{g}}]^2[\overline{\mathbf{g}}^2] - 6[\overline{\mathbf{g}}^4]\Big). && (9.100e)
\end{aligned}
$$

These are elementary symmetric polynomials. The Cayley-Hamilton theorem imposes

$$I_{p+1} = I_0, \tag{9.101}$$

and sets all higher I_n to zero. Note that

$$I_1 = \overline{g}^0{}_0 + \overline{g}^A{}_A, \tag{9.102a}$$

$$I_2 = \overline{g}^0{}_0\overline{g}^A{}_A - \overline{g}^A{}_0\overline{g}^0{}_A + \frac{1}{2}\left[\overline{g}^A{}_A\overline{g}^B{}_B - \overline{g}^A{}_B\overline{g}^B{}_A\right] \tag{9.102b}$$

9.4 Generalization away from orthogonal mappings

Our aim here is to use the previous sections results and ideas to construct a theory for whom the orthogonality of the mappings (2.3) doesn't hold. We will keep to the notion of a material metric, k_{AB}, and coordinates ϕ^A on the material space; we carry on using (2.2) to define the configuration gradients:

$$\phi^A{}_{,a} = \psi^A{}_a. \tag{9.103}$$

The pull-back of the material metric to space-time is

$$k_{ab} = k_{AB}\phi^A{}_{,a}\phi^B{}_{,b}. \tag{9.104}$$

The energy-momentum tensor is

$$T_{ab} = \mathcal{L}g_{ab} - 2L_{ab}, \tag{9.105}$$

where

$$L_{ab} \equiv \frac{\partial \mathcal{L}}{\partial g^{ab}}. \tag{9.106}$$

It follows that L_{ab} can be thought of as the pull-back of some material-manifold tensor L_{AB}

$$L_{ab} = L_{AB}\phi^A{}_{,a}\phi^B{}_{,b}, \qquad L_{AB} \equiv \frac{\partial \mathcal{L}}{\partial k^{AB}}. \tag{9.107}$$

Hence, the energy-momentum tensor is

$$T_{ab} = \mathcal{L} g_{ab} - 2L_{AB}\phi^A{}_{,a}\phi^B{}_{,b}. \tag{9.108}$$

Suppose we wanted to compute $u^a u^b L_{ab}$. Then

$$u^a u^b L_{ab} = L_{AB} u^a \phi^A{}_{,a} u^b \phi^B{}_{,b}, \tag{9.109}$$

but

$$u^A = u^a \phi^A{}_{,a} \tag{9.110}$$

is the push-forward of u^a. In coordinate free notation the push-forward takes contravariant tensors in space-time and returns a contravariant tensor on the material space via a schematic expression of the form $B^{AB\cdots} = \psi_\star B^{ab\cdots}$, as we explained in the discussion leading up to (2.6b). Hence

$$u^a u^b L_{ab} = u^A u^B L_{AB}. \tag{9.111}$$

Similarly, if we errected a vector l^a in space-time that is orthonormal to u^a, i.e., $l^a u_a = 0$ and $l^a l_a = 1$, then

$$l^a L_{ab} = l^A L_{AB} \phi^B{}_{,b} \tag{9.112}$$

where

$$l^A = l^a \phi^A{}_{,a} \tag{9.113}$$

is the push-forward of l^a. We will also introduece a tensor γ^{AB} on the material manifold, to be the tensor for whom l^A is an eigenvector,

$$l^A \gamma^B{}_A = l^B. \tag{9.114}$$

We can now use this to obtain a complete decomposition of the allowed freedom in L_{AB}:

$$L_{AB} = \left(u_C u_D L^{CD}\right) u_A u_B + 2\left(u_C l_D L^{CD}\right) u_{(A} l_{B)} + \gamma_{A(C}\gamma_{D)B} L^{CD}. \tag{9.115}$$

In the hyper-elastic category one has the splitting

$$\left(\phi^A{}_{,a}\right) \longrightarrow \left(\phi^0{}_{,a}, \phi^{\overline{A}}{}_{,a}\right) \tag{9.116}$$

in which

$$u^a \phi^0{}_{,a} \neq 0, \qquad u^a \phi^{\overline{A}}{}_{,a} = 0. \tag{9.117}$$

This means that the pull-back of the material metric splits up as

$$k_{ab} = k_{00}\phi^0{}_{,a}\phi^0{}_{,b} + 2k_{\overline{A}0}\phi^0{}_{,a}\phi^{\overline{A}}{}_{,b} + k_{\overline{A}\overline{B}}\phi^{\overline{A}}{}_{,a}\phi^{\overline{B}}{}_{,b}. \tag{9.118}$$

There are three projections of the pulled-back material metric:

$$u^a u^b k_{ab} = u^a u^b k_{00}\phi^0{}_{,a}\phi^0{}_{,b}, \tag{9.119a}$$

$$u^a \gamma^b{}_c k_{ab} = 2u^a \gamma^b{}_c k_{\overline{A}0} \phi^0{}_{,a} \phi^{\overline{A}}{}_{,b}, \tag{9.119b}$$

$$\gamma^a{}_c \gamma^b{}_d k_{ab} = \gamma^a{}_c \gamma^d{}_b k_{\overline{A}\overline{B}} \phi^{\overline{A}}{}_{,a} \phi^{\overline{B}}{}_{,b}. \tag{9.119c}$$

It is convenient, and does not lose generality, to set

$$\mu_a \equiv \phi^0{}_{,a}, \qquad l_a \equiv k_{\overline{A}0} \phi^{\overline{A}}{}_{,a}, \qquad \overline{k}_{ab} \equiv k_{\overline{A}\overline{B}} \phi^{\overline{A}}{}_{,a} \phi^{\overline{B}}{}_{,b}, \tag{9.120}$$

in which

$$u^a l_a = 0, \qquad u^a \overline{k}_{ab} = 0. \tag{9.121}$$

The pull-back of the material metric now naturally splits up as

$$k_{ab} = \mu_a \mu_b + 2\mu_{(a} l_{b)} + \overline{k}_{ab}. \tag{9.122}$$

Suppose we wrote $\overline{k}_{ab}$ as the pull-back with respect to some new map $\widehat{\psi}$ of some new tensor, $\overline{k}_{ab} = \widehat{\psi}^{\star} \overline{k}_{AB}$, where in coordinates

$$\overline{k}_{ab} = \overline{k}_{\widehat{A}\widehat{B}} \widehat{\phi}^{\widehat{A}}{}_{,a} \widehat{\phi}^{\widehat{B}}{}_{,b}, \tag{9.123}$$

and the components of this new configuration gradient satisfy

$$u^a \widehat{\phi}^{\widehat{A}}{}_{,a} = 0. \tag{9.124}$$

10 Final discussion

- Whats the point?
- what have we learnt?
- where next?

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