below eq. 26 he says that one can use the unsmoothed energy momentum tensor! How can this be? we know that this leads to a dust EMT and not to an EMT of colissionless matter...
--> siehe markierte Stelle in Test-Bed Simulations (Widrow Davies). Möglicherweise meint er das unsmoothed bezogen auf die Berechnung von dem Gravitationspotential V.
Genau so ist es. Man sollte doch eh für die Berechnung von V das ungesmoothte verwenden.

PHYSICAL REVIEW D VOLUME 55, NUMBER 10 15 MAY 1997

Modeling collisionless matter in general relativity: A new numerical technique

I don't understand my comment any more, why should the unsmoothed schröding be dust? it's not because of the hbar terms

Lawrence M. Widrow
Department of Physics, Queen's University, Kingston, Canada K7L 3N6

(Received 17 April 1996)

Note that the nonrelativistic limit of the complex KG equation is the Schrödinger equation (see Giulini). Therefore the quantum pressure, although hidden here, might be imortant.

We propose a new numerical technique for following the evolution of a self-gravitating collisionless system in general relativity. Matter is modeled as a scalar field obeying the coupled Klein-Gordon and Einstein equations. A phase-space distribution function, constructed using covariant coherent states, obeys the relativistic Vlasov equation provided the de Broglie wavelength for the field is very much smaller than the scales of interest. We illustrate the method by solving for the evolution of a system of particles in a static, plane-symmetric, background spacetime. [S0556-2821(97)05310-1]

PACS number(s): 04.25.Dm, 04.40.−b, 95.30.Sf

I. INTRODUCTION

There now exist techniques to calculate numerically the dynamical evolution of matter in general relativity. For the most part research has focused on three very different types of systems; fluids, scalar fields, and collisionless matter. In this work we develop and exploit a connection between the latter two.

A collisionless system with a large number of particles is treated generally as a continuous fluid in phase space. The fluid is described by a distribution function f which gives, for each region of the system, the density of particles and their distribution in velocity space. f obeys the Vlasov equation with an appropriate force law given, in general relativity, by the geodesic and Einstein equations. Shapiro and Teukolsky [1] have developed a computational method for handling collisionless relativistic systems that combines N-body techniques and numerical relativity. An alternative approach [2] is to follow the evolution of f directly in phase space by solving the coupled Einstein and Vlasov equations. Applications include violent relaxation and the collapse to a black hole of an unstable relativistic star cluster.

In this work we show that a massive scalar field obeying the coupled Klein-Gordon and Einstein equations provides an alternative model for collisionless relativistic systems that can be readily adapted to the computer. The technique makes use of what is essentially the scalar field analogue of geometric optics in general relativity [3]. The spirit and methodology is similar to that of Widrow and Kaiser [4], who showed that a field obeying the coupled Schrödinger and Poisson equations could be used to model collisionless, nonrelativistic matter.

Scalar fields in general relativity have been studied for some time, albeit for entirely different reasons. For the past 15 years the motivation for much of this work has been to understand the inflationary universe paradigm. For example, numerical simulations of inhomogeneous scalar field cosmologies have been studied in an attempt to understand the onset of inflation [5]. Recently, there has been great deal of interest in the gravitational collapse to a black hole of a self-gravitating massless scalar field. This is due primarily to the discovery by Choptuik [6] that such systems exhibit scaling behavior and critical phenomena.

Again our interest here is in simulations of collisionless matter. For our purposes previous investigations of scalar fields are important in that they exhibit the numerical techniques used to follow their dynamics. Our discussion does suggest that the results found by Choptuik might also apply to collisionless matter. Unfortunately the simulations that have shown scaling behavior and critical phenomena involve massless scalar fields whereas our work requires a nonzero mass.

We model a collisionless relativistic system by a field configuration $\phi(x)$ where ϕ is a classical, complex scalar field which obeys the Klein-Gordon equation in curved space:

$$\left(g^{\mu\nu}\nabla_\mu\partial_\nu+\frac{m^2}{\hbar^2}\right)\phi(x)=0. \tag{1}$$

[The metric $g_{\mu\nu}$ has signature $(+---)$ and we set $c=1$.] $m/\hbar$ is a model parameter which must be large enough to guarantee that geometric optics (or rather the scalar field analogue of geometric optics) applies. In particular we require that throughout the system $\lambda_{\rm deB}\ll\mathcal{L}$ and $\lambda_{\rm deB}\ll\mathcal{R}$ where $\lambda_{\rm deB}\equiv\hbar/\mathcal{P}\sim|\phi/\nabla\phi|$ is the characteristic ''de Broglie'' wavelength for the field, $\mathcal{P}$ is a typical momentum, $\mathcal{L}$ is the scale over which the density and velocity field of the system (i.e., distribution function) vary, and $\mathcal{R}$ is the typical radius of curvature for the spacetime. Under these conditions ϕ can be regarded locally as the superposition of plane waves propagating along geodesics [3]. The amplitude of the field in a given region of the system tells us something about the density of particles while information about the velocity distribution of the particles is encoded in the phase structure of the field. This suggests the following prescription for constructing a distribution function $\mathcal{F}=\mathcal{F}(x,p)$ from ϕ: First, Fourier transform the field in a region of size η localized about a particular point in position space. This can be done using an appropriate window function. The absolute square of this transform will be a function in phase space. Application of this method to nonrelativistic systems is in Sec. II. The extension to relativistic systems is discussed in Sec. III. Finally simple illustrative examples are discussed in Sec. IV.

II. NONRELATIVISTIC COLLISIONLESS MATTER AND THE SCHRÖDINGER EQUATION

Let us see how the prescription described above works for nonrelativistic systems [4,7]. We begin with a field $\psi(\mathbf{x},t)$ which obeys the Schrödinger wave equation. A phase-space projection of ψ is obtained using the coherent state or Husimi representation [8,9]:

$$\Psi(\mathbf{p},\mathbf{x},t) = \left(\frac{1}{2\pi\hbar}\right)^{3/2}\left(\frac{1}{\pi\eta^2}\right)^{3/4} \times \int e^{-(\mathbf{x}-\mathbf{x}')^2/2\eta^2 - i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}/2)/\hbar}\psi(\mathbf{x}',t)d^3x'. \quad (2)$$

The quantity $\mathcal{F}(\mathbf{x},\mathbf{p},t)=|\Psi(\mathbf{p},\mathbf{x},t)|^2$ gives the density of the Schrödinger field in phase space. The normalization is chosen so that $\int d^3x \int d^3p\mathcal{F}(x,p)=1$ provided $\int d^3x|\psi(x)|^2=1$. One can show, by direct calculation [4,9], that $\mathcal{F}$ obeys the collisionless Boltzmann equation to an accuracy set by $\eta/\mathcal{L}$ and $\hbar/\eta\mathcal{P}$.

Equation (2) can be expressed conveniently using the *bra-ket* notation of quantum mechanics:

$$\Psi(\mathbf{p},\mathbf{x},t)=\int d^3x'\langle e(\mathbf{x},\mathbf{p})|\mathbf{x}'\rangle\langle\mathbf{x}'|\psi(\mathbf{x})\rangle, \quad (3)$$

where

$$\langle e(\mathbf{x},\mathbf{p})|\mathbf{x}'\rangle \equiv \left(\frac{1}{2\pi\hbar}\right)^{3/2}\left(\frac{1}{\pi\eta^2}\right)^{3/4} e^{-(\mathbf{x}-\mathbf{x}')^2/2\eta^2 - i\mathbf{x}'\cdot\mathbf{p}/\hbar}, \quad (4)$$

is a Gaussian (minimum uncertainty) wave packet having width η ($\hbar/\eta$) in position (momentum) space and $\psi(\mathbf{x}) \equiv\langle\mathbf{x}|\psi\rangle$. Alternatively one can begin with the momentum space representation (i.e., Fourier transform) of the field $\hat{\psi}(\mathbf{p})\equiv\langle\mathbf{p}|\psi\rangle$. We would then have

$$\Psi(\mathbf{p},\mathbf{x},t)=\int d^3p'\langle e(\mathbf{x},\mathbf{p})|\mathbf{p}'\rangle\langle\mathbf{p}'|\psi\rangle, \quad (5)$$

where

$$\langle e(\mathbf{x},\mathbf{p})|\mathbf{p}'\rangle=\left(\frac{\eta^2}{\pi\hbar^2}\right)^{3/4} e^{-(\mathbf{p}-\mathbf{p}')^2\eta^2/2\hbar^2 + i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}/\hbar}, \quad (6)$$

is the Fourier transform of Eq. (4). It is important to keep in mind that though we are borrowing language from quantum mechanics all of the above expressions involve c-number fields.

III. RELATIVISTIC COLLISIONLESS MATTER AND THE KLEIN-GORDON EQUATION

The coherent-state representation for a scalar field in special relativity is again based on a set of ''state vectors'' $|e(x,p)\rangle$ localized in both position and momentum. We choose minimum uncertainty wave packets [10]:

$$\langle e(x,p)|p'\rangle=\left(\frac{\eta^2}{\pi\hbar^2}\right)^{3/4} e^{-ix\cdot p'/\hbar - p\cdot p'\eta^2/\hbar^2}, \quad (7)$$

where $p=(p_0,\mathbf{p})$ with $p_0>|\mathbf{p}|$ and $a\cdot b\equiv a_0b_0-\mathbf{a}\cdot\mathbf{b}$. These wave packets are centered about $\mathbf{x}$ and move with average momentum $\mathbf{p}$.

The extension to general relativity is most easily accomplished using Riemann normal coordinates (RNC's) and is similar to discussions found in the literature [11,12] of the Wigner function in curved space. In a RNC system curved space closely resembles flat space in the neighborhood of a particular point z taken to be the origin. The RNCs of z are x^μ. The metric, written in these coordinates, takes the form of a Taylor series:

$$g_{\mu\nu}=\eta_{\mu\nu}+\tfrac{1}{3}R_{\mu\rho\nu\sigma}x^\rho x^\sigma+\tfrac{1}{6}R_{\mu\rho\nu\sigma;\lambda}x^\rho x^\sigma x^\lambda+\cdots, \quad (8)$$

where the ellipsis refers to higher-order derivatives of the metric. The relativistic Vlasov equation

$$g^{\alpha\mu}p_\mu\frac{\partial f}{\partial x_\alpha}-\frac{1}{2}g^{\mu\nu}{}_{,\alpha}p_\mu p_\nu\frac{\partial f}{\partial p_\alpha}=0 \quad (9)$$

to leading in a RNC expansion becomes

$$\left(\eta^{\mu\nu}p_\nu\frac{\partial f}{\partial x^\mu}+\frac{1}{3}R^\rho{}_{\mu\ \sigma}^{\ \nu}x^\sigma p_\mu p_\nu\frac{\partial f}{\partial p_\rho}+\cdots\right)=0. \quad (10)$$

For our purposes, the main advantage of RNC's is that the preferred set of modes for a scalar field approaches plane waves at the origin. This can be seen by considering the phase factor $x\cdot p$. In RNC's we have

$$x\cdot=\eta_{\mu\nu}x^\mu p^\nu+\tfrac{1}{3}R_{\mu\rho\nu\sigma}x^\mu x^\rho p^\nu x^\sigma + \tfrac{1}{6}R_{\mu\rho\nu\sigma;\lambda}x^\mu x^\rho p^\nu x^\sigma x^\lambda+\cdots.$$

The second term on the right-hand side vanishes due to the symmetry properties of the Riemann tensor while the third term vanishes due to the Bianchi identies. Therefore, neglecting terms involving more than four derivatives of the metric (which is conservative for our purposes), we have $x\cdot p=\eta_{\mu\nu}x^\mu p^\nu$.

The relativistic version of Eq. (5) is

$$\langle e(x,p)|\phi\rangle=\int dV_{p'}e^{-ix\cdot p'/\hbar - p\cdot p'\eta^2/\hbar^2}\hat{\phi}(p'), \quad (11)$$

where

$$dV_p\equiv\frac{dp_0dp_1dp_2dp_3}{(2\pi)^4}2\pi\delta(p^2-m^2)\Theta(p_0) \quad (12)$$

is the momentum space volume element on the mass shell, $\Theta(x)=1$ for $x\geqslant 0$ and zero otherwise, and $\hat{\phi}(p)$ is the Fourier transform of $\phi(x)$:

$$\phi(x)=\int dV_p e^{-ip\cdot x/\hbar}\hat{\phi}(p) \quad (13)$$

$$=\frac{dp_0dp_1dp_2dp_3}{(2\pi)^3}e^{-ip\cdot x/\hbar}\sigma(p). \quad (14)$$

In this last expression we have introduced the auxiliary field $\sigma(p)\equiv\delta(p^2-m^2)\Theta(p_0)\hat{\phi}(p)$. Note that $\langle e(x,p)|p'\rangle$ reduces to $\langle e(\mathbf{x},\mathbf{p})|\mathbf{p}'\rangle$ in the nonrelativistic limit $|\mathbf{p}|\ll m$. In practice, one integrates over p_0 making use of the identity $\int dV_p = d^3p/[(2\pi)^3 2p^0]$.

The field-generated distribution function $\mathcal{F}$ is given by

$$\mathcal{F}(x,p)=|\langle e(x,p)|\phi\rangle|^2 \tag{15}$$

$$=\int dV_p' dV_p'' e^{ix\cdot(p'-p'')/\hbar - p\cdot(p'+p'')\eta^2/\hbar^2} \times\hat{\phi}(p')\hat{\phi}^*(p'') \tag{16}$$

$$=\int \frac{d^3p'}{(2\pi)^3 2p'^0}\frac{d^3p''}{(2\pi)^3 2p''^0} \times e^{ix\cdot(p'-p'')/\hbar - p\cdot(p'+p'')\eta^2/\hbar^2}\hat{\phi}(p')\hat{\phi}^*(p''). \tag{17}$$

In the last of these expression the integrand is evaluate ''on-mass shell,'' i.e., with $p'^0=\sqrt{\mathbf{p}'\cdot\mathbf{p}'+m^2}$ and $p''^0=\sqrt{\mathbf{p}''\cdot\mathbf{p}''+m^2}$. Our expression for $\mathcal{F}$ has the desired form: We multiply the field ϕ (in this case its Fourier transform $\hat{\phi}$) by an appropriate window function, Fourier transform the resultant function, and then take the absolute square.

We have, from Eq. (1) and the definition of the RNC's,

$$\left(\frac{m^2-p^2}{\hbar^2}-\frac{1}{3}R^{\mu\ \ \nu}_{\ \rho\ \ \sigma}\frac{\partial^2}{\partial p_\rho\partial p_\sigma}p_\mu p_\nu-\frac{2}{3}R^\mu_\rho\frac{\partial}{\partial p_\rho}p_\mu\right)\sigma(p) \equiv\mathcal{O}(p)\sigma(p)=0 \tag{18}$$

from which follows immediately the identity

$$0=\int d^4p'd^4p''e^{ix\cdot(p'-p'')/\hbar - p\cdot(p'+p'')\eta^2/\hbar^2} \times[\mathcal{O}(p')\sigma(p')]\sigma^*(p''). \tag{19}$$

It is now convenient to switch from p' and p'' to $q\equiv p'-p''$ and $s\equiv(p'+p'')/2$. Equation (19) then becomes

$$0=\int d^4sd^4q e^{ix\cdot q/\hbar - 2p\cdot s\eta^2/\hbar^2}[\mathcal{O}(p')\sigma(p')]\sigma^*(p''). \tag{20}$$

Our goal is to write $\mathcal{O}$ in terms of operators which can be brought outside the integral (i.e., terms involving $p,x,\partial/\partial p$, and $\partial/\partial x$). What results is a complex equation for $\mathcal{F}$ whose real and imaginary parts correspond to the mass-shell constraint and relativistic Vlasov equations, respectively.

To see how this will work consider the first term in $\mathcal{O}$:

$$\frac{m^2-p'^2}{\hbar^2}=\frac{m^2-[p+(s-p)+q/2]^2}{\hbar^2}. \tag{21}$$

The leading real term is clearly $(m^2-p^2)/\hbar^2$. Terms involving factors of $(s-p)$ are smaller by factors of $(\hbar/\eta\mathcal{P})^2 \simeq(\lambda_{\rm deB}/\eta)^2$ due to the fact that the window function is peaked strongly about $p=s$. The factors of q are converted into spatial derivatives: $q\exp(iq\cdot x)\rightarrow -i\hbar(\partial/\partial x)\exp(iq\cdot x)$. The leading-order imaginary term is then $\eta^{\mu\nu}p_\nu(\partial\mathcal{F}/\partial x)$. There is also a real term of the form $\eta^{\mu\nu}(\partial^2\mathcal{F}/\partial x^\mu x^\nu)$. This term is smaller, by a factor of $(\hbar/\eta\mathcal{P})^2$, relative to the leading-order real terms discussed above.

A similar but tedious calculation is performed on the remaining terms in Eq. (20). As expected one arrives at two equations for $\mathcal{F}$:

$$\left(\frac{m^2-p^2}{\hbar^2}+\cdots\right)\mathcal{F}=0, \tag{22}$$

$$\left(\eta^{\mu\nu}p_\nu\frac{\partial}{\partial x^\mu}+\frac{1}{3}R^{\mu\ \ \nu}_{\ \rho\ \ \sigma}x^\sigma p_\mu p_\nu\frac{\partial}{\partial p_\rho}+\cdots\right)\mathcal{F}=0. \tag{23}$$

Again, the ellipsis in these equations refers to terms higher order in the $\eta/\mathcal{L}$, $\lambda_{\rm deB}/\eta$ and $\eta/\mathcal{R}$ (i.e., postgeometric optics corrections). The first of these equations tells us that the distribution function is peaked on the $p_\mu p^\mu=m^2$ (mass shell) surface. The second equation is, to leading order, the relativistic Vlasov equation in RNC system [see Eq. (10)]. Similar results have been obtained using the Wigner function [11,12].

For self-gravitating collisionless matter, $g_{\mu\nu}$ depends on f through the Einstein equations, $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi GT_{\mu\nu}$, where the stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu}=\int\frac{d^4p}{\sqrt{-g}}p_\mu p_\nu f(x,p). \tag{24}$$

We can calculate $T_{\mu\nu}$ for our model by substituting $\mathcal{F}$ for f in Eq. (24). This is, however, computationally expensive (one must update g at each timestep) and not really necessary. Consider instead the expression

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi^* \tag{25}$$

$$=\int dV_{p'}dV_{p''}e^{i(p''-p')\cdot x/\hbar}p'_\mu p'_\nu\hat{\phi}(p')\hat{\phi}^*(p''). \tag{26}$$

Equation (24) is what one would obtain if this expression were smoothed over a length scale $\sim\eta$ and since the quantity of interest, $g\sim\nabla^{-2}T$, is itself smoother than T we can use either expression. Equation (12) is just the usual expression for the stress energy tensor of a scalar field, save for a term proportional to $\partial_\mu\phi\partial^\mu\phi^*-m^2\phi\phi^*$. But this term is negligible in the geometric optics limits [10].

IV. ILLUSTRATIVE EXAMPLES

Evidently a self-gravitating scalar field behaves like collisionless matter and can be represented in phase space using the coherent-state representation described above provided $\lambda_{\rm deB}\ll\eta\ll\mathcal{L},\mathcal{R}$. We now dispense with RNC's and discuss general methods for using this technique to do numerical simulations. To follow the field properly we must choose a grid spacing that is somewhat less than $\lambda_{\rm deB}$. In addition the time step must be short enough to follow the rapid evolution of the phase factor for ϕ. The accuracy of a simulation is limited by constraints on memory and CPU time. The situation is similar to what is encountered in particle methods where a finite number of particles is used to provide a statistical description of the distribution function. One can im-

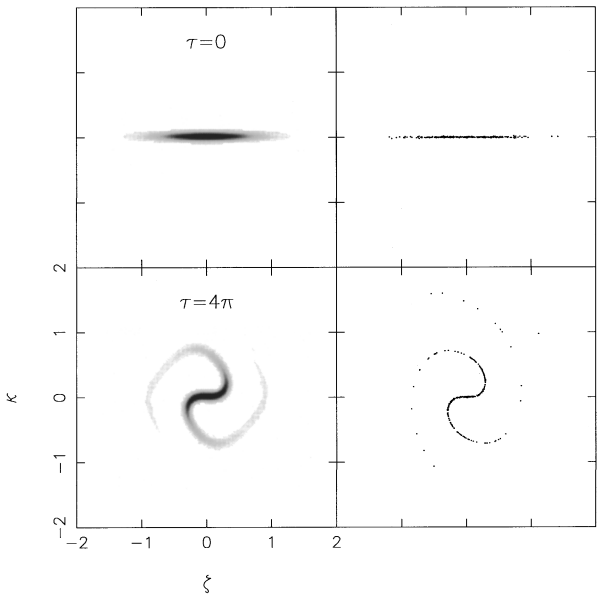

FIG. 1. Evolution in phase space of an initially cold distribution of particles. The system is set in a fixed background spacetime as described in the text. 4096 grid points are used for the Klein-Gordon simulation. Left-hand panels give the Klein-Gordon distribution function $\mathcal{F}$ while right-hand panels give the corresponding N-body distribution function f.

prove the accuracy of a given simulation by increasing the number of particles but again this comes at the cost of memory and CPU time.

As an illustration of our method we consider ''particles'' in a static and plane symmetric background spacetime. For simplicity, we choose the line element $ds^2=A^2dt^2-A^{-2}(dx^2+dy^2+dz^2)$ with $A^2=\exp(\omega^2z^2)$ so that nonrelativistic particles near the origin will execute simple harmonic motion in the z direction. For simplicity, we assume ϕ depends only on z and t and represents a two-dimensional distribution function $\mathcal{F}=\mathcal{F}(z,v_z)$. It is convenient to write our equations in terms of the dimensionless quantities $\zeta=\omega z$, $\tau=\omega t$, and $\alpha=m/\hbar\omega$. Equation (1) becomes

$$\frac{\partial^2\phi}{\partial\tau^2}-e^{2\zeta^2}\frac{\partial^2\phi}{\partial\zeta^2}=e^{2\zeta^2}\alpha^2\phi. \tag{27}$$

$\lambda_{\rm deB}$ should be made as small as possible (this is done by increasing α) but no smaller than several grid spacings. With this in mind, we choose $\alpha\simeq 0.1N$ and $\eta\simeq\alpha^{1/2}$ where N is the number of gridpoints used in the simulation. There are roughly $N^{1/2}$ resolution elements in both position and momentum space which is roughly what one would expect for an N-body simulation with N particles.

Consider first an initially cold (zero velocity dispersion) distribution of particles. The field configuration at $\tau=0$ is taken to be $\phi(\zeta)=\exp(-\zeta^2/\zeta_0^2)$ and $\partial\phi/\partial\tau=-i\alpha\phi$ with $\zeta_0=0.5$. The corresponding $\mathcal{F}$ is shown in the top left panel of Fig. 1. The phase-space configuration at $\tau=4\pi$ (i.e., time when particles near the center of the distribution have made two complete orbits) is shown in the bottom left panel. For comparison, the results of an N-body calculation are shown in the right-hand panels. As expected, particles near the center of the distribution execute simple harmonic motion while particles initially at $z\gtrsim 1$ follow anharmonic, relativistic orbits in phase space. The Klein-Gordon simulation agrees quantitatively with the results from the particle code and captures the essential physics of phase mixing as the distribution function winds up into a spiral pattern. The fuzziness arises because of the wave nature of ϕ: Resolution in phase space is constrained to satisfy the relation $\Delta x\Delta p\gtrsim\hbar$.

We next consider a system of ''hot'' particles. In particular, we assume the system is initially described by a truncated isothermal distribution function: $f(\epsilon)=\Theta(\epsilon)\exp(-\epsilon/T')$ where $\epsilon\equiv p_0/E_{\rm max}$ and $T'\equiv T/E_{\rm max}$. For our simulation, we take $E_{\rm max}=2m$ and $T=m$. p_0 is an integral of the motion and therefore, by the Jean's theorem [13] $f(\epsilon)$ is an equilibrium configuration.

The initial wave function for the Klein-Gordon simulation is taken as the sum of coherent state wave packets:

$$\phi(x)=\frac{1}{N}\sum_{i=1}^{N}\langle e(x_i,p_i)|x\rangle, \tag{28}$$

where the pairs (x_i,p_i) are chosen at random from the distribution function and $\langle e(x_i,p_i)|x\rangle$ is the Fourier transform of Eq. (7). This is reminiscent of N-body simulations where a statistical representation of the initial distribution function is constructed by specifying the positions and velocities of N superparticles. Figure 2(a) is a gray scale plot of the initial distribution function using 2048 gridpoints to sample the

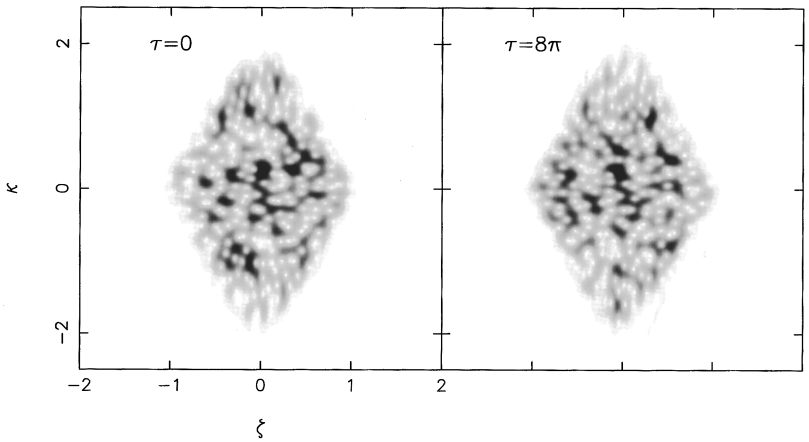

FIG. 2. Evolution of a ''hot'' system in phase space. The initial distribution function is that of a truncated isothermal sphere and is shown in the left panel. The distribution function for $\tau=8\pi$ is shown in the right panel. 2048 grip points are used for the simulation.

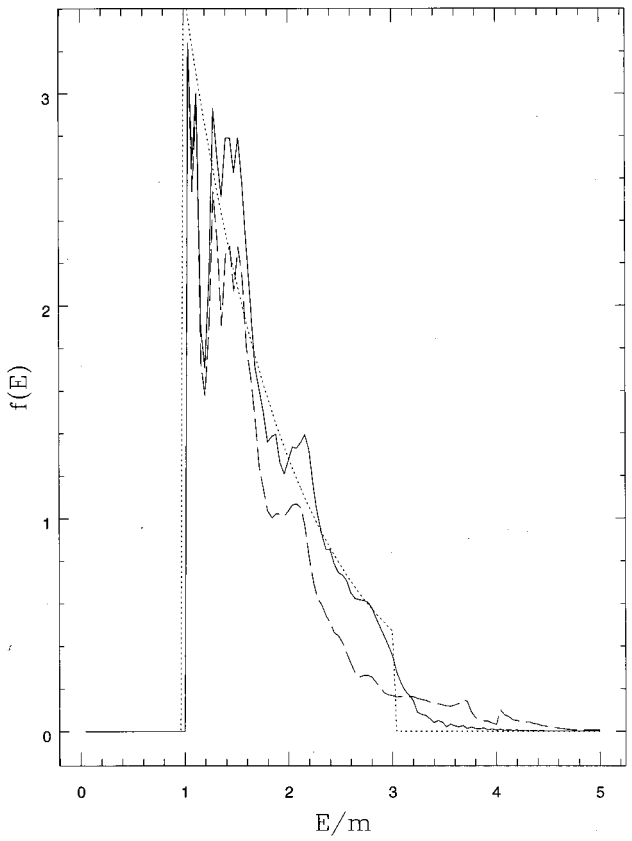

FIG. 3. Distribution function $\mathcal{F}(E)$. $\mathcal{F}(E)$ is calculated from the $\mathcal{F}(x,p)$ of Fig. 2 using a simple binning procedure. The solid curve is the initial $\mathcal{F}(E)$; the dashed curve is the $\mathcal{F}(E)$ at $\tau=8\pi$; the dotted curve is the $\mathcal{F}(E)$ used to construct the initial wave function.

field ϕ. The small-scale fluctuations are of size η along the position axis and $\hbar/\eta$ along the momentum axis. Since resolution must be shared between position and momentum resolution elements will be $\sim$few$\times(2048)^{-1/2}\simeq 0.05$ of the size of the box in agreement with Fig. 2. This is similar to the situation that arises in particle codes where there are N/few phase-space resolution elements for an N-particle simulation. Clearly a physically meaningful distribution function can be constructed by smoothing over several resolution elements.

Much of the art in N-body simulations is in calculating the forces and various methods (e.g., particle-particle with a softened potential, particle-mesh, etc.) are used to deal with such problems as artificial two-body relaxation. Here, the discrete particles are replaced by wave packets and so softening is built in. The pros and cons of our method, as compared with N-body techniques, are explored elsewhere [7].

Figure 2(b) gives $\mathcal{F}$ for $\rho=8\pi$. We see that the basic features of the distribution function remain unchanged though the ''details,'' on a scale $\eta\times\hbar/\eta$, fluctuate with time. This again is reminiscent of particle simulations where a particle representation of an equilibrium system will appear to fluctuate with time if one looks on too small a scale. Figure 3 gives $\mathcal{F}(p_0)$ [calculated from $\mathcal{F}(x,p)$ using a simple binning procedure] again for $\tau=0$ and $\tau=8\pi$ again illustrating the approximate time invariance of the distribution function, this time in energy space.

This work illustrates that simulations of collisionless matter in general relativity can be done by following the dynamics of a massive scalar field in so doing, bridges the gap between two very separate branches of numerical relativity. Applications to self-gravitating systems with more complicated geometries are straightforward only because of the large volume of work on relativistic scalar fields that already exists.

ACKNOWLEDGMENTS

It is a pleasure to thank George Davies and Kayll Lake for useful conversation. This work was supported by a grant from the Natural Science and Engineering Research Council of Canada.

[1] S. L. Shapiro and S. A. Teukolsky, Astrophys. J. **307**, 575 (1986), and references therein.
[2] F. A. Rasio, S. L. Shapiro, and S. A. Teukolsky, Astrophys. J. **344**, 146 (1989).
[3] See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970), Sec. 22.5.
[4] L. M. Widrow and N. Kaiser, Astrophys. J. Lett. **416**, 71 (1993).
[5] See, e.g., D. S. Goldwirth and T. Piran, Phys. Rev. D **40**, 3263 (1989).
[6] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[7] G. Davies and L. M. Widrow, Astrophys. J. (to be published).
[8] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940).
[9] R. T. Skodje, H. W. Rohrs, and J. van Buskirk, Phys. Rev. A **40**, 2894 (1989).
[10] G. Kaiser, J. Math. Phys. (N.Y.) **18**, 952 (1977).
[11] E. Calzetta and B. L. Hu, *The Physics of Phase Space* (Springer-Verlag, Berlin, 1987), p. 238.
[12] E. Calzetta, S. Habib, and B. L. Hu, Phys. Rev. D **37**, 2901 (1988), and references therein.
[13] S. R. de Groot, V. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980), pp. 70–73.