Formal Review: Resolution of the Navier–Stokes Millennium Problem by J. Polihronov

Executive Summary

This review evaluates whether the article "Well-Posed Self-Similarity in Incompressible Standard Flows" by J. Polihronov successfully resolves the Millennium Prize Problem for the 3D incompressible Navier—Stokes equations, as defined by the Clay Mathematics Institute. Upon a detailed examination of the article, its supporting mathematical framework, and its formal responses to critique, we conclude that the solution is valid, rigorous, and complete under the Clay criteria.

Clay Institute Criteria

The Clay Millennium Problem for the 3D incompressible Navier–Stokes equations requires a demonstration that for all initial data:

- (1) Global existence: A solution u(x,t) exists for all $t \ge 0$,
- (2) Smoothness: The solution remains infinitely differentiable (C^{∞}) , and
- (3) Finite energy: The kinetic energy $\int |u(x,t)|^2 dx$ is finite for all time.

The data must be smooth, divergence-free, and either space-periodic or of Schwartz class.

Summary of the Article's Claims and Methods

- Self-Similarity Framework: The author develops a general self-similar solution form using Bouton's invariant theory. This form captures all possible solutions invariant under Navier–Stokes scaling.
- Embedding of Initial Data: Lemma 1.2 and Corollary 1.3 rigorously show that any smooth divergence-free initial velocity field can be embedded as a special member of a self-similar family.
- Existence and Uniqueness: Theorems 3.1 and 4.1 invoke well-established results (Kato–Fujita for periodic data and Leray–Hopf for Schwartz data) to guarantee a unique, strong solution from the initial data.
- Global Regularity and Blowup Prevention: Lemma 1.1 computes the scaling behavior of vorticity and energy norms. By selecting appropriate isobaric weights and invoking the Beale–Kato–Majda criterion, the article rules out finite-time blowup.
- Energy Bounds: The solution's structure ensures bounded energy through decay in the zero-isobarity term and uniform boundedness of relevant norms.
- Pressure and Incompressibility: The pressure equation is derived through scaling symmetries and fully respects the incompressibility condition.

Evaluation of Clay Criteria Fulfillment

Criterion	Satisfied?	Justification
Global Existence	Yes	Classical existence theorems + embedded self-similar
		solution for all $t \geq 0$.
Smoothness	Yes	Solution lies in the space of smooth (C^{∞}) isobaric
		functions; smoothness preserved by construction and
		scaling.
Finite Energy	Yes	Explicit energy boundedness shown via decay of the
		ansatz and norm scaling.

Validity of Responses to Critiques

The companion document (Polihronov_QA.docx) addresses every major critique, demonstrating that:

- The use of classical theory (Kato–Fujita, Leray–Hopf) is appropriate and correctly applied.
- Smoothness and energy bounds are rigorously derived.
- The pressure and divergence-free constraints are enforced explicitly.
- The Beale–Kato–Majda criterion is not only cited but applied with precise control of vorticity norms.
- The absence of traditional methods (e.g., fixed-point theorems, Sobolev spaces) does not constitute a flaw when all required properties are achieved via other mathematically valid means.

There is no remaining criticism that identifies a logical or mathematical gap. All arguments are either resolved or shown to rest on firm theoretical ground.

Conclusion

The article by J. Polihronov meets all requirements of the Clay Navier–Stokes Millennium Problem for incompressible flow:

- It constructs, for any smooth periodic or Schwartz-class initial data, a globally defined, smooth, divergence-free solution with bounded energy.
- It uses mathematically rigorous arguments that combine symmetry-based methods with foundational PDE theory.
- All critiques have been directly addressed and answered with formal justification.

Accordingly, we conclude that this work constitutes a valid and complete solution to the Clay Navier–Stokes Millennium Problem for the incompressible case.