Solutions to Tasksheet 2

Task 1:

I have chosen to use the C++ Programming Language for this class. The program I wrote for Task 1 is under the src folder in this directory. The code is as follows:

```
//LanguageDeclaration.cpp
#include <iostream>

int main()
{
  std::cout << "Hello World!" << std::endl;
  std::cout << "I have decided to use the C++ Programming Language for
this class." << std::endl;
  return 0;
}
```

- The code was compiled with the command: `g++ LanguageDeclaration.cpp`
- The compilation resulted in the creation of an executable named `a.out`
- I executed the file `a.out` with the following command: `./a.out`
- That resulted in the printing of the following message to the screen:

```
Hello World!
I have decided to use the C++ Programming Language for this class.
```

Task 2:

I have edited my `README.md` file to contain an introduction for the repository. In addition, there are links to my hw_toc and my software manual.

Task 3:

I have created a hw_toc folder to hold the table of contents for the class tasksheets. In addition, I have a local version of my math4610 repository up and running. I have cloned it and updated it with the `git pull` command.

Task 4:

Taylor series expansion of a function, f(x), around a point x = a, looks like

$$f(a+h) = f(a) + hf'(a) + h^2\frac{f''(a)}{2!} + h^3\frac{f'''(a)}{3!} + \cdots$$

When rearranging the terms, you get

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

This is the approximation of the derivative of the equation f(x). In this case, it is known as the forward difference approximation.

In the above case, a + h is used. If we replace that with a-h during times of h<0, we get the backward difference approximation

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

The forward difference approximation and the backward difference approximation have a first order approximation.

When you subtract the backward difference approximation from the forward difference approximation, like this:

$$f(a+h)-f(a-h) = 2hf'(a) + h^3 \frac{f'''(c_1)+f'''(c_2)}{3!}$$

you get:

$$\frac{f(a+h)-f(a-h)}{2h} - f'(a) = h^2\left(\frac{f'''(c_1)+f'''(c_2)}{12}\right)$$

The right-hand side demonstrates the order of accuracy with h^2. After rearranging to find the first derivative, you get the centered difference approximation:

$$f'(a) = \frac{f(a+h)-f(a-h)}{2h}$$

<u>The centered difference approximation is a second order approximation because if h is decrease by a factor of 2, the error will decrease by a factor of 2^2.</u>

Task 5:

The order of accuracy of the given central difference approximation of the second derivative can be identified through Taylor series expansions:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \frac{h^4 f^{(4)}(c_1)}{4!} \cdots$$

$$(x-h) = f(x) - hf'(x) + \frac{h^2 f''(x)}{2!} - \frac{h^3 f'''(x)}{3!} + \frac{h^4 f^{(4)}(c_2)}{4!} \cdots$$

Adding these expansions together to get closer to the central difference approximation:

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + h^4 \frac{\left(f^4(c_1)+f^4(c_2)\right)}{24}$$

Then, with some manipulation:

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2} - f''(x) = h^2\frac{\left(f^4(c_1)+f^4(c_2)\right)}{24}$$

From here, we can identify the order of approximation through the h^2 from the right-hand side of the equation. This central difference approximation of the second derivative is of second-order accuracy. The equation in question can be found from the previous equation through further manipulation:

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

The source code can be found in the math4610/src/SecDerivAppr.cpp

Task 6:

There are three finite difference approximations mentioned in my findings. Those are the forward difference approximation, the backward difference approximation, and the central/centered difference approximation (Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems – Chapter 1). Big O notation is often used in Taylor Series Approximations to express the order of accuracy, such as O(h) for first order and $O(h^2)$ for second order. In addition, higher-order approximations can usually be found using similar manipulation techniques as with the first and second orders (Fundamentals of Engineering Numerical Analysis – Chapter 2, pg 14~).

Some examples of finite difference approximations of different orders (Numerical Differentiation: Finite Differences):

$O(\Delta x^2)$ centered difference approximations:

$$f'(x): \quad \big\{f(x+\Delta x) - f(x-\Delta x)\big\}/(2\Delta x)$$
$$f''(x): \quad \big\{f(x+\Delta x) - 2f(x) + f(x-\Delta x)\big\}/\Delta x^2$$

$O(\Delta x^2)$ forward difference approximations:

$$f'(x): \quad \big\{-3f(x) + 4f(x+\Delta x) - f(x+2\Delta x)\big\}/(2\Delta x)$$
$$f''(x): \quad \big\{2f(x) - 5f(x+\Delta x) + 4f(x+2\Delta x) - f(x+3\Delta x)\big\}/\Delta x^3$$

$O(\Delta x^2)$ backward difference approximations:

$$f'(x): \quad \big\{3f(x) - 4f(x-\Delta x) + f(x-2\Delta x)\big\}/(2\Delta x)$$
$$f''(x): \quad \big\{2f(x) - 5f(x-\Delta x) + 4f(x-2\Delta x) - f(x-3\Delta x)\big\}/\Delta x^3$$

$O(\Delta x^4)$ centered difference approximations:

$$f'(x): \quad \big\{-f(x+2\Delta x) + 8f(x+\Delta x) - 8f(x-\Delta x) + f(x-2\Delta x)\big\}/(12\Delta x)$$
$$f''(x): \quad \big\{-f(x+2\Delta x) + 16f(x+\Delta x) - 30f(t) + 16f(x-\Delta x) - f(x-2\Delta x)\big\}/(12\Delta x^2)$$

For the purpose of continuity: $\Delta x = h$