Structures III – Tensors as Multilinear Maps

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In [1], we studied linear transformations between vector spaces. A linear map

$$T : E \to F$$

is a function satisfying, for all $x, y \in E$ and all $\alpha \in K$,

$$T(x +_E y) = T(x) +_F T(y), \qquad T(\alpha \circ x) = \alpha \circ T(x).$$

Linearity encodes the idea that the map respects the algebraic structure of the vector space. Many constructions in mathematics, however, depend on functions that take *several* vector arguments simultaneously, while remaining linear in each argument when the others are held fixed.

This observation leads naturally to the notion of multilinearity.

Let $E_1, E_2, \dots, E_n$ and F be vector spaces over the same field K.

Definition 1 (Multilinear map)**.** A map

$$T : E_1 \times E_2 \times \cdots \times E_n \to F$$

is called *multilinear* if it is linear in each argument separately. That is, for each index $i \in \{1, \dots, n\}$, and for all vectors

$$x_i, y_i \in E_i, \quad \alpha \in K,$$

we have

$$T(x_1, \dots, x_i +_{E_i} y_i, \dots, x_n) = T(x_1, \dots, x_i, \dots, x_n) +_F T(x_1, \dots, y_i, \dots, x_n),$$

and

$$T(x_1, \dots, \alpha \circ x_i, \dots, x_n) = \alpha \circ T(x_1, \dots, x_i, \dots, x_n),$$

with all other arguments held fixed.

When $n = 1$, a multilinear map is simply a linear map. Thus multilinear maps are not a new kind of object, but a direct generalization of linear transformations.

Multilinear maps arise naturally in many contexts: bilinear products, pairings between vectors and linear functionals, and higher-order interactions that cannot be reduced to a single input. Their systematic study leads to the concept of tensors.

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Informally, a tensor is an object designed to encode multilinear behavior in an intrinsic and coordinate-free way.

There are two equivalent viewpoints that will be used throughout this document. The first treats tensors directly as multilinear maps.

Let E be a vector space over K, and let E^* denote its dual space, that is, the vector space of linear maps from E to K.

Definition 2 (Tensor as a multilinear map)**.** A *tensor of type* (k, ℓ) on E is a multilinear map

$$T : \underbrace{E^* \times \cdots \times E^*}_{k \text{ times}} \times \underbrace{E \times \cdots \times E}_{\ell \text{ times}} \longrightarrow K.$$

Such a tensor takes k linear functionals and ℓ vectors as inputs, and produces a scalar. Linearity is required in each argument separately.

This definition makes precise several familiar cases:

- A scalar in K can be viewed as a tensor of type $(0, 0)$.
- A vector in E corresponds to a tensor of type $(0, 1)$.
- A linear functional in E^* is a tensor of type $(1, 0)$.
- A bilinear form on E is a tensor of type $(0, 2)$.
- A linear map $T : E \to E$ can be identified with a tensor of type $(1, 1)$.

Thus vectors and matrices do not disappear in tensor theory; they reappear as the lowest-order cases of a single unified framework.

At this level of abstraction, no coordinates, arrays, or geometric interpretations are involved. A tensor is defined entirely by how it acts on vectors and linear functionals, and by the multilinearity of this action.

The definition of tensors as multilinear maps is conceptually clear, but it has a practical limitation: multilinear maps do not form a vector space in a natural way that is compatible with composition and algebraic manipulation. To overcome this, multilinearity is reduced to linearity by means of a universal construction.

This construction is the *tensor product.*

Let $E_1, \dots, E_n$ be vector spaces over the same field K. Consider the set of all multilinear maps

$$T : E_1 \times \cdots \times E_n \longrightarrow F$$

into an arbitrary vector space F. The tensor product is defined so that every such multilinear map corresponds uniquely to a linear map defined on a single vector space.

Definition 3 (Tensor product)**.** The *tensor product* of the vector spaces $E_1, \dots, E_n$ is a vector space, denoted

$$E_1 \otimes \cdots \otimes E_n,$$

together with a multilinear map

$$\otimes : E_1 \times \cdots \times E_n \longrightarrow E_1 \otimes \cdots \otimes E_n,$$

satisfying the following universal property:
For every vector space F and every multilinear map

$$T : E_1 \times \cdots \times E_n \longrightarrow F,$$

there exists a unique linear map

$$\widetilde{T} : E_1 \otimes \cdots \otimes E_n \longrightarrow F$$

such that

$$T(x_1, \dots, x_n) = \widetilde{T}(x_1 \otimes \cdots \otimes x_n)$$

for all $x_i \in E_i$.

This property characterizes the tensor product completely. The elements $x_1 \otimes \cdots \otimes x_n$ are called *simple tensors*. Every element of the tensor product space is a finite linear combination of simple tensors.

The tensor product is therefore not defined by a formula, but by what it *does*: it converts multilinear problems into linear ones in a canonical way.

As a consequence, the study of multilinear maps is equivalent to the study of linear maps defined on tensor products. Specifically,

$$\text{Multilinear maps } E_1 \times \cdots \times E_n \to F \quad \longleftrightarrow \quad \text{Linear maps } E_1 \otimes \cdots \otimes E_n \to F.$$

This equivalence is the conceptual foundation of tensor theory. It explains why tensors can be manipulated algebraically, composed with linear maps, and represented concretely once bases are chosen.

Returning to the case of a single vector space E, a tensor of type (k, ℓ) may equivalently be viewed as a linear map

$$T : \underbrace{E^* \otimes \cdots \otimes E^*}_{k} \otimes \underbrace{E \otimes \cdots \otimes E}_{\ell} \longrightarrow K.$$

Thus tensors are not mysterious higher-dimensional objects. They are linear maps defined on tensor products, and their apparent complexity arises only when coordinates are introduced.

Note: Once bases are chosen, tensor products become finite-dimensional vector spaces, and tensors admit coordinate representations as multidimensional arrays. These representations depend on the chosen bases and do not define the tensor itself.

Note. This memo is Part III of an ongoing three-part work on abstract linear and geometric structures. An AI language model was used as a support tool for discussion, clarification of concepts, and stylistic refinement. All mathematical choices and final wording are the responsibility of the author.

References

[1] Portillo, Jose, *Structures I: Linear Transformations, Matrices, and Composition*, 2025.