Structures II – Geometry from Linear Structure

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In [1], vector spaces were introduced as purely algebraic objects. A vector was an abstract element of a set equipped with addition and scalar multiplication, and linear transformations were functions preserving this structure. No notion of length, angle, distance, or orthogonality was assumed or required.
This absence was intentional.

Algebra alone allows us to decide whether vectors are independent, whether a set spans a space, or whether a linear transformation is injective or surjective. However, it does not allow us to ask geometric questions. There is no intrinsic way, within a bare vector space, to compare the size of two vectors or to determine whether they point in perpendicular directions.

Geometry begins only when additional structure is imposed.

A central theme of this document is that geometry is not a replacement for linear structure, but an enrichment of it. The vector spaces introduced earlier remain unchanged; what changes is that they are now equipped with a rule that allows vectors to interact in a way that produces scalar quantities.

Informally, this rule takes two vectors and returns a scalar. From this single operation, notions such as length, angle, and distance emerge naturally.

To make this interaction precise, we introduce the notion of a bilinear map.

Let E and G be vector spaces over the same field K. A mapping

$$B : E \times G \longrightarrow K$$

is called *bilinear* if it is linear in each argument separately. That is, for fixed $y \in G$, the map

$$x \longmapsto B(x, y)$$

is a linear functional on E, and for fixed $x \in E$, the map

$$y \longmapsto B(x, y)$$

is a linear functional on G.

Explicitly, for all $x_1, x_2 \in E$, $y_1, y_2 \in G$, and all scalars $\alpha \in K$,

$$B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y), \qquad B(\alpha x, y) = \alpha B(x, y),$$

$$B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2), \qquad B(x, \alpha y) = \alpha B(x, y).$$

When $E = G$, such a mapping is called a *bilinear form* on E.

At this level, no geometric interpretation is assumed. A bilinear form is merely a rule that assigns a scalar to an ordered pair of vectors, subject to linearity in each argument. It need not be symmetric, positive, or related to any notion of length or angle.

Nevertheless, bilinear forms play a fundamental structural role. They provide a systematic way to associate vectors with linear functionals and to construct scalar quantities from pairs of vectors.

Only after additional conditions are imposed on a bilinear form do familiar geometric notions emerge. These special cases will be introduced later.

Once such a rule is fixed, vectors acquire measurable properties. Two vectors may be declared orthogonal. A vector may be assigned a length. The distance between two points may be defined in terms of the vectors connecting them.
None of these concepts exist prior to the introduction of this additional structure.

As in [1], coordinates play no fundamental role. Geometric quantities must be independent of any particular reference system. A formula that changes when the basis is changed describes a representation, not a geometric object.

Only after a basis is chosen do familiar coordinate expressions appear. Sums of squares, dot products, and quadratic equations arise as coordinate-level manifestations of the underlying structure, not as definitions of it.

This point of view explains why objects such as circles, spheres, and ellipsoids can be described by algebraic equations, yet retain a geometric meaning that is invariant under changes of coordinates. The equations depend on the chosen basis; the geometry does not.

The goal is therefore not to introduce new algebraic machinery, but to show how geometry emerges from linear structure once a suitable interaction between vectors is specified.

In later parts of this document, this interaction will be generalized further. When the geometric rule itself varies from point to point, the resulting structures lead naturally to curved spaces and manifolds. For now, the focus remains on understanding how familiar geometric notions arise from linear spaces endowed with additional structure.

The central object of this work is the linear transformation. Before introducing geometric notions such as length, angle, and distance, we make explicit a fundamental construction associated with every linear transformation: its transpose.

Let E and G be vector spaces over the same field K, and let

$$\psi : \mathsf{E} \longrightarrow \mathsf{G}$$

be a linear transformation. Recall that the dual spaces E^* and G^* consist of all linear functionals on E and G, respectively.

For each $\mathsf{g}^* \in \mathsf{G}^*$, the composition

$$\mathsf{g}^* \circ \psi : \mathsf{E} \longrightarrow \mathsf{K}$$

is a linear functional on E. This defines a mapping

$$\psi^* : \mathsf{G}^* \longrightarrow \mathsf{E}^*, \qquad \psi^*(\mathsf{g}^*) = \mathsf{g}^* \circ \psi.$$

The mapping ψ^* is linear and is called the *transpose* (or dual transformation) of ψ. The direction of the arrow is reversed:

$$\mathsf{E} \xrightarrow{\psi} \mathsf{G} \quad \Longrightarrow \quad \mathsf{G}^* \xrightarrow{\psi^*} \mathsf{E}^*.$$

This construction is intrinsic: it depends only on the linear transformation ψ and does not require any choice of basis, coordinates, or inner product.

When bases are chosen for E and G, the linear transformation ψ is represented by a matrix A, and the transpose transformation ψ^* is represented by the transposed matrix A^T. Thus, the familiar matrix transpose is the coordinate expression of the dual transformation ψ^*.

Only after additional geometric structure is introduced — specifically, an inner product identifying vectors with linear functionals — can the transpose be interpreted as an operator acting on the original vector spaces themselves. This identification marks the transition from algebraic structure to geometry.

The interaction between vectors that gives rise to geometry is not arbitrary. Its algebraic prototype is the notion of a bilinear form.

Let E be a vector space over a field K. A *bilinear form* on E is a function

$$B : E \times E \longrightarrow K$$

that is linear in each argument separately. That is, for all $x, x', y, y' \in E$ and all $\lambda \in K$,

$$B(x + x', y) = B(x, y) + B(x', y), \qquad B(\lambda x, y) = \lambda B(x, y),$$

and similarly in the second argument.

Bilinear forms arise naturally from linear transformations. If

$$\psi : E \longrightarrow E^*$$

is a linear mapping, then the formula

$$B(x, y) = \psi(x)(y)$$

defines a bilinear form on E. Conversely, every bilinear form determines such a mapping. Thus, bilinear forms encode a controlled way in which vectors may interact to produce scalars.

At this level, no geometry is present. A bilinear form may be degenerate, asymmetric, or indefinite. It allows vectors to interact, but it does not yet measure anything.

Geometry appears when the bilinear form satisfies additional constraints.

An *inner product* on a vector space E over $K = \mathbb{R}$ or $\mathbb{C}$ is a bilinear (or sesquilinear) form

$$\langle \cdot, \cdot \rangle : E \times E \longrightarrow K$$

that is symmetric (or Hermitian), positive, and non-degenerate. These properties ensure that

$$\langle x, x \rangle > 0 \quad \text{for all } x \neq 0.$$

Once an inner product is fixed, vectors acquire length. The norm of a vector $x \in E$ is defined by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Orthogonality also becomes meaningful. Two vectors $x, y \in E$ are said to be orthogonal if

$$\langle x, y \rangle = 0.$$

The order of the vectors is irrelevant. Symmetry of the inner product implies

$$\langle x, y \rangle = \langle y, x \rangle,$$

so orthogonality is a mutual relation.

The zero vector is orthogonal to every vector in E. Indeed, for any $v \in E$,

$$\langle 0, v\rangle = \langle 0+0, v\rangle = \langle 0, v\rangle + \langle 0, v\rangle,$$

which forces $\langle 0, v\rangle = 0$. Moreover, the zero vector is the only vector with this property.

Theorem (Pythagorean Theorem). Let E be a vector space with inner product. If $x, y \in E$ are orthogonal, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. If x and y are orthogonal, then $\langle x, y\rangle = 0$. Using bilinearity of the inner product,

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle.$$

The middle terms vanish by orthogonality, yielding

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

□

This identity marks the moment where geometry truly enters. Lengths, angles, and distances are no longer imposed by coordinates, but emerge from the algebraic interaction encoded by the inner product.

Thus, inner products are not primitive geometric notions. They are structured bilinear forms, added to a vector space to extend algebra into geometry without altering its linear foundation.

Once a norm has been introduced, certain geometric sets arise as direct consequences of the structure and require no additional assumptions.

Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|x\| = \sqrt{\langle x, x\rangle}$ be the associated norm. For a fixed vector $x_0 \in E$ and a scalar $r > 0$, the *closed ball* of radius r centered at x_0 is defined by

$$B_r(x_0) = \{\, x \in E \mid \|x - x_0\| \le r \,\}.$$

The corresponding *sphere* is the boundary of this set,

$$S_r(x_0) = \{\, x \in E \mid \|x - x_0\| = r \,\}.$$

These definitions are intrinsic and do not depend on any choice of coordinates or basis.

More generally, let $B : E \times E \to K$ be a symmetric positive-definite bilinear form. The set

$$\mathcal{E} = \{\, x \in E \mid B(x - x_0, x - x_0) \le r^2 \,\}$$

is called an *ellipsoid*. Thus, an ellipsoid is the unit ball associated with a bilinear form, or equivalently, with a modified inner product on E.

When a basis is chosen in which the bilinear form is diagonal, this invariant definition reduces to a familiar coordinate expression. In $\mathbb{R}^n$, one obtains

$$\mathcal{E} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \frac{(x_i - a_i)^2}{h_i^2} \le r^2 \right\}.$$

Written componentwise, this inequality reads

$$\left(\frac{x_1 - a_1}{h_1}\right)^2 + \left(\frac{x_2 - a_2}{h_2}\right)^2 + \cdots + \left(\frac{x_n - a_n}{h_n}\right)^2 \le r^2.$$

The equation depends on the chosen coordinates; the ellipsoid itself does not.

Remark (Degenerate cases). The geometric nature of the set defined above depends on the coefficients $h_1, \dots, h_n$. If $h_1 = \cdots = h_n$, the ellipsoid reduces to an n-dimensional sphere. If one or more of the parameters h_i vanish, the defining quadratic form becomes degenerate and the ellipsoid collapses into a lower-dimensional object: a disk, a line segment, or, in the extreme case, a single point.

From the structural point of view, these degenerations correspond to a loss of rank in the associated quadratic or bilinear form. Thus, changes in the algebraic structure translate directly into changes in geometric dimension.

Conceptual summary. Vector spaces provide the stage. Linear transformations describe allowable changes of state. Geometry enters only when vectors are permitted to interact in a way that produces scalars. From this interaction, lengths, angles, distances, and geometric shapes emerge. The transition from algebra to geometry is therefore not a leap, but a controlled extension of the structures already introduced.

Note. This memo is Part II of an ongoing three-part work on abstract linear and geometric structures. An AI language model was used as a support tool for discussion, clarification of concepts, and stylistic refinement. All mathematical choices and final wording are the responsibility of the author.

References

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