Notes on Linear Transformations

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$$\left\langle \langle K, +_g, \cdot \rangle,\ \langle V, +_v \rangle,\ \circ \right\rangle$$

The system above is called a *vector space* over the field K. Elements of K are called *scalars*, and elements of V are called *vectors*. The operation $\circ : K \times V \to V$ is scalar multiplication, and $+_v$ is vector addition. In what follows, symbols $\alpha_1, \ldots, \alpha_k$ always denote scalars in K, and symbols $x_1, \ldots, x_k$ always denote vectors in V.

A vector space like this is defined over a field K, where $\langle K, +_g, \cdot \rangle$ satisfies the usual *field axioms*: for all $\alpha, \beta, \gamma \in K$, addition $+_g$ is associative and commutative, has an identity $0 \in K$ and inverses $-\alpha$, multiplication $\cdot$ is associative and commutative, has an identity $1 \in K$, every nonzero element has a multiplicative inverse, and multiplication distributes over addition. Vector addition $+_v$ on V is associative and commutative, has an identity vector 0_V, and every $x \in V$ has an additive inverse $-x$. Scalar multiplication $\circ$ satisfies: $\alpha \circ (x +_v y) = \alpha \circ x +_v \alpha \circ y$, $(\alpha +_g \beta) \circ x = \alpha \circ x +_v \beta \circ x$, $(\alpha \cdot \beta) \circ x = \alpha \circ (\beta \circ x)$, and $1 \circ x = x$ for all $\alpha, \beta \in K$ and $x, y \in V$.

A finite set of vectors $\{b_1, \ldots, b_n\} \subset V$ is called a *basis* of the vector space V if every vector $x \in V$ can be written uniquely as a linear combination $x = \alpha_1 \circ_E b_1 + \alpha_2 \circ_E b_2 + \cdots + \alpha_n \circ_E b_n, \quad \alpha_1, \ldots, \alpha_n \in K$. A basis therefore provides a minimal and non-redundant set of building blocks for the entire space. Once a basis is fixed, every vector is completely determined by its coefficients relative to that basis.

Given vectors $x_1, \ldots, x_k \in V$ and scalars $\alpha_1, \ldots, \alpha_k \in K$, an expression of the form

$$x_t = \alpha_1 \circ_E x_1 + \alpha_2 \circ_E x_2 + \cdots + \alpha_k \circ_E x_k$$

is called a *linear combination* of the vectors $x_1, \ldots, x_k$.

By closure of vector addition and scalar multiplication, every linear combination is itself a vector in V. Linear combinations therefore describe the permitted way to construct new vectors from given ones inside the same vector space.

Let $\left\langle \langle K, +_g, \cdot \rangle,\ \langle E, +_E \rangle,\ \circ_E \right\rangle$ and $\left\langle \langle K, +_g, \cdot \rangle,\ \langle G, +_G \rangle,\ \circ_G \right\rangle$ be vector spaces over the same field K.

A function $\psi : E \to G$ is called a *linear transformation* if it preserves linear combinations. That is, for all vectors $x_1, \ldots, x_k \in E$ and all scalars $\alpha_1, \ldots, \alpha_k \in K$,

$$\psi(\alpha_1 \circ_E x_1 + \alpha_2 \circ_E x_2 + \cdots + \alpha_k \circ_E x_k) = \alpha_1 \circ_G \psi(x_1) + \alpha_2 \circ_G \psi(x_2) + \cdots + \alpha_k \circ_G \psi(x_k).$$

This equation expresses the defining property of linearity: vectors may be combined first and then transformed, or transformed first and then combined, with identical results. Linear transformations therefore preserve the algebraic structure of vector spaces independently of any choice of coordinates.

Let $\psi : E \to G$ be a linear transformation, and let $B_E = \{e_1, \ldots, e_n\} \subset E, B_G = \{g_1, \ldots, g_m\} \subset G$ be bases of the domain and codomain, respectively. For each basis vector e_i, the image $\psi(e_i)$ can be written uniquely as a linear combination of the basis vectors of G: $\psi(e_i) = a_{1i} \circ_G g_1 + a_{2i} \circ_G g_2 + \cdots + a_{mi} \circ_G g_m$. The scalars $a_{ji} \in K$ assemble into an $m \times n$ matrix

$$A = (a_{ji}),$$

called the *matrix of ψ relative to the bases B_E and B_G*. The matrix A represents the linear transformation ψ in coordinates. Different choices of bases generally lead to different matrices, but they all represent the same underlying linear transformation.

Matrices thus do not define linear transformations; rather, they are concrete representations of linear transformations once bases (coordinate systems) have been chosen.

This abstract formulation is independent of geometry, yet it underlies most quantitative models in science and engineering. In modern machine learning, for example, embeddings and weight matrices are linear transformations on high-dimensional vector spaces, composed with non-linear activations. Even where non-linearity dominates, linear transformations provide the structural backbone—the universality of a man-made linear structure, even though the universe itself is not linear.

References

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2. Cohen, Leon and Ehrlich, Gertrude, *The Structure Of The Real Number System*, D. Van Nostrand Company, 1963.