Math topics in AMS Euler fonts

Jose Portillo & chatGPT

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Abstract

This document demonstrates the use of the Euler math font, paired with elegantly typeset theorems and proofs in the spirit of D. E. Knuth.

Some mathematical symbols using AMS Euler

Here is a lovely Euler-style formula using the classic integral:

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1}$$

$$\int_{a}^{b} f(x)\,dx \text{ } - \text{ } \frac{1}{2\pi i}\oint_{z=r} \frac{g(z)\,dz}{z^n}$$

```
\[
\int_{a}^{b} f(x)\,dx \text{  } - \text{  } \frac{1}{2\pi i}\oint_{z=r}  \frac{g(z){z^n}
\]
```

And the continued fraction with Knuth's Volume 2 Seminumerical Algorithms compact notation:

$$\frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{\dots + \frac{1}{x_{n-1} + \frac{1}{x_n}}}}} = 1 / \left(x_1 + 1 / (x_2 + 1/ (\dots / (x_{n-1} + 1 / x_n) \dots))\right) = //x_1, x_2, \dots, x_n// \tag{2}$$

```
\begin{equation}
\frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{\dots + \frac{1}{x_{n-1} + \frac{1}{x_n}}}}} =
1 / \left(x_1 + 1 / (x_2 + 1/ (\dots / (x_{n-1} + 1 / x_n) \dots))\right) = \KnuthSlashes{x_1, x_2, \dots, x_n}
\end{equation}
```

A Case of N-Sphere

Metric: A Generalized Distance Function

In mathematics, a **metric** is a function that defines a distance between elements of a set. A **metric space** is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function satisfying the following properties for all $x, y, z \in X$:

- **Non-negativity:** $d(x, y) \geq 0$
- **Identity of indiscernibles:** $d(x, y) = 0$ if and only if $x = y$
- **Symmetry:** $d(x, y) = d(y, x)$
- **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

The standard Euclidean metric in $\mathbb{R}^n$ is defined as:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

```
\[
d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}
\]
```

This formula, which we recognize as the Euclidean distance, has deep roots in classical geometry. It reflects the natural "straight-line" distance between two points in Euclidean space and directly invokes **Pythagoras' Theorem**, which asserts that in a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides. In the case of Euclidean distance, the distance between two points in space can be interpreted as the generalized form of the Pythagorean Theorem — summing squared differences across multiple dimensions, and then taking the square root. This concept is the foundation of how we measure straight-line distances in any n-dimensional space.
More generally, a metric can be induced by a *quadratic form*, such as:

$$d_A(x, y) = \sqrt{(x-y)^\top A^{-1}(x-y)}$$

where A is a symmetric positive definite matrix. This defines a *warped* geometry, such as an ellipsoid. In physics — especially in General Relativity — a more general kind of metric called a **metric tensor** $g_{\mu\nu}$ is used to measure intervals in curved spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Unlike the Euclidean case, this can produce zero or even negative values for ds^2, reflecting the Lorentzian structure of spacetime.

Surface vs. Solid

The difference between the boundary (surface) and the filled-in shape (solid) — crucial when working with spheres and their higher-dimensional analogs:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \quad \text{(Surface, } n-1\text{-dimensional)}$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2 \quad \text{(Solid, } n\text{-dimensional)}$$

The n-Sphere and Generalization to Ellipsoids

An n-sphere of radius r is defined as:

$$S^{n-1} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^{n} \;\middle|\; x_1^2 + x_2^2 + \dots + x_n^2 = r^2 \quad \text{(i.e., } \sum_{i=1}^n x_i^2 = r^2\text{)} \right\} \tag{3}$$

```
\begin{equation}
S^{n-1} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^{n} \;\middle|\;
x_1^2 + x_2^2 + \dots + x_n^2 = r^2 \quad \text{(i.e., }
\sum_{i=1}^n x_i^2 = r^2\text{)} \right\}
\end{equation}
```

or equivalently, as an ellipsoid centered at $h = (h_1, h_2, \dots, h_n)$ with principal semi-axes a_i:

$$S^{n-1} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \; \middle|\; \frac{(x_1 - h_1)^2}{a_1^2} + \frac{(x_2 - h_2)^2}{a_2^2} + \dots + \frac{(x_n - h_n)^2}{a_n^2} = r^2 \quad \text{(i.e., } \sum_{i=1}^{n} \frac{(x_i - h_i)^2}{a_i^2} = r^2, \; \text{where} \; h_i = 0, \; a_i = 1 \; \forall i\text{)} \right\} \tag{4}$$

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\begin{equation}
S^{n-1} = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \; \middle|\;
\frac{(x_1 - h_1)^2}{a_1^2} + \frac{(x_2 - h_2)^2}{a_2^2} + \dots + \frac{(x_n - h_n)^2}{a_n^2} = r^2 \quad \text{(i.e., }
\sum_{i=1}^{n} \frac{(x_i - h_i)^2}{a_i^2} = r^2, \; \text{where} \; h_i = 0, \; a_i = 1 \; \forall i\text{)} \right\}
\end{equation}
```

This formulation is a generalization of the Pythagorean theorem to ellipsoidal surfaces and is another example of how distance is defined geometrically by sums of squares. The relationship between the squared components of the points on the surface embodies the extension of Pythagoras' insight to multiple dimensions, where each additional dimension adds a new term to the sum of squares, much like the familiar two-dimensional case.
A more general and compact form of the n-sphere (or ellipsoid) equation is expressed via a quadratic form:

$$(x-h)^\top A^{-1}(x-h) = r^2, \tag{5}$$

where

- $x \in \mathbb{R}^n$ is the coordinate vector of a point on the surface.
- $h \in \mathbb{R}^n$ is the center of the ellipsoid.
- A is a symmetric positive-definite matrix that encodes the shape and orientation of the ellipsoid.
- A^{-1} is the inverse of A, which appears in the quadratic form defining the surface.
- The left-hand side, $(x-h)^\top A^{-1}(x-h)$, acts as a generalized squared distance from the center with respect to the ellipsoid geometry.
- r^2 is the squared "radius" (or scale) determining the size of the ellipsoid.

Understanding the role of the matrix A:

1. The matrix A governs how the unit sphere is transformed into the ellipsoid by stretching and rotating the space.
2. If A is the identity matrix I, then the ellipsoid reduces to a perfect sphere with equal axes.
3. Generally, A may have off-diagonal entries, indicating that the ellipsoid's principal axes are not aligned with the standard coordinate axes — the ellipsoid can be tilted or skewed.
4. To reveal the principal axes and their lengths, we diagonalize A: find a matrix P of eigenvectors and a diagonal matrix D of eigenvalues such that
$$A = PDP^\top$$
5. The columns of P are the eigenvectors of A, pointing along the ellipsoid's principal directions (the natural axes).
6. The diagonal entries λ_i of D (the eigenvalues) quantify how the ellipsoid is stretched along each principal axis; specifically, the semi-axis lengths a_i relate to these eigenvalues by
$$a_i = \frac{r}{\sqrt{\lambda_i}}$$

By changing coordinates to $y = P^{-1}(x-h)$, the ellipsoid equation becomes
$$y^\top D^{-1} y = r^2,$$
which corresponds to an axis-aligned ellipsoid where each axis length is explicitly seen and the shape is simplified.
Matrix A encodes the essence of the a_i's and more:

- The a_i^2 (squared semi-axis lengths), but in a more generalized way.
- Rotation information — through off-diagonal terms (correlations between axes).
- Complete geometry of the ellipsoid — both its orientation and its deformation.

Diagonalizing A allows us to fully understand the ellipsoid's shape and orientation:

- $A = PDP^\top$, where:
  - P is an orthogonal matrix whose columns are the eigenvectors of A, representing the principal directions of the ellipsoid.
  - D is a diagonal matrix of eigenvalues λ_i, each related to a semi-axis by $a_i = \frac{1}{\sqrt{\lambda_i}}$ (assuming $r = 1$). For general r, $a_i = \frac{r}{\sqrt{\lambda_i}}$.

Hence, diagonalizing A recovers both:

- The lengths of the semi-axes (via the eigenvalues),
- The directions of the semi-axes (via the eigenvectors).

Example in $\mathbb{R}^2$:
Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix},$$

with $h = \mathbf{0}$ and $r^2 = 1$. Then the ellipsoid is defined by:

$$x^\top A^{-1} x = 1,$$

which describes a rotated ellipse. Diagonalizing A yields eigenvalues $\lambda_1 = 3 + \sqrt{2}$ and $\lambda_2 = 3 - \sqrt{2}$, giving semi-axis lengths:

$$a_1 = \frac{1}{\sqrt{3+\sqrt{2}}}, \quad a_2 = \frac{1}{\sqrt{3-\sqrt{2}}},$$

and directions aligned with the corresponding eigenvectors of A.

Final Thought: Whether modeling data distributions, optimizing constrained problems, or exploring curved spacetime, the geometry encoded by A — via its eigenstructure — is a recurring mathematical motif. The interplay of distance, shape, and orientation lies at the heart of understanding structure in $\mathbb{R}^n$ and beyond.

The Ellipsoid in Hyperspherical Coordinates

Using hyperspherical coordinates, any point $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ can be expressed as

$$\begin{cases} x_1 = r \cos\theta_1, \\ x_2 = r \sin\theta_1 \cos\theta_2, \\ x_3 = r \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ \quad \vdots \\ x_{n-1} = r \left(\prod_{j=1}^{n-2} \sin\theta_j \right) \cos\theta_{n-1}, \\ x_n = r \left(\prod_{j=1}^{n-2} \sin\theta_j \right) \sin\theta_{n-1}, \end{cases}$$

where

$$r \geq 0, \quad \theta_1, \dots, \theta_{n-2} \in [0, \pi], \quad \theta_{n-1} \in [0, 2\pi).$$

Substituting the expressions for x_i into the ellipsoid equation yields

$$\frac{r^2 \cos^2\theta_1}{a_1^2} + \frac{r^2 \sin^2\theta_1 \cos^2\theta_2}{a_2^2} + \frac{r^2 \sin^2\theta_1 \sin^2\theta_2 \cos^2\theta_3}{a_3^2} + \cdots + \frac{r^2 \left(\prod_{j=1}^{n-2} \sin^2\theta_j\right) \cos^2\theta_{n-1}}{a_{n-1}^2} + \frac{r^2 \left(\prod_{j=1}^{n-2} \sin^2\theta_j\right) \sin^2\theta_{n-1}}{a_n^2} = 1$$

Finally, we can write the ellipsoid as a set of hyperspherical coordinate tuples:

$$\mathbb{E}^{n-1} = \left\{ (r, \theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^+ \times [0,\pi]^{n-2} \times [0, 2\pi) \;\middle|\; \sum_{i=1}^{n} \frac{x_i^2(r,\theta)}{a_i^2} = 1 \right\},$$

where each $x_i(r, \theta)$ is defined above.

Frobenius Norm: A Metric for Matrices

In addition to the standard Euclidean distance and the generalized quadratic form $d_A(x, y) = \sqrt{(x-y)^\top A^{-1}(x-y)}$, we can define a specific norm for matrices, known as the **Frobenius norm**, which acts as a natural distance function for matrix spaces. Given two matrices A and B of the same dimensions $m \times n$, the *Frobenius norm* is defined as:

$$\|A - B\|_F = \left\{ \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} - b_{ij}|^2} = \sqrt{(a_{11} - b_{11})^2 + (a_{12} - b_{12})^2 + \cdots + (a_{1n} - b_{1n})^2 + \cdots + (a_{m1} - b_{m1})^2 + \cdots + (a_{mn} - b_{mn})^2} \right\} \tag{6}$$

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{\footnotesize
\begin{equation}
\|A - B\|_F = \left\{ \sqrt{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} - b_{ij}|^2 } =
\sqrt{(a_{11} - b_{11})^2 + (a_{12} - b_{12})^2 + \dots +
(a_{1n} - b_{1n})^2 + \dots + (a_{m1} - b_{m1})^2 + \dots + (a_{mn} - b_{mn})^2 } \right\}
\end{equation}
}
```

where m and n represent the row and column dimensions of matrices A and B, respectively.
This is a specific case of the quadratic form distance, where the matrix A in the quadratic form is replaced by the identity matrix (assuming we are working in standard Euclidean space).

Properties of the Frobenius Norm

The Frobenius norm satisfies all the defining properties of a metric:

- **Non-negativity:** The Frobenius norm is always non-negative, i.e., $\|A - B\|_F \geq 0$.
- **Identity of indiscernibles:** It equals zero if and only if $A = B$.
- **Symmetry:** $\|A - B\|_F = \|B - A\|_F$.
- **Triangle inequality:** The Frobenius norm obeys the triangle inequality, making it a valid metric for matrices.

Thus, the Frobenius norm serves as a distance function in the space of matrices, much like the Euclidean distance serves as a distance function in the space of vectors.

Application of the Frobenius Norm in Metric Spaces

The Frobenius norm can be seen as a natural extension of the Euclidean distance. Instead of comparing vectors, it compares matrices by treating them as vectors in $\mathbb{R}^{m \times n}$. When applied to the space of matrices, it provides a means to define a metric space on matrices. This is particularly useful in fields such as *machine learning*, *optimization*, and *signal processing*, where matrix approximation and error minimization often occur.
In this sense, the Frobenius norm serves as a **generalized metric** in the study of matrix spaces, offering a way to quantify the "distance" between matrices, much like the Euclidean distance quantifies the distance between vectors. For example, in matrix factorization problems, the Frobenius norm can be used to minimize the reconstruction error between an approximation and the true matrix.

Manifolds, Metric Tensors, and the Geometry Playground

In the grand scheme of geometry, the **manifold** is the "space" where all the geometric drama unfolds. Formally, a manifold is a set that locally resembles $\mathbb{R}^n$ — meaning that if we zoom in close enough at any point, it looks like flat Euclidean space, even if globally it might be curved, twisted, or have a more complicated shape.
Think of the manifold as the big stage or playground. On this stage live various geometric objects: curves, surfaces, volumes, and their higher-dimensional analogs, such as our familiar n-sphere or n-ellipsoid. These shapes are subsets or surfaces embedded inside the manifold.

Two Perspectives on Manifolds. There are two equally valid but conceptually distinct ways to understand what a **manifold** is.

- **Intrinsic viewpoint:** In modern differential geometry, a manifold is defined abstractly as a topological space that is locally homeomorphic to $\mathbb{R}^n$, and equipped with smooth transition functions between coordinate charts. It is not necessarily a subset of any ambient space — rather, it is an independent structure whose local resemblance to Euclidean space allows us to apply calculus and geometry. This is the viewpoint used in general relativity and in the work of Hawking and Penrose.
- **Extrinsic viewpoint:** In classical geometry and topology, manifolds are often considered as smooth subsets embedded in a Euclidean space $\mathbb{R}^n$. For example, a 2-sphere S^2 embedded in $\mathbb{R}^3$ is a 2-dimensional manifold because it locally looks like $\mathbb{R}^2$. This approach is commonly found in geometric analysis and in older literature, such as Morton L. Curtis's 1962 paper on continua in Euclidean space.

These two viewpoints are not in conflict. By the Whitney Embedding Theorem, any smooth n-dimensional manifold can be embedded in $\mathbb{R}^{2n}$, meaning the intrinsic picture can always be realized extrinsically — though the converse is not always necessary. Both perspectives are valuable depending on context.

But to understand geometry on this stage, we need a way to measure distances and angles — this is where the **metric tensor** $g_{\mu\nu}$ enters the scene.

- At each point p on the manifold, $g_{\mu\nu}(p)$ is a symmetric, non-degenerate matrix — basically a local ruler or measuring tape.
- Together, these matrices form a *field* over the manifold, smoothly varying from point to point. In other words, $g_{\mu\nu}$ is a *matrix-valued function* that assigns a metric matrix to every point on the manifold.
- This algebraic structure allows you to measure infinitesimal distances via the line element:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu,$$

  capturing how "lengths" and "angles" are defined locally, even if the space itself is curved or warped.
- Beyond measuring, $g_{\mu\nu}$ enables algebraic operations like raising and lowering indices, contraction, and manipulation of tangent vectors and tensors — the very tools that let us define curvature, geodesics, and the intrinsic geometry of the manifold.

In essence, the metric tensor is a smooth field of matrices—an algebraic structure equipped with all the operations you'd expect from tensor algebra, but with a dose of calculus (since it varies continuously over the manifold). Fancy speak: it's *matrix math on steroids* combined with differential geometry.

The matrix A is essentially a special case of a metric tensor — a fixed, constant matrix encoding how we stretch or skew distances in a flat, linear space. The metric tensor generalizes this idea to curved spaces and varies smoothly with position, describing more complex geometries like the curved surface of a sphere or the fabric of spacetime itself.

Putting it all together:

- The **manifold** is the space — the stage where geometric objects live.
- The **metric tensor** $g_{\mu\nu}$ is the local ruler, smoothly assigned everywhere, defining how distances and angles behave.
- The **matrix** A is a "flat-space" cousin of $g_{\mu\nu}$, encoding geometry in simpler, often linear algebraic contexts.
- Your n-sphere or n-ellipsoid is a shape drawn on this stage, its geometry influenced by the ambient metric.

In short: the manifold is the playground, and the metric tensor (along with matrices like A) are the tools to understand the shapes, distances, and curvatures of everything playing on it.

This beautiful interplay between algebra, calculus, and geometry is the heartbeat of modern mathematical physics and differential geometry — whether you're modeling data ellipsoids or the warped fabric of spacetime itself.

Sidebar – Hawking, the Metric, and the Geometry of Spacetime

In his foundational writings on general relativity, Stephen Hawking described the metric as a symmetric tensor g of type $(0, 2)$, written locally as

$$g = g_{ab}\, du^a \otimes du^b,$$

where the tensor product $\otimes$ is often left implicit, giving rise to the notation ds^2. This expression defines a scalar product on each tangent space T_p, turning the metric into a smoothly varying field of bilinear forms.

Crucially, Hawking emphasizes that the metric must be *non-degenerate* — meaning no nonzero vector $X \in T_p$ can be orthogonal to all vectors $Y \in T_p$. In matrix terms, the component matrix (g_{ab}) must be invertible at every point.

This is precisely the same metric tensor $g_{\mu\nu}$ we've been exploring: a field of symmetric, non-degenerate matrices defined over a manifold. It's an algebraic structure — a smoothly varying assignment of inner products — that encodes the very geometry (and curvature) of the space or spacetime itself.

Sidebar – Exotic Spheres and the Wild World of Smoothness

In 1956, John Milnor discovered that the 7-dimensional sphere S^7 admits multiple smooth structures — now known as *exotic spheres*. These are manifolds that are topologically equivalent (homeomorphic) to the standard 7-sphere, but not diffeomorphic to it.

This astonishing result opened up the field of *differential topology*, revealing that even "simple" manifolds like spheres can hide deep geometric complexity beneath their smooth surfaces.

In the context of our article: exotic spheres still live in the same manifold universe we're exploring — with metric tensors $g_{\mu\nu}$, tangent spaces, and all. But they show us that the choice of differentiable structure can fundamentally change the geometry without altering the underlying shape.

Note: The Unified Philosophy of Normalization and Distance

In vector spaces:

$$\text{Normalization} = \text{take a direction and set its length to } 1 \quad \Rightarrow \quad \text{It lives on the unit sphere}$$

In data plotting:

$$\text{Normalization} = \text{scale data to a target range (like } [0,1]) \quad \Rightarrow \quad \text{So your chart isn't a chaotic mess}$$

In statistics:

$$\text{Normalization (e.g., z-scores)} = \text{shift and scale data to have mean } 0 \text{ and standard deviation } 1 \quad \Rightarrow \quad \text{So different datasets become comparable}$$

All of these share the same spirit:

Strip away scale. Focus on structure, direction, and relationships.

Whether it's vectors pointing at you from hyperspace, plot lines hugging the axes, or data distributions lining up for comparison—it's all the same clever trick: *rescale without distorting.*

Here's the beautiful connection:

- **Normalizing data** = Rescaling to fit a certain magnitude or range.
- **Normalizing a vector** = Rescaling to fit a unit length.
- **Normal distribution** = A curve centered at the mean, with unit spread, appearing naturally in random processes.

Different contexts, same philosophical core:

Let's keep things in proportion.

In high dimensions:

- Sampling from a multivariate normal distribution gives directions with no preference—it's isotropic (same in all directions).
- Normalizing such a random vector projects it onto the unit sphere.

So: the normal distribution gives you the raw direction; normalization gives you the pure direction vector. Two sides of the same geometric coin.

Historical tangent: The name "normal" was popularized in the 19th century by Karl Pearson. He called it the *normal distribution* because it appeared everywhere. Mathematicians observed:

- Measurement errors often followed this curve.
- Averages of random variables tended toward it (by the Central Limit Theorem).
- It's symmetric, smooth, and mathematically convenient.

So "normal" became shorthand for "ubiquitous and nice."

Pearson's Chi-Square and the Sphere: Long-lost Cousins?
Let's write the version Knuth mentions more generally:

$$V = \sum_{s=2}^{12} (Y_s - np_s)^2$$

That's the sum of squared deviations between observed counts Y_s and expected counts np_s, where n is the number of trials and p_s the probability of outcome s.
This resembles the classic sphere equation:

$$x_1^2 + x_2^2 + \cdots + x_k^2 = r^2$$

Are they connected?

Yes—conceptually!

Both express a kind of **distance** in a space:

Chi-Square Statistic = Distance in Statistical Space:

$$\chi^2 = \sum_{s=2}^{12} \frac{(Y_s - np_s)^2}{np_s}$$

This is:

- A squared distance between vectors: $\vec{Y}$ and $\vec{E} = n\vec{p}$
- Scaled by np_s to account for how "rare" or "expected" each value is
- A way to quantify how far the observed data strays from expected randomness

Sphere Equation = Distance in Geometric Space:

$$\sum_{i=1}^{n} x_i^2 = r^2$$

This measures how far a point $\vec{x}$ lies from the origin in Euclidean space.

Deeper Insight:

- Both rely on squared norms: $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$
- Both define a "shell" of equal-distance points from a center
- The chi-square test is a metric in probability space; the sphere is a metric in geometric space

Bonus Twist: If $Z_i \sim \mathcal{N}(0, 1)$, then:

$$\chi_k^2 = \sum_{i=1}^{k} Z_i^2$$

This defines the **chi-square distribution** with k degrees of freedom — which is literally the squared distance of a random point from the origin in standard normal space.
Sound familiar?

It's the probabilistic cousin of the sphere's radius squared.

Summary Table:

Concept	Formula	Interprets...
Sphere Equation	$\sum x_i^2 = r^2$	Fixed-radius in Euclidean space
Chi-Square Statistic	$\sum \frac{(Y-E)^2}{E}$	Distance between observed and expected (normalized)
Chi-Square Random Variable	$\sum Z_i^2$	Distance from origin of random normal vector

Conclusion: Whether you're testing dice for fairness, plotting data for clarity, or projecting points on an N-sphere, the underlying game is the same:

Distance. Geometry. Proportionality. Normalization.

These aren't isolated techniques — they're recurring patterns that unify geometry, probability, and statistical reasoning. Recognize one, and you're halfway to understanding the rest.

Relation Between the Electric Field and the 3-Dimensional Ball Equation

In classical electromagnetism, Maxwell's equations describe the behavior of electric and magnetic fields. A key feature of these fields is their radial symmetry, particularly in the case of point sources such as electric charges. The electric field $\vec{E}$ due to a point charge q is described by Coulomb's law:

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r},$$

where r is the radial distance from the charge, and $\hat{r}$ is the unit vector pointing radially outward from the charge. This equation indicates that the electric field strength diminishes with the square of the distance from the charge. The $\frac{1}{r^2}$ dependence arises from the fact that the field spreads out over the surface of a sphere, and the surface area of a sphere grows with r^2.
On the other hand, the equation for a **3-dimensional ball** in $\mathbb{R}^3$ is given by:

$$B^3 = \left\{(V_1, V_2, V_3) \in \mathbb{R}^3 \,\middle|\, V_1^2 + V_2^2 + V_3^2 \le r^2\right\}.$$

Here, r^2 defines the volume inside a ball of radius r, centered at the origin. This equation describes a **solid ball** in three-dimensional space, representing all points that are within a distance r from the origin. Both equations share a fundamental geometric property: they describe **radial symmetry** around a central point (the origin). In Coulomb's law, the electric field spreads out from a point charge, and the field strength decreases as the **radius r** increases, following a $\frac{1}{r^2}$ dependence. This can be conceptually linked to the 3-dimensional ball equation, where r^2 determines the size of the ball. The electric field in Coulomb's law spreads over a spherical surface, and as the radius increases, the field strength diminishes because the field is distributed over a larger area.

Similarly, the volume of the 3-dimensional ball increases as r^3, but the relevant factor here is the **surface area** (which grows as r^2) when discussing the field's propagation. The analogy helps us understand the **spatial distribution** of the field in a more intuitive way, as the field's strength depends on the distance from the source, just as the volume and surface area of the ball depend on r.

Thus, the r^2 in Coulomb's law reflects how the electric field's strength diminishes with the **increase of distance** from the point charge, while the r^2 in the 3-ball equation describes how the **geometric volume** grows with distance. The connection between these two r^2's lies in the concept of radial spread—whether it's a field spreading out over a spherical surface, or the growth of a 3-dimensional region. Both phenomena highlight the importance of distance in shaping how fields and regions expand in space.

Historical Note. Although James Clerk Maxwell did not use the term "3-ball" in his formulation of electromagnetism, the geometric concept was inherently present in his reasoning. When evaluating electric flux or field strength at a distance r from a point charge, Maxwell considered spherical symmetry and computed integrals over spherical surfaces. In modern terms, these surfaces are the boundaries of solid regions defined as:

$$B^3 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 \le r^2\right\},$$

which is the definition of a closed 3-dimensional ball in $\mathbb{R}^3$.
Therefore, while the formal notion and terminology of the "3-ball" came later in mathematical history, Maxwell's physical models and mathematical techniques were already operating within the geometric structure we now recognize as the 3-ball. His treatment of radial fields, spherical surfaces, and symmetry were deeply aligned with the core intuition behind this concept.

We define the n-dimensional ball (or n-ball) generally as:

$$B^n = \left\{ (V_1, \ldots, V_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} V_i^2 \leq r^2 \right\}.$$

In particular, the case $n = 3$ — the familiar 3-ball — is expressed more concretely as:

$$B^3 = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \leq r^2 \right\},$$

highlighting the geometric intuition behind the algebraic structure.

Radial Symmetry in Physics: The N-Ball Hidden Behind Fundamental Forces

In the heart of classical physics lie some of its most famous laws—Newton's law of gravitation, Coulomb's law of electrostatics, and the Laplace equation for potential fields. Though these equations come from seemingly different domains, they all share a common geometric soul: **radial symmetry** around a point in space. This symmetry naturally invokes the mathematics of the **n-ball** and the **$(n-1)$-sphere**.

The Common Geometry of Inverse-Square Laws

Both Newton's gravitational law and Coulomb's electrostatic law follow the same mathematical form:

$$\text{Coulomb's Law:} \qquad F_e = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

$$\text{Newton's Law of Gravitation:} \quad F_g = G \cdot \frac{m_1 m_2}{r^2}$$

Why $\frac{1}{r^2}$? Because in three-dimensional space, the *surface area* of a sphere centered at the origin is:

$$A = 4\pi r^2.$$

So, any field (gravitational or electric) that *radiates uniformly outward* from a point must distribute its effect over a growing spherical surface. The $\frac{1}{r^2}$ scaling reflects how much that influence *dilutes* with distance—*it's just geometric dilution across the 2-sphere.*

Potentials and Laplace's Equation

In fields with radial symmetry, potential functions (gravitational or electrostatic) often take the form:

$$\phi(r) = \frac{C}{r},$$

which is the solution to **Laplace's equation** in spherical coordinates:

$$\nabla^2 \phi = 0.$$

The Laplacian encodes the way scalar values (like potential energy) spread in space, and when the source has spherical symmetry, the solutions reflect this through r-based expressions—again drawing directly from the properties of spheres and balls in $\mathbb{R}^3$.

The N-Ball Perspective

Let's step back. The **3-ball** in $\mathbb{R}^3$ is:

$$B^3 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq r^2\right\}.$$

It contains *all points* at a distance r or less from the origin.
Now compare this with a physical field: at each radius r, the field's effect is defined over the surface ∂B^3 (the sphere), and the total influence within radius r accumulates through the *volume* of the ball. So we get:

- **Force laws** → tied to the *surface area* of ∂B^3: $\frac{1}{r^2}$,
- **Potential functions** → tied to the *integration* over the ball's volume: $\frac{1}{r}$,
- **Energy densities and field lines** → often discussed in terms of *flux through spheres*.

TL;DR: Physics Flows Through the Ball

Law / Concept	Depends on r	Ball/Sphere Link	Meaning
Coulomb's Law	$\frac{1}{r^2}$	Surface of sphere	Electric field dilutes over area
Newton's Gravitation	$\frac{1}{r^2}$	Surface of sphere	Gravity spreads radially
Potential (Gravity/Electric)	$\frac{1}{r}$	Integral over sphere	Accumulated field effect
Laplace's Equation	r-based	Radial symmetry in space	Governs behavior in vacuum
N-Ball Geometry	$r^2 \leq \dots$	Geometry of region	Region containing source or effect

Conclusion: Geometry and Physics Are Old Friends

These equations aren't just coincidentally similar—they reflect a **deep unity between physics and geometry**. The reason so many natural laws involve r, r^2, or $\frac{1}{r^2}$ is because we live in a space shaped like $\mathbb{R}^3$, and the way influence propagates in that space is governed by its geometric structure. The **3-ball and 2-sphere** show up not as metaphors, but as **underlying realities** behind how the universe behaves.

The N-Ball and the Quest for a Theory of Everything

The search for a *Theory of Everything* (TOE) — an ultimate framework unifying all fundamental interactions in nature — is one of the greatest quests in modern physics. If such a master theory exists, the humble N-Ball, a geometric object defined by

$$B^N = \left\{\vec{x} \in \mathbb{R}^N \mid \|\vec{x}\|^2 \leq r^2\right\},$$

will almost certainly play a starring role. Here is why.

Geometry at the Core of Physics

Modern physics teaches us that geometry is the language of nature. Einstein's General Relativity interprets gravity as the curvature of spacetime itself. To describe spacetime, particularly in theories extending beyond our familiar 4 dimensions (such as string theory's 10 or 11 dimensions), mathematicians and physicists use higher-dimensional geometric objects — the N-Balls and spheres naturally arise as fundamental domains and symmetry spaces in these theories.

Four Fundamental Forces and Their Geometric Underpinnings

Nature exhibits four fundamental forces:

- **Gravity:** The curvature of spacetime, described by the metric tensor on a (pseudo-)Riemannian manifold. Its geometry dictates how matter and light move.

- **Electromagnetism:** Governed by Maxwell's equations, which can be understood as fields living on spaces that often have spherical symmetry, linked closely to 3-dimensional balls and their boundaries (spheres).
- **Strong Nuclear Force:** Described by Quantum Chromodynamics (QCD), a gauge theory with symmetry groups acting on internal spaces that can be modeled using higher-dimensional geometric objects like spheres and balls.
- **Weak Nuclear Force:** Another gauge theory, unified with electromagnetism in the electroweak theory, also reliant on symmetry groups and geometric structures analogous to those seen in N-dimensional spaces.

The Role of the N-Ball

The N-Ball serves as a fundamental building block in these theories:

- *Symmetry and Group Actions:* Lie groups such as $SO(N)$, which describe rotational symmetries, act naturally on N-Balls and their boundaries (spheres), enabling the elegant mathematical formulation of force symmetries.
- *Locality and Integration Domains:* Field theories require integration over finite regions of space or spacetime; the N-Ball often provides a clean, well-defined domain for such operations.
- *Normalization and Probability:* Quantum theories work with normalized states—points on unit spheres in high dimensions (the boundaries of N-Balls)—making the N-Ball central to the probabilistic interpretation of the universe.
- *Higher Dimensions and Compactifications:* Theories beyond four dimensions compactify extra dimensions into small geometric shapes, often modeled on generalized balls or spheres, making the N-Ball indispensable in describing these hidden spaces.

In Conclusion

While Maxwell and even Einstein did not explicitly think in terms of N-Balls, the language of modern theoretical physics — from quantum fields to string theory — is drenched in the geometry of spheres and balls in multiple dimensions.
The N-Ball is not just a mathematical curiosity; it is a fundamental geometric lens through which the universe's deepest secrets may be glimpsed. Should a Theory of Everything ever be written down, it will almost certainly include N-Balls quietly shaping the fabric of reality, reminding us that the universe, in all its complexity, is ultimately bound by elegant geometry.
—
So yes: when hunting for the master key to nature, keep an eye on the N-Ball — the shape that might just hold the shape of everything.

Theorem 1. *Let f be differentiable on its domain. Then f is continuous.*

Proof. Differentiability at a point implies the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists, which leads directly to

$$\lim_{h\to 0} f(a+h) = f(a).$$

Thus, f is continuous at every point in its domain. □

Corollary 2. *All polynomial functions are continuous on $\mathbb{R}$.*

Proof. Polynomials are differentiable everywhere. By the theorem, they are also continuous everywhere. □

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{7}$$

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \cdot \frac{z^k}{k!} \tag{8}$$

Apendix A. Conclusions on the n-Dimensional Sphere and its Surface

In this section, we summarize the key points regarding the relationship between the dimension of the solid sphere and its surface:

- In n-dimensional space, the solid sphere (or ball) is an n-dimensional object, denoted as B^n. The solid sphere includes all points inside and on the surface, and the object dimension matches the space dimension.
- The surface of the n-dimensional sphere, denoted S^{n-1}, is an $(n-1)$-dimensional object. Despite being embedded in n-dimensional space, the surface is intrinsically $(n-1)$-dimensional, as only $n-1$ independent coordinates are needed to describe any point on the surface.
- The surface S^{n-1} is the boundary of the solid sphere B^n, and it represents the $n-1$-dimensional shell that encloses the entire solid ball. This distinction illustrates how the object designation is always one dimension lower than the space in which it resides, i.e., B^n is n-dimensional, while S^{n-1} is $(n-1)$-dimensional.
- These relationships hold in any n-dimensional space, with the object dimension of the surface always being one less than the space dimension.

For example, in 3-dimensional space, the equations for the solid sphere and its surface are:

- The equation for the **solid sphere** (3D ball):

  $$x^2 + y^2 + z^2 \leq r^2$$

  This represents all points inside and on the surface of a sphere of radius r in 3D space.
- The equation for the **surface of the sphere** (2D surface in 3D space):

  $$x^2 + y^2 + z^2 = r^2$$

  This represents all the points exactly on the surface of the sphere.

Apendix B. The Fascinating World of Tensors, Neural Networks, and the Brain

In the world of artificial intelligence (AI) and neural networks, the concept of **tensors** plays a central role. A tensor is a mathematical object that generalizes scalars, vectors, and matrices to higher dimensions, representing data across multiple axes. Tensors are the building blocks of modern deep learning models, allowing them to process and learn from vast amounts of data.

Tensors in AI and Neural Networks

In AI, particularly deep learning, tensors are the primary data structures used to represent and manipulate data.

- A **rank-0 tensor** is a scalar.
- A **rank-1 tensor** is a vector.
- A **rank-2 tensor** is a matrix.
- Higher rank tensors (rank-3 and beyond) are used in advanced machine learning models, where data is represented in higher dimensions.

Neural networks are built to process these tensors in layers, performing mathematical operations like matrix multiplication, addition, and nonlinear activation functions at each layer. Each layer extracts higher-level features from the data, just as the brain processes information in a hierarchical, multi-dimensional manner.

Neural Networks in the Human Brain

The human brain processes information in a way that bears similarities to how AI models function. Neurons in the brain act as the "units" or "nodes" of a network, while synapses are the connections between these neurons. These connections have **weights** that influence the strength of the signal passed from one neuron to another, similar to the weights in an artificial neural network.
As information flows through the brain, certain neurons are activated based on the strength of incoming signals, which enables the brain to process complex data and make decisions. Just like a neural network in AI, the brain processes multi-dimensional data at multiple levels, adapting and learning from experience.

Neurotransmitters as Activation Functions

In AI, **activation functions** are used to decide whether a signal (or piece of information) should pass through a neural network. In the human brain, this role is played by **neurotransmitters**. Neurotransmitters such as dopamine, serotonin, and glutamate regulate communication between neurons by modulating the strength of the signal at the synapse.
These neurotransmitters can be thought of as biological **activation functions** that help the brain determine which neural pathways should be activated in response to stimuli. In this way, both artificial neural networks and biological brains use activation-like mechanisms to regulate the flow of information.

Learning and Adaptation: AI vs. The Brain

Both neural networks and the human brain are capable of learning and adapting. In artificial neural networks, learning occurs through **backpropagation**, where the model adjusts its weights based on feedback (error signals) to improve its performance.
Similarly, the human brain exhibits **neuroplasticity**, the ability to rewire itself and form new connections in response to learning. When we learn new things or have new experiences, the synapses in our brain strengthen or weaken, enabling the brain to adapt and improve its ability to process information.
Thus, both systems, though vastly different in nature (biological vs. artificial), share remarkable similarities in how they process information, learn, and adapt.

Apendix C. Ellipsoids and Quadratic Forms

We can now connect the ideas of dot products and matrix multiplication to geometry in higher dimensions. A very common structure in optimization, statistics, and control theory is the quadratic form:

$$(x-h)^\top A^{-1}(x-h) = r^2$$

This equation defines a geometric object known as a **hyperellipsoid**. Here:

- $x \in \mathbb{R}^n$ is a point (a column vector),
- $h \in \mathbb{R}^n$ is the center of the ellipsoid,
- A is a diagonal matrix containing the squares of the semi-axis lengths:

  $$A = \mathrm{diag}(a_1^2, a_2^2, \dots, a_n^2),$$

  and its inverse is

  $$A^{-1} = \mathrm{diag}\left(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \dots, \frac{1}{a_n^2}\right).$$

The expression

$$(x-h)^\top A^{-1}(x-h)$$

is just a fancier way of writing the sum

$$\sum_{i=1}^{n} \frac{(x_i-h_i)^2}{a_i^2},$$

which you have already seen in the expanded form of the hyperellipsoid equation.
Thus, the matrix form is equivalent to the traditional equation:

$$\frac{(x_1 - h_1)^2}{a_1^2} + \frac{(x_2 - h_2)^2}{a_2^2} + \cdots + \frac{(x_n - h_n)^2}{a_n^2} = r^2.$$

Why use the matrix form?

- It is concise and generalizes naturally to any number of dimensions.
- It fits seamlessly into linear algebra frameworks used in applied mathematics, physics, and engineering.
- It reveals how the matrix A encodes the shape and scaling of the ellipsoid.

Special Case: Unit Hypersphere If all semi-axes are equal, $a_i = 1$, then $A = I$, the identity matrix, and the equation simplifies to

$$(x - h)^\top (x - h) = r^2,$$

which describes a hypersphere of radius r centered at h.

Geometric Interpretation This quadratic form measures a weighted squared distance from the point x to the center h, where the weights are determined by the inverse squares of the ellipsoid's semi-axes. Points x satisfying the equation lie exactly on the hyperellipsoid's surface.

Matrix form as a generalization The equation

$$(x - h)^\top A^{-1} (x - h) = r^2$$

is a more general and compact way of writing the expanded sum

$$\frac{(x_1 - h_1)^2}{a_1^2} + \frac{(x_2 - h_2)^2}{a_2^2} + \cdots + \frac{(x_n - h_n)^2}{a_n^2} = r^2.$$

The expanded form corresponds to the special case where A is a diagonal matrix with entries $a_1^2, a_2^2, \ldots, a_n^2$, representing ellipsoids aligned with the coordinate axes.
However, the matrix form allows A to be any positive definite matrix, which means the ellipsoid can be rotated or skewed in any direction. Thus, the matrix notation not only generalizes the shape but also provides a powerful tool for working with ellipsoids in arbitrary orientations and dimensions.

Apendix D. An Overview of Mathematical Spaces

In mathematics, a *space* is fundamentally a set equipped with additional structure. These structures govern the operations and properties relevant to the objects (points, vectors, functions, etc.) within the space. The structures may include:

- Distance
- Angles
- Smoothness
- Curvature
- Topological features (e.g., connectedness, holes)
- Algebraic operations such as addition or scalar multiplication

Depending on which of these features are present, we obtain different types of mathematical spaces. Below is a summary of some of the most prominent ones.

1. Euclidean Space ($\mathbb{R}^n$)

The classical example:

- Points are represented as vectors.
- Distances are measured using the Pythagorean theorem.
- Inner products define angles and orthogonality.
- The geometry is flat and linear.

Euclidean space serves as the local model for differentiable manifolds.

2. Metric Spaces

A metric space is a set equipped with a distance function d satisfying:

$$d(x, y) \geq 0 \qquad \text{(non-negativity)}$$
$$d(x, y) = 0 \iff x = y \qquad \text{(identity of indiscernibles)}$$
$$d(x, y) = d(y, x) \qquad \text{(symmetry)}$$
$$d(x, z) \leq d(x, y) + d(y, z) \qquad \text{(triangle inequality)}$$

Examples include both Euclidean space and spaces with non-standard metrics like the taxicab (Manhattan) metric.

3. Topological Spaces

A topological space provides a notion of *closeness* without requiring distance. It is defined by a collection of open sets satisfying union, finite intersection, and inclusion of the entire set and the empty set. Topological spaces are foundational in studying continuity, compactness, and connectedness.

4. Vector Spaces

A vector space over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) includes:

- Vector addition and scalar multiplication.
- Linear operations satisfying associativity, commutativity, distributivity, and existence of additive identities and inverses.

Examples include $\mathbb{R}^n$, function spaces, and polynomial spaces.

5. Normed and Inner Product Spaces

Adding structure to vector spaces:

- A **normed space** has a norm $\|x\|$, defining length.
- An **inner product space** has an inner product $\langle x, y\rangle$, defining both length and angle.

Euclidean space is both a normed and inner product space.

6. Hilbert Spaces

A Hilbert space is a complete inner product space, possibly infinite-dimensional. It generalizes Euclidean space to accommodate function spaces. These are central in quantum mechanics and Fourier analysis.

7. Riemannian Manifolds

A Riemannian manifold is a smooth manifold where each tangent space is equipped with an inner product varying smoothly across the manifold. This allows measurement of lengths, angles, and curvature on curved surfaces.

8. Symplectic Manifolds

These are even-dimensional smooth manifolds equipped with a closed, non-degenerate 2-form ω. They model phase space in classical mechanics and are foundational in Hamiltonian dynamics.

9. Banach Spaces

A Banach space is a complete normed vector space. Unlike Hilbert spaces, Banach spaces may lack an inner product. They are widely used in functional analysis and differential equations.

10. Projective Spaces ($\mathbb{P}^n$)

Projective space identifies points along the same line through the origin in $\mathbb{R}^{n+1}$. It is essential in algebraic geometry and perspective geometry. In $\mathbb{P}^2$, parallel lines intersect at a point at infinity.

11. Affine Spaces

Affine spaces resemble vector spaces but lack a canonical origin. They support affine combinations and translations, making them a natural setting for geometry without fixed reference points.

12. More Exotic Spaces

Advanced fields of mathematics define more abstract spaces, such as:

- **Topos theory** (logic as geometry)
- **Moduli spaces** (parametrizing geometric structures)
- **Loop spaces** (mapping circles into manifolds)
- **Galois spaces**, etc.

These spaces often arise in category theory, algebraic geometry, and homotopy theory.

Summary Table of Space Properties

Space Type	Distance	Angles	Curvature	Infinite-Dimensional?
Euclidean ($\mathbb{R}^n$)	Yes	Yes	Flat	No
Metric Space	Yes	No	None	Yes
Topological Space	No	No	None	Yes
Vector Space	No	No	None	Yes
Inner Product Space	Yes (from $\langle\cdot,\cdot\rangle$)	Yes	None	Yes
Riemannian Manifold	Locally Yes	Locally Yes	Yes	Optional
Hilbert Space	Yes	Yes	Flat (but infinite)	Yes

Resume

The ball of radius r in $\mathbb{R}^n$, often called the n-dimensional ball B^n, consists of all points whose norm is less than or equal to r. Using the **norm** $\|\cdot\|$, we can define the ball as:

$$B^n = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \le r \quad \text{(i.e., } \sum_{i=1}^{n} x_i^2 \le r^2 \quad \forall i\text{)and } \mathbb{R} = \big(\mathbb{R}, +, \cdot, \le, \sup, d, \tau, \|\cdot\|, \langle\cdot,\cdot\rangle\big) \right\} \tag{9}$$

or equivalently, as an ellipsoid centered at $h = (h_1, h_2, \ldots, h_n)$ with principal semi-axes a_i:

$$B^n = \left\{ \begin{array}{l} (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \\ \quad \sqrt{\dfrac{(x_1-h_1)^2}{a_1^2} + \dfrac{(x_2-h_2)^2}{a_2^2} + \cdots + \dfrac{(x_n-h_n)^2}{a_n^2}} \le r, \\ \quad \text{(i.e., } \displaystyle\sum_{i=1}^{n} \frac{(x_i-h_i)^2}{a_i^2} \le r^2, \text{ where} \\ \quad h_i = 0, \quad a_i = 1 \quad \forall i\text{),} \\ \quad \text{and } \mathbb{R} = \big(\mathbb{R}, +, \cdot, \le, \sup, d, \tau, \|\cdot\|, \langle\cdot,\cdot\rangle\big) \end{array} \right\} \tag{10}$$

Now, by using hyperspherical coordinates, any point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be expressed as

$$\begin{cases} x_1 = r\cos\theta_1, \\ x_2 = r\sin\theta_1\cos\theta_2, \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \\ \quad\vdots \\ x_{n-1} = r\left(\prod_{j=1}^{n-2}\sin\theta_j\right)\cos\theta_{n-1}, \\ x_n = r\left(\prod_{j=1}^{n-2}\sin\theta_j\right)\sin\theta_{n-1}, \end{cases}$$

where

$$r \geq 0, \quad \theta_1, \ldots, \theta_{n-2} \in [0, \pi], \quad \theta_{n-1} \in [0, 2\pi).$$

Substituting the expressions for x_i into the ellipsoid equation yields

$$\sqrt{\frac{r^2\cos^2\theta_1}{a_1^2} + \frac{r^2\sin^2\theta_1\cos^2\theta_2}{a_2^2} + \frac{r^2\sin^2\theta_1\sin^2\theta_2\cos^2\theta_3}{a_3^2} + \cdots + \frac{r^2\left(\prod_{j=1}^{n-2}\sin^2\theta_j\right)\cos^2\theta_{n-1}}{a_{n-1}^2} + \frac{r^2\left(\prod_{j=1}^{n-2}\sin^2\theta_j\right)\sin^2\theta_{n-1}}{a_n^2}} \leq r$$

Finally, we can write the ellipsoid as a set of hyperspherical coordinate tuples:

$$\mathbb{E}^{n-1} = \left\{ (r, \theta_1, \ldots, \theta_{n-1}) \in \mathbb{R}^+ \times [0,\pi]^{n-2} \times [0, 2\pi) \;\middle|\; \sum_{i=1}^{n} \frac{x_i^2(r,\theta)}{a_i^2} \leq r^2 \quad \text{and } \mathbb{R} = (\mathbb{R}, +, \cdot, \leq, \sup, d, \tau, \|\cdot\|, \langle\cdot,\cdot\rangle) \right\},$$

where each $x_i(r, \theta)$ is defined above.

We now extend the real-valued model into the complex domain. The complex number system is defined as:

$$\mathbb{C} = (\mathbb{C}, +, \cdot, \mathrm{Re}, \mathrm{Im}, \bar{z}, |z|, \arg z, \|\cdot\|, \langle\cdot,\cdot\rangle),$$

where: $Re(z)$ is the real part, $Im(z)$ is the imaginary part, $\bar{z}$ is the complex conjugate, $|z| = \sqrt{z\bar{z}}$ is the modulus, $\arg z$ is the argument (angle), $\|\cdot\|$ is the induced norm, $\langle z, w\rangle = z\overline{w}$ is the Hermitian inner product.

The unit ball in $\mathbb{C}^n$, scaled by a real radius $r > 0$, is given by:

$$B_{\mathbb{C}}^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} |z_i|^2 \leq r^2 \right\} \tag{11}$$

A more general form is the **complex ellipsoid**, centered at $h = (h_1, \ldots, h_n) \in \mathbb{C}^n$ with principal semi-axes $a_i > 0$, defined as:

$$B_{\mathbb{C}}^n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{|z_i - h_i|^2}{a_i^2} \leq r^2, \text{and } \mathbb{C} = (\mathbb{C}, +, \cdot, \mathrm{Re}, \mathrm{Im}, \bar{z}, \|\cdot\|, \arg, \langle\cdot,\cdot\rangle) \right\} \tag{12}$$

In analogy with hyperspherical coordinates from $\mathbb{R}^n$, where the angles θ_i parametrize directions on the real unit sphere, a point in $\mathbb{C}^n$ can be represented using a set of moduli ρ_i and phase angles ϕ_i as:

$$z_i = r \cdot \rho_i \cdot e^{i\phi_i}, \quad \text{with } \rho_i \in [0,1],\ \phi_i \in [0, 2\pi),\ \text{subject to } \sum_{i=1}^{n} \rho_i^2 = 1.$$

Here, the ρ_i parametrize a point on the real unit sphere S^{n-1}, while the ϕ_i represent the complex phases.

Substituting into the ellipsoid equation yields:

$$\sum_{i=1}^{n} \frac{|z_i|^2}{a_i^2} = \sum_{i=1}^{n} \frac{r^2\rho_i^2}{a_i^2} \leq r^2 \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{\rho_i^2}{a_i^2} \leq 1.$$

Thus, the complex ellipsoid may be defined in spherical-polar coordinates as:

$$\mathbb{E}_{\mathbb{C}}^{n-1} = \left\{ (r, \rho_1, \ldots, \rho_n, \phi_1, \ldots, \phi_n) \in \mathbb{R}^+ \times [0,1]^n \times [0, 2\pi)^n \;\middle|\; \sum_{i=1}^{n} \rho_i^2 = 1, \quad \sum_{i=1}^{n} \frac{\rho_i^2}{a_i^2} \leq 1, \text{and } z_i = r\rho_i e^{i\phi_i} \in \mathbb{C} \right\} \tag{13}$$

This generalization provides a compact geometric model in $\mathbb{C}^n$, forming the basis of the Hilbert space formalism in quantum mechanics, and is deeply connected to complex projective geometry and higher-dimensional physics such as string theory and complex manifolds.