From Apostol and Curtis to Affine Geometry:
Lines, Cosets, and the Formal Axioms of Affine Spaces

A friendly tour for the math enthusiast

Abstract

This article connects two complementary viewpoints you encountered in Apostol's *Calculus, Vol. 1* (analytic geometry via points and direction vectors) and Curtis's *Abstract Linear Algebra* (cosets of subspaces in a vector space). We show that these are two faces of the same underlying structure: *affine geometry.* After a careful motivation grounded in Apostol's parametric description of lines and Curtis's coset viewpoint, we present a complete, formal definition of an affine space (as a set with a free and transitive action of a vector space), develop the basic theory of affine subspaces, affine combinations, and affine maps, and tie each concept back to the Apostol/Curtis framework. No prior background beyond linear algebra at the level of subspaces and linear maps is assumed.

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1 Motivation: Apostol meets Curtis

1.1 Apostol's parametric lines

In Apostol's exposition of analytic geometry, a *line* in $\mathbb{R}^n$ is described by a *point* $P \in \mathbb{R}^n$ and a *direction vector* $A \in \mathbb{R}^n$:

$$L = \{\, P + tA : \ t \in \mathbb{R} \,\}.$$

More generally, a *plane* (or k-dimensional flat) is described using k independent direction vectors $A_1, \ldots, A_k$:

$$P + \operatorname{span}\{A_1, \ldots, A_k\} = \{\, P + t_1 A_1 + \cdots + t_k A_k : \ t_1, \ldots, t_k \in \mathbb{R} \,\}.$$

The essential idea is: *start at a point, then move along directions.*

1.2 Curtis's cosets of subspaces

Curtis abstracts this by emphasizing linear-algebraic structure. A k-dimensional *linear* subspace $V \leq \mathbb{R}^n$ encodes "pure direction" through the origin. Sliding (translating) that subspace by a vector $v \in \mathbb{R}^n$ gives a *coset* $v + V = \{\, v + w : w \in V \,\}$, which is exactly a k-dimensional *affine* subspace (a k-flat). In particular:

- A line is a coset of a 1-dimensional subspace.
- A plane is a coset of a 2-dimensional subspace.
- A single point is a coset of the 0-subspace $\{0\}$.

Thus, Curtis's language of *cosets* repackages Apostol's "point + directions" picture.

1.3 Why these pictures are the same

If $V = \operatorname{span}\{A\}$ is 1-dimensional, then $v + V = \{v + tA : t \in \mathbb{R}\}$, which is exactly Apostol's parametric line with basepoint $P = v$ and direction A. If $V = \operatorname{span}\{A_1, \ldots, A_k\}$, then $v + V = \{v + \sum t_i A_i\}$, Apostol's parametric k-flat. The two viewpoints coincide perfectly: *"point + directions"* equals *"coset of a subspace"*.

These observations inspire the formal notion of an *affine space*: geometry built from points and directions, but with no privileged origin. This is what we now define.

2 Affine spaces: the formal definition

2.1 Difference vectors and the action viewpoint

The key operational idea in both Apostol and Curtis is that while we cannot (and should not) add two points $P + Q$, we *can* subtract points to obtain a vector: given P, Q we can form a "difference" $\overrightarrow{PQ}$, and we can translate a point P by a vector v to get a new point $P + v$.

This motivates the standard modern definition: an affine space is a set of points on which a vector space acts freely and transitively.

Definition 2.1 (Affine space)**.** *Let $\mathbb{F}$ be a field and let $\mathcal{V}$ be a (nonzero) vector space over $\mathbb{F}$. An* affine space *over $\mathbb{F}$* modeled on $\mathcal{V}$ *is a nonempty set $\mathcal{A}$ together with a map*

$$\mathcal{A} \times \mathcal{V} \longrightarrow \mathcal{A}, \qquad (P, v) \mapsto P + v,$$

such that for all $P \in \mathcal{A}$ and $u, v \in \mathcal{V}$:

1. $P + 0 = P$ *(identity element),*

2. $(P + u) + v = P + (u + v)$ *(associativity of the action),*

and the action is free and transitive*:*

(F) *If* $P + v = P$ *for some* $P \in \mathcal{A}$*, then* $v = 0$ *(freeness).*

(T) *For any* $P, Q \in \mathcal{A}$*, there exists a* unique $v \in \mathcal{V}$ *with* $P + v = Q$ *(transitivity).*

The vector space $\mathcal{V}$ *is called the* model space *(or* direction space*) of* $\mathcal{A}$*.*

Remark 2.2 (Difference vectors). By (T) the difference map

$$\mathcal{A} \times \mathcal{A} \to \mathcal{V}, \qquad (P, Q) \mapsto \overrightarrow{PQ}$$

is well-defined by the rule $P + \overrightarrow{PQ} = Q$, and uniqueness in (T) ensures $\overrightarrow{PQ}$ is unique. This abstracts the familiar $\overrightarrow{PQ} = Q - P$ from $\mathbb{R}^n$ while avoiding any reference to an origin in $\mathcal{A}$ itself.

Example 2.3 (The standard affine space $\mathbb{R}^n$). Let $\mathcal{A} = \mathbb{R}^n$ and $\mathcal{V} = \mathbb{R}^n$ with the action $(P, v) \mapsto P + v$ given by vector addition. This is the classical setting of Apostol: points and vectors *look* the same as raw tuples, but conceptually we distinguish them by role. Curtis's cosets appear as $P + \mathcal{W}$ for linear subspaces $\mathcal{W} \leq \mathcal{V}$.

2.2 Affine subspaces (flats)

Definition 2.4 (Affine subspace)**.** *Let* $\mathcal{A}$ *be an affine space modeled on* $\mathcal{V}$*. A nonempty subset* $S \subseteq \mathcal{A}$ *is an* affine subspace *(or* flat*) if there exists a point* $P \in \mathcal{A}$ *and a linear subspace* $\mathcal{W} \leq \mathcal{V}$ *such that*

$$S = P + \mathcal{W} = \{\, P + w : \; w \in \mathcal{W} \,\}.$$

The linear subspace $\mathcal{W}$ *is called the* direction subspace *of* S *and is denoted* $\overrightarrow{S}$*.*

Proposition 2.5 (Intrinsic characterization)**.** *A nonempty subset* $S \subseteq \mathcal{A}$ *is an affine subspace if and only if for all* $P \in S$ *and all* $w \in \overrightarrow{S}$ *one has* $P + w \in S$*, where*

$$\overrightarrow{S} \;=\; \operatorname{span}\{\overrightarrow{PQ} : \; P, Q \in S\} \;\leq \mathcal{V}.$$

Moreover, if S *is an affine subspace, then for any* $P \in S$*,* $S = P + \overrightarrow{S}$*.*

Proof. If $S = P_0 + \mathcal{W}$, then $\overrightarrow{S} = \mathcal{W}$ (because differences of points in $P_0 + \mathcal{W}$ lie in $\mathcal{W}$, and conversely all vectors in $\mathcal{W}$ arise as such differences), and $P + w \in P_0 + \mathcal{W}$ holds for all $P \in S$, $w \in \mathcal{W}$.

Conversely, suppose S is nonempty and closed under translations by $\overrightarrow{S}$. Fix $P_0 \in S$. For any $Q \in S$, $\overrightarrow{P_0Q} \in \overrightarrow{S}$, hence $Q = P_0 + \overrightarrow{P_0Q} \in P_0 + \overrightarrow{S}$. Thus $S \subseteq P_0 + \overrightarrow{S}$. The reverse inclusion $P_0 + \overrightarrow{S} \subseteq S$ follows from closure, yielding $S = P_0 + \overrightarrow{S}$. □

Definition 2.6 (Lines, planes, and dimension). *If* $\dim(\overrightarrow{S}) = 1$, *then* S *is a* line*; if* $\dim(\overrightarrow{S}) = 2$, *a* plane*; and in general* $\dim(\overrightarrow{S}) = k$ *means* S *is a* k-flat*. The dimension of the whole affine space* $\mathcal{A}$ *is defined to be* $\dim(\mathcal{V})$.

Remark 2.7 (Parallelism). Two affine subspaces $S, T \subseteq \mathcal{A}$ are said to be *parallel* if $\overrightarrow{S} = \overrightarrow{T}$. This matches analytic geometry: parallel lines share the same direction.

2.3 Affine combinations and affine independence

The notion of "averaging" points without an origin is captured by *affine combinations.*

Definition 2.8 (Affine combination). *Let* $P_0, \ldots, P_m \in \mathcal{A}$ *and scalars* $\lambda_0, \ldots, \lambda_m \in \mathbb{F}$ *with* $\sum_{i=0}^m \lambda_i = 1$. *Choose any basepoint* P_0 *and set*

$$\sum_{i=0}^{m} \lambda_i P_i \ := \ P_0 + \sum_{i=1}^{m} \lambda_i \, \overrightarrow{P_0P_i}.$$

One checks that this is independent of the choice of basepoint. The set of all affine combinations of $\{P_0, \ldots, P_m\}$ *is the* affine hull $\operatorname{aff}\{P_0, \ldots, P_m\}$.

Proposition 2.9 (Affine hull is the smallest affine subspace). *For any finite subset* $S \subseteq \mathcal{A}$, $\operatorname{aff}(S)$ *is an affine subspace containing* S*, and it is contained in every affine subspace that contains* S.

Definition 2.10 (Affine independence and affine frames). *Points* $P_0, \ldots, P_m \in \mathcal{A}$ *are* affinely independent *if the difference vectors* $\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_0P_m}$ *are linearly independent in* $\mathcal{V}$. *An* affine frame *(or* affine basis*) of an* n*-dimensional affine space is a set of* $n+1$ *affinely independent points; every point of* $\mathcal{A}$ *is an affine combination of the frame with coefficients summing to* 1.

2.4 Affine maps and their linear parts

Definition 2.11 (Affine map)**.** *Let $\mathcal{A}$ be an affine space modeled on $\mathcal{V}$, and $\mathcal{B}$ an affine space modeled on $\mathcal{W}$. A map $f : \mathcal{A} \to \mathcal{B}$ is* affine *if there exists a linear map $L : \mathcal{V} \to \mathcal{W}$ such that*

$$f(P + v) = f(P) + L(v) \qquad \text{for all } P \in \mathcal{A},\ v \in \mathcal{V}.$$

The linear map L is called the linear part *(or* differential*) of f and is uniquely determined by f.*

Proposition 2.12 (Uniqueness of the linear part)**.** *If f is affine, the linear map L with $f(P + v) = f(P) + L(v)$ is unique. Moreover, for all $P, Q \in \mathcal{A}$,*

$$\overrightarrow{f(P)f(Q)} = L\big(\overrightarrow{PQ}\big).$$

Proof. Fix P. For any $v \in \mathcal{V}$, $L(v) = \overrightarrow{f(P)f(P + v)}$. This does not depend on P: if $Q = P + u$, then

$$\overrightarrow{f(Q)f(Q + v)} = \overrightarrow{f(P + u)\, f(P + u + v)} = L(v),$$

by the defining property. Hence L is uniquely determined and satisfies the difference-vector identity. □

Example 2.13 (Affine transformations in $\mathbb{R}^n$)*.* In the standard affine space $\mathbb{R}^n$, a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is affine iff $f(x) = Ax + b$ with A linear (matrix) and b a fixed vector. The linear part is A, and $b = f(0)$ once an origin is chosen for convenience. This recovers the familiar "matrix plus translation" picture from analytic geometry and matches Curtis's idea of sliding subspaces by vectors.

3 Revisiting Apostol and Curtis inside the formalism

3.1 Apostol's parametric sets as affine subspaces

Given $P \in \mathcal{A}$ and linearly independent $A_1, \ldots, A_k \in \mathcal{V}$, the set

$$P + \operatorname{span}\{A_1, \ldots, A_k\}$$

is a k-flat with direction subspace $\operatorname{span}\{A_1, \ldots, A_k\}$. This is exactly Apostol's description: *start at a point, move along k directions.* In the standard model $\mathcal{A} = \mathbb{R}^n$, these are the usual parametric formulas for lines and planes.

3.2 Curtis's cosets as affine subspaces

Fix a linear subspace $\mathcal{W} \leq \mathcal{V}$ ("pure direction through the origin"). For any basepoint $P \in \mathcal{A}$, the coset $P + \mathcal{W}$ is an affine subspace with direction $\mathcal{W}$. Thus Curtis's language of *cosets* is precisely the general notion of affine subspace. In $\mathbb{R}^3$:

- Lines are cosets of 1-dimensional subspaces.
- Planes are cosets of 2-dimensional subspaces.
- Points are cosets of $\{0\}$.

3.3 Parallelism and difference vectors

Two lines $L_1 = P_1 + \text{span}\{A\}$ and $L_2 = P_2 + \text{span}\{A\}$ are parallel because they share the same direction subspace. The difference vector $\overrightarrow{P_1P_2} \in \mathcal{V}$ witnesses that L_2 is a translate of L_1 by that vector, again echoing both Apostol's "point plus direction" and Curtis's "coset shift."

4 Basic results and useful tools

4.1 Intersections and sums of flats

Proposition 4.1. *The intersection of two affine subspaces is either empty or an affine subspace whose direction subspace is the intersection of the direction subspaces:*

$$(P + \mathcal{W}) \cap (Q + \mathcal{U}) \ \ \textit{is either empty or equals} \ \ R + (\mathcal{W} \cap \mathcal{U})$$

for some R (necessarily in the intersection if it is nonempty).

Idea of proof. If the intersection is nonempty, pick R in it. Differences of any two points in the intersection lie in both $\mathcal{W}$ and $\mathcal{U}$, so the direction is $\mathcal{W} \cap \mathcal{U}$, and Proposition 2.5 applies. □

Proposition 4.2 (Affine hull via differences)**.** *For a finite set $S = \{P_0, \ldots, P_m\} \subseteq \mathcal{A}$,*

$$\text{aff}(S) \ = \ P_0 + \text{span}\{\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_0P_m}\}.$$

4.2 Change of affine frame; barycentric coordinates

If $\{P_0, \ldots, P_n\}$ is an affine frame of an n-dimensional affine space, then every $P \in \mathcal{A}$ can be written uniquely as

$$P = \sum_{i=0}^{n} \lambda_i P_i, \qquad \sum_{i=0}^{n} \lambda_i = 1.$$

The scalars $(\lambda_0, \ldots, \lambda_n)$ are called the *barycentric coordinates* of P in that frame. This is the affine analogue of coordinates in a linear basis and gives a coordinate-free way to speak about "weights" on points (familiar from centroids in geometry).

5 Examples connecting back to the books

Example 5.1 (Lines and planes in $\mathbb{R}^3$ (Curtis)). Let $\mathcal{V} = \mathbb{R}^3$ and let $\mathcal{W} = \operatorname{span}\{A\}$ with $A \neq 0$. Then any line is $P + \mathcal{W} = \{P + tA : t \in \mathbb{R}\}$. If $\mathcal{U} = \operatorname{span}\{A, B\}$ with A, B independent, then any plane is $P + \mathcal{U} = \{P + sA + tB : s, t \in \mathbb{R}\}$. Points are $P + \{0\} = \{P\}$. This is exactly Curtis's "cosets of subspaces" description.

Example 5.2 (Parametric description (Apostol)). Pick $P = (2, 1, 0)$ and $A = (3, 4, 1)$. The line

$$L = \{(2, 1, 0) + t(3, 4, 1) : t \in \mathbb{R}\}$$

is an affine 1-flat with direction subspace $\operatorname{span}\{A\}$. Replace A by (A, B) to obtain a plane with direction $\operatorname{span}\{A, B\}$. This is Apostol's "point plus direction(s)" presentation.

Example 5.3 (Solution sets of linear equations). In $\mathbb{R}^n$, the solution set of a consistent linear system $Ax = b$ is either empty or an affine subspace: pick one solution x_0; then all solutions are $x_0 + \ker(A)$. Thus the homogeneous solution space provides the direction subspace (Curtis), and the particular solution is the basepoint (Apostol).

6 Affinization of linear results

6.1 Linear vs. affine maps

Every affine map $f : \mathcal{A} \to \mathcal{B}$ decomposes into "a translation plus a linear map" once we choose basepoints. Concretely, fix $P_0 \in \mathcal{A}$ and $Q_0 = f(P_0) \in \mathcal{B}$. For $P \in \mathcal{A}$,

$$f(P) = Q_0 + L\big(\overrightarrow{P_0P}\big),$$

where L is the unique linear part from Proposition 2.12. Changing P_0 shifts Q_0 accordingly but leaves L unchanged.

6.2 Rigid motions and similarities

Within $\mathbb{R}^n$ with its usual inner product, an *isometry* is an affine map whose linear part is orthogonal; a *similarity* has linear part a scalar multiple of an orthogonal map. These familiar transformations are special cases of affine maps, showing how classical Euclidean motions sit naturally in the affine framework.

7 Summary: the unifying picture

- **Apostol (analytic geometry):** A k-flat is "point + k directions": $P + \operatorname{span}\{A_1, \ldots, A_k\}$.
- **Curtis (cosets):** A k-flat is a coset $P + \mathcal{W}$ of a k-dimensional linear subspace $\mathcal{W}$.
- **Affine geometry (formal):** An affine space $\mathcal{A}$ modeled on $\mathcal{V}$ is a set with a free, transitive action of $\mathcal{V}$. Affine subspaces are translates of linear subspaces, lines/planes are 1/2-flats, and affine maps are exactly those with $f(P + v) = f(P) + L(v)$ for a linear L.

These are not different subjects but different *languages* describing the same structure. Apostol shows you the engine; Curtis shows you the blueprint; affine geometry names the machine.

Appendix: Axioms-only presentation (optional perspective)

One can axiomatize an affine space $(\mathcal{A}, \mathcal{V})$ entirely in terms of points and difference vectors without mentioning a group action:

- For $P, Q \in \mathcal{A}$, there is a vector $\overrightarrow{PQ} \in \mathcal{V}$.
- For $P \in \mathcal{A}$ and $v \in \mathcal{V}$, there is a point $P + v \in \mathcal{A}$.

These satisfy:

1. $\overrightarrow{PP} = 0$ and $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.
2. $P + 0 = P$ and $(P + u) + v = P + (u + v)$.
3. $P + \overrightarrow{PQ} = Q$ for all $P, Q \in \mathcal{A}$.
4. For each $P \in \mathcal{A}$, the map $\mathcal{V} \to \mathcal{A}$, $v \mapsto P + v$, is a bijection.

This axiomatization is equivalent to the action definition used above and may feel close in spirit to Apostol's parametric constructions while preserving Curtis's emphasis on vectors and subspaces.

Final word. The analytic (Apostol) and coset (Curtis) pictures are two windows into the same landscape. Affine geometry is the frame that holds both windows in place.