A Generalized Probabilistic Gibbard-Satterthwaite Theorem

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April 27, 2014

Importance of Elections

- Political voting
- Electing board members
- Shareholders voting on company issues
- Artificial intelligent agent decision
- Search engine page-ranking

Why are elections important?

They can be applied any time independent agents need to come to a consensus.

Modeling an Election

- Each voter ranks the alternatives (*preference list*)
- All the preference lists make up a *preference profile*
- A *voting rule* (election system) chooses a winner based on a profile

The interesting part is the voting rules.

This field of study is called social choice theory.

Fairness in Election Systems

- Everyone should have an equal say
- Winner should accurately represent the group preference
- Simple with 2 alternatives; complicated with many

With two alternatives the one prefered by the most voters should win.

First studied by Condorcet

He proposed that the winning candidate be the candidate who would win a head-to-head election against each of the other candidates.

Such a winner is known as the Condorcet winner.

Unfortunately, Condorcet also proved that a Condorcet winner does not always exist.

But this was one of the first fairness criterion.

Manipulation

A voter can get a better result by voting strategically, rather than voting his actual preferences.

Real Preferences

You	Others		Win
L	**D**	**R**	**D**
R	R	D	
D	L	L	

Manipulation

You	Others		Win
R	**D**	**R**	**R**
L	R	D	
D	L	L	

Manipulation is an enemy of fairness because the manipulative voter gets more influence than others.

Assume a simple plurality/first-past-the-post system.

In the first case it would be a tie, but lets assume our arbitrary tie breaking technique will choose the Democrats.

This is essentially the “wasted vote” problem.

This is a very simplistic example, but almost all voting systems are susceptible to manipulation.

Gibbard-Satterthwaite Theorem

Voting rules are manipulable if they satisfy:

Non-dictatorship No single voter always dictates the group preference.

Non-imposition Every alternative has the possibility of winning.

We would like to devise an unmanipulable voting rule, but this theorem says it's impossible.

They must satisfy both of these.

Circumventing the Gibbard-Satterthwaite Theorem

- Bartholdi, Tovey, and Trick studied the computational difficulty of finding a winner for various voting rules
- Computational aspect of voting rules is significant
- The Dodgson method is infeasible to manipulate because finding the winner is NP-hard
- Voting rules need to resist manipulation *and* make it feasible to find the winner
- People began to search for a computational barrier to manipulation

Random Manipulation

- Friedgut, Kalai, and Nisan studied random manipulation
- Succeeds with non-negligible probability
- Shows the limits of a computational barrier to manipulation
- Only proved results for elections with 3 alternatives
- This is the work I attempted to generalize

If you alternative is not winning, randomly permute your preference list.

Independent Work

- Isaksson, Kindler, and Mossel have, independently and during the writing of this thesis, published a brilliant generalization of the original theorem of Friedgut, Kalai, and Nisan and even improved slightly upon the results. In their paper “The geometry of manipulation: A quantitative proof of the gibbard-satterthwaite theorem.”
- Translating their results into the terminology we have been using, they proved that for a neutral social choice function f with $m \geq 4$ alternatives and n voters that is ϵ-far from dictatorship, a uniformly chosen profile will be manipulable with probability at least $2^{-1}\epsilon^2 n^{-4} m^{-6} (m!)^{-3}$.

Before a formal look at my work, as a disclaimer, others have published an independent generalization before I was finished.

This is unfortunately for me, but fortunate for the field of social choice theory as a whole.

My work is useful as:

- A simpler proof
- A proof that closely follows the original
- An alternative proof

Notation

- $C = \{1, \ldots, m\}$ is the set of alternatives.
- A preference list is a total ordering of the alternatives.
- The set of all preference lists is $L(C)$.
- A preference profile is a sequence of n preference lists.
- The set of all preference profiles is $P = L(C)^n$.
- A voting rule is a function that chooses a winning alternative from a profile, i.e. $f : P \to C$.
- An election is a voting rule paired with a profile: (f, p).

m is the number of alternatives

n is the number of voters

Voting rule == social choice function

$L(C)^n$ is the Cartesian product of $L(C)$ with itself n times, or the set of all n-tuples of elements of $L(C)$

Restricted Preference Profiles

- For a preference list v, $v|_D$ means v *restricted to* D, i.e. v with the alternatives from D removed.
- For a preference profile, p, $p|_D$ means p with each preference list restricted to D.

p

1	3	2
2	2	3
3	1	1

$p|_{\{1,2\}}$

1	2	2
2	1	1

Proof Summary

Friedgut's proof is in three steps:

- Step 1 Application of a quantitative version of Arrow's impossibility theorem.
- Step 2 Reduction from an SCF with low dependence on irrelevant alternatives to a GSWF with a low paradox probability.
- Step 3 Reduction from low manipulation power to low dependence on irrelevant alternatives.

Generalized Steps

Friedgut, was able to generalize Step 1 and Step 2 as follows:

Lemma (Generalized Step 1)

For every fixed m and $\epsilon > 0$ there exists $\delta > 0$ such that if $F = f^{\otimes \binom{m}{2}}$ is a neutral IIA GSWF over m alternatives with $f : \{0,1\}^n \to \{0,1\}$, and $\Delta(f, DICT) > \epsilon$, then F has probability of at least $\delta \geq (C\epsilon)^{\lfloor m/3 \rfloor}$ of not having a Generalized Condorcet Winner, where $C > 0$ is an absolute constant.

Lemma (Generalized Step 2)

For every fixed m there exists $\delta > 0$ such that for all $\epsilon > 0$ the following holds. Let f be a neutral SCF among m alternatives such that $\Delta(f, DICT) > \epsilon$. Then for all (a, b) we have $M^{a,b}(f) \geq \delta$.

So we only need to generalize step 3 to get the whole proof.

Step 3

The original Step 3 was:

Lemma (Non-General Step 3)

For every SCF f on 3 alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot 6$

And the generalization we attempt to prove is:

Lemma (Generalized Step 3)

For every SCF f on m alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot m!$

Main Result

When we put together all 3 generalized steps we get our main result:

Theorem (Main Result)

There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If f is a neutral SCF for n voters over m alternatives and $\Delta(f, g) > \epsilon$ for any dictatorship g, then f has total manipulability: $\sum_{i=1}^{n} M_i(f) \geq \frac{(C\epsilon)^{\lfloor m/3 \rfloor}}{m!}$.

Step 3 is comprised of Lemma 6, Lemma 7, and Lemma 8 which we will generalize one at a time.

Statement of Lemma 6

Lemma (Original Lemma 6)

$$M^{a,b}(f) = \mathrm{E}_{q \in L(C)^n}\left[\frac{|A(q)|}{3^n} \cdot \frac{|B(q)|}{3^n}\right],$$

where q is chosen uniformly at random.

Lemma (Generalized Lemma 6)

$$M^{a,b}(f) = \mathrm{E}_{q \in L(C)^n}\left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}\right],$$

where q is chosen uniformly at random.

Define A and B Functions

Let $a, b \in C$ be the first two alternatives, let $p \in L(C)^n$ be a preference profile. We define

$$A(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = a\}$$
$$B(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = b\}.$$

Define $M^{a,b}$

Recall the definition of $M^{a,b}(f)$ from Friedgut:

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

where p, p' are chosen at random in $L(C)^n$ with $p|_{\{a,b\}} = p'|_{\{a,b\}}$.

Size of Profiles

For any preference profile $p \in P$ there are $(\frac{m!}{2})^n$ profiles x such that $x|_{\{a,b\}} = p|_{\{a,b\}}$. This is because there are $m!$ possible preference lists; half of them will have the preference between a and b that agrees with $p|_{\{a,b\}}$ and half will disagree. This gives $\frac{m!}{2}$ possible preference lists for each voter, so there are $(\frac{m!}{2})^n$ profiles comprised of these preference lists.

Proof of Generalized Lemma 6

First we fix a profile q. Then

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n}$$

is the probability that a randomly chosen profile, p, satisfying $p|_{\{a,b\}} = q|_{\{a,b\}}$ also satisfies $f(p) = a$. This is because there are $(\frac{m!}{2})^n$ profiles that agree with $q|_{\{a,b\}}$, and $|A(q)|$ is the number of those for which the outcome is a.

Since $p|_{\{a,b\}} = q|_{\{a,b\}}$ and $p'|_{\{a,b\}} = q|_{\{a,b\}}$, clearly we have that $p|_{\{a,b\}} = p'|_{\{a,b\}}$.

Since $f(p) = a$ and $f(p') = b$ are independent events, the joint probability is

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}.$$

Proof of Generalized Lemma 6

So we can rewrite

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

as

$$M^{a,b}(f) = \mathrm{E}_{q \in L(C)^n} \left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n} \right],$$

$\square$

Statement of Lemma 7

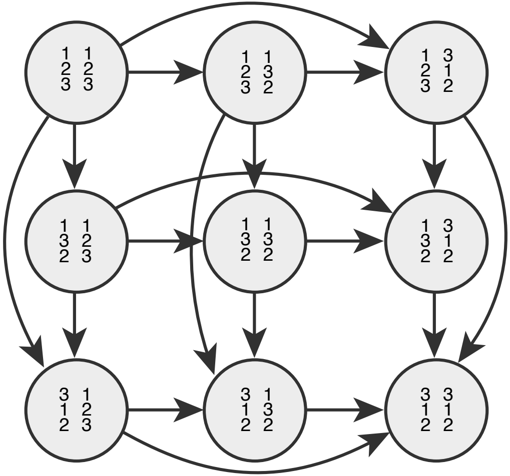
Lemma (Original Lemma 7)

$$\sum_i M_i(f) \geq \frac{1}{6} 3^{-n}\, \mathrm{E}_p \left[|\partial A(p)| + |\partial B(p)| \right]$$

Lemma (Generalized Lemma 7)

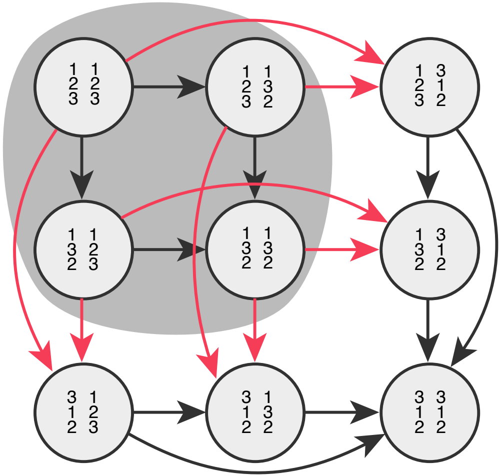
$$\sum_i M_i(f) \geq \frac{1}{m!} \left(\frac{m!}{2} \right)^{-n} \mathrm{E}_p \left[|\partial A(p)| + |\partial B(p)| \right]$$

Profile Lattice



Upper Edge Border

- Set of edges whose tail is in $A(p)$ and whose head is not in $A(p)$
- Denoted $\partial A(p)$
- Edge notation: (x_{-i}, x_i, x_i') as shorthand for $((x_{-i}, x_i), (x_{-i}, x_i'))$



x_{-i} with the ith index removed.

Formal Definition of Upper Edge Border

$$\partial_i A(p) = \{(x_{-i}, x_i, x_i') \mid (x_{-i}, x_i) \in A(p),$$
$$(x_{-i}, x_i') \notin A(p),$$
$$x_i|_{\{a,b\}} = x_i'|_{\{a,b\}},$$
$$x_i <_s x_i'\}$$

$$\partial A(p) = \bigcup_j \partial_j A(p)$$

For all $i \in \{1, \ldots, n\}$

We define the upper edge border of $B(p)$ analogously.

Edges Correspond to Manipulations

Lemma

Then each $(x_{-i}, x_i, x_i') \in \partial_i A(p) \cup \partial_i B(p)$ *corresponds to at least one successful manipulation.*

We don't have time to show the proof, but it's in my paper.

Proof of Generalized Lemma 7

- Randomly choose p and p'_i
- $M_i(f)$ is the probability that p'_i is a successful manipulation
- We wish to come up with a lower bound for $M_i(f)$
- We can think of these as two distinct profiles, p and p', where $p' = (p_{-i}, p'_i)$
- Clearly $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$
- We have $p_i|_{\{a,b\}} = p'_i|_{\{a,b\}}$ with probability $\frac{1}{2}$, and we condition the following on this being the case
- So $p|_{\{a,b\}} = p'|_{\{a,b\}}$

Proof of Generalized Lemma 7

- Since every edge in $\partial_i A(p) \cup \partial_i B(p)$ corresponds to at least one manipulation, we can lower bound $M_i(f)$ by the probability that an edge is in $\partial_i A(p) \cup \partial_i B(p)$
- The total number of possible edges of the form (x_{-i}, x_i, x_i') is

$$(m!)^{n-1} \cdot m! \cdot m!$$

- But all edges in $\partial_i A(p) \cup \partial_i B(p)$ must agree with $p|_{\{a,b\}}$. The total number of possible edges agreeing with $p|_{\{a,b\}}$ is

$$\left(\frac{m!}{2}\right)^{n-1} \cdot \frac{m!}{2} \cdot \frac{m!}{2} = \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

- Since $\partial_i A(p)$ and $\partial_i B(p)$ are disjoint, no edge can be in both sets and so we have

$$|\partial_i A(p) \cup \partial_i B(p)| \leq \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

Proof of Generalized Lemma 7

- Therefore, the probability that a randomly chosen edge is in either $\partial_i A(p)$ or $\partial_i B(p)$ is
$$\frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right]$$
- We conditioned our analysis on $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$, so our lower bound becomes
$$M_i(f) \geq \frac{1}{2} \cdot \frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$
- Simplifying gives
$$M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$
- Summing over i gives
$$\sum_i M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial A(p)| + |\partial B(p)|\right].$$

Lemma 8

Lemma (Original Lemma 8)

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{1}{3}\right)^n |A(p)| \cdot |B(p)|$$

Lemma (Generalized Lemma 8)

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{2}{m!}\right)^n |A(p)| \cdot |B(p)|$$

I got close, but wasn't able to complete the proof of this lemma.

In my thesis I give a partial proof and a detailed description of the things that would be required to make it work.

Because of lack of time, I won't go into it here.

Combining Lemma 6, 7, and 8

Restatement of the lemmas:

$$M^{a,b} = \mathrm{E}[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$L_7 \cdot \mathrm{E}[|\partial A| + |\partial B|] \leq \sum_i M_i \qquad \text{lemma 7}$$

$$\frac{1}{L_8} \cdot (|\partial A| + |\partial B|) \geq |A||B| \qquad \text{lemma 8}$$

Now we can solve for the result of step 3:

$$M^{a,b} = \mathrm{E}[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$M^{a,b} \leq \mathrm{E}[|\partial A| + |\partial B|] \cdot \frac{L_6}{L_8} \qquad \text{by lemma 8}$$

$$M^{a,b} \leq \sum_i M_i \cdot \frac{L_6}{L_7 L_8} \qquad \text{by lemma 7}$$

$A \equiv A(p)$ because it's *much* easier to read Also, $M^{a,b} \equiv M^{a,b}(f)$ and $M_i \equiv M_i(f)$

Combining Lemma 6, 7, and 8

The variables have the following values:

$$L_6 = \left(\frac{m!}{2}\right)^{-2n}$$

$$L_7 = \frac{1}{m!}\left(\frac{m!}{2}\right)^{-n}$$

$$L_8 = \left(\frac{m!}{2}\right)^{-n}$$

Substituting becomes:

$$\begin{aligned}\frac{L_6}{L_7 L_8} &= \left(\frac{m!}{2}\right)^{-2n} \cdot m!\left(\frac{m!}{2}\right)^{n} \cdot \left(\frac{m!}{2}\right)^{n} \\ &= \left(\frac{m!}{2}\right)^{-2n} \cdot m! \cdot \left(\frac{m!}{2}\right)^{2n} \\ &= m!\end{aligned}$$

Step 3 Result

The final result for step 3 is:

$$M^{a,b}(f) \leq \sum_i M_i(f) \cdot m!$$

By combining it with Friedgut's step 1 and 2 we get the generalized main theorem:

Theorem (Main Result)

There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If f is a neutral SCF for n voters over 3 alternatives and $\Delta(f, g) > \epsilon$ for any dictatorship g, then f has total manipulatiblity: $\sum_{i=1}^{n} M_i(f) \geq \frac{(C\epsilon)^{\lfloor m/3 \rfloor}}{m!}$.

Questions