# A Generalized Probabilistic Gibbard-Satterthwaite Theorem

Jonathan Potter

Rochester Institute of Technology

April 23, 2014

# Importance of Elections

- Political voting
- Electing board members
- Shareholders voting on company issues
- Artificial intelligent agent decision
- Search engine page-ranking

# Desirable Election Systems

- Fairness
  - Everyone should have an equal say
  - Winner should accurately represent the group
  - Simple with 2 alternatives; complicated with many
  - First studied by Condorcet

# Arrow's Impossibility Theorem

- Unrestricted domain (universality)
- Independence of irrelevant alternatives
- Pareto principle (unanimity)
- Non-dictatorship

A voter can get a better result by voting strategically, rather than voting his actual preferences.

**Real Preferences**

| You | Others | | Win |
|---|---|---|---|
| **L** | **D** | **R** | **D** |
| R | R | D | |
| D | L | L | |

**Manipulation**

| You | Others | | Win |
|---|---|---|---|
| **R** | **D** | **R** | **R** |
| L | R | D | |
| D | L | L | |

# Gibbard-Satterthwaite Theorem

Voting rules are manipulable if they satisfy:

- **Non-dictatorship**: No single voter always dictates the group preference.
- **Non-imposition**: Every alternative has the possibility of winning.

# Notation

- $C = \{1, \dots, m\}$ is the set of alternatives.
- A preference list is a total ordering of the alternatives.
- The set of all preference lists is $L(C)$.
- A preference profile is a sequence of $n$ preference lists.
- The set of all preference profiles is $P = L(C)^n$.
- A voting rule is a function that chooses a winning alternative from a profile, i.e. $f : P \to C$.
- An election is a voting rule paired with a profile: $(f, p)$.

# Restricted Preference Profiles

- For a preference list $v$, $v|_D$ means $v$ *restricted to* $D$, i.e. $v$ with the alternatives from $D$ removed.
- For a preference profile, $p$, $p|_D$ means $p$ with each preference list restricted to $D$.

$p$

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 2 | 3 |
| 3 | 1 | 1 |

$p|_{\{1,2\}}$

| | | |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 1 | 1 |

# Proof Summary

Friedgut's proof is in three steps:

- Step 1: application of a quantitative version of Arrow's impossibility theorem.
- Step 2: reduction from an SCF with low dependence on irrelevant alternatives to a GSWF with a low paradox probability.
- Step 3: reduction from low manipulation power to low dependence on irrelevant alternatives.

# Generalized Steps

Friedgut, was able to generalize Step 1 and Step 2 as follows:

**Lemma (Generalized Step 1)**

*For every fixed $m$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $F = f^{\otimes \binom{m}{2}}$ is a neutral IIA GSWF over $m$ alternatives with $f : \{0,1\}^n \to \{0,1\}$, and $\Delta(f, DICT) > \epsilon$, then $F$ has probability of at least $\delta \geq (C\epsilon)^{\lfloor m/3 \rfloor}$ of not having a Generalized Condorcet Winner, where $C > 0$ is an absolute constant.*

**Lemma (Generalized Step 2)**

*For every fixed $m$ there exists $\delta > 0$ such that for all $\epsilon > 0$ the following holds. Let $f$ be a neutral SCF among $m$ alternatives such that $\Delta(f, DICT) > \epsilon$. Then for all $(a, b)$ we have $M^{a,b}(f) \geq \delta$.*

# Step 3

The original Step 3 was:

**Lemma (Non-General Step 3)**

*For every SCF $f$ on 3 alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot 6$*

And the generalization we attempt to prove is:

**Lemma (Generalized Step 3)**

*For every SCF $f$ on $m$ alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot m!$*

# Main Result

When we put together all 3 generalized steps we get our main result:

**Theorem (Main Result)**

*There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If $f$ is a neutral SCF for $n$ voters over $m$ alternatives and $\Delta(f, g) > \epsilon$ for any dictatorship $g$, then $f$ has total manipulability:* $\sum_{i=1}^{n} M_i(f) \geq \frac{(C\epsilon)^{\lfloor m/3 \rfloor}}{m!}$.

Step 3 is comprised of Lemma 6, Lemma 7, and Lemma 8 which we will generalize one at a time.

# Statement of Lemma 6

## Lemma (Original Lemma 6)

$$M^{a,b}(f) = E_{q\in L(C)^n}\left[\frac{|A(q)|}{3^n}\cdot\frac{|B(q)|}{3^n}\right],$$

*where $q$ is chosen uniformly at random.*

## Lemma (Generalized Lemma 6)

$$M^{a,b}(f) = E_{q\in L(C)^n}\left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n}\cdot\frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}\right],$$

*where $q$ is chosen uniformly at random.*

# Define $A$ and $B$ Functions

Let $a, b \in C$ be the first two alternatives, let $p \in L(C)^n$ be a preference profile. We define

$$A(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = a\}$$
$$B(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = b\}.$$

# Define $M^{a,b}$

Recall the definition of $M^{a,b}(f)$ from Friedgut:

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

where $p, p'$ are chosen at random in $L(C)^n$ with $p|_{\{a,b\}} = p'|_{\{a,b\}}$.

# Size of Profiles

For any preference profile $p \in P$ there are $(\frac{m!}{2})^n$ profiles $x$ such that $x|_{\{a,b\}} = p|_{\{a,b\}}$. This is because there are $m!$ possible preference lists; half of them will have the preference between $a$ and $b$ that agrees with $p|_{\{a,b\}}$ and half will disagree. This gives $\frac{m!}{2}$ possible preference lists for each voter, so there are $(\frac{m!}{2})^n$ profiles comprised of these preference lists.

# Proof of Generalized Lemma 6

First we fix a profile $q$. Then

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n}$$

is the probability that a randomly chosen profile, $p$, satisfying $p|_{\{a,b\}} = q|_{\{a,b\}}$ also satisfies $f(p) = a$. This is because there are $(\frac{m!}{2})^n$ profiles that agree with $q|_{\{a,b\}}$, and $|A(q)|$ is the number of those for which the outcome is $a$.
Since $p|_{\{a,b\}} = q|_{\{a,b\}}$ and $p'|_{\{a,b\}} = q|_{\{a,b\}}$, clearly we have that $p|_{\{a,b\}} = p'|_{\{a,b\}}$.
Since $f(p) = a$ and $f(p') = b$ are independent events, the joint probability is

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}.$$

# Proof of Generalized Lemma 6

So we can rewrite

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

as

$$M^{a,b}(f) = E_{q \in L(C)^n}\left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}\right],$$

$\square$

# Statement of Lemma 7

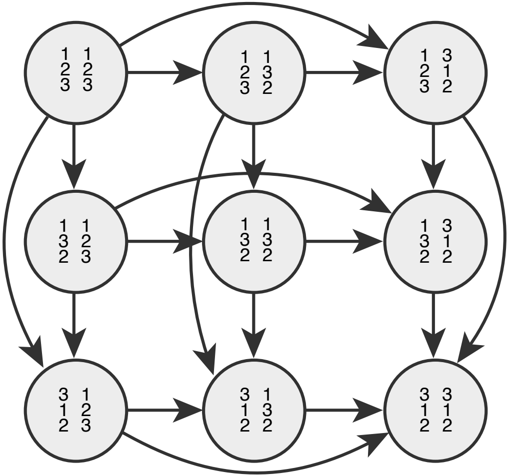
**Lemma (Original Lemma 7)**

$$\sum_i M_i(f) \geq \frac{1}{6} 3^{-n} E_p \left[ |\partial A(p)| + |\partial B(p)| \right]$$

**Lemma (Generalized Lemma 7)**

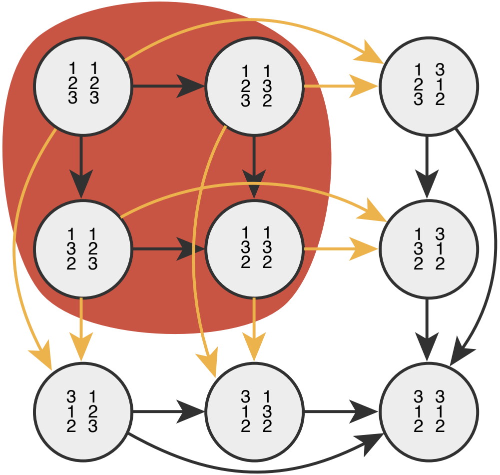
$$\sum_i M_i(f) \geq \frac{1}{m!} \left( \frac{m!}{2} \right)^{-n} E_p \left[ |\partial A(p)| + |\partial B(p)| \right]$$

# Profile Lattice



# Upper Edge Border

- Set of edges whose tail is in $A(p)$ and whose head is not in $A(p)$
- Denoted $\partial A(p)$
- Edge notation: $(x_{-i}, x_i, x_i')$ as shorthand for $((x_{-i}, x_i), (x_{-i}, x_i'))$



# Formal Definition of Upper Edge Border

$$\begin{aligned}\partial_i A(p) = \{(x_{-i}, x_i, x_i') \mid (x_{-i}, x_i) &\in A(p),\\ (x_{-i}, x_i') &\notin A(p),\\ x_i|_{\{a,b\}} &= x_i'|_{\{a,b\}},\\ x_i &<_s x_i'\}\end{aligned}$$

$$\partial A(p) = \bigcup_j \partial_j A(p)$$

# Edges Correspond to Manipulations

**Lemma**

*Then each* $(x_{-i}, x_i, x_i') \in \partial_i A(p) \cup \partial_i B(p)$ *corresponds to at least one successful manipulation.*

# Proof of Generalized Lemma 7

- Randomly choose $p$ and $p'_i$
- $M_i(f)$ is the probability that $p'_i$ is a successful manipulation
- We wish to come up with a lower bound for $M_i(f)$
- We can think of these as two distinct profiles, $p$ and $p'$, where $p' = (p_{-i}, p'_i)$
- Clearly $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$
- We have $p_i|_{\{a,b\}} = p'_i|_{\{a,b\}}$ with probability $\frac{1}{2}$, and we condition the following on this being the case
- So $p|_{\{a,b\}} = p'|_{\{a,b\}}$

# Proof of Generalized Lemma 7

- Since every edge in $\partial_i A(p) \cup \partial_i B(p)$ corresponds to at least one manipulation, we can lower bound $M_i(f)$ by the probability that an edge is in $\partial_i A(p) \cup \partial_i B(p)$
- The total number of possible edges of the form $(x_{-i}, x_i, x_i')$ is

$$(m!)^{n-1} \cdot m! \cdot m!$$

- But all edges in $\partial_i A(p) \cup \partial_i B(p)$ must agree with $p|_{\{a,b\}}$. The total number of possible edges agreeing with $p|_{\{a,b\}}$ is

$$\left(\frac{m!}{2}\right)^{n-1} \cdot \frac{m!}{2} \cdot \frac{m!}{2} = \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

- Since $\partial_i A(p)$ and $\partial_i B(p)$ are disjoint, no edge can be in both sets and so we have

$$|\partial_i A(p) \cup \partial_i B(p)| \leq \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

# Proof of Generalized Lemma 7

- Therefore, the probability that a randomly chosen edge is in either $\partial_i A(p)$ or $\partial_i B(p)$ is
$$\frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot E_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right]$$
- We conditioned our analysis on $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$, so our lower bound becomes
$$M_i(f) \geq \frac{1}{2} \cdot \frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot E_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$
- Simplifying gives
$$M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot E_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$
- Summing over $i$ gives
$$\sum_i M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot E_p\left[|\partial A(p)| + |\partial B(p)|\right].$$

□

# Lemma 8

**Lemma (Original Lemma 8)**

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{1}{3}\right)^n |A(p)| \cdot |B(p)|$$

**Lemma (Generalized Lemma 8)**

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{2}{m!}\right)^n |A(p)| \cdot |B(p)|$$

# Combining Lemma 6, 7, and 8

Restatement of the lemmas:

$$M^{a,b} = E[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$L_7 \cdot E[|\partial A| + |\partial B|] \leq \sum_i M_i \qquad \text{lemma 7}$$

$$\frac{1}{L_8} \cdot (|\partial A| + |\partial B|) \geq |A||B| \qquad \text{lemma 8}$$

Now we can solve for the result of step 3:

$$M^{a,b} = E[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$M^{a,b} \leq E[|\partial A| + |\partial B|] \cdot \frac{L_6}{L_8} \qquad \text{by lemma 8}$$

$$M^{a,b} \leq \sum_i M_i \cdot \frac{L_6}{L_7 L_8} \qquad \text{by lemma 7}$$

# Combining Lemma 6, 7, and 8

The variables have the following values:

$$L_6 = \left(\frac{m!}{2}\right)^{-2n}$$

$$L_7 = \frac{1}{m!}\left(\frac{m!}{2}\right)^{-n}$$

$$L_8 = \left(\frac{m!}{2}\right)^{-n}$$

Substituting becomes:

$$\begin{aligned}\frac{L_6}{L_7 L_8} &= \left(\frac{m!}{2}\right)^{-2n} \cdot m!\left(\frac{m!}{2}\right)^{n} \cdot \left(\frac{m!}{2}\right)^{n} \\ &= \left(\frac{m!}{2}\right)^{-2n} \cdot m! \cdot \left(\frac{m!}{2}\right)^{2n} \\ &= m!\end{aligned}$$

# Step 3 Result

The final result for step 3 is:

$$M^{a,b} \leq \sum_i M_i \cdot m!$$

# Independent Work

- Placeholder

# Questions

- Placeholder