A Generalized Probabilistic Gibbard-Satterthwaite Theorem

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Importance of Elections

- Political voting
- Electing board members
- Shareholders voting on company issues
- Artificial intelligent agent decision
- Search engine page-ranking

Modeling an Election

- Each voter ranks the alternatives (*preference list*)
- All the preference lists make up a *preference profile*
- A *voting rule* (election system, social choice function) chooses a winner based on a profile

The interesting part is the voting rules.

This field of study is called social choice theory.

Fairness in Election Systems

- Everyone should have an equal say
- Winner should accurately represent the group preference
- Simple with 2 alternatives; complicated with many

With two alternatives the one preferred by the most voters should win.

Manipulation

A voter can get a better result by voting strategically, rather than voting his actual preferences.

Real Preferences

You	Others		Win
L	**D**	**R**	**D**
R	R	D	
D	L	L	

Manipulation

You	Others		Win
R	**D**	**R**	**R**
L	R	D	
D	L	L	

Manipulation is an enemy of fairness because the manipulative voter gets more influence than others.

Assume a simple plurality/first-past-the-post system.

In the first case it would be a tie, but lets assume our arbitrary tie breaking technique will choose the Democrats.

This is essentially the “wasted vote” problem.

This is a very simplistic example, but almost all voting systems are susceptible to manipulation.

Gibbard-Satterthwaite Theorem

Voting rules are manipulable if they satisfy:

Non-dictatorship No single voter always dictates the group preference.

Non-imposition Every alternative has the possibility of winning.

We would like to devise an unmanipulable voting rule, but this theorem says it's impossible.

Circumventing the Gibbard-Satterthwaite Theorem

- Searching for a computational barrier to manipulation
- Bartholdi, Tovey, and Trick studied the computational difficulty of finding a winner for various voting rules
- The Dodgson method is infeasible to manipulate because finding the winner is NP-hard
- Voting rules need to resist manipulation *and* make it feasible to find the winner

Many people have followed this line of research to great effect.

Random Manipulation

- Friedgut, Kalai, and Nisan studied random manipulation
- Succeeds with non-negligible probability
- Shows the limits of a computational barrier to manipulation
- Only proved results for elections with 3 alternatives
- This is the work I attempted to generalize

If your alternative is not winning, randomly permute your preference list.

Independent Work

- Isaksson, Kindler, and Mossel have independently published a generalization in their paper *The geometry of manipulation: A quantitative proof of the gibbard-satterthwaite theorem*
- They proved that for a neutral social choice function, a uniformly chosen profile will be manipulable with probability at least $2^{-1}\epsilon^2 n^{-4} m^{-6} (m!)^{-3}$
- Where ϵ is the distance from dictatorship

As a disclaimer before a formal look at my work, others have published an independent generalization before I finished.

This is unfortunate for me, but fortunate for the field of social choice theory as a whole.

My work is still useful as:

- A simpler proof
- A proof that closely follows the original
- An alternative proof

Notation

- $C = \{1, \ldots, m\}$ is the set of alternatives.
- A preference list is a total ordering of the alternatives.
- The set of all preference lists is $L(C)$.
- A preference profile is a sequence of n preference lists.
- The set of all preference profiles is $L(C)^n$.
- A voting rule is a function that chooses a winning alternative from a profile, i.e. $f : L(C)^n \to C$.
- An election is a voting rule paired with a profile: (f, p).

$L(C)^n$ is the Cartesian product of $L(C)$ with itself n times, or the set of all n-tuples of elements of $L(C)$

Restricted Preference Profiles

- For a preference list v, $v|_D$ means v *restricted to* D, i.e. v with the alternatives not in D removed.
- For a preference profile, p, $p|_D$ means p with each preference list restricted to D.

p

1	3	2
2	2	3
3	1	1

$\Rightarrow$

$p|_{\{1,2\}}$

1	2	2
2	1	1

Manipulation Power

- Manipulation power $M_i(f)$, of voter i on a social choice function f is the probability that p'_i is a profitable manipulation by voter i
- Where p is a profile and p'_i is a preference list which are both chosen uniformly at random

Manipulation Potential

$$M^{a,b}(f) = \mathrm{P}[f(p) = a, f(p') = b]$$

- Describes the scenario where all voters together attempt to manipulate f to be b rather than a
- p and p' are chosen uniformly at random
- Voters don't alter their preference between a and b

This definition does not require that anyone in particular gain from this, just that something "unexpected" happens.

Proof Summary

Friedgut's proof is in three steps:

- **Step 1** Application of a quantitative version of Arrow's impossibility theorem.
- **Step 2** Reduction from an SCF with low dependence on irrelevant alternatives to a GSWF with a low paradox probability.
- **Step 3** Reduction from low manipulation power to low dependence on irrelevant alternatives.

Generalized Steps

Friedgut, was able to generalize Step 1 and Step 2 as follows:

Lemma (Generalized Step 1)

For every fixed m and $\epsilon > 0$ there exists $\delta > 0$ such that if $F = f^{\otimes \binom{m}{2}}$ is a neutral IIA GSWF over m alternatives with $f : \{0,1\}^n \to \{0,1\}$, and $\Delta(f, DICT) > \epsilon$, then F has probability of at least $\delta \geq (C\epsilon)^{\lfloor m/3 \rfloor}$ of not having a Generalized Condorcet Winner, where $C > 0$ is an absolute constant.

Lemma (Generalized Step 2)

For every fixed m there exists $\delta > 0$ such that for all $\epsilon > 0$ the following holds. Let f be a neutral SCF among m alternatives such that $\Delta(f, DICT) > \epsilon$. Then for all (a, b) we have $M^{a,b}(f) \geq \delta$.

So we only need to generalize step 3 to get the whole proof.

Step 3

Lemma (Non-General Step 3)

For every SCF f on 3 alternatives and every $a, b \in C$,

$$M^{a,b}(f) \leq \sum_i M_i(f) \cdot 6$$

Lemma (Generalized Step 3)

For every SCF f on m alternatives and every $a, b \in C$,

$$M^{a,b}(f) \leq \sum_i M_i(f) \cdot m!$$

Combining Steps

- Step 3 is comprised of Lemma 6, Lemma 7, and Lemma 8 which we will generalize one at a time
- When we put together all 3 generalized steps we get our main result

Statement of Lemma 6

Lemma (Original Lemma 6)

$$M^{a,b}(f) = \mathrm{E}_p\left[\frac{|A(p)|}{3^n} \cdot \frac{|B(p)|}{3^n}\right],$$

where $p \in L(C)^n$ *is chosen uniformly at random.*

Lemma (Generalized Lemma 6)

$$M^{a,b}(f) = \mathrm{E}_p\left[\frac{|A(p)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(p)|}{\left(\frac{m!}{2}\right)^n}\right],$$

where $p \in L(C)^n$ *is chosen uniformly at random.*

Define A and B Functions

Let $a, b \in C$ be the first two alternatives, let $p \in L(C)^n$ be a preference profile. We define

$$A(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = a\}$$
$$B(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = b\}.$$

Recall $M^{a,b}$ Definition

Recall the definition of $M^{a,b}(f)$:

$$M^{a,b}(f) = \mathrm{P}[f(p) = a, f(p') = b]$$

where p, p' are chosen at random in $L(C)^n$ with $p|_{\{a,b\}} = p'|_{\{a,b\}}$.

Size of Profiles

For any preference profile $p \in L(C)^n$ there are $(\frac{m!}{2})^n$ profiles x such that $x|_{\{a,b\}} = p|_{\{a,b\}}$ because:

- There are $m!$ possible preference lists
- Half of them will have the preference between a and b that agrees with $p_i|_{\{a,b\}}$, for any i
- This gives $\frac{m!}{2}$ possible preference lists for each voter
- So there are $(\frac{m!}{2})^n$ profiles comprised of these preference lists

Proof of Generalized Lemma 6

- First we fix a profile q
- There are $(\frac{m!}{2})^n$ profiles that agree with $q|_{\{a,b\}}$
- $|A(q)|$ is the number of those for which the outcome is a
- Randomly choose a profile, p, satisfying $p|_{\{a,b\}} = q|_{\{a,b\}}$
- So the probability that $f(p) = a$ is

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n}$$

Proof of Generalized Lemma 6

- Likewise, randomly choose a profile, p', satisfying $p'|_{\{a,b\}} = q|_{\{a,b\}}$
- And the probability that $f(p') = b$ is

$$\frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}$$

Proof of Generalized Lemma 6

Since $p|_{\{a,b\}} = q|_{\{a,b\}}$ and $p'|_{\{a,b\}} = q|_{\{a,b\}}$, clearly we have that $p|_{\{a,b\}} = p'|_{\{a,b\}}$. Since $f(p) = a$ and $f(p') = b$ are independent events, the joint probability is

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}.$$

Proof of Generalized Lemma 6

So we can rewrite

$$M^{a,b}(f) = \mathrm{P}[f(p) = a, f(p') = b]$$

as

$$M^{a,b}(f) = \mathrm{E}_q \left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n} \right],$$

□

Statement of Lemma 7

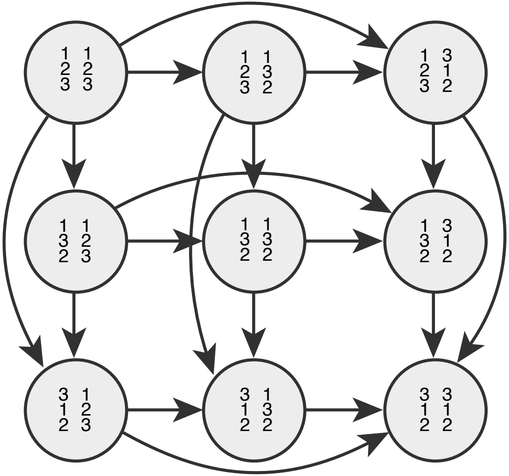
Lemma (Original Lemma 7)

$$\sum_i M_i(f) \geq \frac{1}{6} 3^{-n} \, \mathrm{E}_p \left[|\partial A(p)| + |\partial B(p)| \right]$$

Lemma (Generalized Lemma 7)

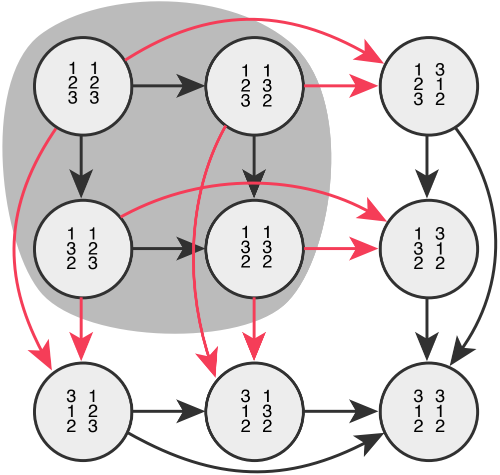
$$\sum_i M_i(f) \geq \frac{1}{m!} \left(\frac{m!}{2} \right)^{-n} \mathrm{E}_p \left[|\partial A(p)| + |\partial B(p)| \right]$$

Profile Lattice



Upper Edge Border

- Set of edges whose tail is in $A(p)$ and whose head is not in $A(p)$
- Denoted $\partial A(p)$
- Edge notation: (x_{-i}, x_i, x_i') as shorthand for $((x_{-i}, x_i), (x_{-i}, x_i'))$



x_{-i} is x without the ith index.

Formal Definition of Upper Edge Border

$$\begin{aligned}\partial_i A(p) = \{(x_{-i}, x_i, x_i') \mid (x_{-i}, x_i) &\in A(p), \\ (x_{-i}, x_i') &\notin A(p), \\ x_i|_{\{a,b\}} &= x_i'|_{\{a,b\}}, \\ x_i &<_s x_i'\}\end{aligned}$$

$$\partial A(p) = \bigcup_j \partial_j A(p)$$

We don't have time to go into more of the lattice work I've done, including $<_s$.

$<_s$ means something like being close to the $(1, 2, 3)$ ordering.

We define the upper edge border of $B(p)$ analogously.

Edges Correspond to Manipulations

Lemma

Each $(x_{-i}, x_i, x_i') \in \partial_i A(p) \cup \partial_i B(p)$ *corresponds to at least one successful manipulation.*

We don’t have time to show the proof, but it’s in my paper.

Proof of Generalized Lemma 7

- Randomly choose p and p'_i
- $M_i(f)$ is the probability that p'_i is a successful manipulation
- We wish to come up with a lower bound for $M_i(f)$
- We can think of these as two distinct profiles, p and p', where $p' = (p_{-i}, p'_i)$
- Clearly $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$
- We have $p_i|_{\{a,b\}} = p'_i|_{\{a,b\}}$ with probability $\frac{1}{2}$, and we condition the following on this being the case
- So $p|_{\{a,b\}} = p'|_{\{a,b\}}$

Proof of Generalized Lemma 7

- Since every edge in $\partial_i A(p) \cup \partial_i B(p)$ corresponds to at least one manipulation, we can lower bound $M_i(f)$ by the probability that an edge is in $\partial_i A(p) \cup \partial_i B(p)$
- The total number of possible edges of the form (x_{-i}, x_i, x_i') is

$$(m!)^{n-1} \cdot m! \cdot m!$$

- But all edges in $\partial_i A(p) \cup \partial_i B(p)$ must agree with $p|_{\{a,b\}}$. The total number of possible edges agreeing with $p|_{\{a,b\}}$ is

$$\left(\frac{m!}{2}\right)^{n-1} \cdot \frac{m!}{2} \cdot \frac{m!}{2} = \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

- Since $\partial_i A(p)$ and $\partial_i B(p)$ are disjoint, no edge can be in both sets and so we have

$$|\partial_i A(p) \cup \partial_i B(p)| \leq \frac{m!}{2}\left(\frac{m!}{2}\right)^{n}$$

Proof of Generalized Lemma 7

- Therefore, the probability that a randomly chosen edge is in either $\partial_i A(p)$ or $\partial_i B(p)$ is
$$\frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right]$$
- We conditioned our analysis on $p_{-i}|_{\{a,b\}} = p'_{-i}|_{\{a,b\}}$, so our lower bound becomes
$$M_i(f) \geq \frac{1}{2} \cdot \frac{2}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$

Proof of Generalized Lemma 7

- Simplifying gives

$$M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial_i A(p)| + |\partial_i B(p)|\right].$$

- Summing over i gives

$$\sum_i M_i(f) \geq \frac{1}{m!}\left(\frac{2}{m!}\right)^n \cdot \mathrm{E}_p\left[|\partial A(p)| + |\partial B(p)|\right].$$

$\square$

Lemma 8

Lemma (Original Lemma 8)

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{1}{3}\right)^n |A(p)| \cdot |B(p)|$$

Lemma (Generalized Lemma 8)

$$|\partial A(p)| + |\partial B(p)| \geq \left(\frac{2}{m!}\right)^n |A(p)| \cdot |B(p)|$$

I got close, but wasn't able to complete the proof of this lemma.

In my thesis I give a partial proof and a detailed description of the things that would be required to make it work.

Because of lack of time, I won't go into it here.

Combining Lemma 6, 7, and 8

Restatement of the lemmas:

$$M^{a,b} = \mathrm{E}[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$L_7 \cdot \mathrm{E}[|\partial A| + |\partial B|] \leq \sum_i M_i \qquad \text{lemma 7}$$

$$\frac{1}{L_8} \cdot (|\partial A| + |\partial B|) \geq |A||B| \qquad \text{lemma 8}$$

Now we can solve for the result of step 3:

$$M^{a,b} = \mathrm{E}[|A||B|] \cdot L_6 \qquad \text{lemma 6}$$

$$M^{a,b} \leq \mathrm{E}[|\partial A| + |\partial B|] \cdot \frac{L_6}{L_8} \qquad \text{by lemma 8}$$

$$M^{a,b} \leq \sum_i M_i \cdot \frac{L_6}{L_7 L_8} \qquad \text{by lemma 7}$$

Because it's *much* easier to read:

- $A \equiv A(p)$
- $M^{a,b} \equiv M^{a,b}(f)$
- $M_i \equiv M_i(f)$

Combining Lemma 6, 7, and 8

The variables have the following values:

$$L_6 = \left(\frac{m!}{2}\right)^{-2n}$$

$$L_7 = \frac{1}{m!}\left(\frac{m!}{2}\right)^{-n}$$

$$L_8 = \left(\frac{m!}{2}\right)^{-n}$$

Substituting becomes:

$$\begin{aligned}\frac{L_6}{L_7 L_8} &= \left(\frac{m!}{2}\right)^{-2n} \cdot m!\left(\frac{m!}{2}\right)^{n} \cdot \left(\frac{m!}{2}\right)^{n} \\ &= \left(\frac{m!}{2}\right)^{-2n} \cdot m! \cdot \left(\frac{m!}{2}\right)^{2n} \\ &= m!\end{aligned}$$

Step 3 Result

The final result for step 3 is:

$$M^{a,b}(f) \le \sum_i M_i(f) \cdot m!$$

By combining it with Friedgut's step 1 and 2 we get the generalized main theorem:

Theorem (Main Result)

There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If f is a neutral SCF for n voters over m alternatives and $\Delta(f, g) > \epsilon$ for any dictatorship g, then f has total manipulatiblity:

$$\sum_{i=1}^{n} M_i(f) \ge \frac{(C\epsilon)^{\lfloor m/3 \rfloor}}{m!}.$$

In this case C is a constant, and not the set of the alternatives.

Questions