A Generalized Probabilistic Gibbard-Satterthwaite Theorem

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Importance of Elections

- Political voting
- Electing board members
- Shareholders voting on company issues
- Artificial intelligent agent decision
- Search engine page-ranking

Desirable Election Systems

- Fairness
  - Everyone should have an equal say
  - Winner should accurately represent the group
  - Simple with 2 alternatives; complicated with many
  - First studied by Condorcet

Arrow's Impossibility Theorem

- Unrestricted domain (universality)
- Independence of irrelevant alternatives
- Pareto principle (unanimity)
- Non-dictatorship

Manipulation

A voter can get a better result by voting strategically, rather than voting his actual preferences.

Real Preferences

You	Others		Win
L	**D**	**R**	**D**
R	R	D	
D	L	L	

Manipulation

You	Others		Win
R	**D**	**R**	**R**
L	R	D	
D	L	L	

Gibbard-Satterthwaite Theorem

Voting rules are manipulable if they satisfy:

- **Non-dictatorship**: No single voter always dictates the group preference.
- **Non-imposition**: Every alternative has the possibility of winning.

Notation

- $C = \{1, \ldots, m\}$ is the set of alternatives.
- A preference list is a total ordering of the alternatives.
- The set of all preference lists is $L(C)$.
- A preference profile is a sequence of n preference lists.
- The set of all preference profiles is $P = L(C)^n$.
- A voting rule is a function that chooses a winning alternative from a profile, i.e. $f : P \to C$.
- An election is a voting rule paired with a profile: (f, p).

Restricted Preference Profiles

- For a preference list v, $v|_D$ means v *restricted to* D, i.e. v with the alternatives from D removed.
- For a preference profile, p, $p|_D$ means p with each preference list restricted to D.

p

1	3	2
2	2	3
3	1	1

$p|_{\{1,2\}}$

1	2	2
2	1	1

Proof Summary

Friedgut's proof is in three steps:

- Step 1: application of a quantitative version of Arrow's impossibility theorem.
- Step 2: reduction from an SCF with low dependence on irrelevant alternatives to a GSWF with a low paradox probability.
- Step 3: reduction from low manipulation power to low dependence on irrelevant alternatives.

Generalized Steps

Friedgut, was able to generalize Step 1 and Step 2 as follows:

Lemma (Generalized Step 1)

For every fixed m and $\epsilon > 0$ there exists $\delta > 0$ such that if $F = f^{\otimes \binom{m}{2}}$ is a neutral IIA GSWF over m alternatives with $f : \{0,1\}^n \to \{0,1\}$, and $\Delta(f, DICT) > \epsilon$, then F has probability of at least $\delta \geq (C\epsilon)^{\lfloor m/3 \rfloor}$ of not having a Generalized Condorcet Winner, where $C > 0$ is an absolute constant.

Lemma (Generalized Step 2)

For every fixed m there exists $\delta > 0$ such that for all $\epsilon > 0$ the following holds. Let f be a neutral SCF among m alternatives such that $\Delta(f, DICT) > \epsilon$. Then for all (a, b) we have $M^{a,b}(f) \geq \delta$.

Step 3

The original Step 3 was:

Lemma (Non-General Step 3)

For every SCF f on 3 alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot 6$

And the generalization we attempt to prove is:

Lemma (Generalized Step 3)

For every SCF f on m alternatives and every $a, b \in A$, $M^{a,b} \leq \sum_i M_i \cdot m!$

Main Result

When we put together all 3 generalized steps we get our main result:

Theorem (Main Result)

There exists a constant $C > 0$ such that for every $\epsilon > 0$ the following holds. If f is a neutral SCF for n voters over m alternatives and $\Delta(f, g) > \epsilon$ for any dictatorship g, then f has total manipulability: $\sum_{i=1}^{n} M_i(f) \geq \frac{(C\epsilon)^{\lfloor m/3 \rfloor}}{m!}$.

Step 3 is comprised of Lemma 6, Lemma 7, and Lemma 8 which we will generalize one at a time.

Statement of Lemma 6

Lemma (Original Lemma 6)

$$M^{a,b}(f) = E_{q \in L(C)^n}\left[\frac{|A(q)|}{3^n} \cdot \frac{|B(q)|}{3^n}\right],$$

where q is chosen uniformly at random.

Lemma (Generalized Lemma 6)

$$M^{a,b}(f) = E_{q \in L(C)^n}\left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}\right],$$

where q is chosen uniformly at random.

Define A and B Functions

Let $a, b \in C$ be the first two alternatives, let $p \in L(C)^n$ be a preference profile. We define

$$A(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = a\}$$
$$B(p) = \{x \in L(C)^n \mid x|_{\{a,b\}} = p|_{\{a,b\}}, f(x) = b\}.$$

Define $M^{a,b}$

Recall the definition of $M^{a,b}(f)$ from Friedgut:

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

where p, p' are chosen at random in $L(C)^n$ with $p|_{\{a,b\}} = p'|_{\{a,b\}}$.

Size of Profiles

For any preference profile $p \in P$ there are $(\frac{m!}{2})^n$ profiles x such that $x|_{\{a,b\}} = p|_{\{a,b\}}$. This is because there are $m!$ possible preference lists; half of them will have the preference between a and b that agrees with $p|_{\{a,b\}}$ and half will disagree. This gives $\frac{m!}{2}$ possible preference lists for each voter, so there are $(\frac{m!}{2})^n$ profiles comprised of these preference lists.

Proof of Generalized Lemma 6

First we fix a profile q. Then

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n}$$

is the probability that a randomly chosen profile, p, satisfying $p|_{\{a,b\}} = q|_{\{a,b\}}$ also satisfies $f(p) = a$. This is because there are $(\frac{m!}{2})^n$ profiles that agree with $q|_{\{a,b\}}$, and $|A(q)|$ is the number of those for which the outcome is a.
Since $p|_{\{a,b\}} = q|_{\{a,b\}}$ and $p'|_{\{a,b\}} = q|_{\{a,b\}}$, clearly we have that $p|_{\{a,b\}} = p'|_{\{a,b\}}$.
Since $f(p) = a$ and $f(p') = b$ are independent events, the joint probability is

$$\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n}.$$

Proof of Generalized Lemma 6

So we can rewrite

$$M^{a,b}(f) = \mathrm{P}(f(p) = a, f(p') = b)$$

as

$$M^{a,b}(f) = E_{q \in L(C)^n} \left[\frac{|A(q)|}{\left(\frac{m!}{2}\right)^n} \cdot \frac{|B(q)|}{\left(\frac{m!}{2}\right)^n} \right],$$

□

Independent Work

- Placeholder

Questions

- Placeholder