Week 8: Hypothesis Testing, Correlation vs. Causation

DSUA111: Data Science for Everyone, NYU, Fall 2020

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- This slideshow: https://jjacobs.me/dsua111-sections/week-08 (https://jjacobs.me/dsua111-sections/week-08)
- All materials: https://github.com/jpowerj/dsua111-sections (https://github.com/jpowerj/dsua111-sections)

Outline

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3. Null vs. Alternative Hypotheses
4. Test Statistics
5. The Normal Distribution
6. Correlation vs. Causation

Hypothesis Testing Overview

tl;dr

- If your theory was true, what would the data look like?
- Now compare that to the actual, **observed** data

Testing Coins

Example: I walk up to you and say "Hey, wanna gamble? We'll each put in a dollar, then I'll flip this coin. Heads I get the $2, tails you get the \$2"

- Xavier's Theory: I think the coin is fair. Heads and tails will come up **about the same** number of times
- Yasmin's Theory: I don't trust this guy, I think **heads will come up more often than tails**

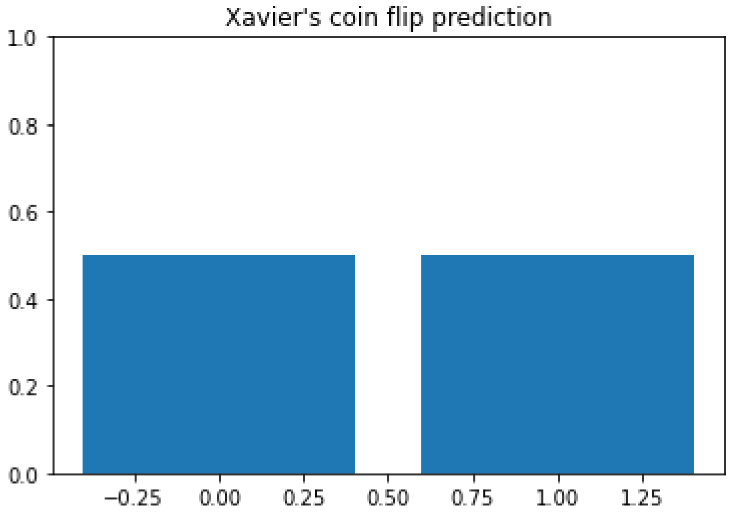
"Suit yourself -- here, I'll let you flip it **100 times** and then draw your own conclusions about whether it's fair or not!"

What do the two theories predict in terms of the outcome of a series of coin flips?

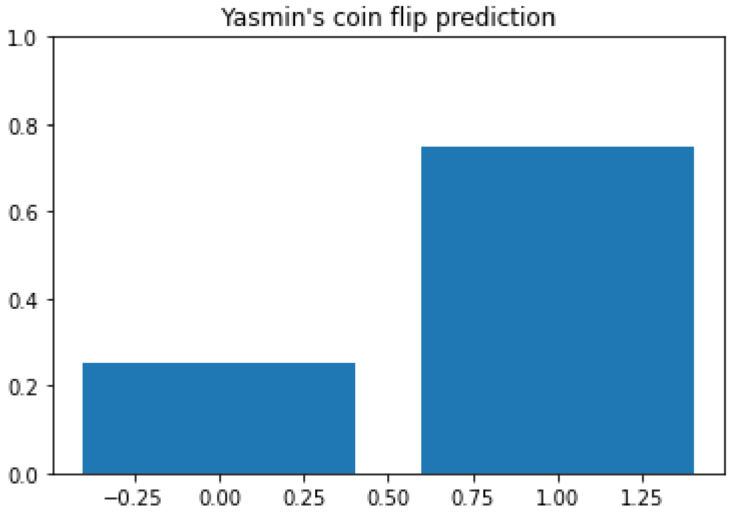
In [1]:
```
x_predictions = [0.5, 0.5]
y_predictions = [0.25, 0.75]

import matplotlib.pyplot as plt
def plot_prediction(prediction, who):
    plt.bar([0,1], prediction)
    plt.ylim([0,1])
    plt.title(f"{who}'s coin flip prediction")
    plt.show()
```

In [2]:

```
plot_prediction(x_predictions, "Xavier")
```



In [3]:
```
plot_prediction(y_predictions, "Yasmin")
```



- Data:

```
In [4]:
from collections import Counter
import pandas as pd
import numpy as np
import secret_coin
np.random.seed(123)

def do_coin_flips(N):
    coin_flips = np.array([secret_coin.flip() for i in range(N)])
    return coin_flips

def get_flip_distributions(x_predictions, y_predictions, results):
    flip_counter = Counter(results)
    flip_counts = np.array([flip_counter[0], flip_counter[1]]) / len(results)
    flip_df = pd.DataFrame({'outcome':["Tails","Heads"],'p':flip_counts,'which':['Actual
outcome','Actual outcome']})
    x_pred_df = pd.DataFrame({'outcome':["Tails","Heads"],'p':x_predictions,'which':["Xa
vier's prediction","Xavier's prediction"]})
    y_pred_df = pd.DataFrame({'outcome':["Tails","Heads"],'p':y_predictions,'which':["Ya
smin's prediction","Yasmin's prediction"]})
    full_df = pd.concat([flip_df, x_pred_df, y_pred_df]).reset_index()
    return full_df
```

In [5]:
```
flips_100 = do_coin_flips(100)
flips_100
```

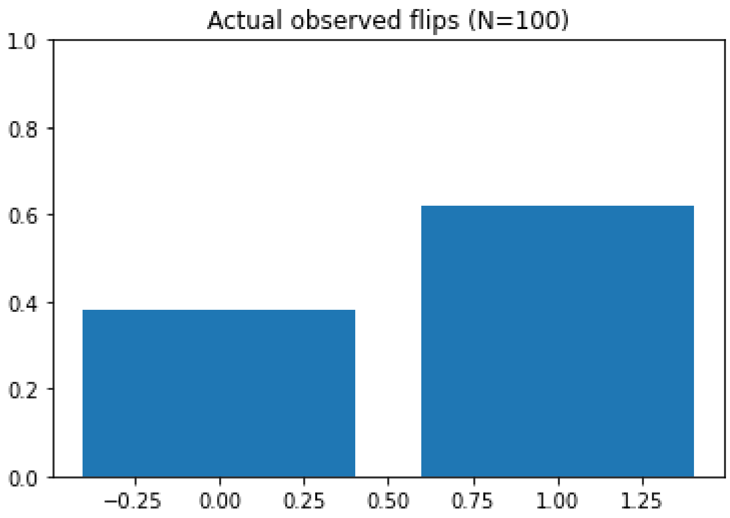
Out[5]:
```
array([0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0,
       0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1,
       0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1,
       0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1,
       1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1])
```

In [6]:
```
dist_df = get_flip_distributions(x_predictions, y_predictions, flips_100)
dist_df
```

Out[6]:

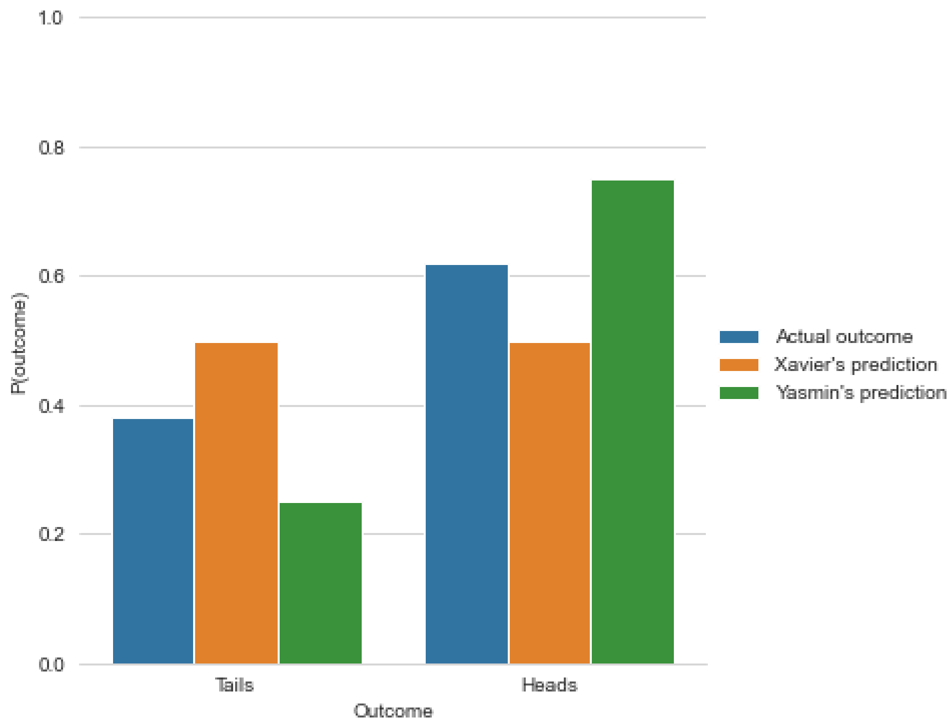
	index	outcome	p	which
0	0	Tails	0.38	Actual outcome
1	1	Heads	0.62	Actual outcome
2	0	Tails	0.50	Xavier's prediction
3	1	Heads	0.50	Xavier's prediction
4	0	Tails	0.25	Yasmin's prediction
5	1	Heads	0.75	Yasmin's prediction

In [7]:
```
plt.bar([0,1], [dist_df.iloc[0]['p'],dist_df.iloc[1]['p']])
plt.title("Actual observed flips (N=100)")
plt.ylim([0,1])
plt.show()
```

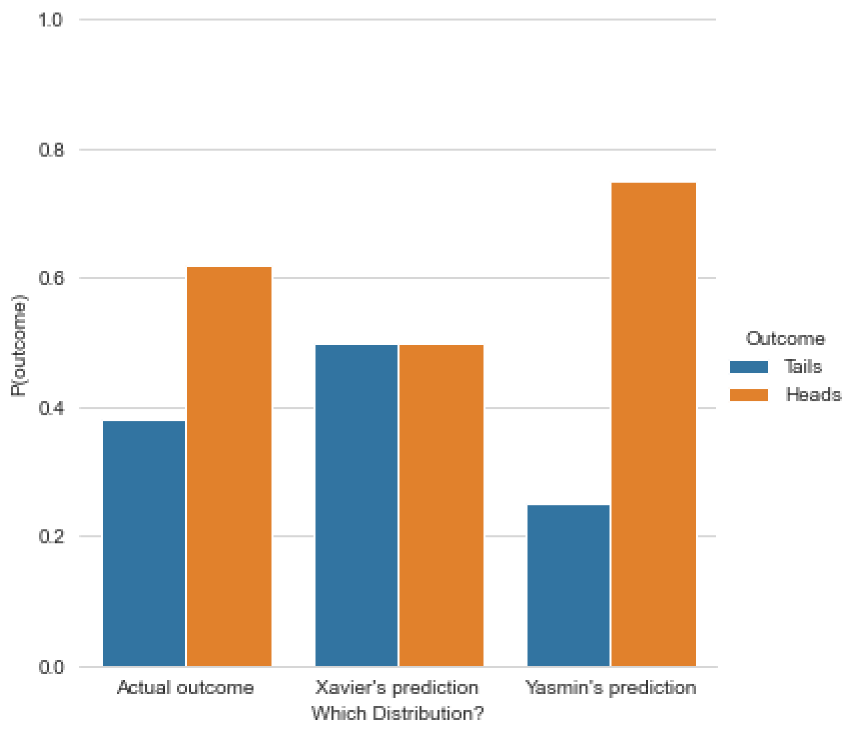


But... **how** wrong is our prediction?

In [8]:
```
import seaborn as sns; sns.set_style("whitegrid")
def plot_distributions_a(dist_df):
    g = sns.catplot(data=dist_df, kind="bar",x="outcome", y="p", hue="which")
    g.despine(left=True)
    g.set_axis_labels("Outcome", "P(outcome)")
    g.legend.set_title("")
    g.set(ylim=(0,1))
plot_distributions_a(dist_df)
```



In [9]:
```
def plot_distributions_b(dist_df):
    g = sns.catplot(
        data=dist_df, kind="bar",
        x="which", y="p", hue="outcome"
    )
    g.despine(left=True)
    g.set_axis_labels("Which Distribution?", "P(outcome)")
    g.legend.set_title("Outcome")
    g.set(ylim=(0,1))
plot_distributions_b(dist_df)
```



Hmm... that actual outcome still looks kinda sketchy. Let's try one more time

In [10]:
```
flips_100_2 = do_coin_flips(100)
flips_100_2
```

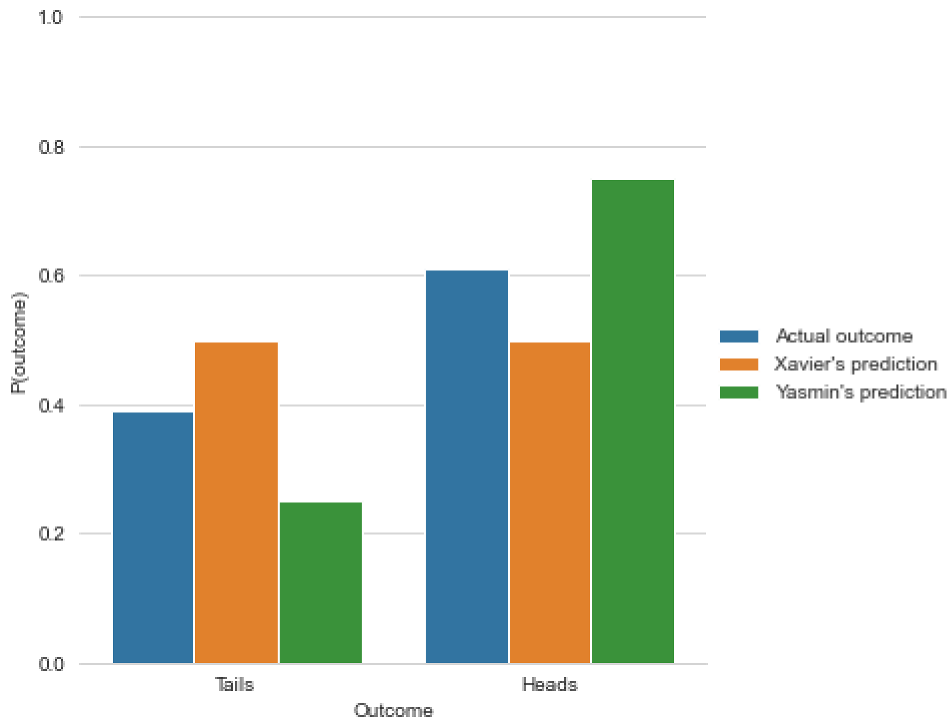
Out[10]:
```
array([1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1,
       1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1,
       1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 0,
       1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1,
       0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1])
```

In [11]:
```
dist_df2 = get_flip_distributions(x_predictions, y_predictions, flips_100_2)
dist_df2
```

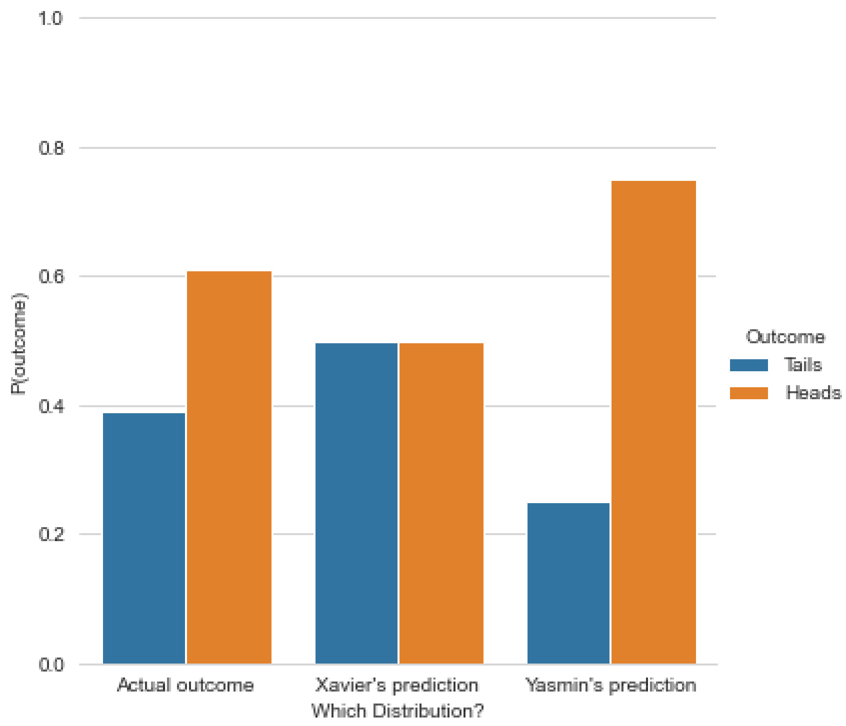
Out[11]:

	index	outcome	p	which
0	0	Tails	0.39	Actual outcome
1	1	Heads	0.61	Actual outcome
2	0	Tails	0.50	Xavier's prediction
3	1	Heads	0.50	Xavier's prediction
4	0	Tails	0.25	Yasmin's prediction
5	1	Heads	0.75	Yasmin's prediction

In [12]:
```
plot_distributions_a(dist_df2)
```



In [13]:
```
plot_distributions_b(dist_df2)
```



So, we need to **formalize** how to measure **how good or bad** a prediction is.

Enter... statistics!

Null vs. Alternative Hypotheses

- **Null Hypothesis** (H_0): The skeptical hypothesis... "Nothing interesting is going on here, any patterns were simply due to chance"
    - The coin is not weird. $P(\text{heads}) = 0.5$
- **Alternative Hypothesis** (H_A): Something *other than* chance is generating the pattern we observe
    - The coin is loaded! $P(\text{heads}) \neq 0.5$

ONLY TWO POSSIBLE CONCLUSIONS FROM YOUR HYPOTHESIS TEST

1. "We **reject** the null hypothesis"
    - if it seems sufficiently unlikely that the patterns in the data were produced simply due to chance
2. "We **fail to reject** the null hypothesis"
    - otherwise

Test Statistic

- Computed from the **observed** data
- A measure of how reasonable our alternative hypothesis is for explaining this data
- I think of it like: a measure of "weirdness" -- should get larger the more "suspicious" the data is
- e.g., for a sequence of dice rolls, should be small if most sides come up ~1/6th of the time, but very high if only 6 ever comes up

Unfair Coin Detection Statistic

- Idea: (number of heads) - (number of tails)

This is the number we were looking for before! It allows us to measure "unfairness" of the coin:

- Fair coins should produce test statistics close to 0 on average, while
- Coins biased towards heads will produce test statistics larger than 0 on average

So, how bad were our coin flip predictions?

In [14]:

```
def test_stat_a(coin_flips):
    # Num heads - num tails
    num_heads = len([f for f in coin_flips if f == 1])
    num_tails = len([f for f in coin_flips if f == 0])
    return num_heads - num_tails
```

In [15]:

```
print(test_stat_a(do_coin_flips(100)))
print(test_stat_a(do_coin_flips(100)))
```

```
24
26
```

What would this test statistic look like if the coin was *actually* fair?

In [16]:
```
def fair_coin_flips(N):
    return np.array([np.random.binomial(1, 0.5) for i in range(N)])
```

In [17]:
```
print(test_stat_a(fair_coin_flips(100)))
print(test_stat_a(fair_coin_flips(100)))
```

```
-12
8
```

Testing Procedure

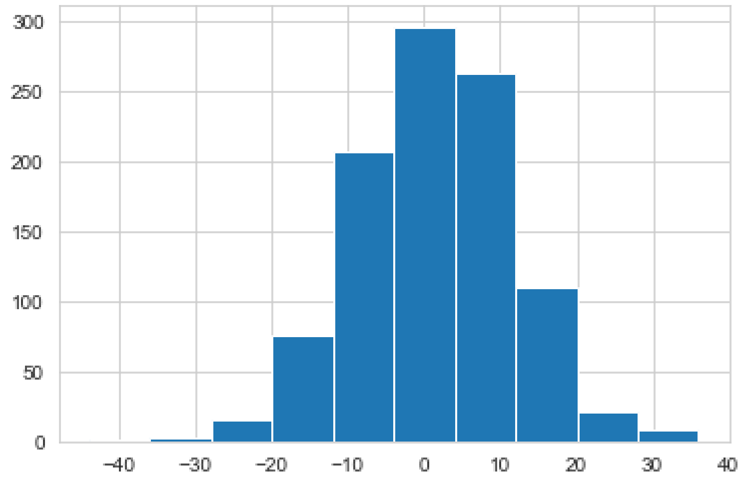
- We'll consider a **trial** to be a sequence of $N = 100$ coin flips.
- We'll perform **1000 trials** with a **fair** coin, and for each trial we'll record what the test statistic is
- Once we finish, we'll look at the **distribution** of test statistics that were generated from flips of the fair coin

In [18]:

```
test_stats = []
for trial_num in range(1000):
    test_stats.append(test_stat_a(fair_coin_flips(100)))
# Turn it into a NumPy array so we can do fancy math stuff with it
test_stats = np.array(test_stats)
```

In [19]:

```
plt.hist(test_stats)
plt.show()
```



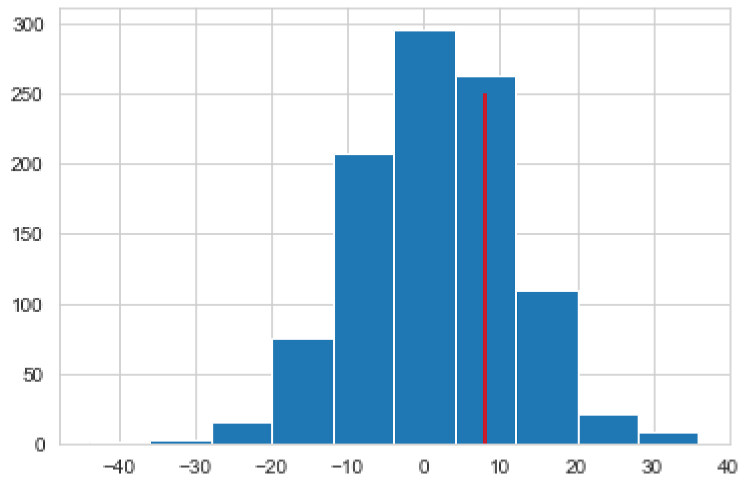
So... now we do 100 flips of our secret coin and see where it falls on this distribution!

In [20]:
```
secret_coin_results = do_coin_flips(100)
```

In [21]:
```
sc_stat = test_stat_a(secret_coin_results)
sc_stat
```

Out[21]: 8

In [22]:
```
plt.hist(test_stats)
plt.vlines(sc_stat, 0, 250, color='red')
plt.show()
```



p-values

- You should always have this plot in your head when thinking about **p-values**.
- The p-value is just the proportion of test statistics **generated from a true null hypothesis** (in this case, with a fair coin) that are as extreme or more extreme than the test statistic generated from the **observed data**
- In this case,

In [23]:
```
sum(test_stats >= sc_stat) / len(test_stats)
```

Out[23]:
```
0.246
```

246 out of the 1000 trials of 100 *fair* coin flips, 24.6%, produced test statistics as extreme or more extreme than 8, so our **p-value** is 0.246

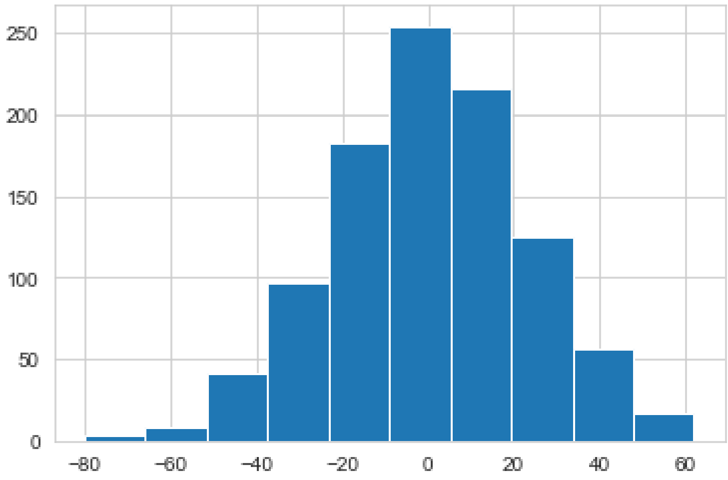
Increasing the Sample Size

- As of now, the test statistic for our observed sequence of secret-coin flips seems like it could easily be generated by chance.
- If we want to be more confident, we'll need to increase the sample size. This time, let's consider a trial to be *500* coin flips, and re-do our testing procedure

In [24]:
```
test_stats_500 = []
for i in range(1000):
    coin_flips = fair_coin_flips(500)
    test_stat = test_stat_a(coin_flips)
    test_stats_500.append(test_stat)
test_stats_500 = np.array(test_stats_500)
```

In [25]:

```
plt.hist(test_stats_500)
plt.show()
```

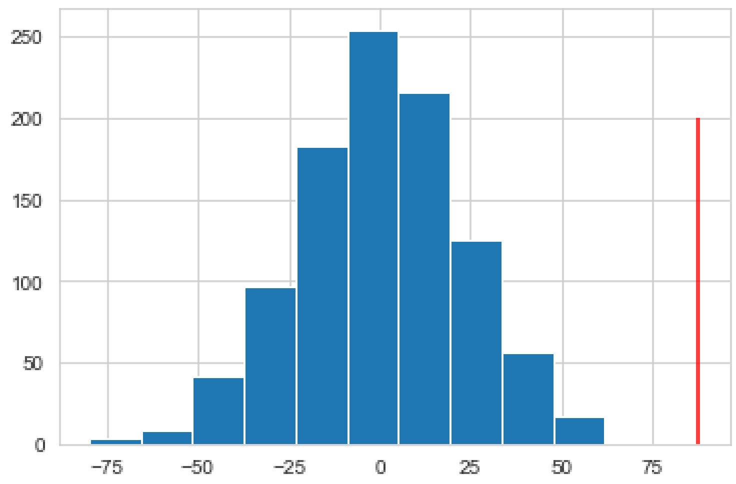


And now we do 500 flips of the secret coin, compute the test statistic for this sequence, and see where it lies on that histogram

In [26]:
```
secret_coin_results_500 = do_coin_flips(500)
sc_stat_500 = test_stat_a(secret_coin_results_500)
sc_stat_500
```

Out[26]:
```
88
```

In [27]:
```
plt.hist(test_stats_500)
plt.vlines(sc_stat_500, 0, 200, color='red')
plt.show()
```



In [28]:

```
sum(test_stats_500 >= sc_stat_500) / len(test_stats_500)
```

Out[28]:

```
0.0
```

Ok, **now** it looks suspicious -- the test statistic for the 500 secret coin flips is *way* higher than any test statistic produced by 500 flips of a known-fair coin

🤔🤔🤔

So our **p-value** here would be **0**: 0 out of the 1000 fair-coin trials produced test statistics this high or higher!

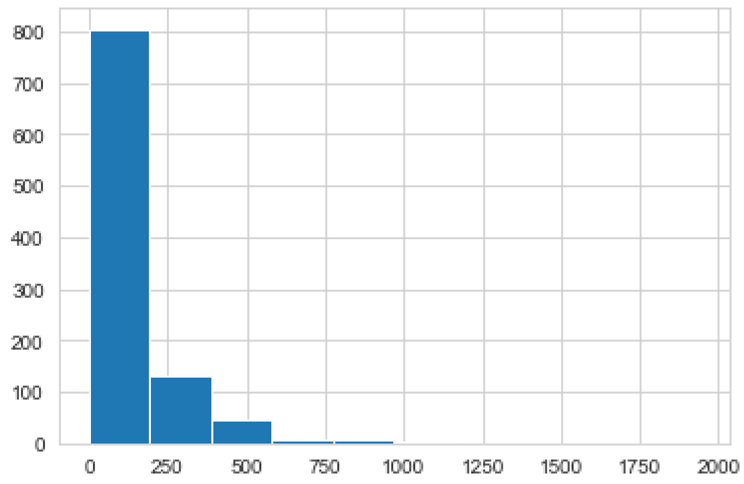
Alternative Test Statistics

- Remember: the statistic we used is **not** the only possible test statistic!
- We *make up*/choose the test statistic that can best help us "detect" not-by-chance data

In [29]:
```
def test_stat_b(coin_flips):
    # The squared difference in predicted probabilities
    num_heads = len([f for f in coin_flips if f == 1])
    num_tails = len([f for f in coin_flips if f == 0])
    return (num_heads - num_tails) ** 2
```

In [30]:
```
test_stats_b = []
for trial_num in range(1000):
    test_stats_b.append(test_stat_b(fair_coin_flips(100)))
test_stats_b = np.array(test_stats_b)
```

In [31]:
```
plt.hist(test_stats_b)
plt.show()
```

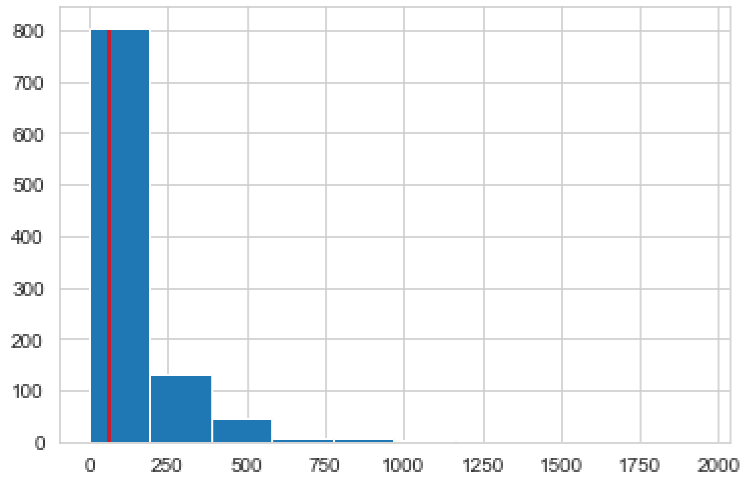


Now let's place our **observed** data on this plot

In [32]:
```
sc_stat_b = test_stat_b(secret_coin_results)
sc_stat_b
```

Out[32]: 64

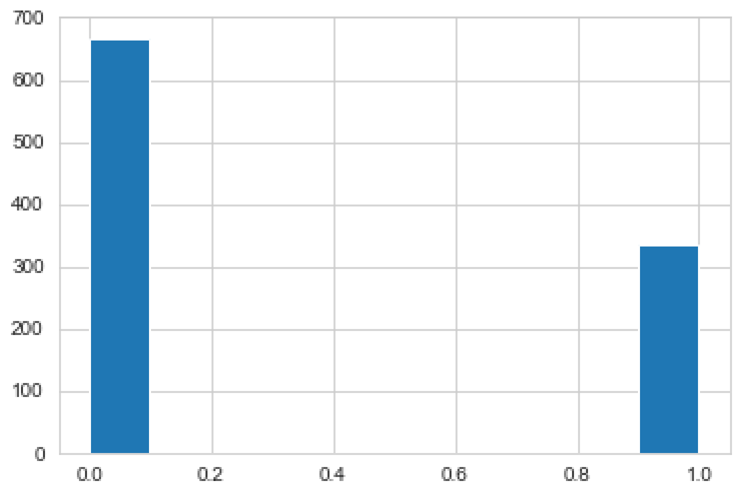
In [33]:
```
plt.hist(test_stats_b)
plt.vlines(sc_stat_b, 0, 800, color='red')
plt.show()
```



In [34]:
```
def test_stat_c(coin_flips):
    # 1 if it's within 5, 0 otherwise
    num_heads = len([f for f in coin_flips if f == 1])
    num_tails = len([f for f in coin_flips if f == 0])
    diff = abs(num_heads - num_tails)
    return 1 if diff <= 5 else 0
```

In [35]:
```
test_stats_c = []
for trial_num in range(1000):
    test_stats_c.append(test_stat_c(fair_coin_flips(100)))
test_stats_c = np.array(test_stats_c)
```

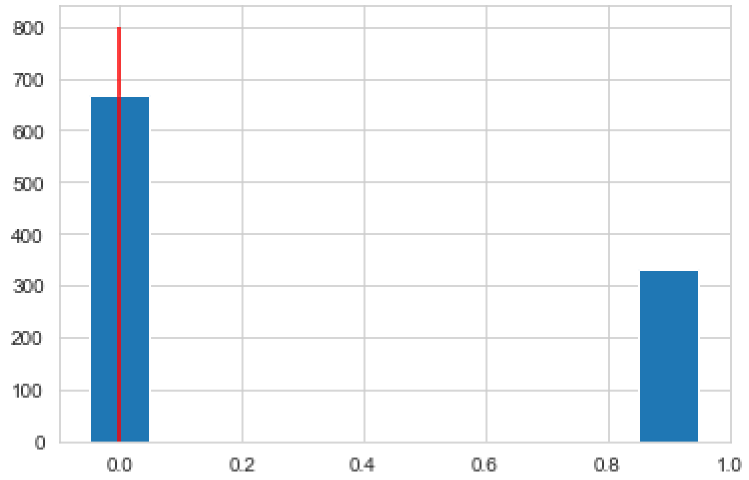
In [36]:
```
plt.hist(test_stats_c)
plt.show()
```



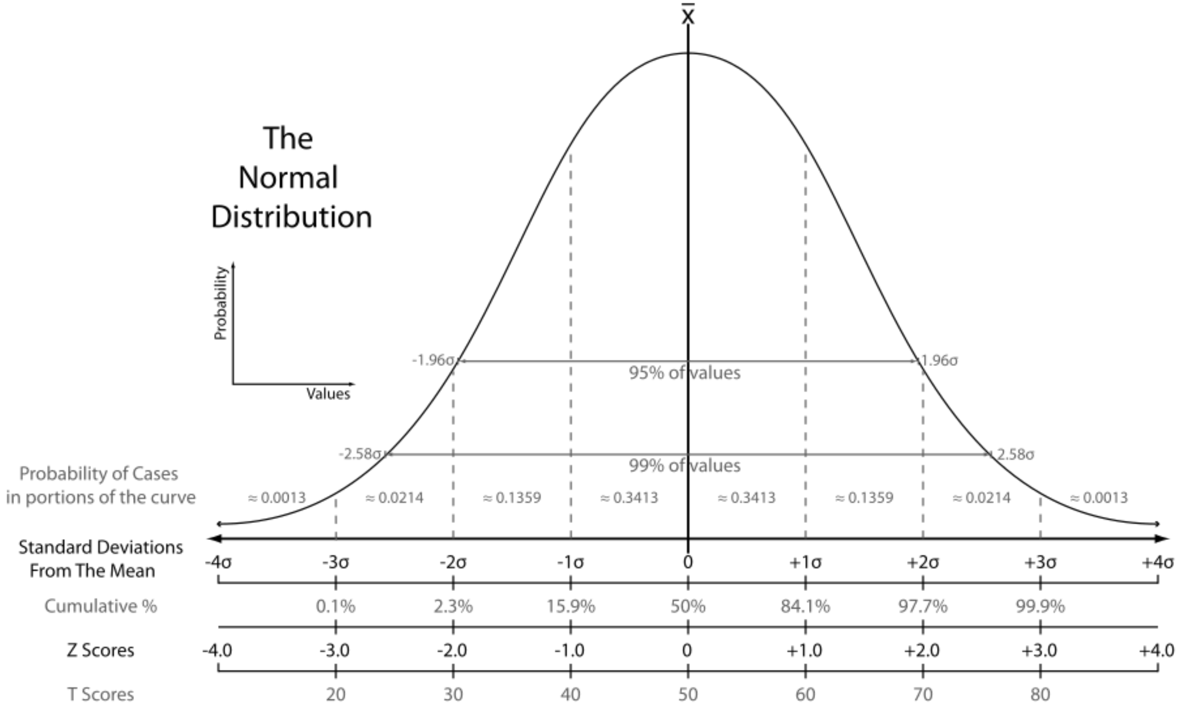
In [37]:
```python
sc_stat_c = test_stat_c(secret_coin_results)
sc_stat_c
```

Out[37]:
```
0
```

In [38]:
```python
plt.hist(test_stats_c, align='left')
plt.vlines(sc_stat_c, 0, 800, color='red')
plt.show()
```



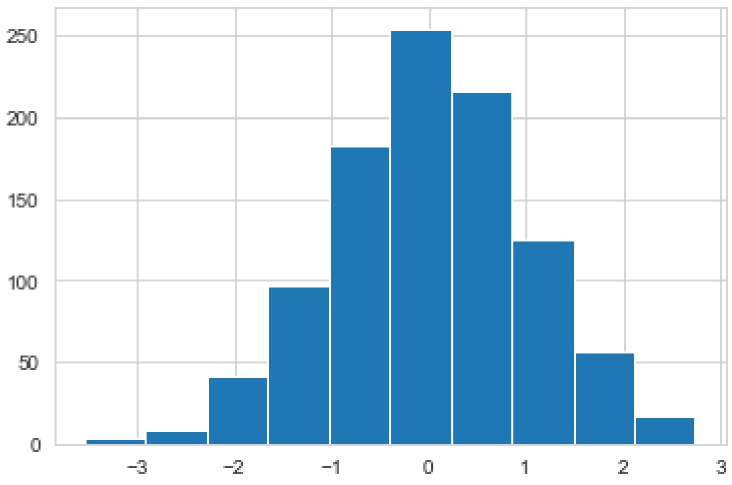
The Normal Distribution



Returning to our 100-flip test statistics, we can *see* "how normal" they are:

In [39]:
```
test_stat_zscores = (test_stats_500 - test_stats_500.mean()) / test_stats_500.std()
```

In [40]:
```
plt.hist(test_stat_zscores)
plt.show()
```



In [41]:

```
def print_std_dev_props(zscores):
    # Within 1 std dev
    for i in range(1,4):
        prop_within = len([ts for ts in zscores if (ts <= i) and (ts >= -i)]) / len(zsco
res)
        print(f"% within {i} standard deviations: {prop_within * 100}")
```

In [42]:

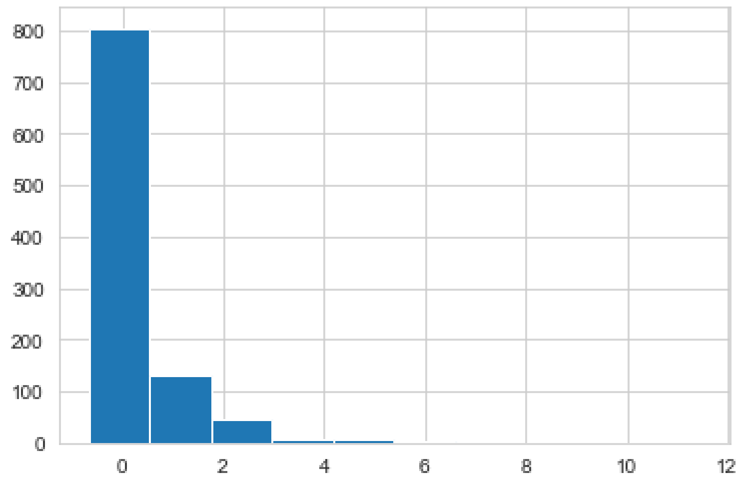
```
print_std_dev_props(test_stat_zscores)
```

```
% within 1 standard deviations: 69.69999999999999
% within 2 standard deviations: 95.1
% within 3 standard deviations: 99.7
```

Not always the case! Recall our alternative test stat (test stat B):

In [43]:
```
test_stat_zscores_b = (test_stats_b - test_stats_b.mean()) / test_stats_b.std()
```

In [44]:
```
plt.hist(test_stat_zscores_b)
plt.show()
```



In [45]:
```
print_std_dev_props(test_stat_zscores_b)
```

```
% within 1 standard deviations: 91.2
% within 2 standard deviations: 95.7
% within 3 standard deviations: 98.2
```

Correlation