deal.II Users and Developers Training

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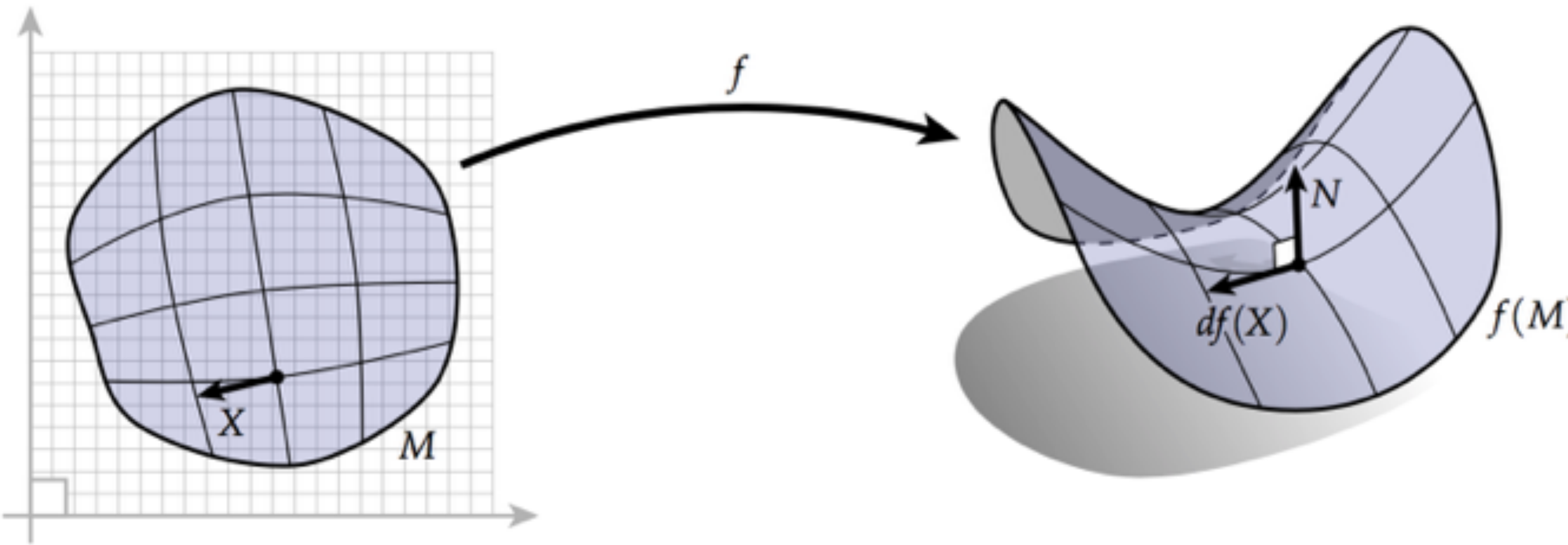
Making computations on Manifolds

- Exterior calculus of differential forms:
  - modern language of differential geometry
  - modern language of mathematical physics

- Key Aspect: Geometry of Surfaces
  - usually requires **local charts**
  - **fundamental** to understand the deal.II concept of **Manifolds** and of **Mappings**
  - necessary to formulate problems on surfaces

- A (small) portion of a surface in three dimensions can be represented (using charts) as the image of a map:

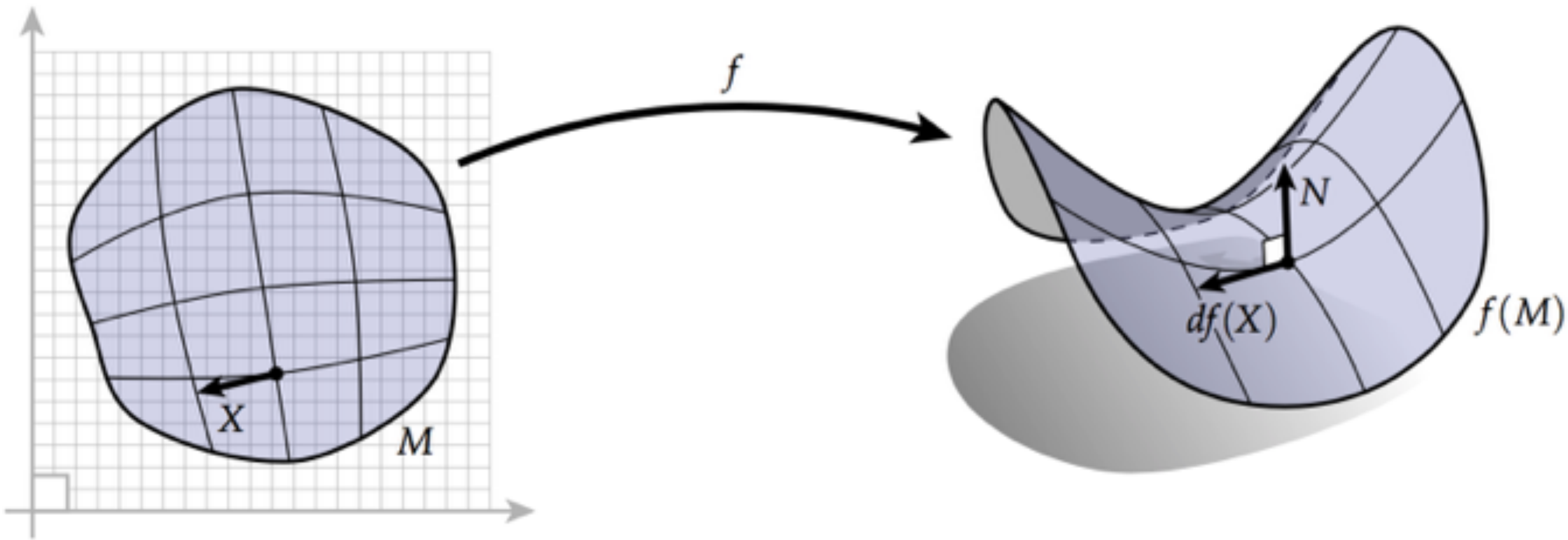
$$f : M \to \mathbf{R}^3$$



Crash course on Geometry

- The **differential (df)** of such map, tells how a **vector X** on M transforms to a vector df(X) tangent to the manifold:

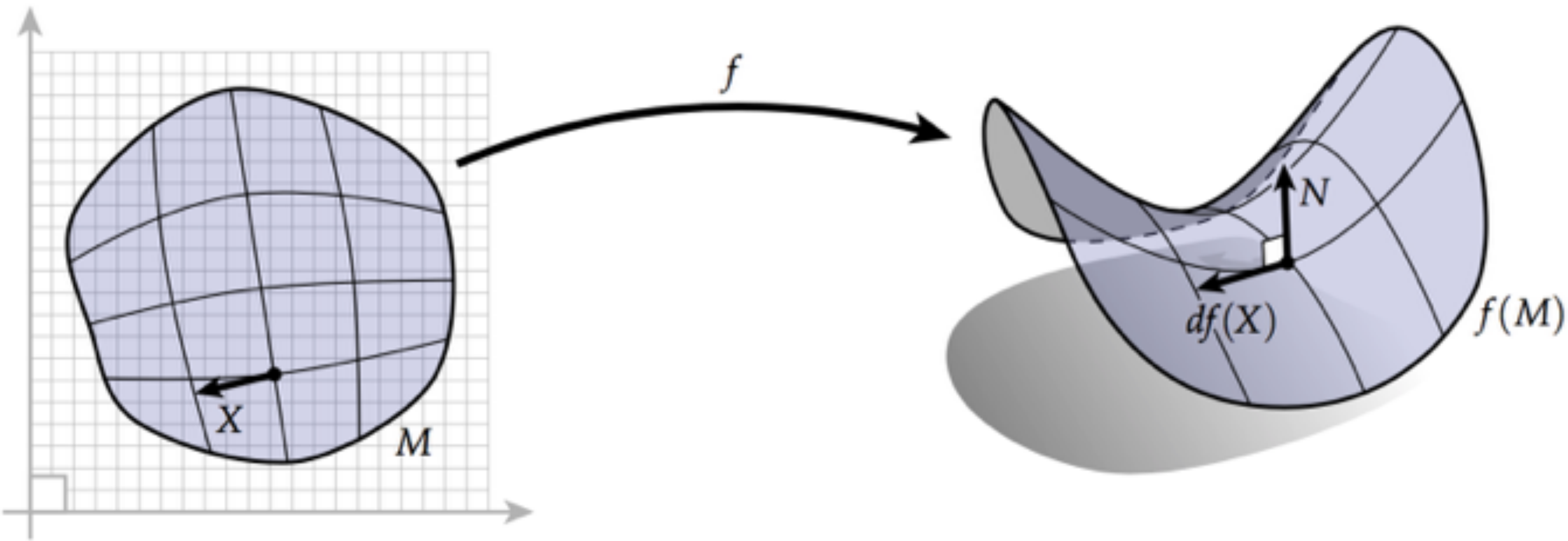
$$df : TM \to Tf(M)$$



Crash course on Geometry

❖ The **normal vector N** at a point p, is the **unit** vector **N** such that:

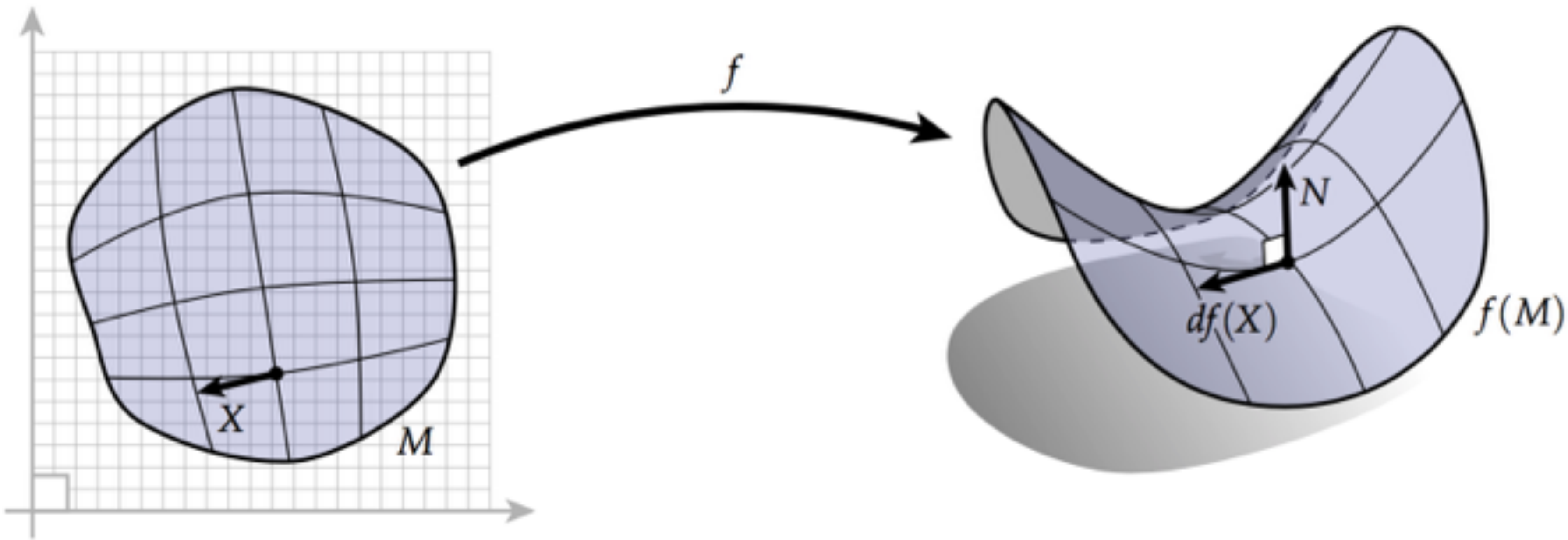
$$\mathbf{N} \cdot df(\mathbf{X}) = 0 \qquad \forall \mathbf{X} \in TM$$



Crash course on Geometry

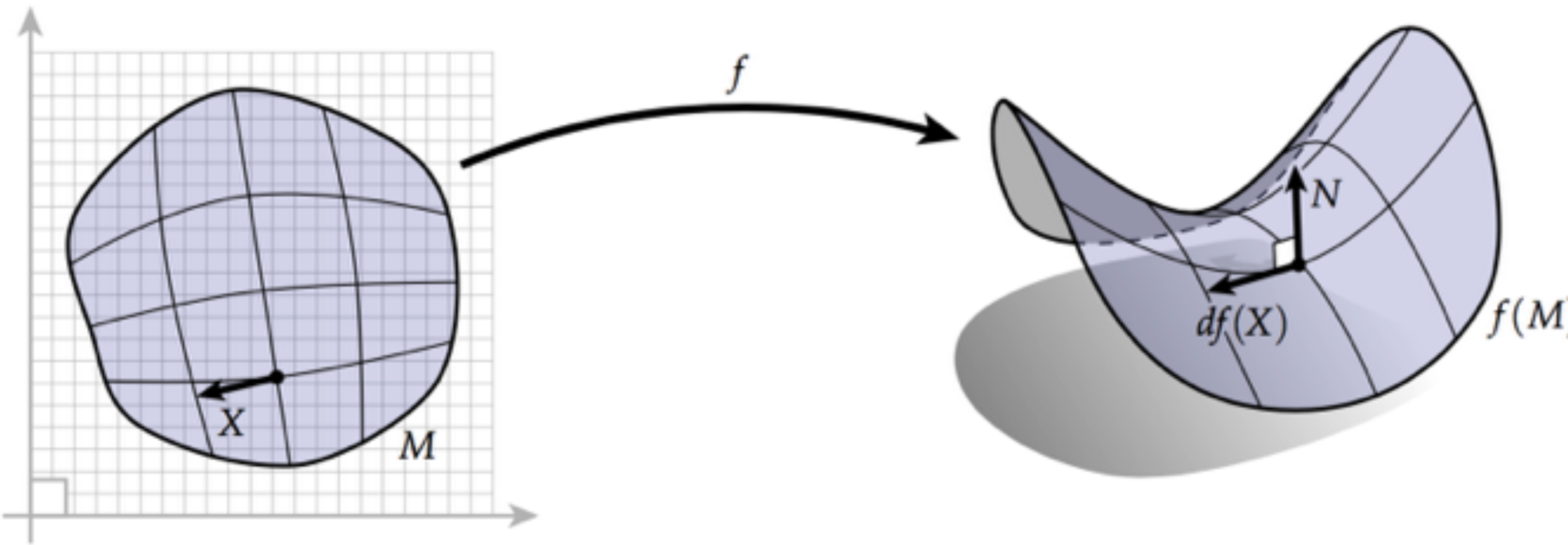
- The differential tells **how to stretch out,** or **push forward** vectors as you go from M to f(M)

$$\sqrt{\mathbf{X} \cdot \mathbf{X}} \quad \to \quad \sqrt{df(\mathbf{X}) \cdot df(\mathbf{X})}$$



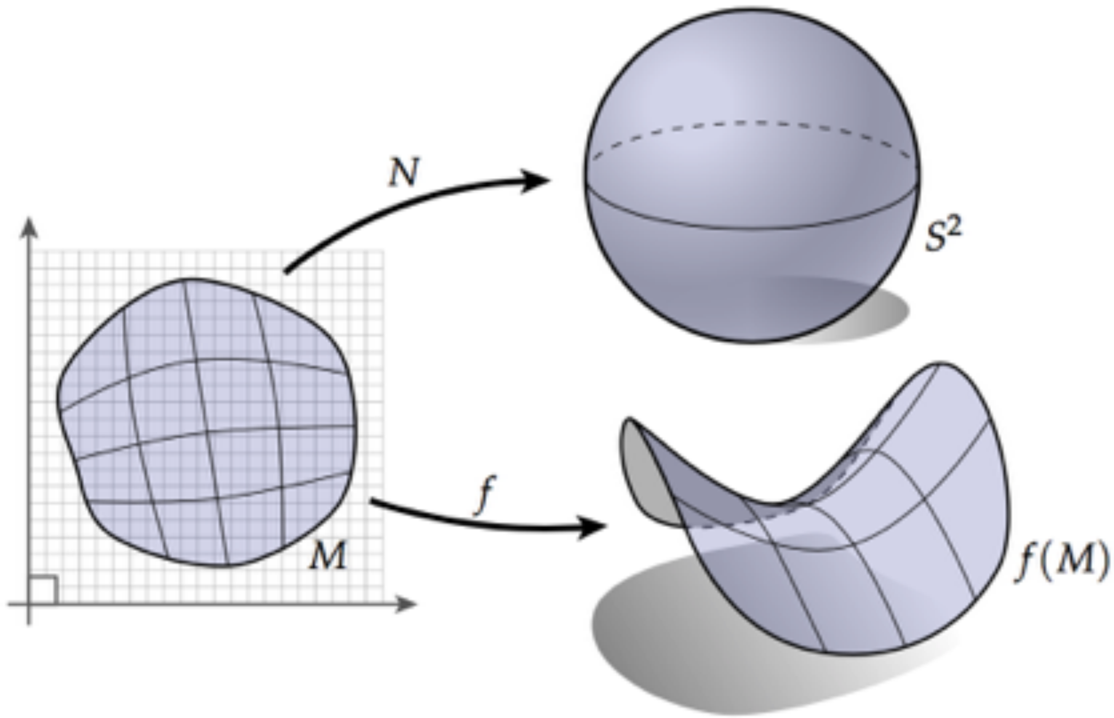
❖ If we generalise this to two vectors, we get the **metric induced by f (or First Fundamental Form):**

$$g(\mathbf{X}, \mathbf{Y}) := df(\mathbf{X}) \cdot df(\mathbf{Y})$$



- For **orientable** surfaces, we have a continuous map from **M** to **S2**, and **dN** is called the **Weingarten map** (it tells how the normal direction changes on **M**)

$$\mathbf{N} : M \rightarrow S^2$$



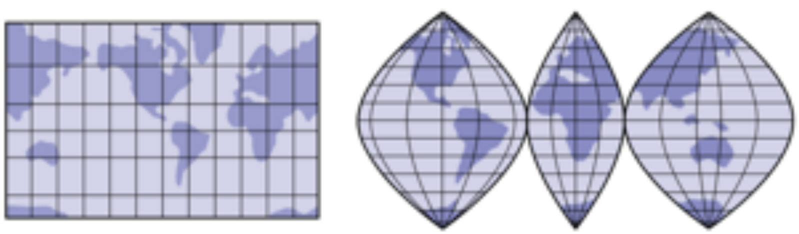
- Important facts:
  - Isometries rarely exist (we almost always stretch vectors for non trivial maps)

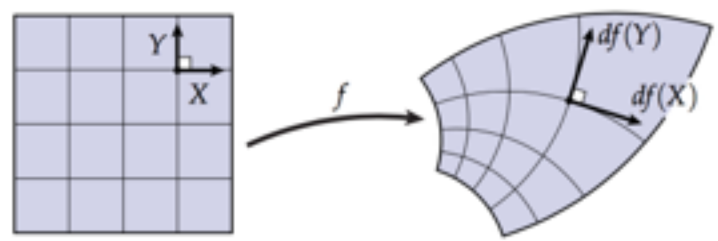
$$|\mathbf{X}| = |df(\mathbf{X})|$$

  - A conformal map always exists (it preserves the angles)

$$\exists u : M \to \mathbf{R} \text{ such that}$$

$$df(\mathbf{X}) \cdot df(\mathbf{X}) = e^{u}\mathbf{X} \cdot \mathbf{X} \qquad \forall \mathbf{X}, \mathbf{Y} \in TM$$

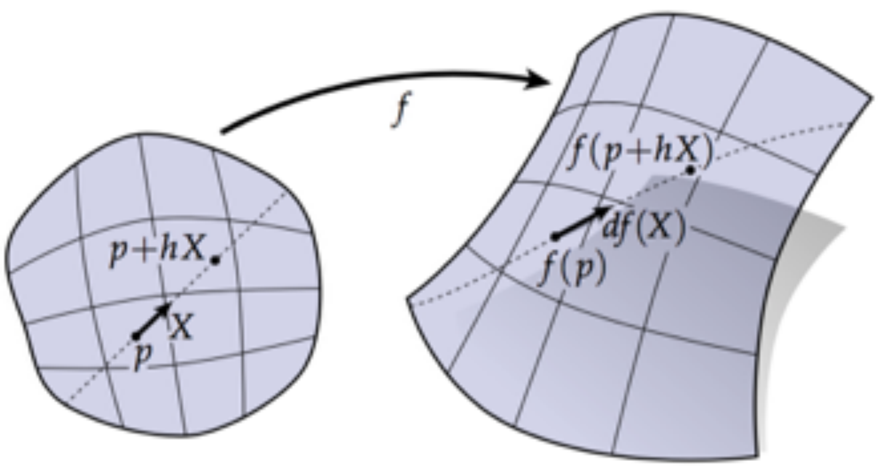




Geometry in Coordinates

❖ Defining df in terms of limits:

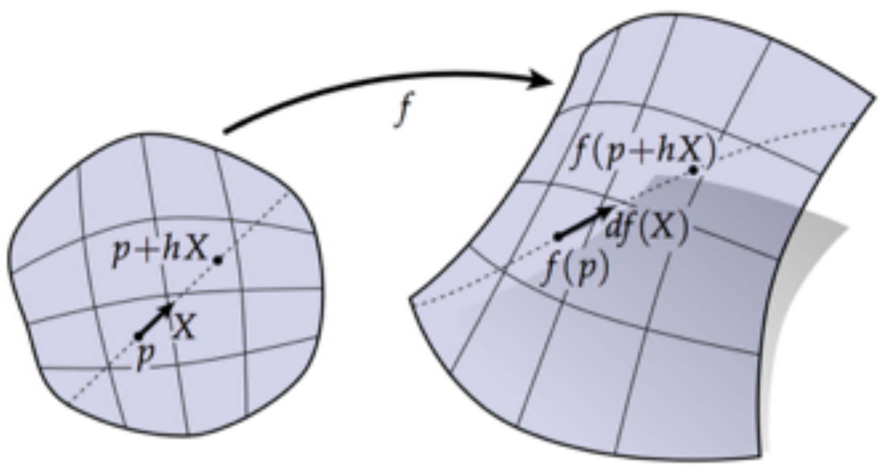
$$df_p(\mathbf{X}) = \lim_{h \to 0} \frac{f(p + h\mathbf{X}) - f(p)}{h}$$



Geometry in Coordinates

- In general, if we fix a **coordinate system** on M, we get that df is the Jacobian matrix:

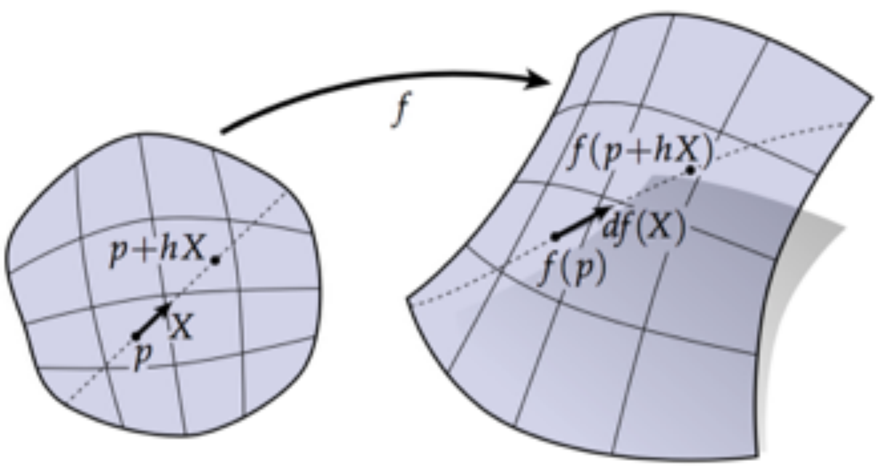
$$\mathsf{J} = \begin{bmatrix} \partial f^1/\partial x^1 & \partial f^1/\partial x^2 \\ \partial f^2/\partial x^1 & \partial f^2/\partial x^2 \\ \partial f^3/\partial x^1 & \partial f^3/\partial x^2 \end{bmatrix}$$



Geometry in Coordinates

- And the **metric (or First Fundamental Form)** is the matrix:

$$g_{ij} := g(\mathbf{e}_i, \mathbf{e}_j) = df(\mathbf{e}_i) \cdot df(\mathbf{e}_j) = (J^T J)_{ij}$$



Derivatives on Manifolds

Why is this relevant for Finite Elements?

- If a function is defined only on a surface (for example, the temperature on the surface of a soap bubble), what does it mean to take a "**gradient on the surface**" or "**surface gradient**" of this function?
- Start from the other way around!
- Assume we know a (scalar) function in a box, which includes f(M), take the gradient of the function, and "**restrict it in an appropriate way**" to f(M)
- How do we **restrict?**

Derivatives on Manifolds

What should we ask for? Our goals:

- A "surface gradient", should be the same for any two **different** functions of R^3, which are the same when restricted to f(M)
- If I use two **different** parametrisations of M (say f1(M) and f2(M)), the gradient of the function should not change (it is defined in R^3, **not** depending on f(M))
- Only possibility: **remove normal component**

$$u : \mathbf{R}^3 \to \mathbf{R}$$

$$\nabla_S u := \nabla u - \mathbf{n}(\mathbf{n} \cdot \nabla u) \qquad \text{on } S = f(M)$$

Derivatives on Manifolds

What if we only have the values of u **on the surface**?

- Use the previous definition, by taking an arbitrary extension of u in a tubular section of **S=f(M)**
- Theorem: the **surface gradient** defined in this way is independent on the parametrisation f(M), and on the extension of u

Derivatives on Manifolds

Is there a more explicit definition, in terms of f(M) and u?

$$u : \mathbf{R}^3 \to \mathbf{R}$$

$$\tilde{u} : M \to \mathbf{R}$$

$$\tilde{u}(p) := u(f(p)) \qquad \forall p \in M$$

$$\frac{\partial \tilde{u}}{\partial p_\alpha} = \frac{\partial u}{\partial x_j} \frac{\partial f_j}{\partial p_\alpha}$$

$$\nabla_p \tilde{u} = J^T \nabla_x u$$

Derivatives on Manifolds

In general, the matrix J^T is **not** invertible. This is to be expected, since I should **not** be able to recover the full gradient of u, starting from one (possible) restriction on a surface gamma, **unless** the **dimensionality** of M is the **same** of the embedding space

$$df(\mathbf{X})\cdot\mathbf{n} = (J\mathbf{X})\cdot\mathbf{n} = 0 \qquad \forall\mathbf{X}\in TM$$

$$\Longrightarrow J_{ij}\mathbf{e}_j n_i = 0 \qquad \forall\mathbf{X}\in TM$$

$$\Longrightarrow J^T\mathbf{n} = 0$$

$$\Longrightarrow J^T\nabla u = J^T(\nabla u - \mathbf{n}(\mathbf{n}\cdot\nabla u)) := J^T\nabla_S u$$

Derivatives on Manifolds

In the previous setting, J^T **becomes** invertible on the tangent space, and we get

$$\nabla_S u|_{x=f(p)} := J(J^T J)^{-1} \nabla_p u|_p \qquad \forall p \in M$$

- If we have **local basis functions** (u), and a **mapping** (f), this applies directly to finite elements!
- This definition of (surface) gradient is also valid for standard gradients if the dimension of M is the same as the embedding space!
- We can generalise this to a **surface divergence**, and to a **surface Laplacian**

Derivatives on Manifolds

- For a function in R^3, the surface Laplacian is given by

$$
\begin{aligned}
\Delta_S u &:= \nabla_S \cdot \nabla_S u \\
&= \Delta u - \mathbf{n}^T(\nabla(\nabla u))\mathbf{n} - (\mathbf{n}\cdot\nabla u)(\nabla\cdot\mathbf{n} - \mathbf{n}^T\nabla\mathbf{n}\mathbf{n})
\end{aligned}
$$

- This is necessary to compute **manufactured solutions**

Now … Exercise Time (step-38)!