Barra Risk Model: Formulas and Explanations

1. Daily Total Covariance Matrix

The daily total covariance matrix combines systematic (factor) risk and idiosyncratic risk into one estimate of cross-stock risk for day t.

$$\Sigma_t = B_t \, F_t \, B_t^\top + D_t \tag{1}$$

Variables:

- Σ_t: *Total* covariance matrix on day t, size $N \times N$.
- B_t: Factor loading matrix, size $N \times K$. $\beta_{t,ij}$ in B_t is stock i's exposure to factor j on day t.
- F_t: Factor covariance matrix, size $K \times K$. $(F_t)_{j\ell} = \mathrm{Cov}(f_{u,j}, f_{u,\ell})$ over a historical window.
- D_t: Idiosyncratic variance matrix, diagonal $N \times N$. Diagonal entry $D_{t,ii} = \sigma^2_{\epsilon_i}$ is the variance of residuals for stock i.

2. Portfolio Risk Calculation

Given a portfolio weight vector w_t on day t, this model computes the portfolio's variance and volatility.

$$\mathrm{Var}_t(R_{\mathrm{port}}) = w_t^\top \, \Sigma_t \, w_t, \tag{2}$$

$$\sigma_t = \sqrt{w_t^\top \, \Sigma_t \, w_t}, \tag{3}$$

$$\sigma_t^{\mathrm{ann}} = \sigma_t \, \sqrt{D_{\mathrm{days}}}. \tag{4}$$

Variables:

- w_t: Weight vector (N-dimensional), $\sum_i w_{t,i} = 1$.
- R_{port}: Portfolio return on day t.
- $\mathrm{Var}_t(\cdot)$: Conditional variance given Σ_t.
- σ_t: Daily volatility (standard deviation).
- D_{days}: Number of trading days per year (e.g. 244).
- σ_t^{ann}: Annualized volatility.

3. Single-Stock Risk Attribution

This section splits total portfolio risk into contributions from each stock.

3.1 Marginal Risk Contribution (MRC)

$$\mathrm{MRC}_{t,i} = \frac{\partial \sigma_t}{\partial w_{t,i}} = \frac{\big[\Sigma_t w_t\big]_i}{\sigma_t}, \tag{5}$$

$$\big[\Sigma_t w_t\big]_i = \sum_{k=1}^{N} \Sigma_{t,ik} w_{t,k}. \tag{6}$$

Purpose: How much σ_t changes if you infinitesimally increase $w_{t,i}$.
Variables:

- $\mathrm{MRC}_{t,i}$: Marginal contribution to portfolio volatility from stock i.
- $\big[\Sigma_t w_t\big]_i$: Covariance-weighted sum for stock i.

3.2 Risk Contribution (RC)

$$\mathrm{RC}_{t,i} = w_{t,i} \mathrm{MRC}_{t,i}, \qquad \sum_{i=1}^{N} \mathrm{RC}_{t,i} = \sigma_t. \tag{7}$$

Purpose: Absolute risk each stock brings to the portfolio.

4. Factor Risk Attribution

This section attributes portfolio risk to underlying factors.

4.1 Portfolio Factor Exposure

$$\beta_{\mathrm{port},t} = B_t^\top w_t \in \mathbb{R}^K, \tag{8}$$

$$\beta_{\mathrm{port},t,j} = \sum_{i=1}^{N} \beta_{t,ij} w_{t,i}. \tag{9}$$

Purpose: Compute portfolio's net exposure to each factor.

4.2 Factor Risk Multiplier

$$\big[F_t \beta_{\mathrm{port},t}\big]_j = \sum_{\ell=1}^{K} F_{t,j\ell} \beta_{\mathrm{port},t,\ell}. \tag{10}$$

Purpose: Measures how factor j's risk interacts with portfolio's factor exposures.

4.3 Factor Risk Contribution

$$\mathrm{RC}_{t,j}^{\mathrm{fac}} = \beta_{\mathrm{port},t,j} \;\times\; \big[F_t\, \beta_{\mathrm{port},t}\big]_j. \tag{11}$$

Purpose: Absolute contribution of factor j to portfolio volatility.

4.4 Total Idiosyncratic Risk

$$\sum_{i=1}^{N} w_{t,i}^2\, D_{t,ii}. \tag{12}$$

Purpose: Sum of stock-specific (residual) risk contributions.

5. Risk Constraints and Optimization

5.1 Minimum Variance Portfolio

$$\min_{w_t} \quad w_t^\top \Sigma_t\, w_t, \tag{13}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} w_{t,i} = 1, \quad w_{t,i} \geq 0. \tag{14}$$

Purpose: Find weights that minimize portfolio variance subject to budget and long-only constraints.

5.2 Risk Parity

$$w_{t,i}\, \big[\Sigma_t\, w_t\big]_i = w_{t,j}\, \big[\Sigma_t\, w_t\big]_j, \qquad \forall\, i, j. \tag{15}$$

Purpose: Allocate risk equally across all assets.

5.3 Factor Exposure Limits

$$-c_j \leq \beta_{\mathrm{port},t,j} \leq c_j, \quad j = 1, \ldots, K. \tag{16}$$

Purpose: Constrain portfolio's exposure to each factor within $\pm c_j$.

6. Performance Attribution

Decomposes realized portfolio return into factor-driven and idiosyncratic components.

$$R_{t,\mathrm{port}} = \underbrace{\beta_{\mathrm{port},t}^\top f_t}_{\text{Factor return}} + \underbrace{\epsilon_{t,\mathrm{port}}}_{\text{Idiosyncratic P\&L}}. \tag{17}$$

Variables:

- $f_t \in \mathbb{R}^K$: Vector of factor returns on day t.
- $\epsilon_{t,\mathrm{port}}$: Portfolio residual return (alpha or noise).

7. Alpha Signal Generation

Builds predictive signals from factor forecasts.

7.1 Cross-Sectional Regression

$$r_{t,i} = \sum_{j=1}^{K} \beta_{t,ij}\, f_{t,j} + \epsilon_{t,i}. \tag{18}$$

Purpose: Estimate daily factor returns $f_{t,j}$ via OLS.

7.2 Forecasting and Signals

$$\hat{f}_{t+1,j} = \mathrm{TimeSeriesModel}(\{f_{u,j}\}_{u \le t}), \tag{19}$$

$$\hat{R}_{t+1,i} = \sum_{j=1}^{K} \beta_{t,ij}\, \hat{f}_{t+1,j}. \tag{20}$$

Purpose: Generate stock return predictions (alpha signals) from factor forecasts.

1. 因子协方差矩阵 F.

$(f_{t,j})$.

$$\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_K \end{array}\begin{bmatrix} s_1 & s_2 & \cdots & s_N \\ \beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\ - & - & - & - \\ \vdots & & & \\ - & - & - & - \end{bmatrix}\begin{array}{c} \\ f_1 \\ f_2 \\ \\ f_K \end{array}$$

$\Downarrow$

$$\begin{bmatrix} f_{t1} \\ f_{t2} \\ f_{t3} \\ \vdots \\ f_{tK} \end{bmatrix}$$ at t.

expand $\Rightarrow$

$$\boxed{r_{t,i} = \beta_i f_t + \varepsilon_{t,i}} \qquad \boxed{\beta_i : \frac{x_i - \mu}{\sigma}}$$

Factor Exposure.

$$T\begin{bmatrix} & a_1 & a_2 & \cdots & a_K \\ t_1 & f_{t_1 a_1} & f_{t_1 a_2} & \cdots & f_{t_1 a_K} \\ t_2 & f_{t_2 a_1} & \cdots & & \\ \vdots & & & & \\ t_T & f_{t_T a_T} & \cdots & & f_{t_T a_K} \end{bmatrix} = \textcircled{F}.$$
K

multiple period.

$= \{f_{t,j}\}$.

covariance: $F_{jl} = \frac{1}{T-1}\sum_{t=1}^{T}(f_{t,j} - \bar{f_j})(f_{t,l} - \bar{f_l})$.

$$K\begin{bmatrix} & a_1 & a_2 & \cdots & a_K \\ a_1 & F_{11} & F_{12} & \cdots & F_{1K} \\ a_2 & F_{21} & F_{22} & \cdots & F_{2K} \\ \vdots & & & & \vdots \\ a_K & F_{K1} & F_{K2} & \cdots & F_{KK} \end{bmatrix} \textcircled{?} = F_{(K\times K)}$$
K

可选滚动动态 F_t.

3. 因子暴露矩阵. at t.

$$\underset{(N\times K)}{B_t} = N\begin{bmatrix} & a_1 & a_2 & \cdots & a_K \\ s_1 & \beta_{11} & \beta_{12} & \cdots & \beta_{1K} \\ s_2 & \beta_{21} & \beta_{22} & \cdots & \beta_{2K} \\ \vdots & & & & \\ s_N & \beta_{N1} & \beta_{N2} & \cdots & \beta_{NK} \end{bmatrix}$$
K

$$\boxed{\Sigma_t = B_t F B_t^T + D_t}$$

2. 特异性方差矩阵 D ; 在 t 这天.

$r_{t,i} = \sum_{j=1}^{K} \beta_{ij} f_{t,j} + \varepsilon_{t,i}$

i: stock
j: factor.

$\Rightarrow [\varepsilon_{t,1}, \varepsilon_{t,2}, \cdots \varepsilon_{t,N}]$. at t

$$T\begin{bmatrix} & s_1 & s_2 & \cdots & s_N \\ t_1 & \varepsilon_{t_1,1} & \varepsilon_{t_1,2} & \cdots & \varepsilon_{t_1,N} \\ t_2 & \varepsilon_{t_2,1} & \varepsilon_{t_2,2} & & \\ \vdots & \vdots & & & \\ t_T & \varepsilon_{t_T,1} & \cdots & & \varepsilon_{t_T,N} \end{bmatrix}$$
N

$$\boxed{\sigma^2_{\varepsilon_i} = \frac{1}{\underbrace{T-K-1}_{T.df}}\sum_{t=1}^{T}(\varepsilon_{t,i} - \bar{\varepsilon_i})^2}$$

$$\Rightarrow N\begin{bmatrix} & s_1 & s_2 & \cdots & s_N \\ s_1 & \sigma^2_{\varepsilon_1} & 0 & & \\ s_2 & 0 & \sigma^2_{\varepsilon_2} & & \\ \vdots & & & \ddots & \\ s_N & 0 & & & \sigma^2_{\varepsilon_N} \end{bmatrix}$$
N

$= D_t \ (N\times N)$

Barra 模型常见用法：

1. 组合风险：

$w = (w_1, w_2, \ldots, w_N)^T$ $1\times N$

全市场，无权重则为0，归一化

$$Var_t = w_t^T \Sigma_t w_t$$

$$\sigma_t = \sqrt{w_t^T \Sigma_t w_t}$$

$$\sigma_t^{ann} = \sigma_t \times \sqrt{\text{total trading days}}$$

年化化组合波动，

2. 单股票风险归因

$$\Sigma_t = \begin{array}{c|c} & \begin{array}{cccc} S_1 & S_2 & \cdots & S_N \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ \vdots \\ S_N \end{array} & \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \\ \Sigma_{21} & \Sigma_{22} & \cdots & \\ \vdots & & \ddots & \\ & & & \Sigma_{NN} \end{bmatrix} \end{array} \quad N \times N$$

* 边际风险贡献

$$MRC_{t,i} = \boxed{\frac{\sum_{j=1}^{N} \Sigma_{t,ij} w_{t,j}}{\sqrt{w_t^T \Sigma_t w_t}}}$$

$$= \frac{[\Sigma_t w_t]_i}{\sigma_t} = \boxed{\frac{\partial \sigma_t}{\partial w_{t,i}}}$$ 波动随权重（个股）的变化率..

* $RC_{t,i} = \boxed{w_{t,i} MRC} = \boxed{w_{t,i} \frac{\partial \sigma_t}{\partial w_{t,i}}}$, $\boxed{\sum_{i=1}^{N} RC_{t,i} = \sigma_t}$

绝对风险贡献：股票 i 对组合波动的绝对贡献

3. 因子风险归因.

组合因子暴露：$\beta_{port\,t,j} = \boxed{\sum_{i=1}^{N} \beta_{t,ij} w_{t,i}}$ $\boxed{\beta_{port} = B_t^T w_t}$
（因子 j）

* 因子风险贡献：

$$RC_{t,j}^{fac} = \boxed{\beta_{port\,t,j} [F_t \beta_{port}]_j} = \cancel{\beta_{port\,t,j} \sum_{l=1}^{K} F_{t,jl} \beta_{port,t,l}}$$

$$\beta_{port\,t,j} \sum_{l=1}^{k} F_{t,jl} \beta_{port,t,l}$$

$$\Rightarrow RC_{t,j}^{fac} = \boxed{\sum_{i=1}^{N} B_{t,ij} w_{t,i} \times \left(\sum_{l=1}^{K} F_{t,jl} \sum_{k=1}^{N} B_{t,kl} w_{t,k}\right)}$$

且 $\sum_{j=1}^{K} RC_{t,j}^{fac} + \sum_{i=1}^{N} w_{t,i}^2 D_{t,ii} = \sigma_t^2$

4. 风险限额与优化.

* 最小方差组合：$\boxed{\min_{w_t} w_t^T \Sigma_t w_t}$ s.t. $\sum_{i=1}^{N} w_{t,i} = 1$, $w_{t,i} \geq 0$.

* 风险平价：目标：$\boxed{w_{t,i} [\Sigma_t w_t]_i = w_{t,j} [\Sigma_t w_t]_j}$ $\forall i,j$.

$$w_{t,i} \sum_{k=1}^{N} \Sigma_{ik} w_{tk} = w_{t,j} \sum_{k=1}^{N} \Sigma_{t,jk} w_{t,k}.$$

s.t. $\sum_{i=1}^{N} w_{t,i} = 1$. $w_{t,i} \geq 0$.

* 因子敞口约束：$\boxed{-c_j \leq \beta_{port,t,j} \leq c_j}$, for $j = 1, \ldots, K$.

5. 业绩归因:

note $\beta_{port} = B_t^T w_t$

$$\boxed{R_{t,port} = \beta_{port}^T f_t + \varepsilon_{t,port}}$$

组合当日实际收益. 因子贡献 特异性收益.

注意, $R_{t,port} = \sum_{i=1}^{N} w_{t,i} r_{t,i}$, $\beta_{port}^T f_t = \sum_{j=1}^{K} \beta_{port,t,j} f_{t,j}$

组合收益 (加权) 因子暴露 × 贡献

$\varepsilon_{t,port}$ 为剩下的残差收益.

$\varepsilon_{t,port}$ 为正, => 有特异性收益.

为负, => alpha 失效.