One-homogeneous Minplus Dynamic Systems, Negatively Weighted Petri Nets and Traffic Applications

N. Farhi, M. Goursat & J.-P. Quadrat

Abstract

We show that car traffic in a town can be modeled using a deterministic Petri net extension where edges may have negative weights (the firing of a transition may consume tokens in downstream places). The dynamics of deterministic temporized Petri nets with weights can be described by composing a standard matrix with a minplus matrix. The corresponding dynamics $x^{k+1} = f(x^k)$ is nonlinear in minplus or standard algebra (f is piecewise linear concave in standard algebra). Moreover, the microscopic traffic models used are homogeneous of degree one in minplus algebra $f(\lambda \otimes x) = \lambda \otimes f(x)$ where $\otimes$ denotes the standard addition. Possibly depending on the initial condition, the state trajectories of homogeneous systems may be periodic or have a chaotic behavior around a linear drift or may explode exponentially. The traffic model has a linear drift which has the interpretation of the average car flow. This average flow, as a function of the car density, is called the fundamental traffic diagram.

The example of two circular roads with a junction is studied in details. Thanks to the ergodic theorems we can derive an existence result of the average flow $\chi = \lim_k x^k/k$. It differs but is close to the minplus eigenvalue λ of the system ($\lambda \otimes x = f(x)$) which exists and can be computed explicitly. The fundamental diagram, thus obtained, presents four phases which have traffic interpretations. These four phases still exist in the traffic of general regular towns.

It is useful to compose input-output Petri nets with the standard operations: parallel ($\boxplus$), series ($\boxtimes$) and feedback ($\boxdot$) compositions. The compositions of homogeneous systems with these operations gives homogeneous systems but neither the associativity nor the distributivity hold true. Nevertheless, we give the operations, corresponding to the three compositions, on the matrix pairs defining a system. Using these compositions, the dynamics of a regular town can be built from elementary simple systems, and thus, the asymptotic vehicle distribution and the fundamental diagram can be computed.

I. INTRODUCTION

The traffic on a road has been studied from different points of view at macroscopic level. For example:

- The Lighthill-Whitham-Richards Model [26] is the most standard one. It expresses the mass conservation of cars seen as a fluid:
$$\begin{cases} \partial_t \rho + \partial_x \varphi = 0 \ , \\ \varphi = f(\rho), \end{cases}$$
where: $\varphi(x,t)$ denotes the flow at time t and position x on the road, $\rho(x,t)$ denotes the density, and f is a given function called the fundamental traffic diagram. For traffic, this diagram plays the role of the perfect gas law for the fluid dynamics. It has been estimated using experimental datas. Its behavior is quite different of standard gas at high density. We can see the "traffic on a circular road" subsection to get an intuition of its shape.
- The kinetic model (Prigogine-Herman [35]) gives the evolution of the density of particles $\rho(t,x,v)$ as a function of t, x and v, where v is the speed of particles. The model is given by:
$$\partial_t \rho + v \partial_x \rho = C(\rho, \rho) \ ,$$
where $C(\rho,\rho)$ is an interacting term in general quadratic in ρ which model in a simple way the driver behavior. From the distribution ρ we can derive all the useful quantities, for example the average speed $\bar{v}(t,x) = \int v\rho(t,x,v)dv$.

The integration of the second model is more time consuming and therefore is not used in practice. The first one supposes the knowledge of the function f. This function, called *fundamental traffic diagram*, has been studied experimentally or theoretically using simple microscopic models (exclusion processes, for example [14], [7], cellular automata [3], [9], or simulation of individual car dynamics). The main interest of the Prigogine-Herman equation is that we can derive macroscopic laws like the fundamental diagram from its resolution. Here, we will recall a way, to derive an approximation of this diagram. This derivation consists in computing the eigenvalue of a simple minplus linear system counting the number of cars entered in a road section.

The main purpose of this paper is to generalize this fundamental law to the 2D case where the roads cross each other. In statistical physics, a lot of numerical work has been done on idealized towns see for example [32], [12], [31], [5], [8] (as well as the good surveys [9], [19]). These works analyze numerical experimentations based on various stochastic models with or without turning possibilities and show mainly the existence of a density at which the system blocks suddenly. The particular case of one junction is studied in [20], [21] where precise results in the stochastic case without turning possibilities are given. Here we present a deterministic model, with turning possibilities, based on Petri nets and minplus algebra. The minplus linear model on a unique road can be described in term of event graph which is a subclass of Petri nets. The presence of junctions prevents the extension of this model to the 2D case. We propose a way to solve the difficulty by using Petri nets with negative weights. Thanks to such weights the firing of a transition can consume tokens in downstream places. The usage of these weights improve a lot the modeling power of Petri nets. This possibility is used to model the authorization to enter in the junctions. The dynamics of general Petri nets with negative weights can be written easily but is neither linear in minplus algebra nor monotone. Nevertheless, in the traffic applications given here, dynamics are always homogeneous of degree one ($x^{k+1} = f(x^k)$ with $f(\lambda \otimes x) = \lambda \otimes f(x)$, where $\otimes$ denotes the minplus multiplication that is the standard addition). For such systems the eigenvalue problem (computation of λ such that $\lambda \otimes x = f(x)$) can be reduced to a fixed point problem. The existence of the growth rate $\chi = \lim_k x^k/k$ comes back to the existence of a Birkhoff average. When f is also monotone, λ and χ coincides but this is not true in the general case. The monotone case has been studied carefully in [24], [30]. In traffic examples of roads with junctions the dynamics are not monotone. We show an example of a "1-homogeneous system" which may have a chaotic behavior. Moreover for this example the growth rate is not an eigenvalue if we do not start with an eigenvector.

We study always systems of roads on a torus without entries in such a way that the number of cars stay constant in the system. It represents an idealization of constant densities for more realistic system. To maintain constant a density in an open system a new vehicle has to enter each time another one leaves the system. This is mathematically equivalent to consider circular roads. If we want maintain constant a local density in a large system, it is the same problem: each time a car leaves the local subsystem studied another one has to enter.

The particular case of two circular roads with one junction is studied into details. For this system, a result on the existence of the growth rate is obtained by using the nondecreasing trajectory (starting from zero) property of the states from which is deduced the fact that distances between the states stay bounded. The eigenvalue problem can be solved explicitly. On simulation, we see that the eigenvalue and the growth rate do not coincide, but are very close on all the density domain and for all the relative size of the two roads. Therefore, the simple formulas obtained for the eigenvalue give a good approximation of the traffic fundamental diagram. This fundamental diagram presents four phases: – a *free phase* when the density is low where the vehicles do not interact, – a *saturation phase* where the junction is saturated but the places downstream the junction are free when a car wants to leave it, – a *recession phase* where the places downstream the junction are sometimes crowded when a car wants to leave it, – a *blocking phase* where the cars cannot move. Preliminary results have been presented in [16], [17]. In [18], written in french, we can find many developments, complementary results and other examples completely solved. The theorems given here on the eigenvalue problem and the growth rate complete some of the main results of this thesis by relaxing some hypotheses and clarifying the growth rate existence problem.

We also show that a system theory can be developed for the class of concave polyhedral 1-homogeneous systems. Indeed, they are closed by parallel, series and feedback compositions. Moreover, their dynamics are characterized by the composition of a standard matrix (having the sum of the entries in each line equal to one) with a minplus matrix. We can compute easily the transformation on these two matrices corresponding to the three composition operations. The interest of this "system theory" can be shown by building the traffic dynamics of a regular town starting from three elementary systems. We do not detail the construction, but we show a stationary car distribution obtained for a simple regular town with a dynamics built in that way. In this case, the fundamental diagrams still present the four phases observed in the simple case of two roads with one junction. All the numerical simulations have been done using the ScilabGtk software [37].

The paper is divided into three parts. In the first part we recall the basic results of minplus algebra. We present a system theory for polyhedral concave 1-homogeneous systems and discuss their growth rate and eigenvalue problems. In a second part we present Petri nets with negative weights. In a third part we give applications to the computation of the fundamental traffic diagram. In this part we consider, first, the case of one circular road which has a minplus linear dynamics, then we study into details the case of two circular roads with a junction, lastly we explain briefly the way to build the dynamics using the system theory described in the first part.

II. Minplus Algebra and extensions

A. Recalls on minplus algebra

In this section, to give a comprehensive presentation, we recall the main definitions and properties of the minplus algebra. An in-depth treatment of the subject is in [4].

The structure $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ is defined by the set $\mathbb{R} \cup \{+\infty\}$ endowed with the operations $\min$ (denoted by $\oplus$, called minplus *sum*) and $+$ (denoted by $\otimes$, called minplus *product*). The element $\varepsilon = +\infty$ is the *zero* element, it satisfies $\varepsilon \oplus x = x$ and is *absorbent* $\varepsilon \otimes x = \varepsilon$. The element $e = 0$ is the *unit* element, it satisfies $e \otimes x = x$. The main difference with respect to the conventional algebra is the idempotency of the addition $x \oplus x = x$ and the fact that the addition cannot be *simplified*, that is: $a \oplus b = c \oplus b \not\Rightarrow a = c$. It is called *minplus algebra*. We will call $\overline{\mathbb{R}}_{\min}$ the completion of $\mathbb{R}_{\min}$ by $-\infty$ with $-\infty \otimes \varepsilon = \varepsilon$.

This minplus structure on scalars induces an idempotent semiring structure on $m \times m$ square matrices with the element-wise minimum denoted by $\oplus$ and the matrix product defined by $(A \otimes B)_{ik} = \min_j (A_{ij} + B_{jk})$, where the zero and the unit matrices are still denoted by ε and e. We associate to a square matrix A a precedence graph $\mathcal{G}(A)$ where the nodes correspond to the columns (or the rows) of the matrix A and the edges to the non zero ($\neq \varepsilon$) entries (the *weight* of the edge going from i to j being the non zero entry A_{ji}). We define the weight of a path p in $\mathcal{G}(A)$, which we denote by $|p|_w$, as the minplus product of the weights of the edges composing the path (that is the standard sum of weights). The number of edges of a path p is denoted by $|p|_l$. We will use the three fundamental results (see [4]).

Theorem 1: Given A a $m \times m$ minplus matrix, if the weights of all the circuits of $\mathcal{G}(A)$ are positive, then the equation $X = A \otimes X \oplus B$ admits a unique solution: $X = A^* \otimes B$ where

$$A^* = \bigoplus_{n=0}^{\infty} A^n = \bigoplus_{n=0}^{m-1} A^n .$$

Theorem 2: If the graph $\mathcal{G}(A)$ associated with the $m \times m$ minplus matrix A is strongly connected, then the matrix A admits a unique *eigenvalue* $\lambda \in \mathbb{R}_{\min} \setminus \{\varepsilon\}$:

$$\exists\ X \in \mathbb{R}_{\min}^m, X \neq \varepsilon : A \otimes X = \lambda \otimes X \ \text{ with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l} , \tag{1}$$

where $\mathcal{C}$ is the set of circuits of $\mathcal{G}(A)$.

Theorem 3: The minplus linear dynamic system $X^{k+1} = A \otimes X^k$ associated with the minplus matrix A, with strongly connected graph $\mathcal{G}(A)$, is *asymptotically periodic*:

$$\exists T, K, \lambda : \forall k \geq K : A^{k+T} = \lambda^T \otimes A^k \ .$$

B. A generalized matrix calculus

In Petri net two kinds of operations appear, accumulation of resources in the places, synchronization in the transitions. The first one can be modeled by an addition, the second one by a min or max (a task can start at the maximum of the arrival instants of the resources needed by the task). Matrix notations can be generalized to such kinds of situations.

For that, we consider the set of $m \times m$ matrices where the rows and columns are partitioned in two sets the standard and the minplus (here the m' first columns [resp. rows] and the last m" columns, [resp. rows]) with entries in $\mathbb{R} \cup \{+\infty, -\infty\}$, equipped with the two operations $\boxplus$ and $\boxtimes$ defined by:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxplus \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A + A' & B + B' \\ C \oplus C' & D \oplus D' \end{bmatrix} \ ,$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxtimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} AA' + BC' & AB' + BD' \\ C \otimes A' \oplus D \otimes C' & C \otimes B' \oplus D \otimes D' \end{bmatrix} \ .$$

Since the entries are in an extension of $\mathbb{R}$, we have to precise the scalar addition and multiplication table:

$$0 \times \pm\infty = \pm\infty \times 0 = 0, \quad +\infty \otimes (-\infty) = +\infty - \infty = +\infty \ .$$

These choices have been done to preserve the absorption properties for the multiplication of the null elements of the standard algebra (0) and the minplus algebra ($\varepsilon = +\infty$). This absorption property is useful to model the arc absence in the precedence graph $\mathcal{G}(A)$ associated to a square matrix A (defined in the same way as in the pure minplus case).

The addition $\boxplus$ is associative, commutative and has the null element $\begin{bmatrix} 0 & 0 \\ \varepsilon & \varepsilon \end{bmatrix}$ still denoted ε.

The multiplication $\boxtimes$ has no neutral element. It is neither associative nor commutative nor distributive with respect to the addition.

Nevertheless, this multiplication represents in a simple way the transformation that we can represent by the graph $\mathcal{G}(A)$ where we have two kinds of nodes: the ones corresponding to the standard $+$ operation, the others corresponding to $\oplus$ operation and two kinds of edges: the edges which operate multiplicatively ($\times$) and the others which operate additively ($\otimes$).

Example 1: Let us consider the graph $\mathcal{G}(A)$ associated to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ given in Figure-1 with one node associated to the standard algebra and one node to the minplus algebra. Then $y = A \boxtimes x$, where y and x are two vectors with two entries, means:

$$y_1 = ax_1 + bx_2, \quad y_2 = \min(c + x_1, d + x_2) \ .$$

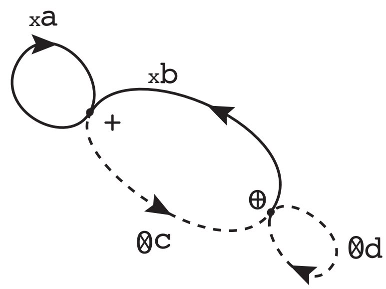

Fig. 1. The incidence graph associated to the matrix A with one $\oplus$node and one +node.

We will adopt the following conventions to solve some ambiguities in the formula notations:

– In a set of formulas as soon as appears a minplus symbol, all the operations must be understood in the minplus sense at the exception of the exponent that must be understood in the standard sense. For example $x^{a/b} \oplus 1$ or $\sqrt[b]{x^a} \oplus 1$ must be understood as $\min((a/b)x, 1)$ and not as $\min((a-b)x, 1)$ (in minplus sense $a/b = a - b$ since it is the solution[1] of $b \otimes x = a$ which means $b + x = a$).

– In a minplus formula, the rational numbers (written with figures) are denoted in the standard algebra : $\frac{1}{2}x \oplus 1$ (instead of $\sqrt{1}x \oplus 1$) means $\min(0.5 + x, 1)$ and not $\min(-1 + x, 1)$, but $(a/b)x \oplus 1$ means $\min(a - b + x, 1)$.

[1]For the reader familiar with the residuation, the minplus division used here means the standard minus operator with the convention previously given for infinite elements. This choice is incompatible with the residuation which chooses the smallest solution of $b \otimes x = a$.

– A non zero element in minplus sense means a finite element in the usual sense but positive element will be used always in the standard sense.

With these conventions the formulas stay compact and often clear.

C. A generalized system theory

We can develop a generalized system theory based on the two operations $\boxplus$ and $\boxtimes$. For that we partition the states [resp. inputs, outputs] in two classes the standard states [resp inputs, otputs] and the minplus states [resp. inputs, outputs]. Then the dynamics can be written:

$$\begin{bmatrix} X^{k+1} \\ Y^{k+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & \epsilon \end{bmatrix} \boxtimes \begin{bmatrix} X^k \\ U^k \end{bmatrix}$$

This dynamics denoted by S, defined by the matrices (A, B, C), associates to the input signals $(U^k)_{k\in\mathbb{N}}$ the output signals $(Y^k)_{k\in\mathbb{N}}$: $Y = S(U)$.

Let us remark that $\boxtimes$ used here corresponds to the definition given previously up to a permutation of rows and columns, since $[X^k\ U^k]'$ [resp. $[X^k\ Y^k]'$)] is not written in the canonical form since all the standard entries are not followed by all the minplus entries, but by the standard states, the minplus states, the standard inputs [resp. outputs], the minplus inputs [resp. outputs]. See the Petri net application 6 for developed formulas.

On these systems we define the following operations:

- **Parallel Composition.** Given two systems S_1 and S_2 (with the same numbers of inputs and outputs), we define the system $S = S_1 \boxplus S_2$ obtained by using the same entries and by adding[2] the outputs. The dynamics of S is defined by:

$$A = \begin{bmatrix} A_1 & \epsilon \\ \epsilon & A_2 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} .$$

- **Series Composition.** Given two systems S_1 and S_2 (where the output numbers of S_2 equal the input numbers of S_1), we define the system $S = S_1 \boxtimes S_2$ obtained by composition of the two systems: $S(U) = S_1(S_2(U))$. Using the new state $[X_1\ X_2\ Y_2]'$, the dynamics of S

[2]The standard output nodes are added in the standard algebra and the minplus output nodes in the minplus algebra.

is defined by:

$$A = \begin{bmatrix} A_1 & \epsilon & B_1 \\ \epsilon & A_2 & \epsilon \\ \epsilon & C_2 & \epsilon \end{bmatrix},\ B = \begin{bmatrix} \epsilon \\ B_2 \\ \epsilon \end{bmatrix},\ C = \begin{bmatrix} C_1 & \epsilon & \epsilon \end{bmatrix}.$$

- **Feedback.** Given a system S, we define the feedback system $S^{\boxdot}$ as the solution in Y of $Y = S(U \boxplus Y)$. Using the new state $[X\ Y]'$, its dynamics is defined by:

$$A^{\boxdot} = \begin{bmatrix} A & B \\ C & \epsilon \end{bmatrix},\ B^{\boxdot} = \begin{bmatrix} B \\ \epsilon \end{bmatrix},\ C^{\boxdot} = \begin{bmatrix} C & \epsilon \end{bmatrix}.$$

D. Additively homogeneous autonomous dynamic systems

Let us discuss now one-homogeneous minplus dynamical systems which is a large class appearing very often in applications when there is a conservation for example probability mass or tokens in Petri nets. But since we accept negative entries it is a generalization to the case of measure not necessarily positive with a total mass equals to one or conservation of tokens in situation where negative ones may appear. Let us start with an academic example. The traffic modeling will give more concrete examples in the following sections.

Example 2: Let us go back to Example1. Assuming that $a+b=1$, adding λ to each component of x (that we can write $\lambda \otimes x$) implies that the two components of y are augmented by λ. We have

$$A \boxtimes (\lambda \otimes x) = \lambda \otimes (A \boxtimes x).$$

We say that the system is *additively homogeneous of degree 1* or more simply homogeneous. Indeed, using the minplus notation $y = A \boxtimes x$ can be written:

$$y_1 = (x_1)^{\otimes a} \otimes (x_2)^{\otimes b}, \quad y_2 = c \otimes x_1 \oplus d \otimes x_2 ,$$

which is clearly homogeneous of degree 1 in the minplus algebra as soon as $a+b=1$. Moreover, if a and b are nonnegative then the transformation is *nondecreasing*.

To simplify the notations we will write the transformation in the following way:

$$y_1 = (x_1)^a (x_2)^b, \quad y_2 = cx_1 \oplus dx_2 .$$

More generally we say that the function $f : \overline{\mathbb{R}}^n_{\min} \mapsto \overline{\mathbb{R}}^n_{\min}$ is *homogeneous* if

$$f(\lambda \otimes x) = \lambda \otimes f(x).$$

E. Eigenvalues of homogeneous systems

The *eigenvalue problem* of such a function can be formulated as finding $x \in \mathbb{R}^n_{\min}$ non zero, and $\lambda \in \overline{\mathbb{R}}_{\min}$ such that:

$$\lambda \otimes x = f(x) \; .$$

Since f is homogeneous, we can suppose without loss of generality that if x exists then $x_1 \neq \varepsilon$. The eigenvalue problem becomes:

$$\begin{cases} \lambda & = f_1(x/x_1) \; , \\ x_2/x_1 & = (f_2/f_1)(x/x_1) \; , \\ \cdots & = \cdots \\ x_n/x_1 & = (f_n/f_1)(x/x_1) \; , \end{cases}$$

where the division is in minplus sense, that is the subtraction. Denoting $y = (x_2/x_1, \cdots, x_n/x_1)$ and $g_{i-1}(y) = (f_i/f_1)(0, y)$, the problem is reduced to the computation of the fixed point problem $y = g(y)$. This fixed point gives the normalized eigenvector from which the eigenvalue is deduced by: $\lambda = f_1(0, y)$. We remark that g is a general minplus function of y not homogeneous.

The fixed point problem has not always a solution, but nevertheless, there are cases where we are able to find one:

- *f is affine in standard algebra*. In this case $f(x) = Ax + b$. The homogeneity[3] implies that the kernel of $A - I_d$ is not empty. When this kernel has one dimension, the eigenvalue is equal to $\lambda = pb$ where p is the normalized ($p\bar{1} = 1$) left standard eigenvector of A. But even in this case, all the standard eigenvalues do not have a module necessarily smaller than one, and the dynamic system may be unstable. We remark that when all the entries of the matrix A are nonnegative, f is monotone nondecreasing but when there are positive entries and negative entries, the system is not monotone.
- *f is minplus linear*: $f(x) = A \otimes x$. In this case the system is monotone.
- *f corresponds to stochastic control*. In this case $f(x) = D \otimes (Hx)$ where H is a standard matrix such that the rows define discrete probability laws (we call stochastic such kind of matrices). Then the dynamics $x^{k+1} = f(x^k)$ has the interpretation of a dynamic programming equation associated to a stochastic control problem and the eigenvalue is the optimal

[3] $A\bar{1} = \bar{1}$ with $\bar{1}$ the vector with all its entries equal to 1.

average cost of the corresponding stochastic control problem. We remark that the x^k are components of the dynamics:

$$y^{k+1} = A \boxtimes y^k, \text{ with } A = \begin{bmatrix} 0 & H \\ D & \varepsilon \end{bmatrix} .$$

- *f corresponds to stochastic games*. In this case $f = D_1 \odot (D_2 \otimes (Hx))$ where $\odot$ denotes the maxplus product (obtained by changing min in max in the maxplus matrix product) and H is a stochastic matrix. This case corresponds to dynamic programming equations associated to stochastic games. In this case, f is monotone.
- *f has a particular triangular structure* for example:

$$\begin{cases} x^{k+1} = A \otimes x^k \ , \\ y^{k+1} = B(x^k) \otimes y^k \ , \end{cases}$$

with $B(x)$ additively homogeneous of degree 0 but not necessarily monotone. For such systems, it is easy to find the eigenvalue and eigenvector by applying the minplus algebra results. In this case f is not always monotone. See [18] for discussions and generalizations.

In the general case, it is possible to compute the fixed point by the Newton's method[4] which corresponds to the policy iteration when the dynamic programming interpretation hold true. For stochastic control problems, the policy iteration is globally stable. In the game case it is only locally stable.

We may have unstable fixed points which are not accessible by integrating the dynamics. In this case, the eigenvalue, computable by the Newton's method, gives no information on the time asymptotic of the system. When all the fixed points are unstable, we may have a linear growth of the state trajectories. This point is illustrated by the chaotic tent dynamics example given in the next section.

F. Growth rate of homogeneous systems

We define the *growth rate* of a dynamic system $x^{k+1} = f(x^k)$, with $x \in \overline{\mathbb{R}}^n_{\min}$, as $\chi(f)$ the common limit $\lim_k x_i^k/k$ of all the components i when it exists. In [24] it has been proven with a special definition of connection (satisfied for a system defined by $f(x) = A \boxtimes x$ as soon as the

[4]we have to solve a piecewise linear system of equations

graph $\mathcal{G}(A)$ is strongly connected[5] that the growth rate and the eigenvalue of an homogeneous and monotone system exist and are equal. Let us show, on a system which is only homogeneous and strongly connected, that chaos may appear and that eigenvalues and growth rate which exist are not equal.

Let us consider the homogeneous dynamic system where k is the time index:

$$\begin{cases} x_1^{k+1} = x_2^k \ , \\ x_2^{k+1} = (x_2^k)^3/(x_1^k)^2 \oplus 2(x_1^k)^2/x_2^k \ . \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} \lambda x_1 = x_2 \ , \\ \lambda x_2 = x_2^3/x_1^2 \oplus 2x_1^2/x_2 \ , \end{cases}$$

where the minplus power exponent must not be confused with a time index.

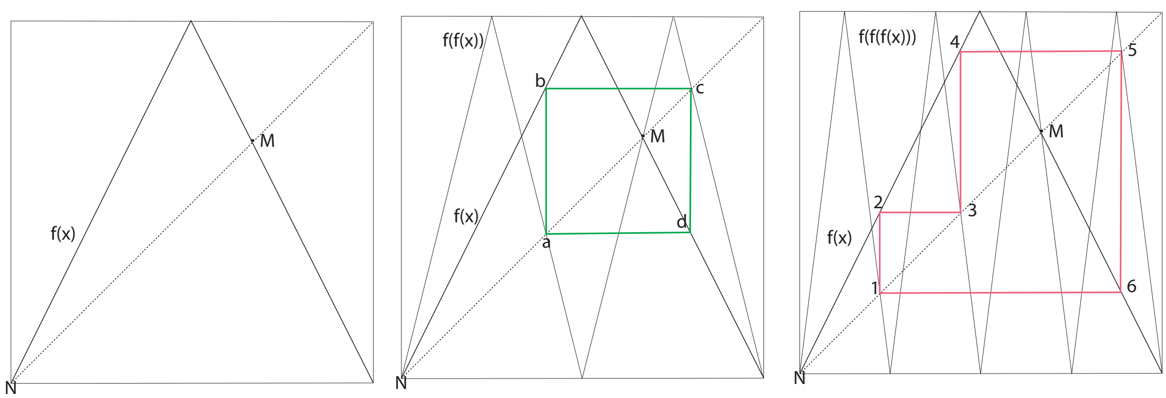

Fig. 2. Cycles of the tent transformation. The abscissa x_M of M is a fixed point of f: $x_M = f(x_M)$. The pair (x_a, x_c) is a cycle of f composed of two fixed points of $f \circ f$: $x_c = f(x_a)$, $x_a = f(x_c) = f(f(x_a))$. The triplet (x_1, x_3, x_5) is a circuit of f is composed of three fixed points of $f \circ f \circ f$: $x_3 = f(x_1)$, $x_5 = f(x_3) = f(f(x_1))$, $x_1 = f(x_5) = f(f(f(x_1)))$.

The solution is $\lambda = y$ with $y = x_2/x_1$ satisfying the equation

$$y = y^2 \oplus 2/y^2,$$

which has the solutions $y = 0$ and $y = \frac{2}{3} = 0.66...$. These two solutions are unstable fixed points of the transformation $f(y) = y^2 \oplus 2/y^2$. But the system $y^{k+1} = f(y^k)$ is chaotic since f is the

[5]Here there is an edge from i towards j in $\mathcal{G}(A)$ if $A_{ji} \neq \varepsilon$) if it is a minplus edge or $A_{ji} \neq 0$ if it is a standard one.

tent transform (see [6] for a clear discussion of this dynamics). In Figure-2 we show the graph of $x \mapsto f(x)$, $x \mapsto f(f(x))$, $x \mapsto f(f(f(x)))$, their fixed points, and periodic trajectories.

In Figure-3 we show a trajectory for an initial condition chosen randomly with the uniform law on the set $\{(i-1)/10^5, i = 1, \cdots, 10^5\}$. The diagonal line in the picture is a decreasing sort applied to the set $\{y^k,\ k = 1, \cdots, 10^5\}$. It shows that the invariant empirical density is approximatively uniform (in fact with these initial conditions the trajectories are periodic with a possible long period). It has been proved that the tent iteration has a unique invariant measure

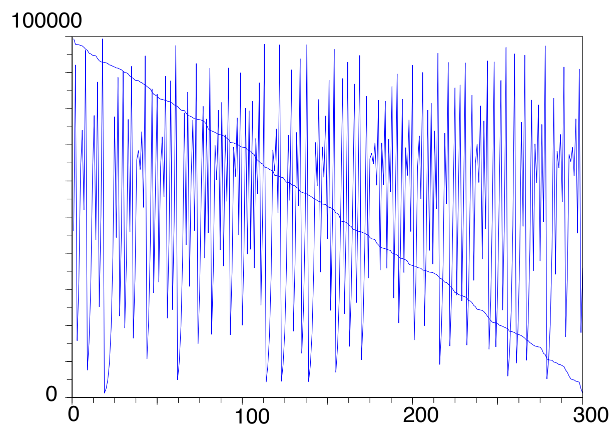

Fig. 3. A tent iteration trajectory $(10^5 y^k)_{k=1,.,300})$ and its arrangement in decreasing order (almost diagonal line) .

absolutely continuous with respect to the Lebesgue measure: the uniform law on $[0, 1]$. Therefore, the system is ergodic, the growth rate:

$$x_1^N/N = \frac{1}{N}\sum_{k=1}^{N}(x_1^k - x_1^{k-1}) \ ,$$

can be computed, by averaging with respect to the uniform law an increase in one step: $f_1(x)-x_1$ with the standard notations (that is $f_1(x)/x_1 = y$ with the minplus notations). Therefore $\chi(f) = \int_0^1 y dy = 0.5$, for almost all initial condition, which is different from the eigenvalues (0 and $\frac{2}{3}$).

More generally, for an homogeneous system, we can write the system dynamics:

$$x_1^{k+1}/x_1^k = f_1(x^k)/x_1^k = h(y^k), \ \ y^{k+1} = g(y^k),$$

with $y_{i-1}^k = x_i^k/x_1^k$ and $g_{i-1} = f_i/f_1$ for $i = 2, \cdots, n$. As soon as the y^k belong to a bounded closed (compact) set for all k, we remark (after Kryloff and Bogoliuboff) that the set of measures:

$$\left\{P_{y^0}^N = \frac{1}{N}\left(\delta_{y^0} + \delta_{g(y_0)} + \cdots + \delta_{g^{N-1}(y^0)}\right),\ n \in \mathbb{N}\right\} ,$$

(where δ_a denotes the Dirac mass on a) is tight. Therefore we can extract convergent subsequences which converge towards invariant measures Q_{y^0} that we will call Kryloff-Bogoliuboff invariant measure. Then we can apply the ergodic theorem at the sequence $(y^k)_{k\in\mathbb{N}}$ which shows that for almost all new initial condition chosen randomly according to the Q_{y^0} we have [6]:

$$\chi(f) = \frac{1}{N}(x_1^N - x_1^0) = \lim_N \frac{1}{N}\left(\sum_{k=0}^{N-1} h(y^k)\right) = \int h(y) dQ_{y^0}(y) \ . \tag{2}$$

We can also see [2] for invariant measures construction of stochastic recursions.

Coming back to the tent example, according to the initial value y^0, the tent iterations y^k stay in circuits or follow trajectories without circuit (possibly dense in $[0,1]$). For example, assuming that the initial condition is such that $y = \frac{2}{5}$, the trajectory is periodic of period 2. The invariant measure is $Q_{y^0} = \frac{1}{2}(\delta_{\frac{2}{5}} + \delta_{\frac{6}{5}})$. The growth rate is $\frac{4}{5}$ which is once more different from the eigenvalues 0 and $\frac{2}{3}$. Moreover it can be shown that for all initial conditions with a finite binary development (this set contains all the float numbers of computers) the trajectory stays in the unstable fixed point 0 after a finite number of steps. That is for a dense set of initial conditions the invariant measure is δ_0 and the growth rate is 0.

III. PETRI NET DYNAMICS

A. *Autonomous Petri nets*

Let us give in min-plus-times algebra a presentation of timed continuous Petri nets with weights. The weight can be negative and the numbers of tokens are not necessarily integer (in continuous Petri nets, what we call tokens are in fact fluid amounts).

[6] It may happen that the initial condition y_0 is transient therefore it is in the attractive basin of Q_{y^0} not be in the support of Q_{y^0}. It would be very useful to prove that y^0 is generic (in the sense of Furstenberg[23]) that is the limit exists for y^0. A priori homogeneous systems have not the uniform continuity property necessary to prove the convergence of the Cesaro means for all y^0. In the case where the compact set is finite, we can apply the ergodicity results on Markov chains with a finite state number to show the convergence of $P_{y^0}^N$ towards Q_{y^0}. Instead of the subsequence convergence, this convergence proves the genericity of y^0.

A Petri net $\mathcal{N}$ is a graph with two sets of nodes: the *transitions* $\mathcal{Q}$ (with $|\mathcal{Q}|$ elements) and the *places* $\mathcal{P}$ (with $|\mathcal{P}|$ elements) and two sorts of edges, the *synchronization edges* (from a place to a transition) and the *production edges* (from a transition to a place).

A minplus $|\mathcal{Q}| \times |\mathcal{P}|$ matrix D, called *synchronization*[7] *matrix* is associated to the synchronization edges. $D_{qp} = a_p$ if there exists an edge from the place $p \in \mathcal{P}$ to the transition $q \in \mathcal{Q}$, and $D_{qp} = \varepsilon$ elsewhere, where a_p is the *initial marking* of the place p which is, graphically, the number of tokens in p. We suppose here that the sojourn time in all the places is one time unit[8].

A standard algebra $|\mathcal{P}| \times |\mathcal{Q}|$ matrix H, called *production*[9] *matrix* is associated with the production edges. It is defined by $H_{pq} = m_{pq}$ if there exists an edge from q to p, and 0 elsewhere, where m_{pq} is the *multiplicity* of the edge [10].

Therefore a Petri net is characterized by the quadruple:

$$(\mathcal{P}, \mathcal{Q}, H, D) \ .$$

It is a dynamic system in which the token (fluid) evolution is partially defined by the transition firings saying that a transition can fire as soon as all its upstream places contain a positive quantity of tokens (fluid) having stayed at least one unit of time. When a transition fires, it consumes a quantity of tokens (fluid) equal to the minimum of all the available quantities being in the upstream places. Cumulating the firings done up to present time defines the *cumulated transition firing* of the transition. The firing produces a quantity of tokens (fluid) in each downstream place equal to the firing of the transition multiplied by the multiplicity of the corresponding production edge. If the multiplicity of a production edge, going from q to p, is negative, the firing of q consumes tokens (fluid) of p.

A general Petri Net defines constraints on the transition firing. Denoting by q^k the cumulated

[7]Decision matrix in stochastic control.

[8]When different integer sojourn times are considered, an equivalent Petri net with a unique sojourn time can be obtained by adding places and transitions and solving the implicit relations.

[9]Hazard matrix in stochastic control

[10]Here the multiplicity appears only with the output transition edges. The multiplicity of input transition edges is supposed to be always equal to one. Look at the more general case [10] we see that we don't lose generality by doing so (the dynamics class obtained is the same)

firings of transitions $q \in \mathcal{Q}$ up to instant k, they satisfy the constraints:

$$\min_{p \in q^{in}} \left[a_p + \sum_{q \in p^{in}} m_{pq} q^{k-1} - \sum_{q \in p^{out}} q^k \right] = 0, \ \forall q \in \mathcal{Q}, \ \forall k, \tag{3}$$

where: $p \in \mathcal{P}$ is a place of the Petri Net, $q^{in} \subset \mathcal{P}$ [resp. $q^{out} \subset \mathcal{P}$] denotes set of places upwards [resp. downwards] the transition q and $p^{in} \subset \mathcal{Q}$ [resp. $p^{out} \subset \mathcal{Q}$] denotes the set of transitions upwards [resp. downwards] the place p.

Indeed, being at time k, we know (from the firing definition of transitions) that after the firing (which is instantaneous) at least one place upwards any transition is empty of token entered before time $k-1$. For each transition q, the equation computes the number of token in each place $p \in q^{in}$ which have stayed at least one unit of time and express that at least one is 0 the other being nonnegative.

As soon as there is more than one edge leaving a place, the trajectory of the system is not well defined because we don't know the path of a token leaving this place.

In the case of a *deterministic* Petri net, where all the places only have one downstream edge, the dynamics is well defined, which means, there is no token consumption conflict between the downstream transitions[11]. Then, denoting $Q = (q^k)_{q \in \mathcal{Q}, k \in \mathbb{N}}$ the vector of sequences of cumulated firing quantities of transitions and $P = (p^k)_{p \in \mathcal{P}, k \in \mathbb{N}}$ the vector of sequences of cumulated token quantities arrived in the places at time k, we have:

$$\begin{bmatrix} P^{k+1} \\ Q^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & H \\ D & \varepsilon \end{bmatrix} \boxtimes \begin{bmatrix} P^{k+1} \\ Q^k \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} HQ^k \\ D \otimes P^{k+1} \end{bmatrix} . \tag{4}$$

From this dynamics we can deduce the dynamics of the cumulated firing quantities by eliminating the places variables and the dynamics of the cumulated token quantities by eliminating the transition variables:

$$Q^{k+1} = D \otimes (HQ^k), \quad P^{k+1} = H(D \otimes P^k) .$$

In the case of *event graphs* (particular deterministic Petri nets where all the multiplicities m_{pq} are equal to 1 and all the places have exactly one edge upstream) the dynamics is linear in the

[11]In the non-deterministic case, we have to specify the rules which resolve the conflicts by, for example, giving priorities to the consuming transitions or by imposing ratios to be respected. As soon as these rules are added, the initial non-deterministic Petri net becomes a deterministic one.

minplus sense. It is:

$$Q^{k+1} = A \otimes Q^k,$$

where $A_{q'q} = a_p$ with p the unique place between q and q'.

B. Deterministic Petri nets

Making clear a Petri net dynamics, in such a way that the trajectories are uniquely defined, comes down to find another Petri net (compatible with the dynamics constraints expressed by the initial net) having only one edge leaving each place. Let us discuss these points more precisely on the simple system given in the first picture of Figure-4.

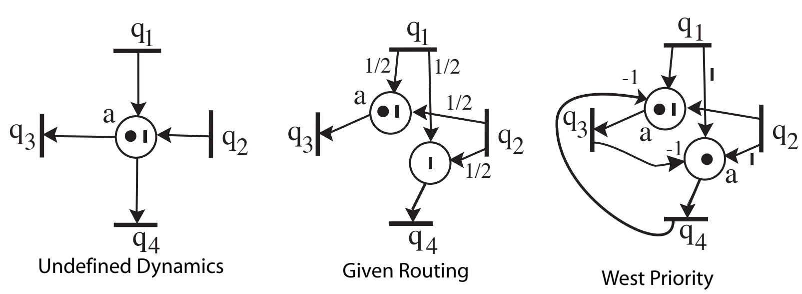

Fig. 4. The nondeterministic Petri net, given in the left figure, is made clear by: – choosing a routing policy: 1/2 towards q_3, 1/2 towards q_4 in the central figure, – giving top priority to q_3 against q_4 in the right figure (where the time shift given by the sticks are not anymore only in the places but also on the edges).

The incomplete dynamics of this system can be written in minplus algebra[12]:

$$q_4^k q_3^k = a q_1^{k-1} q_2^{k-1} \ . \tag{5}$$

Clearly q_3 and q_4 are not defined uniquely. For example, we can complete the dynamics, in the two following ways useful for the traffic application that we will see in the following sections:

- By specifying the routing policy (for example we choose arbitrarily that half of the total tokens available are given to q_3 and half to q_4, see [10] for results on the general routing case) [13]:

$$q_4^k = \sqrt{q_1^{k-1} q_2^{k-1}}, \quad q_3^k = a q_4^k \ .$$

[12]Which means in standard algebra: $q_4^k + q_3^k = a + q_1^{k-1} + q_2^{k-1}$.

[13]Which means in standard algebra: $q_4^k = \frac{1}{2}\left(q_1^{k-1} + q_2^{k-1}\right), \quad q_3^k = a + q_4^k$.

The minplus product of the two equations gives the constraint (5).

- By choosing a priority rule (top priority to q_3 against q_4)[14]:

$$q_3^k = aq_1^{k-1}q_2^{k-1}/q_4^{k-1}, \quad q_4^k = aq_1^{k-1}q_2^{k-1}/q_3^k \ .$$

The last equation implies that the initial constraint (5) is satisfied and that q_3^k is the largest possible (priority) considering the dynamics constraints and the increasing properties of the trajectories. We see that the negative weights on q_4^{k-1} and on q_3^k are essential to express this priority. We remark also that these weights can be seen as negative proportions in a routing policy.

In the two cases we obtain a *degree one homogeneous minplus* system.

C. *Input-Output Petri nets*

We can define a Petri net with inputs and outputs in the following way. We partition the transition set in three parts $(\mathcal{V}, \mathcal{Q}, \mathcal{Z})$: the input set $\mathcal{V}$, the state set $\mathcal{Q}$ and the output set $\mathcal{Z}$. We do the same thing for the place set and we get the three parts $(\mathcal{U}, \mathcal{P}, \mathcal{Y})$. The inputs are the transitions [resp. places] without upstream edges. The outputs are the ones without output edges. Then the dynamics can be rewritten:

$$\begin{bmatrix} P^{k+1} \\ Q^{k+1} \\ Y^{k+1} \\ Z^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & A & 0 & B \\ C & \varepsilon & D & \varepsilon \\ 0 & E & 0 & 0 \\ F & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \boxtimes \begin{bmatrix} P^{k+1} \\ Q^k \\ U^{k+1} \\ V^k \end{bmatrix} \triangleq \begin{bmatrix} AQ^k + BV^k \\ C \otimes P^{k+1} \oplus D \otimes U^{k+1} \\ EQ^k \\ F \otimes P^{k+1} \end{bmatrix} . \tag{6}$$

This dynamics denoted by S, defined by the matrices (A, B, C, D, E, F), associates to the input signals $(U^k, V^k)_{k\in\mathbb{N}}$ the output signals $(Y^k, Z^k)_{k\in\mathbb{N}}$: $(Y, Z) = S(U, V)$. These systems can be considered as a special case, where some blocks are null, of an extension of generalized system described in the minplus section to the case of system with implicit part.

The three operations (parallel, series, and feedback compositions) described previously can be extended easily to this case.

[14]Which means in standard algebra: $q_3^k = a + q_1^{k-1} + q_2^{k-1} - q_4^{k-1}, \quad q_4^k = a + q_1^{k-1} + q_2^{k-1} - q_3^k$.

IV. Traffic Application

A. *Traffic on a circular road*

Let us recall the simplest model to derive the fundamental traffic diagram on a single road. The simplest way is to study the stationary regime on a circular road with a given number of vehicles[15]. We present two ways to obtain this diagram: – by logical deduction from an exclusion process point of view, – by computing the eigenvalue of a minplus system derived from a simple Petri net modeling of the road. In the following, the road is cut in m sections which can contain at most one vehicle.

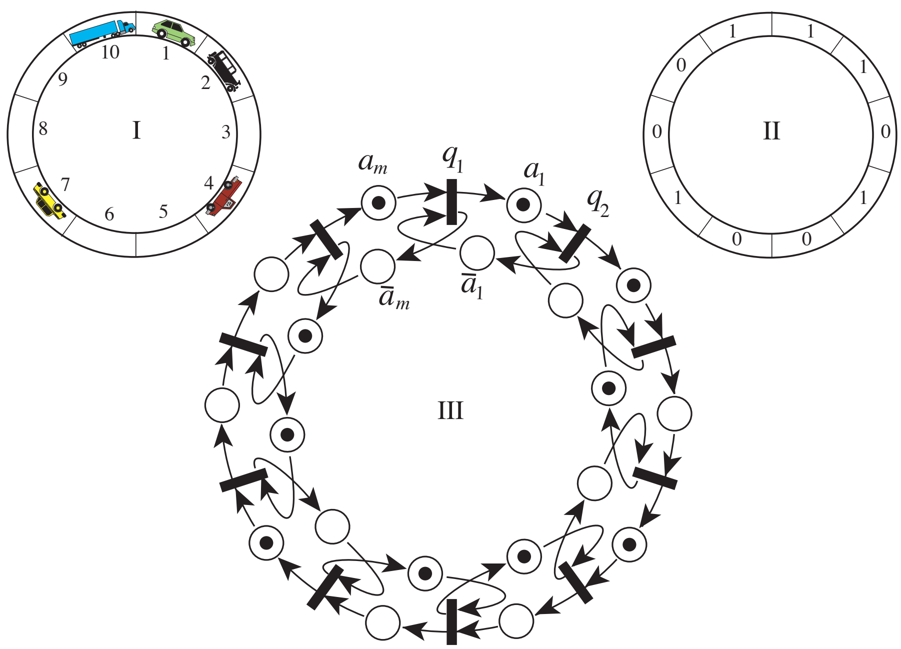

Fig. 5. A circular road (top-left) cut in sections, its exclusion process where 1 means that the section is occupied (top-right) and its Petri net (middle) representation where the ticks (1 time delay) present in each place is not represented.

B. *Exclusion process modeling*

Following [7] we can consider the dynamic system defined by the rule $10 \to 01$ applied to a binary word w. The word w_k describes the car positions at instant k on a road cut in sections

[15]We consider that the stationary regime of the circular road is reached locally on a standard road when its density is constant at the considered zone.

(each bit representing a section, 1 meaning occupied and 0 meaning free, see II in Figure-5). Let us take an example:

$$\begin{aligned} &w_1 = 1101001001, \quad w_2 = 1010100101, \\ &w_3 = 0101010011, \quad w_4 = 1010101010, \\ &w_5 = 0101010101. \end{aligned}$$

Let us define: – the *density* ρ by the number of vehicles n divided by the number of sections m: $\rho = n/m$, – the flow $\varphi(t)$ at time t by the number of vehicles going one step forward at time t divided by the number of sections. Then the *fundamental traffic diagram* gives the relation between $\varphi(t)$ and ρ.

If $\rho \leq 1/2$, after a transient period, all the vehicle groups split off, and then all the vehicles can move forward without other vehicles in the way, and we have:

$$\varphi(t) = \varphi = n/m = \rho \ .$$

If $\rho \geq 1/2$, the free place groups split off after a finite time and move backward without other free places in the way. Then $m - n$ vehicles move forward and we have:

$$\varphi(t) = \varphi = (m-n)/m = 1 - \rho \ .$$

Theorem 4:

$$\exists T : \ \forall t \geq T \quad \varphi(t) = \varphi = \begin{cases} \rho & \text{if} \ \ \rho \leq 1/2 \ , \\ 1 - \rho & \text{if} \ \ \rho \geq 1/2 \ . \end{cases}$$

Proof: This result has been proven in [7]. Let us give the idea of the proof. We only have to prove that after a finite number of steps the separations of vehicles or holes appear.

Assuming that the density is not greater than 1/2, let us look what happens at a cluster of at least two vehicles denoted by A. There are two cases:

- The cluster behind A is separated from A by one hole, we have the configuration 10A0. Then at the next step we have 0A01. The cluster A has gone backward of one place.
- The cluster behind is separated from A by more than one hole, we have the configuration 00A0. Then at the next step the size of A has been reduced of one.

Then after a finite number of steps, bounded by the number of places, the size of the clusters of vehicles are reduced strictly since, going backward, the density being not greater than 1/2, and individual vehicles going forward, there is cluster of vehicles that must meet a cluster of at least two holes.

For the density larger than 1/2, we follow the cluster of holes instead of vehicles. We can show by the same arguments that their sizes decrease or they go forward. Therefore after a finite number of steps the holes are separated by vehicles. ■

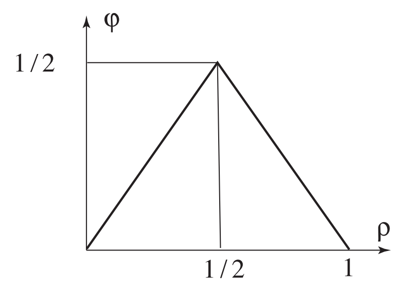

Fig. 6. The fundamental traffic diagram giving the dependance on the average flow with respect of the vehicle density.

C. Event Graph modeling

Consider the Petri net given in III of Figure-5 which describes in a different way the same dynamics. In fact, this Petri net is an event graph and therefore its dynamics is linear in minplus algebra. The vehicle number entered in the section s before time k is denoted q_s^k. The initial vehicle position is given by the booleans a_s which take the value 1 when the cell contains a vehicle and 0 otherwise.

We use the notation $\bar{a} = 1 - a$, then the dynamics is given by:

$$q_s^{k+1} = \min\{a_{s-1} + q_{s-1}^k, \bar{a}_s + q_{s+1}^k\} ,$$

which can be written linearly in minplus algebra:

$$q_s^{k+1} = a_{s-1} q_{s-1}^k \oplus \bar{a}_s q_{s+1}^k ,$$

where the index addition is done modulo m.

Theorem 5: The average transition speed (car flow) φ depends to the car density ρ according to the law:

$$\varphi = \min(\rho, 1-\rho) \ .$$

Proof: This event graph has three kinds of elementary circuits: – the outside circuit with average weight n/m, – the inside circuit with average weight $(m-n)/m$, – the circuits corresponding to make some step forward and coming back, with average mean 1/2. Therefore using Theorem-2, its eigenvalue is

$$\varphi = \min(n/m, (m-n)/m, 1/2) = \min(\rho, 1-\rho) \ ,$$

which gives the average speed as a function of the car density since the minplus eigenvalue is equal to $\lim_t x_i(t)/t = \varphi$ for all i.

D. Traffic on two roads with one junction

Before discussing the dynamics of the town, let us study into details the case of two circular roads with a junction given in Figure-7 (top-right).

A first trial is to consider the Petri net given in Figure-7 (middle). This Petri net is not an event graph. It is a general non deterministic Petri Net.

We can write the dynamics of this Petri net using Theorem-3, but these equations do not determine uniquely the trajectories of the system. We have two places with two outgoing edges: – at place a_n we have to specify the *routing policy* giving the proportion of cars going towards y_2 and the proportion going towards y_3 – at place $\bar{a}_n$ we may follow *the first arrived the first served rule with the right priority* when two cars want to enter in the junction simultaneously. Using Petri net with negative weights we obtain the Petri Net Figure-7 with the junction described precisely in the top-left part of the figure.

The corresponding dynamics can be written with minplus notations:

$$\begin{cases} q_i^{k+1} = a_{i-1} q_{i-1}^k \oplus \bar{a}_i q_{i+1}^k, \ i \neq 1, n, n+1, n+m, \\ q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k \oplus a_{n-1} q_{n-1}^k \ , \\ q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1} \oplus a_{n+m-1} q_{n+m-1}^k \ , \\ q_1^{k+1} = a_n \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_1 q_2^k \ , \\ q_{n+1}^{k+1} = a_{n+m} \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_{n+1} q_{n+2}^k \ , \end{cases} \tag{7}$$

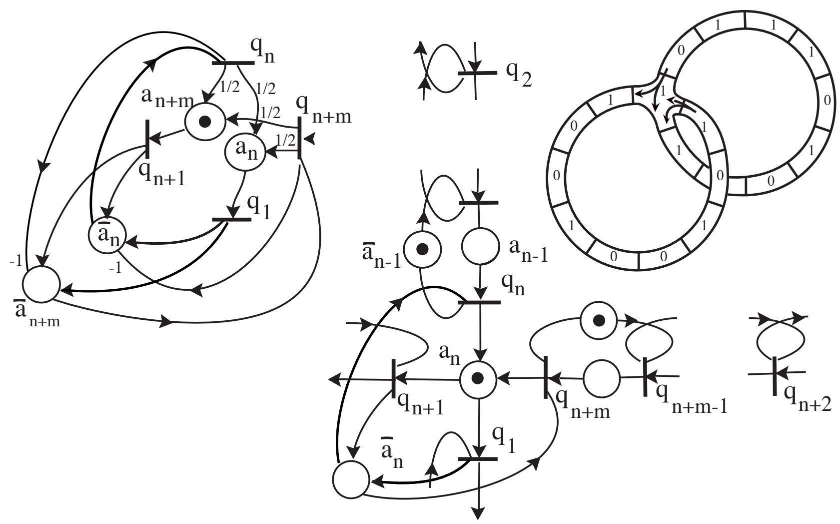

Fig. 7. A junction with two circular roads cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left). The ticks representing the time delays present in each place are not represented.

where the entries satisfy the following constraints (written with the standard notations):

- $0 \leq a_i \leq 1$ for $i = 1, \cdots, n+m$, these initial markings give the presence 1 or absence 0 of a car in the road sections. But cars are seen here as a fluid and we can relax this integer constraint. Moreover the transition firing cuts the tokens and if we want see the systems after some firings there is not necessarily an integer number of tokens in a place. Therefore it is better to accept, there, real numbers belonging to the $[0, 1]$ interval;
- $\bar{a}_i = 1 - a_i$ for $i \neq n, n+m$ they give the initial free spaces in the places;
- $a_n + a_{n+m} \leq 1$ the maximum number of cars in the junction is 1;
- $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$ give the free place in the junction.

Let us study the existence of a growth rate. For that, we use the two following results.

Theorem 6: The trajectories of the states of the junction dynamics (7), $(q_i^k)_{k \in \mathbb{N}}$ starting from 0, are nonnegative and nondecreasing for all i.

Proof: Computing q^1 using the fact that all the a_i and $\bar{a}_i$ are nonnegative, it is clear that $q^1 \geq 0$. Let us prove by induction that the trajectories are nondecreasing. It is true for $k = 1$. We suppose that it is true for k, that is $q^k \geq q^{k-1}$, and let us prove that $q^{k+1} \geq q^k$. Let us rewrite (7) $q_i^{k+1} = f_i(q^k)$ for $i = 1, \cdots, n+m$. The functions f_i for $i \neq n, n+m$ are nondecreasing.

Therefore, for a such i, we have:

$$q_i^{k+1} = f_i(q^k) \geq f_i(q^{k-1}) \geq q_i^k \ ,$$

using first the induction hypothesis and then the dynamics definition.

Let us prove the nondecreasing property of q_n.

- If $q_n^{k+1} = a_{n-1}q_{n-1}^k$ we have

$$q_n^{k+1} = a_{n-1}q_{n-1}^k \geq a_{n-1}q_{n-1}^{k-1} \geq f_n(q^{k-1}) = q_n^k \ .$$

- If $q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$, using the dynamics we have $q_{n+m}^k \leq \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1} / q_n^k$ and therefore

$$q_n^{k+1} \geq \bar{a}_n q_n^k q_1^k q_{n+1}^k / \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1}$$

  which gives $q_n^{k+1} \geq q_n^k$ using the induction hypothesis and the assumption $\bar{a}_n = \bar{a}_{n+m}$.

The nondecreasing property of q_{n+m} is proven in the same way. If $q_{n+m}^{k+1} = a_{n+m-1}q_{n+m-1}^k$ we have $q_{n+m}^{k+1} = a_{n+m-1}q_{n+m-1}^k \geq a_{n+m-1}q_{n+m-1}^{k-1} \geq f_{n+m}(q^{k-1}) = q_{n+m}^k$. If $q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1}$, using the dynamics we have $q_n^{k+1} \leq \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$ and therefore $q_{n+m}^{k+1} \geq \bar{a}_{n+m} q_{n+m}^k / \bar{a}_n$, which gives the result using $\bar{a}_n = \bar{a}_{n+m}$. ■

Theorem 7: The distances between any pair of states stay bounded.

$$\exists c_1 : \sup_k |q_i^k - q_j^k| \leq c_1, \forall i, j.$$

Moreover

$$\forall T, \exists c_2 : \sup_k |q_i^{k+T} - q_i^k| \leq c_2 T, \forall i.$$

Proof: This result comes from the following inequalities (written in minplus algebra) deduced from the dynamics and the nondecreasing property of the trajectories:

$$q_i^{k+1} \leq a_{i-1} q_{i-1}^k \ , i \neq 1, n+1$$

$$q_1^{k+1} \leq a_n \sqrt{q_n^k q_{n+m}^k} \leq a_n \sqrt{q_n^{k+n} q_{n+m}^k} \leq a_n \sqrt{b_1^{n-1} q_1^{k+1} q_{n+m}^k} \Rightarrow q_1^{k+1} \leq a_n b_1^n q_{n+m}^k \ ,$$

with $b_j^k = \bigotimes_{i=j}^k a_i$.

$$q_{n+1}^{k+1} \leq a_{n+m} \sqrt{q_n^k q_{n+m}^k} \leq a_{n+m} \sqrt{q_n^k q_{n+m}^{k+m}} \leq a_{n+m} \sqrt{q_n^k b_n^{n+m-1} q_{n+1}^{k+1}} \Rightarrow q_{n+1}^{k+1} \leq a_{n+m} b_n^{n+m} q_n^k \ .$$

Therefore we have:

$$q_n^k \leq a_{n-1} q_{n-1}^{k-1} \leq \cdots \leq (b_1^n)^2 q_{n+m}^{k-n} \leq (b_1^n)^2 a_{n+m-1} q_{n+m-1}^{k-n-1} \leq \cdots \leq (b_1^{n+m})^2 q_n^{k-n-m}.$$

The result follows from these inequalities which give bounds for all the the distances between two states and between the same state but at different times. ■

Using this theorem and (2) we have shown an existence theorem of the growth rate of (7) which has the interpretation of the average traffic flow.

Theorem 8: It exists initial distribution on $(q_j^0/q_1^0)_{j=2,n+m}$ the Kryloff-Bogoljuboff invariant measure such that the average flow

$$\chi = \lim_k q_i^k/k, \ \ \forall i \ ,$$

of the dynamical system (7) exists almost everywhere.

This result is not completely satisfactory. We would like to have the existence of the growth rate for the initial condition $q_i^0 = 0$ for all i. Nevertheless, we remark that the numerical approximation obtained by simulation of the growth rate always exists. Indeed in this case, the number of the approximated states obtained by floating number approximation is finite since they belong to a bounded set[16] and in this case the growth rate existence can be proved using the Cesaro-convergence towards invariant measure of the probability to be in a state of finite Markov chains. At this point it is also useful to recall that the continuous Petri net model used is an approximation of a discrete state process. The continuous model has been used because it has the nice homogeneity property.

E. Eigenvalue Existence of the Junction Dynamics

Let us consider the eigenvalue problem associated to the dynamics (7). It is defined as finding λ and q such that:

$$\begin{cases} \lambda q_i = a_{i-1}q_{i-1} \oplus \bar{a}_i q_{i+1}, \ i \neq 1, n, n+1, n+m, \\ \lambda q_n = \bar{a}_n q_1 q_{n+1}/q_{n+m} \oplus a_{n-1}q_{n-1} \ , \\ \lambda q_{n+m} = \bar{a}_{n+m} q_1 q_{n+1}/(\lambda q_n) \oplus a_{n+m-1}q_{n+m-1} \ , \\ \lambda q_1 = a_n \sqrt{q_n q_{n+m}} \oplus \bar{a}_1 q_2 \ , \\ \lambda q_{n+1} = a_{n+m} \sqrt{q_n q_{n+m}} \oplus \bar{a}_{n+1} q_{n+2} \ , \end{cases} \tag{8}$$

[16]We suppose here that the approximation does not destroy the fact that the states stay in a bounded set .

with: $0 \leq a_i \leq 1$ for $i = 1, \cdots, n+m$, $\bar{a}_i = 1 - a_i$ for $i \neq n, n+m$, $a_n + a_{n+m} \leq 1$ and $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$.

The eigenvalue problem can be solved explicitly.

Theorem 9: The nonnegative eigenvalues λ are solutions of the equation (written with the standard notations):

$$0 = \max\left\{-\lambda, \min\left\{(1-\rho)\, d - \lambda, \frac{1}{4} - \lambda, r - (1-\rho)\, d - (2r - 1 + 2\rho)\, \lambda\right\}\right\} \,, \tag{9}$$

with $N = n + m$, $\rho = 1/N$, $r = m/N$ and $d = \left(\sum_{i=1}^{n+m} a_i\right)/(N-1)$ the density of vehicles.

When N is large, if $r > 1/2$, the formula (9) has a unique solution with a simple approximation:

$$\lambda \simeq \max\left\{0,\ \min\left\{d, \frac{1}{4}, \frac{r-d}{2r-1}\right\}\right\}.$$

Proof: We will give here only a sketch of the proof. Eight pages of computation are necessary to explicit all the details. The proof has two parts. The first part consists in reducing the problem to a generalized eigenvalue problem in a four dimension space. The second part consists in a verification of the generalized eigenvalue equations since we will give explicit formulas for all the eigenelements. The formulas of the eigenelements have been obtained empirically by trial and error after observing the phases on numerical simulations, and remarking that the generalized eigenvalue problem must have a solution depending in a piecewise linear of the density, since it is mainly a solution of a dynamic programming equation depending piecewise linearly on the car density.

After verifying that $\lambda \leq 1/4$, by elimination of $q_i,\ i \neq 1, n, n+1, n+m$, thanks to the minplus linearity of the first equation of (8), we obtain the closed set of equations defining q_i, $i = 1, n, n+1, n+m$:

$$\begin{cases} q_n = (\bar{a}_n/\lambda) q_1 q_{n+1}/q_{n+m} \oplus (b_n/\lambda^{n-1}) q_1 \,, \\ q_{n+m} = (\bar{a}_{n+m}/\lambda^2) q_1 q_{n+1}/q_n \oplus (b_m/\lambda^{m-1}) q_{n+1} \,, \\ q_1 = (a_n/\lambda)\sqrt{q_n q_{n+m}} \oplus (\bar{b}_n/\lambda^{n-1}) q_n \,, \\ q_{n+1} = (a_{n+m}/\lambda)\sqrt{q_n q_{n+m}} \oplus (\bar{b}_m/\lambda^{m-1}) q_{n+m} \,, \end{cases} \tag{10}$$

where $b_n = \bigotimes_{i=1}^{n-1} a_i$ is the number of cars in the street with priority, $\bar{b}_n = \bigotimes_{i=1}^{n-1} \bar{a}_i$ is the number of free places in the street with priority, $b_m = \bigotimes_{i=n+1}^{n+m-1} a_i$ is the number of cars in the street without priority and $\bar{b}_m = \bigotimes_{i=n+1}^{n+m-1} \bar{a}_i$ is the number of free places in the street without priority.

Remarking after numerical simulations that four phases exist, it is possible to precise their domains analytically, and by observing the asymptotic regimes to give their physical interpretations:

- *Free moving* When the density is small, $0 \leq d \leq \alpha$ with $\alpha = \frac{1}{4(1-\rho)}$, after a finite time, all the cars move freely.
- *Saturation* When $\alpha \leq d \leq \beta$ with $\beta = \frac{1}{2}\frac{r+1/2-\rho}{1-\rho}$ the junction is used at its maximal capacity without being bothered by downstream cars.
- *Recession* When $\beta < d < \gamma$ with $\gamma = \frac{r}{1-\rho}$ the crossing is fully occupied but cars sometimes cannot leave it because the road where they want to go are crowded. When $\gamma < \beta$, on the interval $[\gamma, \beta]$ three eigenvalues exist. In this case the system is in fact blocked.
- *Blocking* When $\gamma \leq d \leq 1$, the road without priority is full of cars, no car can leave it and one car wants to enter.

Table I gives for each phase the eigenvalue and eigenvector formulas as a function of the density d. In order to complete the proof we have only to verify the eigenvalue equation. We will not explicit this verification which is too long to be given here (32 inequalities have to be verified).

	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta,\gamma) < d < \max(\beta,\gamma)$	$\gamma \leq d \leq 1$
λ	$(1-\rho)d$	$\frac{1}{4}$	$\frac{r-(1-\rho)d}{2r-1+2\rho}$	0
q_n	$b_n - \lambda(2n-2)$	$b_n - \lambda(2n-2)$	$b_n - \lambda(2n-2)$	$\bar{a}_{n+m} + \bar{b}_m$
q_{n+m}	$2(\lambda - a_n) - b_n$	$\frac{1}{2} - 2a_n - b_n$	$2(\lambda - a_n) - b_n$	$-2a_n - \bar{a}_{n+m} - \bar{b}_m$
q_1	$-\lambda(n-1)$	$-\lambda(n-1)$	$-\lambda(n-1)$	0
q_{n+1}	$a_{n+m} - a_n - \lambda(n-1)$	$a_{n+m} - a_n - \lambda(n-1)$	$(5-n)\lambda - 1 + a_{n+m} - a_n$	$-2a_n - \bar{a}_{n+m}$

TABLE I

THE EIGENVALUES AND EIGENVECTORS AS FUNCTION OF THE CAR DENSITIES.

After to have verified the eigenvalue formulas given in I, we obtain (9) by computing the boundaries of half spaces to which all the eigenvalues belong in the d, λ plane. We give it only for its concision. The Table I, giving explicitly the eigenvalues, is more useful in practice. ■

We show in Figure-8 the fundamental diagram obtained by simulation (using the maxplus arithmetic of the ScilabGtk software [37]) for a particular relative size r of the two roads and the eigenvalue λ given in Table I. We see clearly the four phases described in the proof of the eigenvalue formula.

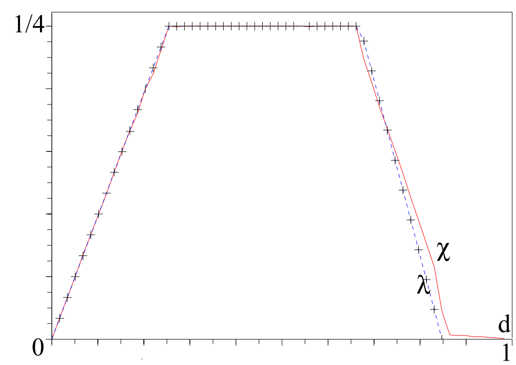

Fig. 8. The traffic fundamental diagram $\chi(d)$ when $r = 5/6$ (continuous line) obtained by simulation and its comparison with the eigenvalue $\lambda(d)$ given in Table I.

F. Regular City Modeling.

To derive the fundamental diagram of a regular town on a torus described in Figure-9 (left), we can generalize the modeling approach used in the case of one junction. For that we can build the model by combining three elementary systems (representing one section, a junction input and a junction output) with the composition operators described in the generalized system theory subsection of the the minplus section. The details can be found in [18].

The asymptotic car repartition for a small town composed of two North-South, South-North, East-West and West-East avenues is given in Figure-9 (right). The fundamental diagram presents a four phases shape analogous to the two roads with one junction case. In this more general case, the role of the non priority road is plaid by a circuit of non priority roads which, when it is full, blocks the complete system.

V. Conclusion

The Petri net modeling of traffic in town can be done thanks to the introduction of negative weights on output transition edges. The dynamics has a nice degree one homogeneous minplus

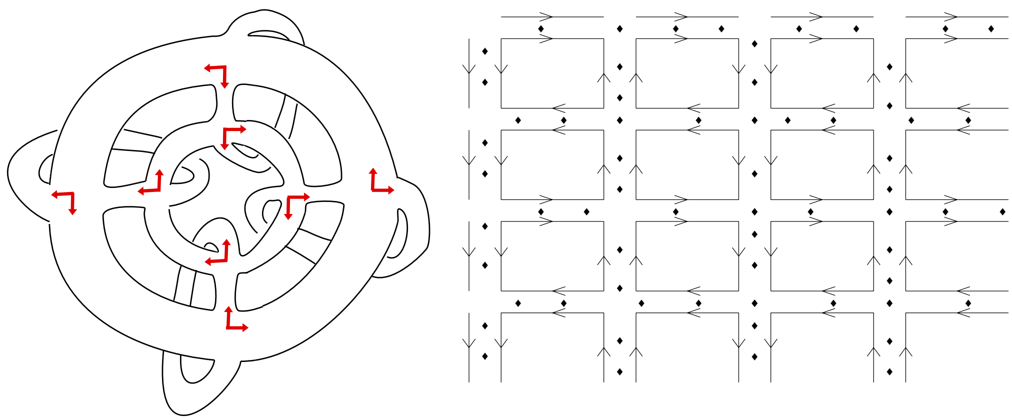

Fig. 9. Roads on a torus of 4×2 streets with its authorized turn at junctions (left) and tht asymptotic car repartition in the streets on a torus of 4×4 streets obtained by simulation.

property, but is not monotone anymore. This lost of monotonicity implies that the eigenvalue and the growth rate are not equal anymore. Experimental results show that they are close in the case of two roads with one crossing where we are able to solve explicitly the eigenvalue problem and compute numerically the growth rate. The fundamental traffic diagram which gives the dependance of the growth rate on the density presents four phases which have traffic interpretations.

This set of 1-homogeneous minplus systems seems to be a good class of systems that we can describe by two matrices: one in the standard algebra and one in the minplus algebra. The standard compositions of these systems can be easily described in terms of these two matrices. These compositions of simple systems are useful to build the dynamics of large systems like the regular town traffic one.

In a future paper, more traffic oriented, we will describe in more details the traffic interpretation of the four phases valid also for more general systems like regular towns. The influence of the control of traffic light on the fundamental diagram will be also studied.

REFERENCES

[1] M. Akian, S.Gaubert, R. Nussbaum, *The Collatz-Wielandt theorem for order preserving maps and cones* Preprint 2007.

[2] V. Anantharam, T. Konstantopoulos: *Stationnary solutions of stochastic recursions describing discrete event systems,* Stochastic Processes and their applications 68, 1997, pp. 181-194, Elsevier.

[3] Cécile Appert and Ludger Santen *Modélisation du trafic routier par des automates cellulaires*, Actes INRETS 100, Ecole d'automne de Modélisation du Trafic Automobile 2002.

[4] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat: *Synchronization and Linearity*, Wiley (1992).

[5] R. Barlovic, T. Huisinga, A. Schadschneider, and M. Schreckenberg: *Adaptive Traffic Light Control in the ChSch Model for City Traffic* in Proceedings of the "Traffic and Granular Flow 03" Conference, Springer-Verlag, 2005.

[6] N. Berglund: *Geometrical Theory of Dynamical Systems* ArXiv:math (2001).

[7] M. Blank: *Variational principles in the analysis of traffic flows*, Markov Processes and Related Fields, pp.287-305, vol.7, N.3 (2000).

[8] E. Brokfeld, R. Barlovic, A. Schadschneider, M. Schreckenberg:*Optimizing traffic lights in a cellular automaton model for city traffic*, Physical Review E, volume 64, 2001.

[9] D. Chowdhury, L. Santen, A. Shadschneider: *Statistical physics of vehicular traffic and some related systems.* Physics Report 329, pp. 199-329, 2000.

[10] G. Cohen, S. Gaubert, J.-P. Quadrat: *Asymptotic Throughput of Continuous Petri Nets* Proceedings of the 34th CDC New Orleans Dec. 1995.

[11] J. Cochet-Terrasson, S. Gaubert: *A policy iteration algorithm for zero sum games with mean payoff* C.R.A.S. 343(5): 377-382, 2006.

[12] J.A. Cuesta, F.C. Martinez, J.M. Molera, A. Sanchez: *Phase transition in two dimensional traffic-flow models* Physical Review E, Vol. 48, N.6, pp. R4175-R4178, 1993.

[13] R. David, H. Alla: *Discrete, Continuous and Hybrid Petri Nets* Springer, 2005.

[14] B. Derrida: *An exactly soluble non-equilibrium system: the asymmetric simple exclusion process*, Physics Reports 301, 65-83 (1998).

[15] C. Diadaki, M. Papageorgiou, K. Aboudolas: *A Multivariable regulator approach to traffic-responsive network-wide signal control* Control Eng. Practice N. 10, pp. 183-195, 2002.

[16] N. Farhi, M. Goursat, J.-P. Quadrat: *Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling*, in Proceedings IEEE-CDC, 2005, Seville (2005).

[17] N. Farhi, M. Goursat, J.-P. Quadrat: *Fundamental Traffic Diagram of Elementary Road Networks algebra and Petri net modeling*, in Proccedings ECC-2007, Kos, Dec. 2007.

[18] N. Farhi: *Modélisation minplus et commande du trafic de villes régulière*, thesis dissertation, Université de Paris 1 Panthéon - Sorbonne, 2008.

[19] D. Helbing: *Traffic and related self-driven many-particle systems*, Reviews of modern physics, Vol. 73, pp.1067-1141, October 2001.

[20] Fukui M., Ishibashi Y.: *Phase Diagram on the Crossroad II: the Cases of Different Velocities*, Journal of the Physical Society of Japan, Vol. 70, N. 12, pp. 3747-3750, 2001.

[21] Fukui M., Ishibashi Y.: *Phase Diagram on the Crossroad*, Journal of the Physical Society of Japan, Vol. 70, N. 9, pp. 2793-2797, 2001.

[22] Fukui M., Ishibashi Y.: *Phase Diagram for the traffic on Two One-dimensional Roads with a Crossing*, Journal of the Physical Society of Japan, Vol. 65, N. 9, pp. 2793-2795, 1996.

[23] H. Furstenbeg : *Strict Ergodicity and Transformation of the Torus*, American Journal of Mathematics, Vol. 83, No. 4, pp. 573-601, Oct. 1961.

[24] S. Gaubert and J. Gunawerdena: *The Perron-Frobenius theorem for homogeneous monotone functions*, Transacton of AMS, Vol. 356, N. 12, pp. 4931-4950, 2004.

[25] L. Libeaut: *Sur l'utilisation des dioïdes pour la commande des systèmes à événements discrets*, Thèse, Laboratoire d'Automatique de Nantes (1996).

[26] J. Lighthill, J. B. Whitham: *On kinetic waves: II) A theory of traffic Flow on long crowded roads*, Proc. Royal Society A229 p. 281-345 (1955).

[27] P. Lotito, E. Mancinelli and J.P. Quadrat: *A Minplus Derivation of the Fundamental Car-Traffic Law*, Inria Report Nov. 2001 and in IEEE Automatic Control V.50, N.5, p.699-705 May 2005.

[28] K. Petersen, *Ergodic theory*, Cambridge University Press, 1983.

[29] E. Mancinelli, Guy Cohen, S. Gaubert, J.-P. Quadrat, E. Rofman: "On Traffic Light Control of Regular Towns" INRIA Report Sept. 20001.

[30] J. Mallet-Paret, R. Nussbaum: *Eigen values for a class of homogenous cone maps arising from max-plus operators*. Discrete and Continuous Dynamical Systems 8(3):519-562, July 2002.

[31] F.C. Martinez, J.A. Cuesta, J.M. Molera, R. Brito: *Random versus deterministic two-dimensional traffic flow models* Physical Review E, Vol. 51, N. 2, pp. R835-R838, 1995.

[32] J.M. Molera, F.C. Martinez, J.A. Cuesta, R. Brito: *Theoretical approach to two-dimensional traffic flow models* Physical Review E, Vol.51, N.1, pp. 175-187, 1995.

[33] T. Murata: *Petri Nets: Properties, Analysis and Applications* Proceedings of the IEEE, Vol. 77, No. 4, pp. 541-580, 1989.

[34] K. Nagel, M. Schreckenberg: *A cellular automaton model for free way traffic*, Journal de Physique I, Vol. 2, No. 12, pp. 2221-2229, 1992.

[35] I. Prigogine, R. Herman: *Kinetic Theory of Vehicular Traffic*, Elsevier (1971).

[36] J.-P. Quadrat, Max-Plus Working Group: *Min-Plus Linearity and Statistical Mechanics*, Markov Processes and Related Fields, Vol.3, N.4, p.565-587, 1997.

[37] http://www.scilabgtk.org/