EXPLICIT COMPUTATION OF A MAXPLUS LYAPUNOV EXPONENT GIVING THE AVERAGE SPEED ON A CIRCULAR TRAFFIC LINE WITHOUT OVERTAKING

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ABSTRACT. We give a stochastic maxplus model of the traffic on a circular road without overtaking. The average speed is a maxplus Lyapounov exponent. We obtain a complete characterization of the stationary regime. Based on this characterization we obtain an explicit formula for the average speed and a very simple asymptotic result when the number of cars grows to infinity. We present numerical simulations of the evolution of the system using the maxplus toolbox of Scilab which confirm the theoretical results.

1. INTRODUCTION

We consider N cars in a circular road of unitary length. The car are allowed to move at velocities w and v (with $w < v$) chosen randomly and independently with probability (μ, λ) ($\mu \stackrel{\text{def}}{=} 1 - \lambda$). Because we are in the discrete time case, the velocities w and v represent also the distance covered during a time step. As the cars cannot make more than one turn at each step we suppose that $v < 1$. Without lost of generality we can suppose that $w = 0$.

We consider the case where each car is not allowed to overtake other cars. We compute explicitely the mean speed of the cars which turns out to be the Lyapunov exponent of a stochastic maxplus linear system. Numerical results confirm the validity of the formula obtained.

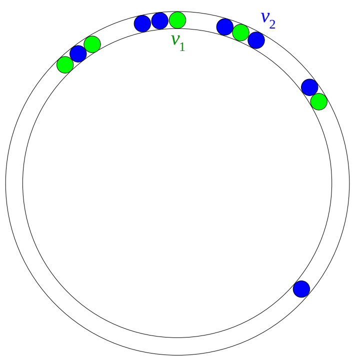

FIGURE 1. Traffic line without overtaking.

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In general it is very difficult to compute a Lyapounov exponent. In our case we are able to characterize completely the stationary regime composed of $1/v$ clusters of cars with a uniform distribution of the N cars in these clusters. From this characterization it is straightforward to obtain the Lyapounov exponent.

2. MODELLING

Let x_n^t be the position of the n-car at time t measured over the circle from a fixed starting point and v_n^t the wanted-speed (a wanted-speed becomes effective only if no car ahead prevents to reach this speed) of the n-car at the time t (0 or v). In this model we consider that the cars have 0 size. Therefore two cars may be in the same position. The dynamic of the system is

$$x_n^{t+1} = \begin{cases} \min\left(v_n^t + x_n^t\,,\; x_{n+1}^{t+1}\right)\;, & \text{if } n < N\,, \\ \min\left(v_N^t + x_N^t\,,\; 1 + x_1^{t+1}\right)\;, & \text{if } n = N\,. \end{cases} \tag{1}$$

This system is linear in the sense of the minplus algebra.

The minplus algebra is by definition [1] the set $\mathbb{R} \cup \{+\infty\}$ together with the laws min (denoted by $\oplus$) and $+$ (denoted by $\otimes$). The element $\epsilon = +\infty$ satisfies $\epsilon \oplus x = x$ and $\epsilon \otimes x = \epsilon$ (ϵ acts as zero). The element $e = 0$ satisfies $e \otimes x = x$ (e is the unit). The main discrepancy with the conventional algebra is that $x \oplus x = x$. We denote $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}\,, \oplus, \otimes)$ this structure. $\mathbb{R}_{\min}$ is a special instance of diod (semiring whose addition is idempotent). This minplus stucture on scalars induces a dioid structure on square matrices with matrix product $A \otimes B$ for two compatible matrices with coefficients in $\mathbb{R}_{\min}$ defined by $(A \otimes X)_i = \min_j \left(A_{ij} + B_{jk}\right)$. Then the unit matrix is denoted E.

Within this algebra the formula (1) becomes

$$x_n^{t+1} = \begin{cases} v_n^t \otimes x_n^t \oplus x_{n+1}^{t+1}\;, & \text{if } n < N\,, \\ v_N^t \otimes x_N^t \oplus 1 \otimes x_1^{t+1}\;, & \text{if } n = N\,. \end{cases} \tag{2}$$

Defining

$$X^t = \begin{pmatrix} x_1^t \\ \vdots \\ x_N^t \end{pmatrix}, \qquad A = \begin{pmatrix} \epsilon & e & & \\ & \ddots & \ddots & \\ & & \ddots & e \\ 1 & & & \epsilon \end{pmatrix}, \qquad B^t = \begin{pmatrix} v_1^t & & & \\ & v_2^t & & \\ & & \ddots & \\ & & & v_N^t \end{pmatrix}$$

where the coefficients not stated are ϵ , we can rewrite the equations in a vectorial way

$$X^{t+1} = A \otimes X^{t+1} \oplus B^t \otimes X^t\;. \tag{3}$$

From the minplus algebra theory we know that if there are no circuits with negative weight in $G(A)$, the incidence graph of A, then the series $A^* = E \oplus A \oplus A^2 \cdots = E \oplus A \oplus A^2 \cdots \oplus A^N$. In our case A^* is easy to calculate

$$A^* = \begin{pmatrix} e & e & \cdots & e \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & e \\ 1 & \cdots & 1 & e \end{pmatrix}.$$

Then

$$X^{t+1} = C^t \otimes X^t \tag{4}$$

with $C^t = A^* \otimes B^t$

Using the fact that the matrices C^t are all irreducibles we know by Cor. 7.31 of [1] that :

$$\lim_t x_n^t / t = \bar{v}, \forall n \ .$$

Then $\bar{v}$ is called the Lyapounov exponent of the stochastic maxplus matrix C (whose C^t are independent samples).

Computing explicitely the Lyapounov exponent is a difficult task. In [4] explicit formulas involving computation of expectations are given. Here we are able to characterize the stationary regime of X^t and to compute explicitely the expectation appearing in $\bar{v}$.

3. Jam regime

In order to represent the state of the system we use diagrams where :

- each segment outside the outer circle represents the amount of cars in that position;
- the black [resp. grey]segments betwen the circles are proportional to the proportion of cars with wanted speed 0 [resp. v];
- the cars numbered 1, $N/2$ and N are represented by a light-grey, grey and dark dot respectively.

In the Figure 2 we show the evolution of the system for 100 cars with speeds 0 and $v = 1/3$, at times $t = 0, 10, 100, 1000$

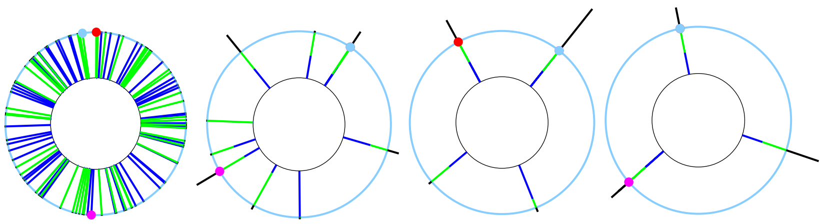

FIGURE 2. Example of evolution of the system (v=1/3).

In the Figure 2 we show the evolution of the system for 50 cars with speeds 0 and $v = 0.3$, at times $t = 0, 10, 100, 500$.

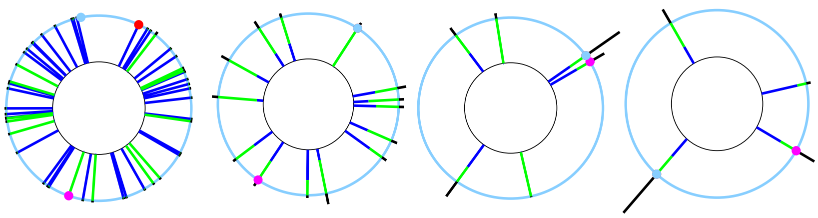

FIGURE 3. Example of evolution of the system (v=0.3).

DEFINITION 1.

1. A *jam state* is a state where the cars are concentrated in a number k of clusters (where $k = \lceil \frac{1}{v} \rceil$ is the upper round of $1/v$) $\{\pi_1, \cdots \pi_k\}$ with $\pi_{i+1} - \pi_i = v$ for $i = 1, \ldots, k-1$. When $1/v \in \mathbb{N}$ we say that the jam state is *regular*.
2. When for all $t \geq T$ the system stays in jam states we say that after T the system is in a *jam regime*.

PROPOSITION 2. *A jam state is characterized by* $d_v(x) = 0$ *with :*

$$d_v(x) = \min_h \left(\sum_{j \neq h} \left\{ \frac{x_{j+1} - x_j}{v} \right\} \right); \ \textit{with} \quad \{x\} = x - [x] \ , \tag{5}$$

where $[x]$ *denotes the integer part of* x *and therefore* $\{x\}$ *denotes the decimal part of* x*. For non jam states we have* $d_v(x) > 0$*. Moreover*

$$d_v(X^T) = 0 \Rightarrow d_v(X^t) = 0, \ \forall t \geq T \ ,$$

that is after to be entered in a jam state we stay in a jam regime.

Proof. It is easy to see that $d_v(x) = 0$ for a jam state x. The question is then to show the converse. Let us suppose that $d_v(x) = 0$ by definition of d_v there is an h_0 such that

$$\sum_{j \neq h_0} \left\{ \frac{x_{j+1} - x_j}{v} \right\} = 0 \, ,$$

then for every $j \neq h_0$ we have that

$$\left\{ \frac{x_{j+1} - x_j}{v} \right\} = 0 \, .$$

So we are in the jam state.

After having reached a jam state the system stays in a jam regime because the wanted deplacement size of the cars are v or 0. In a jam state only two clusters atmost h and $h+1$ may be at a distance different of v. After a deplacement of only one car there is only two possibilities : – the cluster stays in the same position – the cluster $h+1$ change in such a way that the distance of cluster h and $h+1$ becomes v. Then the new state is still a jam state with one cluster in a new position $\pi_{h+1} - \pi_h = v$ and $\pi_{h+2} - \pi_{h+1} \leq v$. □

The function $d_v(x)$ can be seen as a kind of distance to a jam regime.

Some numerical experiments (Figure 4) suggest the following result which is not proved.

Conjecture : The sequence $\left(d_v(X^t)\right)_t$ is non increasing.

THEOREM 3. *A jam regime is reached with probability one.*

Proof. In order to prove that jam regime is attainable we construct a finite sequence of independent event with positive probability after which the system reaches a jam state. Then this finite sequence will appear with probability one in an infinite sequence of events.

The dynamic of the system is given by the matrix $C(\omega) = A^* B(v(\omega))$ where B is the diagonal matrix of car wanted-speeds chosen randomly and independently among 0 and v. Let us consider the matrix C_j associated to the speed $v = (0 \cdots 0, v, 0 \cdots 0)$ with v in position j. All the matrices C_j, $j = 1 \cdots N$ have a strictly positive probability of occurrence.

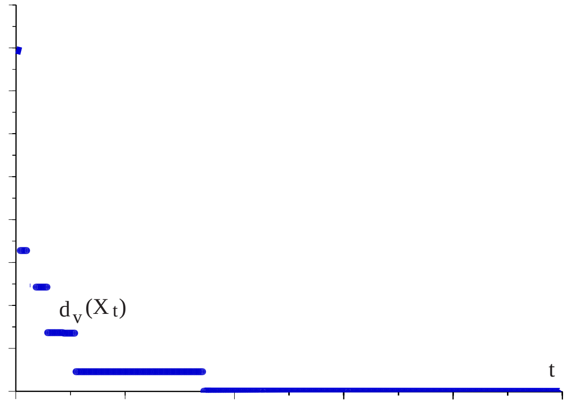

FIGURE 4. Evolution of $d_v(X^t)$ with t.

Consider the finite sequence of independent events associated to the following matrix product

$$C_1^k C_2^k ......C_{N-2}^k C_{N-1}^k \ ,$$

it is easy to understand that after these events all the cars are together in only one cluster which is a jam state. Then the proposition gives the result. $\square$

This proof suggest that the time needed to reach a jam regime is very long. In fact the system meet other jam states much more probable. The particular jam state in the proof has only the property to be easily characterized.

4. THE STATIONARY DISTRIBUTION OF THE CARS

Let us determine the stationary distribution of the population of cars in the k clusters denoted $\mathcal{N} = (N_1 \cdots N_k)$.

THEOREM 4. *The stationary distribution of $\mathcal{N}$ is uniform on the simplex :*

$$S = \left\{ (n_1 \cdots n_k) \ | \ \sum_{i=1,k} n_i = N, \ n_i \in \mathbb{N} \right\} .$$

Proof. Let us consider the Markov chain where the states belongs to the set solutions of the previous diophantic equation. Then we have $\mathbb{C}_N^{N+k-1}$ nodes, where k is the number of clusters. Let us show that for each outcoming arc in a node with transition probability p there is an incoming arc with the same transition probability (which show that the transition matrix is bistochastic). This property is clearly a local balance property.

To prove this local balance property let us consider the state $(n_1 \cdots n_k)$ then all the possible transitions following it are of the form

$$(n_1 - d_1 + d_k, \cdots, n_k - d_k + d_{k-1}) \quad \text{with } 0 \le d_j \le N_j$$

this means that there are d_j cars that leave the cluster j to the cluster $j+1$. The probability of that event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(d_j, n_j)} \text{ where } \phi(d_j, n_j) = \begin{cases} 0 & \text{if } d_j = n_j \\ 1 & \text{otherwise} \end{cases} .$$

If we consider now the state $(n_1 - d_1 + d_2 \cdots n_k - d_k + d_1)$ then we can reach the state $(n_1 \cdots n_k)$ making leave d_2 cars from the cluster 1, d_3 cars from the cluster 3 and so on until the last one in which we make leave d_1 cars. Now the probability of this event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(d_{j+1}, n_j - d_j + d_{j+1})}$$

but $\phi(d_{j+1}, n_j - d_j + d_{j+1}) = \phi(d_j, n_j)$and so we have the same probability.

To finish the proof we have to show that this construction which associates to each output arc an input one is a bijective mapping. For that, since the correspondance is injective, let us show that the number of outgoing arcs to a particular state $\mathcal{N} = (n_1 \cdots n_k)$ is equal to the number of incoming arcs to this state. The number outgoing arcs of $\mathcal{N}$ is the number of elements of the set

$$\{(d_1 \cdots d_k) \ | \ 0 \leq d_i \leq n_i, \ i = 1 \cdots k\} \ .$$

The number of incoming arcs of $\mathcal{N}$ is the number of elements of the set

$$\left\{(d_1 \cdots d_k) \ | \ 0 \leq n_i' - d_i + d_{i-1} \leq n_i, \ n_i' - d_i \leq 0, \ i = 1 \cdots k\right\} \ .$$

These two numbers are equal because

$$0 \leq n_i' - d_i + d_{i-1} \leq n_i, \quad n_i' - d_i \leq 0, \quad 0 \leq n_i' \ \Leftrightarrow \ 0 \leq d_{i-1} \leq n_i \ .$$

□

5. Computation of the mean speed

Knowing the stationary measure we are able to compute explicitely the mean speed.

THEOREM 5. *If* $k \stackrel{\text{def}}{=} 1/v \in \mathbb{N}$ *then the mean speed satisfies*

$$\bar{v}_\lambda(N, k) = \frac{\lambda v}{N\mu}(k - S_k(N))$$

with

$$(N + k)S_k(N + 1) = k - 1 + (N + 1)\lambda S_k(N), \ \ S_k(0) = k, \ \forall k, N \in \mathbb{N} \ .$$

Moreover for large N *we have the asymptotic*

$$\bar{v}_\lambda(N, k) = \frac{\lambda}{N\mu} + o(1/N) \ .$$

EXAMPLE 6. 1. $\bar{v}_\lambda(3, 3) = v(6\lambda + 3\lambda^2 + \lambda^3)/10$.
2. $\bar{v}_\lambda(4, 4) = v(\lambda^4 + 4\lambda^3 + 10\lambda^2 + 20\lambda)/35$.

In Figure 5 we show a plot of the mean speed as a function of N and k when $\lambda = 0.5$.

Proof. Knowing the distribution of $\mathcal{N}$ let us compute the mean speed in the following way : the first car of a queue leaves with probability λ increasing the mean speed of v/N, then the second car leaves this queue with probability λ^2 increasing the mean speed of v/N and so on. Then the mean speed $\overline{v} = \mathbb{E}(V)$ with

$$V = \sum_{s=1}^{k} \left(\sum_{j=1}^{N_s} \lambda^j \frac{v}{N} \right) ,$$

that is

$$V = \lambda \frac{v}{N} \sum_{s=1}^{k} \frac{1 - \lambda^{N_s}}{1 - \lambda} = \frac{\lambda}{\mu} \frac{v}{N} \left(k - \sum_{s=1}^{k} \lambda^{N_s} \right) .$$

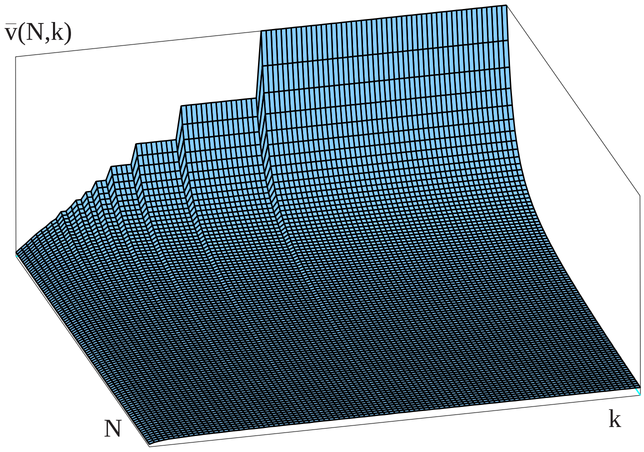

FIGURE 5. $\bar{v}_{0.5}(N,k)$.

If $k=1$ we easily obtain that

$$V=\frac{\lambda-\lambda^{N+1}}{1-\lambda}\frac{v}{N}.$$

Let us assume that $k\geq 2$ and let us denote

$$S_k\left(N\right)=\mathbb{E}\left(\sum_{s=1}^{k}\lambda^{N_s}\right)=\frac{1}{\complement_N^{N+k-1}}\sum_{\sum_s N_s=N}\lambda^{N_s}=\frac{1}{\complement_N^{N+k-1}}\sum_{s=1}^{k}\sum_{h=0}^{N}\sum_{\sum_{j\neq s}N_j=N-h}\lambda^h .$$

Then counting we obtain :

$$S_k(N)=\frac{Z(N)}{\complement_N^{N+k-1}}\text{ with }Z(N)=\sum_{h=0}^{N}\complement_{N-h}^{N+k-h-2}\lambda^h .$$

If we call $z=1/\lambda$ and D_z the derivative with respect to z we obtain that

$$Z(N)=\frac{\lambda^N}{(k-2)!}\sum_{h=0}^{N}D_z^{k-2}\,z^{N+k-h-2}\ . \tag{6}$$

Therefore

$$Z(N+1)=\frac{\lambda^{N+1}}{(k-2)!}\sum_{h=0}^{N+1}D_z^{k-2}\,z^{N+1+k-h-2}\,,$$

$$Z(N+1)=\frac{\lambda^{N+1}}{(k-2)!}D_z^{k-2}\left(z^{N+1+k}+\sum_{h=0}^{N}z^{N+k-h-2}\right),$$

$$Z(N+1)=\frac{\lambda^{N+1}}{(k-2)!}\left(D_z^{k-2}z^{N-1+k}+\sum_{h=0}^{N}D_z^{k-2}\,z^{N+k-h-2}\right),$$

but from (6) we have that

$$Z(N+1)=\frac{\lambda^{N+1}}{(k-2)!}\left(D_z^{k-2}z^{N-1+k}+\complement_N^{N+k-1}(k-2)!\lambda^{-N}S_k(N)\right)\ .$$

Computing the derivative and simplifying we obtain

$$S_k(N+1) = \frac{k-1}{N+k} + \frac{N+1}{N+k}\lambda S_k(N).$$

which proves the first part of th theorem.

To find the asymptotic we make the approximation $\mathcal{N}$ is always ∞ in the computation of $S_k(N)$. □

In Figure 6 we show a simulation of X_t/t converging towards the computed Lyapounov exponent $\bar{v}$.

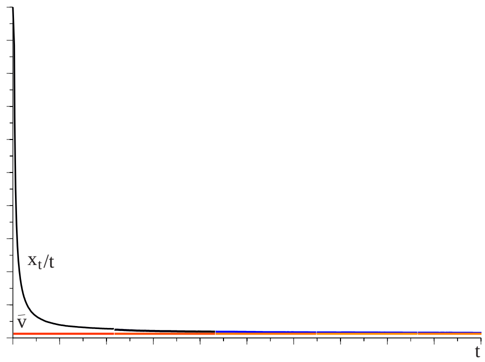

FIGURE 6. Convergence of X_t/t towards $\bar{v}$.

6. CONCLUSION

The model presented here can be extented to the case where the cars have a non negligible size l. The model is still minplus linear :

$$x_n^{t+1} = \begin{cases} v_n^t\, x_n^t \oplus m x_{n+1}^{t+1}\ , & \text{if } n < N \\ v_N^t x_N^t \oplus d x_1^{t+1}\ , & \text{if } n = N \end{cases}$$

with $m = -l$ and $d = 1 - l$. The same kind of analysis can be developped.

The formula giving the mean speed can be extended to the case where $1/v \notin \mathbb{N}$ and when overtaking is allowed. These extensions will be done in future works.

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