THE DISCRETE STATE LINEAR QUADRATIC PROBLEM

Jean-Pierre QUADRAT
INRIA, Domaine de Voluceau, BP 105, 78153 Rocquencourt, France

April 30, 2010

Abstract

We pose and solve the control problem for Discrete State Markov chains when : — the control influence in an affine way the transition probabilities, — the cost is quadratic with respect to the state and the control.

In this case the dynamic programming equation becomes a Riccati one. This type of problem appears in particular when we discretize the Hamilton Jacobi Bellman equation (HJB). Then we discuss the order of approximation of some schemes of discretization of the HJB equation in connection with the previous results.

Introduction

The Riccati equation appears classicaly in two cases : — continuous time, linear system, quadratic cost; — discrete time, linear system, quadratic cost.

We discuss here the control of a discrete Markov chain, where — the transition probabilities are affine functions of the control — the cost is quadratic with respect to the control and the state.

The starting point of this discussion comes from the difficulty to solve numerically HJB equation. It appears (up to our knowledge) that the LQ Discrete state problem have not been studied in the Markov chains litterature. But in this particular case a system of equations of the Riccati type solves completely the problem. More precisely the system is composed of a Riccati equation and of two other ones describing the evolution of the affine term which is present in general.

The affine dependence of the transition with respect to the control can induce some difficulties with the necessary positivity of the transition probabilities. We shall see on an example that the relaxation of this constraint leads to a natural continuation of our initial problem which does not induce any change to the solution — in the domain where the positivity is verified. It is equivalent to add a boundary condition which does not destroy the structure of the solution of the Bellman equation, the domain being chosen in such a way that the positivity is guaranted.

Such a kind of Markov chains appears when we discretize a stochastic control problem to solve numerically the dynamic programming equation. The previous result induces to think that sometimes this discrete problem can be solved explicitely thus on these examples we can discuss the order of approximation more precisely. Thus we review the classical ways of discretizing the HJB equation and propose some modifications of some classical schemes in a such way that the continuous version and the discrete one coincides. That means more generally that we have obtained a scheme of approximation of order 2 which does not depend of the two first derivatives of the unknown. The positivity of the transition is essential to assure the stability of the scheme. We discuss schemes for which the positivity is assured. In the degenerate case (small noise) we have to choose between the order 2 and the stability. We can avoid the difficulty and obtain a 2 order precision by adding one iteration of a fixed point based on a modified operator obtained by adding noise (viscosity).

1 The discrete state linear quadratic problem

1.1 The model

Let us define precisely a control problem for discrete Markov chains which plays a role similar to the LQ problem.

Let us consider : — the state space R^n, — the control space R^m, — a set of vectors of R^n noted $[c^1, c^2, \ldots, c^k] = C$, — a (k, n) matrix A such that $\mathbf{1}'A = 0$, — a (k, m) matrix B such that $\mathbf{1}'B = 0$, — a k vector p such that $\mathbf{1}'p = 1$.

The transition matrix M of a Markov chain living in $R^n \quad (x \in R^n, x' \in R^n)$ is then defined by :

$$M_{x,x+c^i}(u) = (Bu + Ax + p)_i, i = 1, \ldots, k \tag{1}$$

This means that there are k possible jumps of "length" $(c^1, \ldots, c^k)$ with affine transition probabilities with respect to the state and the control.

In principle the matrix m must have its coefficients, ≥ 0, this is compatible with (1) only on a subset of R^n. Let us suppose for the moment that the Markov chain is defined only on this subset (we shall discuss more precisely this point later).

We want to minimize a criterion with an instantaneous quadratic cost $u'Qu + x'Rx$ on a finite horizon, where Q is a (m, m) positive definite matrix and R a (n, n) positive one. Thus we want minimize :

$$\bigwedge_u E \sum_{t=0}^{T-1} (U_t'QU_t + X_t'RX_t) \tag{2}$$

Let us denote by $v(t, x)$ the Bellman function associated to the problem (2) :

$$v(t,x) = \bigwedge_{u_{t+1},\dots,u_T} E\left\{\sum_{s=t+1}^{T} U_s' Q U + X_s' R X_s \mid X_t = x\right\} \tag{3}$$

It satisfies the HJB equation

$$v(t,x) = \bigwedge_u \left\{u'Qu + x'Rx + \sum_{i=1}^{k} M_{x,x+c^i}(u) v(t+1, x+c^i)\right\} \tag{4}$$

that is :

$$\left\{\begin{array}{rcl} v(t,x) & = & \bigwedge_u (x'Rx + u'Qu \\ & & + \sum_{i=1}^{k} (Ax+Bu+p)_i v(t+1,x+c^i)\ ) \\ v(T,x) & = & 0 \end{array}\right. \tag{5}$$

1.2 The Riccati equation

Let us show that the solution of (5) can be written :

$$v(t,x) = x'P_t x + f_t' x + g_t \tag{6}$$

with P_t (n,n)matrix, $f_t \in R^n$ and $g_t \in R$.

Theorem 1 *$v(t,x)$ solution of (5) has the structure (6) with P_t, f_t, g_t satisfying the recurrent equations :*

$$\begin{array}{rcl} P_t & = & P_{t+1} + R - P_{t+1}CBQ^{-1}B'C'P_{t+1} \\ & & + A'C'P_{t+1} + P_{t+1}CA \qquad (7) \\ P_T & = & 0 \\ f_t' & = & f_{t+1}' - (f_{t+1}'C + s')BQ^{-1}B'C'P_{t+1} + s'A \\ & & + 2p'C'P_{t+1} + f_{t+1}'CA \qquad (8) \\ f_T & = & 0 \\ g_t & = & g_{t+1} - \frac{1}{4}(f_{t+1}'C' + s')BQ^{-1}B'(Cf_{t+1} + s) \\ & & + f_{t+1}'Cp + s'p \qquad (9) \\ g_T & = & 0 \end{array}$$

where s is the diagonal of the matrix $C'P_{t+1}C$ seen as a vector.

Proof The result is obtained by induction on t starting with T. Let us suppose that the result is true for $t+1,\dots,T$, thus we have :

$$\begin{array}{l} \sum_i (Ax+Bu+p)_i v(t+1,x+c^i) \\ = x'(P_{t+1} + A'C'P_{t+1} + P_{t+1}CA)x \\ +2u'\left[B'C'P_{t+1}x + \frac{1}{2}B's + \frac{1}{2}B'C'f_{t+1}\right] \\ +x'\left[A's + f_{t+1} + A'C'f_{t+1} + 2P_{t+1}Cp\right] \\ +g_{t+1} + f_{t+1}'Cp + s'p \end{array}$$

because ($1'A = 0, 1'B = 0$) $\Rightarrow$ (all the terms of degree 3, $u'B'1f_{t+1}'x$ and $x'A'1f_{t+1}'x$ are equal to zero).

Thus the optimal control satisfies :

$$u^* = -Q^{-1}\left[B'C'P_{t+1}x + \frac{1}{2}s + \frac{1}{2}B'C'f_{t+1}\right].$$

Thus we obtain (7),(8),(9).

Corollary 1 *On the conditions :*

$$\sum_q c_i^q c_j^q b_q^l = 0 \quad \forall i,j = 1,\dots,n, \forall l = 1,\dots,m \tag{10}$$

$$\sum_q c_i^q c_j^q a_q^l = 0 \quad \forall i,j,l = 1,\dots,n \tag{11}$$

$$\sum_q c_i^q p_q = 0 \quad \forall i = 1,\dots,n \tag{12}$$

$v(t,x) = x'P_t x + g_t$ with P_t et g_t given by (7) and (9)

Proof On the condition (10), (11), (12) by integration of (8) we see that $f_t = 0$.

Remark We shall see later that the conditions (10), (11), (12) are satisfied for some symetric discretization of the HJB equation.

1.3 Remark on the positivity of the transition matrix

To obtain the result we need to have a special structure in particular the transition matrix must be linear in$x$. Moreover the optimal control will be linear in x. x belongs in principle to the whole space thus $m_{x,x'+c^i}$ cannot in general stay positive everywhere. Thus what is the meaning of the problem solved ?. To understand what happens let us consider the simplest problem having the difficulty (even without control) :

$$v_t = \left(\frac{1}{2} - \varepsilon x\right) v_t(x+1) + \left(\frac{1}{2} + \varepsilon x\right) v_t(x-1) + x^2, \quad v_T = 0 \tag{13}$$

where : $C = [1,-1], A' = [-\varepsilon, +\varepsilon], p = [\frac{1}{2}, +\frac{1}{2}], B = 0, Q = 0, R = 1, n = 1, m = 0, k = 2$. The solution of (13) is

$$\left\{\begin{array}{rcl} P_t & = & P_{t+1} - 4\varepsilon P_{t+1} + 1 \\ g_t & = & g_{t+1} + P_{t+1} \end{array}\right.$$

But $\alpha = \frac{1}{2} + \varepsilon x \geq 0$ and $\beta = \frac{1}{2} - \varepsilon x \geq 0$ for $|x| \leq \frac{1}{2\varepsilon}$.

Now let us suppose that $\varepsilon = \frac{1}{2N}$, N integer then α et $\beta \geq 0$ for $|x| \leq N$ and the Markov chain is defined on $[-N,N]$ and it has a probability 0 to leave this zone. The values outside can be seen as a continuation of the value function without any infuence on the zone $[-N,N]$.

In some general case when the “negative” zone can have a small influence. In this case we can imagine that we stop the Markov chains, when we leave the “positive” zone, with a quadratic cost — which is the continuation of the quadratic cost completely defined by the zone of positivity.

1.4 Example

Let us consider a problem in dimension 2 which comes from the discretization of a diffusion process that we shall discuss in more details in the next part :

$$\begin{aligned} v_t(x,y) = &\alpha[v_{t+1}(x+1,y) + v_{t+1}(x,y+1) + v_{t+1}(x-1,y) \\ &+v_{t+1}(x,y-1)] \quad + \quad (1-4\alpha)v_{t+1}(x,y) + (x+y)^2 \\ &+ \bigwedge_u \{(-x+u_1)\beta(v_{t+1}(x+1,y) - v_{t+1}(x-1,y)) \\ &\qquad +(-y+u_2)\beta(v_{t+1}(x,y+1) - v_{t+1}(x,y-1)) \\ &\qquad +u_1^2 + u_2^2\} \end{aligned}$$

we have :

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad p' = [\alpha, \alpha, \alpha, \alpha, 1-4\alpha]$$

$$B = \begin{bmatrix} \beta & 0 \\ -\beta & 0 \\ 0 & \beta \\ 0 & -\beta \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -\beta & 0 \\ \beta & 0 \\ 0 & -\beta \\ 0 & \beta \\ 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad n=2, m=2, k=5$$

We are in the case of the corollary because (10), (11), (12) are verified thus $f_t = 0\ \forall t$, let us compute the equation (8) and (9) we obtain :

$$\begin{cases} P_t &= P_{t+1}(1-4\beta) - 4\beta^2 P_{t+1}^2 + R \\ g_t &= g_{t+1} + 2\alpha \text{ trace } (P_{t+1}) \end{cases}$$

2 The discretization of the HJB equation

We consider the classical control of diffusion processes :

- $dX_t = b(X_t, U_t)dt + \sqrt{2\varepsilon}dW_t, X(t) \in R^n, U_t \in R^m, W_t \in R^n$ independant brownian motions,
- $\bigwedge_U E \int_0^\infty e^{-\lambda t} c(X_t, U_t)dt$ a discounted cost to be optimized.

The Bellman function defined by :

$$v(x) = \bigwedge_U E \left\{ \int_0^\infty e^{-\lambda t} c(X_t, U_t) dt \mid X(0) = x \right\}$$

satisfies the HJB equation :

$$-\lambda v + \bigwedge_u (b(x,u).\nabla v + c(x,u)) + \varepsilon^2 \Delta v = 0 \quad \forall x \in R^n \quad (14)$$

Sometimes we stop the process when it leaves an open bounded set σ of R^n. Let us call τ_σ the corresponding stopping time. This once the criterion to optimize is $\bigwedge_U E \left(\int_0^{\tau_\sigma} e^{-\lambda t} c(X_t, U_t) dt \right)$ and the HJB equation becomes : (14) $\forall x \in \sigma$, with the boundary conditions $v \mid_{\partial\sigma} = 0$, we have $\partial\sigma$ denotes the boundary of σ.

In the particular case : — linear system $b(x,u) = Ax + Bu$ — quadratic cost, $c(x,u) = u'Qu + x'Rx$ (LQG problem)(14) admits at least one quadratic solution $x'Px + g$ where P satisfies the Riccati equation :

$$\begin{cases} -\lambda P + A'P + PA - PBQ^{-1}B'P + R = 0 \\ -\lambda g + 2\varepsilon^2 tr(P) = 0 \end{cases} \quad (15)$$

It must be clear that (15) does not define the only solution of (14) independently of the unicity or not of the solution of (15). For example, the particular case in dimension one :

$$-\frac{1}{4}v_x^2 - v = 0$$

admits the solutions 0 and $-(c+x)^2$. The solution of (15) in this case is $-P - P^2 = 0$ which implies $P = 0$ or $P = -1$ thus the only positive solution of (15) is $P = 0$, which is the only one of interest for the control problem.

For the problem :

$$v + \frac{1}{2}(v_x)^2 = \frac{1}{2}x^2 \quad (16)$$

it is more difficult to compute explicitely the solution, nevertheless we can define a series solving (16), indeed by successive differentiation of (16) we obtain :

$$\begin{cases} v_2v_1 + v_1 = x \\ v_3v_1 + v_2^2 + v_2 = 1 \\ v_4v_1 + 3v_3v_2 + v_3 = 0 \\ v_5v_1 + 4v_4v_2 + 3v_3^2 + v_4 = 0 \\ v_6v_1 + 5v_5v_2 + 10v_4v_3 + v_5 = 0 \\ v_7v_1 + 6v_6v_2 + 15v_5v_3 + 10v_4^2 + v_6 = 0 \\ v_8v_1 + 7v_7v_2 + 21v_6v_3 + 35v_5v_4 + v_7 = 0 \end{cases} \quad (17)$$

where v_i denotes the i^{th} derivative of v. We see that for *an arbitrary* $v(0)$ *such that* $v_1(0) \neq 0$ we can defined $v(x)$ by the series $\sum_{i=0}^\infty x^i \frac{v_i(0)}{i!}$. Indeed on these hypotheses (16) defines $v_1(0)$, (17.1) defines $v_2(0)$, (17.2) defines $v_3(0)$ and so on. The Riccati equation is the particular case where $v_3(0) = 0$ indeed (17.3) in this case gives $v_2^2 + v_2 = 1$ which is the Riccati equation. Nevertheless in the following we shall be interested only by the quadratic, ≥ 0, solution of the HJB equation which will be the limit of the well posed backward Cauchy problem :

$$\frac{\partial v^T}{\partial t} - \lambda v^T + \bigwedge_u (b(x,u).\nabla v^T + c(x,u)) + \varepsilon^2 \Delta v^T = 0, \quad (18)$$

$$v^T(T,x) = 0, \quad (19)$$

$$v(x) = \lim_{T\to\infty} v^T(0,x). \quad (20)$$

Moreover, in the bounded domain case, it is easy, by functional analysis method, to prove the existence and unicity of a global solution A. Bensoussan [2].

In general we cannot solve explicitely (14). One important exception is the LQG problem already discussed. To solve approximatively the problem we discretize it by different ways

— finite differences, — finite elements, — discretization of the stochastic control problem.

We are mainly concerned in this part with the quality of this discretization (the order of the approximation and the stability of the schemes). The speed to compute the solution has been discussed for example in M. Akian, JP. Chancelier, JP. Quadrat [1]. Moreover, after the discretization of a LQG problem we obtain a DLQ one — discussed in the first part. In this particular case we are able to explicit the quadratic solution and thus to well understand the quality of this approximation. Our purpose is to choose, between the different points of view, based on this LQG case.

2.1 The three point of view for the discretization of the HJB equation

Let us explain these three ways of discretization on a simple example :

$$-v - xv' + \bigwedge_u (uv' + u^2) + v'' + x^2 = 0 \tag{21}$$

with or without boundary condition of Dirichlet type :

$$v(0) = v(1) = 0 \tag{22}$$

Computing the minimum in (21) we obtain :

$$-v - xv' - \frac{1}{4}(v')^2 + v'' + x^2 = 0$$

Its quadratic, ≥ 0, solution in the R domain case can be computed explicitely :

$$\begin{cases} v(x) = px^2 + g, \quad \text{with :} \\ 3p + p^2 = 1, \quad f = 0, \quad g = 2p. \end{cases} \tag{23}$$

2.1.1 The finite difference point of view

Its consists in substituting, in the HJB equation, the derivative by finite differences that is

$$\begin{aligned} v'(x) \quad &\rightarrow \quad \frac{v(x+h) - v(x-h)}{2h} = \nabla_h v \quad h \in R_*^+, \\ v''(x) \quad &\rightarrow \quad \frac{v(x+h) - 2v(x) + v(x-h)}{h^2} = \Delta_h v. \end{aligned}$$

We obtain the Discrete HJB equation :

$$-v_h - x\nabla_h v_h + \bigwedge_u (u\nabla_h v_h + u^2) + \Delta_h v_h + x^2 = 0 \tag{24}$$

computing the minimum we obtain

$$-v_h - x\nabla_h v_h - \frac{1}{4}(\nabla_h v_h)^2 + \Delta_h v_h + x^2 = 0$$

It exists a quadratic solution thanks to the first part and its extension to the discounted case. We can verify that *its quadratic positive solution is exactly (23).*

2.1.2 The finite element point of view

Let us explain it in the one dimensional bounded case.

First we take a finite dimensional space V_h of functions approximating the Sobolev space

$$V = H_0^1(\sigma) = \{v \in L^2(\sigma), v' \in L^2(\sigma), v(0) = v(1) = 0\}.$$

Let us denote by : $\mathcal{B}_x = \{\phi_i, i = 1, \dots, N\}$ a basis of V_h the set of "hat" functions (continuous piecewise linear functions $\phi_i(ih) = 1, \phi_i(jh) = 0, \forall j \neq i$).

We want an approximation $v_h = \sum_i v_h^i \phi_i(x)$ of the solution of the HJB equation. For that, in general, we need and approximation of the control space. Let us denote by W_h a piecewise constant approximation of $L^2(\sigma)$ the control space. A basis will be $\mathcal{B}_u = \{\psi_j, j = 0, \dots, N\}$

$$\psi_j(x) = \begin{cases} \frac{1}{h}, & \forall x \in]jh, (j+1)h] \\ 0, & \textit{elsewhere.} \end{cases}$$

and $u_h = \sum_j u_h^j \psi_j(x)$.

Then we define the approximated problem by :

$$\begin{aligned} &(v_h, \phi) + (\nabla v_h, \nabla \phi) + (x\nabla v_h, \phi) \\ &- \bigwedge_{u_h \in W_h} (u_h \nabla v_h + u_h^2, \phi) - (x^2, \phi) = 0, \qquad \forall \phi \in \mathcal{B}_x \end{aligned} \tag{25}$$

We can prove that v_h solving (25) is defined uniquely and that $| v_h - v |_{L^2(\sigma)} \leq h^2$ when

$$v \in H^2(\sigma) = \{v, \quad v \in H^1(\sigma), \quad v'' \in L^2(\sigma)\}.$$

(25) can be computed explicitely it gives after a numerical integration maintaining the order of approximation :

$$\begin{aligned} &- \frac{v_h^{i+1} + 4v_h^i + v_h^{i-1}}{6} + \frac{v_h^{i+1} - 2v_h^i + v_h^i}{h^2} \\ &- \frac{1}{2}x_{i+\frac{1}{2}} \frac{v_h^{i+1} - v_h^i}{h} - \frac{1}{2}x_{i-\frac{1}{2}} \frac{v_h^i - v_h^{i-1}}{h} \\ &+ \frac{1}{2} \bigwedge_{u_h^i} \left[u_h^i \left(\frac{v_h^{i+1} - v_h^i}{h} \right) + (u_h^i)^2 \right] \\ &+ \frac{1}{2} \bigwedge_{u_h^{i-1}} \left[u_h^{i-1} \left(\frac{v_h^i - v_h^{i-1}}{h} \right) + (u_h^{i-1})^2 \right] \\ &+ \frac{1}{2} \left[\left(x_{i+\frac{1}{2}} \right)^2 + \left(x_{i-\frac{1}{2}} \right)^2 \right] = 0 \end{aligned} \tag{26}$$

(27) is also a discrete linear quadratic problem and its solution can be computed explicitely if we don't take into account the boundary condition used only to simplify the explanations of the method :

$$v_h^i = p(ih)^2 + g_h \tag{27}$$

with p defined by (23) thus exact, and :

$$g_h = p \left[2 - h^2 \left(\frac{5}{6} + \frac{p}{4} \right) \right] + \frac{h^2}{4} \tag{28}$$

which *confirm the h^2 error of this scheme.*

2.1.3 The discretization of the stochastic control problem point of view

The initial control problem leading to the HJB equation (23) can be discretized by considering a Markov chain closed of the diffusion process — in the sense of the weak topology of measures — and optimizing a discrete approximation of the cost Kushner [6], Goursat-Quadrat [5].

To build this Markov chain we have mainly to choose a transient measure $\pi_u^x(dy)$ such that :

$$\int (y-x)\pi_u^x(dy) = (u-x)\frac{h^2}{4} + o(h^2) \tag{29}$$

$$\int (y-x)^2\pi_u^x(dy) = \frac{h^2}{2} + o(h^2) \tag{30}$$

where h will be a step of discretization on R used to define the π_u^x only on a discrete mesh of R.

Let us define π by :

$$\begin{aligned}\pi_u^x(dy) &= \delta_{x+h}(y)\left[\frac{1}{4}+\frac{h}{4}(x_-+u_+)\right]\\ &\quad +\delta_x(y)(\frac{1}{2}-\frac{h}{4}(|x|+|u|))\\ &\quad +\delta_{x-h}(y)\left[\frac{1}{4}+\frac{h}{4}(x_++u_-)\right]\end{aligned} \tag{31}$$

and approximate the continuous cost by the following discrete one :

$$\bigwedge_u E\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+\frac{h^2}{4}}\right)^{t+1}\frac{u_t^2+x_t^2}{2}\right] \tag{32}$$

Then the dynamic programming equation associated with the Markov chain control problem (29), (30), (31), (32) with a discounted cost is :

$$\begin{aligned}\left(1+\frac{h^2}{4}\right)v_h^i = \bigwedge_u &\left[\left(\frac{1}{4}+\frac{h}{4}(x_-+u_+)\right)v_h^{i+1}\right.\\ &+(\frac{1}{2}-\frac{h}{4}(|x|+|u|))v_h^i\\ &\left.+\left(\frac{1}{4}+\frac{h}{4}(x_++u_-)\right)v_h^{i-1}+\frac{x^2+u^2}{2}\right]\end{aligned} \tag{33}$$

Using the a priori knowledge of the sign of u, (33) becomes :

$$\begin{cases} -v_h^i & +\Delta_h v_h - x\frac{v_h^i-v_h^{i-1}}{h} \\ & +\bigwedge_u\left[u\frac{v_h^i-v_h^{i-1}}{h}+u^2\right]+x^2=0 \text{ sur } x>0 \\ -v_h^i & +\Delta_h v_h - x\frac{v_h^{i+1}-v_h^i}{h} \\ & +\bigwedge_u\left[u\frac{v_h^{i+1}-v_h^i}{h}+u^2\right]+x^2=0 \text{ sur } x<0 \end{cases} \tag{34}$$

In $x=0$ we have in principle to write the corresponding (33) equation. But to have a piecewise quadratic solution we suppose that in this point the function is obtained by continuation.

With these simplifications we are able to give explicitely the solution of (34). It is :

$$v_h(x) = px^2 + f_h|x| + g_h \tag{35}$$

with p given by (23) and thus is exact,

$$f_h = \frac{ph+p^2h}{2+p}\ , \qquad g_h = 2p - \frac{1}{4}(ph-f_h)^2.$$

Thus *the scheme is of order h* as we already know because this point of view have led to a well known decentered scheme. Its only interest is *its stochastic interpretation whathever the relative order of the drift term and the diffusion term is.*

From this first discussion we can conclude that *the best scheme is the finite difference one* ; indeed, its stability can be assured by chosing h small enough.

2.2 The order of the schemes

To study the quality of the approximation it exists different methods adapted to the point of view used to make the approximation.

For the finite difference method we suppose that the solution is as regular as we want and we use a Taylor development to compute the difference between the discrete operator A_h and the continuous one. Thus the regularity assumptions done are very strong.

This drawback is avoided in the finite element case because the error is computed in the Sobolev space needed to define uniquely the solution. Thus the existence and the error comes together. The counter part is a loss of simplicity of the scheme and the proof.

The stochastic point of view has not led to many studies on the approximation order obtained. The reason is perhaps the difficulty to study the speed of convergence of measures for the weak topology.

Let us use here the finite difference point of view and let us see what happens on the example of the previous part :

$$-v - xv' + v'' - \frac{1}{4}(v')^2 + x^2 = 0 \tag{36}$$

approximated by

$$-v_h - x\nabla_h v_h + \Delta_h v_h - \frac{1}{4}(\nabla_h v_h)^2 + x^2 = 0 \tag{37}$$

developping up to the order 4 $v(x+h)$ and $v(x-h)$ — where v is solution of (36) — substituting in the operator part of (37) we obtain :

$$\begin{aligned}-v - x\nabla_h v + \Delta_h v - \frac{1}{4}(\nabla_h v)^2 + x^2 &= \\ \frac{h^2}{6}(-xv^{(3)} + \frac{v^{(4)}}{2} - \frac{1}{2}v^{(3)}) + o(h^2)&\end{aligned} \tag{38}$$

where in the o definition appears derivatives of v to an order larger than 3. Then if the solution of the continuous problem is quadratic the one of the discrete problem is the same and thus is also quadratic. This remark, well known by numerical analyst, is another proof of the result of the first part of this paper — on this particular example.

From such kind of analysis we can understand why the schemes are not exact in the quadratic case and propose, eventually, some improvements.

The scheme coming from the finite element point of view becomes exact if we substitute the term $\dfrac{v^{i-1}+4v_h^i+v_h^{i+1}}{6}$ by $\dfrac{v_h^{i-1}+6v_h^i+v_h^{i+1}}{8}$.

The stochastic control scheme must be recentered to become exact up to the order 2. The following scheme is exact in the quadratic case or degenerated case (($\varepsilon=0$) thus without the Laplacian term) :

$$-\left(\frac{3}{8}v_h^{i+1}+\frac{3}{4}v_h^i-\frac{1}{8}v_h^{i-1}\right)+\Delta_h v_h - x_{i+\frac{1}{2}}\left(\frac{v_h^{i+1}-v_h^i}{h}\right)$$
$$+\bigwedge_u\left(u\frac{v_h^{i+1}-v_h^i}{h}+u^2\right)+(x_{i+\frac{1}{2}})^2=0 \qquad (39)$$

for $x_i<0$ and a similar equation — denoted (39') — for $x_i>0$. In $x_i=0$ we can write the equation (39) + (39'). With this scheme we lose the stochastic interpretation — the coefficient of v_h^{i+1} may become negative in the degenerate case. In the non quadratic case we can maintain the order 2 to the price of a Laplacian recentering that we shall not do here.

2.3 Stability or stochastic interpretation of the scheme

At this point of the discussion we have concluded that the best scheme is the finite difference one. It is of order 2 in general and exact for the quadratic problem. This conclusion is acceptable when there are enough noise in the system — $\varepsilon\neq 0$, and practically, large enough with respect to the drift term. Let us discuss now the "degenerate" case $\varepsilon=0$. For that let us take a typical example in dimension 1.

$$-v-xv'+\min_u(uv'+u^2)+x^2=0 \qquad (40)$$

that is :

$$-v-xv'-\frac{1}{4}(v')^2+x^2=0 \qquad (41)$$

the solution is $v(x)=px^2$ with p solution of the Riccati equation $+3p+p^2=1$, that is the equation (23). The noise translates of a constant the Bellman function in the quadratic case.

The finite difference scheme :

$$-v_h-x\nabla_h v_h-\min_u(u\nabla v_h+u^2)+x^2=0 \qquad (42)$$

is still exact but the scheme has not anymore the stochastic interpretation. Indeed if we want write (42) in a form similar to (33) from which the stochastic interpretation is clear we obtain :

$$v_h^i=\min_u[v^{i+1}(-x+u)-v_h^{i-1}(-x+u)+(u^2+x^2)h] \qquad (43)$$

and one of the coefficient of v_h^{i+1} or v_h^{i-1} is negative and thus we cannot solve (43) by classical iterative methods. Thus this h^2 scheme is not acceptable. The finite element scheme has the same has drawback. The "Markov chain" scheme is of order h and stable. The modified Markov chain is of order h^2 but not stable also. In dimension 2 the difficulty is more dramatic — the necessity of centering kill the positivity of at least one coefficient.

Now a classical numerical analysis remedy for the lost of positivity of the scheme (42) is the adding of a "viscosity term" $\alpha h\Delta_h v_h$ term to (42) which becomes :

$$-v_h+\alpha h\Delta_h v_h-x\nabla_h v_h+\bigwedge_u(u\nabla_h v_h+u^2)+x^2=0. \qquad (44)$$

with α a positive coefficient depending of the domain and of the value of the control ($\alpha\geq\sup_{x\in\sigma}|-x+u(x)|$). The solution of (44) is px^2+g_h with $g_h=2\alpha hp$ and thus the error is a translation of order h.

Now we can compare (44) and the Markov chain scheme, which is, when $x<0$:

$$-v_h^i-x\frac{v_h^{i+1}-v_h^i}{h}+\bigwedge_u\left[u\frac{v_h^{i+1}-v_h^i}{h}+u^2\right]+x^2=0. \qquad (45)$$

The solution of (45) is $px^2+f_h|x|+g_h$ with p exact

$$f_h=\frac{ph+p^2h}{2+p},\, g_h=-\frac{1}{4}(ph-f_h)^2.$$

The error of this scheme is affine and of order h. Thus the scheme (44) is better because in the quadratic case the control is exact for (44) and is of order h for (45).

A first conclusion is that the finite difference is better than the Markov chain also in the small noise case. *It is better to add a viscosity term and center the scheme than to decenter to obtain the stability.*

The last question is how we can do to improve the "centered with viscosity" scheme in such a way that it becomes of order 2. The remedy is clear, in the quadratic case, we have to compute the translation that is $v_h(0)$). In a more general situation we have to compensate the $hv''(x)$ term which is the only one which destroys the order 2. This can be done (and once more the solution is well known in the numerical analysis litterature). We have to make 2 iterations of the fixed point :

$$\begin{cases} -v_h^{(n+1)}-x\nabla_h v_h^{(n+1)}-\frac{1}{4}(\nabla_h v_h^{(n+1)})^2+x^2+\alpha h\Delta_h v_h^{(n+1)} \\ \qquad\qquad =\alpha h\Delta_h v_h^{(n)} \\ v_h^{(0)}=0 \end{cases} \qquad (46)$$

In conclusion the best way to solve the HJB equation is to — discretize by finite difference method, — if the noise is small, to add viscosity and make 2 iterations of the fixed point (46). We obtain by this way a stable algorithm and a solution precise up to h^2 when the solution is enough regular.

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