ON TRAFFIC LIGHT CONTROL OF REGULAR TOWNS

G. COHEN, S. GAUBERT, E. MANCINELLI, J.P. QUADRAT, E. ROFMAN

ABSTRACT. We present a hierarchical way to design the light control of the traffic of a town. We first describe a model of traffic light synchronization based on Petri net modelling and max-plus algebra. Based on this modelling we decompose the problem in three parts : -computation of the cycle length of each traffic light, -computation of the starting time of each traffic light cycle, -computation of the proportion of the green and red phases. The example of the Bahia Blanca argentinian town is given.

1. INTRODUCTION.

We describe a way to compute a traffic light plan in regular towns.

In a traffic light plan three quantities must be computed for each traffic light : -the cycle length of lights, -the starting time of the ligtht cycle, -the proportion of red and green in a cycle.

We call regular a town which possesses some symmetry at least approximatively. Here we consider the example of the center of Bahia Blanca whose map is given in Figure 5 which is approximatively invariant by 2 translations. One which is North-South of length 2 blocks of house. One which is East-West of length 2 blocks.

For such a symmetric town we build a Petri net describing a cooordination between all the lights of the town based on virtual circulation of cars at a given speed and a simple description of the flows of these cars.

By an analysis of this model we can show a decomposition between the flow evaluation and the time spent in the system.

To optimize the time spent in a system by a car we have to design "green waves". If we choose correctly the light cycle length, we show that it is possible to design four systems of compatible green waves which assure that we can join two points in the town at prescribe speed meeting at most one red light. This result is valid only in unsaturated situation when the flow in all the street are smaller than the virtual car flow.

The maximal virtual car flows is given by saturation of the slowest resource which are in our case the junctions. They are reached when the junction are always occupied whatever is the phase and therefore are easily computable.

In practise we propose to adapt the phase proportion to the real flow and to maintain the coordination between the lights given by the four systems of green waves.

The regular hypothesis about the town is not so restrictive. It can be reached more often by adapting the speed on each portion of street in such a way that the times to cover each block be equal.

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2. MAX-PLUS MODELLING OF PETRI NETS.

Let us explain the way to compute the troughput of a quite general class of Petri nets which can be interpreted in term of stochastic control. For more details see [1].

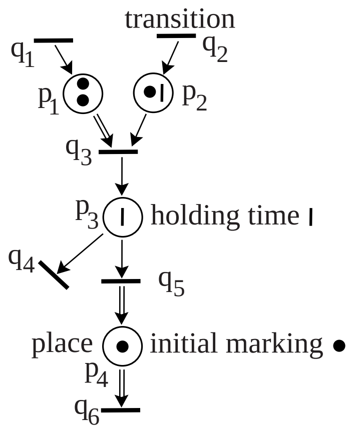

FIGURE 1. A Petri net.

DEFINITION 1. A continuous Petri net is defined by

$$\mathcal{N} = (\mathcal{P}, \mathcal{Q}, M, \rho, m, \tau) \ ,$$

where :

1. $\mathcal{P}$ is a finite set whose elements are called *places* ;
2. $\mathcal{Q}$ is a finite set whose elements are called *transitions* ;
3. $M \in \left(\mathbb{R}^+\right)^{\mathcal{P}\times\mathcal{Q}\cup\mathcal{P}\times\mathcal{Q}}$ are the *arc multipliers* that is M_{pq} (respect. M_{qp}) denotes the number of arcs from transition q to place p (respect. from place p to transition q);
4. $\rho : \mathcal{Q} \times \mathcal{P} \to \mathbb{R}^+$ verifying :
$$\sum_{q\in p^{out}} \rho_{qp} = 1, \ \forall p \in \mathcal{P} \ ,$$
is the *routing policy* which gives the imposed proportion of tokens going from the place p to transition q with respect to quantity of fluid entering in the place p ;
5. $m \in \left(\mathbb{R}^+\right)^{\mathcal{P}}$ is the *initial marking*, that is : m_p is the amount of fluid available at place p at starting time ;
6. $\tau \in \left(\mathbb{R}^+\right)^{\mathcal{P}}$ is the *holding time* which is the time that a molecule of fluid has to stay in place p before leaving.

The dynamic of the system is determined by the *firing* of transitions. A transition fire as soon as there are fluids available by the routing process having spent a time equal to the holding time in all places $p \in q^{in}$ upstream[1] the transition q. The cumulated quantity consumed by q at place p at time t is $Z_q(t)M_{qp}$ (where the amount of firing at q is denoted Z_q). The cumulated amount of fluid produced at time t in $p \in q^{out}$ is $M_{pq}Z_q(t)$. The firing process works as fast as possible. The cumulated amount of fluid which has entered in place p at time t is denoted $Z_p(t)$.

[1]Given a node r (resp s) (place or transition) is upstream (resp. downstram) a node s if $M_{sr} \neq 0$. We denote r^{out} (resp. r^{in}) the set of nodes s downstream (resp. upstream) r.

Defining :

$$\mu_{pq} \triangleq M_{pq},\ \mu_{qp} \triangleq M_{qp}^{-1},\ \tilde{\mu}_{qp} \triangleq \mu_{qp}\rho_{qp}\ ,$$

The dynamic of the system is completely defined by :

$$\begin{cases} Z_q(t) & = \min_{p\in q^{in}} \tilde{\mu}_{qp} Z_p\left(t-\tau_p\right), \\ Z_p(t) & = m_p + \sum_{q\in p^{in}} \mu_{pq} Z_q(t)\ . \end{cases}$$

Eliminating the variable Z_p we obtain a dynamic programming equation defining Z_q :

$$Z_q(t) = \min_{p\in q^{in}} \left[\tilde{\mu}_{qp} \left(m_p + \sum_{q'\in p^{in}} \mu_{pq'} Z_{q'}\left(t-\tau_p\right) \right) \right]\ . \tag{1}$$

This equation may be interpreted as a dynamic programming equation for stochastic control problem with a discounted cost.

Under certain condition described by the following theorem this stochastic control problem (1) is undiscounted.

THEOREM 2. *If there exists $v \in \left(\mathbb{R}^+\right)^{\mathcal{Q}}$ such that :*

$$\sum_{q\in p^{out}} v_q M_{qp} = \sum_{\tilde{q}\in p^{in}} M_{pq'} v_{q'},\ \forall p \in \mathcal{P}\ , \tag{2}$$

equation (1) has the interpretation of an undiscounted stochastic control with Bellman function :

$$W_q \triangleq \frac{Z_q}{v_q}\ .$$

In particular the condition of the theorem is fullfiled when :

$$\sum_{q\in p^{out}} M_{qp} = \sum_{\tilde{q}\in p^{in}} M_{p\tilde{q}},\ \forall p \in \mathcal{P}\ ,$$

that is when for all places there is an equal number of arc entering and leaving the place with a routing uniform that $\rho_{qp} = 1/|p^{out}|$ (where $|A|$ of a finite set A denotes is cardinality). In this cas we have $v = \mathbf{1}$.

With this remark we are able to define the througput of those Petri nets having a stochastic interpretation (which are the only living and stable Petri nets, the other ones exploding or dying after a finite time).

THEOREM 3. *Denoting*

$$P_{qq'}^p = v_q^{-1} \tilde{\mu}_{qp} \mu_{pq'} v_{q'} \textit{ and } \nu_q^p = m_p \tilde{\mu}_{qp}\ ,$$

the troughput $\lambda \triangleq \lim_{t\to\infty} Z_q(t)/t$ of a strongly connected Petri net, satisfying (2), exists, is independent of q and is solution of the dynamic programming equation :

$$w_q = \min_{p\in q^{in}} (\nu^p - \lambda\tau_p + P^p w)_q,\ \forall q \in \mathcal{Q}\ .$$

Using this theorem we can compute the troughput of such Petri net by the Howard algorithm (see [2] for a recent reference) whose complexity is experimentally almost linear in the number of arcs of the Petri net.

3. MODELLING TRAFFIC LIGHT SYNCHRONIZATION BY PETRI NET.

In the sequel we shall propose models in term of Petri net of the circulation of an ideal regular town. The purpose of theses Petri nets will be to model the coordination between the authorizations given to the cars by the light. What we will call *cars* in the following will not be real car but virtual cars which travel like real cars on the town. A real car can decide or not to follow the speed of a virtual car. These virtual cars are useful to study an ideal coordination. The flows of real cars cannot be larger than the virtual ones but they can be smaller. Probabilistic links between virtual and real car will be studied in future work. Here we define only an signal environment for the circulation of real cars.

3.1. MODELLING A JUNCTION.

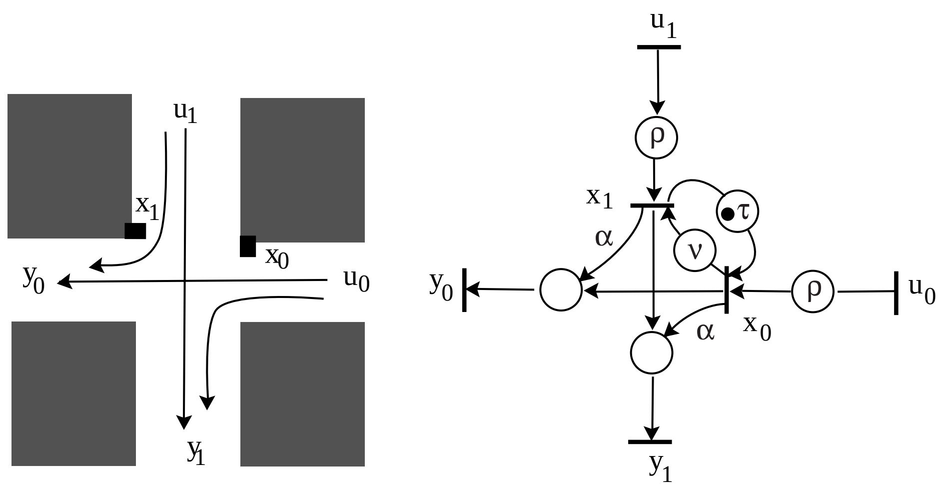

FIGURE 2. The Petri net of a junction.

The Petri net associated to a junction is given in Figure 2. We denote by $x_0(t)$ and $x_1(t)$ the cumulated number of green phases that have happened at each of the two lights up until date t. The green phase lengths of the two lights are denoted respectively τ and ν. We suppose that the amount of cars that can cross the junction is proportional to the length of the corresponding green phase with a coefficient that we choose equal to one. We suppose that at each junction a proportion of vehicules equal to α turn in the only turning direction available at the junction. We denote by $u_0(t)$ and $u_1(t)$ the cumulated amount of cars arrived at the junction up until time t and by $y_0(t)$ and $y_1(t)$ the cumulated amount of cars which have left the junction up until time t.

The relation between the inputs u and the output y is a stochastic dynamic programming equation where the Bellman function is x :

$$x = a \otimes x \oplus b \otimes u, \;\; y = cx \;,$$

where

$$a = \begin{bmatrix} \epsilon & \gamma\delta^{\tau} \\ \delta^{\nu} & \epsilon \end{bmatrix}, \; b = \begin{bmatrix} \delta^{\rho}/\nu & \epsilon \\ \epsilon & \delta^{\rho}/\tau \end{bmatrix}, \; c = \begin{bmatrix} (1-\alpha)\nu & \alpha\tau \\ \alpha\nu & (1-\alpha)\tau \end{bmatrix} \,,$$

Where $\oplus$ denotes the matrix max-plus addition (max element wise), $\otimes$ the max-plus matrix multiplication (substitution of plus by max and times by plus in the matricial product), δ is the unit shift in timing ($\delta v(t) = v(t-1)$) and γ is the unit shift in numbering ($\gamma v(t) = 1 + v(t)$).

3.2. Modelling a block of junctions.

Let us consider a regular town such as the one depicted in Figure 4 composed of squares separated by one way streets (with opposite direction of circulation for successive streets).

To determine the dynamic of this system it is useful to determine first the dynamic of a block composed of four squares. On the city map it can be observed a regularity characterized by an invariance by horizontal and vertical translation of size this basic block.

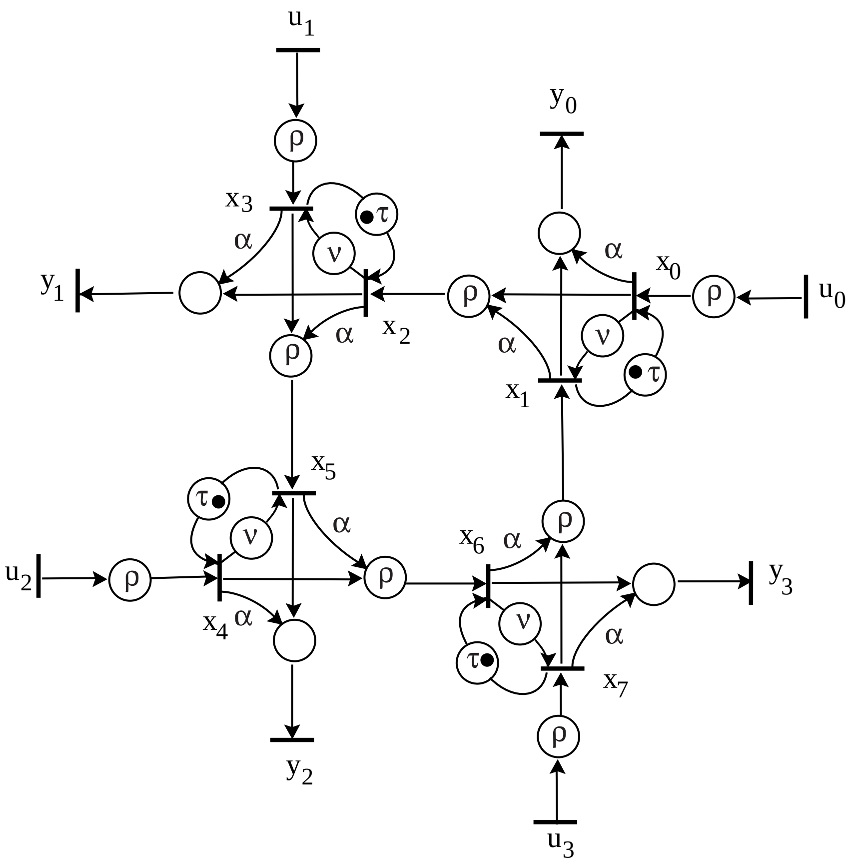

FIGURE 3. The Petri net of a block of 4 squares.

The dynamic of a block is defined by the Petri net given in Figure 3. The corresponding max-plus equations[2] are :

$$\chi_i = a \otimes \chi_i \oplus b \otimes \pi^i \otimes c\chi_{i-1} \oplus b \otimes \pi^{i-1} \otimes u_i , \;\; y_i = \pi^i \otimes c\chi_i , \;\; i = 0, 1, 2, 3 ,$$

[2]The computation on the index i is done here modulo 4 and we have added artificial empty inputs and outputs to symmetrize the formulas.

where

$$\chi_i = \begin{bmatrix} x_{2i} \\ x_{2i+1} \end{bmatrix}, \quad \pi^1 = \pi^3 = \begin{bmatrix} e & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \quad \pi^0 = \pi^2 = \begin{bmatrix} \epsilon & \epsilon \\ \epsilon & e \end{bmatrix} .$$

This a nonlinear max-plus system with 8 states 4 inputs sand 4 outputs system that we can written formally as :

$$x = Ax \oplus B \otimes u, \; y = Cx \;,$$

where A is the 8×8 nonlinear operator :

$$A = \begin{bmatrix} a & \epsilon & \epsilon & b\pi^0 c \\ b\pi^1 c & a & \epsilon & \epsilon \\ \epsilon & b\pi^0 c & a & \epsilon \\ \epsilon & \epsilon & b\pi^1 c & a \end{bmatrix} .$$

In order to have an undiscounted stochastic control interpretation of this dynamic programming equation it is sufficient that

$$(1-\alpha)\nu + \alpha\tau = \nu, \; (1-\alpha)\tau + \alpha\nu = \tau \;,$$

which implies $\alpha = \tau$.

In fact the α, ν and τ can depend of the junction and in this case the sufficient condition becomes :

$$\alpha_{q-1}\tau_{q-1} + (1-\alpha_{q-2})\nu_{q-2} = \nu_q, \; q \text{ even} \;,$$

$$\alpha_{q-3}\nu_{q-3} + (1-\alpha_{q-2})\tau_{q-2} = \tau_q, \; q \text{ odd} \;.$$

3.3. Modelling of a regular town.

A regular town is composed of blocks that we can index with a couple (I, J) where I is the east-west (E-W) coordinate of the block and J the south-north (S-N) coordinate.

The dynamic of the inside of a complete town can be written :

$$x_{IJ} = Ax_{IJ} \oplus \mathcal{A}_0 x_{I+1,J} \oplus \mathcal{A}_1 x_{I,J+1} \oplus \mathcal{A}_2 x_{I-1,J} \oplus \mathcal{A}_3 x_{I,J-1} \;,$$

where

$$\mathcal{A}_0 = \begin{bmatrix} \epsilon & b\pi^1 c & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & b\pi^0 c & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & b\pi^1 c \\ \epsilon & \epsilon & \epsilon & \epsilon \end{bmatrix}, \quad \mathcal{A}_3 = \begin{bmatrix} \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon \\ b\pi^0 c & \epsilon & \epsilon & \epsilon \end{bmatrix}.$$

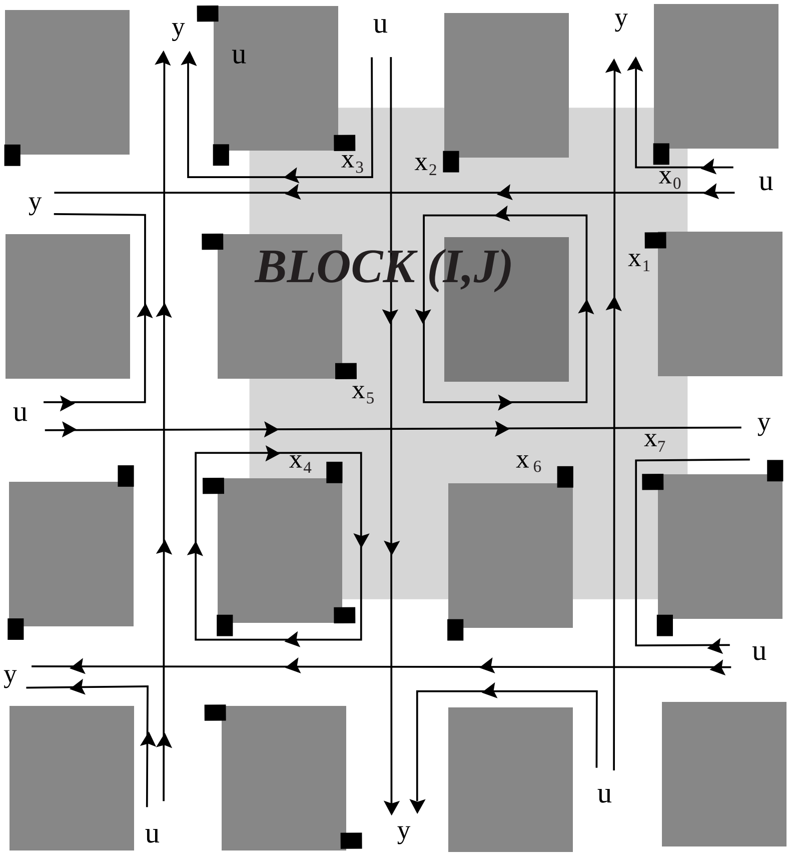

FIGURE 4. A regular town.

4. DECOMPOSITION OF THE TRAFFIC LIGHT CONTROL PROBLEM.

As we have seen in the first section, the previous type of system can be solved efficiently. But on this system we have to optimize some parameters. For example we have to decide the initial markings in the places corresponding to the streets (the ones containing ρ in Figure 3), the length of the green and red phases of each light. In an unsaturated situation, by a good choice of these quantities, we can achieve - a throughput only limited by the full occupation of the junctions by cars - obtained a system of green waves which allows a travel between two points of the town meeting, at most, one red light.

4.1. THROUGHPUT.

The interpretation of the throughput λ of a Petri net having the undiscounted stochastic control interpretation is

$$\lambda = \min_{\phi \in \mathcal{F}} \frac{r^{\phi}.\tilde{m}^{\phi}}{r^{\phi}.\tau^{\phi}} \ ,$$

where $\mathcal{F}$ denotes the set of achievable final classes of the Markov chains obtained by choosing only one place upstream each transition, r^ϕ the corresponding invariant measure and

$$\tilde{m}_p \triangleq \begin{cases} m_p & \text{if } p \text{ is a } \tau \text{ or } \nu \text{ place ,} \\ m_p/\nu & \text{if } p \text{ is a } \rho \text{ place arriving in an even transition ,} \\ m_p/\tau & \text{if } p \text{ is a } \rho \text{ place arriving in an odd transition .} \end{cases}$$

Among all the possible final classes there are the ones with only the two arcs representing the two light phases of a junction with a troughput equal to $1/(\tau+\nu)$. If we increase the number of tokens in the places corresponding to the street after some amount the optimal final classes will be the ones associated to lights. If we have not put enough tokens in these places the effective light cycle can be slowed down (two lights of the same junction simulatenously can stay red simultaneously).

4.2. Travel time.

We have to design the system in such a way that the throughput is maximal (collective objective). There are many ways to achieve this objective. It is sufficient that there are everywhere and always cars waiting at all lights. Among all these possibilities we may want minimized the travel times (individual objectives).

The maximal flow by cycle at a light, let us say at an E-W street, is ν. Then because the travelling time at maximal speed in a street is ρ the Little formula tells us the number of cars m_p needed in the street too achieve the wanted flow :

$$\rho\nu = m_p \ .$$

With this number of cars there is no waiting time, that is there are green waves with the maximal authorized speed.

A manufacturing way to say that is : $m_p = \rho\nu$ is the minimal number of pallets necessary in an E-W street to achieve the optimal speed given by the slowest machines which are here the junctions seen as machines serving successively E-W and N-S streets.

Clearly for S-N street we have $\rho\tau = m_p$.

4.3. Orthogonal system of green waves.

The green wave problem can be seen as compatiblity conditions between systems of equations. This compatibility condition will determine the phase difference between the lights and the cycle time of each light.

It is more easy to consider a continuous version of the problem. For that let us denote by $c(t,x,y)$ the color of a (x,y) point of a town at time t (each point is supposed to have a light). Let us suppose that there is a propagation of the green color with speed $v(x,y)$ along the x axis then c would satisfy the equation :

$$\partial_t c + v\partial_x c = 0 \ . \tag{3}$$

Moreover if we suppose that there is another green wave along the y axis with speed $w(x,y)$ then c would also satisfy the equation :

$$\partial_t c + w\partial_y c = 0 \ . \tag{4}$$

Clearly the system of equations (3) and (4) have not always a non trivial solution. In order to have a solution v and w must satisfy compatibility conditions. This kind of questions is well studied. A good reference is [5].

THEOREM 4. *The equations (3) and (4) are compatible iff it exists a potential* $\psi(x, y)$ *such that :*

$$\begin{bmatrix} v \\ w \end{bmatrix} = \mathrm{grad}(\psi) \ . \tag{5}$$

Proof. NECESSARY CONDITION. Let us differentiate $(3/v)$ with respect to y and $(4/w)$ with respect to x, we obtain

$$(1/v)\partial_{ty}c + \partial_{xy}c + \partial_t c\partial_y(1/v) = 0, \ \ (1/w)\partial_{tx}c + \partial_{xy}c + \partial_t c\partial_x(1/w) = 0 \ .$$

Moreover differentiating $(3/v)$ and $(4/w)$ with respect to t, we obtain

$$\partial_{xt}c + (1/v)\partial_{tt}c = 0, \ \ \partial_{yt}c + (1/w)\partial_{tt}c = 0 \ .$$

It follows that

$$\partial_y(1/v) = \partial_x(1/w) \ ,$$

which implies the wanted condition.

SUFFICIENT CONDITION If (5) is satisfied $\psi(x - vt, y) = \psi(x, y - wt)$ is satisfied and $c(t, x, y) \triangleq \psi(x - vt, y)$ is a solution of the two equations and therefore they are compatible[3]. □

The interpretation of this result is easy. A given stationary[4] light plan defines speeds in x and y direction at which one see the light always green (the difference of phase of two successive lights divided by the distance between these two lights). But this speed is not everywhere constant in x and y. Therefore this light plan is itself an acceptable potential from which derives the speeds v and w.

From these considerations, the phase difference of the lights in N-S, E-W streets can be choosen in such a way that we have fixed speed green waves along all those streets (if we don't consider the light of the opposite direction (S-N and W-E streets)). Clearly we can also realize the green wave for (S-N,W-E) streets.

4.4. LIGHT CYCLE LENGTH.

We have taken in the previous section a system of two orthogonal green waves. This can be generalized to two non orthogonal fields of speeds. But when the two fields becomes dependent it does not exist anymore compatibility condition of the two partial differential equation. But it is possible that the solution of two equations coincide at some point of a mesh. Let us analyze that on the simplest possible situation.

Let us consider a two directions street. It is easy to see that there is no nontrivial solution to :

$$\partial_t c + v\partial_x c = 0, \ \ \partial_t c - v\partial_x c = 0 \ .$$

But consider a mesh of lights of coordinates $\{x_i\}$ and let us denote ρ_i^+ (resp. ρ_i^-) the time for a car to go from x_i to x_{i+1} (resp. from x_i to x_{i-1}) and let us denote by T_i the i length of all the lights. As soon as

$$\exists k, k' \in \mathbb{N} : kT_i = \rho_i^+ + \rho_{i+1}^-; \ \ k'T_i = \rho_i^- + \rho_{i-1}^+ \ ,$$

it exists a green wave in the two directions. In the particular cases of a regular mesh and a constant speed we obtain $kT = 2\rho$. In practice, the smallest k ($k = 1$) is acceptable (speed $26km/h$ square of side length $130m$ gives for $k = 1$, $T = 36s$).

[3] $c(t, x, y) \triangleq t + \psi(x, y)$ where ψ is the potential of $(1/, 1/w)$ is also a solution of the two equations.

[4] The phase difference between two lights stay constant in time.

For a 2D regular town we arrive at the same conclusion by considering two cycles which differ from two parallel portions of streets corresponding to a square. Supposing that there exist green waves on these two streets then at each light common to the two cycles we have $kT = 2\rho$ by computing the difference of the time length to cover the two cycles. Moreover, to have a cycle length long enough in practice, we have to choose $k = 1$ and the light cycle length is determined and is equal to 2ρ.

With this light cycle length the four systems of green waves E-W, W-E, S-N and N-S are compatible. In this town we can go from any two points meeting at least one red light.

5. Bahia Blanca example.

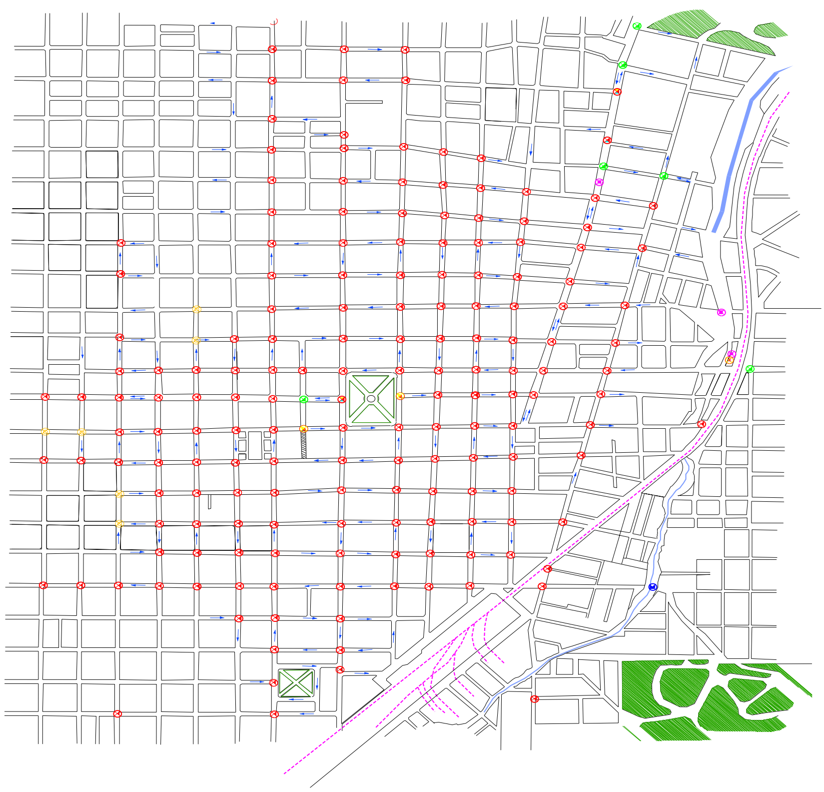

FIGURE 5. The Bahia Blanca map.

The previous discussion has been inspired by the Bahia Blanca town which is an argentina town with a very regular design. Clearly this town is enough regular to apply the previous results. The flow of real cars in the street are always changing therefore we popose to adapt the proportion of green and red lengths according to the real flow without changing the light cycle length and the difference of phase between the lights which can be seen as a coordination between all the lights.

We can summarize our proposition of light plan.

1. Determine a reasonable speed v for the green wave in the town. It must be compatible with a light cycle length equal to $T = 2\rho$, with $\rho = d/v$ where d is the average length of a square of the town.

2. Using this ideal time ρ to cover a square length compute the initial starting time of each light cycle. Practically, this means that two successive lights in the same street must have a difference of phase of ρ.
3. At each junction adapt the proportion of green and red length according to the flow in each direction. If ϕ and ψ denote the two average flows arriving at the junction and τ and $T-\tau$ the length of the corresponding green phase we must have
$$\phi/(\phi+\psi)=\tau/T\ .$$

With such a policy in unsaturated situation the average speed of circulation is everywhere approximatively v.

This model is valid only in unsaturated situation.

The policy proposed here implies an observation of the flow of cars at all the junctions. This observation is expensive. Traffic modelling like in [6] can be useful to interpolate the traffic observed at some points and to avoid the installation of some observators. This possibility will be explored in future works.

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INRIA-ROCQUENCOURT, B.P. 105, 78153 LE CHESNAY CEDEX, FRANCE