PRODUCT FORM AND OPTIMAL LOCAL FEEDBACK FOR A MULTIINDEX MARKOV CHAIN *

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ABSTRACT For a network of queues of Jackson type, with derivation from the entry to the exit when the capacity of storage of a mean is saturated, we prove the product form property for the equilibrium probability law. We give an algorithm to compute the optimal local feedback for the controled system in the case where the control is the service rate of each individual queue. We prove the convergence of the algorithm to a strategy optimal player by player.

INTRODUCTION

When we want to compute the optimal feedback for a large scale stochastic control problem we meet the "curse of dimensionality" difficulty which indicates that the number of parameters to compute grows exponentially with the dimension of the system. One way to avoid this difficulty is to limit our ambition and compute the optimum only for a subclass of feedbacks for example for the class of local feedbacks. It appears very quickly that in general this last problem is more difficult than the former one. But for some particular systems there is a gain to impose this limitation on the admissible strategies. For example when the dynamic of the system is decoupled, with perhaps a coupling criterion we can see that a player by player optimum strategy is reachable when the global optimum is not computable. The purpose of this paper is to generalize this situation to system for which the equilibrium probability has the product form, that is this equilibrium probability can be written as a product of independant marginal laws of probability. In the first part we describe some finite valued Markov chain with this property. They are of Jackson type (Jackson [1], B.C.M.P. [2], Kelly [3], F.K.A.S [4], Schassberger [617]) with a derivation of the input of customer, to the output whenever the queue is full. Then we suppose that the output of each queue is a control variable. We give a necessary and sufficient condition for a control to be optimum player by player where a player is a server of a queue. And we give an iteration policy algorithm (Howard [8], Miller, Veinott [7]) to compute this control.

I. A FINITE VALUED MARKOV CHAIN WITH THE PRODUCT FORM PROPERTY (Fig. 2)

We consider a set of interconnected queues, with finite capacities, and derivation of the input to the output when a queue is full. We suppose that the output rate is exponentially distributed and that the input of the system depends of the total number of customers. This system is of Jackson type with derivation. First we give a precise model and then give the equilibrium probability law.

I.1. The model

We denote by $E = \{1, 2, \ldots, m\}$ the set of the index of the queues. The index $\{o\}$ represents the exterior considered as a special queue and $\bar{E} = E \cup \{o\}$.

If n_i, $i \in E$ is the number of customers in the queue i we denote by $n(n_o = \sum_{i=1}^{m} n_i, n_1, \ldots, n_m)$. $\check{n}_i$, $i \in E$ is the capacity of the queue i. We use the notations $\check{n} = (\check{n}_o = \sum_i \check{n}_i, \check{n}_1, \check{n}_2, \ldots, \check{n}_m)$ and $N = \{n, o \le n_i \le \check{n}_i, i \in E\}$.

The service rate is denoted by μ:

(i) $\mu_i : [o, \check{n}_i] \to \mathbb{R}^+$, $i \in \bar{E}$.

We suppose that $\mu_i(n_i) \ge \mu$ $\forall i$, $\forall n \in N$, and $\mu_i(o) = o$. $\forall i \in E, \mu_o(\check{n}_o) = o$.

We call r the routage matrix :

(ii) $r : \bar{E} \times \bar{E} \to [0, 1]$ r, being a stochastic matrix associated to an irreducible markov chain. r_{ij} indicates the proportion of customers leaving the queue i going into the queue j. r_{oj} is the proportion of the input going into the queue j and r_{jo} the proportion of the output of the queue j leaving the system.

$$i \quad j \quad r_{ij}$$

When a queue is full the input of the queue is derived to the output of the queue and the corresponding customer should been trying to enter in a queue until it finds a place free. If $F(n) \subset E$, is a set of queues which have a place free, when the state of the system is n, we can describe this phenomenon using the routage matrix r^F (r^F_{ij} = the probability starting from a queue belonging to $i \in E$ ending in the queue $j \in F$ (the first queue with a free place))

$$r^F : \bar{E} \times F \to [0,1].$$
$$i \quad j \quad r^F_{ij}$$

Using the notations : $\bar{F} = F \cup \{o\}$, $F^c = C^F_E$, $r^F = \begin{array}{c} \\ \bar{F} \\ F^c \end{array}\begin{array}{c} \bar{F} \\ \begin{bmatrix} r^F_1 \\ r^F_2 \end{bmatrix} \end{array}$ $r = \begin{array}{c} \\ \bar{F} \\ F^c \end{array}\begin{array}{c} \begin{array}{cc} \bar{F} & F^c \end{array} \\ \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \end{array}$

$$s = \sum_{i=o}^{+\infty} r_4^i.$$

We have : $r^F_1 = r_1 + r_2 s r_3$, $r^F_2 = s r_3$.

We remark that s is meaningfull because the chain is irreductible, so there is a path connecting F^c to F, therefore r_4 has no eigenvalue of modulus 1.

We need the introduction of the following operators :

$$T_{ij} : N \to \bar{N}$$

$$(n_o, n_1, \ldots, n_m) \to T_{ij}(n)$$

with

$T_{ij}(n) = (n_o, n_1, \dots, n_i - 1, \dots, n_j + 1, \dots, n_m)$ for $i \in E$, $j \in E$;
$T_{io}(n) = (n_o - 1, n_1, \dots, n_i - 1, \dots, n_m)$ for $i \in E$;
$T_{oi}(n) = (n_o + 1, n_1, \dots, n_i + 1, \dots, n_m)$ for $i \in E$.

and

$$\bar{N} = \{(n_o, n_1, \dots, n_m) \mid n_o = \sum_{i>o} n_i, \ -1 \le n_i \le \check{n}_i + 1, \ i \in E\}$$

Now it is possible to define the transition rates of the system denoted by q

$$q : N \times N \to \mathbb{R}^+$$
$$(n,n') \ \ q(n, n').$$

The non zero terms of q are :

$$q(n, T_{ij}n) = \mu_i(n_i) \ r_{ij}^{F_i(n)}, \text{ for } i \in E, \ j \in F_i(n), \ n \in N,$$

with $F_i(n) = \bar{F}(n) \cup \{i\}$.

I.2. A product form result

The equilibrium probability p(n) satisfies the global balance equation

$$\text{(1.1)} \qquad \sum_{i \in E, j \in E} p(T_{ij}^{-1}n) \ q(T_{ij}^{-1}n, n) = p(n) \sum_{i \in \bar{E}, j \in \bar{E}} q(n, T_{ij}n).$$

We remark that $T_{ij}^{-1} = T_{ji}$ for $j \in E$, $i \in E$; and because $T_{oi}T_{io} = T_{io}T_{oi} = I$ we have $T_{oi}^{-1} = T_{io}$ and $T_{io}^{-1} = T_{oi}$, so (1.1) can be written also as

$$\text{(1.1')} \qquad \sum_{i \in \bar{E}, j \in \bar{E}} p(T_{ij} \ n) \ q(T_{ij} \ n, n) = p(n) \sum_{i \in \bar{E}, j \in \bar{E}} q(n, T_{ij}n)$$

Theorem 1 The system described in I.1 admits a unique invariant measure of probability

$$\text{(1.2)} \qquad p(n) = C \prod_{i=1}^{m} \left[\prod_{k=1}^{n_j} (e_i/\mu_i(k)) \right] \prod_{k=o}^{n_o - 1} (\mu_o(k)/e_o), \ n \in N \text{ where}$$

the vector e satisfies

$$\text{(1.3)} \qquad e \ r = e$$

and C is a constant of normalization

Proof

We verify that (1.2) satisfies the following partial balance equation

$$\text{(1.4)} \qquad p(n) \sum_{j \in \bar{E}, j \neq i} q(n, T_{ij}n) = \sum_{j \in \bar{E}, j \neq i} p(T_{ij}n) \ q(T_{ij}n, n), \ i \in E.$$

Let us first verify (1.4) for $o \le n < \check{n}$. In this case

$q(n, T_{ij}n) = \mu_i(n_i) \ r_{ij}$
$q(T_{ij}n, n) = \mu_j(n_j + 1) \ r_{ji}$, $j \in E$
$q(T_{io}n, n) = \mu_o(n_o - 1) r_{oi}$.

But $p(T_{ij}n)/p(n) = (\mu_i(n_i)/e_i)(e_j/\mu_j(n_j+1))$ for $j \in E$, $j \neq i$;

$p(T_{io}n)/p(n) = (e_o/\mu_o(n_o-1))(\mu_i(n_i)/e_i)$ and using (1.3)(1.4) is easily verified.

Let us verify (1.4) when $C_E F(n) \neq \emptyset$. In this case we have :

$q(n, T_{ij}n) = \mu_i(n_i) r_{ij}^{F_i(n)} \; 1^{(j)}_{F_i(n)}$, for $j \in E$;

$q(T_{ij}n, n) = \mu_j(n_j+1)\; r_{ji}^{F_i(n)} \; 1^{(j)}_{F_i(n)}$ because $F_j(T_{ij}n) = F(n) \cup \{i\}$ when $j \in F_i(n)$;

$q(T_{io}n, n) = \mu_o(n_o-1)\; r_{oi}^{F_i(n)} \; 1^{(o)}_{F_i(n)}$, because o always belongs to $F_i(n)$.

Therefore, the proof is the same as the case $0 \leqslant n < \check{n}$ provides we con prove that

$$\sum_{j \in F_i(n)} e_j \; r_{ji}^{F_i(n)} = e_i. \tag{1.5}$$

Using the notation $e_{F_i} = \{e_k,\ k \in F_i\}$ $\quad e_{F_i^c} = \{e_k,\ k \in F_i^c\}$ from (1.3) we have :

$$e_{F_i} r_1 + e_{F_i^c} r_2 = e_{F_i^c}$$

$$e_{F_i} r_3 + e_{F_i^c} r_4 = e_{F_i}$$

which prooves that

$$e_{F_i} r_1 + e_{F_i} r_3 s r_2 = e_{F_i} \quad \text{which is (1.5).}$$

Therefore p(n) is an equilibrium probability. The uniqueness comes from (i) and (ii) which guarantees that it is possible to reach every point of $\mathcal{N}$ from every other point.

From the theorem the following result is easily deduced.

<u>Corollary</u>. <u>The invariant measure of the system described in (1.1)</u>

$p(n_1, \ldots, n_m) = C \prod_{i=1}^{m} p_i(n_i)\; p_o(\sum_{i \in E} n_i - 1)$ <u>where</u> $p_i(n_i)$, $i \in E$ <u>is the invariant measure of the queue i, in isolation with input</u> $e_i 1^{(n_i)}_{n_i < \check{n}_i}$ <u>and</u> $p_o(n_o)$ <u>is the invariant system measure of an aggregate queue, describing the total number of customers in the system with input</u> $\mu_o(n_o)$ <u>and output</u> $e_o 1^{(n_o)}_{n_o} \geqslant (n_o)$.

<u>Remark</u> This kind of boundary behaviour is not exactly the truncation described in Kelly [3], but it preserves also the partial balance condition and accordingly the product form property.

II. OPTIMAL LOCAL FEEDBACK OF JACKSONIAN QUEUES

II.1. The control problem

Now we need to make one more hypothesis. We assume that:

(iii) μ_o is independant of n_o $\quad N = \prod_{i=1}^{m} \{o, \check{n}_i\}$.

Given the network of queues described in I.1, (where now the service rates μ are control variables) and a cost function c:

$c : N \times [\hat{\mu}, \check{\mu}]^m \to \mathbb{R}^+$ continuous, we want to optimize the objective function:

$$(2.1) \qquad J(\mu) = \lim_{T\to\infty} 1/T \sum_{t=o}^{T-1} c(n(t), \mu(t))$$

in the class of local feedback S_L that is :

$$S_L = \{\mu | \ \mu_i : \{1, \check{n}_i\} \to U \subset [\hat{\mu}, \check{\mu}]\} .$$

In this paragraph we take advantage of the product form property to characterize strategies belonging to S_L, which are optimal player by player (i.e. each μ_i is one of these players). An equivalent formulation is :

$$(2.1.') \qquad \min_{\mu\in S_L} J(u) = \sum_{n\in N} c(n, \mu(n))\, p_\mu(n)$$

where $p_\mu(n)$ is given by (1.1), because $\forall\mu \in S_L$ the system described in I.1 has a unique invariant measure $p_\mu(n)$.

Let us introduce the following notations:

$g(n_i) = e_i \ 1^{(n_i)}_{n_i < \check{n}_i}$ where e_i is defined by (1.3). $\forall i \in E$, $A_i(\mu_i)$ is a matrix characterized by :

$$A_i(\mu_i)_{k,k+1} = g_i(k) \quad \forall k,\ o \le k \le \check{n}_i$$
$$A_i(\mu_i)_{k,k-1} = \mu_i(k)$$
$$A_i(\mu_i)_{k,k} = -g_i(k) - \mu_i(k)$$
$$A_i(\mu_i)_{k,\ell} = o \text{ elsewhere}$$

$A*$ is the transpose of A,

$W \in \mathbb{R}^+$ shall be interpreted as the optimal cost,

$V_i : [o, \check{n}_i] \to \mathbb{R}$ could be interpreted as a Bellman function.

The next result is the characterization of a Nash point $\mu^* \in S_L$ that is μ^* which satisfies :

$$J(\mu_1^*, \mu_2^*, \ldots, \mu_m^*) \le J(\mu_1^*, \mu_2^*, \ldots, \mu_i, \ldots, \mu_m^*) \quad \forall \mu_i : \{1, \ldots, \check{n}_i\} \to [\hat{\mu}, \check{\mu}].$$

Under differentiability properties for $\mu\in S_L \to J(\mu) \in \mathbb{R}$, such a point is a local minimum. Under convexity and differentiability properties for J such a point is a global minimum.

II.2. Characterization of Nash-optimal strategies

Theorem 2. A necessary and sufficient condition for W to be a Nash optimal cost is that $(\mu_i^*, p_i, V_i, W, i \in E)$ satisfying

$$(2.2) \qquad A_i^*(\mu_i^*) p_i = 0, \quad \sum_{k=o}^{n_i} p_i(k) = 1$$

$$(2.3) \qquad \underset{\mu_i}{\mathrm{Min}} \ \{A_i(\mu_i) V_i + c_i(\mu_i)\} = W$$

$$(2.4) \qquad \mu_i^* \in \mathrm{Arg} \underset{\mu_i}{\mathrm{Min}} \ \{A_i(\mu_i) V_i + c_i(\mu_i)\}$$

where $c_i : \{0, \ldots, \check{n}_i\} \times [\hat{\mu}, \check{\mu}] \to \mathbb{R}^+$
$\qquad\qquad k \qquad\qquad\qquad \nu \qquad\quad c_i(k,\nu)$

$$(2.5) \qquad c_i(k,\nu) = \mathbb{E}_p \{c(n,\mu(n)) \mid n_i = k, \mu_i(n_i) = \nu, \mu_j = \mu_j^*, j \neq i\} \quad \text{and}$$

$$p = \prod_{i=1}^{m} p_i, \text{ exists.}$$

Proof The invariant measure $p(\mu)$ associated to the strategy $\mu \in S_L$ can be written as $p^\mu = \prod_{j=1}^{m} p_i^{\mu_i}$ where $p_i^{\mu_i}$ is the solution of $A_i^*(\mu_i) p_i^{\mu_i} = o$, $\sum_{k=o}^{\check{n}_i} p_i^{\mu_i}(k) = 1$.

The criterium that we have to optimize can be written as:

$$\underset{\mu}{\mathrm{Min}} \ J(\mu) = \sum_n c(n,\mu(n)) \prod_{i=1}^{m} p_i^{\mu_i}(n_i).$$

Now if μ^* is a Nash point and if we denote by $S_{\mu_i}^i$ the strategy $(\mu_1^*, \mu_2^*, \ldots, \mu_i, \mu_{i+1}^*, \ldots, \mu_m^*)$

$$(2.6) \qquad J(\mu^*) \leq J(S_{\mu i}^i) \quad \forall \mu_i : [0, \check{n}_i] \to [\hat{\mu}, \check{\mu}]$$

then (2.6) can be written as:

$$(2.7) \qquad J(\mu^*) = \underset{\mu_i}{\mathrm{Min}} \ \mathbb{E}_{p_i}^{\mu_i} c_i(n_i, \mu_i)$$

$$\text{with } c_i(n_i,\mu_i) = \sum_{\{n \mid n_i \text{ given}\}} c(n,\mu(n)) \prod_{j \neq i} p_j^{\mu_j^*}(n_j)$$

and because the n_i is an irreducible Markov chain $\forall \mu_i$, using the undiscounted cost dynamic programming theory, a necessary and sufficient condition for $W = J(\mu^*)$ to be the minimum cost is that it satisfies :

$$\underset{\mu_i}{\mathrm{Min}} \ \{A(\mu_i) V_i + c_i(\mu_i)\} = W, \ \forall i.$$

The existence of a Nash point is guaranteed by the continuity of the function $\mu_i \to p_i^{\mu_i}$ (which a consequence of the continuity of the unique eigen vector associated to the eigen value o of the operator $A_i(\mu_i)$ Kato [20]. The uniqueness of p_i follows from the fact that the Markov chain $n_i(t)$ has only one recurrent class).

The function $\mu \to c(\mu)$ is continuous, the admissible set of strategies is compact by hypothesis. Thus the optimal cost in the class of local feedbacks, which is a Nash point, exists.

II.3. A relaxation algorithm to compute the Nash optimal strategy

To compute $(\mu_i^*, p_i, V_i, W, i \in E)$ solution of (2.2) to (2.5) we can use the following policy improvement algorithm :

1) start with $\mu \in S_L$, compute p^μ, put $j = n = 1$.

2) compute $c_j(\mu_j)$
compute by a classical improvement algorithm V_j, W^n, μ_j^* the solution of
$$\operatorname*{Min}_{\mu_j} \{A_j(\mu_j) V_j + c_j(\mu_j)\} = W^n, \quad \mu_j^* \in \operatorname*{ArgMin}_{\mu_j} \{A_j(\mu_j)V_j + c_j(\mu_j)\}.$$

3) μ becomes $(\mu_1, \mu_2, \ldots, \mu_j^*, \mu_{j+1}, \ldots, \mu_m)$, compute $p_j^{\mu_j^*}$.

4) n becomes n+1, j=j+1 modulo n, and return to 2) until W^n converges.

Clearly the sequence W^n is decreasing and bounded by o from below, thus it converges to W_* but we are not assured that W_* is Nash optimal cost, excepte when the set of control is finited valued Sandell [12]. Powell's counter example [13], for example, shows the existence of limit cycles. To avoid this difficulty we can modify the step (3) which changes to (3') :

μ becomes $(\mu_1, \mu_2, \ldots, \mu_j^*, \mu_{j+1}, \ldots, \mu_m)$ only if $W^{n-1} - W^n > \varepsilon$ where ε is a given number.

Then clearly after a finite number of steps we obtain a point μ^ε which is ε-Nash optimal that is

$$J(\mu^\varepsilon) \le J(\mu_1^\varepsilon, \ldots, \mu_{i-1}^\varepsilon, \nu, \mu_{i+1}^\varepsilon, \ldots, \mu_m^\varepsilon) + \varepsilon, \ \forall \nu \in [o, \check{n}_i] \to [\hat{\mu}, \check{\mu}].$$

Now having obtained μ^ε, ε becomes $\varepsilon/2$, and we reiterate using the same algorithm with initial condition μ^ε to obtain $\mu^{\varepsilon/2}$, and so on. Clearly a subsequence of $\{\mu^{\varepsilon/2^n}, n \in N\}$ is convergent and converges to a Nash point, following the continuity of $\mu \to J(\mu)$. Nevertheless this new algorithm cannot be implemeted on a computer because the step 2 needs an ∞ number of iterations to be achieved. So we have to solve the step 2 only approximatively if we want to avoid this new difficulty.

Before giving an convergent algorithm let us introduce the notations

$$J_j : S_L \to \mathbb{R}^+ \{o, \ldots, \check{n}_j\} \times [\hat{\mu}, \check{\mu}]$$
$$(\mu_1, \ldots, \mu_m) \quad C_j \text{ defined by } (2.5)$$

$SL_{j,\mu} : [\hat{\mu},\check{\mu}]^{\{o,\ldots,\check{n}_j\}} \to \mathbb{R}^{\check{n}_j+2}$

$\mu_j \qquad (V_j, W)$ a solution of $A_j(\mu_j)\, V_j + c_j(\mu_j) = W$
with $c_j = J_j(\mu)$.

A consequence of (i) and (ii) is that W is uniquely defined.

$M_{j,\mu} : \mathbb{R}^{\check{n}_j+1} \longrightarrow [\hat{\mu},\check{\mu}]^{\{o,\ldots,n_j\}}$

$V_j \qquad \mu_j$

$\mu_j \in \operatorname{Arg\,Min}_{\mu} A_j(\mu)\, V_j + c_j(\mu)$

Fig. 1 gives now the flowchart of the algorithm in which appears two supplementary indices appear, ε which was previously defined , h which is an index for an iteration of the Howard algorithm. μ, V and W are indexed by this two variables.

'Test' is a variable which counts the number of players which cannot improve their cost more than an ε, by an iteration of the Howard algorithm.

<u>Theorem 3</u> $\{W_\varepsilon \mid \varepsilon = 1/2^n\}$ <u>is a positive decreasing sequence converging to W^* which is cost optimal player by player.</u>

From $\{\mu_\varepsilon \mid \varepsilon = 1/2^n\}$ <u>we can extract a subsequence which is convergent, and all convergent subsequences converge to a policy optimal player by player of cost W^*.</u>

<u>Proof</u> $W_\varepsilon = J(\mu_\varepsilon)$ and so it is always positive. Each time we go through the point G of the flowchart W decreases by ε. So in a finite number of iteration we certainly go through the point F of the flowchart. At this point, whatever the player, it is impossible to improve the cost by ε. So at this point the following relations are verified:

$$A_j(\mu_{j,\varepsilon})\, V_{j,\varepsilon} + c_j(\mu_{j,\varepsilon}) = W_\varepsilon \tag{2.6}$$

$$A_j(\mu'_{j,\varepsilon})\, V'_{j,\varepsilon} + c_j(\mu'_{j,\varepsilon}) = W'_\varepsilon \qquad \forall j \in E \tag{2.7}$$

$$W'_\varepsilon \geq W_\varepsilon - \varepsilon \tag{2.8}$$

$$A_j(\mu'_{j,\varepsilon})\, V_{j,\varepsilon} + c_j(\mu'_{j,\varepsilon}) \leq A_j(\mu_j)\, V_{j,\varepsilon} + c_j(\mu_j) \tag{2.9}$$

$$\forall \mu_j : [o,\check{n}_j] \to [\hat{\mu},\check{\mu}]$$

Denoting by : $A'_{j,\varepsilon} = A(\mu'_{j,\varepsilon})$, $c'_{j,\varepsilon} = c(\mu'_{j,\varepsilon})$

from (2.6) we have $A'_{j,\varepsilon} V_{j,\varepsilon} + c'_{j,\varepsilon} + (A_{j,\varepsilon} - A'_{j,\varepsilon})V_{j,\varepsilon} + c_{j,\varepsilon} - c'_{j,\varepsilon} = W_\varepsilon$,

which can be written as :

$$\operatorname*{Min}_{\mu} \{A_j(\mu)\, V_{j,\varepsilon} + c_j(\mu)\} + \xi_{j,\varepsilon} = W_\varepsilon \tag{2.10}$$

with

$$\xi_{j,\varepsilon} = (A_{j,\varepsilon} - A'_{j,\varepsilon})\, V_{j,\varepsilon} + c_{j,\varepsilon} - c'_{j,\varepsilon} \geq 0$$

Let us now give an estimate for $\sup_{n_j} \xi_{j,}(n_j)$.

From (2.6), (2.7):

$$A'_j(V_{j,\varepsilon} - V'_{j,\varepsilon}) + \xi_{j,\varepsilon} = W_\varepsilon - W'_\varepsilon$$

and so using (2.8) we have $W_\varepsilon - W'_\varepsilon = \sum_{n_j} \xi_{j,\varepsilon}(n_j)\, p_j^{\mu'_{j,\varepsilon}}(n_j) \leq \varepsilon$

which implies that

$$(2.11) \qquad \sup_{n_j} \xi_{j,\varepsilon}(n_j) \leq \varepsilon / \operatorname{Inf}_{n_j} p_j^{\mu'_{j,\varepsilon}}(n_j)$$

but because (i) and (ii).

$\exists\, \alpha > o \;! \operatorname{Inf}_{j,\mu,n_j} p_j^{\mu_j}(n_j) \geq 1/\alpha$ (the hypotheses (i) and (ii) were introduced for this reason).

Thus we have obtained that :

$$(2.12) \qquad |\operatorname{Min}_\mu \{A_j(\mu)\, V_{j,\varepsilon} + c_j(\mu)\} - W_\varepsilon| \leq k\varepsilon .$$

Now because $\mu'_{j,\varepsilon}$ belongs to a compact set we can extract a subsequence of $\{\mu'_{j,\varepsilon} \mid \varepsilon = 1/2^q\}$ which is convergent. The index of this subsequence is denoted by q.

Because of (i) and (ii) $V_{j,\varepsilon}$ for all j and , we can always choose V_j, in such away that it is uniformly bounded and so we can extract a subsequence of $(V_{j,\varepsilon} \mid \varepsilon = 1/2^q)$ which is convergent .The corresponding subsequence is once more indexed by q.

The sequence $(W_\varepsilon \mid \varepsilon = 1/2^q,\ q \in \mathbb{N})$ is convergent and then using the continuity property of $\mu_j \to A_j(\mu_j)$, $\mu \to c_j(\mu_j)$ and the fact that $1/2^q$ converges to zero, (2.12) and the definition of $\mu'_{j,\varepsilon}$ gives the equation (2.2) to (2.5) and we have the result.

<u>Remark</u> This policy algorithm gives a constructive existence result of a solution of (2.2) to (2.5).

CONCLUSION

In this paper we have proved that a network of queues of Jackson type with finite capacity of storage and derivation of the input to the output (when there is saturation of this storage capacity) has the product form property. Then we have studied the control problem where the control variables are the output rates as a feedback on the number of customers waiting in the corresponding queue. We have given a necessary and sufficient condition for a strategy to be optimal player by player and an algorithm to compute this strategy. The extension to more general systems of queues for example B.C.M.P. [2] queues is currently under investigation.

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The same kind of result can be obtained for a system of diffusion processes QUADRAT-VIOT [10].

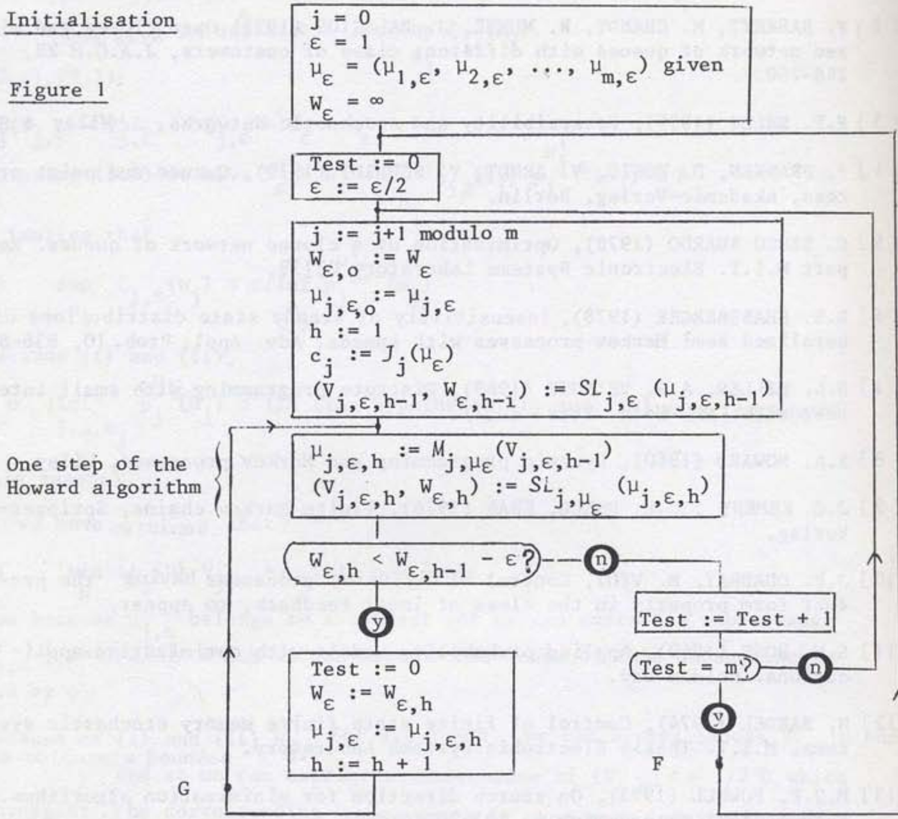

Figure 1

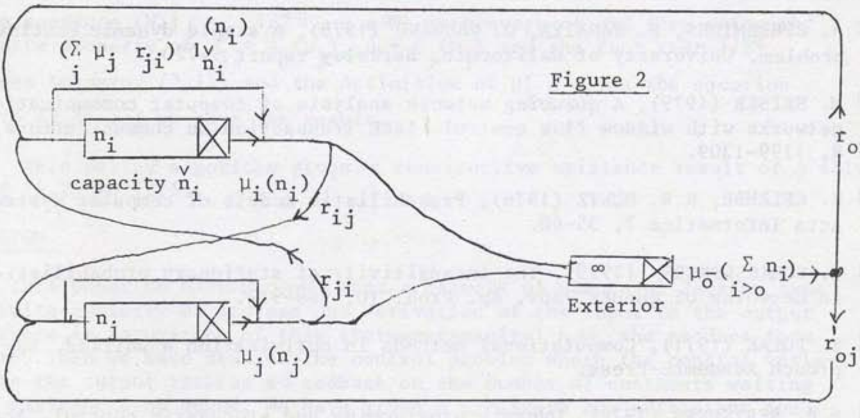

Figure 2

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