

4

Asymmetric Cryptography

In this chapter, you will be introduced to the concepts and practical aspects of **public key cryptography**, also called **asymmetric cryptography** or **asymmetric key cryptography**. We will start with the theoretical foundations of public key cryptography and will gradually build on the concepts, with relevant practical exercises with OpenSSL. Please refer to the previous chapter for an OpenSSL installation guide, if you have not already. After this, we will introduce some new and advanced cryptography constructs.

Along the way, we will discuss the following topics:

- Foundational mathematics
- Asymmetric cryptography
- Introducing RSA
- Introducing **elliptic curve cryptography** (ECC)
- Digital signatures
- Cryptographic constructs and blockchain technology

Before discussing cryptography further, some mathematical terms and concepts need to be explained in order to build a foundation for fully understanding the material provided later in this chapter.

Foundational mathematics

As the subject of cryptography is based on mathematics, this section will introduce some basic concepts that will help you understand the concepts presented later. An explanation with proofs and relevant background for all of these terms would require somewhat complex mathematics, which is beyond the scope of this book. More details on these topics can be found in any standard number theory, algebra, or cryptography book:

- **Modular arithmetic:** Also known as clock arithmetic, numbers in modular arithmetic wrap around when they reach a certain fixed number. This fixed number is a positive number called **modulus** (sometimes abbreviated to **mod**), and all operations are performed concerning this fixed number. In other words, this type of arithmetic deals with the remainders after the division operation. For example, $50 \bmod 11$ is 6 because $50/11$ leaves a remainder of 6.

- **Sets:** These are collections of distinct objects, for example, $X = \{1, 2, 3, 4, 5\}$.
- **Groups:** A group is a commutative set with an operation that combines two elements of the set. The group operation is closed and associated with a defined identity element. Additionally, each element in the set has an inverse. **Closure** (closed) means that if, for example, elements **A** and **B** are in the set, then the resultant element after performing an operation on the elements is also in the set. **Associative** means that the grouping of elements does not affect the result of the operation. Four group axioms must be satisfied for a set to qualify as a group. These group axioms include closure, associativity, an identity element, and an inverse element:
 - **Cyclic group:** A type of group that can be generated by a single element called the group generator.
- **Fields:** A field is a set in which all its elements form an additive or multiplicative group. It satisfies specific axioms for addition and multiplication. For all group operations, the **distributive law** is also applied. The law dictates that the same sum or product will be produced, even if any of the terms or factors are reordered:
 - A **finite field** is one with a finite set of elements. Also known as **Galois fields**, these structures are of particular importance in cryptography as they can be used to produce accurate and error-free results of arithmetic operations.
 - A **prime field** is a finite one with a prime number of elements. It has specific rules for addition and multiplication, and each non-zero element in the field has an inverse. Addition and multiplication operations are performed mod p , that is, modulo of a prime number.
 - **Order:** The number of elements in a field. It is also known as the cardinality of the field.

This completes a basic introduction to some mathematical concepts involved in cryptography. In the next section, you will be introduced to cryptography concepts.

Asymmetric cryptography

Asymmetric cryptography refers to a type of cryptography where the key that is used to encrypt the data is different from the key that is used to decrypt the data. These keys are called public and private keys, respectively, which is why asymmetric cryptography is also known as **public key cryptography**. It uses both public and private keys to encrypt and decrypt data, respectively. Various asymmetric cryptography schemes are in use, including **RSA** and **ElGamal** encryption.

Public and private keys

A **private key**, as the name suggests, is a randomly generated number that is kept secret and held privately by its users. Private keys need to be protected and no unauthorized access should be granted to those keys; otherwise, the whole scheme of public key cryptography is jeopardized, as this is the key that is used to decrypt messages. Private keys can be of various lengths, depending on the type and class of algorithms used. For example, in **RSA**, typically, a key of 1,024 bits or 2,048 bits is used. The 1,024-bit key size is no longer considered secure, and at least a 2,048-bit key size is recommended.

A **public key** is freely available and published by the private key owner. Anyone who would then like to send the publisher of the public key an encrypted message can do so, by encrypting the message using the published public key and sending it to the holder of the private key. No one else can decrypt the message because the corresponding private key is held securely by the intended recipient. Once the public key-encrypted message is received, the recipient can decrypt the message using the private key. There are a few concerns, however, regarding public keys. These include authenticity and identification of the publisher of the public keys.

A generic depiction of public - key cryptography is shown in the following diagram:

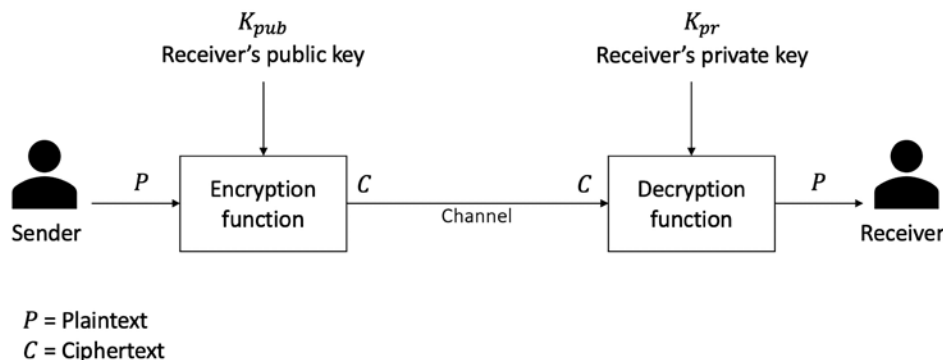


Figure 4.1: Encryption/decryption using public/private keys

The preceding diagram illustrates how a sender encrypts data **P** using the recipient's public key and encryption function, and produces an output of encrypted data **C**, which is then transmitted over the network to the receiver. Once it reaches the receiver, it can be decrypted using the receiver's private key by feeding the **C**-encrypted data into the decryption function, which will output plaintext **P**. This way, the private key remains on the receiver's side, and there is no need to share keys to perform encryption and decryption, which is the case with symmetric encryption.

The following diagram shows how the receiver uses public key cryptography to verify the integrity of the received message. In this model, the sender signs the data using their private key and transmits the message across to the receiver. Once the message is received, it is verified for integrity by the sender's public key.

It's worth noting that there is no encryption being performed in this model. It is simply presented here to help you thoroughly understand the material on message authentication and validation that will be provided later in this chapter:

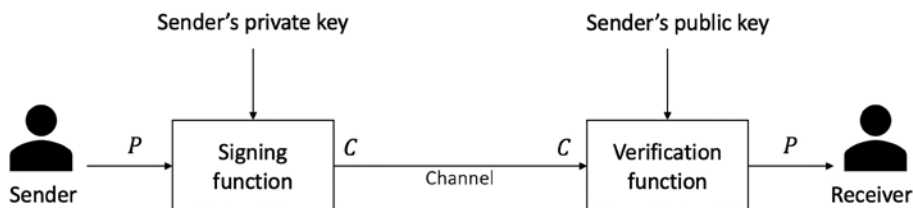


Figure 4.2: Model of a public-key cryptography digital signature scheme

The preceding diagram shows that the sender digitally signs the plaintext **P** with their private key using signing function **S**, and produces data **C**, which is sent to the receiver, who verifies **C** using the sender's public key and function **V** to ensure the message has indeed come from the sender.

Security mechanisms offered by public key cryptosystems include key establishment, digital signatures, identification, encryption, and decryption.

Key establishment mechanisms are concerned with the design of protocols that allow the setting up of keys over an insecure channel. Non-repudiation (defined in *Chapter 3, Symmetric Cryptography*, as the assurance that an action, once taken by someone, cannot be denied later) services, a very desirable option in many scenarios, can be provided using **digital signatures**. Sometimes, it is important not only to authenticate a user but also to identify the entity involved in a transaction. This can also be achieved by a combination of digital signatures and **challenge-response protocols**. Finally, the encryption mechanism to provide confidentiality can also be obtained using public key cryptosystems, such as RSA, ECC, and ElGamal.

Asymmetric cryptography algorithms

Public key algorithms are slower in terms of computation than symmetric key algorithms. Therefore, they are not commonly used in the encryption of large files or the actual data that requires encryption. They are usually used to exchange keys for symmetric algorithms. Once the keys are established securely, symmetric key algorithms can be used to encrypt the data.

Public key cryptography algorithms are based on various underlying mathematical functions. The three main **categories of asymmetric algorithms** are described here.

Integer factorization

Integer factorization schemes are based on the hard problem that large integers are extremely hard to factor. RSA is a prime example of this type of algorithm.

Discrete logarithm

A **discrete logarithm scheme** is based on a problem in modular arithmetic. It is easy to calculate the result of a modulo function, but it is computationally impractical to find the exponent of the generator. In other words, it is extremely difficult to find the input from the result (output). This is called a one-way function.

For example, consider the following equation:

$$3^2 \bmod 10 = 9$$

Now, given 9, the result of the preceding equation, finding 2, which is the exponent of the generator (3) in the equation, is extremely hard to determine. This difficult problem is commonly used in **Diffie-Hellman** key exchange and digital signature algorithms.

Elliptic curves

The **elliptic curve algorithm** is based on the discrete logarithm problem discussed previously, but in the context of elliptic curves. An **elliptic curve** is an algebraic cubic curve over a field, which can be defined by an equation, as shown here. The curve is non-singular, which means that it has no cusps or self-intersections. It has two variables a and b , as well as a point of infinity:

$$y^2 = x^3 + ax + b$$

Here, a and b are integers whose values are elements of the field on which the elliptic curve is defined. Elliptic curves can be defined over reals, rational numbers, complex numbers, or finite fields. For cryptographic purposes, an elliptic curve over prime finite fields is used instead of real numbers. Additionally, the prime should be greater than 3. Different curves can be generated by varying the values of a and/or b .

The most prominently used cryptosystems based on elliptic curves are the **Elliptic Curve Digital Signatures Algorithm (ECDSA)** and the **Elliptic Curve Diffie-Hellman (ECDH)** key exchange.

Integrated encryption scheme

An **integrated encryption scheme (IES)** is a hybrid encryption mechanism that combines public-key schemes with symmetric key schemes to achieve convenience and efficiency. Public-key schemes are convenient as there is no need to follow the cumbersome process of secret key sharing. On the other hand, symmetric-key schemes are more efficient than public-key schemes for data encryption. So, hybrid encryption schemes combine the best of both worlds to achieve efficiency and convenience.

Hybrid schemes are composed of two mechanisms, firstly a key encapsulation mechanism, which is a public key cryptosystem, and secondly a data encapsulation mechanism, which is a symmetric key encryption mechanism. Protocols such as TLS and SSH utilize a hybrid encryption scheme. An IES has two variants: a **discrete logarithm integrated encryption scheme (DLIES)** and an **elliptic curve integrated encryption scheme (ECIES)**.

In the following sections, we will introduce two examples of asymmetric key cryptography: **RSA** and **ECC**. RSA is the first implementation of public key cryptography, whereas ECC is used extensively in blockchain technology.

Introducing RSA

RSA was invented in 1977 by Ron Rivest, Adi Shamir, and Leonard Adelman, hence the name RSA. This type of public key cryptography is based on the integer factorization problem, where the multiplication of two large prime numbers is easy, but it is difficult to factor the product (the result of the multiplication) back into the two original numbers.

The crux of the work involved with the RSA algorithm happens during the key generation process. An RSA key pair is generated by performing the following steps:

1. Modulus generation:
 - Select p and q , which are very large prime numbers.

- Multiply p and q , $n=p.q$ to generate modulus n .
2. Generate the co-prime:
 - Assume a number called e .
 - e should satisfy a certain condition; that is, it should be greater than 1 and less than $(p-1)(q-1)$. In other words, e must be a number such that no number other than 1 can be divided into e and $(p-1)(q-1)$. This is called a **co-prime**, that is, e is the co-prime of $(p-1)(q-1)$.
 3. Generate the public key:
 - The modulus generated in *step 1* and co-prime e generated in *step 2* is a pair that is a public key. This part is the public part that can be shared with anyone; however, p and q need to be kept secret.
 4. Generate the private key:
 - The private key is called d here and is calculated from p , q , and e . The private key is basically the inverse of e modulo $(p-1)(q-1)$. As an equation, it is as follows:

$$ed = 1 \bmod (p-1)(q-1)$$

Usually, an extended Euclidean algorithm is used to take p , q , and e and calculate d . The key idea in this scheme is that anyone who knows p and q can easily calculate private key d by applying the extended Euclidean algorithm. However, someone who does not know the value of p and q cannot generate d . This also implies that p and q should be large enough for the modulus n to become extremely difficult (computationally impractical) to factor.



The original RSA paper is available here: <http://web.mit.edu/6.857/OldStuff/Fall03/ref/rivest78method.pdf>.

Rivest, R.L., Shamir, A., and Adleman, L., 1978. A method for obtaining digital signatures and public-key cryptosystems. Communications of the ACM, 21(2), pp.120–126.

Now, let's see how encryption and decryption operations are performed using RSA. RSA uses the following equation to produce ciphertext:

$$C = P^e \bmod n$$

This means that plaintext P is raised to the power of e and then reduced to modulo n . Decryption in RSA is provided in the following equation:

$$P = C^d \bmod n$$

This means that the receiver who has a public key pair (n, e) can decipher the data by raising C to the value of the private key d , and then reducing to modulo n .

Encrypting and decrypting with RSA

The following example illustrates how RSA public and private key pairs can be generated using the OpenSSL command line.

First, the RSA private key can be generated using OpenSSL as follows:

```
$ openssl genpkey -algorithm RSA -out privatekey.pem -pkeyopt rsa_keygen_
bits:1024
.....++++++
.....++++++
```

After executing the command, a file named `privatekey.pem` is produced, which contains the generated private key, as follows:

```
$ cat privatekey.pem
-----BEGIN PRIVATE KEY-----
MIICdgIBADANBgkqhkiG9w0BAQEFAASCAmAwggJcAgEAAoGBAKJ0FBzPy2v0d6em Bk/
UGrZdy7TVgDYnYxBfiEJId/r+eYMt/F14k2fDToVwxXaXTxiQgD+BKuiey/69
9itnrqW/xy/pocDMvobj8QCngEntOdNoVSan+t0f9nRM3iVM94mz3/C/v4vXvoac
PyPkr/0jhIV0woCurXGTghgqIbHRAgMBAAECgYEA1B3s/N41Jh011TkOSYunWtzt
6isnNkr7g1WrY9H+rG9xx4kP5b1DyE3SvxBLJA6xgBle8JVQMzm3sKJrJPFZzzT5
NNNnugCxairxcF1mPzJAP3aqpcSjxKpTv4qqqYevwgW1A0R3xKQZzBKU+bT02hXV
D1oHxu75mDY3xCwqSAECQQDUYV04wNSEjEy9tYJ0zaryDACvd/VG2/U/6qiQGajB
eSpSqoEESigbusKku+wVtRYgWWEomL/X58t+K01eMMZZAKEAw6PUR9YLebsm/
Sji i0ShV4AKuFdi7t7DYWE5Ulb1uqP/i28zN/ytt4BXKIs/KcFykQGeAC6LDHzyycy
ntDIOQJAVqrE1/wYvV5jkqcXbYlgV5YA+KYD0b9Y/ZRM5UETVKCVXNanf5Cjfw1h
MMhfNxyGwvy2YVK0Nu8oY3xYPI+5QQJAUGcm0Re4w6Cs12JUJ5p+zG0s+rG/URhw
B7djTXm7p6b6wR1EWYAZDM9MArenj8uXAA1AGCcIsmiDqHfU7lgz0QJAE9mOdNGW
7qRppgm0E5nuEbxDKSI70qHYb0LuwfCjHzJBrSgqyi6pj9/9CbXJrZPgNDwdLEb
GgpDKtZs9gLv3A==
-----END PRIVATE KEY-----
```



It is OK to reveal the private key here because we are simply using it for our example. However, in production systems, safeguarding the private key is of utmost importance. Make sure that the private key is always kept secret. Also, remember that the preceding key is just being used here as an example; do not reuse it.

As the private key is mathematically linked to the public key, it is also possible to generate or derive the public key from the private key. Using the example of the preceding private key, the public key can be generated as follows:

```
$ openssl rsa -pubout -in privatekey.pem -out publickey.pem
writing RSA key
```

The public key can be viewed using a file reader or any text viewer:

```
$ cat publickey.pem
-----BEGIN PUBLIC KEY-----
MIGfMA0GCSqGSIb3DQEBAQUAA4GNADCBiQKBgQCiTThQcz8trznenpgZP1Bq8w8u0
74A2J2MQX4hCSHF6/hMjLfxdeJNnw0zlcMV2l08YkIA/gSronsV+vfYrZ66lv8cv 6aHAzL6G4/
EAp4BJ7TnTaFUmjfrdH/Z0TN4lTPeJs9/wv7+L176GnD8j5K/9I4SF dMKArq1xk4IYKiGx0QIDAQAB
-----END PUBLIC KEY-----
```

In order to see more details of the various components, such as the modulus, prime numbers that are used in the encryption process, or exponents and coefficients of the generated private key, the following command can be used (only part of the output is shown here as the actual output is very long):

```
$ openssl rsa -text -in privatekey.pem
Private-Key: (1024 bit)
modulus:
```

Similarly, the public key can be explored using the following commands. Public and private keys are base64-encoded:

```
$ openssl pkey -in publickey.pem -pubin -text
-----BEGIN PUBLIC KEY-----
MIGfMA0GCSqGSIb3DQEBAQUAA4GNADCBiQKBgQCiTThQcz8trznenpgZP1Bq8w8u0
74A2J2MQX4hCSHF6/hMjLfxdeJNnw0zlcMV2l08YkIA/gSronsV+vfYrZ66lv8cv 6aHAzL6G4/
EAp4BJ7TnTaFUmjfrdH/Z0TN4lTPeJs9/wv7+L176GnD8j5K/9I4SF dMKArq1xk4IYKiGx0QIDAQAB
-----END PUBLIC KEY-----
Public-Key: (1024 bit) Modulus:
00:a2:4e:14:1c:cf:cb:6b:ce:77:a7:a6:06:4f:d4:
1a:bc:c3:cb:b4:ef:80:36:27:63:10:5f:88:42:48:
77:fa:fe:13:23:2d:fc:5d:78:93:67:c3:4c:e5:70:
c5:76:97:4f:18:90:80:3f:81:2a:e8:9e:cb:fe:bd:
f6:2b:67:ae:a5:bf:c7:2f:e9:a1:c0:cc:be:86:e3:
f1:00:a7:80:49:ed:39:d3:68:55:26:8d:fa:dd:1f:
f6:74:4c:de:25:4c:f7:89:b3:df:f0:bf:bf:8b:d7:
be:86:9c:3f:23:e4:af:fd:23:84:85:74:c2:80:ae:
ad:71:93:82:18:2a:21:b1:d1
Exponent: 65537 (0x10001)
```

Now, the public key can be shared openly, and anyone who wants to send you a message can use the public key to encrypt the message and send it to you.

Taking the private key we generated in the previous example, the command to encrypt a text file `message.txt` can be constructed, as shown here:

```
$ echo datatoencrypt > message.txt
```



```
$ openssl rsautl -encrypt -inkey publickey.pem -pubin -in message.txt -out
message.rsa
```

This will produce a file named `message.rsa`, which is in binary format. If you display `message.rsa`, it will show some scrambled data, as shown below:

```
$ cat message.rsa
s???c?ngJ[Lt!?!LgC\[f?L?1?^q?r?
a?????Da?=??m??_?P?Y???KE
```

In order to decrypt the RSA-encrypted file, the following command will employ the corresponding private key to decrypt the file:

```
$ openssl rsautl -decrypt -inkey privatekey.pem -in message.rsa -out message.
dec
```

Now, if the file is read using `cat`, decrypted plaintext can be seen, as shown here:

```
$ cat message.dec
datatoencrypt
```

With this example, we complete our introduction to RSA. Next, let's explore elliptic curve cryptography.

Introducing ECC

ECC is based on the discrete logarithm problem, founded upon elliptic curves over finite fields (Galois fields). The main benefit of ECC over other types of public key algorithms is that it requires a smaller key size, while providing the same level of security as, for example, RSA. Two notable schemes that originate from ECC are ECDH for key exchange and ECDSA for digital signatures.

ECC can also be used for encryption, but it is not usually used for this purpose in practice. Instead, key exchange and digital signatures are used more commonly. As ECC needs less space to operate, it is becoming very popular on embedded platforms and in systems where storage resources are limited. By comparison, the same level of security can be achieved with ECC when only using 256-bit operands as compared to 3,072 bits in RSA.

Mathematics behind ECC

To understand ECC, a basic introduction to the underlying mathematics is necessary. An elliptic curve is basically a type of polynomial equation known as the **Weierstrass equation**, which generates a curve over a finite field. The most commonly used field is where all arithmetic operations are performed modulo a prime number p . Elliptic curve groups consist of points on the curve over a finite field.

An elliptic curve is defined in the following equation:

$$y^2 = x^3 + Ax + B \bmod P$$

Here, A and B belong to a finite field \mathbb{Z}_p or \mathbb{F}_p (a prime finite field), along with a special value called the point of infinity. The point of infinity, ∞ , is used to provide identity operations for points on the curve.

Furthermore, a condition also needs to be met that ensures that the equation mentioned earlier has no repeated roots. This means that the curve is non-singular.

The condition is described in the following equation, which is a standard requirement that needs to be met. More precisely, this ensures that the curve is non-singular:

$$4a^3 + 27b^2 \neq 0 \pmod{p}$$

To construct the discrete logarithm problem based on elliptic curves, a large enough cyclic group is required. First, the group elements are identified as a set of points that satisfy the previous equation. After this, group operations need to be defined on these points.

Basic group operations on elliptic curves are point addition and point doubling. **Point addition** is a process where two different points are added, and **point doubling** means that the same point is added to itself.

Point addition

Point addition is shown in the following diagram. This is a geometric representation of point addition on elliptic curves. In this method, a diagonal line is drawn through the curve that intersects the curve at two points shown below P and Q , which yields a third point between the curve and the line.

This point is mirrored as $P+Q$, which represents the result of the addition as R . This is shown as $P+Q$ in the following diagram:

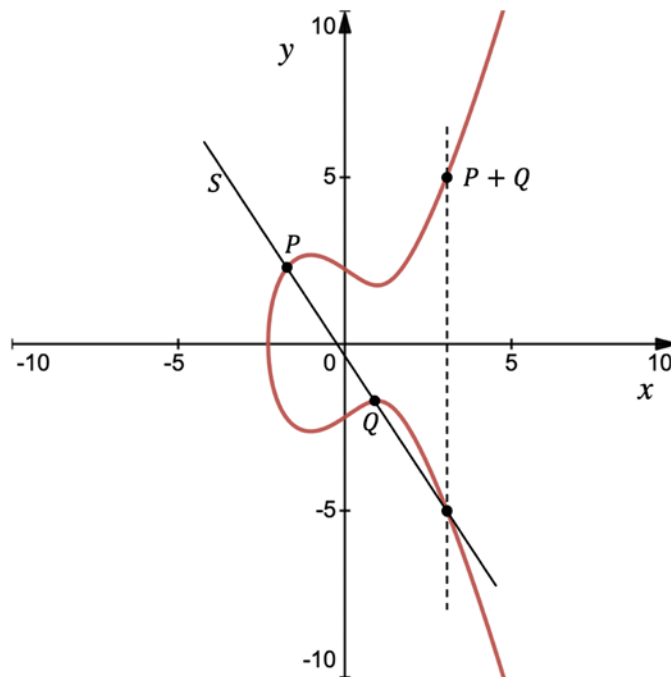


Figure 4.3: Point addition over R

The group operation denoted by the + sign for addition yields the following equation:

$$P + Q = R$$

More precisely, this means that coordinates are added, as shown in the following equation:

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$$

So, if $P \neq Q$, then it means point addition. To add these points, see as follows:

$$S = \frac{(y_1 - y_2)}{(x_1 - x_2)} \bmod p$$

S depicts the line going through P and Q .

$$x_3 = S^2 - x_1 - x_2 \bmod p$$

$$y_3 = S(x_1 - x_3) - y_1 \bmod p$$

So far, we've learned how the point addition group operation works and seen the analytical expressions of the addition group operation.

Next, let's look at a complete example of point addition. This example shows the addition and solution for the equation over a finite field F_{23} , which contains 23 elements.

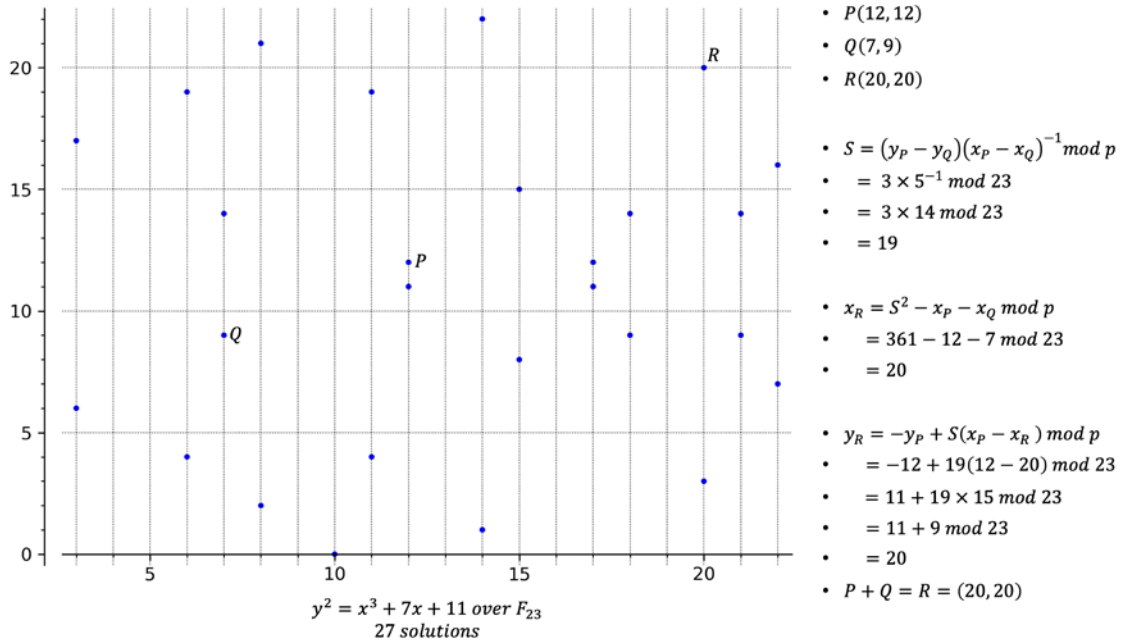


Figure 4.4: Example of point addition

In the preceding example, the graph on the left-hand side shows the points that satisfy this equation:

$$y^2 = x^3 + 7x + 11$$

There are 27 solutions to the equation shown earlier over the finite field F_{23} . P and Q are chosen to be added to produce point R . Calculations are shown on the right-hand side, which calculates the third point R . Note that in the preceding graph, in the calculation shown on the right-hand side, l is used to depict the line going through P and Q .

As an example, to show how the equation is satisfied by the points shown in the graph, a point (x, y) is picked up where $x = 3$ and $y = 6$. This point can be visualized in the graph shown here. Notice the point at coordinate $(3, 6)$, indicated by an arrow:

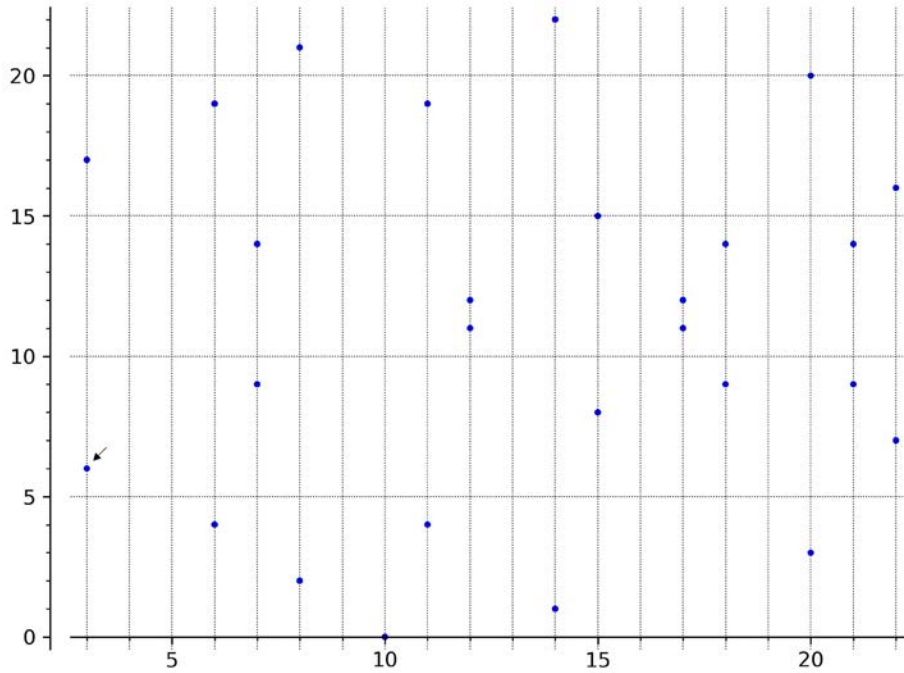


Figure 4.5: Point at $(3, 6)$ shown with an arrow

Using these values shows that the equation is indeed satisfied:

$$y^2 \bmod 23 = x^3 + 7x + 11 \bmod 23$$

$$6^2 \bmod 23 = 3^3 + 7(3) + 11 \bmod 23$$

$$36 \bmod 23 = 59 \bmod 23$$

$$13 = 13$$

The next section introduces the concept of point doubling, which is another operation that can be performed on elliptic curves.

Point doubling

The other group operation on elliptic curves is called **point doubling**. This is a process where P is added to itself. This is the case when P and Q are at the same point, so effectively the operation becomes adding the point to itself and is therefore called point doubling. In this method, a tangent line is drawn through the curve, as shown in the following graph. The second point is obtained, which is at the intersection of the tangent line drawn and the curve. This point is then mirrored to yield the result, which is shown as $2P = P + P$:

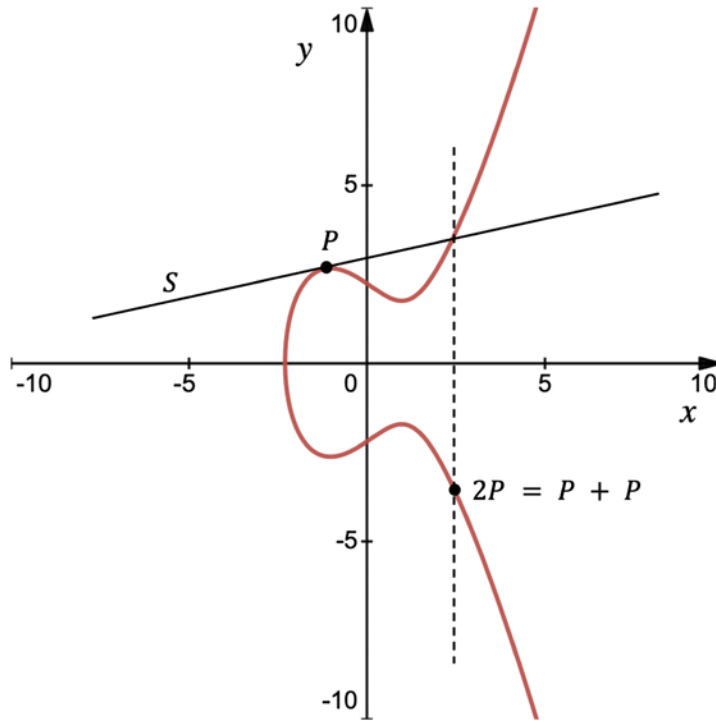


Figure 4.6: Graph representing point doubling

In the case of point doubling, the equation becomes:

$$S = \frac{3x_1^2 + a}{2y_1}$$

$$x_3 = s^2 - x_1 - x_2 \bmod p$$

$$y_3 = s(x_1 - x_3) - y_1 \bmod p$$

Here, S is the slope of the tangent line going through P , which is the line shown at the top in the preceding graph. In the preceding example, the curve is plotted as a simple example, and no solution to the equation is shown.

The following example shows the solutions and point doubling of elliptic curves over the finite field F_{23} .

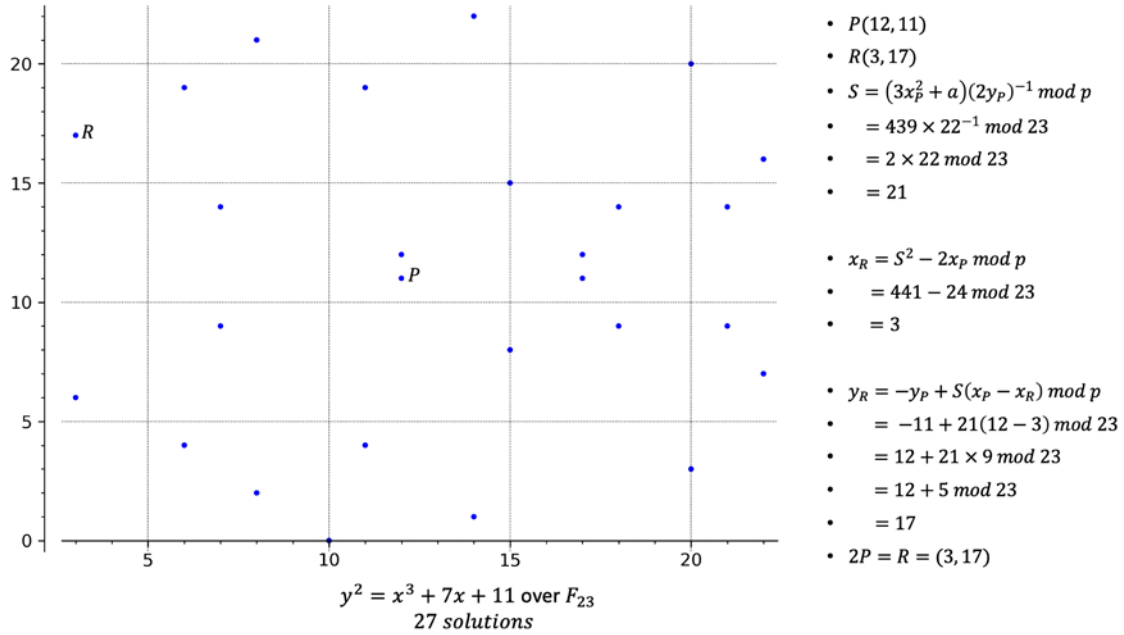


Figure 4.7: Example of point doubling

The graph on the left-hand side shows the points that satisfy the equation:

$$y^2 = x^3 + 7x + 11$$

As shown on the right-hand side of the preceding graph, the calculation finds R , after P is added to itself (point doubling). There is no Q shown here because the same point P is used for doubling.

Using addition and doubling, we can construct another operation, called point multiplication, which we introduce next.

Point multiplication

This operation is also called scalar point multiplication. It is used to multiply points on elliptic curves by a given integer. Let's call this integer d and the point P . We get dP by repeatedly adding P , d times. This operation can be described as follows:

$$P + P + \dots + P = dP, \text{ where } P \text{ is added } d \text{ times.}$$

Any point on the curve can be added multiple times to itself. The result of this operation is always another point on the curve.

The addition operation is not efficient for large values of d . However, point multiplication operation can be made efficient by utilizing the **double and add algorithm**, which combines point addition and doubling operations to achieve exponential performance gain.

For example, if using addition only, to get $9P$, we must do $P + P + P + P + P + P + P + P + P$, which can become infeasible quickly if the number of P s increases. To address this, we can use the double-and-add mechanism where we first convert nine into binary, then starting from the most significant bit, for each bit that is 1 (high) perform a double-and-add operation, and for each 0, perform only the double operation. We do not perform any operation on the most significant bit. As 9 is 1001 in binary, we get for each bit, starting from left to right, P , $2P$, $4P$, and $8P+P$. This process produces $9P$ only with three double operations and one addition operation, instead of nine addition operations.

In this example, point doubling and addition have been used to construct an efficient operation of scalar multiplication. Now consider that dP results in producing another point on the curve; let's call that point T . We can say that with all these doublings and additions we have computed a point called T . We multiplied a point P on the curve with a number d to compute another point T .

Here's the key idea now: even if we know points P and T , it is computationally infeasible to reconstruct the sequence of all the double-and-add operations that we did to figure out the number d . In other words, even if someone knows P and T , it is almost impossible for them to find d . This means that it is a one-way function, and it is the basis of the **elliptic curve discrete logarithm problem (ECDLP)**. We describe the discrete logarithm problem in more detail next.

The discrete logarithm problem

The discrete logarithm problem in ECC is based on the idea that, under certain conditions, all points on an elliptic curve form a cyclic group.

On an elliptic curve, the public key is a random multiple of the generator point, whereas the private key is a randomly chosen integer used to generate the multiple. In other words, a private key is a randomly selected integer, whereas the public key is a point on the curve. The discrete logarithm problem is used to find the private key (an integer) where that integer falls within all points on the elliptic curve. The following equation shows this concept more precisely.

Consider an elliptic curve E , with two elements P and T . The discrete logarithmic problem is to find the integer d , where $1 \leq d \leq \#E$, such that:

$$P + P + \dots + P = dP = T$$

Here, T is the public key (a point on the curve, (x, y)), and d is the private key. In other words, the public key is a random multiple of the generator P , whereas the private key is the integer that is used to generate the multiple. $\#E$ represents the order of the elliptic curve, which means the number of points that are present in the cyclic group of the elliptic curve. A **cyclic group** is formed by a combination of points on the elliptic curve and the point of infinity.

The initial starting point P is a public parameter, and the public key T is also published, whereas d , the private key, is kept secret. If d is not known, it is impractical to calculate it by only having the knowledge of T and P , thus creating the hard problem on which ECDLP is built.

A key pair is linked with the specific domain parameters of an elliptic curve. Domain parameters include the field size, the field representation, two elements from the fields a and b , two field elements X_g and Y_g , order n of point G , which is calculated as $G = (X_g, Y_g)$, and the cofactor $h = \#E(F_q)/n$. A practical example using OpenSSL will be described later in this section.

Various parameters are recommended and standardized so they can be used as curves with ECC. An example of the secp256k1 specification is shown here. The following figure is an excerpt taken from the standard specification from <http://www.secg.org/sec2-v2.pdf>. It is used in Bitcoin.

The elliptic curve domain parameters over F_p , associated with a Koblitz curve secp256k1, are specified by the sextuple $T = (p, a, b, G, n, h)$, where the finite field F_p is defined by:

$$P = \text{FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE FFFFFFFF FFFFFFFF FFFFFFFE FFFFFFFC2F}$$

$$= 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1$$

The curve $E: y^2 = x^3 + ax + b$ over F_p is defined by:

$$a = 00000000 00000000 00000000 00000000 00000000 00000000 00000000 00000000$$

$$b = 00000000 00000000 00000000 00000000 00000000 00000000 00000000 00000007$$

The base point G in compressed form, this is:

$$G = 02\ 79BE667E\ F9DCBBAC\ 55A06295\ CE870B07\ 029BFCDB\ 2DCE28D9\ 59F2815B\ 16F81798$$

In the uncompressed form is:

$$G = 04\ 79BE667E\ F9DCBBAC\ 55A06295\ CE870B07\ 029BFCDB\ 2DCE28D9\ 59F2815B\ 16F81798$$

$$483ADA77\ 26A3C465\ 5DA4FBFC\ 0E1108A8\ FD17B448\ A6855419\ 9C47D08F\ FB10D4B8$$

Finally, the order n of G and the cofactor are:

$$n = \text{FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE BAAEDCE6 AF48A03B BFD25E8C D0364141}$$

$$h = 01$$

Here, we see:

- P is the prime p that specifies the size of the finite field.
- a and b are the coefficients of the elliptic curve equation.
- G is the base point that generates the required subgroup, also known as the generator. The base point can be represented in either compressed or uncompressed form. There is no need to store all points on the curve in a practical implementation. The compressed generator works because the points on the curve can be identified using only the x coordinate and the least significant bit of the y coordinate.
- n is the order of the subgroup.
- h is the cofactor of the subgroup.

We can also use the OpenSSL command line to view these parameters of secp256k1. This can be seen here:

```
$ openssl ecparam -param_enc explicit -text -noout -name secp256k1
```


This command will show the output as follows:

```
Field Type: prime-field
```



```
Prime:
  00:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:
  ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:fe:ff:
  ff:fc:2f
A:  0
B:  7 (0x7)
Generator (uncompressed):
  04:79:be:66:7e:f9:dc:bb:ac:55:a0:62:95:ce:87:
  0b:07:02:9b:fc:db:2d:ce:28:d9:59:f2:81:5b:16:
  f8:17:98:48:3a:da:77:26:a3:c4:65:5d:a4:fb:fc:
  0e:11:08:a8:fd:17:b4:48:a6:85:54:19:9c:47:d0:
  8f:fb:10:d4:b8
Order:
  00:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:
  ff:fe:ba:ae:dc:e6:af:48:a0:3b:bf:d2:5e:8c:d0:
  36:41:41
Cofactor: 1 (0x1)
```

This output can be readily compared and verified using the SECP256K1 specification shown earlier.



Note that there are different types of curves, and they must be chosen carefully in order to ensure cryptographic security guarantees. There are also curves that have been standardized by the NIST in the US and published in the FIPS 186 document.

You can find more about safe curves and their selection criteria at <https://safecurves.cr.yp.to>.

As a summary, a quick comparison of RSA and ECC cryptography is shown below.

Feature	ECC	RSA
Key size	Smaller	Larger
Key generation speed	Faster	Slower
Encryption speed	Faster	slower
Decryption speed	Slower	faster
Quantum-resistant	No	No

In the following section, examples of using OpenSSL will be shown to help you understand the practical aspects.

Generating keys with ECC

OpenSSL provides a very rich library of functions to perform ECC. In this section, first, an example will be presented that demonstrates the creation of a private key using the ECC functions available in the OpenSSL library.

ECC is based on domain parameters defined by various standards. You can see the list of all available standards defined and the recommended curves available in OpenSSL using the following command (once again, only part of the output is shown here):

```
$ openssl ecparam -list_curves
.
.
.
brainpoolP384r1: RFC 5639 curve over a 384 bit prime field brainpoolP384t1: RFC
5639 curve over a 384 bit prime field brainpoolP512r1: RFC 5639 curve over a
512 bit prime field brainpoolP512t1: RFC 5639 curve over a 512 bit prime field
```

Here, a private key based on SECP256K1 will be generated:

```
$ openssl ecparam -name secp256k1 -genkey -noout -out ec-privatekey.pem
```

This command will produce a file named `ec-privatekey.pem`, which we can view using the command shown here:

```
$ cat ec-privatekey.pem
-----BEGIN EC PRIVATE KEY-----
MHQCAQEEIJHUIm9NZAgfpUrSxUk/iINq1ghM/ewn/RLNreuR52h/oAcGBSuBBAK
oUQDQgAE0G33mCZ4PKbg5EtwQjk6ucv9Qc9DTr8JdcGXYGxHdzr0Jt1INaYE0GG
ChFMT5pK+wfvSLkY15ul0oczwwKjng==
-----END EC PRIVATE KEY-----
```

The file named `ec-privatekey.pem` now contains the elliptic curve private key that is generated based on the SECP256K1 curve. In order to generate a public key from a private key, issue the following command:

```
$ openssl ec -in ec-privatekey.pem -pubout -out ec-pubkey.pem
read EC key
writing EC key
```

Reading the file produces the following output, displaying the generated public key:

```
$ cat ec-pubkey.pem
-----BEGIN PUBLIC KEY-----
MFYwEAYHkoZiZj0CAQYFK4EEAAoDQgAE0G33mCZ4PKbg5EtwQjk6ucv9Qc9DTr8J
dcGXYGxHdzr0Jt1INaYE0GGChFMT5pK+wfvSLkY15ul0oczwwKjng==
-----END PUBLIC KEY-----
```

Now, the `ec-pubkey.pem` file contains the public key derived from `ec-privatekey.pem`. The private key can be further explored using the following command:

```
$ openssl ec -in ec-privatekey.pem -text -noout
read EC key
Private-Key: (256 bit) priv:
```

```

00:91:d4:22:6f:4d:64:08:1f:a5:4a:d2:c5:49:3f:
88:83:6a:d6:08:4c:fd:ec:27:fd:12:cd:ad:eb:91: e7:68:7f
pub:

04:d0:6d:f7:98:26:78:3c:a6:e0:e4:4b:70:42:39:
3a:b9:cb:fd:41:cf:43:4e:bf:09:75:c1:97:60:6c:
47:77:3a:f4:26:dd:4d:22:76:98:13:41:86:0a:11:
4c:4f:9a:4a:fb:07:ef:48:b9:18:97:9b:a5:d2:87:
33:c1:62:a3:9e
ASN1 OID: secp256k1

```

Similarly, the public key can be further explored with the following command:

```

$ openssl ec -in ec-pubkey.pem -pubin - text -noout
read EC key
Private-Key: (256 bit) pub:
04:d0:6d:f7:98:26:78:3c:a6:e0:e4:4b:70:42:39:
3a:b9:cb:fd:41:cf:43:4e:bf:09:75:c1:97:60:6c:
47:77:3a:f4:26:dd:4d:22:76:98:13:41:86:0a:11:
4c:4f:9a:4a:fb:07:ef:48:b9:18:97:9b:a5:d2:87:
33:c1:62:a3:9e
ASN1 OID: secp256k1

```

It is also possible to generate a file with the required parameters—in this case, SECP256K1—and then explore it further to understand the underlying parameters:

```

$ openssl ecparam -name secp256k1 -out secp256k1.pem
$ cat secp256k1.pem
-----BEGIN EC PARAMETERS-----
BgUrgQQACg==
-----END EC PARAMETERS-----

```

The file now contains all the SECP256K1 parameters, and it can be analyzed using the following command:

```

$ openssl ecparam -in secp256k1.pem -text -param_enc explicit -noout

```

This command will produce output similar to the one shown here:

```

Field Type: prime-field Prime:
00:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:
ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:fe:ff: ff:fc:2f
A: 0
B: 7 (0x7)
Generator (uncompressed): 04:79:be:66:7e:f9:dc:bb:ac:55:a0:62:95:ce:87:

```

```

0b:07:02:9b:fc:db:2d:ce:28:d9:59:f2:81:5b:16:
f8:17:98:48:3a:da:77:26:a3:c4:65:5d:a4:fb:fc:
0e:11:08:a8:fd:17:b4:48:a6:85:54:19:9c:47:d0:
8f:fb:10:d4:b8 Order:
00:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:ff:
ff:fe:ba:ae:dc:e6:af:48:a0:3b:bf:d2:5e:8c:d0:
36:41:41
Cofactor: 1 (0x1)

```

The preceding example shows the prime number used and the values of A and B, along with the generator, order, and cofactor of the SECP256K1 curve domain parameters. With this, our introduction to public key cryptography from an encryption and decryption perspective is complete. Next, we introduce digital signatures.

Digital signatures

Digital signatures provide a means of associating a message with an entity from which the message has originated. Digital signatures are used to provide data origin authentication and non-repudiation.

Various schemes, such as RSA, DSA, and ECDSA-based digital signature schemes, are used in practice. RSA is the most common; however, with the traction of ECC, ECDSA-based schemes have become quite popular. This is beneficial in blockchains because ECC provides the same level of security that RSA does, but it uses less space. Also, the generation of keys is much faster in ECC compared to RSA, so it helps with the overall performance of the blockchain. The following table shows that ECC can provide the same level of cryptographic strength as an RSA-based system with a smaller key size:

RSA key size (bits)	Elliptic curve key size (bits)
1,024	160
2,048	224
3,072	256
7,680	384
15,360	521

Digital signatures are calculated in two steps. As an example, the high-level steps of the RSA digital signature scheme are as follows.

RSA digital signature algorithms

RSA-based digital signature algorithms are calculated using the two steps listed here. Fundamentally, the idea is to first compute the hash of the data and then sign it with the private key:

- **Calculate the hash value of the data packet:** This will provide the data integrity guarantee, as the hash can be computed at the receiver's end again and matched with the original hash to check whether the data has been modified in transit. Technically, message signing can work without hashing the data first, but that is not considered secure.

- **Sign the hash value with the signer's private key:** As only the signer has the private key, the authenticity of the signature and the signed data is ensured.

Digital signatures have some important properties, such as authenticity, unforgeability, and non-reusability:

- **Authenticity** means that digital signatures are verifiable by a receiving party.
- The **unforgeability** property ensures that only the sender of the message can use the signing functionality using the private key. Digital signatures must provide protection against **forgery**. Forgery means an adversary fabricating a valid signature for a message without any access to the legitimate signer's private key. In other words, unforgeability means that no one else can produce the signed message produced by a legitimate sender. This is also called the property of **non-repudiation**.
- **Non-reusability** means that the digital signature cannot be separated from a message and used again for another message. In other words, the digital signature is firmly bound to the corresponding message and cannot be simply cut from its original message and attached to another.

The operation of a generic digital signature function is shown in the following diagram:

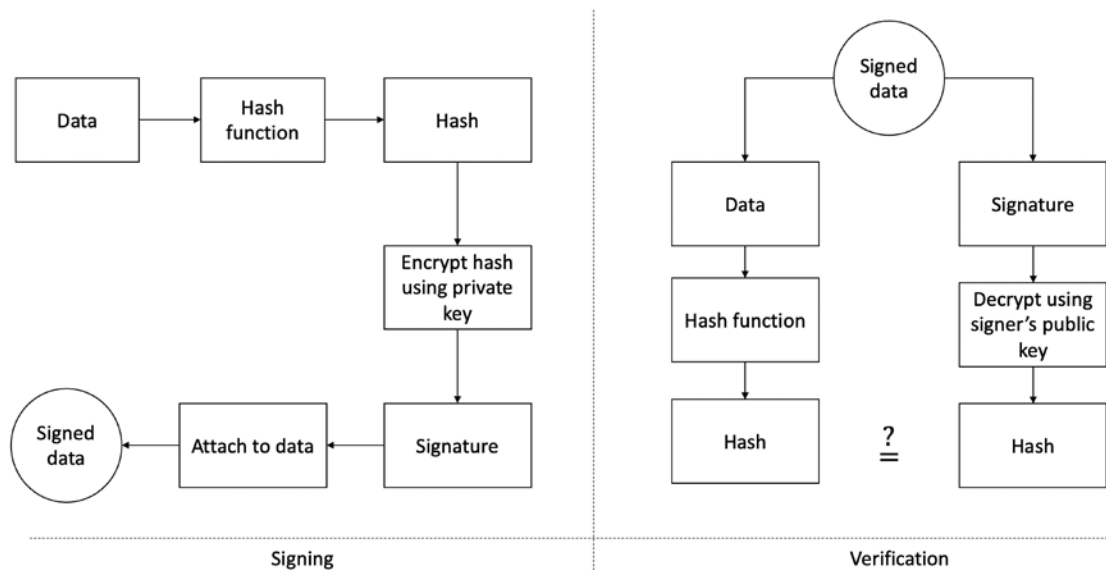


Figure 4.8: Digital signing (left) and verification process (right) (example of RSA digital signatures)

If a sender wants to send an authenticated message to a receiver, there are two methods that can be used:

- **Sign then encrypt:** With this approach, the sender digitally signs the data using the private key, appends the signature to the data, and then encrypts the data and the digital signature using the receiver's public key. This is considered a more secure scheme compared to the **encrypt then sign** scheme described next.
- **Encrypt then sign:** With this method, the sender encrypts the data using the receiver's public key and then digitally signs the encrypted data.



In practice, a digital certificate that contains the digital signature is issued by a **certificate authority (CA)** that associates a public key with an identity.

Various practical examples will now be shown that demonstrate how the RSA digital signature can be generated, used, and verified using OpenSSL.

Generating RSA digital signatures

The first step is to generate a hash of the message file. Note that the SHA-256 hash is just chosen as an example:

```
$ openssl dgst -sha256 message.txt
SHA256(message.txt)=
eb96d1f89812bf4967d9fb4ead128c3b787272b7be21dd2529278db1128d559c
```

Both hash generation and signing can be done in a single step, as shown here. Note that `privatekey.pem` was generated in the steps provided previously:

```
$ openssl dgst -sha256 -sign privatekey.pem -out signature.bin message.txt
```

Now, let's display the directory showing the relevant files:

```
$ cat signature.bin
```

To verify the signature, the following operation can be performed:

```
$ openssl dgst -sha256 -verify publickey.pem -signature signature.bin message.txt
Verified OK
```

Similarly, if some other signature file that is not valid is used, the verification will fail, as shown here:

```
$ openssl dgst -sha256 -verify publickey.pem -signature someothersignature.bin message.txt
Verification Failure
```

Now that we understand how the RSA digital signature scheme works, in the next section, we'll introduce ECDSA, which is another popular digital signature scheme.

The elliptic curve digital signature algorithm

The ECDSA is a DSA based on elliptic curves. The DSA is a standard for digital signatures. It is based on modular exponentiation and the discrete logarithm problem. It is used on the Bitcoin and Ethereum blockchain platforms to validate messages and provide data integrity services. Now, we'll describe how the ECDSA works.

To sign and verify using the ECDSA scheme, the first key pair needs to be generated:

1. First, define an elliptic curve E with the following:

- Modulus P
- Coefficients a and b
- Generator point A that forms a cyclic group of prime order q

2. An integer d is chosen randomly so that $0 < d < q$.

3. Calculate public key B so that $B = dA$:

- The public key is a sextuple in the form shown here:

$$Kpb = (p, a, b, q, A, B)$$

- The private key is a randomly chosen integer d in step 2:

$$Kpr = d$$

- Now, the signature can be generated using the private and public keys.
- An ephemeral key K_e is chosen, where $0 < K_e < q$. It should be ensured that K_e is truly random and that no two signatures have the same key; otherwise, the private key can be calculated.
- Another value R is calculated using $R = K_e A$ —that is, by multiplying A (the generator point) and the random ephemeral key.
- Initialize a variable r with the x coordinate value of point R so that $r = xR$.
- The signature can be calculated as follows:

$$S = (h(m) + dr)K_e^{-1} \bmod q$$

4. Here, m is the message for which the signature is being computed, and $h(m)$ is the hash of the message m .

5. Signature verification is carried out by following this process:

- Auxiliary value w is calculated as $w = s^{-1} \bmod q$
- Auxiliary value $u_1 = w \cdot h(m) \bmod q$
- Auxiliary value $u_2 = w \cdot r \bmod q$
- Calculate point P , $P = u_1A + u_2B$

6. Verification is carried out as follows:

- r, s is accepted as a valid signature if the x coordinate of point P calculated in step 4 has the same value as the signature parameter $r \bmod q$ —that is:

$$X_p = r \bmod q \text{ means valid signature}$$

$$X_p \neq r \bmod q \text{ means valid signature}$$

Next, an example will be presented that shows how OpenSSL can be used to perform ECDSA-related operations based on ECC.

Generating ECDSA digital signatures

In this example we'll see how an ECDSA digital signature can be generated using OpenSSL. First, the private key is generated using the following commands:

```
$ openssl ecparam -genkey -name secp256k1 -noout -out eccprivatekey.pem
$ cat eccprivatekey.pem
-----BEGIN EC PRIVATE KEY-----
MHQCAQEEIMVmyrnED0s7SYxS/AbXoIwqZqJ+gND9Z2/nQyzcpaPBoAcGBSuBBAK
oUQDQgAEEKKS4E4+TATIEBX8o2J6PxKkjcoWrXPwNRo/k4Y/CZA4pXvlyTgH5LYm
QbU0qUtPM7dAEzOsaoXmetqB+6cM+Q==
-----END EC PRIVATE KEY-----
```

Next, the public key is generated from the private key:

```
$ openssl ec -in eccprivatekey.pem -pubout -out eccpublickey.pem
read EC key
writing EC key
$ cat eccpublickey.pem
-----BEGIN PUBLIC KEY-----
MFYwEAYHKoZIzj0CAQYFK4EEAAoDQgAEEKKS4E4+TATIEBX8o2J6PxKkjcoWrXPw
NRo/k4Y/CZA4pXvlyTgH5LYmQbU0qUtPM7dAEzOsaoXmetqB+6cM+Q==
-----END PUBLIC KEY-----
```

Now, suppose a file named `testsign.txt` needs to be signed and verified. This can be achieved as follows:

1. Create a test file:

```
$ echo testing > testsign.txt
$ cat testsign.txt
testing
```

2. Run the following command to generate a signature using a private key for the `testsign.txt` file:

```
$ openssl dgst -ecdsa-with-SHA1 -sign eccprivatekey.pem testsign.txt >
ecsign.bin
```

3. Finally, the command for verification can be run, as shown here:

```
$ openssl dgst -ecdsa-with-SHA1 -verify eccpublickey.pem -signature
ecsign.bin testsign.txt
Verified OK
```


A certificate can also be produced by using the private key generated earlier by using the command shown here:

```
$ openssl req -new -key eccprivatekey.pem -x509 -nodes -days 365 - out  
eccertificate.pem
```

Enter the appropriate parameters to generate the certificate:

```
Country Name (2 letter code) []:GB  
State or Province Name (full name) []:Montgomery  
Locality Name (eg, city) []:London  
Organization Name (eg, company) []:BAK  
Organizational Unit Name (eg, section) []:Research  
Common Name (eg, fully qualified host name) []:masteringblockchain.com  
Email Address []:sonic@xmail.com
```

Once generated, the certificate can be explored using the following command:

```
$ openssl x509 -in eccertificate.pem -text -noout
```

The output shows the certificate:

```
Certificate:  
  Data:  
    Version: 1 (0x0)  
    Serial Number: 9248556263013038870 (0x805978eb961b3b16)  
    Signature Algorithm: ecdsa-with-SHA256  
    Issuer: C=GB, ST=Montgomery, L=London, O=BAK, OU=Research,  
    CN=masteringblockchain.com/emailAddress=sonic@xmail.com  
    Validity  
      Not Before: Apr 23 21:21:40 2022 GMT  
      Not After : Apr 23 21:21:40 2023 GMT  
    Subject: C=GB, ST=Montgomery, L=London, O=BAK, OU=Research,  
    CN=masteringblockchain.com/emailAddress=sonic@xmail.com  
    Subject Public Key Info:  
      Public Key Algorithm: id-ecPublicKey  
      Public-Key: (256 bit)  
      pub:  
        04:d3:11:d6:9d:6a:f6:bf:0c:11:3c:32:05:66:49:  
        c0:b0:06:ec:0f:1a:9f:28:47:19:06:61:b7:7e:d5:  
        e8:df:26:85:07:7a:e1:f5:a3:f9:9a:ea:61:52:b3:  
        4e:d7:bd:4b:ab:21:12:db:9e:4f:41:cf:b4:85:50:  
        2a:d2:08:3c:57
```

```
ASN1 OID: secp256k1
Signature Algorithm: ecdsa-with-SHA256
30:46:02:21:00:be:ba:b3:65:74:12:33:86:1e:1c:8b:cf:82:
19:5e:34:b9:f1:28:26:bf:84:3b:ba:34:8c:64:87:f2:16:29:
a2:02:21:00:ec:98:8f:fd:37:d5:38:4b:e2:15:f1:87:06:f4:
d9:4a:fd:b4:fd:a1:d7:ff:9b:38:6d:e0:7a:73:f0:05:f8:75
```

In this section, we covered digital signature schemes and some practical examples. Next, we'll introduce different types of digital signatures.

Different types of digital signatures

Now, we'll provide an overview of different types of digital signatures and their relevance and applications in blockchain technology. There are different types of digital signature schemes based on the underlying mathematics they use. Families of signatures schemes are:

- RSA-based signature and public keys
- Discrete log-based digital signatures

RSA signatures and public keys are at least 256 bytes in size, which is not suitable for blockchains as they require more space. They are, however, fast to verify but due to the unnecessary space requirements, they are not used in blockchains.

Discrete log signatures are commonly used in blockchains. They are small, usually 48 or 64 bytes with 32-byte public keys. This small size makes them useful for blockchains, and they are used in Bitcoin, Ethereum, and quite a few other blockchains. This family includes Schnorr and ECDSA signatures.

Digital signatures are used in blockchains, where transactions are digitally signed by senders using their private key before the sender broadcasts the transaction to the network. This digital signing proves that the sender is the rightful owner of the asset, for example, of bitcoins, and is authorized. These transactions are verified again by other nodes on the network to ensure that the funds indeed belong to the node (user) who claims to be the owner. They are also used in consensus protocols to sign votes, for example, in BFT consensus schemes.

Some other types based on the difference in their application and mechanics are described below.

Blind signatures

The **blind signature scheme** was invented by David Chaum in 1982. It is based on public key digital signature schemes, such as RSA. The key idea behind blind signatures is to get the message signed by the signer, without actually revealing the message. This is achieved by disguising or blinding the message before signing it, hence the name **blind signatures**.

This blind signature can then be verified against the original message, just like a normal digital signature. Blind signatures were introduced as a mechanism to allow the development of digital cash schemes. Specifically, a blind signature can provide unlinkability and anonymity services in a distributed system, such as a blockchain.



The original paper from David Chaum on blind signatures is available at <http://blog.koehtopp.de/uploads/Chaum.BlindSigForPayment.1982.PDF>

Chaum, David. "Blind signatures for untraceable payments." In *Advances in cryptology*, pp. 199–203. Springer, Boston, MA, 1983.

The blind signature scheme is also used to build Blindcoin, which is used in the Mixcoin protocol for Bitcoin to hide user addresses from the mixing service. The Mixcoin protocol allows anonymous payments in the Bitcoin network, but the user addresses are still visible to the mixing service. Blindcoin is a protocol that addresses this limitation.



More information on this scheme is available in the following paper:

Valenta, L. and Rowan, B., January. "Blindcoin: Blinded, accountable mixes for bitcoin." In *International Conference on Financial Cryptography and Data Security*, pp. 112–126. Springer, Berlin, Heidelberg, 2015.

Multisignatures

In the **multisignature scheme**, a group of entities signs a single message. In other words, multiple unique keys held by their respective owners are used to sign a single message.



This scheme was introduced in 1983 by Itakura et al. in their paper "A Public-key Cryptosystem Suitable for Digital Multisignatures, vol. 71." *Nec Research & Development*, pp. 474–480, 1983.

Multisignatures are also sometimes called multiparty signatures in literature. In blockchain implementations, multisignatures provide the ability to allow transactions to be signed by multiple users, which results in increased security. This is also called multi-sig and has been implemented in Bitcoin. These schemes can be used in such a way that the requirement of a number of signatures can be set in order to authorize transactions. For example, a 1-of-2 multisignature can represent a joint account where either one of the joint account holders is required to authorize a transaction by signing it.

In another variation, for example, a 2-of-2 multisignature can be used in a scenario where both joint account holders' signatures are required to sign the transaction. This concept is also generalized as m of n signatures, where m is the minimum number of required signatures and n is the total number of signatures.

This concept can be visualized with the following diagram:

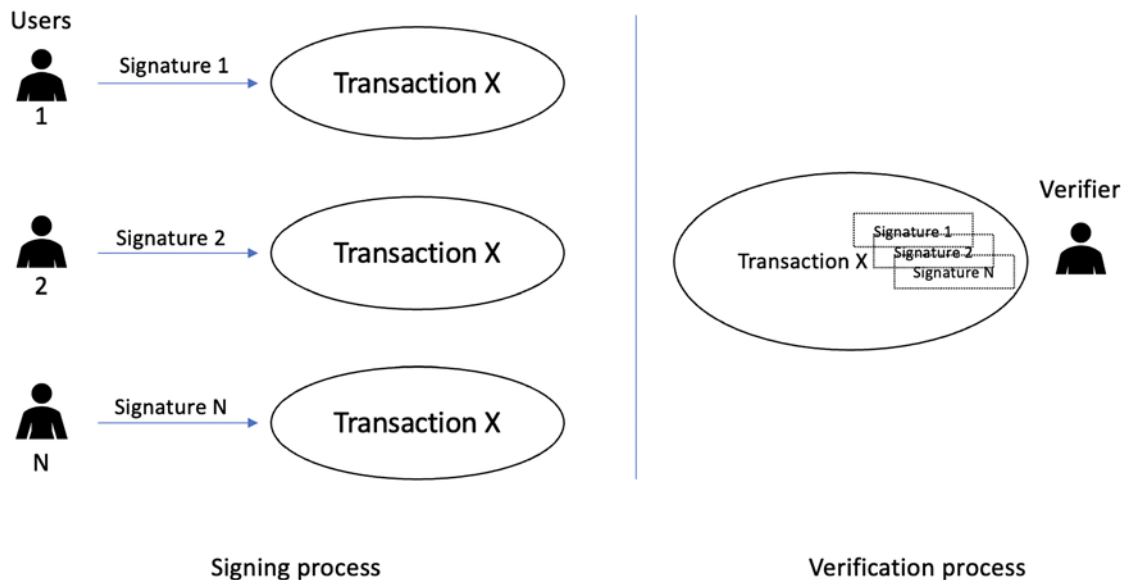


Figure 4.9: Multisignature scheme

The preceding diagram shows the signing process on the left-hand side, where m is the number of different users, and holding m unique signatures signs a single transaction. When the validator or verifier receives it, all the signatures in it need to be individually verified.



The Openchain and Multichain blockchains make use of multisignature schemes.

More information on Openchain is available at <https://docs.openchain.org/en/latest/general/overview.html>.

More information regarding Multichain's multisignature scheme is available at <https://www.multichain.com/developers/multisignature-transactions/>.

Threshold signatures

This scheme is a specific type of multisignature. It does not rely on users to sign the message with their own unique keys; instead, it requires only one public key and one private key, and results in only one digital signature. In multisignature, the resultant message contains digital signatures from all signers and requires verification individually by the verification party, but in threshold signatures, the verifier verifies only one digital signature. The key idea of the scheme is to split the private key into multiple parts, and each signer keeps their own share of the private key. The signing process requires each user to use their respective share of the private key to sign the message. In this scheme, only a subset of signers can produce the signature with their share, and there is no need for all members to collaborate to produce the signature.

The communication between signers is governed by a specific communication protocol. In contrast with multisignatures, the threshold signatures result in a smaller transaction size and are quicker to verify. A threshold signature increases the availability and resilience of the scheme as shares of the signature are stored by distinct signers (servers).

A slight limitation, however, is that for threshold signatures to work, all signers involved in the signing process must remain online to participate in an interactive protocol to generate the signature, whereas in multisignatures, the signature can be provided asynchronously; that is, users can provide signatures whenever they are available.

There are also variants that are non-interactive, where each user computes its share of the signature without any online communication with other users (servers). From another angle, there could be a scenario in multisignature schemes where a user could withhold their signature maliciously, which could result in denial of service. With threshold signatures, it is possible to construct the required signature only with any authorized subset of the entire group, i.e., $t + 1 \leq n$, but at least those $t + 1$ participants must be online.

Threshold signatures can also be used to provide anonymity services in a blockchain network. This is so because threshold signature schemes do not reveal the members in the threshold group that have signed to produce the signature. This is in contrast with multisignature schemes, which do reveal the identities of all the signers. Moreover, in permissioned networks, threshold signatures can be used in scenarios where a threshold of nodes (users) is required to agree on an operation.

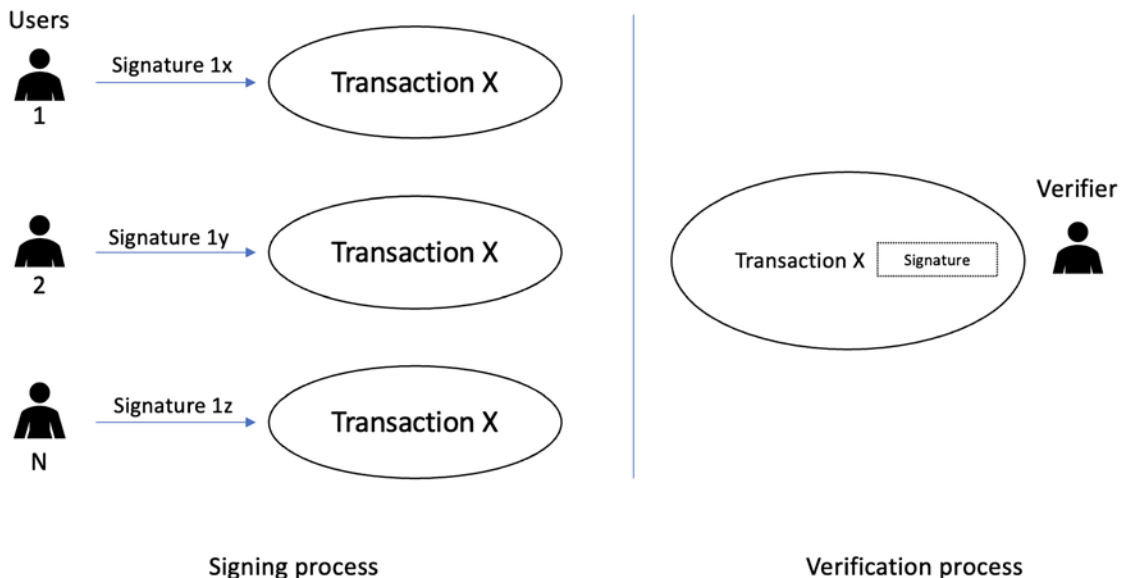


Figure 4.10: Threshold signature scheme

The preceding diagram shows the signing process on the left-hand side, where m number of different users, holding different parts (shares) of a digital signature, sign a single transaction. When the validator or verifier receives it, only one signature needs to be verified.

Aggregate signatures

Aggregate signatures are used to reduce the size of digital signatures. This scheme is particularly useful in scenarios where multiple digital signatures are in use. The core idea is to aggregate multiple signatures into a single signature, without increasing the size of the signature of a single message. It is simply a type of digital signature that supports aggregation. The small aggregate signature is enough to provide verification to the verifier that all users did sign their original messages. Aggregate signatures are commonly used to reduce the size of messages in network and security protocols. For example, the size of digital certificate chains in **public key infrastructure (PKI)** can be reduced significantly by compressing all signatures in the chain into a single signature. **Boneh–Lynn–Shacham (BLS)** aggregate signatures are a popular example of an aggregate signature.



More information regarding aggregate BLS signatures is available in the paper here: <https://crypto.stanford.edu/~dabo/papers/agggreg.pdf>.

Boneh, D., Gentry, C., Lynn, B. and Shacham, H., Aggregate and Verifiably Encrypted Signatures from Bilinear Maps.

More information on BLS and specifically its usage in blockchains to reduce the size of the blockchain is available in an excellent paper here: <https://eprint.iacr.org/2018/483.pdf>

Boneh, D., Drijvers, M. and Neven, G., 2018, December. Compact multi-signatures for smaller blockchains. In International Conference on the Theory and Application of Cryptology and Information Security (pp. 435–464). Springer, Cham.

Aggregation allows you to compress many signatures from many different users into a single 48-byte signature. Instead of writing x number of signatures for x number of transactions, you can compress and write only one signature on the blockchain. This property makes BLS signatures very useful to blockchains due to the space-saving benefit. Moreover, it's quick to verify, which makes it even more suitable for blockchains. BLS signatures allow you to build m out of n type threshold schemes quite easily, where a threshold of m users is required to sign a transaction out of n users.

BLS signatures are used in Ethereum and a few other blockchains. BLS signatures are 48 bytes in length; they can be aggregated and allow you to build threshold schemes. Most new blockchains use BLS signatures due to their smaller size and aggregation feature.

Ring signatures

Ring signatures were introduced in 2001 by Ron Rivest, Adi Shamir, and Yael Tauman.



The original paper is available at: https://link.springer.com/content/pdf/10.1007/3-540-45682-1_32.pdf

Rivest, R.L., Shamir, A. and Tauman, Y., 2001, December. How to leak a secret. In International Conference on the Theory and Application of Cryptology and Information Security (pp. 552–565). Springer, Berlin, Heidelberg.

Ring signature schemes allow a mechanism where any member of a group of signers can sign a message on behalf of an entire group. The key requirement here is that the identity of the actual signer who signed the message must remain unknown (computationally infeasible to determine) to an outside observer. It looks equally likely that anyone in the trusted group of signers could have signed the message, but it is not possible to figure out who actually signed the message. Each member of the ring group keeps a public key and a private key. Ring signatures can be used to provide privacy (anonymity)-preserving services. A variation of ring signatures called traceable ring signatures so that double spending can be prevented is used in **CryptoNote**. **Monero** cryptocurrency also uses ring signatures.

In this section, we covered different types of digital signatures, their operations, and their relevance to blockchains. Next, we'll present some more advanced cryptography topics.

Cryptographic constructs and blockchain technology

Now, we'll present some advanced topics in cryptography that not only are important on their own but are also relevant to blockchain technology due to their various applications in this space.

Homomorphic encryption

An encryption algorithm is homomorphic if it can apply some operation on encrypted data without decrypting it. Public key cryptosystems, such as RSA, are multiplicative homomorphic or additive homomorphic, such as the Paillier cryptosystem, and are called **partially homomorphic** systems. Additive **partially homomorphic encryptions** (PHEs) are suitable for e-voting and banking applications due to their ability to perform additional operations on encrypted data. For example, in voting, they can sum up votes even if they are encrypted.

Until recently, there has been no system that supported both operations, but in 2009, a **fully homomorphic** system was discovered by Craig Gentry. As these schemes enable the processing of encrypted data without the need for decryption, they have many different potential applications, especially in scenarios where maintaining privacy is required, but data is also mandated to be processed by potentially untrusted parties, for example, cloud computing and online search engines.

Recent developments in homomorphic encryption have been very promising, and researchers are actively working to make it efficient and more practical. This is of particular interest in blockchain technology, as described later in this book, as it can solve the problem of confidentiality and privacy in the blockchain.

Secret sharing

Secret sharing is the mechanism of distributing a secret among a set of entities. All entities within a set get a unique part of the secret after it is split into multiple parts. The secret can be reconstructed by combining all or some parts (a certain number or threshold) of the secret. The individual secret shares/parts, on their own, do not reveal anything about the secret.

This concept can be visualized through the following diagram:

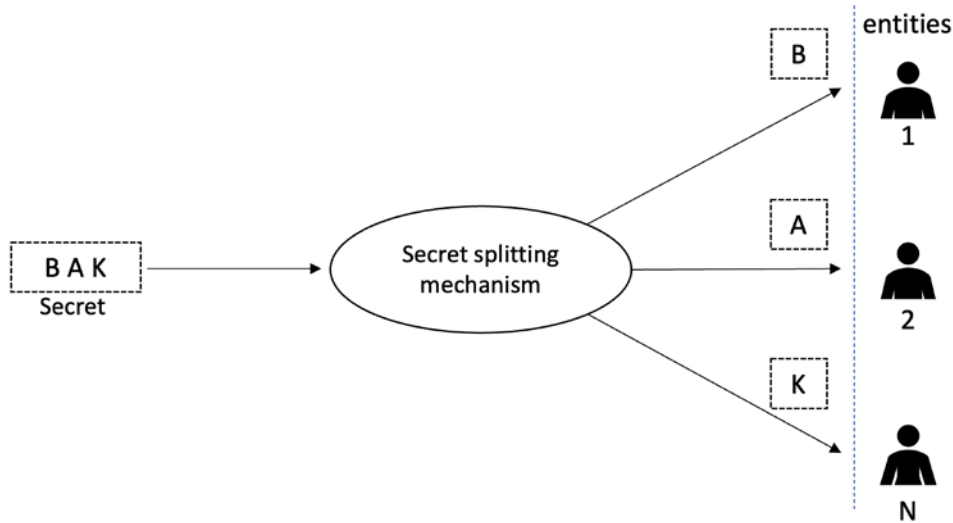


Figure 4.11: Secret sharing scheme

Commitment schemes

Commitment schemes are usually described as a digital cryptographic equivalent of a sealed envelope. A commitment itself does not reveal any information about the actual value inside it. This scheme runs in two phases, namely:

- **Commit phase:** The commit phase provides two security features—that is, **hiding** and **binding**:
 - The **hiding** property ensures that an adversary cannot find any information or reveal the committed value before the open phase.
 - The **binding** property ensures that once the sender has committed to a value, that value or message cannot be changed.
- **Open phase:** The open phase (also called the unveil phase) is where the receiver receives some additional information from the sender. This allows the receiver to establish the value hidden (concealed) by the commitment with proof that the sender has not modified the value after the commitment— that is, that the sender has not deceived during the commit phase.

Commitment schemes play a vital role in building privacy-preserving blockchains. They have applications in building zero-knowledge protocols and range-proof protocols. Blockchains such as **Monero** and **Zcash** make use of commitment schemes to introduce privacy.

The key idea behind commitment schemes is that they allow someone to secretly commit to a value with the ability to reveal it later. These schemes prevent parties from changing their committed value later after they've committed to a value. A prime example of a commitment scheme is the **Pedersen commitment scheme**.



The original paper on the Pedersen commitment scheme is available here: https://link.springer.com/content/pdf/10.1007/3-540-46766-1_9.pdf.

Pedersen, T.P., 1991, August. Non-interactive and information-theoretic secure verifiable secret sharing. In *Annual international cryptology conference* (pp. 129–140). Springer, Berlin, Heidelberg.

In the next section, we'll introduce **Zero-Knowledge Proofs (ZKPs)**, which is an exciting subject and a very ripe area for research. Already, various major blockchains and cryptocurrencies make use of zero-knowledge proofs to provide privacy. A prime example is **Zcash** (<https://z.cash>).

Zero-knowledge proofs

ZKPs were introduced by Goldwasser, Micali, and Rackoff in 1985. These proofs are used to prove the validity of an assertion without revealing any information whatsoever about the assertion. There are three properties of ZKPs that are required: completeness, soundness, and the zero-knowledge property:

- **Completeness** ensures that if a certain assertion is true, then the verifier will be convinced of this claim by the prover.
- The **soundness** property makes sure that if an assertion is false, then no dishonest prover can convince the verifier otherwise.
- The **zero-knowledge property**, as the name implies, is the key property of ZKPs, whereby it is ensured that absolutely nothing is revealed about the assertion except whether it is true or false.

ZKPs are probabilistic instead of deterministic. The probabilistic nature of ZKPs is because these protocols require several rounds to achieve a higher level of assurance gradually, with each round, which allows the verifier to accept that the prover indeed knows the secret.

In literature, usually, an analogy known as **Ali Baba's Cave** is used to explain ZKPs. The following diagram shows a variant of this analogy of how the zero-knowledge protocol works. It shows two characters called **Peggy** and **Victor** whose roles are *Prover* and *Verifier*, respectively. Peggy knows a secret magic word to open the door in a cave, which is shaped like a ring. It has a split entrance and a magic door in the middle of the ring that blocks the other side. We have also labeled the left and right entrances as **A** and **B**.

Victor wants to know the secret, but Peggy does not want to reveal it. However, she would like to prove to Victor that she does know the secret:

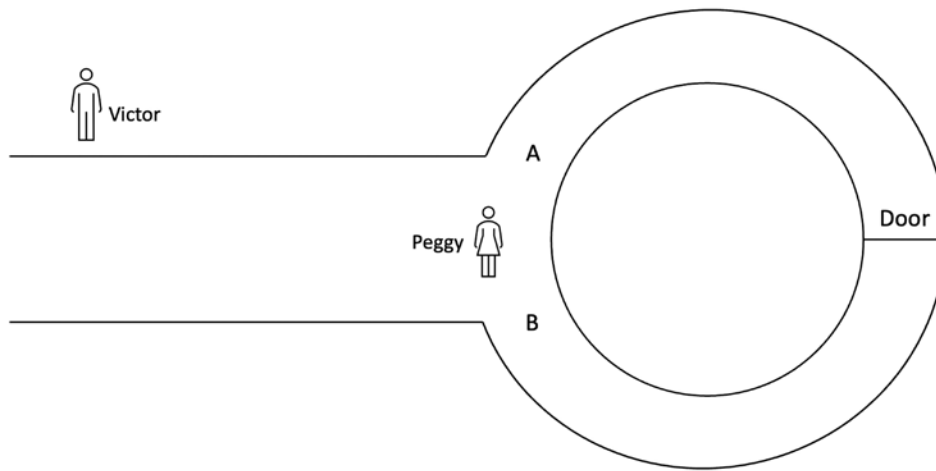


Figure 4.12: Analogy of the zero-knowledge mechanism

Now, there are several steps that are taken by both Peggy and Victor to reach a conclusion regarding whether Peggy knows the secret or not:

- First, Victor waits outside the main cave entrance and Peggy goes in the cave.
- Peggy randomly chooses either the A or B entrance to the cave.
- Now, Victor enters the cave and shouts either A or B randomly, asking Peggy to come out of the exit he named.
- Victor records which exit Peggy comes out from.

Now, suppose Victor asked Peggy to come out from exit A and she came out from exit B. Victor then knows that Peggy does not know the secret. If Peggy comes out of exit A, then there is a 50% chance that she does know the secret, but this also means that she may have gotten lucky and chosen A to enter the cave in the first place, and now has just returned without needing to go through the magic door at all. However, if this routine is performed several times, and given that Victor is choosing A or B at random, with each run (round) of this routine (protocol), the chances of Peggy getting lucky diminish. If Peggy repeatedly manages to emerge from the entrance that Victor has named, then it is highly probable that Peggy does know the secret to open the magic door.



Remember, we said earlier that zero-knowledge protocols are probabilistic. Peggy and Victor repeatedly performing this routine proves knowledge of the secret with high probability, as the probability of guessing or getting lucky reduces to the point of being negligible after *some* rounds.

Note that during this process, Peggy has not revealed the secret at all, but still managed to convince Victor with high probability that she does know the secret.

A ZKP comprises the following phases:

- **Witness phase:** In this phase, the prover sends proof of the statement and sends it to the verifier.
- **Challenge phase:** In this phase, the verifier chooses a question (challenge) and sends it to the prover.
- **Response phase:** In this phase, the prover generates an answer and sends it as a response to the verifier. The verifier then checks the answer to ascertain whether the prover really knows the statement.

This scheme can also be visualized using the following diagram, which shows how the zero-knowledge protocol generally works and what steps the prover and verifier take to successfully run the protocol:

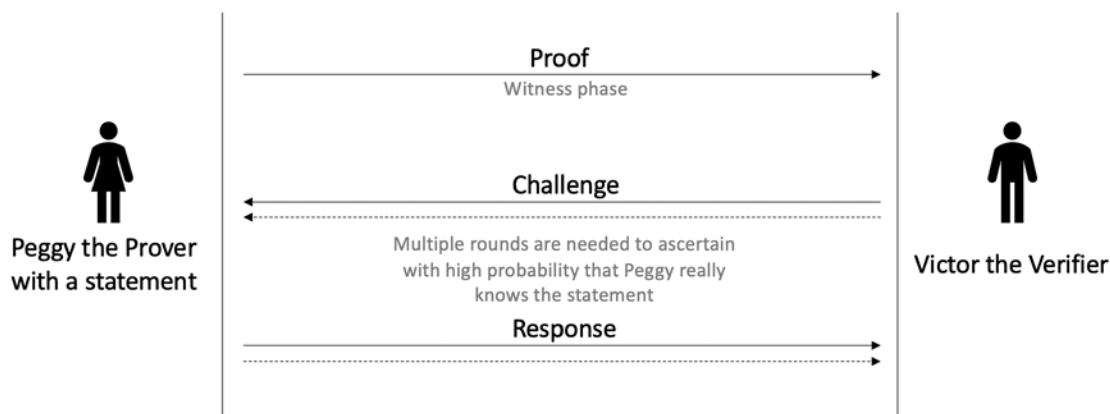


Figure 4.13: A ZKP scheme

Even though **Zero Knowledge (ZK)** is not a new concept, these protocols have sparked a special interest among researchers in the blockchain space due to their privacy properties, which are very much desirable in finance and many other fields, including law and medicine. A prime example of the successful implementation of a ZKP mechanism is the **Zcash** cryptocurrency. In Zcash, the **zero-knowledge Succinct Non-interactive ARGument of Knowledge (zk-SNARK)** is implemented to provide anonymity and confidentiality.

Applications of ZKP include proof of ownership, that is, proof that the prover is the owner of some secret or a private key without revealing the nature of it. ZKPs can also be used to prove that the prover is a member of some organization or group, without revealing their identity. Another use case could be proof of age, where the prover proves that they are, for example, older than 25 years, but does not want to reveal their exact age. Another application could be where citizens can pay taxes without revealing their income.

zk-SNARKs

Zero-knowledge protocols are usually interactive as they require repeated interactions between the prover and the verifier, but there are protocols that do not require any interaction between a prover and a verifier. These types of ZKPs are called non-interactive types of proofs. A prominent example is the zk-SNARK.

Before the introduction of zk-SNARKs, ZKPs were considered not very practical because of the complexity that arises from the requirements of repeated interaction between the prover and the verifier and a large proof size.

There has been some very influential work published on non-interactive zero-knowledge proofs over the years. The most prominent of these, and the one that proposed the first universal design, is provided by the paper by Eli Ben-Sasson et al., which is available at <https://www.usenix.org/system/files/conference/usenixsecurity14/sec14-paper-ben-sasson.pdf>

Ben-Sasson, E., Chiesa, A., Tromer, E., and Virza, M., 2014. Succinct non-interactive zero knowledge for a von Neumann architecture. In 23rd {USENIX} Security Symposium ({USENIX} Security 14) (pp. 781–796).



Other previous notable works that contributed to the development of zk-SNARKs include:

Gennaro, R., Gentry, C., Parno, B., and Raykova, M., “Quadratic span programs and succinct NIZKs without PCPs.” In Annual International Conference on the Theory and Applications of Cryptographic Techniques, pp. 626–645, Springer, Berlin, Heidelberg, 2013.

(https://link.springer.com/content/pdf/10.1007/978-3-642-38348-9_37.pdf)

Groth, J. and Sahai, A., “Efficient non-interactive proof systems for bilinear groups.” In Annual International Conference on the Theory and Applications of Cryptographic Techniques, pp. 415–432, Springer, Berlin, Heidelberg, 2008.

(https://link.springer.com/content/pdf/10.1007/978-3-540-78967-3_24.pdf)

zk-SNARKs have several properties, which are described here:

- **zk:** This property allows a prover to convince a verifier that an assertion (statement) is true without revealing any information about it. A statement in this context is any computer program that terminates and does not take too long to run, that is, not too many cycles.
- **Succinct (S):** This property allows for a small-size proof that is quick to verify.
- **Non-interactive (N):** This property allows the prover to prove a statement by only sending just a single message (proof) to the verifier.
- **ARguments (AR):** These are the arguments to convince the verifier that the prover’s assertion is true. Remember that we discussed soundness and completeness earlier. In the case of arguments, the prover is computationally bounded, and it is computationally infeasible for it to cheat the verifier.
- **Knowledge (K):** This property means that the prover has the evidence that they indeed know the statement they claim knowledge of.

zk-SNARKs have been implemented in different blockchains. The most prominent example is Zcash, which uses them for its shielded transactions feature to provide confidentiality and anonymity. Support for cryptographic primitives required for zk-SNARKs has also been introduced in the Ethereum blockchain with the Byzantium release.

zk-SNARK proofs are generated by following several different steps. Generating SNARKs for a program (a statement) is not a simple process as it requires the program to be converted into a circuit with a very specific format. This specific form is called the **quadratic arithmetic program (QAP)**. The process of proof generation is as follows:



Figure 4.14: zk-SNARK construction

The process takes the following steps:

- **Arithmetic circuit:** The first step in zk-SNARK construction is to convert a logical step of a **computation** into the smallest possible units comprised of basic arithmetic operations. Arithmetic operations include addition, subtraction, multiplication, and division. In the arithmetic circuit, the computation is presented as wires and gates, representing the flow of inputs and arithmetic operations performed on the inputs. It is a **directed acyclic graph (DAG)** that evaluates a polynomial by taking inputs and performing arithmetic operations on it.
- **R1CS:** R1CS is the abbreviation of **Rank 1 Constraint System**. Basically, this system is a set of constraints that allows all the steps in the arithmetic circuit to be verified and confirms that, at the end of the process, the output is as expected.
- **QAP:** The prover makes use of QAPs to construct a proof of the statement. In R1CS, the verifier must check many different constraints, but with a QAP representation of the circuit, all the different constraints can be bundled into only a single constraint.
- Finally, QAP is used in the **zk-SNARK** protocol to prove the assertion through the prover and verifier.

zk-SNARKs are not a silver bullet to privacy problems. Instead, like many other technologies, there are pros and cons.

The biggest criticism of zk-SNARKs is the initial trusted setup. This trusted setup can be influenced and compromised. However, if done correctly, it does work, but there is, however, always a chance that the initial setup was compromised, and no one will ever be able to find out. This is the issue that has been addressed by zk-STARKs.

zk-STARKs

Zero-Knowledge Scalable Transparent ARGuments of Knowledge (zk-STARKs) are a new type of ZKP that has addressed several limitations in zk-SNARKs.



This scheme was designed by Eli-Ben Sasson et al. The original paper on the subject is available here:

<https://eprint.iacr.org/2018/046.pdf>

Ben-Sasson, E., Bentov, I., Horesh, Y., and Riabzev, M., “Scalable, transparent, and post-quantum secure computational integrity. “IACR Cryptology ePrint Archive, pp. 46, 2018

The key differences between the zk-STARK and zk-SNARK schemes are listed in the following table:

Properties	zk-SNARKs	zk-STARKs
Scalability	Less scalable	More scalable
Initial trusted setup	Required	Not required—has a publicly verifiable mechanism
Post-quantum-resistant	Not resistant to attacks from quantum computers	Resistant to attacks from quantum computers
Construction techniques	Rely on elliptic curves and pairings	Based on hash functions and concepts from information theory

Even though this scheme is more efficient, post-quantum-resistant, scalable, and transparent, the biggest limitation is its size of proof, which is a few hundred kilobytes compared to zk-SNARKs’ 288 bytes. This is seen as a problem in public blockchains where large proofs may pose a problem to scalability and performance.

Zero-knowledge range proofs

Zero-knowledge range proofs (ZKRPs) are used to prove that a given number is within a certain range. This can be useful when, for example, someone does not want to reveal their salary but is willing to only reveal the range between which their salary lies. Range proofs can also be used for age checks without requiring the prover to reveal their exact age.

In this section, we have covered the very interesting topic of ZKPs and their relevant techniques and developments. ZKPs is a very active area of research, especially in the context of blockchains, where this technology can provide much needed privacy and confidentiality services on public blockchain networks.

In the next section, we will introduce encoding schemes and some relevant examples.

Encoding schemes

Other than cryptographic primitives, binary-to-text **encoding schemes** are also used in various scenarios. The most common use is to convert binary data into text so that it can either be processed, saved, or transmitted via a protocol that does not support the processing of binary data. For example, images can be stored in a database encoded in base64, which allows a text field to be able to store a picture. Another encoding, named base58, was popularized by its use in Bitcoin.

Base64

The base64-encoding scheme is used to encode binary data into printable characters.

We can do a quick experiment using the OpenSSL command line:

```
$ openssl rand 16 -base64
4ULD5sJtGeoSnogIniHp7g==
```

The preceding command has generated a random sequence of 16 bits and then, using the `-base64` switch, it has converted that into a base64 text string. The base64 converts from 8 bits to a 6-bit ASCII representation. This is useful for storage and transmission, especially in cases where binary data handling could lead to incompatibility between systems. This mechanism is a flexible way to represent binary data as ASCII, which can be easily and universally stored and transmitted.

base58

The base58 scheme was first introduced with Bitcoin and is used to encode integers into alphanumeric strings. The key idea behind this encoding scheme is to avoid non-alphanumeric characters and also those characters that look similar and could lead to ambiguity; for example, a lowercase L (l) may look like the number one (1). This feature is especially useful because Bitcoin addresses must not have any confusion about the character representation; otherwise, it could lead to wrongly sending bitcoins to some non-existent or incorrect address, which is clearly a financial loss. This ingenious encoding scheme avoids this type of situation by ignoring similar-looking characters.

We will explore base58 and its role in generating Bitcoin addresses in detail in *Chapter 6, Bitcoin Architecture*.

Verifiable random functions

A **verifiable random function (VRF)** is a keyed hash function that uses public cryptography instead of symmetric key cryptography, as in MACs. It is a public key variation of a keyed hash function. In this scheme, the hash is computed by the holder of the private key, which is publicly verifiable, with the public key used to check if the hash is correct.



VRF was introduced by Micali et.al in:

Micali, Silvio, Michael Rabin, and Salil Vadhan. "Verifiable random functions." In Proceedings of the 40th Annual Symposium on the Foundations of Computer Science (FOCS '99), pp. 120–130. New York: IEEE Computer Society Press, 1999.

The original paper is available here: <http://nrs.harvard.edu/urn-3:HUL.InstRepos:5028196>.

A VRF is a random number generator. It takes an arbitrary input and produces a random output and a proof. The proof is used to verify the correctness of the output. More formally, it comprises three algorithms, KeyGen, Evaluate, and Verify. KeyGen takes a random input and generates a verification key and a secret key.

The Evaluate function takes the private key, a message, and produces a random output and a proof. This output is unique and pseudorandom. Finally, the Verify algorithm takes the verification key, the message, the output, and the proof and verifies if the output produced by the evaluating algorithm is indeed correct.

VRFs are used in blockchains for block proposer selection in proof-of-stake consensus protocols. Some examples of such protocols include **Algorand**, **Cardano Ouroboros**, and **Polkadot BABE**.

Cryptography is a vast field, and this section has introduced some of the main concepts that are essential to understanding cryptography in general, specifically from a blockchain and cryptocurrency point of view.

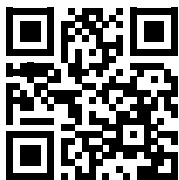
Summary

This chapter started with an introduction to some basic mathematics concepts and asymmetric key cryptography. We discussed various constructs such as RSA and ECC. We also performed some experiments using OpenSSL to see how theoretical concepts could be implemented practically. We also looked at some advanced and modern concepts such as zero-knowledge proofs and relevant constructions, along with different types of digital signatures.

In the next chapter, we will explore the captivating world of distributed consensus, which is central to the integrity of any blockchain and is a very active area of research.

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