3. Mechanical resonance

A. Objectives

- Determine the natural frequency, decay rate, and quality factor of a cantilever oscillator with measurements of its free decay.
- Quantitatively study the driven oscillation of a cantilever oscillator as a function of drive frequency. Measure the ratio of the displacement amplitude to the drive amplitude and the phase difference between the displacement oscillation and the drive oscillation as functions of frequency. The frequencies should include the range from well below resonance to well above resonance, with enough points near resonance to clearly see the resonance lineshape.
- Compare your results for the steady state oscillation amplitude ratio and phase difference to theoretical predictions. Compare the frequency and quality factor that you obtained with the driven oscillations to those you obtained with the free decay.

B. Equipment Required

1. Breadboard and foam pad
2. Diode laser module, mount, and power supply
3. Laser beam blocks
4. Damped cantilever assembly
5. Drive coil assembly, two patch cables with banana plug ends, and one DIN-to-banana plug adapter cable
6. Optical polarizer in rotatable mount
7. 10 cm focal length lens in mount
8. Pasco light sensor and connecting cable
9. Mounting components, hex key wrenches and ball driver (take from bins as needed)
10. Computer data acquisition system

C. Introduction

Free decay of damped harmonic oscillator

Consider the mechanical oscillator shown in Figure 3.1. A mass *m* hangs from a spring of spring constant *K*. If the mass is displaced from its equilibrium position by a distance *x*, the spring exerts a restoring force $F_s = -Kx$ on the mass according to Hooke's law. In addition, an extension from the mass is connected to a dashpot. The dashpot resists the motion of the mass with a *damping force* $F_d = -b\dfrac{dx}{dt}$ that is proportional to the negative of the velocity dx/dt. This damping arises from the viscous force that oil in the dashpot exerts on a small cylinder connected to the mass.

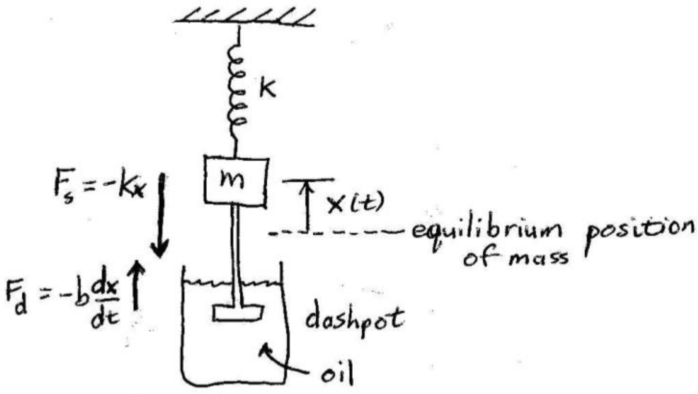


Figure 3.1. Damped harmonic oscillator

The equation of motion of the mass is given by Newton's second law as

$$m\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt} \tag{3.1}$$

This equation can be rewritten as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 = 0 \tag{3.2}$$

where $\omega_0 = \sqrt{\dfrac{K}{m}}$ is the *natural vibration frequency* of the oscillator, and $\gamma = \dfrac{b}{m}$ is the *damping constant*. The damping constant γ has units of inverse time, so it is also referred to as the *damping rate.*

Suppose now that we displace the mass by a distance x_0 from its equilibrium position and let go. The subsequent motion of the mass will be given by equation (3.2) subject to the initial conditions $x(t=0) = x_0$, and $\left.\dfrac{dx}{dt}\right|_{t=0} = 0.$ The solution to this initial value problem, for the *underdamped* case $\omega_0 > \dfrac{\gamma}{2}$, is

$$x(t) = Ae^{-\frac{\gamma}{2}t}\cos\left(\omega_0' t + \phi\right) \tag{3.3}$$

where $\omega_0' = \sqrt{\omega_0{}^2 - \left(\frac{\gamma}{2}\right)^2}$, $\phi = \tan^{-1}\left(\frac{\gamma}{2\omega_0'}\right)$, and $A = \frac{x_0}{\cos(\phi)}$.

In this lab, we'll focus only on the strongly underdamped case, in which case $\omega_0' \approx \omega_0$, $\phi \approx 0$, $A \approx x_0$, and

$$x(t) \approx x_0 e^{-\frac{\gamma}{2}t} \cos(\omega_0 t) \tag{3.4}$$

Equation (3.4) gives the displacement *vs*. time of a *freely decaying damped harmonic oscillator,* in the weakly damped limit, as shown in Figure 3.2(a).

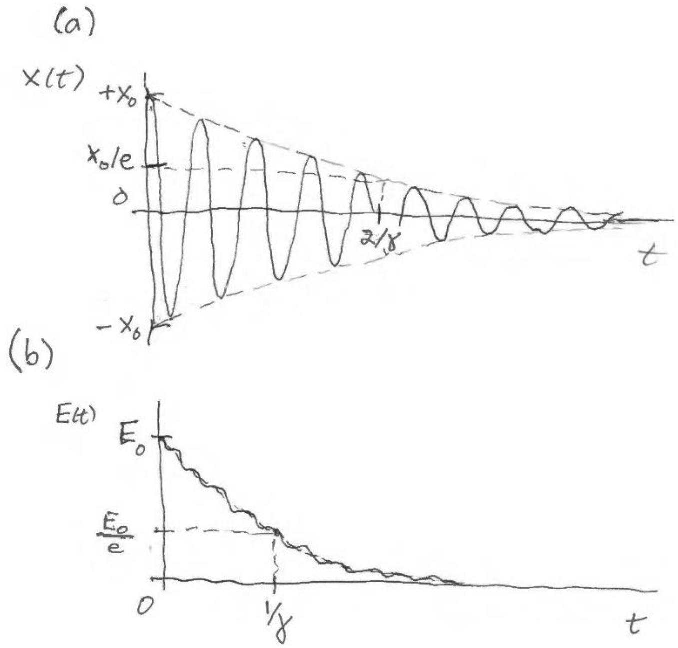


Figure 3.2. Free decay of a weakly damped oscillator. (a) Displacement of the oscillator *vs.* time. (b) Energy of the oscillator *vs.* time.

In the above equations, we have followed common practice and defined the damping constant γ as the rate for the *energy* of the oscillator to decay. To see this, note that the energy of the oscillator is

$$E(t) = \frac{1}{2}K\left(x(t)\right)^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \approx E_0 e^{-\gamma t}\left(1 + \frac{\gamma}{2\omega_0}\sin\left(2\left(\omega_0' t + \phi\right)\right)\right) \approx E_0 e^{-\gamma t} \tag{3.5}$$

where $E_0 = \frac{1}{2}Kx_0{}^2$ is the energy of the oscillator at time $t = 0$, and the approximations again refer to the weakly damped limit. This result for $E(t)$ is shown in Figure 3.2(b). The decay rate of the *energy E* is γ, and is twice as large as the decay rate $\gamma/2$ of the *amplitude* of $x(t)$. This is due to the fact that the energy is proportional to the square of the amplitude. The *damping time constant* of the oscillator is defined as $\tau = 1/\gamma$. This is the time for the energy of the oscillator to decay by a factor of *e* (base of natural logarithm).

In reality, all oscillators are damped, even if no deliberate damping mechanism like the dashpot is present. Damping mechanisms include friction between moving parts, viscous damping of components moving through a gas or fluid, inelasticity in the deformation of materials, transmission of vibrational energy through mechanical supports, radiation of power to the surroundings (for instance radiated sound waves or electromagnetic waves), resistive losses of currents in metals, and eddy current damping of moving magnetic parts. Thus, all real oscillators will decay as shown in Figure 3.2.

Quality factor *Q* of an oscillator

The *quality factor Q* of an oscillator is defined as

$$Q = \frac{\omega_0}{\gamma} \tag{3.6}$$

$Q/2\pi$ is equal to the number of cycles of oscillation in one damping time:

$$\frac{\tau}{T} = \frac{1}{\gamma}\frac{\omega_0}{2\pi} = \frac{Q}{2\pi} \tag{3.7}$$

(Here $T = 2\pi/\omega_0$ is the period of the oscillation.) Thus, a high-*Q* oscillator will oscillate for many cycles before it decays appreciably, whereas a lower *Q* oscillator may only oscillate for a few cycles. The vibrating metal bars you studied in last week's lab can have a *Q* as high as a few thousand. Some other examples of real-world *Q* values are given in Table 3.1.

System	Typical *Q*
Automobile suspension	1
Vibration mode of the earth	250
Guitar string	1700
Quartz crystal in a watch	10,000
L-*C* circuit	30
Microwave resonator	100 to 10,000
Superconducting microwave resonator	10^7 to 10^{11}
Optical resonance in a single, laser-cooled, trapped ion	10^{14} (world record laboratory *Q*, not typical)

Table 3.1. Typical quality factors for some physical oscillators.

Driven, damped oscillator

Let's suppose now that we apply an additional, oscillating force $F = F_0 \cos(\omega t)$ to the mass shown in Figure 3.1. Such an oscillator is a *driven, damped harmonic oscillator*. (Of course, the drive need not be harmonic, but that's the only case we'll consider in this lab.) The equation of motion of the oscillator is now

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 = \frac{F_0}{m}\cos(\omega t) \tag{3.8}$$

Note that we do *not* assume that the frequency of oscillation ω of the force is equal to the natural frequency ω_0. We can choose ω to be any frequency.

When you're working in the laboratory, you'll often encounter the situation where you turn the oscillating force on at some time t_0, and watch the oscillation after that time. This results in the general behavior shown Figure 3.3. For a few time constants τ after time t_0, the oscillator exhibits a time-dependent oscillation amplitude, and may show beating effects. This behavior is described by the *transient solution* of equation (3.8). We won't concern ourselves with this behavior in this lab (except that you may need to wait for it to end when making measurements). However, if you wait long enough, you'll reach the *steady state* regime, in which amplitude and phase of the oscillation no longer depend on time. This corresponds to the *steady state solution* of equation (3.8).

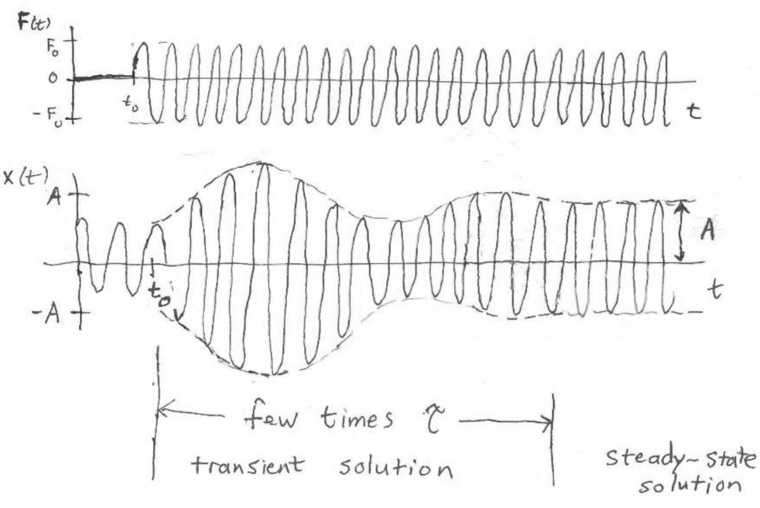


Figure 3.3. Displacement of an oscillator after a harmonic driving force is turned on at time t_0.

The steady state solution of equation (3.8) is

$$x(t) = A\cos(\omega t - \delta) \tag{3.9}$$

where $A = A(\omega) = \frac{F_0}{m}\sqrt{\frac{1}{\left(\omega^2 - \omega_0{}^2\right)^2 + \omega^2\gamma^2}}$ (3.10)

and $\delta = \delta(\omega) = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0{}^2 - \omega^2}\right)$ (3.11)

Both the amplitude *A* and the phase difference δ depend on the drive frequency ω; the general dependence is shown in Figure 3.4.

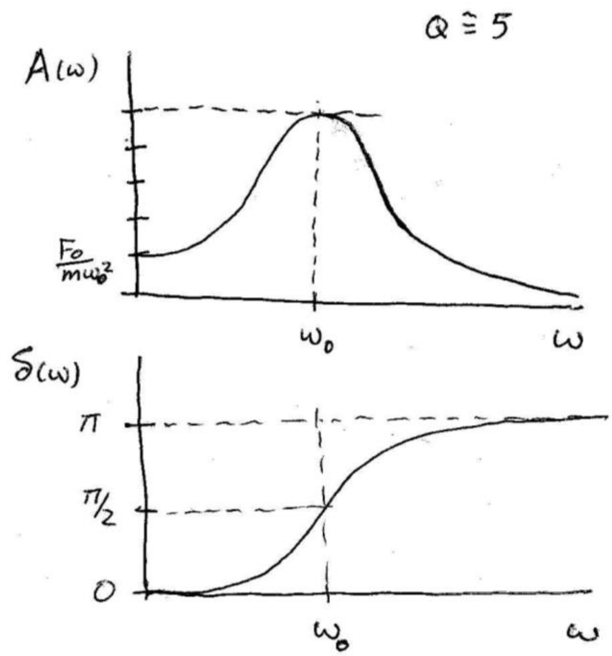


Figure 3.4. Dependence of amplitude *A* and phase shift δ on drive frequency ω, for a harmonic oscillator with a *Q* of about 5.

From this solution, we see that

- In the steady state, the mass oscillates at the frequency ω of the driving force, not at its natural frequency ω_0.
- The amplitude A is a function of frequency, and is largest when the drive frequency is near *resonance*; *i.e.* when $\omega \approx \omega_0$. The amplitude has limiting value $F_0 / m\omega_0{}^2$ as $\omega \to 0$, and limiting value 0 as $\omega \to \infty$.
- There is a difference δ between the phase of the oscillator and the drive. The two oscillate nearly in phase $(\delta \approx 0)$ when $\omega \ll \omega_0$, and they oscillate nearly out-of-phase $(\delta \approx \pi)$ when $\omega \gg \omega_0$. At exact resonance $(\omega = \omega_0)$, the oscillation of the mass *lags* the oscillation of the drive by $\pi / 2$ $(\delta = \pi / 2)$.

The oscillator also absorbs power P from the drive, and this power depends on frequency. In the limit of high Q, and for frequencies not too far from resonance, this absorbed power can be written approximately as

$$P(\omega) \approx \frac{F_0^{\,2}}{2m\gamma} \frac{\left(\frac{\gamma}{2}\right)^2}{(\omega-\omega_0)^2+\left(\frac{\gamma}{2}\right)^2} \tag{3.12}$$

This function is plotted in Figure 3.5, and is called a *Lorentzian lineshape function.* This lineshape occurs frequently in physics. We can define a width referred to as the full-width-at-half-maximum (FWHM) for this lineshape as follows. We find the frequencies ω_+ and ω_- on either side of ω_0 at which $P(\omega)$ has reached $1/2$ of its maximum value. Then, $\text{width(FWHM)} = \omega_+ - \omega_-$. From equation (3.12), we see that for the Lorentzian lineshape, $\text{width(FWHM)} = \gamma$. From this result, we have an alternate definition of Q:

$$Q = \frac{\text{resonance frequency}}{\text{resonance width}} = \frac{\omega_0}{\gamma} \tag{3.13}$$

Note that it's important to use these results only for the lineshape of *power P vs.* frequency. The function $A(\omega)$ has a width different from γ, since $P(\omega) \propto |A(\omega)|^2$.

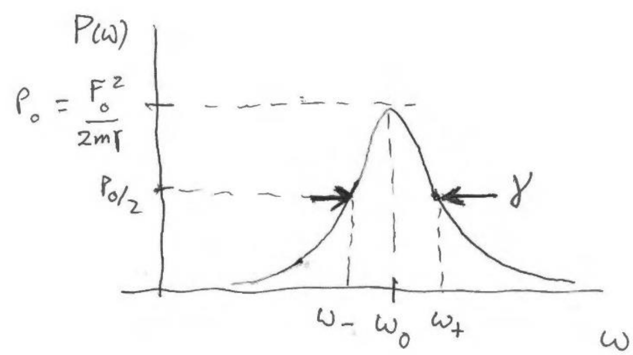


Figure 3.5. Power absorbed from the drive as a function of frequency ω, with ω close to resonance, showing a Lorentzian lineshape.

Damped cantilever

A cantilever is a bar supported at one end and free at the other. A cantilever strongly tends to vibrate in its fundamental mode, with maximum displacement at its free end. This week, we will study such vibrations of a damped cantilever, constructed as shown in Figure 3.6. The cantilever is made of a thin metal strip that is 0.5" wide and either 2.5" or 3" long. It is secured at one end to a ½" post, which will be held in place with a post holder. The free end of the cantilever is weighted with a magnet and magnet holder. The magnet will allow you to drive the motion of the cantilever with a coil. The free end also includes a bent piece of metal that functions as a "detection flap" as described below.

Without deliberate introduction of damping, the Q of the cantilever is between 500 and 1,000. This is a bit high for our purposes, because it makes the ratio of resonant to non-resonant response very large, and that makes it difficult to study both resonant and non-resonant response. It also takes a long time for the transient response to die out. To lower the Q, a layer of Sorbothane is glued to the top of the metal strip. Sorbothane is a polymer which dissipates vibrational energy. (Its deformations are *inelastic*.) With the Sorbothane layer, the Q of the cantilever is reduced to a value in the range from 50 to 350.

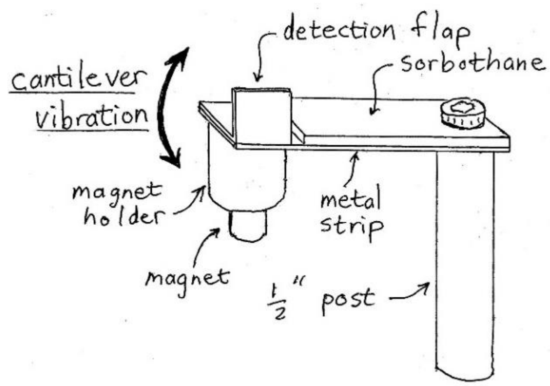


Figure 3.6. Cantilever used in this week's experiment.

Optical detection of cantilever displacement

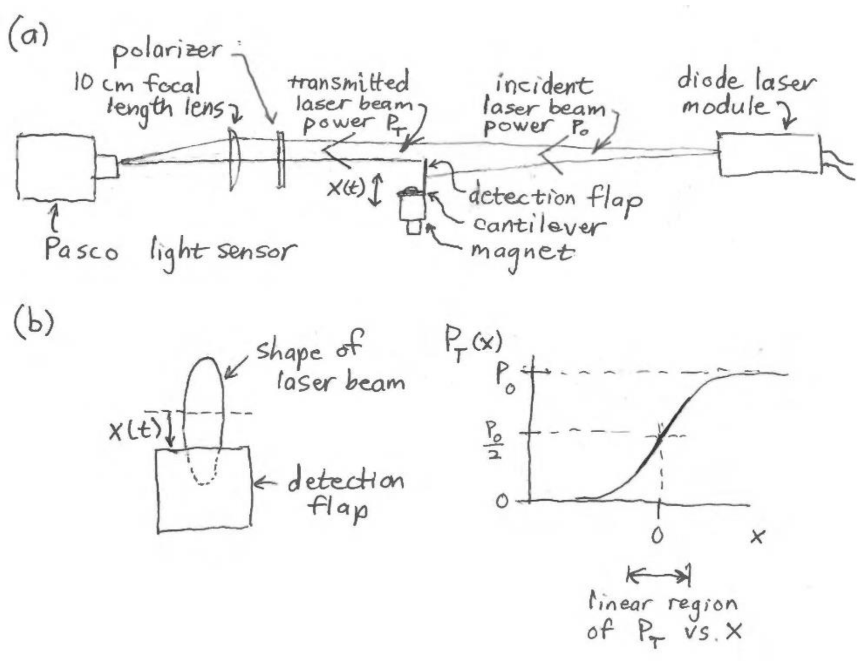


Figure 3.7. Optical method to detect the cantilever displacement. (a) Arrangement of components. (b) Detail illustrating the dependence of transmitted laser power on cantilever displacement $x(t)$.

In order to detect the cantilever vibrational displacement, we will use the optical method shown in Figure 3.7. A "detection flap" attached to the cantilever end has a vertical surface with a sharp upper edge. We will illuminate this edge with a laser beam from a diode laser module with an adjustable focus. The diode laser produces an elliptical beam which is partly intercepted by the flap, as shown in Figure 3.7(b). The laser power P_T transmitted past the flap is collected by a lens and focused onto a light sensor. As the cantilever moves, the fraction of the laser power intercepted by the flap changes; this produces the dependence of P_T on displacement x shown in Figure 3.7(b). Over the central part of the laser beam, this dependence of P_T on x is linear. The change in P_T is detected with the light sensor.

Driving coil

Vibrations of the cantilever will be driven with the coil and magnet arrangement shown in Figure 3.8. The magnet is a ¼" diameter by ½" long Nd-Fe-B rare earth (N42 or N40) cylindrical magnet, magnetized in the direction of the cylinder axis. It is glued into the magnet holder and secured to the cantilever with a #8-32 screw. The coil consists of 120 turns of #22 magnet wire wound onto a black plastic (delrin) coil form. The coil is attached to a ½" diameter post that can be mounted securely to the breadboard with the supplied mounting components.

We will drive current through the coil with a function generator output of the Pasco 850 interface. When the current passes through the coil, the coil produces a magnetic field that has a strong gradient near the ends of the coil. In one direction of the current, the magnet and the coil attract each other; in the other direction they repel each other. Thus, an AC current passing through the coil will produce an alternating force between the coil and the magnet. This will allow you to produce an alternating force on the cantilever at the same frequency as the AC current. The force is maximum with the magnet pushed just a little inside the bore of the coil, as depicted in Figure 3.8. The magnet should be centered in the coil bore, not touching the coil form.

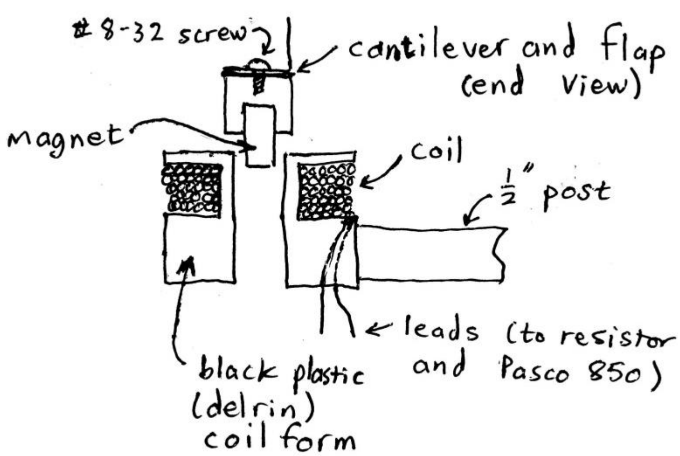


Figure 3.8. Magnet and coil arrangement for driving vibrations of the cantilever.

D. Experimental Procedure

Safe use of the diode laser module

In the following experiments, we will use a diode laser module. It emits laser light with a wavelength of 635 nm and a power of 4.5 mW. This module requires special care.

The diode laser will damage your eye if the beam shines directly into it. Do not allow this to happen. One key strategy to prevent the laser beam from entering your eye is to confine the beam to a work area. If the work area is such that you cannot reasonably place your eye into it, and if no beams leave that area, you should be pretty safe.

To implement this safety step, **don't turn the laser on unless it is in its mount, the mount is secured to the breadboard with a screw, and the laser is pointing down the length of the breadboard**. Also, **place one beam block just off each end of the breadboard opposite the laser**. After you turn on the laser, check with a piece of paper at the laser beam level all the way around your apparatus. **Block any beams you find that are leaving the area of the apparatus**. (You probably won't find any besides the end ones this week.) **Repeat this step after you have completely set up the experiment,** since the positions of any stray beams will have changed.

Also, get in the habit that you **never look at your apparatus with your eye at the same level as the laser beam**. This will further minimize the chance of a stray beam entering your eye.

Diode lasers tend to burn out easily if they experience any glitch in their current. This can occur if a laser is suddenly plugged in to its power supply, or suddenly unplugged, or if the power supply is unplugged with the laser running. To avoid burning out the laser, **the laser should be turned on and off <u>only</u> with the switch on its power supply.** Do not unplug the laser from the supply, or the supply from its power, with the laser running. And, make sure that the supply is off before you connect a laser to it.

Basic experiment set-up

The experimental set-up is illustrated in Figure 3.9. You'll use the breadboard and the mounting hardware that we introduced in experiment 2, including posts, post holders, screws, and bases. You may want to use the pedestal-style post-holder and clamping fork for the lens and /or light sensor, as illustrated in Figure 3.9. This mount type has the advantage that it can easily be positioned anywhere on the breadboard.

When working with lasers and optics, **secure all optical components to the board. Do not leave any optical elements in mounts that are loose.** Leaving mounts loose will result in non-reproducible results and wasted time when the loose item gets knocked out of position. It might also result in reduced experimental stability. The loose items can also be a potential safety hazard, since the laser beam can be inadvertently deflected by the loose item. We recommend that you do not screw post holders directly into the breadboard for most items. If you do this, you will have too little flexibility in the position of the holder. Use the pedestal bases or standard bases instead.

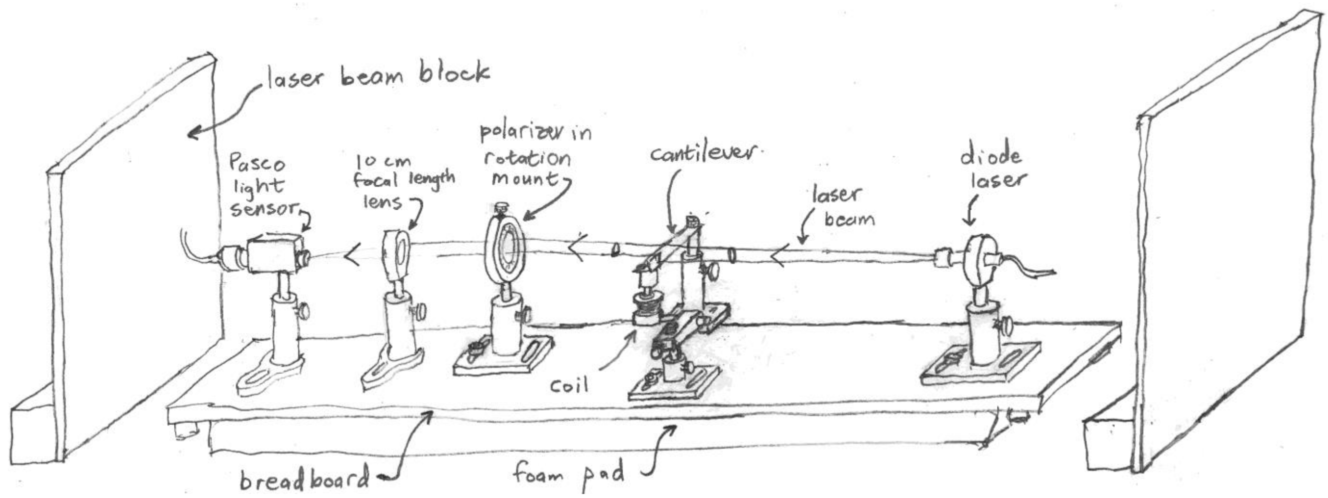


Figure 3.9. Experimental set-up.

There is considerable room for variation in the exact arrangement of components in an experiment like this. There are pros and cons of different ways of using the mounts, including differences in stability and in the flexibility of positioning. Specific suggestions for mounting the cantilever and coil are illustrated in Figure 3.10.

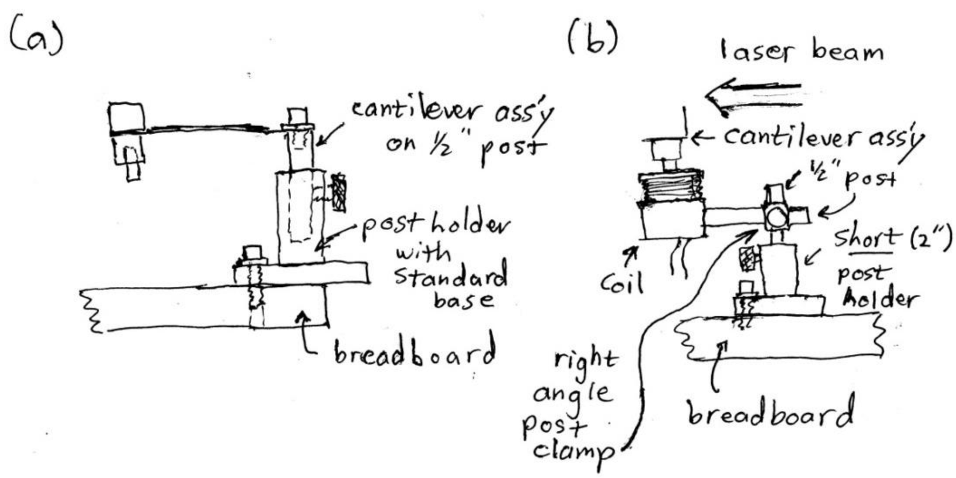


Figure 3.10 (a) Suggested mounting of the cantilever assembly. (b) Suggested method to mount the coil below the cantilever.

Your first step should be to **place the foam pad under the breadboard**, as shown in Figure 3.9. Make sure that the breadboard feet don't sit on the foam – you want the breadboard to sit flat on the foam. This foam pad is important, because the vibrations of the cantilever can couple to vibrations of the breadboard. The foam pad reduces the vibrations of the breadboard and the undesirable effects of this coupling.

Next, mount the laser at one end of the breadboard, with the beam directed towards the opposite end of the board. The beam should run approximately down the board centerline. You may turn the laser on once the mount is secure and you have the beam blocks in place. **The beam height should be about 6" above the board surface**. This is important for two reasons. One is that we have a variety of heights of post holders (2", 3" and 4") and posts. Most combinations of posts, holders, and optical items are compatible with a 6" beam height. If you set the beam much lower or much higher, you may find that the posts and holders you need for that height are scarce. The other reason is that you'll need to place the coil below the cantilever as shown in Figure 3.10(b). If the laser beam is too low, you won't have room to do that.

Next, position the Pasco light sensor at the other end of the breadboard with the laser beam shining into the sensor. A pedestal-style post holder or a mount with a right-angle post clamp will make it easier to move the light sensor to the correct position. Connect the light sensor to the Pasco 850 channel A, start up Capstone, and set it up to monitor the light sensor signal in a Scope window. The light sensor signal is linear with laser power up to some point, but then remains constant ("saturates") for higher laser power. Capstone displays the signal level as a percentage of this saturation value – i.e. it displays a value between 0 and 100. For this reason, **adjust the Scope window to display signal levels over the full range from 0 to 100**. Your laser power is high enough to easily drive the sensor to the saturation level. For this reason, set the gain switch on the sensor to its lowest, "x1" setting.

Next mount the lens to the breadboard about 10 cm in front of the light sensor. The lasers put out a beam that is elliptical, with an adjustable focus. We have oriented the lasers so that the long axis of the ellipse is vertical, as shown in Figure 3.7(b). We have also adjusted the focus of the lasers so that the beam diverges. The beam divergence should bring the beam size to be about 6-8 mm high in the center of the board, and roughly 12-16 mm high at the other end of the board. The beam should fill up much of the clear aperture of the lens, but not spill over onto the lens mount. If you think your laser beam divergence or orientation needs to be adjusted, ask your instructor to do that for you.

Next, position the lens and detector so that the detector lies at the focus of the beam. At this point, you should see the signal level pegged near 100, because the laser power is above the saturation power. So, you need to reduce the power. To do this, put the polarizer in place just before the lens, such that the laser beam is transmitted through the clear area of the polarizer. Then, loosen the set-screw on the polarizer mount. The polarizer functions as an adjustable laser power attenuator. (You'll learn more about how this works in an upcoming lab; for now just think of the polarizer angle adjustment as a laser power attenuation adjustment.) Rotate the polarizer until you see the light sensor signal register below 100. Set it to about 80, then tighten the set screw.

Check that your laser beam is hitting the middle of the detector. To do this, move the detector slight left and right, and slightly up and down. You should be able to move the detector by a couple of mm in any direction with negligible change in signal level. If you cannot do this, that means you are not at the focus of the laser beam, where the laser beam size is small compared to the detector size. Adjust the detector position as needed so you are at the focus, hitting the middle of the detector.

Now, mount the cantilever in the middle of the breadboard, such that it can be raised into the laser beam. One way to set up the cantilever mount is shown in Figure 3.10(a). It may be necessary to let the standard base hang a little over the side of the breadboard, but you should be able to get a secure mounting with two screws on the opposite side of the base. Next, adjust the cantilever height so that it blocks about 50% of the laser beam, *i.e.* so that the light sensor signal registers to be about 40. This sets the apparatus so that you are in the range where the light signal vs. cantilever displacement is approximately linear, as shown in Figure 3.7(b).

Monitor the time dependence of the signal. At this point, you may see some oscillation of the signal even after it has been undisturbed for several damping times. If so, this is happening because there is some noise in the room, and this noise can be picked up by the cantilever and set it into a weak vibration. This noise should be small (a few percentage units or less). If it is not small, consult with your lab instructor.

The cantilevers are numbered and are all different in one or more ways – metal strip material (phosphor bronze or stainless steel), thickness (various, from 0.010" to 0.020"), and/or length (2.5" or 3"), magnet holder weight, and sorbothane hardness and/or thickness. For this reason, the parameters of ω_0, γ, and Q for each cantilever will be different. For this reason, **be sure to record the number of your cantilever and to include it in your report.** You may expect that your cantilever's resonance frequency $f_0 = \omega_0 / 2\pi$ will be between 10 Hz and 30 Hz, and that its damping constant γ will be between $0.3\ \mathrm{s}^{-1}$ and $2\ \mathrm{s}^{-1}$.

You should **set your sampling rate to give at least 10 measurements per cycl**e. Since your cantilever could have a frequency as high as 30 Hz, a sampling rate of 1 kHz is a reasonable choice.

Measurement of the free decay and driven oscillation of a damped cantilever.

Next, manually displace the cantilever, let go, and observe the time-dependent signal from the light sensor. You should be able to observe a damped oscillation. Record several examples of the damped oscillation. One way to do this is to open a Graph window, select the light sensor on the vertical axis, and set the window to continuous mode. Then, displace the cantilever and immediately press "record". Stop the recording after the signal has decayed. This method has the advantage of recording <u>only</u> the decaying oscillation, which makes the fitting easier. Once you have the recorded signal, <u>fit your data to a damped sine function</u> $Ae\wedge(-Bt)(\sin(\omega t+\varphi))+C$. Do this directly in Capstone using the fitting tool along the top row of the window (the sloped pink line with the blue dots around it). (This tool will only be available in the Graph window.) In order to get a good fit, you'll want the light sensor signal to stay in the range where it is linear with displacement – roughly 20 to 60 on the vertical scale. One way to do this is to start with a large oscillation amplitude, and to restart the data collection when the oscillation amplitude has decayed to a suitable range.

By comparing this fitting function to equation (3.4), we see that the best fit value for the parameter ω gives the measured cantilever resonance frequency ω_0 (in radians per second), and that the best fit value for the parameter B gives the measured value of $\gamma / 2$.

From your recorded waveforms and fits, determine the resonance frequency ω_0 of your cantilever, its damping rate γ, and its Q.

It is possible that you may see a signal that does not look like a damped sine wave, but has a "wavy" appearance. If so, there are two possible causes. The most likely one is that you are seeing an effect of digital sampling called aliasing. This happens if you set your sampling rate too low. If so you should be able to eliminate it by setting your sampling rate to at least 1 kHz. It is also possible that you're seeing "beats" due to the coupling of your cantilever to some other mode of vibration. If you think this may be happening, consult with your instructor.

Measurement of the driven oscillation of a damped cantilever.

Next, position the coil below the cantilever assembly as shown in Figure 3.10(b). You will probably need to use one of the short (2" tall) post holders for this. The magnet should protrude slightly into the magnet bore, and be centered in that bore. The magnet and coil form should not be touching each other at any time.

To drive current through the coil, we will use analog output 1 of the Pasco 850 interface. This output is designed to drive high currents through low impedance loads like your coil. However, the impedance of the coil by itself is less than 1 Ohm, which is a bit too low for the Pasco 850. For this reason, you should connect the coil as shown in Figure 3.11, with the 5 Ohm resistor in series with the coil. This results in a cleaner and better controlled current sine wave than a direct connection to the coil.

When you do this lab, you should keep the voltage amplitude to be always less than 5 Volts, and most of the time less than 1 Volt. This will limit the current amplitude to always less than 1 Amp and most of the time less than 0.2 Amps. There are several reasons why this is important. One is that this will limit the power dissipated into the resistor to $I^2 R / 2 = 1^2 \times 5 / 2 = 2.5$ W. This is safely below the 10 W maximum power rating of the resistor, so you won't risk burning the resistor out. Another reason to limit the voltage is that if you try to drive with amplitudes much more than 5 Volts, you may get a clipped sine wave due to an output current limit of the Pasco 850. This is definitely not what you want. Finally, currents of 1 Amp or more can easily overdrive the cantilever, so that it swings into the nonlinear detection region (see Figure 3.7(b)). **Most of the time, a small fraction of an Amp will be the appropriate current amplitude** – enough to give a good signal without overdriving the cantilever. Near resonance, you may want to use a voltage amplitude below 0.1 Volts.

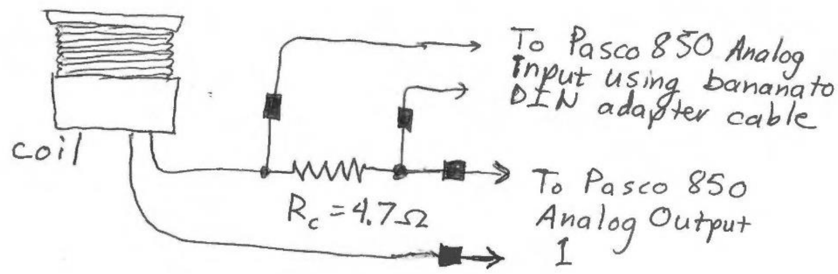


Figure 3.11. Electrical connections of the coil to the Pasco 850 interface box. Black rectangles represent banana plug connections.

Pasco's sensors use DIN connectors with the pin-out shown in Figure 3.12. Each input (A, B, C, or D) connects to a differential amplifier, which measures the voltage *difference* between two pins, sig+ and sig-. The remaining pins are used to supply power to the sensors. Aside from this power supply function, all the Pasco box is really doing is to record the voltage difference between sig+ and sig-. You can use the Pasco 850 to measure a voltage difference between any two points by connecting directly to sig+ and sig-. This can be done with the cables in the lab that have DIN connectors at one and, and banana plugs at the other – these are wired to connect the banana plugs to sig+ and sig-.

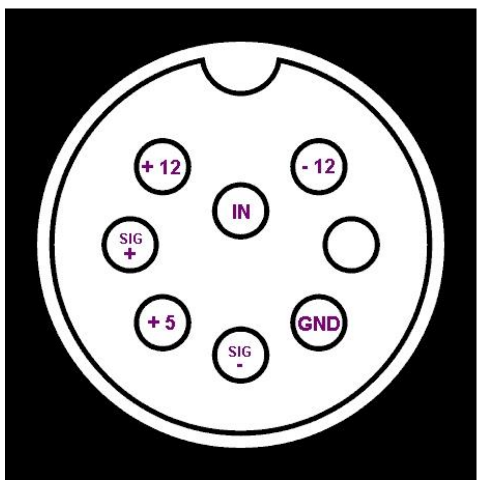


Figure 3.12. Pinout of the DIN connectors according to Pasco's standard. This shows the pinout looking into the connector on a detector. The pinout looking into the PASCO 850's connectors is a mirror reflection of this pinout in the vertical plane (e.g. sig+ is the right-most pin).

In this experiment, you'll measure the voltage difference across the resistor as shown in Figure 3.11. This voltage is proportional to the current through the coil, and therefore to the force applied by the coil to the magnet. This is the best way to measure the force. For instance, it would not work as well to use Capstone to monitor the voltage of analog output 1. The reason is that the coil has reactive (inductive) impedance. Therefore the current through the coil does not necessarily have the same phase as the voltage across the coil.

Connect the two ends of the resistor to the Pasco 850's input B using the banana to DIN connector. Set that channel up as a "Voltage Sensor" in Capstone. This voltage will be proportional to the force on the cantilever. Set up a Scope window to display this voltage.

Set the Pasco 850 analog output 1 to output a sine wave of 1 Volt amplitude and 8 Hz frequency. You can do this with controls accessible through the left hand column. Your scope window should now display a sine wave with an amplitude that is similar to the Pasco output amplitude. Reduce the frequency to 1 Hz. You should be able to see the cantilever moving in response to the applied magnetic force.

Add the light sensor display to the scope using the "add new y-axis display" tool at the top of the scope. At this point, you should be able to observe the coil current and light sensor signal simultaneously. I found it useful for the following measurements to trigger the display from the voltage signal.

Experiment with the cantilever by varying the frequency and amplitude of the voltage applied to the coil. See if you can observe the expected qualitative features – non-zero response at very low frequency, a large enhancement in the vibration amplitude near resonance, very small amplitudes at high frequency, a linewidth of amplitude *vs*. frequency that is similar to the γ you observed from free decay. You will need to turn down the driving amplitude close to resonance. Also, see if you can observe the change in relative phase between the resistor voltage and the displacement signal as you tune through resonance.

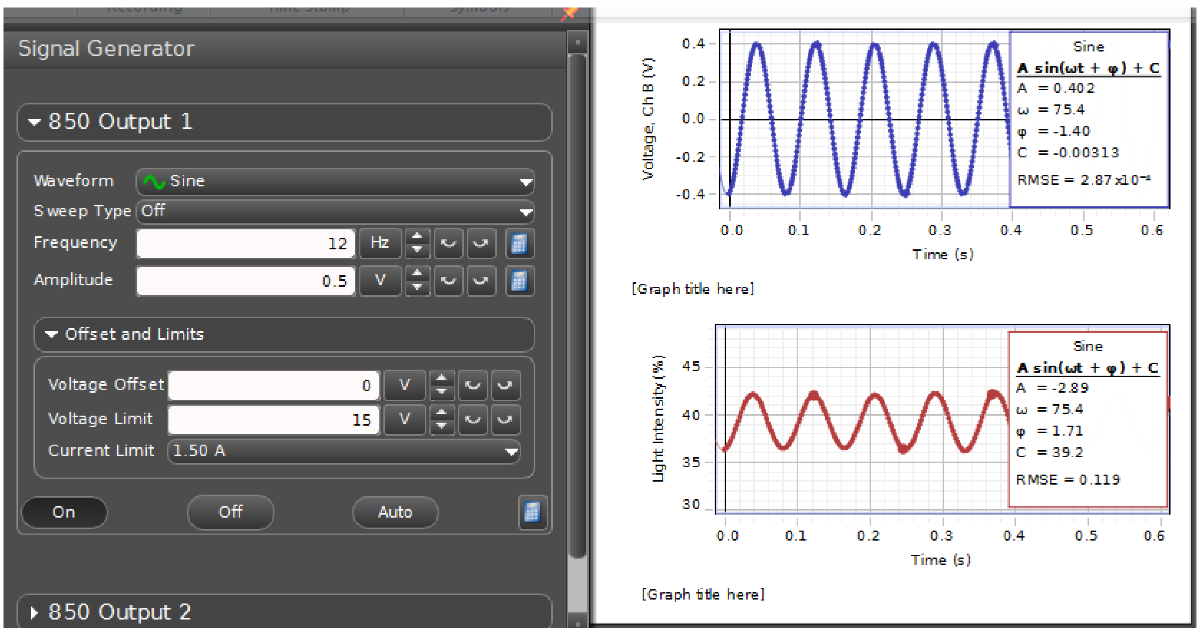


Figure 3.13. Capstone display when the Signal Generator is operating, signals have been recorded from the voltage and light sensor in two open Graph windows, and fits to a sine wave have been applied to each set of data. The force applied to the cantilever is proportional to the voltage signal recorded in the upper window, and the displacement of the cantilever is proportional to the light intensity signal recorded in the lower window.

Experiment with fitting a pair of these curves. To do this, close your scope windows, open two "Graph" windows, and select the voltage sensor for one and the light sensor for the other. Record a brief period of the oscillation (one second to a few seconds). Use the "Apply selected curve fits" tool to fit for a sine wave $A\sin(\omega t+\phi)+C$ in each window. If you have everything working correctly, your Capstone display should look like that shown in Figure 3.13.

Leave these windows open as you take data – the data and fits should continue to automatically update with each new sweep, which is quite convenient. From these fits, you can obtain the ratio of the light signal amplitude to the resistor voltage amplitude (proportional to the ratio of displacement amplitude to force amplitude), and the phase difference between the light sensor (displacement) oscillation and the voltage (force) oscillation.

Once you have the experiment running, take a series of measurements of the driven oscillation at different frequencies. At each frequency record (i) the amplitude of the voltage signal, (ii) the amplitude of the light sensor signal (iii), the phase of the voltage signal, and (iv) the phase of the light sensor signal. You will obtain these quantities from the sinusoidal fits for parameters *A* and φ to your oscillation data as shown in Figure 3.13. At each frequency, calculate the amplitude ratio and the phase difference. Remember that phases are defined only modulu 2π. So, if you get one phase that is 6.14 radians, and another phase that is 0.02 radians, it might make more sense to convert the second phase to $0.02+2\pi=6.30$ radians. Also, remember that a negative amplitude is equivalent to a phase shift of π. For instance, if your fit gives
$A=-0.050$ Volts, ϕ=2.80 radians, that is the same as
$A=+0.050$ Volts, ϕ=2.80 +π=5.95 radians.

Don't take evenly spaced frequencies. Instead, make your measurements close together in frequency near resonance, such that you have at least five points as you scan over the peak. Include at least one point at a very low frequency, and then include a few points intermediate between the low frequency and resonance. Above resonance, include a few points at substantial multiples of the resonance frequency (something like $2\omega_0, 3\omega_0, 5\omega_0, 10\omega_0$).

Fit your data for the amplitude ratio to equation (3.10). (Leave the overall amplitude factor $F_0/m\omega_0^2$ as a free parameter in the fit.) Guidance for carrying out this fit is given in the appendix of this manual section. Determine ω_0, γ, and Q from your fits. Compare them to the values you obtained from the free decay measurement. Also, use your fitted values for ω_0 and γ to make a plot of the theoretical vs. measured phase shifts, as a function of drive frequency ω.

In your report, comment on the degree to which you were able to confirm the driven oscillator theory in the introduction. Were you able to observe the qualitative features discussed in the bullet points following Figure 3.4? How well do the *Q* values from linewidth agree with those from free decay? How well does your data match the theoretical expressions (3.10) and (3.11)? If there are areas of significant discrepancy, can you think of experimental explanations for that?

Please remember to include the number of your cantilever in your report. That way, we can determine whether your measurements are correct for your particular cantilever.

Appendix: Fitting your data

You will need to do a nonlinear least-squares fit of your data to equation (3.10). You can do this with any programming environment you like. It might work to use Capstone's "fit user-defined function" feature, or you might prefer some other program.

If you are not familiar with non-linear least-squares fits like this, it may help to see a very similar example of such a fit. This appendix provides an example of such a fit using Mathematica. You can use any fitting program you like, and the procedure for other programs will be similar to this one. If you want to use Mathematica, it will be acceptable for this lab to cut-and-paste the following code and modify it to fit your data. If you want to go that route, Mathematica is available for download to any student in this course (link here), or in PMCL. If you do want to do that, I recommend cutting from the plain text version of the program, which I've also posted. (Minor formatting issues in text cut from Word and PDF documents often cause instructions to be entered incorrectly in Mathematica.) Once you have the program entered into an open Mathematica window, you can execute it with shift-enter.

The following "data" is not real, and uses an oscillator frequency and *Q* that is different than those of the cantilevers from your lab. Since I picked "data" values that closely fit the correct function, the quality of the fit is better than what you'll probably get from real data. However, the fitting functions and general procedure are the same as what you'll need.

Purely hypothetical experiment and fitting:

Joe has a lab with a damped oscillator of resonance frequency f_0 and quality factor Q. Joe doesn't know the resonance frequency and quality factor ahead of time. One goal of the experiment is to measure these. This measurement will be accomplished through the fit of a theoretical function to his data.

The oscillator displacement is $x(t)$. The lab has a device that produces a (voltage) signal proportional to $x(t)$, *i.e.* $V_D(t) = \beta x(t)$, where β is some constant. The subscript "D" stands for "displacement." The experiment is also equipped with a transducer that produces a force F on the oscillator that is proportional to an input voltage V_F, *i.e.* $F(t) = \alpha V_F(t)$, where α is some constant. The subscript "F" stands for "force."

Joe is instructed to apply a sinusoidal voltage

$$V_F(t) = V_{F0} \sin\left(2\pi ft + \phi_F\right) \tag{3.14}$$

to the transducer at a series of different frequencies f, and to measure the steady-state response of $V_D(t)$ to the applied force. Since the steady state response of a driven, damped oscillator to a single frequency force is an oscillation at that same frequency, Joe knows that $V_D(t)$ must be of the form

$$V_D(t) = V_{D0} \sin\left(2\pi ft + \phi_D\right). \tag{3.15}$$

This means the displacement is $x(t) = x_0 \sin\left(2\pi ft + \phi_D\right)$, where $x_0 = V_{D0} / \beta$. Joe records samples of $V_F(t)$ and $V_D(t)$ at a bunch of different frequencies *f*, and fits each to obtain V_{F0}, ϕ_F, V_{D0}, and ϕ_D. For each frequency, he also calculates $R = \dfrac{V_{D0}}{V_{F0}}$ and $\Delta\phi = \phi_F - \phi_D$. His data is given in the table below.

Frequency (Hz)	V_{F0}	ϕ_F *	V_{D0}	ϕ_D *	$R = \dfrac{V_{D0}}{V_{F0}}$	$\Delta\phi = \phi_F - \phi_D$
1.0	1.0	-	0.0741	-	0.0741	0.0027
10	1.0	-	0.0862	-	0.0862	0.094
20	1.0	-	0.0876	-	0.0876	0.035
40	1.0	-	0.1651	-	0.1651	0.094
50	0.2	-	0.0808	-	0.404	0.322
52	0.2	-	0.1112	-	0.556	0.53
54	0.2	-	0.1706	-	0.853	0.87
55	0.2	-	0.2182	-	1.091	1.35
56	0.2	-	0.2134	-	1.067	1.69
57	0.2	-	0.172	-	0.860	2.05
58	0.2	-	0.1392	-	0.696	2.54
60	0.2	-	0.0944	-	0.472	2.80
64	1.0	-	0.223	-	0.223	2.89
70	1.0	-	0.142	-	0.142	3.08
80	1.0	-	0.0677	-	0.0677	3.01
100	1.0	-	0.0361	-	0.0361	3.16
140	1.0	-	0.0155	-	0.0155	3.02
200	3.0	-	0.0180	-	0.00600	3.10
400	3.0	-	0.00453	-	0.00151	3.25
600	3.0	-	0.00195	-	0.00065	3.14

* I didn't bother making up numbers for these quantities.

The primary goal of the experiment is to see if theory correctly describes the driven motion of the oscillator. Theory gives

$$x_0 = \frac{V_{D0}}{\beta} = \frac{F_0}{m}\sqrt{\frac{1}{\left(\omega^2 - {\omega_0}^2\right)^2 + \omega^2\gamma^2}} = \frac{\alpha V_{F0}}{m}\sqrt{\frac{1}{\left(\omega^2 - {\omega_0}^2\right)^2 + \omega^2\gamma^2}} \tag{3.16}$$

Thus

$$R = \frac{V_{D0}}{V_{F0}} = \frac{\dfrac{\alpha\beta}{m}}{\sqrt{\left(\omega^2 - {\omega_0}^2\right)^2 + \omega^2\gamma^2}} = \frac{\dfrac{\alpha\beta}{(2\pi)^2 m}}{\sqrt{\left(f^2 - {f_0}^2\right)^2 + f^2\left(\dfrac{\gamma}{2\pi}\right)^2}} \tag{3.17}$$

Here we are using the convention that ω is measured in rad/s, and *f* in Hz, so that $f = \omega / 2\pi.$ Joe made no measurements related to α or $\beta;$ thus he must take the factor $\frac{\alpha\beta}{(2\pi)^2 m}$ as a undetermined parameter in his fitting function. Let's call that parameter *a*. Joe also wants the *data* to tell him what f_0 and $\gamma / 2\pi$ are; thus he wants to take those as fit parameters, too. He calls those fit parameters *b* and *c*, respectively. Thus his fitting function is

$$R = \frac{a}{\sqrt{\left(f^2 - b^2\right)^2 + f^2 c^2}} \tag{3.18}$$

The theoretical value for the phase difference is

$$\Delta\phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^{\ 2} - \omega^2}\right) = \tan^{-1}\left(\frac{cf}{f_0^{\ 2} - f^2}\right) \tag{3.19}$$

In a real research experiment we'd probably want to fit to equations (3.18) and (3.19) simultaneously, but that would be overkill for a teaching lab experiment. Fitting to equation (3.19) can get a little tedious because of problems involving the domain of the $\tan^{-1}$ function and non-meaningful phase differences of $2\pi.$ Thus Joe will fit only his amplitude data to equation (3.18).

The following Mathematica program carries out the necessary fit, outputs the best fit parameters, and then makes a few plots showing how well the theory fits the data. (Pretty well, since I made up the data to do that.) The comments within the program give some guidance on what the various lines of code are doing.

```
(* This program enters some (fake) data values for amplitudes
and phase shifts vs. frequency from an "experiment" on a driven
damped harmonic oscillator, finds the best fit values of the
parameters of a theoretical function to the amplitude data, and
makes some plots showing the best fit theoretical curve and data
for the amplitudes. It also makes some plots of the theoretical
curve and data for the phase shifts, using the parameters from
the amplitude fit. Comments following each line give some
indication that line's function. *)

Clear[a,b,c,f]

(* If you've been working with Mathematica, and entered values
previously for parameters, it will keep those parameters set to
those values under some conditions. This can sometimes cause
unexpected errors. This statement clears the values of the
parameters we want to use. It's often a good idea to do this. *)

Rdata={{1,0.0741},{10,0.0862},{20,0.0876},{40,0.1651},{50,0.404},{52,0
.556},{54,0.853},{55,1.091},{56,1.067},{57,0.860},{58,0.696},{60,0.472
},{64,0.223},{70,0.142},{80,0.0677},{100,0.0361},
{140,0.0155},{200,0.00600},{400,0.00151},{600,0.00065}};
```

(* This statement enters the values for our experimental frequencies and experimental R values as a two-dimensional array named Rdata. The semicolon at the end of the line tells Mathematica not to output the array when the instruction executes. *)

```
phidata =
{{1,0.0027},{10,0.094},{20,0.035},{40,0.094},{50,0.322},{52,0.53},{54,
0.87},{55,1.35},{56,1.69},{57,2.05},{58,2.54},{60,2.80},
{64,2.89},{70,3.08},{80,3.01},{100,3.16},{140,3.02},{200,3.10},
{400,3.24},{600,3.14}};
```

(* This statement enters the values for our experimental frequency and experimental delta-phi values as a two dimensional array named phidata. *)

```
FindFit[Rdata,a/Sqrt[(f^2-b^2)^2+f^2 c^2],{a,{b,2},{c,56}},f]
```

(* This statement finds the parameters a, b, and c, for which the squares of the differences between the values of R in Rdata and the values of the function a/Sqrt[(f^2-b^2)^2+f^2 c^2] are minimized. In other words, it carries out a nonlinear least squares fit of the data to the theoretical function. The notation {a,{b,2},{c,56}},f tells the program that f is the independent variable (first column of Rdata) and that a, b, and c are the fit parameters. The notation {b,2} tells the program to start searching for the best fit with b set to 2, and similarly for {c,56}. Sometimes you have to tell the fitting routine where to start searching like this, or the fit won't "converge", i.e. won't find the best fit parameters. Note that these will not be appropriate starting values for your fit, so you may need to change them. Or it may work to just delete them, i.e take {a,{b,2},{c,56}} -> {a,b,c}. When this line executes, it will also output the best fit values for a, b, and c. *)

```
disp[f_]=a/Sqrt[(f^2-b^2)^2+f^2 c^2]/.%;
```

(* This statement defines the function disp{f} as the theoretical function a/Sqrt[(f^2-b^2)^2+f^2 c^2]. The notation /.% tells Mathematica to use the values of a,b, and c from the last executed instruction, i.e. the values that resulted from the least squares fit to the data. *)

```
phi[f_]= Piecewise[{{Pi+ArcTan[c f/(b^2-f^2)],c f/(b^2-f^2)<0}},
ArcTan[c f/(b^2-f^2)]]/.%%;
```

(* This statement defines the function phi[f] as the correct theoretical function for the phase shift. ArcTan is a periodic function with period Pi. In Mathematica, the presumed domain of this function is the interval [-Pi/2,Pi/2]. However, it is conventional to plot phase shift data for an oscillator on the domain [0,Pi]. The Piecewise instruction in this line tells Mathematica to add Pi to the -ArcTan function if it is in the negative part of the presumed domain. This is valid due to the

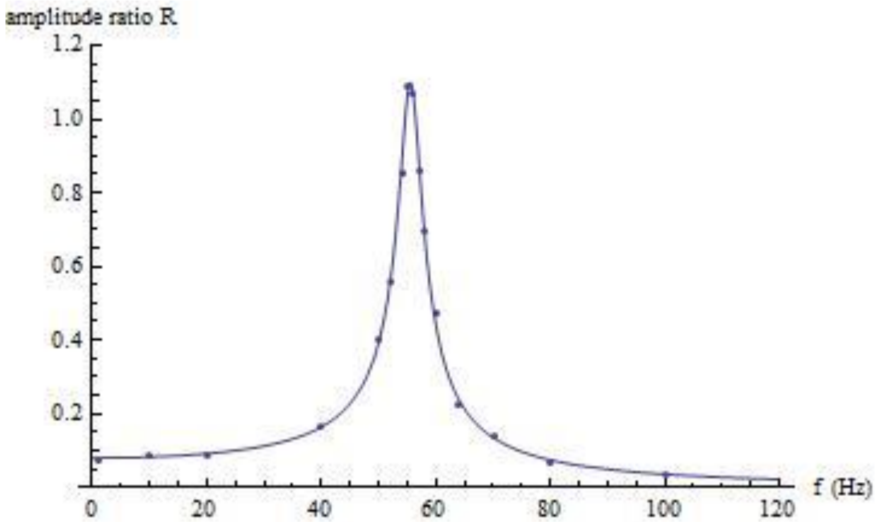
```
Pi periodicity of the function, and has the effect of shifting
the domain of Mathematica's ArcTan function to match the domain
used for the data. The notation /.%% tells Mathematica to use
the values of a,b, and c from the instruction executed two
lines previously, i.e. the values that resulted from the least
squares fit to the data. *)

Show[Plot[disp[f],{f,0,120},PlotRange ->{0,1.2}],
ListPlot[Rdata],AxesLabel->{"f (Hz)","amplitude ratio R"}]

(* This makes a plot of the theoretical function disp[f] along
with the data. Show is an instruction that tells Mathematica to
combine graphical objects. In this case one graphical object is
the Plot (plot of the function disp), and the other graphical
object is the ListPlot (plot of the data points) *)

Show[Plot[disp[f],{f,100,600}, PlotRange->{0,0.05}],
ListPlot[Rdata],AxesLabel->{"f (Hz)","amplitude ratio R"}]

(* This shows the same quantities near the high end of the
frequency range so you can see the result better. *)

Show[Plot[phi[f],{f,0,600},PlotRange ->{0,3.2}],
ListPlot[phidata], AxesLabel ->{"f (Hz)", "delta phi"}]

(* This instruction makes a plot of the theoretical and
experimental phase shift over the full frequency range. *)

Show[Plot[phi[f],{f,40,70},PlotRange ->{0,3.2}],
ListPlot[phidata], AxesLabel ->{"f (Hz)", "delta phi"}]

(* This instruction makes a plot of the theoretical and
experimental phase shift over the frequency range near resonance
so you can see it better. *)
```

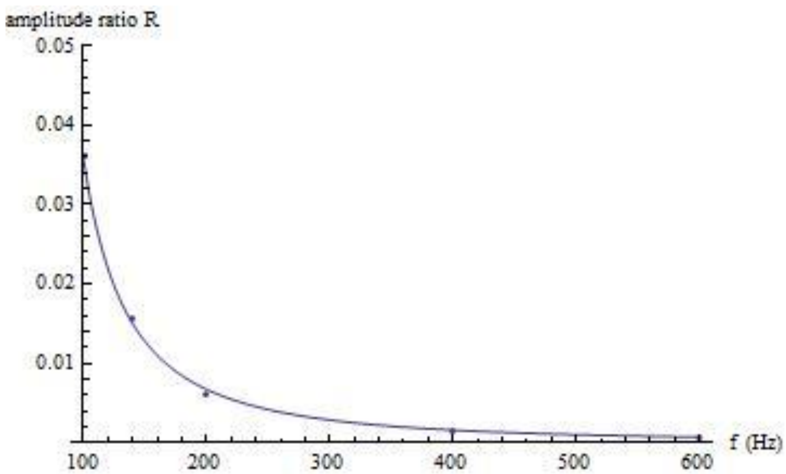
Here is the output of the program:

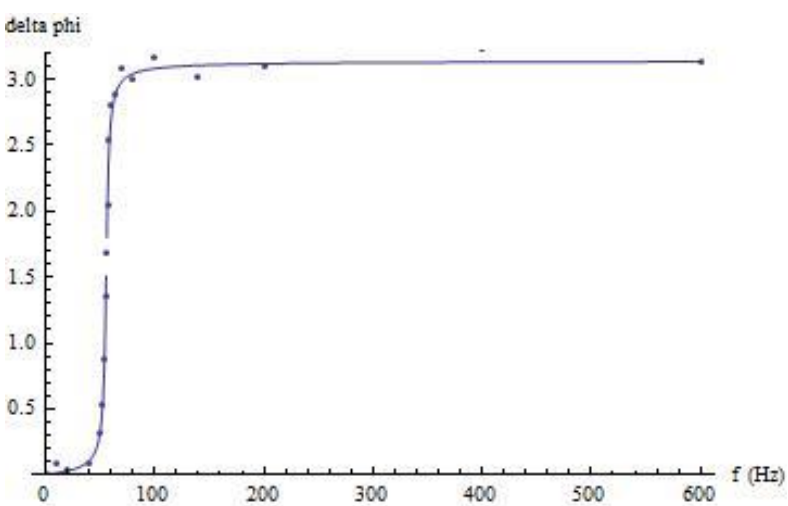
```
{a->246.827,b->-55.6057,c->4.04369}
```

This tells us the best fit values of *a*, *b*, and *c*. The value of *a* doesn't mean much here. The value of *b* tells us that the resonant frequency is $f_0 = 55.61$ Hz . The value of *c* tells us that the decay parameter is $\gamma / 2\pi = 4.04$ Hz. The *Q* of the oscillator is therefore 55.61/4.04 = 13.89. (Again, these are not what you'll find for your cantilever.)

And, here are the plots:







Note that as always, both axes of all graphs are labelled, with correct units where appropriate. In a research experiment, you would also go further and calculate the statistical errors in the fit parameters. However, we will not expect you to do this for this particular experiment. (You should still say something about the errors in your results, though.)