Ensuring a safe perpetual stew

Consider a pot of "perpetual stew" that remains on a heat source forever. Each day, we remove some stew to eat and replace the volume with fresh ingredients. How can we ensure that our stew mostly contains recently-added ingredients?

Theory

Suppose that in each daily cycle we replace proportion $0 < p < 1$ of the stew. We consider these new ingredients to be 0 days old. In general, the volume of stew that is n days old "survived" n daily cycles – it was part of the $(1-p)$ proportion not removed in each cycle. We compute this as

$$V(n \text{ days old}) = p(1-p)^n.$$

The leading p term accounts for the fact that the n-day-old ingredients made up proportion p when they were 0 days old. To double-check our work, we verify that

$$\begin{aligned} \sum_{n=0}^{\infty} V(n \text{ days old}) &= p \sum_{n=0}^{\infty} (1-p)^n \\ &= p \, \frac{1}{1-(1-p)} = 1, \end{aligned} \tag{1}$$

where we used the well-known infinite geometric series formula. Now suppose we define any food older than N days as unsafe, and we wish to ensure that the stew contains a proportion no greater than ϵ of unsafe food (where ϵ is very small). We have

$$\begin{aligned} V(\geq N \text{ days old}) &= \sum_{n=N}^{\infty} V(n \text{ days old}) \\ &= p \sum_{n=N}^{\infty} (1-p)^n \\ &= p(1-p)^N \sum_{n=0}^{\infty} (1-p)^n \\ &= (1-p)^N, \end{aligned} \tag{2}$$

where we reused the geometric series result from (1). Therefore, to ensure $V(\geq N \text{ days old}) < \epsilon$, it is sufficient to ensure that

$$(1-p)^N < \epsilon \iff p > 1 - \epsilon^{1/N}. \tag{3}$$

Example

The United States Food and Drug Administration (FDA) *Food Defect Levels Handbook* specifies acceptable levels of food contamination in various food products. We will take wheat as an example because its threshold is given directly as a proportion: up to 9 milligrams per kilogram of "rodent filth" is permitted. If we assume that old stew is no more dangerous than "rodent filth", this translates to $\epsilon = 9 \times 10^{-6}$.

It is common in United States households to keep fresh vegetables in a refrigerator for up to one week, cook them, and then keep cooked leftovers for up to one more week. If we assume that the high temperature in our stew pot is at least as good at preventing bacterial growth as this process, this translates to $N = 14$.

Evaluating equation (3) with these values, we find $p \geq 0.564$ is sufficient. This gives us a guideline for perpetual stew planning: we should choose a stew pot that is a little less than twice as large as the volume of stew we expect to eat each day.

Future work

It is unrealistic to assume that the proportion of stew removed each day is always exactly the same. A more sophisticated approach would be to assume that p is drawn independently each day from a known probability distribution, and use concentration inequalities to derive a high-probability bound of the type

$$\Pr\{V(\geq N \text{ days old}) < \epsilon\} \geq 1 - \delta.$$