Apex coordinates in dipole magnetic field

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Abstract

This document presents some fundamental equations for a magnetic dipole model, and the equations for apex base vectors in a dipole field for a spherical Earth. The purpose of this is to be able to easily use the equations developed for apex coordinates with a dipole magnetic field.

1 Definitions

- λ is the dipole latitude.
- ϕ is the dipole longitude
- Subscript ma is used to denote Modified Apex (MA) coordinates
- Subscript qd is used to dentoe Quasi-Dipole (QD) coordinates
- r is the radius (same for dipole, MA, and QD coordinates)
- B_0 is the reference magnetic field, equal to the norm of a vector formed by the dipole Gauss coefficients: $B_0 = \sqrt{(g_0^1)^2 + (g_1^1)^2 + (h_1^1)^2}$
- R is the modified apex reference radius
- $R_E = 6371.2$ km is the Earth radius

2 Equation for a dipole

The equation for a dipole magnetic field in dipole coordinates is

$$\mathbf{B}(r, \lambda) = B_0 \left(\frac{R_E}{r}\right)^3 (-2\sin\lambda \hat{\mathbf{e}}_r + \cos\lambda \hat{\mathbf{e}}_\lambda). \tag{1}$$

The field magnitude is

$$B(r, \lambda) = B_0 \left(\frac{R_E}{r} \right)^3 \sqrt{4 - 3\cos^2 \lambda} \tag{2}$$

The equation for a dipole field line is:

$$r(\lambda) = r_{eq} \cos^2 \lambda \tag{3}$$

which implies that the apex radius of the field line at (r, λ) is

$$r_{eq}(r, \lambda) = r / \cos^2 \lambda \tag{4}$$

Equation 1 can be written as $\mathbf{B} = -\nabla V$ where the magnetic potential V is:

$$V = -B_0 \frac{R_E^3}{r^2} \sin \lambda \tag{5}$$

3 Conversion

Below are equations for converting between dipole, modified apex, and quasi-dipole coordinates for a dipole field. The longitudes are all equal:

$$\boxed{\phi = \phi_{qd} = \phi_{ma}} \tag{6}$$

3.1 Modified apex

Equation 4 can be used directly in the equation for modified apex latitude:

$$\boxed{\lambda_{ma}(r, \lambda) = \pm \cos^{-1} \sqrt{\left(\frac{R}{r} \right) \cos^2 \lambda}.} \tag{7}$$

where $\pm$ refers to the Northern $(+)$ and Southern $(-)$ hemisphere. The opposite conversion, from λ_{ma} to λ is

$$\boxed{\lambda(r, \lambda_{ma}) = \pm \cos^{-1} \sqrt{\left(\frac{r}{R} \right) \cos^2 \lambda_{ma}}.} \tag{8}$$

3.2 Quasi-dipole

For quasi-dipole coordinates, the above equations are the same except that instad of R we trace back to r. All the ratios in the parentheses above are 1, so that

$$\boxed{\lambda_{qd} = \lambda \qquad\qquad \forall\, r} \tag{9}$$

4 The base vectors

From the equations above, we can define base vectors similar to those in Richmond [1995], only that in this case they hold for a dipole magnetic field, and thus they can be found analytically. I will express the vectors in (E, N, U)-directions, which here refer to dipole coordinates. All equations below are derived for $\lambda \in [0°, 90°]$, so the sign of λ should be changed for points in the Southern hemisphere. $\pm$ and $\mp$ are used to keep track of the sign of each hemisphere (North on top).

4.1 Modified apex base vectors

In a dipole field, the modified apex base vectors are everywhere perpendicular, but they are unit length only at $r = R$. Only at $r = R$ is $\mathbf{d}_i = \mathbf{e}_i$, although they are parallel everywhere, and $\mathbf{d}_i \cdot \mathbf{e}_j = \delta_{ij}$ holds. This is because they are defined to scale differently with the magnetic field.

The modified apex base vectors are defined as (Eqs. 3.8–3.9 in Richmond [1995])

$$\mathbf{d}_1 = R \cos \lambda_{ma} \nabla \phi_{ma} \tag{10}$$

$$\mathbf{d}_2 = -R \sin I_{ma} \nabla \lambda_{ma} \tag{11}$$

$$\mathbf{d}_3 = \frac{-\nabla V}{BD} = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{d}_2\|^2} \tag{12}$$

where the last expression for $\mathbf{d}_3$ can be found using (3.13) and (3.15) in Richmond [1995].

4.1.1 $\mathbf{d}_1$

The $\mathbf{d}_1$ base vector (Equation 10) depends on the gradient of $\phi_{ma} = \phi$. Using (7) to replace $\cos \lambda_{ma}$, we get:

$$\mathbf{d}_1(r, \lambda) = R \left(\frac{R}{r}\right)^{3/2} \cos \lambda \frac{1}{r \cos \lambda} \frac{\partial \phi}{\partial \phi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

$$\boxed{\mathbf{d}_1(r, \lambda) = \left(\frac{R}{r}\right)^{3/2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \tag{14}$$

4.1.2 $\mathbf{d}_2$

The $\mathbf{d}_2$ base vector can be found by calculating the gradient of Eq. 7 and inserting the result into the definition (11). We also need to express $\sin I_{ma}$ in terms of (r, λ). I start by the latter:

$$\begin{aligned} \sin I_{ma} &= 2\sin\lambda_{ma}(4 - 3\cos^2\lambda_{ma})^{-1/2} \\ &= \pm 2\sqrt{\frac{1 - \frac{R}{r}\cos^2\lambda}{4 - 3\frac{R}{r}\cos^2\lambda}} \end{aligned} \tag{15}$$

where Equation 7 was used.

The gradient of λ_{ma} can be written

$$\begin{aligned} \nabla\lambda_{ma} &= \pm \begin{pmatrix} 0 \\ \frac{1}{r}\frac{\partial}{\partial\lambda} \\ \frac{\partial}{\partial r} \end{pmatrix} \left[\cos^{-1}\sqrt{\left(\frac{R}{r}\right)\cos^2\lambda}\right] \\ &= \pm \begin{pmatrix} 0 \\ \sqrt{\frac{R}{r}}\frac{\sin\lambda}{r\sqrt{1-\frac{R}{r}\cos^2\lambda}} \\ \sqrt{\frac{R}{r}}\frac{\cos\lambda}{2r\sqrt{1-\frac{R}{r}\cos^2\lambda}} \end{pmatrix} = \pm\frac{\sqrt{R/r}}{r\sqrt{1-\frac{R}{r}\cos^2\lambda}} \begin{pmatrix} 0 \\ \sin\lambda \\ \cos\lambda/2 \end{pmatrix} \end{aligned} \tag{16}$$

Inserting this expression, and the expression for $\sin I_{ma}$ into the definition of $\mathbf{d}_2$ (11) we get

$$\boxed{\mathbf{d}_2(r,\lambda) = \mp\left(\frac{R}{r}\right)^{3/2}\left(4 - 3\frac{R}{r}\cos^2\lambda\right)^{-1/2}\begin{pmatrix} 0 \\ 2\sin\lambda \\ \cos\lambda \end{pmatrix}} \tag{17}$$

This expression can be used to confirm that $\|\mathbf{d}_2(r = R, \lambda)\| = 1$, as it should be for a dipole field and spherical Earth.

4.2 $\mathbf{d}_3$

The last base vector can be found by crossing $\mathbf{d}_1$ and $\mathbf{d}_2$. Multiplication of the scalar coefficients give:

$$\mp\left(\frac{R}{r}\right)^3\left(4 - 3\frac{R}{r}\cos^2\lambda\right)^{-1/2},$$

and crossing the vector parts give

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2\sin\lambda \\ \cos\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\cos\lambda \\ 2\sin\lambda \end{pmatrix} \tag{18}$$

We calculate the quantity $D = \|\mathbf{d}_1 \times \mathbf{d}_2\|$:

$$D = \left(\frac{R}{r}\right)^3 \sqrt{\frac{4 - 3\cos^2\lambda}{4 - 3\frac{R}{r}\cos^2\lambda}} \tag{19}$$

The $\mathbf{d}_3$ base vector is

$$\boxed{\mathbf{d}_3(r,\lambda) = \left(\frac{r}{R}\right)^3 \frac{\sqrt{4 - 3\frac{R}{r}\cos^2\lambda}}{4 - 3\cos^2\lambda} \begin{pmatrix} 0 \\ \cos\lambda \\ -2\sin\lambda \end{pmatrix}} \tag{20}$$

which is parallel to the dipole field. B_{e_3} is supposed to be constant along a field line, so we can now check that this is true, by calculating

$$B_{e_3} = \mathbf{B} \cdot \mathbf{d}_3 = B_0 \left(\frac{R_E}{R}\right)^3 \sqrt{4 - 3R/r_{eq}} \tag{21}$$

which is constant. B_{e_3} should be equal to B/D. Let's check that also:

$$B/D = \frac{B_0 \left(\frac{R_E}{r}\right)^3 \sqrt{4 - 3\cos^2\lambda}}{\left(\frac{R}{r}\right)^3 \sqrt{\frac{4-3\cos^2\lambda}{4-3\frac{R}{r}\cos^2\lambda}}} = B_0 \left(\frac{R_E}{R}\right)^3 \sqrt{4 - 3R/r_{eq}} \tag{22}$$

as expected. Also, at $r = R$, B_{e_3} is the magnetic field strength. This can be seen by replacing r_{eq} with $r/\cos^2\lambda$, and r by R.

References

A. D. Richmond. Ionospheric electrodynamics using magnetic apex coordinates. *J. Geomag. Geoelectr.*, 47:191–212, 1995. doi: 10.5636/jgg.47.191.