Bin 1 Problems

December 6, 2023

This document holds problems that fit into the Bin 1 according to the prelim syllabus.

Contents

1 Prelim Problems

The problems are listed in most recent first. IE the first question is from August 2023, second is from January 2023, etc. In addition, the proofs provided are my own to complement the ones posted by the committees of previous years.

1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$.
   Prove that $\dim U - \dim V \leq \dim \operatorname{null} ST - \dim \operatorname{null} S$.

Solution:
Solution with intuition building sprinkled in
This problem uses the rank nullity theorem (Axler calls this the fundamental theorem of linear maps).

We can get this intuition from the fact that we are being asked to relate the dimension of spaces to the dimension of the range/null space of a linear map FROM these spaces.

Reminder of the rank nullity theorem:

Theorem 1. *If U, V are vector spaces, and $T \in \mathcal{L}(U, V)$, then*

$$\mathsf{dim}\, U = \mathsf{dim\,range}\, T + \mathsf{dim\,null}\, T \tag{1}$$

Well, this would introduce a dimension based on ranges. But we'll hope it goes away through some algebra or hiding it behind the $\leq$ operator.

Since we want to have $\dim \operatorname{null} ST$ and $\dim \operatorname{null} S$ present, we should try to use theorem 1 in a way that would give us these operators on the right side.

See that because T maps from U to V and S maps from V to W, then ST maps from U to W. This means by using theorem 1, we get

$$\dim U = \dim \operatorname{null} ST + \dim \mathsf{range}\, ST$$

Next, we can use theorem 1 to get that

$$\dim V = \dim \operatorname{null} S + \dim \mathsf{range}\, S$$

By subtracting these two equations, we have

$$\dim U - \dim V = \dim \operatorname{null} ST + \dim \mathsf{range}\, ST - (\dim \operatorname{null} S + \dim \mathsf{range}\, S)$$

So if we group together our range and nullspace dimensions we are close to what we need

$$\dim U - \dim V = \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \mathsf{range}\, ST - \dim \mathsf{range}\, S)$$

So, we are really close to what we want. All that is left is just try try to get rid of our range dimensions through some kind of algebra. Since we are wanting a $\leq$ in our final statement, if we can argue that $(\dim \mathsf{range}\, ST - \dim \mathsf{range}\, S) \leq 0$ then we would be done.

So, we are essentially trying to argue $\dim \mathsf{range}\, ST \leq \dim \mathsf{range}\, S$.

Before talking about how we would approach this, recall that $\mathsf{range}\, ST$ is the set of all vectors in W that $S(T(u))$ can produce for any $u \in U$, while $\mathsf{range}\, S$ is the set of all vectors in W that $S(v)$ can produce for any $v \in V$.

We also know from theorem 3.19 from Axler that $\mathsf{range}\, T$ is a subspace of V. This means that if we look at $\mathsf{range}\, ST$ we are looking at what S can map to from $\mathsf{range}\, T$. So this means we are mapping a potentially smaller (but never larger) set inside V to W meaning that $\mathsf{range}\, ST \leq \mathsf{range}\, S$

So we can conclude that

$$\begin{aligned}
\dim U - \dim V &= \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \mathsf{range}\, ST - \dim \mathsf{range}\, S) \\
&\leq \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \mathsf{range}\, S - \dim \mathsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S
\end{aligned}$$

as desired.

<u>Proof written up properly for the prelim</u>

From the definition of S and T in the problem description, we know that $ST : U \to W$. From the rank-nullity theorem, we know the following

$$\begin{aligned}
\dim U &= \dim \operatorname{null} ST + \dim \mathsf{range}\, ST \\
\dim V &= \dim \operatorname{null} S + \dim \mathsf{range}\, S
\end{aligned}$$

We also know from Axler that $\mathsf{range}\, T$ is a subspace of V thus $\mathsf{range}\, ST$ is a subspace of $\mathsf{range}\, S$ because we are mapping a subset of V to W with ST. This allows us to say that

$$\dim \mathsf{range}\, ST \leq \dim \mathsf{range}\, S$$

Combining all of the above results we have

$$\begin{aligned}
\dim U - \dim V &= \dim \operatorname{null} ST + \dim \mathsf{range}\, ST - (\dim \operatorname{null} S + \dim \mathsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \mathsf{range}\, ST - \dim \mathsf{range}\, S) \\
&\leq \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \mathsf{range}\, S - \dim \mathsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S
\end{aligned}$$

which is our desired result

2. Suppose U, W are subspaces of a finite-dimensional vector space V.
   (a) Show that $\dim\, (U \cap W) = \dim U + \dim W - \dim\, (U + W)$
   (b) Let $n = \dim V$. Show that if $k < n$, then an intersection of k subspaces of dimension $n - 1$ always has dimension at least $n - k$.

3. Prelim Aug 22 Part 1.

(a) Let v_1, v_2, v_3 be linearly dependent, and v_2, v_3, v_4 are linearly independent

i. Show that v_1 is a linear combination of v_2 and v_3.

ii. Show that v_4 is not a linear combination of v_1, v_2, and v_3.

4. Let V be a vector space of dimension n over a field F. Let $v_1, v_2, \ldots, v_n$ be a basis of V and T be an operator on V.
Prove: T is invertible if and only if $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

2 Axler Problems

1. Suppose
$$U = \left\{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

2. Suppose
$$U = \left\{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

3. Prove or give a counter example: If U_1, U_2, W are subspaces of V such that
$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$
Then $U_1 = U_2$.

3 Other Problems