Bin 1 Problems

December 18, 2023

This document holds problems that fit into the Bin 1 according to the prelim syllabus.

Contents

1 Prelim Problems

The problems are listed in most recent first. IE the first question is from August 2023, second is from January 2023, etc. In addition, the proofs provided are my own to complement the ones posted by the committees of previous years.

1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$.
   Prove that $\dim U - \dim V \leq \dim \operatorname{null} ST - \dim \operatorname{null} S$.

Solution:
Solution with intuition building sprinkled in

This problem uses the rank nullity theorem (Axler calls this the fundamental theorem of linear maps).

We can get this intuition from the fact that we are being asked to relate the dimension of spaces to the dimension of the range/null space of a linear map FROM these spaces.

Reminder of the rank nullity theorem:

Theorem 1. *If U, V are vector spaces, and $T \in \mathcal{L}(U, V)$, then*

$$\mathsf{dim}\, U = \mathsf{dim\,range}\, T + \mathsf{dim\,null}\, T \tag{1}$$

Well, this would introduce a dimension based on ranges. But we'll hope it goes away through some algebra or hiding it behind the $\leq$ operator.

Since we want to have $\dim \operatorname{null} ST$ and $\dim \operatorname{null} S$ present, we should try to use theorem 1 in a way that would give us these operators on the right side.

See that because T maps from U to V and S maps from V to W, then ST maps from U to W. This means by using theorem 1, we get

$$\dim U = \dim \operatorname{null} ST + \dim \mathsf{range}\, ST$$

Next, we can use theorem 1 to get that

$$\dim V = \dim \operatorname{null} S + \dim \mathsf{range}\, S$$

By subtracting these two equations, we have

$$\dim U - \dim V = \dim \operatorname{null} ST + \dim \textsf{range}\, ST - (\dim \operatorname{null} S + \dim \textsf{range}\, S)$$

So if we group together our range and nullspace dimensions we are close to what we need

$$\dim U - \dim V = \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \textsf{range}\, ST - \dim \textsf{range}\, S)$$

So, we are really close to what we want. All that is left is just try try to get rid of our range dimensions through some kind of algebra. Since we are wanting a $\leq$ in our final statement, if we can argue that $(\dim \textsf{range}\, ST - \dim \textsf{range}\, S) \leq 0$ then we would be done.

So, we are essentially trying to argue $\dim \textsf{range}\, ST \leq \dim \textsf{range}\, S$.

Before talking about how we would approach this, recall that $\textsf{range}\, ST$ is the set of all vectors in W that $S(T(u))$ can produce for any $u \in U$, while $\textsf{range}\, S$ is the set of all vectors in W that $S(v)$ can produce for any $v \in V$.

We also know from theorem 3.19 from Axler that $\textsf{range}\, T$ is a subspace of V. This means that if we look at $\textsf{range}\, ST$ we are looking at what S can map to from $\textsf{range}\, T$. So this means we are mapping a potentially smaller (but never larger) set inside V to W meaning that $\textsf{range}\, ST \leq \textsf{range}\, S$

So we can conclude that

$$\begin{aligned}
\dim U - \dim V &= \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \textsf{range}\, ST - \dim \textsf{range}\, S) \\
&\leq \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \textsf{range}\, S - \dim \textsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S
\end{aligned}$$

as desired.

Proof written up more properly for the prelim

From the definition of S and T in the problem description, we know that $ST : U \to W$. From the rank-nullity theorem, we know the following

$$\begin{aligned}
\dim U &= \dim \operatorname{null} ST + \dim \textsf{range}\, ST \\
\dim V &= \dim \operatorname{null} S + \dim \textsf{range}\, S
\end{aligned}$$

We also know from Axler that $\textsf{range}\, T$ is a subspace of V thus $\textsf{range}\, ST$ is a subspace of $\textsf{range}\, S$ because we are mapping a subset of V to W with ST. This allows us to say that

$$\dim \textsf{range}\, ST \leq \dim \textsf{range}\, S$$

Combining all of the above results we have

$$\begin{aligned}
\dim U - \dim V &= \dim \operatorname{null} ST + \dim \textsf{range}\, ST - (\dim \operatorname{null} S + \dim \textsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \textsf{range}\, ST - \dim \textsf{range}\, S) \\
&\leq \dim \operatorname{null} ST - \dim \operatorname{null} S + (\dim \textsf{range}\, S - \dim \textsf{range}\, S) \\
&= \dim \operatorname{null} ST - \dim \operatorname{null} S
\end{aligned}$$

which is our desired result.

2. Suppose U, W are subspaces of a finite-dimensional vector space V.
   (a) Show that $\dim\,(U \cap W) = \dim U + \dim W - \dim\,(U + W)$

   Solution: See the solution provided by the January 2023 committee. This is a good proof that goes through the steps very well. For intuition, we are talking about dimensions of finite dimensional spaces, so trying to something with basis length will be a good start in general.

(b) Let $n = \dim V$. Show that if $k < n$, then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$.

Solution: See the linked solution as well for this part. As far as intuition is concerned, we are asking about a property of an intersection of an arbitrary number of sets so induction is always a good place to start. As far as the method is concerned, this problem statement seems to lead to setting up using the first part, so that is usually a good place to start.

3. Prelim Aug 22 Part 1.

(a) Let v_1, v_2, v_3 be linearly dependent, and v_2, v_3, v_4 are linearly independent

i. Show that v_1 is a linear combination of v_2 and v_3.

Solution:

Intuition build-up

Since we know that v_2, v_3, v_4 are linearly independent, we can use this fact as if v_1 was not a linear combination of v_2 and v_3 then either v_2 or v_3 are a scalar multiple of each other which would make the second given statement untrue.

Solution

Since we know that v_1, v_2, and v_3 are linearly dependent, we know that there exist some $a_1, a_2, a_3 \in \mathbb{F}$ such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \tag{2}$$

where not all are 0.

We will prove the desired statement through contradiction, so assume that v_1 is not a linear combination of v_2 and v_3, then this means that in equation (2), know that $a_1 = 0$. This is because if $a_1 \neq 0$ we have

$$v_1 = \frac{1}{a_1}\left(a_2 v_2 + a_3 v_3\right)$$

which violates our assumption on v_1.

Therefore, we have that v_2 is a scalar multiple of v_3, so there exists some $c \in \mathbb{F}$ such that $v_2 = cv_3$. Now, we will show there exists some $b_2, b_3, b_4 \in \mathbb{F}$ such that

$$b_2 v_2 + b_3 v_3 + b_4 v_4 = 0$$

and not all are 0. Let $b_4 = 0$, $b_3 = 1$ and $b_2 = \frac{-1}{c}$ where c is the aforementioned constant. See:

$$\begin{aligned} b_2 v_2 + b_3 v_3 + b_4 v_4 &= \frac{-1}{c} v_2 + v_3 \\ &= -v_3 + v_3 \\ &= 0 \end{aligned}$$

Thus, v_2, v_3, and v_4 are linearly dependent, which contradicts the assumption that they are linearly independent. So, we have that v_1 must be a linear combination of v_2 and v_3.

ii. Show that v_4 is not a linear combination of v_1, v_2, and v_3.

Solution:

Intuition build-up

Since we have from the previous part that v_1 is a linear combination of v_2 and v_3, and if we also had that v_4 is a linear combination of v_1, v_2, and v_3 then we could show that v_4 is a linear combination of both v_2 and v_3 which would again contradict the fact that v_2, v_3, and v_4 are linearly independent.

Solution

We will prove this statement through contradiction. Assume the given problem statement and that v_4 is a linear combination of v_1, v_2, and v_3.

Since v_4 is a linear combination of v_1, v_2, and v_3, we there exist a_1, a_2, and $a_3 \in \mathbb{F}$ such that

$$v_4 = a_1v_1 + a_2v_2 + a_3v_3 \tag{3}$$

From the previous part, we know that v_1 is a linear combination of v_2 and v_3, so there exist some b_2, and $b_3 \in \mathbb{F}$ such that

$$v_1 = b_2v_2 + b_3v_3 \tag{4}$$

Combining equations (3) and (4) we get that

$$\begin{aligned} v_4 &= a_1v_1 + a_2v_2 + a_3v_3 \\ &= a_1\left(b_2v_2 + b_3v_3\right) + a_2v_2 + a_3v_3 \\ &= \left(a_1b_2 + a_2\right)v_2 + \left(a_1b_3 + a_3\right)v_3 \end{aligned}$$

Thus, v_4 is a linear combination of v_2 and v_3. Following a similar argument as the last part, we have that v_2, v_3, and v_4 are linearly dependent contradicting the problem statement.
Thus, we have shown that v_4 is not a linear combination of v_1, v_2, and v_3 as desired.

4. Let V be a vector space of dimension n over a field F. Let $v_1, v_2, \ldots, v_n$ be a basis of V and T be an operator on V.
Prove: T is invertible if and only if $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

Solution:
Intuition build-up
Since we have an iff statement we are trying to prove, picking the direction that seems the most direct would be a great place to start. Other than that, there aren't many tricks that we can do or take advantage of.

Note that since $v_1, \ldots, v_n$ is a basis of V, V is a finite dimensional vector space of size n. So we need only show injectivity or surjectivity of T to show T^{-1} exists due to Axler 3.69. However the provided solution shows both methods. So we will do the same here as the proof will extend to a infinite dimensional.

Solution
We first show the $\implies$ direction. We do this via contradiction. Assume that $Tv_1, Tv_2, \ldots, Tv_n$ are linearly dependent. IE, $\exists a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1Tv_1 + \cdots + a_nTv_n = \mathbf{0}.$$

Since T is invertible, T^{-1} exists. We now apply this operator to both sides of our above equation and see that

$$\begin{aligned} T^{-1}\left(a_1Tv_1 + \cdots + a_nTv_n\right) &= T^{-1}\mathbf{0} \\ a_1T^{-1}Tv_1 + \cdots + a_nT^{-1}Tv_n &= \mathbf{0} \\ a_1v_1 + \cdots + a_nv_n &= \mathbf{0} \end{aligned}$$

However, since $v_1, \ldots, v_n$ form a basis of V, we have that $a_1 = \cdots = a_n = 0$. Thus, we have that $Tv_1, \ldots, Tv_n$ are linearly independent as desired.

Next, we prove the $\impliedby$ direction. So, we are assuming that $Tv_1, \ldots, Tv_n$ are linearly independent and we will show that T^{-1} exists by showing T is injective and surjective.

First, we show injectivity. Suppose $u_1, u_2 \in V$ such that $Tu_1 = Tu_2$. Since we know that $v_1, \ldots, v_n$

form a basis of V, there exists $a_1, \ldots, a_n \in \mathbb{F}$ and $b_1, \ldots, b_n \in \mathbb{F}$ such that

$$u_1 = \sum_{k=1}^{n} a_k v_k$$
$$u_2 = \sum_{k=1}^{n} b_k v_k$$

We also see that

$$\begin{aligned} Tu_1 = Tu_2 &\implies T\left(\sum_{k=1}^{n} a_k v_k\right) = T\left(\sum_{k=1}^{n} b_k v_k\right) \\ &\implies \sum_{k=1}^{n} a_k T v_k = \sum_{k=1}^{n} b_k T v_k \qquad \implies \sum_{k=1}^{n} (a_k - b_k) T v_k = \mathbf{0}. \end{aligned}$$

However, since $Tv_1, \ldots, Tv_n$ are linearly independent, we know that for $k = 1, \ldots, n$, $a_k - b_k = 0$ meaning our coefficients must be equal. Thus, we have that $u_1 = u_2$, so T is injective.

Next, we show surjectivity. Since we know that $Tv_1, \ldots, Tv_n$ is a linearly independent list of the correct size, this list forms a basis of V. Thus for all $v \in V$, we have that there exists, $c_1, \ldots, c_n \in \mathbb{F}$ such that

$$v = \sum_{k=1}^{n} c_k T v_k$$

However, we also see that

$$\begin{aligned} v &= \sum_{k=1}^{n} c_k T v_k \\ &= T\left(\sum_{k=1}^{n} c_k v_k\right) \end{aligned}$$

See that if we define $u = \sum_{k=1}^{n} c_k v_k$, then we have that for all $v \in V$, there exists some $u \in V$ such that $v = Tu$. So, we have that T is surjective.

Since T is both injective and surjective, we know that T is invertible as desired.

2 Axler Problems

1. Suppose
$$U = \left\{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution: This problem is exactly like the next one, but I included both in case there was another way to approach this problem than what I expected. See the proof of the next problem for the approach and set $W = W_1 \oplus W_2 \oplus W_3$.

2. Suppose
$$U = \left\{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution:

Intuition build-up

Remember the definition of a direct sum, so we need to propose 3 subspaces W_1, W_2, W_3, such that the only way to get

$$u + w_1 + w_2 + w_3 = \mathbf{0}$$

for $u \in U$, and $w_j \in W_j$ respectively is to have all of them be 0. (This is really just grabbing 3 vectors that are linearly independent from the basis vectors of U).

Based on the structure of our problem, we just plug in 1 for x and 0 for y to get the first vector then get vice versa for the second vector.

This means that our basis vectors are given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

Visually inspecting this matrix lets us see that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are linearly independent of each other and of the columns of A. Thus we would define $W_1 = \operatorname{span}(e_1)$, $W_2 = \operatorname{span}(e_2)$, and $W_3 = \operatorname{span}(e_3)$

Then we would be done.

Solution

We need to construct 3 subspaces of $\mathbb{F}^5$ that will direct sum with U and each other. Define

$$W_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), W_2 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), \text{ and } W_3 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

See that these are the spans of the first 3 standard basis vectors ie for $k = 1, 2, 3$ we have $W_k = \operatorname{span}(e_k)$.

Let $u \in U$, and for $k = 1, 2, 3$, $w_k \in W_k$ such that

$$u + w_1 + w_2 + w_3 = \mathbf{0}. \tag{5}$$

We know that u is of the form

$$u = \begin{bmatrix} x \\ y \\ x + y \\ x - y \\ 2x \end{bmatrix}$$

and for each k, w_k is of the form

$$w_k = a_k e_k$$

This means that we can rewrite equation (5) as

$$x + a_1 = 0 \tag{6}$$
$$y + a_2 = 0 \tag{7}$$
$$x + y + a_3 = 0 \tag{8}$$
$$x - y = 0 \tag{9}$$
$$2x = 0 \tag{10}$$

Equation (10) gives us that $x = 0$. Plugging this into equations (6) and (9) gives us that $a_1 = 0$ and $y = 0$ respectively. Finally we plug our knowns into equations (7) and (8) to get that $a_2 = 0$ and $a_3 = 0$ respectively.

This means that $u = w_1 = w_2 = w_3 = \mathbf{0}$. Thus, the sum of these subspaces would be a direct sum. Now, we need only show that that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

In order to do this, we need only show that the direct sum has a basis of length 5 and we are done because all the basis vectors would live inside $\mathbb{F}^5$.

In order to do this, we will first look at U and its basis vectors. From the construction of U, if we set $x = 1$ and $y = 0$ then $x = 0$ and $y = 1$, we get 2 vectors out that look like

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

By setting up the equations we would need to solve to prove these are linearly independent, we would get the same system of equations as listed above, so we have:

$$\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$$

as desired.

3. Prove or give a counter example: If U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

Then $U_1 = U_2$.

Solution:
Intuition build-up
This one turns out to not be true, but where we would get this idea from (if it's not directly obvious to you) would be trial and error of trying to prove it to be true and trying to leverage any issues present. So, we would want to construct a counter example where W is of dimension $\dim V - 1$ and $\dim U_1 = \dim U_2$ but $U_1 \neq U_2$. This is because we can construct two linearly independent vectors that are not equal but would increase the dimension. Since we are providing a counter example, we can do this in a way that makes our lives as easy as possible so let's stick with $V = \mathbb{R}^3$, and make W be simple.

Consider the spaces given as below

$$W = \left\{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\}$$
$$U_1 = \left\{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\right\}$$
$$U_2 = \left\{(0, x, x) \in \mathbb{R}^3 : x \in \mathbb{R}\right\}$$

See that the basis vectors of W are e_1 and e_2, while the basis vector of U_1 is $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^\top$ while the basis vector for U_2 is $\begin{bmatrix}0 & 1 & 1\end{bmatrix}^\top$, so these spaces are not equal but each provide unique information in the third element, so they will direct sum with W, and then our new basis will be of size 3 so we would have a basis of $\mathbb{R}^3$ so we'd be done.

When it comes to showing that we have direct sums, we will do so using theorem 1.45 from Axler, which is given below

Theorem 2. *1.45 Axler*
Suppose U, W are subspaces of some vector space V. Then U and W form a direct sum if and only if $U \cap W = \{\mathbf{0}\}$.

Solution
Consider the vector space $V = \mathbb{R}^3$, and the following subspaces of V

$$\begin{aligned} W &= \left\{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\} \\ U_1 &= \left\{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\right\} \\ U_2 &= \left\{(0, x, x) \in \mathbb{R}^3 : x \in \mathbb{R}\right\} \end{aligned}$$

We will first show that $U_1 \oplus W = V = U_2 \oplus W$. We first show that the provided sums are direct sums using theorem 2.

First consider $U = U_1 \cap W$. Let $u \in U$, this means we can write u as both

$$\begin{aligned} u &= \begin{bmatrix}x & y & 0\end{bmatrix}^\top \\ u &= \begin{bmatrix}z & 0 & z\end{bmatrix}^\top \end{aligned}$$

So each of these vectors must be equal. The last term means that $z = 0$, so u must be $\mathbf{0}$. Thus $U_1 \cap W = \mathbf{0}$ meaning that we have a direct sum. Finally, we need only show that $U_1 \neq U_2$.

See that U_1 contains the vector $\begin{bmatrix}1 & 0 & 1\end{bmatrix}$ while U_2 does not. Therefore these spaces are not the same.

3 Other Problems