Bin 1 Problems

December 6, 2023

This document holds problems that fit into the Bin 1 according to the prelim syllabus.

Contents

1 Prelim Problems

The problems are listed in most recent first. IE the first question is from August 2023, second is from January 2023, etc. In addition, the proofs provided are my own to complement the ones posted by the committees of previous years.

1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$.
   Prove that $\dim U - \dim V \leq \dim \operatorname{null} ST - \dim \operatorname{null} S$.

   Solution:
   Solution with intuition building sprinkled in
   This problem uses the rank nullity theorem (Axler calls this the fundamental theorem of linear maps).

   We can get this intuition from the fact that we are being asked to relate the dimension of spaces to the dimension of the range/null space of a linear map FROM these spaces.

   Reminder of the rank nullity theorem:

   Theorem 1. *If U, V are vector spaces, and $T \in \mathcal{L}(U, V)$, then*

   $$\mathsf{dim}\, U = \mathsf{dim\, range}\, T + \mathsf{dim\, null}\, T \tag{1}$$

   Well, this would introduce a dimension based on ranges. But we'll hope it goes away through some algebra or hiding it behind the $\leq$ operator.

   Since we want to have $\dim \operatorname{null} ST$ and $\dim \operatorname{null} S$ present, we ne

2. Suppose U, W are subspaces of a finite-dimensional vector space V.
   (a) Show that $\dim (U \cap W) = \dim U + \dim W - \dim (U + W)$
   (b) Let $n = \dim V$. Show that if $k < n$, then an intersection of k subspaces of dimension $n - 1$ always has dimension at least $n - k$.

3. Prelim Aug 22 Part 1.
   (a) Let v_1, v_2, v_3 be linearly dependent, and v_2, v_3, v_4 are linearly independent

i. Show that v_1 is a linear combination of v_2 and v_3.

ii. Show that v_4 is not a linear combination of v_1, v_2, and v_3.

4. Let V be a vector space of dimension n over a field F. Let $v_1, v_2, \ldots, v_n$ be a basis of V and T be an operator on V.
   Prove: T is invertible if and only if $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

2 Axler Problems

3 Other Problems