# Practice Prelim

December 28, 2023

Assume that $\mathbb{F}$ is a field of elements either real or complex (ie any proofs using $\mathbb{F}$ should hold for either real or complex numbers)

## Section 1 Problems

1. Let $V$ be a vector space over a field $\mathbb{F}$. Suppose $v_1, \ldots, v_m$ is linearly independent in $V$ and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m)$$

2. Let $V$ be an inner product space over a field $\mathbb{F}$. Suppose that $U$ is a subspace of $V$ and $u_1, \ldots, u_m$ forms a basis of $U$. Furthermore, suppose that $u_1, \ldots, u_m, w_1, \ldots, w_n$ forms a basis of $V$. Prove that if the Gram-Schmidt Procedure is applied to the basis of $V$ above, producing a list $e_1, \ldots, e_m, f_1, \ldots, f_n$, then $e_1, \ldots, e_m$ forms an orthonormal basis of $U$ and $f_1, \ldots, f_n$ is an orthonormal basis of $U^\perp$

   **Note:** If you are unfamiliar with the Gram-Schmidt Procedure, see Theorem 6.31 in Axler. Familiarity with this theorem is expected, but you will probably not be expected to apply it. (However as always ask the committee as they are the only ones qualified to make such statements with exactness).

3. For each of the following 4 statements, give either a counterexample or a reason why it is true.
   (a) For every real matrix $A$ there is a real matrix $B$ with $B^{-1}AB$ diagonal.
   (b) For every symmetric real matrix $A$, there is a real matrix $B$ with $B^{-1}AB$ diagonal.
   (c) For every complex matrix $A$ there is a complex matrix $B$ with $B^{-1}AB$ diagonal.
   (d) For every symmetric complex matrix $A$ there is a complex matrix $B$ with $B^{-1}AB$ diagonal.

4. Suppose $V$ is a real inner product space and $T \in \mathcal{L}(V)$. Prove that $T$ is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of $T$ are orthogonal and

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T),$$

   where $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of $T$.

   **Note:** This is very similar to prooving $(a) \implies (d)$ in Theorem 5.41. So, you cannot use this theorem!

## Section 2 Problems

Pick 2 of the following

5. Define $\mathbb{R}^{n\times n}$ to be the space of all real $n-$by$-n$ matrices, suppose $S \in \mathbb{R}^{n\times n}$, and define the linear mapping
$$\mathcal{T} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \qquad \mathcal{T} : P \mapsto PS + SP$$
   (a) Prove that if $\lambda$ is an eigenvalue of $S$, $u$ is the corresponding eigenvector, and $u \in \operatorname{null} \mathcal{T}P$, then $Pu$ is also an eigenvector of $S$ with eigenvalue $-\lambda$.
   (b) Prove that if $S$ is symmetric positive definite, then the mapping $\mathcal{T}$ is injective.

6. Let $A$ be a real $3 \times 3$ symmetric matrix, whose eigenvalues are $\lambda_1, \lambda_2,$ and $\lambda_3$. Prove the following:
   (a) If the trace of $A$, is not an eigenvalue of $A$, then $(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) \neq 0$
   (b) If $(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) \neq 0$, then the map $T : S \to S$ is an isomorphism, where $T(W) = AW + WA$ and $S$ is the space of $3 \times 3$ real skew-symmetric matrices (if $W^\top = -W$, then $W$ is called skew-symmetric).

7. Let $T$ be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(T) = T^4 + 2T^3 - 2T - I = 0$ where $I$ is the identity operator on $V$. Suppose that $|\text{trace}\,(T)| = 2$ and that $\dim \mathsf{range}\,(T + I) = 2$. Give a Jordan canonical form of $T$.

8. Let $A$ be an $n-$by$-n$ matrix with complex entries. Prove that A is the sum of two nonsingular matrices.