Bin 1 Problems

December 6, 2023

This document holds problems that fit into the Bin 1 according to the prelim syllabus.

Contents

1 Prelim Problems

The problems are listed in most recent first. IE the first question is from August 2023, second is from January 2023, etc. In addition, the proofs provided are my own to complement the ones posted by the committees of previous years.

1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$.
   Prove that $\dim U - \dim V \leq \dim \operatorname{null} ST - \dim \operatorname{null} S$.

2. Suppose U, W are subspaces of a finite-dimensional vector space V.
   (a) Show that $\dim\,(U \cap W) = \dim U + \dim W - \dim\,(U + W)$
   (b) Let $n = \dim V$. Show that if $k < n$, then an intersection of k subspaces of dimension $n - 1$ always has dimension at least $n - k$.

3. Prelim Aug 22 Part 1.
   (a) Let v_1, v_2, v_3 be linearly dependent, and v_2, v_3, v_4 are linearly independent
      i. Show that v_1 is a linear combination of v_2 and v_3.
      ii. Show that v_4 is not a linear combination of v_1, v_2, and v_3.

4. Let V be a vector space of dimension n over a field F. Let $v_1, v_2, \ldots, v_n$ be a basis of V and T be an operator on V.
   Prove: T is invertible if and only if $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

2 Axler Problems

1. Suppose
   $$U = \left\{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
   Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

2. Suppose
   $$U = \left\{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\right\}.$$
   Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

3. Prove or give a counter example: If U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

Then $U_1 = U_2$.

3 Other Problems