

# Bin 1 Problems

December 17, 2023

This document holds problems that fit into the Bin 1 according to the prelim syllabus.

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## 1 Prelim Problems

The problems are listed in most recent first. IE the first question is from August 2023, second is from January 2023, etc. In addition, the proofs provided are my own to complement the ones posted by the committees of previous years.

1. Let  $T$  be a linear map  $T : U \rightarrow V$  and  $S$  be a linear map  $S : V \rightarrow W$ .  
Prove that  $\dim U - \dim V \leq \dim \text{null } ST - \dim \text{null } S$ .

### **Solution:**

#### **Solution with intuition building sprinkled in**

This problem uses the rank nullity theorem (Axler calls this the fundamental theorem of linear maps).

We can get this intuition from the fact that we are being asked to relate the dimension of spaces to the dimension of the range/null space of a linear map FROM these spaces.

Reminder of the rank nullity theorem:

**Theorem 1.** *If  $U, V$  are vector spaces, and  $T \in \mathcal{L}(U, V)$ , then*

$$\dim U = \dim \text{range } T + \dim \text{null } T \quad (1)$$

Well, this would introduce a dimension based on ranges. But we'll hope it goes away through some algebra or hiding it behind the  $\leq$  operator.

Since we want to have  $\dim \text{null } ST$  and  $\dim \text{null } S$  present, we should try to use theorem 1 in a way that would give us these operators on the right side.

See that because  $T$  maps from  $U$  to  $V$  and  $S$  maps from  $V$  to  $W$ , then  $ST$  maps from  $U$  to  $W$ . This means by using theorem 1, we get

$$\dim U = \dim \text{null } ST + \dim \text{range } ST$$

Next, we can use theorem 1 to get that

$$\dim V = \dim \text{null } S + \dim \text{range } S$$

By subtracting these two equations, we have

$$\dim U - \dim V = \dim \text{null } ST + \dim \text{range } ST - (\dim \text{null } S + \dim \text{range } S)$$

So if we group together our range and nullspace dimensions we are close to what we need

$$\dim U - \dim V = \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } ST - \dim \text{range } S)$$

So, we are really close to what we want. All that is left is just try try to get rid of our range dimensions through some kind of algebra. Since we are wanting a  $\leq$  in our final statement, if we can argue that  $(\dim \text{range } ST - \dim \text{range } S) \leq 0$  then we would be done.

So, we are essentially trying to argue  $\dim \text{range } ST \leq \dim \text{range } S$ .

Before talking about how we would approach this, recall that  $\text{range } ST$  is the set of all vectors in  $W$  that  $S(T(u))$  can produce for any  $u \in U$ , while  $\text{range } S$  is the set of all vectors in  $W$  that  $S(v)$  can produce for any  $v \in V$ .

We also know from theorem 3.19 from Axler that  $\text{range } T$  is a subspace of  $V$ . This means that if we look at  $\text{range } ST$  we are looking at what  $S$  can map to from  $\text{range } T$ . So this means we are mapping a potentially smaller (but never larger) set inside  $V$  to  $W$  meaning that  $\text{range } ST \leq \text{range } S$

So we can conclude that

$$\begin{aligned} \dim U - \dim V &= \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } ST - \dim \text{range } S) \\ &\leq \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } S - \dim \text{range } S) \\ &= \dim \text{null } ST - \dim \text{null } S \end{aligned}$$

as desired.

### **Proof written up more properly for the prelim**

From the definition of  $S$  and  $T$  in the problem description, we know that  $ST : U \rightarrow W$ . From the rank-nullity theorem, we know the following

$$\begin{aligned} \dim U &= \dim \text{null } ST + \dim \text{range } ST \\ \dim V &= \dim \text{null } S + \dim \text{range } S \end{aligned}$$

We also know from Axler that  $\text{range } T$  is a subspace of  $V$  thus  $\text{range } ST$  is a subspace of  $\text{range } S$  because we are mapping a subset of  $V$  to  $W$  with  $ST$ . This allows us to say that

$$\dim \text{range } ST \leq \dim \text{range } S$$

Combining all of the above results we have

$$\begin{aligned} \dim U - \dim V &= \dim \text{null } ST + \dim \text{range } ST - (\dim \text{null } S + \dim \text{range } S) \\ &= \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } ST - \dim \text{range } S) \\ &\leq \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } S - \dim \text{range } S) \\ &= \dim \text{null } ST - \dim \text{null } S \end{aligned}$$

which is our desired result.

2. Suppose  $U, W$  are subspaces of a finite-dimensional vector space  $V$ .
  - (a) Show that  $\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$

**Solution:** See the [solution](#) provided by the January 2023 committee. This is a good proof that goes through the steps very well. For intuition, we are talking about dimensions of finite dimensional spaces, so trying to something with basis length will be a good start in general.

- (b) Let  $n = \dim V$ . Show that if  $k < n$ , then an intersection of  $k$  subspaces of dimension  $n - 1$  always has dimension at least  $n - k$ .

**Solution:** See the linked solution as well for this part. As far as intuition is concerned, we are asking about a property of an intersection of an arbitrary number of sets so induction is always a good place to start. As far as the method is concerned, this problem statement seems to lead to setting up using the first part, so that is usually a good place to start.

3. Prelim Aug 22 Part 1.

- (a) Let  $v_1, v_2, v_3$  be linearly dependent, and  $v_2, v_3, v_4$  are linearly independent
- Show that  $v_1$  is a linear combination of  $v_2$  and  $v_3$ .

**Solution:**

**Intuition build-up**

Since we know that  $v_2, v_3, v_4$  are linearly independent, we can use this fact as if  $v_1$  was not a linear combination of  $v_2$  and  $v_3$  then either  $v_2$  or  $v_3$  are a scalar multiple of each other which would make the second given statement untrue.

**Solution**

Since we know that  $v_1, v_2$ , and  $v_3$  are linearly dependent, we know that there exist some  $a_1, a_2, a_3 \in \mathbb{F}$  such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \quad (2)$$

where not all are 0.

We will prove the desired statement through contradiction, so assume that  $v_1$  is not a linear combination of  $v_2$  and  $v_3$ , then this means that in equation (2), know that  $a_1 = 0$ . This is because if  $a_1 \neq 0$  we have

$$v_1 = \frac{1}{a_1} (a_2 v_2 + a_3 v_3)$$

which violates our assumption on  $v_1$ .

Therefore, we have that  $v_2$  is a scalar multiple of  $v_3$ , so there exists some  $c \in \mathbb{F}$  such that  $v_2 = c v_3$ . Now, we will show there exists some  $b_2, b_3, b_4 \in \mathbb{F}$  such that

$$b_2 v_2 + b_3 v_3 + b_4 v_4 = 0$$

and not all are 0. Let  $b_4 = 0$ ,  $b_3 = 1$  and  $b_2 = \frac{-1}{c}$  where  $c$  is the aforementioned constant. See:

$$\begin{aligned} b_2 v_2 + b_3 v_3 + b_4 v_4 &= \frac{-1}{c} v_2 + v_3 \\ &= -v_3 + v_3 \\ &= 0 \end{aligned}$$

Thus,  $v_2, v_3$ , and  $v_4$  are linearly dependent, which contradicts the assumption that they are linearly independent. So, we have that  $v_1$  must be a linear combination of  $v_2$  and  $v_3$ .

- Show that  $v_4$  is not a linear combination of  $v_1, v_2$ , and  $v_3$ .

**Solution:**

**Intuition build-up**

Since we have from the previous part that  $v_1$  is a linear combination of  $v_2$  and  $v_3$ , and if we also had that  $v_4$  is a linear combination of  $v_1, v_2$ , and  $v_3$  then we could show that  $v_4$  is a linear combination of both  $v_2$  and  $v_3$  which would again contradict the fact that  $v_2, v_3$ , and  $v_4$  are linearly independent.

**Solution**

We will prove this statement through contradiction. Assume the given problem statement and that  $v_4$  is a linear combination of  $v_1, v_2$ , and  $v_3$ .

Since  $v_4$  is a linear combination of  $v_1$ ,  $v_2$ , and  $v_3$ , we there exist  $a_1$ ,  $a_2$ , and  $a_3 \in \mathbb{F}$  such that

$$v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3 \quad (3)$$

From the previous part, we know that  $v_1$  is a linear combination of  $v_2$  and  $v_3$ , so there exist some  $b_2$ , and  $b_3 \in \mathbb{F}$  such that

$$v_1 = b_2 v_2 + b_3 v_3 \quad (4)$$

Combining equations (3) and (4) we get that

$$\begin{aligned} v_4 &= a_1 v_1 + a_2 v_2 + a_3 v_3 \\ &= a_1 (b_2 v_2 + b_3 v_3) + a_2 v_2 + a_3 v_3 \\ &= (a_1 b_2 + a_2) v_2 + (a_1 b_3 + a_3) v_3 \end{aligned}$$

Thus,  $v_4$  is a linear combination of  $v_2$  and  $v_3$ . Following a similar argument as the last part, we have that  $v_2$ ,  $v_3$ , and  $v_4$  are linearly dependent contradicting the problem statement. Thus, we have shown that  $v_4$  is not a linear combination of  $v_1$ ,  $v_2$ , and  $v_3$  as desired.

4. Let  $V$  be a vector space of dimension  $n$  over a field  $F$ . Let  $v_1, v_2, \dots, v_n$  be a basis of  $V$  and  $T$  be an operator on  $V$ .  
Prove:  $T$  is invertible if and only if  $Tv_1, Tv_2, \dots, Tv_n$  are linearly independent.

## 2 Axler Problems

1. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace  $W$  of  $\mathbb{F}^5$  such that  $\mathbb{F}^5 = U \oplus W$ .

**Solution:** This problem is exactly like the next one, but I included both in case there was another way to approach this problem than what I expected. See the proof of the next problem for the approach and set  $W = W_1 \oplus W_2 \oplus W_3$ .

2. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces  $W_1, W_2, W_3$  of  $\mathbb{F}^5$  such that  $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$ .

**Solution:**

### Intuition build-up

Remember the definition of a direct sum, so we need to propose 3 subspaces  $W_1, W_2, W_3$ , such that the only way to get

$$u + w_1 + w_2 + w_3 = \mathbf{0}$$

for  $u \in U$ , and  $w_j \in W_j$  respectively is to have all of them be 0. (This is really just grabbing 3 vectors that are linearly independent from the basis vectors of  $U$ ).

Based on the structure of our problem, we just plug in 1 for  $x$  and 0 for  $y$  to get the first vector then get vice versa for the second vector.

This means that our basis vectors are given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

Visually inspecting this matrix lets us see that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are linearly independent of each other and of the columns of  $A$ . Thus we would define  $W_1 = \text{span}(e_1)$ ,  $W_2 = \text{span}(e_2)$ , and  $W_3 = \text{span}(e_3)$

Then we would be done.

### **Solution**

We need to construct 3 subspaces of  $\mathbb{F}^5$  that will direct sum with  $U$  and each other. Define

$$W_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), W_2 = \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), \text{ and } W_3 = \text{span} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

See that these are the spans of the first 3 standard basis vectors ie for  $k = 1, 2, 3$  we have  $W_k = \text{span}(e_k)$ .

Let  $u \in U$ , and for  $k = 1, 2, 3$ ,  $w_k \in W_k$  such that

$$u + w_1 + w_2 + w_3 = \mathbf{0}. \quad (5)$$

We know that  $u$  is of the form

$$u = \begin{bmatrix} x \\ y \\ x + y \\ x - y \\ 2x \end{bmatrix}$$

and for each  $k$ ,  $w_k$  is of the form

$$w_k = a_k e_k$$

This means that we can rewrite equation (5) as

$$x + a_1 = 0 \quad (6)$$

$$y + a_2 = 0 \quad (7)$$

$$x + y + a_3 = 0 \quad (8)$$

$$x - y = 0 \quad (9)$$

$$2x = 0 \quad (10)$$

Equation (10) gives us that  $x = 0$ . Plugging this into equations (6) and (9) gives us that  $a_1 = 0$  and  $y = 0$  respectively. Finally we plug our knowns into equations (7) and (8) to get that  $a_2 = 0$  and  $a_3 = 0$  respectively.

This means that  $u = w_1 = w_2 = w_3 = \mathbf{0}$ . Thus, the sum of these subspaces would be a direct sum. Now, we need only show that that  $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$ .

In order to do this, we need only show that the direct sum has a basis of length 5 and we are done because all the basis vectors would live inside  $\mathbb{F}^5$ .

In order to do this, we will first look at  $U$  and its basis vectors. From the construction of  $U$ , if we set  $x = 1$  and  $y = 0$  then  $x = 0$  and  $y = 1$ , we get 2 vectors out that look like

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

By setting up the equations we would need to solve to prove these are linearly independent, we would get the same system of equations as listed above, so we have:

$$\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$$

as desired.

3. Prove or give a counter example: If  $U_1, U_2, W$  are subspaces of  $V$  such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W$$

Then  $U_1 = U_2$ .

**Solution:**

**Intuition build-up**

This one turns out to not be true, but where we would get this idea from (if it's not directly obvious to you) would be trial and error of trying to prove it to be true and trying to leverage any issues present. So, we would want to construct a counter example where  $W$  is of dimension  $\dim V - 1$  and  $\dim U_1 = \dim U_2$  but  $U_1 \neq U_2$ . This is because we can construct two linearly independent vectors that are not equal but would increase the dimension. Since we are providing a counter example, we can do this in a way that makes our lives as easy as possible so let's stick with  $V = \mathbb{R}^3$ , and make  $W$  be simple.

Consider the spaces given as below

$$W = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$

$$U_1 = \{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$$

$$U_2 = \{(0, x, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$$

See that the basis vectors of  $W$  are  $e_1$  and  $e_2$ , while the basis vector of  $U_1$  is  $[1 \ 0 \ 1]^\top$  while the basis vector for  $U_2$  is  $[0 \ 1 \ 1]^\top$ , so these spaces are not equal but each provide unique information in the third element, so they will direct sum with  $W$ , and then our new basis will be of size 3 so we would have a basis of  $\mathbb{R}^3$  so we'd be done.

When it comes to showing that we have direct sums, we will do so using theorem 1.45 from Axler, which is given below

**Theorem 2.** 1.45 Axler

Suppose  $U, W$  are subspaces of some vector space  $V$ . Then  $U$  and  $W$  form a direct sum if and only if  $U \cap W = \{\mathbf{0}\}$ .

**Solution**

Consider the vector space  $V = \mathbb{R}^3$ , and the following subspaces of  $V$

$$W = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$

$$U_1 = \{(x, 0, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$$

$$U_2 = \{(0, x, x) \in \mathbb{R}^3 : x \in \mathbb{R}\}$$

We will first show that  $U_1 \oplus W = V = U_2 \oplus W$ . We first show that the provided sums are direct sums using theorem 2.

First consider  $U = U_1 \cap W$ . Let  $u \in U$ , this means we can write  $u$  as both

$$u = [x \ y \ 0]^\top$$

$$u = [z \ 0 \ z]^\top$$

So each of these vectors must be equal. The last term means that  $z = 0$ , so  $u$  must be  $\mathbf{0}$ . Thus  $U_1 \cap W = \mathbf{0}$  meaning that we have a direct sum.

### 3 Other Problems