Bin 2 Problems

January 1, 2024

This document holds problems that fit into the Bin 2 according to the prelim syllabus. Or are approached in a way most compatible with Bin 2

Contents

1 Prelim Problems

1. Let u be a unit vector in an $n-$dimensional inner product space V over $\mathbb{R}$. Define $T \in \mathcal{L}(V)$ as:

$$T(x) = x - 2\langle x, u\rangle u, \quad x \in V$$

Show that

(a) T is an isometry

Solution:

Intuition: For this problem, we will need a couple of facts. The first one is that we want to extend u to be an orthonormal basis of V. the reason this would be a good place to start is because we see an inner product in the definition of T and also the definition of an isometry is as follows

Definition 1. Axler 7.37

An operator $S \in \mathcal{L}(V)$ is called an isometry if for all $v \in V$,

$$\|Sv\| = \|v\|$$

Which is an inner product squared. So if we have an orthonormal basis, these equalities are much much easier to deal with.

Now that we have $(u, u_2, \ldots, u_n)$ is an orthonormal basis of V, then we see that

$$\begin{aligned} T(u) &= u - 2\langle u, u\rangle u = -u \\ T(u_k) &= u_k - 2\langle u_k, u\rangle u = u_k \qquad \text{for } k = 2, \ldots, n \end{aligned}$$

At this point, it is possible to be done. If we happen to remember Theorem 7.42 in Axler, we can appeal to parts (a) and (d) on that one and that's it. For this problem, even though this was in the second half, it has 3 parts so we are not trivializing this problem nor making it exceptionally short. However, if this problem was in the first half and it only had 1 or 2 parts, it would be a good idea to not appeal to this theorem. If you are unsure, ask the proctor or err on the side of caution

Theorem 1. *Axler 7.42 Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent.*

1. *S is an isometry*

2. $\langle Su, Sv \rangle = \langle u, v \rangle$ *for all* $u, v \in V$

3. $Se_1, \ldots, Se_n$ *is orthonormal for every orthonormal list of vectors* $e_1, \ldots, e_n$ *in* V

4. *There exists some orthonormal basis* $e_1, \ldots, e_n$ *of* V *such that* $Se_1, \ldots, Se_n$ *is orthonormal*

5. $S^*S = I$

6. $SS* = I$

7. S^* *is an isometry*

8. S *is invertible and* $S^{-1} = S^*$

However, let's say we forget this theorem or we don't feel comfortable appealing to it. Then we would need to show that our operator satsifies the previous definition directly. For ease of notation, we assign $u_1 = u$ in our previous orthonormal basis.

So, let $v \in V$, we aim to show that $\|Tv\| = \|v\|$

Useful theorems for this part are Axler 6.25 and 6.30. These are listed below for convenience

Theorem 2. *Axler 6.25*

If $e_1, \ldots, e_n$ *is an orthonormal list of vectors in* V*, then*

$$\left\| \sum_{k=1}^{n} a_k e_k \right\|^2 = \sum_{k=1}^{n} |a_k|^2$$

Theorem 3. *Axler 6.30*

Suppose $e_1, \ldots, e_n$ *is an orthonormal basis of* V *and* $v \in V$*. Then*

$$v = \sum_{k=1}^{n} \langle v, e_k \rangle e_k$$

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 \tag{1}$$

So we will show that $\|Tv\|^2$ equals equation (1). In order to do this, we will first write out what Tv is in terms of the coefficients of v. See that

$$\begin{aligned} Tv &= T\left(\sum_{k=1}^{n} \langle v, u_k \rangle u_k\right) \\ &= \sum_{k=1}^{n} \langle v, u_k \rangle Tu_k \\ &= -\langle v, u \rangle u + \sum_{k=2}^{n} \langle v, u_k \rangle u_k \end{aligned}$$

Notice that this is almost the exact same as v! In fact, it only has the first term negated. We are also in a form compatible with Theorem 2, so we have that

$$\begin{aligned}
\|Tv\|^2 &= \left\| - \langle v, u \rangle u + \sum_{k=2}^{n} \langle v, u_k \rangle u_k \right\|^2 \\
&= | - \langle v, u \rangle |^2 + \sum_{k=2}^{n} | \langle v, u_k \rangle |^2 \\
&= | \langle v, u \rangle |^2 + \sum_{k=2}^{n} | \langle v, u_k \rangle |^2 \\
&= \sum_{k=1}^{n} | \langle v, u_k \rangle |^2 \\
&= \|v\|^2
\end{aligned}$$

Now, take the square root of both sides, and we get that

$$\|Tv\| = \|v\|$$

Since v was an arbitrary vector in V, this holds for all vectors in V. Thus T is an isometry as desired.

<u>Solution</u>:

<u>Aside</u>: For this one, I am going to show directly that $\|Tv\| = \|v\|$ for all $v \in V$.

First, extend u to an orthonormal basis of V denoted $u_1, \ldots, u_n$ where $u_1 = u$. See that for these basis vectors,

$$\begin{aligned}
T(u) &= u - 2 \langle u, u \rangle u = -u \\
T(u_k) &= u_k - 2 \langle u_k, u \rangle u = u_k \qquad\qquad \text{for } k = 2, \ldots, n
\end{aligned}$$

Now, we aim to show that $\|Tv\| = \|v\|$ for all $v \in V$.

Let $v \in V$. We know from Theorem 6.30 in Axler that

$$\begin{aligned}
v &= \sum_{k=1}^{n} \langle v, u_k \rangle u_k \\
\|v\|^2 &= \sum_{k=1}^{n} |\langle v, u_k \rangle|^2
\end{aligned}$$

This means that we have from above

$$\begin{aligned}
Tv &= T\left(\sum_{k=1}^{n} \langle v, u_k \rangle u_k \right) \\
&= \sum_{k=1}^{n} \langle v, u_k \rangle Tu_k \\
&= \langle v, u \rangle Tu + \sum_{k=2}^{n} \langle v, u_k \rangle Tu_k \\
&= - \langle v, u \rangle u + \sum_{k=2}^{n} \langle v, u_k \rangle u_k
\end{aligned}$$

From Theorem 6.25, we can now say that

$$
\begin{aligned}
\|Tv\|^2 &= \left\| -\langle v, u\rangle u + \sum_{k=2}^{n} \langle v, u_k\rangle u_k \right\|^2 \\
&= |-\langle v, u\rangle|^2 + \sum_{k=2}^{n} |\langle v, u_k\rangle|^2 \\
&= |\langle v, u\rangle|^2 + \sum_{k=2}^{n} |\langle v, u_k\rangle|^2 \\
&= \sum_{k=1}^{n} |\langle v, u_k\rangle|^2 \\
&= \|v\|^2
\end{aligned}
$$

Where the last step is the second equation in Theorem 6.30.

Next, take the square root of both sides, and we get that

$$\|Tv\| = \|v\|$$

Since v was an arbitrary vector in V, this holds for all vectors in V. Thus T is an isometry as desired.

(b) If $A = \mathcal{M}(T)$ is a matrix representation of T, then $\det A = -1$

Solution:

Note: There are two ways to approach this problem. The solution provided by the prelim committee takes a matrix algebra approach. I will briefly justify why they made the claim that they do, but I will take a different approach for this one.

Intuition:

Matrix Algebra Approach:

The committee argues that $\mathcal{M}(T)$ with respect to our above basis is the matrix

$$\mathcal{M}(T) = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

This is a consequence of our equations where we apply T to u and $u_2, \ldots, u_n$. And then the determinant of this matrix is $-1 * 1^{n-1} = -1$. And since we have that the determinant of a matrix is invariant under similarity transformations (change of basis matrices and a standard result in many undergraduate linear algebra courses) we have that the determinant of $\mathcal{M}(T)$ under any basis is -1 as desired.

Axler-esque Approach:

In Axler the determinant of an operator T is defined as follows

Definition 2. Suppose $T \in \mathcal{L}(V)$.

- If $\mathbb{F} = \mathbb{C}$, then the determinant of T is the product of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity
- If $\mathbb{F} = \mathbb{R}$, then the determinant of T is the product of the eigenvalues of $T_{\mathbb{C}}$, with each eigenvalue repeated according to its multiplicity

Where $T_{\mathbb{C}}$ is the complexification of T. Essentially this just means that we allow complex eigenvalues if our operator happened to have them. (But this isn't relevant here, as we will show shortly)

We showed in the previous part that V has an orthonormal basis consisting of eigenvectors of T. $u, u_2, \ldots, u_n$ are eigenvectors where u is associated with eigenvalue -1 and $u_2, \ldots, u_n$ are associated with the eigenvalue 1. This means that T is diagonalizable according to the Real Spectral Theorem. The eigenspace associated with -1 has dimension 1 while the eigenspace associated with 1 has dimension $n-1$ as we have $n-1$ linearly independent vectors living in this space. This means from our above definition,

$$\det T = -1 * 1^{n-1} = -1$$

Since the determinant is a property of the operator and not the matrix representation of the operator (according to Axler and formalized with Theorem 10.42) we know that for any matrix representation of T, denoted A, we have that $\det A = \det T = -1$ as desired.

<u>**Solution**</u>:

(I am only writing up the Axler-esque approach, but the above Matrix Algebra Approach would be sufficient if this approach made more sense to you.)

Since we have that $u, u_2, \ldots, u_n$ is an orthonormal basis of V and

$$\begin{aligned} T(u) &= u - 2\langle u, u\rangle u = -u \\ T(u_k) &= u_k - 2\langle u_k, u\rangle u = u_k \qquad \text{for } k = 2, \ldots, n \end{aligned}$$

we have that u is an eigenvector of T with associated eigenvalue of -1, and $u_2, \ldots, u_n$ are $n-1$ linearly independent eigenvectors of T associated with the eigenvalue 1. Since the eigenspace associated with -1 is of dimension 1 and the eigenspace associated with 1 is of dimension $n-1$, we have that by the definition of the determinant

$$\det T = -1 * 1^{n-1} = -1$$

So by Theorem 10.42, for any matrix representation of T, denoted A, is given by

$$\det A = \det T = -1$$

as desired.

(c) If $S \in \mathcal{L}(V)$ is an isometry with 1 as an eigenvalue, and if the eigenspace of 1 is of dimension $n-1$, then there exists some $w \in V$ where w is a unit vector and for all $x \in V$:

$$S(x) = x - 2\langle x, w\rangle w$$

Solution:

(Note: we are using $*$ to denote the adjoint here. Since we are in a real vector space, this is equivalent to just the transpose).

<u>**Intuition**</u>: We will probably want to follow a similar path as above since this would allow us to "reuse" some of our intuition that we have built up so far. So, we will first consider the eigenspace associated with 1. Let $v_1, \ldots, v_{n-1}$ be an orthonormal basis of $E(1, S)$. Next, we want to be able to use the fact that S is an isometry. From Axler 7.42 (see above intuition section) we know that since S is an isometry, we have $SS^* = I$. So, we would need this to hold for any matrix representation. For matrix representations, we need to have a basis, so we extend $v_1, \ldots, v_{n-1}$ to a basis of V denoted $v_1, \ldots, v_{n-1}, v_n$ (Note: Without Loss of Generality, assume that v_n is a unit vector). We will want to show that v_n is an eigenvector associated with -1.

Let's write out what this matrix representation looks like with respect to this basis. (Note: we don't know anything about v_n yet!)

$$A = \mathcal{M}(S, (v_1, \ldots, v_n)) = \begin{bmatrix} 1 & 0 & \ldots & a_{1n} \\ 0 & 1 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$$

We know the first $n-1$ columns of A are the identity matrix as the vectors $v_1, \dots, v_{n-1}$ are eigenvectors associated with 1. So, let's compute SS^*. (As this is a pain in LaTeX, I am going to do block multiplication for the first part. If anything is confusing, I will happily clarify).

See that $\mathcal{M}(S^*, (v_1, \dots, v_n)) = A^*$, so we need to just compute AA^*.

Let b be the first $n-1$ rows of the n^{th} column of A (This will make writing A as a block matrix much easier in the first couple steps. We will use the actual entries of A before we make any real computations though!).

$$\begin{aligned}
I_n = AA^* &= \begin{bmatrix} 1 & 0 & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & b \\ \mathbf{0} & a_{nn} \end{bmatrix} \begin{bmatrix} I_{n-1} & \mathbf{0} \\ b^\top & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} I_{n-1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \begin{bmatrix} a_{1n} & \dots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} a_{1n}a_{1n} & a_{1n}a_{2n} & \dots & a_{1n}a_{nn} \\ a_{2n}a_{1n} & a_{2n}a_{2n} & \dots & a_{2n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}a_{1n} & a_{nn}a_{2n} & \dots & a_{nn}a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} {a_{1n}}^2 & a_{1n}a_{2n} & \dots & a_{1n}a_{nn} \\ a_{2n}a_{1n} & {a_{2n}}^2 & \dots & a_{2n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}a_{1n} & a_{nn}a_{2n} & \dots & {a_{nn}}^2 \end{bmatrix}
\end{aligned}$$

Now, the first matrix in our sum has $1's$ on the diagonal of the first $n-1$ columns and $0's$ elsewhere. So in order for out final matrix to be the identity, we need the following n equations to hold (we'll worry about off diagonals afterward for the second matrix in our sum).

$$\begin{aligned}
{a_{1n}}^2 &= 0 \\
&\vdots \\
{a_{(n-1)n}}^2 &= 0 \\
{a_{nn}}^2 &= 1
\end{aligned}$$

The first $n-1$ equations give us that for $k = 1, \dots, n-1$,

$$a_{kn} = 0$$

(Note: This means that the off diagonals of our second matrix in the above sum must be 0 as needed)

And the final equation gives us that

$$a_{nn} = \pm 1$$

However, we know that if $a_{nn} = 1$, then we would have that v_n satisfies $Sv_n = v_n$, which is not possible because we would have n linearly independent eigenvectors associated with 1 which violates our assumption that $\dim E(S, 1) = n-1$. So, we know $a_{nn} = -1$.

This means that for our basis vectors, we have

$$\begin{aligned} Sv_1 &= v_1 \\ &\vdots \\ Sv_{n-1} &= v_{n-1} \\ Sv_n &= -v_n \end{aligned}$$

Next, we show that if we assign $w = v_n$, then the desired equation satisfies the above behavior on these basis vectors (which would allow us to conclude that $S(x) = x - 2\langle x, v_n \rangle v_n$ for all $x \in V$).

See that for $k = 1, \ldots, n$,

$$v_k - 2\langle v_k, v_n \rangle v_n = v_k = S(v_k)$$

The first equation is because from Axler 7.22 (as S is diagonalizable, it is clearly Normal) eigenvectors associated with distict eigenvalues are orthogonal. Finally see that

$$v_n - 2\langle v_n, v_n \rangle v_n = v_n - 2v_n = -v_n = S(v_n)$$

So since our equation holds for all basis vectors, it must also hold for all $x \in V$. Thus, we have shown that there exists $w \in V$ such that for all $x \in V$, $S(x) = x - 2\langle x, w \rangle w$ as desired.

Solution:

Let $(v_1, \ldots, v_{n-1}$ denote an orthonormal basis of $E(S, 1)$. We know that such a basis exists because we are assuming that $\dim E(S, 1) = n - 1$. Next, extend this basis to a basis of V. Denote this $v_1, \ldots, v_{n-1}, v_n$. Without loss of generality, assume that $\|v_n\| = 1$. We will now show that v_n is an eigenvector associated with eigenvalue -1.

See that the matrix representation of S with respect to this matrix is given by

$$A = \mathcal{M}\left(S, (v_1, \ldots, v_n)\right) = \begin{bmatrix} 1 & 0 & \ldots & a_{1n} \\ 0 & 1 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$$

We write the final column this way as we don't yet know how v_n relates to the other vectors. Since S is an isometry, Axler 7.42 gives us that $SS^* = I$ where I is the identity operator on V. So, we know that $AA^* = I_n$ where I_n is the n-by-n identity matrix. For ease of notation, let $b = \begin{bmatrix} a_{1n} & \ldots & a_{(n-1)n} \end{bmatrix}^\top$

This means

$$
\begin{aligned}
I_n = AA^* &= \begin{bmatrix} 1 & 0 & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & b \\ \mathbf{0} & a_{nn} \end{bmatrix} \begin{bmatrix} I_{n-1} & \mathbf{0} \\ b^\top & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} I_{n-1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} \begin{bmatrix} a_{1n} & \dots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} a_{1n}a_{1n} & a_{1n}a_{2n} & \dots & a_{1n}a_{nn} \\ a_{2n}a_{1n} & a_{2n}a_{2n} & \dots & a_{2n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}a_{1n} & a_{nn}a_{2n} & \dots & a_{nn}a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} {a_{1n}}^2 & a_{1n}a_{2n} & \dots & a_{1n}a_{nn} \\ a_{2n}a_{1n} & {a_{2n}}^2 & \dots & a_{2n}a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}a_{1n} & a_{nn}a_{2n} & \dots & {a_{nn}}^2 \end{bmatrix}
\end{aligned}
$$

Since we need these two matrices to sum to be I_n, we need the following n equations to hold, which we get from ensuring the diagonal elements are correct in the final sum.

$$
\begin{aligned}
{a_{1n}}^2 &= 0 \\
&\vdots \\
{a_{(n-1)n}}^2 &= 0 \\
{a_{nn}}^2 &= 1
\end{aligned}
$$

The first $n-1$ equations give us that for $k = 1, \dots, n-1$,

$$a_{kn} = 0$$

(Note: This means that the off diagonals of our second matrix in the above sum must be 0 as needed)

And the final equation gives us that

$$a_{nn} = \pm 1$$

However, we know that if $a_{nn} = 1$, then we would have that v_n satisfies $Sv_n = v_n$, which is not possible because we would have n linearly independent eigenvectors associated with 1 which violates our assumption that $\dim E(S, 1) = n - 1$. So, we know $a_{nn} = -1$.

Next, see that

$$
\begin{aligned}
Sv_1 &= v_1 \\
&\vdots \\
Sv_{n-1} &= v_{n-1} \\
Sv_n &= -v_n
\end{aligned}
$$

Next, we show that if we assign $w = v_n$, then the desired equation satisfies the above behavior on these basis vectors (which would allow us to conclude that $S(x) = x - 2\langle x, v_n \rangle v_n$ for all $x \in V$).

See that for $k = 1, \ldots, n$,

$$v_k - 2\langle v_k, v_n\rangle v_n = v_k = S(v_k)$$

The first equation is because from Axler 7.22 (as S is diagonalizable, it is clearly Normal) eigenvectors associated with distict eigenvalues are orthogonal. Finally see that

$$v_n - 2\langle v_n, v_n\rangle v_n = v_n - 2v_n = -v_n = S(v_n)$$

So since our equation holds for all basis vectors, it must also hold for all $x \in V$. Thus, we have shown that there exists $w \in V$ such that w is a unit vector and for all $x \in V$, $S(x) = x - 2\langle x, w\rangle w$ as desired.

2. Let V be a finite dimensional real vector space with basis $e_1, \ldots, e_n$ (the standard basis of $\mathbb{R}^n$)
   (a) Let A be a positive definite bijective matrix in V (This means A is a matrix representation of some invertible linear operator in $\mathcal{L}(V)$). For any $v, w \in V$, expressed as coordinate vectors according to this basis (their standard representation if you were to write them down), define

$$\langle v, w\rangle := v^\top Aw.$$

   Show that this is an inner product.

Solution:
Intuition: This is a computational part. Just need to show the following properties are satisfied
Definition 3. Inner Product Axler 6.3 An operation $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ is an inner product if the following are satisfied.

1. Positivity
   For all $v \in V$, $\langle v, v\rangle \geq 0$
2. Definiteness
   $\langle v, v\rangle = 0 \iff v = 0$
3. Additivity in the first slot
   for all $u, v, w \in V$, we have $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
4. Homogeneity in the first slot
   For all $\lambda \in \mathbb{F}, u, v \in V$, we have $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$
5. Conjugate symmetry
   For all $u, v \in V$, we have $\langle u, v\rangle = \overline{\langle v, u\rangle}$

However, since we are in a real space, we can show this with only $\mathbb{F} = \mathbb{R}$
How we would do this is just directly using the given properties of A
Solution:
First, we show positivity. Let $v \in V$. See that

$$\begin{aligned}\langle v, v\rangle &= v^\top Av \\ &\geq 0\end{aligned}$$

Since A is positive.
Next, we show definiteness. Assume that $v = 0$. See that

$$\langle v, v\rangle = v^\top Av = \mathbf{0}^\top A\mathbf{0} = 0$$

Now, let $v \in V$ such that $\langle v, v\rangle = 0$. This means

$$0 = v^\top Av$$

However, since A is positive definite, v must be 0. Therefore, $\langle v, v \rangle = 0 \iff v = 0$ as desired.
Next, we show additivity in the first slot. Let $u, v, w \in V$. See that:

$$\begin{aligned}\langle u + v, w \rangle &= (u + v)^\top Aw \\ &= (u^\top + v^\top)Aw \\ &= u^\top Aw + v^\top Aw \\ &= \langle u, w \rangle + \langle v, w \rangle\end{aligned}$$

As desired.
Next, we show homogeneity in the first slot. Let $\lambda \in \mathbb{R}$ and $u, v \in V$. See that

$$\begin{aligned}\langle \lambda u, v \rangle &= (\lambda u)^\top Av \\ &= \lambda u^\top Av \\ &= \lambda \langle u, v \rangle\end{aligned}$$

As desired.
Finally, we show conjugate symmetry (Since we are in $\mathbb{R}$, we instead show symmetry). Let $u, v \in V$. See that

$$\begin{aligned}\langle u, v \rangle &= u^\top Av \\ &= (u^\top Av)^\top && \text{Since this is a scalar} \\ &= v^\top A^\top u \\ &= v^\top Au && \text{Since } A \text{ is symmetric due to being positive} \\ &= \langle v, u \rangle\end{aligned}$$

Thus, we have shown that this function is an inner product as desired.

(b) Let $\langle \cdot, \cdot \rangle$ be an inner product in V. Define A to be a matrix such that $A_{ij} = \langle e_i, e_j \rangle$ is a positive bijective matrix such that $\langle v, w \rangle = v^\top Aw$.

Solution:
The idea behind this problem is that if we compute the inner product of all possible combinations of basis vectors, and store them in a matrix (Where the index of the "first" basis vector gives the row and the index of the "second" gives the column). Then for any pair of vectors in our vector space we can instead compute the above mentioned matrix vector product. As far as usefulness, this could potentially be useful in cases where computing the inner product is expensive to do (or in an even more ideal situation, cheap for the basis vectors) and we want to compute a lot of inner products.
This problem has two steps. First is to show that A is positive definite and invertible and the second is to show that if we define A to be this way, then we know that $\langle v, w \rangle = v^\top Aw$. This essentially means that we can write an inner product in this space as the matrix vector multiplication instead.
<u>Intuition</u>: Assuming that we show $\langle v, w \rangle = v^\top Aw$, then the needed properties of A (positive definite and bijective) are nearly immediate consequences. So let $v, w \in V$, See that in the problem statement we write these vectors as coordinates in our giving basis. This means

$$\begin{aligned}v &= \sum_{k=1}^{n} v_k e_k \\ w &= \sum_{k=1}^{n} w_k e_k\end{aligned}$$

So, we have that

$$
\begin{aligned}
\langle v, w \rangle &= \left\langle \sum_{i=1}^{n} v_i e_i, \sum_{j=1}^{n} w_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \left\langle v_i e_i, \sum_{j=1}^{n} w_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i e_i, w_j e_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j \langle e_i, e_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j A_{ij}
\end{aligned}
$$

Next we will show that $v^\top A w$ looks the same.

$$
\begin{aligned}
v^\top A w &= v^\top \begin{bmatrix} \sum_{j=1}^{n} A_{1j} w_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj} w_j \end{bmatrix} \\
&= \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} A_{1j} w_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj} w_j \end{bmatrix} \\
&= \sum_{i=1}^{n} v_i \sum_{j=1}^{n} A_{ij} w_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} v_i w_j A_{ij}
\end{aligned}
$$

Which is the same as $\langle v, w \rangle$. Thus we have that for all $v, w \in V$, $\langle v, w \rangle = v^\top A w$.

Now all that is left to show is that A is positive definite and invertible. For positive, let $v \in V$. Then, $v^\top A v = \langle v, v \rangle \geq 0$.

For definiteness, let $v \in V$ such that $v^\top A v = 0$. See that

$$0 = v^\top A v = \langle v, v \rangle \iff v = \mathbf{0}.$$

Thus, A is positive definite. Now for invertability. We will do this by showing the nullspace is trivial. IE let $v \in V$ such that $Av = \mathbf{0}$. This tells us

$$Av = \mathbf{0} \iff v^\top A v = v^\top \mathbf{0} = 0$$

However recall that $v^\top A v = \langle v, v \rangle$ which is only 0 when $v = \mathbf{0}$. Thus, the only vector in the nullspace of A is the zero vector. Thus, it is invertible as V is a finite vector space.

Solution: First, we will show that for all $v, w \in V$, we have that $\langle v, w \rangle = v^\top A w$. We will do this by expanding both sides to the same form, which will let us conclude that $\langle v, w \rangle = v^\top A w$. See first that

$$\begin{aligned}
\langle v, w\rangle &= \left\langle \sum_{i=1}^n v_i e_i, \sum_{j=1}^n w_j e_j \right\rangle \\
&= \sum_{i=1}^n \left\langle v_i e_i, \sum_{j=1}^n w_j e_j \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle v_i e_i, w_j e_j\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n v_i w_j \langle e_i, e_j\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n v_i w_j A_{ij}
\end{aligned}$$

Next see:

$$\begin{aligned}
v^\top A w &= v^\top \begin{bmatrix} \sum_{j=1}^n A_{1j} w_j \\ \vdots \\ \sum_{j=1}^n A_{nj} w_j \end{bmatrix} \\
&= \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n A_{1j} w_j \\ \vdots \\ \sum_{j=1}^n A_{nj} w_j \end{bmatrix} \\
&= \sum_{i=1}^n v_i \sum_{j=1}^n A_{ij} w_j \\
&= \sum_{i=1}^n \sum_{j=1}^n v_i w_j A_{ij}
\end{aligned}$$

Which is the same as $\langle v, w\rangle$. Thus we have that for all $v, w \in V$, $\langle v, w\rangle = v^\top A w$. Now, we need only show that A is positive definite and invertible. First positivity.
Let $v \in V$. We have, by above,

$$v^\top A v = \langle v, v\rangle \geq 0$$

Next, definiteness. Let $v \in V$ such that $v^\top A v = 0$. See that

$$0 = v^\top A v = \langle v, v\rangle \iff v = \mathbf{0}.$$

We also show the other direction. Let $v = \mathbf{0}$, we have

$$v^\top A v = \mathbf{0}^\top A \mathbf{0} = 0$$

Finally we show that A is invertible. We will do this by showing the nullspace is trivial. IE let $v \in V$ such that $Av = \mathbf{0}$. This tells us

$$Av = \mathbf{0} \iff v^\top A v = v^\top \mathbf{0} = 0$$

However recall that $v^\top A v = \langle v, v\rangle$ which is only 0 when $v = \mathbf{0}$. Thus, the only vector in the nullspace of A is the zero vector. Thus, it is invertible as V is a finite vector space.

3. V be a finite-dimensional inner product space over $\mathbb{C}$. Let T be a normal operator on V. Let $\lambda \in \mathbb{C}$ and

let $v \in V$ be a unit vector (ie $\|v\| = 1$). Prove that T has an eigenvalue λ' such that

$$\|\lambda - \lambda'\| \leq \|Tv - \lambda v\|.$$

Solution:

Intuition: Since we know nothing about T other than it is normal. So we use the one thing that has not failed us yet (eigenvector basis!).

Solution:

Since T is normal, the complex spectral theorem gives us that there exists an orthonormal basis consisting of eigenvectors of T. Denote this basis $(v_1, \ldots, v_n)$ with associated eigenvectors $\lambda_1, \ldots, \lambda_n$.

Next since we have a basis, we know that there exist some $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$v = \sum_{k=1}^{n} a_k v_k$$

Finally, we see that

$$\begin{aligned}
\|Tv - \lambda v\|^2 &= \left\| T\left(\sum_{k=1}^{n} a_k v_k\right) - \lambda\left(\sum_{k=1}^{n} a_k v_k\right)\right\|^2 \\
&= \left\|\sum_{k=1}^{n} a_k T v_k - a_k \lambda v_k\right\|^2 \\
&= \left\|\sum_{k=1}^{n} (\lambda_k - \lambda) a_k v_k\right\|^2 \\
&= \sum_{k=1}^{n} |\lambda_k - \lambda|^2 |a_k|^2 \qquad \text{Axler 6.25} \\
&\geq \sum_{k=1}^{n} \min_{\ell \in \{1,\ldots,n\}} |\lambda_\ell - \lambda|^2 |a_k|^2 \\
&= \min_{\ell \in \{1,\ldots,n\}} |\lambda_\ell - \lambda|^2 \sum_{k=1}^{n} |a_k|^2 \\
&= \min_{\ell \in \{1,\ldots,n\}} |\lambda_\ell - \lambda|^2 \|v\|^2 \qquad \text{Axler 6.25} \\
&= \min_{\ell \in \{1,\ldots,n\}} |\lambda_\ell - \lambda|^2 \qquad v \text{ is a unit vector} \\
&= |\lambda_j - \lambda|^2. \\
&= \|\lambda_j - \lambda\|^2
\end{aligned}$$

The j in the final equation is given by $j = \arg\min_{\ell \in \{1,\ldots,n\}} |\lambda_\ell - \lambda|^2$. Finally, we redo some variable names and take the square root of both sides to get it in the form of the problem statement.

Define $\lambda' = \lambda_j$ and we finally have that

$$\|\lambda - \lambda'\| \leq \|Tv - \lambda v\|$$

as desired.

2 Axler Problems

1. Suppose $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are inner products on V over a field $\mathbb{F}$ such that for all $u, v \in V$, $\langle u, v \rangle_1 = 0 \iff \langle u, v \rangle_2 = 0$. Prove that there is a positive number c such that $\langle u, v \rangle_1 = c \langle u, v \rangle$ for all $u, v \in V$.

Solution:

Preamble:

This problem is a bit difficult to do, and while not exactly in the syllabus of the Prelim (which is why we didn't talk about it in our sessions) but seeing how a linear function can be used may be informative and give some experience using unfamiliar concepts. This should be used as a guideline for how a proof would look (with explainations too). So, we will skip the intuition section and instead only do the more proper solution writeup.

This proof is a slightly modified version of the one provided by Axler. (want to be clear about credit for it).

Solution:

Let $w \in V$ such that $w \neq \mathbf{0}$. Define the linear functionals $\phi : V \to \mathbb{F}$ and $\psi : V \to \mathbb{F}$ as for all $v \in V$:

$$\phi(v) = \langle v, w \rangle_1$$
$$\psi(v) = \langle v, w \rangle_2$$

From the problem statement, we know that $\operatorname{null} \phi = \operatorname{null} \psi$. Recall that ϕ and ψ are elements of the dual space V'. So we can talk about their spans as they are vectors themselves. Next, we also know that:

$$\operatorname{Span}(\phi) = (\operatorname{null} \phi)^0 = (\operatorname{null} \psi)^0 = \operatorname{Span}(psi)$$

This equation comes from the fundamental theorem of linear algebra (or linear maps) along with the definition of the annihilator of a space. Since the spans of these functionals are the same, and they are singular vectors, we have that there exists some $c_w \in \mathbb{F}$ such that

$$\phi = c_w \psi$$

Note: the subscript w denotes the potential reliance on the chosen w. Now, we will show that $c_w \in \mathbb{R}$ and $c_w > 0$. Since above holds for all $v \in V$, we know it also works for w. See

$$\phi(w) = c_w \psi(w) \implies \langle w, w \rangle_1 = c_w \langle w, w \rangle_2 \implies \|w\|_1^2 = c_w \|w\|_2^2 .$$

Where $\|\cdot\|_1$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_1$ and $\|\cdot\|_2$ denotes the norm induced by the inner product $\langle \cdot, \cdot \rangle_2$. Since these are norms and $w \neq 0$, the left hand side of our final equation is a real positive number, and the norm on the right side is c_w multiplied by a real non-negative number. Thus, we have that c_w must be both real and positive.

Next, we show that c_w is independent of our choice of w so we can rename it to be c. Let $w, x \in V$ and $c_w, c_x \in \mathbb{R}$ such that for all $v \in V$ we have:

$$\langle v, w \rangle_1 = c_w \langle v, w \rangle_2 \tag{2}$$
$$\langle v, x \rangle_1 = c_x \langle v, x \rangle_2 \tag{3}$$

We will now show that $c_w = c_x$. Since the above works for all $v \in V$, plug in x into (2) and w into (3). See that this gives us both

$$\langle x, w \rangle_1 = c_w \langle x, w \rangle_2$$
$$\langle w, x \rangle_1 = c_x \langle w, x \rangle_2$$

We can use the conjugate symmetry property of $\langle \cdot, \cdot \rangle_1$ to get that

$$\overline{\langle x, w \rangle_1} = \langle w, x \rangle_1 = c_x \langle w, x \rangle_2$$

which we can take the conjugate of the far left and far right sides to give us

$$\langle x, w\rangle_1 = \overline{c_x \langle w, x\rangle_2} = c_x \overline{\langle w, x\rangle_2}$$

Putting all this together lets us simplify to

$$\begin{aligned} c_w \langle x, w\rangle_2 &= \langle x, w\rangle_1 \\ &= c_x \overline{\langle w, x\rangle_2} \\ &= c_x \langle x, w\rangle_2 \end{aligned}$$

So, we have that $c_w \langle x, w\rangle_2 = c_x \langle x, w\rangle_2$. Since the inner product is a function, we have that $c_w = c_x$. So the choice of c_w is independent of the choice of vector in our previous linear functional, so we drop the subscript and are left with a $c \in \mathbb{R}$ such that $c > 0$ and for all $u, v \in V$ we have

$$\langle u, v\rangle_1 = c \langle u, v\rangle_2$$

as desired.

2. Let V be an inner product space over a field $\mathbb{F}$. Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that
$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$
if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

Solution:

<u>Intuition</u>: This is asking us to prove a little stronger version of Axler 6.25 and 6.30. While the wording is slightly different, it is similar in at least the backwards direction. Due to this, the backwards direction is a little easier to do, so we start there. (Also, we should not appeal to these theorems as they would trivialize the proof in this direction). In addition, we are not given any information on what this norm looks like, so we are assuming it is the norm induced by the inner product on V.

$\Longleftarrow$ direction

Let $v \in \operatorname{span}(e_1, \ldots, e_m)$, and this means that there exists some $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$v = a_1 e_1 + \cdots + a_m e_m = \sum_{k=1}^{m} a_k e_k.$$

Now, we substitute this equation into our norm statement and simplify.

$$\begin{aligned} \|v\|^2 &= \langle v, v\rangle \\ &= \left\langle \sum_{i=1}^{m} a_i e_i, \sum_{j=1}^{m} a_j e_j \right\rangle \\ &= \sum_{i=1}^{m} \left\langle a_i e_i, \sum_{i=j}^{m} a_j e_j \right\rangle \\ &= \sum_{i=1}^{m} a_i \left\langle e_i, \sum_{i=j}^{m} a_j e_j \right\rangle \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \langle e_i, a_j e_j\rangle \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \overline{a}_j \langle e_i, e_j\rangle \end{aligned}$$

However, note that since $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, we know that for all i, j

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

So our double sum can be rewritten as

$$\sum_{i=1}^{m} a_i \overline{a}_i \langle e_i, e_i \rangle = \sum_{i=1}^{m} |a_i|^2$$

which is the exact form we want! Thus, we have shown

$$v \in \operatorname{span}(e_1, \ldots, e_m) \implies \|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

Now, let's show the $\implies$ direction. We will do this through contradiction IE assume that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

and $v \notin \operatorname{Span}(e_1, \ldots, e_m)$.

Since $v \notin \operatorname{Span}(e_1, \ldots, e_m)$ but $v \in V$, we first extend $(e_1, \ldots, e_m)$ to an orthonormal basis of V. Denote this basis $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_n)$. We want to create such a basis so that we can show a contradiction using the norm statement. This means that there exists some $a_1, \ldots, a_m, a_{m+1}, \ldots, a_n \in \mathbb{F}$ where at least one of $a_{m+1}, \ldots, a_n$ non-zero. Let j be the first index such that $j \geq m + 1$ and $a_j \neq 0$ (this will make some later inequalities easier to deal with).

Now, similarly as above, we can write v in terms of a linear combination of this basis. See that

$$v = a_1 e_1 + \cdots + a_n e_n = \sum_{k=1}^{n} a_k e_k.$$

Similarly as above, we rewrite our norm statement and simplify.

$$\begin{aligned}
\|v\|^2 &= \langle v, v \rangle \\
&= \left\langle \sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \left\langle a_i e_i, \sum_{i=j}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} a_i \left\langle e_i, \sum_{i=j}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \langle e_i, a_j e_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a}_j \langle e_i, e_j \rangle \\
&= \sum_{i=1}^{n} |a_i|^2 \\
&\geq \sum_{i=1}^{m} |a_i|^2 + |a_j| \\
&> \sum_{i=1}^{m} |a_i|^2 \\
&= \|v\|^2
\end{aligned}$$

So we have that $\|v\|^2 > \|v\|^2$ which is not possible. Thus we have shown that

$$v \in \operatorname{span}(e_1, \ldots, e_m) \impliedby \|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

So, in all we have shown that

$$v \in \operatorname{span}(e_1, \ldots, e_m) \iff \|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

as desired.

<u>Solution</u>:

First, we prove the $\impliedby$ direction. IE assume that $v \in \operatorname{Span}(e_1, \ldots, e_m)$. This means that there exists some $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$v = a_1 e_1 + \cdots + a_m e_m = \sum_{k=1}^{m} a_k e_k.$$

Next, we compute the norm of v. We assume the norm here is the induced norm based on the inner product on V.

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \left\langle \sum_{i=1}^{m} a_i e_i, \sum_{j=1}^{m} a_j e_j \right\rangle \\
&= \sum_{i=1}^{m} \left\langle a_i e_i, \sum_{i=j}^{m} a_j e_j \right\rangle \\
&= \sum_{i=1}^{m} a_i \left\langle e_i, \sum_{i=j}^{m} a_j e_j \right\rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \langle e_i, a_j e_j\rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \overline{a}_j \langle e_i, e_j\rangle
\end{aligned}$$

However, note that since $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, we know that for all i, j

$$\langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

So our double sum can be rewritten as

$$\sum_{i=1}^{m} a_i \overline{a}_i \langle e_i, e_i\rangle = \sum_{i=1}^{m} |a_i|^2$$

So, we have shown that

$$v \in \operatorname{span}(e_1, \ldots, e_m) \implies \|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

as desired.

Next, we show the $\implies$ direction. We will do this by contradiction. So, assume that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

and that $v \notin \text{Span}(e_1, \ldots, e_m)$.

Extend $e_1, \ldots, e_m$ to be an orthonormal list of vectors in V. We know that we are adding at least one term as if $v \in V$ and $v \notin \text{Span}(e_1, \ldots, e_m)$ then there must be at least one more linearly independent vector in V as otherwise this would be impossible to do. Denote this extended basis as $(e_1, \ldots, e_m, e_{m+1}, \ldots, e_n)$.

Since this is a basis of V, there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = \sum_{k=1}^{n} a_k v_k$$

In addition, we know at least one of $a_{m+1}, \ldots, a_n$ are non-zero as we are assuming $v \notin \text{Span}(e_1, \ldots, e_m)$. For ease of notation, let j be the first index such that $a_j \neq 0$ and $j \geq m+1$.

Similarly as above, we rewrite our norm statement and simplify.

$$\begin{aligned}
\|v\|^2 &= \langle v, v \rangle \\
&= \left\langle \sum_{i=1}^{n} a_i e_i, \sum_{j=1}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \left\langle a_i e_i, \sum_{i=j}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} a_i \left\langle e_i, \sum_{i=j}^{n} a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \langle e_i, a_j e_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a}_j \langle e_i, e_j \rangle \\
&= \sum_{i=1}^{n} |a_i|^2 \\
&\geq \sum_{i=1}^{m} |a_i|^2 + |a_j| \\
&> \sum_{i=1}^{m} |a_i|^2 \\
&= \|v\|^2
\end{aligned}$$

So we have that $\|v\|^2 > \|v\|^2$ which is our contradiction. Thus we have shown that

$$v \in \text{span}(e_1, \ldots, e_m) \impliedby \|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

So, in all we have shown that

$$v \in \text{span}(e_1, \ldots, e_m) \iff \|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

as desired.