# Bin 1 Problems

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This document holds problems that fit into the Bin 4 according to the prelim syllabus. Or are approached in a way most compatible with Bin 4

## Contents



## 1 Prelim Problems

1. Let $V$ be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$
   (a) (5 points) Prove $T$ is invertible
   (b) (15 points) Find the minimal polynomial of $T$.
2. Answer the following
   (a) Is there an $n \times n$ matrix $A$ with $A^{n-1} \neq 0$ and $A^n = 0$? Give an example to show such a matrix exists (and explain why it satisfies both conditions), or disprove it.
   (b) Show that an $n \times n$ upper triangular matrix with $A^n \neq 0$ and $A^{n+1} = 0$ does not exist.
3. Let $T$ be a linear map on a vector space $V$ and $\dim V = n$
   (a) If for some vector $v$, the vectors $v$, $Tv$, $\ldots$, $T^{n-1}v$ are linearly independent, show that every eigenvalue of $T$ has only one corresponding eigenvector up to a scalar multiplication
   (b) If $T$ has $n$ distinct eigenvalues and vector $u$ is a sum of $n$ eigenvectors, corresponding to the distinct eigenvalues, show that $u$, $Tu$, $\ldots$, $T^{n-1}u$ are linearly independent (and thus form a basis of $V$).
4. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $\lambda$ be an eigenvalue of $A$.
   (a) Show that $\lambda^r$ is an eigenvalue of $A^r$ for $r \in \mathbb{N}$.
   (b) Provide an example showing that the multiplicity of $\lambda^r$ as an eigenvalue of $A^r$ may be strictly higher than the multiplicity of $\lambda$ as an eigenvalue of $A$
   (c) Show that $A^\top$ has the same eigenvalues as $A$.
   (d) Show that if $A$ is orthogonal, then $\frac{1}{\lambda}$ is an eigenvalue of $A$.

## 2 Axler Problems

1. Suppose $V$ is an inner product space and $T \in \mathcal{L}(V)$ is normal. Prove that the minimal polynomial of $T$ has no repeated zeros

## 3 Other Problems

These are from the Berkely Problems pdf.

1. (7.5.16) Let $A$ and $B$ denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that $A$ and $B$ share a common eigenvector in $\mathbb{R}^n$.