Bin 4 Problems

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This document holds problems that fit into the Bin 4 according to the prelim syllabus. Or are approached in a way most compatible with Bin 4

Contents

1 Prelim Problems

1. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$

   (a) (5 points) Prove T is invertible

   Solution:
   Intuition Buildup: We know that the roots of the minimal polynomial are exactly the eigenvalues of the associated linear operator, and that a matrix is invertible if and only if 0 is not an eigenvalue (it's nullspace is trivial). This is sufficient as it shows T is an injective linear operator, which is enough as we assume that V is a finite vector space (yes I know the course discusses infinite vector spaces however that is a functional analysis type of approach not Linear Algebra in most cases. If you are unsure, always ask the proctor what V is).
   Solution:
   We will prove that T is invertible by showing that 0 is not an eigenvalue of T. This will let us conclude that null $(T - 0I) = \operatorname{null} T = \{\mathbf{0}\}$. This tells us that T is injective. Next, since $p(z)$ is our minimal polynomial, and the roots of p are exactly the eigenvalues of T, we need only show that 0 is not a root of p. See that

   $$\begin{aligned} p(0) &= 3 + 2 * 0 - 0^2 + 5 * 0^3 + 0^4 \\ &= 3 \\ &\neq 0. \end{aligned}$$

   Thus, 0 is not a root of p, so we conclude that T is invertible as desired.

   (b) (15 points) Find the minimal polynomial of T.

   Solution:
   Intuition Buildup: For this part, we will be using 3 things. The first being that T^{-1} exists, the second that $p(T) = 0$ by definition of the minimal polynomial, and the third is theorem 1. The idea behind approaching it this way is because we want to find a minimal monic polynomial q such

that $q\left(T^{-1}\right)=0$, and all the information we are given that can be useful for this is the minimal polynomial of T.

We put a general outline below after the theorem statement

Theorem 1. *(Axler 8.46) Suppose $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$ if and only if q is a polynomial multiple of the minimal polynomial of T.*

We see that $p(z)$ has degree 4, and we want to try to get this polynomial in terms of T^{-1}. The easiest way to do this is to apply T^{-4} to $p(T)$. This will give us:

$$
\begin{aligned}
p(T)=0 &\iff T^{-4}p(T)=0\\
&\iff T^{-4}\left(3I+2T-T^2+5T^3+T^4\right)=0\\
&\iff 3T^{-4}+2T^{-3}-T^{-2}+5T^{-1}+I=0\\
&\iff T^{-4}+\frac{2}{3}T^{-3}-\frac{1}{3}T^{-2}+\frac{5}{3}T^{-1}+\frac{1}{3}I=0
\end{aligned}
$$

So, now we would define $q(z)=z^4+\frac{2z^3}{3}-\frac{z^2}{3}+\frac{5z}{3}+\frac{1}{3}$
which we have shown is a polynomial multiple of the minimal polynomial due to theorem 1. All that is left at this point is to argue that q is of minimal degree.

The easiest way to do this is to consider a polynomial of degree 3 or less, call it q_1 and then show it would mean T has a polynomial of degree 3 or less, p_1, such that $p_1(T)=0$ which contradicts the assumption that p is the minimal polynomial of T.

Assume that there exists some $q_1 \in \mathcal{P}(\mathbb{F})$ such that $q_1\left(T^{-1}\right)=0$ and q_1 has degree $k \leq 3$. This means that q_1 is of the form

$$
q_1(z)=a_0+\cdots+a_{k-1}z^{k-1}+z^k
$$

Note: We don't need to have $a_k=1$, but we write it in this way to draw attention to the fact that we are picking a minimal polynomial of smaller degree, however the argument follows the same without this restriction.

where $a_0 \neq 0$ as otherwise T^{-1} would not be invertible, which contradicts the definition of T^{-1}. See that we have:

$$
\begin{aligned}
q_1\left(T^{-1}\right)=0 &\iff T^k q_1\left(T^{-1}\right)=0\\
&\iff T^k\left(a_0+\cdots+a_{k-1}T^{-(k-1)}+T^{-k}\right)=0\\
&\iff a_0T^k+\cdots+a_{k-1}T+I=0\\
&\iff T^k+\cdots+\frac{a_{k-1}}{a_0}T+\frac{1}{a_0}I=0
\end{aligned}
$$

Which means the polynomial $p_1(z)=z^k+\cdots+\frac{a_{k-1}z}{a_0}+\frac{1}{a_0}$ is a polynomial multiple of $p(z)$ by theorem 1, however this is impossible because p_1 is of smaller degree than p. So, this means that q_1 cannot have degree smaller than 4.

Thus, we have shown that q is the minimal polynomial of T^{-1} as it is a monic polynomial of minimal degree such that $q\left(T^{-1}\right)=0$.

Solution:

Since we know that the minimal polynomial of T is p, consider the polynomial given by

$$
q(z)=z^4+\frac{2z^3}{3}-\frac{z^2}{3}+\frac{5z}{3}+\frac{1}{3} \tag{1}
$$

See that:

$$\begin{aligned}
3q\left(T^{-1}\right) &= 3\left(T^{-4} + \frac{2}{3}T^{-3} - \frac{1}{3}T^{-2} + \frac{5}{3}T^{-1} + \frac{1}{3}I\right) \\
&= 3T^{-4} + 2T^{-3} - T^{-2} + 5T^{-1} + I \\
&= T^{-4}\left(3 + 2T - T^2 + 5T^3 + T^4\right) && (2) \\
&= T^{-4}p(T) \\
&= 0. && (3)
\end{aligned}$$

Note that step 2 is valid because if T^{-1} exists, we can "factor" out T^{-4}.
Since $q\left(T^{-1}\right) = 0$ we know from theorem 1 that q is a polynomial multiple of the minimal polynomial of T^{-1}. Since it is already monic, we need only show that it is of minimal degree. Assume that there exists some $q_1 \in \mathcal{P}(\mathbb{F})$ such that $q_1\left(T^{-1}\right) = 0$, $q_1 \neq 0$, and where q_1 is of the form

$$q_1(z) = \sum_{l=0}^{k} a_l z^l$$

See that:

$$\begin{aligned}
q_1\left(T^{-1}\right) = 0 &\implies T^k q_1\left(T^{-1}\right) = 0 \\
&\implies T^k \sum_{l=0}^{k} a_l T^{-l} = 0 \\
&\implies \sum_{l=0}^{k} a_l T^{k-l} = 0 \\
&\implies a_0 T^k + \ldots a_k I = 0
\end{aligned}$$

This means the polynomial $p_1(z) = a_0 z^k + \cdots + a_k$ is a polynomial such that $p_1(T) = 0$. This means that by theorem 1 we have that p_1 is a polynomial multiple of p as p is the minimal polynomial of T. However since p_1 has smaller degree than p, which would mean that the minimal polynomial of T would have degree k or lower. But, this contradicts the fact that p is the minimal polynomial of T. Thus, we have that q is of minimal degree therefor, q as given by equation 1 is the minimal polynomial of T^{-1} as desired.

2. Answer the following

(a) Is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ and $A^n = 0$? Give an example to show such a matrix exists (and explain why it satisfies both conditions), or disprove it.

Solution:
Relevant Theorems:
Theorem 2. *Axler 8.11*
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = null\left(T - \lambda I\right)^{dimV}$ (The generalized eigenspace associated with lambda).
Theorem 3. *Axler 8.18*
Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $N^{dimV} = 0$
Intuition Buildup: The phrasing of this problem is very similar to theorem 2, and seems to be phrased in a way to show an example of an operator that we know exists based on this theorem as well. In addition, the kind of matrices to look at are going to be nilpotent as we are looking at a matrix raised to a power becoming 0. In addition, we want to be able to talk about powers of A, so we would want to try the Jordan Form.

Solution: Such a matrix exists. Consider the matrix A such that for all i, j:

$$A_{ij} = \begin{cases} 0 & \text{if } j = i + 1 \\ 1 & \text{otherwise} \end{cases}$$

This matrix is nilpotent, and is already in Jordan Form. Since it is a single Jordan block of size n associated with the eigenvalue 0, it has minimal polynomial $p(z) = z^n$. This means that if we plug in A, we have that $A^n = 0$ by definition. So we satisfy the second condition. Next we also know that A^{n-1} is not 0. This is because if it was, then the minimal polynomial of A would have degree $n - 1$ or less, which contradicts our construction of A. Thus A satisfies both of our conditions as desired.

(b) Show that an $n \times n$ upper triangular matrix with $A^n \neq 0$ and $A^{n+1} = 0$ does not exist.

Solution:
Intuition Buildup: Since we don't start with a lot of information about the matrix, it's a good idea to start with Contradiction. So if we assume such a matrix exists, we want to know things about powers of A an unknown matrix, so it's a good place to start with eigenvalues and eigenvectors. Let's look at what we can say about an eigenpair of a matrix such that $A^{n+1} = 0$. We know that we have eigenvalues and eigenvectors because A is upper triangular. So if (v, λ) is an eigenpair and $A^{n+1} = 0$. Then we have that

$$\begin{aligned} A^{n+1} = 0 &\implies A^{n+1}v = 0 \\ &\implies \lambda^{n+1}v = 0 \\ &\implies \lambda = 0 \text{ or } v = 0. \end{aligned}$$

So we would have that either all eigenvalues are 0 or each eigenvector is 0. The latter is impossible as eigenvectors cannot be 0. So we have an upper triangular matrix with only $0's$ on the diagonal. This means that A is nilpotent. From theorem 3, this means that $A^n = 0$, however this contradicts our assumption that $A^n \neq 0$.

Solution: We will demonstrate this by contradiction. Assume that such an upper triangular matrix exists. Since A is upper triangular, we know that A has eigenvalues, and all the eigenvalues are on the diagonal. Let (λ, v) be an eigenvalue with associated eigenvector. See that

$$Av = \lambda v \implies A^{n+1}v = \lambda^{n+1}v$$

So, we have that by assumption of $A^{n+1} = 0$, that $\lambda^{n+1}v = 0$. This gives us that either $\lambda^{n+1} = 0$ or $v = 0$. Since v is an eigenvector, v cannot be 0. This leaves us with $\lambda^{n+1} = 0$, which means $\lambda = 0$. Since (λ, v) was an arbitrary eigenpair, all the eigenvalues of A are 0. Since A is upper triangular, all it's eigenvalues are on the diagonal. Finally, we have that A is a nilpotent matrix as it fits the form of Theorem 8.19 in Axler, and by theorem 3 we know that $A^n = 0$. However, this contradicts the fact that $A^n \neq 0$. Thus, an upper triangular matrix with $A^n \neq 0$ and $A^{n+1} = 0$ cannot exist.

3. Let T be a linear map on a vector space V and $\dim V = n$

(a) If for some vector v, the vectors v, Tv, $\ldots$, $T^{n-1}v$ are linearly independent, show that every eigenvalue of T has only one corresponding eigenvector up to a scalar multiplication

Solution:
Intuition: For this part, it is a little more informative to take a more Matrix Algebra approach. For starting, we are given a linearly independent list of length equal to the dimension of our vector space, so we are given a basis. Next, we want to look at the matrix representation of this operator

with respect to this basis. Our matrix look like

$$\begin{array}{cc} & \begin{matrix} v & Tv & \dots & T^{n-2}v & T^{n-1} \end{matrix} \\ \begin{matrix} v \\ Tv \\ T^2v \\ \vdots \\ T^{n-2} \\ T^{n-1} \end{matrix} & \begin{bmatrix} 0 & 0 & \dots & 0 & a_{1,n} \\ 1 & 0 & \dots & 0 & a_{2,n} \\ 0 & 1 & \dots & 0 & a_{3,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \\ 0 & 0 & \dots & 1 & a_{n,n} \end{bmatrix} \end{array}$$

We know the first $n-1$ columns of $A = \mathcal{M}(A, (v, Tv, \dots, T^{n-1}))$ based on how the basis vectors are defined. In addition, we don't know what the final column looks like but we can make arguments about what this column looks like once we talk about the eigenvalues and eigenvectors. Let λ be an eigenvalue of T (note: we don't necessarily know that T has eigenvalues, but we need to only make arguments about any eigenvalues that may exist). Next we need to look at null $(A - \lambda I)$ and then argue that we can only have one basis vector of this space.
Consider $A - \lambda I$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & \dots & 0 & a_{1,n} \\ 1 & -\lambda & \dots & 0 & a_{2,n} \\ 0 & 1 & \dots & 0 & a_{3,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda & a_{n-1,n} \\ 0 & 0 & \dots & 1 & a_{n,n} - \lambda \end{bmatrix}$$

If we look at the first $n-1$ columns these are linearly independent. We can see this visually by thinking about how we would cancel out terms of the k^{th} column using other ones. Since each of these columns provides information in the k^{th} and $k+1^{\text{th}}$ column in order to cancel out these terms we would need another vector that will introduce a new term in either the $k-1^{\text{th}}$ or $k+2^{\text{th}}$ element. A statement like this should be mostly sufficient on a prelim to get at least most of the points. However below we will walk through how we would prove this statement anyway. Since the first $n-1$ are linearly independent, then we know that since λ is an eigenvalue, we have that $\dim \text{null } (A - \lambda I) \geq 1$ and $\dim \mathsf{range} \ (A - \lambda I) \geq n - 1$. So, the fundamental theorem of linear maps, we have that

$$n = \dim \text{null } (A - \lambda I) + \dim \mathsf{range} \ (A - \lambda I)$$

. Therefore, both of these dimensions must take their minimal values. As otherwise, the above equation would be impossible to make true. Thus we have that

$$\dim \text{null } (A - \lambda I) = 1.$$

So there can only be one eigenvector associated with this eigenvalue. Since λ was an arbitrary eigenvalue, this must hold for all eigenvalues of A. So, we have that there can only be one eigenvector (up to scalar multiplication) associated with each eigenvalue.

<u>Solution</u>:

Since the vectors $v, Tv, \dots T^{n-1}v$ are linearly independent, and we have n of them, we have a basis of V. Consider the matrix representation of T with respect to this basis, we get that

$$A = \mathcal{M}\left(T, (v, Tv, \dots, T^{n-1}v)\right) = \begin{bmatrix} 0 & 0 & \dots & 0 & a_{1,n} \\ 1 & 0 & \dots & 0 & a_{2,n} \\ 0 & 1 & \dots & 0 & a_{3,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \\ 0 & 0 & \dots & 1 & a_{n,n} \end{bmatrix}.$$

Let λ be an eigenvalue of T. This means that we have $\dim \text{null } (A - \lambda I) \geq 1$. Next, we will argue that this must be exactly 1. We do this by arguing that $\dim \mathsf{range} \ (A - \lambda I) \geq n - 1$. So, consider the matrix $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & \dots & 0 & a_{1,n} \\ 1 & -\lambda & \dots & 0 & a_{2,n} \\ 0 & 1 & \dots & 0 & a_{3,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda & a_{n-1,n} \\ 0 & 0 & \dots & 1 & a_{n,n} - \lambda \end{bmatrix}$$

We will now argue that the first $n-1$ columns of this matrix are linearly independent. Let a_k be the k^{th} column of $A - \lambda I$, and $c_1, \dots, c_{n-1}$ be coefficients such that $\sim_{k=1}^{n-1} c_k a_k = 0$. This leaves us with the following system of equations

$$\begin{aligned} -\lambda c_1 &= 0 \\ c_1 - \lambda c_2 &= 0 \\ &\vdots \\ c_{n-2} - \lambda c_{n-1} & \\ c_{n-1} &= 0. \end{aligned}$$

We can see that if perform backwards substitution that we start with $c_{n-1} = 0$ and as we go up our system, we will get that each of these coefficients must be exactly 0 (Note: we never use the first equation, so we need not consider the case of $\lambda = 0$.

Thus, we have that these columns are linearly independent, so this means that that

$$\text{rank } (A - \lambda I) = \text{number of linearly independent columns of } A \geq n - 1$$

. Since we have that the reank is at least $n-1$ and this is exactly the dimension of the range, we know from the Fundamental Theorem of Linear Maps that rank $(A - \lambda I) = n - 1$ as if it were any larger we would get:

$$n = \dim \mathsf{range}\ (A - \lambda I) + \dim \text{null } (A - \lambda I) \geq n + 1$$

which is impossible to do. Thus, we know that $\dim \text{null } (A - \lambda I) = 1$. So we can only have one eigenvector associated with λ (up to scalar multiplication). Since λ was an arbitrary eigenvalue of T, we know that this holds for all eigenvalues as desired.

(b) If T has n distinct eigenvalues and vector u is a sum of n eigenvectors, corresponding to the distinct eigenvalues, show that u, Tu, ..., $T^{n-1}u$ are linearly independent (and thus form a basis of V).

Solution:
<u>Intuition</u>: For this problem, we first want to write each of the given vectors in terms of the n eigenvectors. From Axler Theorem 5.10, these eigenvectors are linearly independent as they are associated with distinct eigenvectors (It's not worded in the most precise way, but the intent is that the eigenvectors are associated with different eigenvalues. If you are unsure about what a problem means ask the proctor, and they will more than likely answer. If they do not, just be clear about what you are assuming for the problem at the start, and you shouldn't lose too many points).

So, we look at what each of these vectors look like in terms of the eigenvectors $u_1, \dots, u_n$. See that for $k = 1, \dots, n$

$$T^k(u) = \sum_{l=1}^{n} \lambda_l^k u_l$$

Next, we need to show that these vectors are linearly independent. Assume that we have coefficients

$c_1, \ldots, c_n$ such that

$$\sum_{k=1}^{n} c_k T^{k-1} u = 0$$

Simplifying this equation, and grouping together based on the eigenvectors we get

$$\begin{aligned} 0 &= \sum_{k=1}^{n} c_k \sum_{\ell=1}^{n} \lambda_\ell^{k-1} u_\ell \\ &= c_1 \sum_{\ell=1}^{n} u_\ell + c_2 \sum_{\ell=1}^{n} \lambda_\ell u_\ell + \cdots + c_n \sum_{\ell=1}^{n} \lambda_\ell^{n-1} u_\ell \\ &= (c_1 + c_2\lambda_1 + \cdots + c_n\lambda_1^{n-1})u_1 + \cdots + (c_1 + c_2\lambda_n + \cdots + c_n\lambda_n^{n-1})u_n \end{aligned}$$

Since we know that these eigenvectors are linearly independent (Axler 5.10) We know that each of these sums must be 0. So we are left with the following system of equations

$$\begin{aligned} c_1 + c_2\lambda_1 + \cdots + c_n\lambda_1^{n-1} &= 0 \\ &\vdots \\ c_1 + c_2\lambda_n + \cdots + c_n\lambda_n^{n-1} &= 0 \end{aligned}$$

However, we notice that this is actually the same as solving the following matrix vector product

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice also that our matrix is the Vandermonde matrix, which is invertible as long as we have that each of these λ_k values are distinct (which we have by assumption!), and also notice that this matrix vector product is asking us to find the nullspace of an invertible matrix, so that means that each c_k must be exactly 0. thus the vectors $u, Tu, \ldots, T^{n-1}u$ are linearly independent as desired.

Note: Knowing properties about famous matrices like Hilbert, Vandermonde, etc is expected and good to know in general.

Solution: Let $\lambda_1, \ldots, \lambda_n$ be our n distinct eigenvalues, with associated eigenvectors $u_1, \ldots, u_n$ where $u = u_1 + \cdots + u_n$ (ie we are picking the eigenvectors that make up the sum of u). Consider $T^k u$ for $k = 0, \ldots, n-1$, and see that

$$\begin{aligned} T^k u &= \sum_{\ell=1}^{n} T^k u_\ell \\ &= \sum_{\ell=1}^{n} \lambda_\ell^k u_\ell \end{aligned}$$

Next, for $j = 1, \ldots, n$ let c_j be coefficients such that

$$\sum_{j=1}^{n} c_j T^{j-1} u = 0$$

See that by expanding out our above equation we get:

$$\begin{aligned} 0 &= \sum_{j=1}^{n} c_j \sum_{\ell=1}^{n} \lambda_\ell^{j-1} u_\ell \\ &= c_1 \sum_{\ell=1}^{n} u_\ell + c_2 \sum_{\ell=1}^{n} \lambda_\ell u_\ell + \cdots + c_n \sum_{\ell=1}^{n} \lambda_\ell^{n-1} u_\ell \\ &= (c_1 + c_2\lambda_1 + \cdots + c_n\lambda_1^{n-1})u_1 + \cdots + (c_1 + c_2\lambda_n + \cdots + c_n\lambda_n^{n-1})u_n \end{aligned}$$

Since we know that these eigenvectors are linearly independent (Axler 5.10) We know that each of these sums must be 0. So we are left with the following system of equations

$$\begin{aligned} c_1 + c_2\lambda_1 + \cdots + c_n\lambda_1^{n-1} &= 0 \\ &\vdots \\ c_1 + c_2\lambda_n + \cdots + c_n\lambda_n^{n-1} &= 0 \end{aligned}$$

However, we notice that this is actually the same as solving the following matrix vector product

$$\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice also that our matrix is the Vandermonde matrix, which is invertible as long as we have that each of these λ_k values are distinct (which we have by assumption!), and also notice that this matrix vector product is asking us to find the nullspace of an invertible matrix, so that means that each c_k must be exactly 0. thus the vectors $u, Tu, \ldots, T^{n-1}u$ are linearly independent as desired.

4. Let $A \in \mathcal{M}_n(\mathbb{C})$ and λ be an eigenvalue of A.

Solution: For this problem, I am omitting the intuition section, see the committee's solutions for a good intuition building approach. Essentially, we are only given that λ is an eigenvalue of A so we can't do anything aside from trying to apply powers to A or other properties. One exception is part c, but we get the idea to look at the minimal polynomial as its roots are exactly the eigenvalues.

(a) Show that λ^r is an eigenvalue of A^r for $r \in \mathbb{N}$.

Solution: Let $r \in \mathbb{N}$ and v be an eigenvector associated with λ, since λ is an eigenvalue of A, we see that:

$$\begin{aligned} A^r v &= \underbrace{A \ldots A}_{r \text{ times}} v \\ &= \lambda \underbrace{A \ldots A}_{r-1 \text{ times}} v \\ &\vdots \\ &= \lambda^r v \end{aligned}$$

Thus, λ^r is an eigenvalue of A^r as desired.

(b) Provide an example showing that the multiplicity of λ^r as an eigenvalue of A^r may be strictly higher than the multiplicity of λ as an eigenvalue of A

Solution: Consider the permutation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A has eigenvalues -1 and 1. We see this from

$$\begin{aligned}\det(A-\lambda I)=0 &\implies \begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0\\ &\implies \lambda^2-1=0\\ &\implies \lambda^2=1\\ &\implies \lambda=\pm 1\end{aligned}$$

However, if we compute A^2, we get $A^2 = I$ which only has eigenvalue 1. So in A 1 has multiplicity 1 while in A^2, $1^2 = 1$ has multiplicity 2.

(c) Show that $A^\top$ has the same eigenvalues as A.

Solution: Let $p(t)$ be the characteristic polynomial of A. Write $p(z)$ in the form

$$p(z)=\sum_{k=0}^{n} c_k z^k$$

for some coefficients c_k. We also know from the definition of the characteristic polynomial, that $p(A) = 0$. See also that

$$\begin{aligned}p(A)=0 &\implies \sum_{k=0}^{n} c_k A^k = 0\\ &\implies \left(\sum_{k=0}^{n} c_k A^k\right)^\top = 0^\top\\ &\implies \sum_{k=0}^{n} c_k \left(A^k\right)^\top = 0\\ &\implies \sum_{k=0}^{n} c_k \left(A^\top\right)^k = 0\\ &\implies p(A^\top)=0\end{aligned}$$

Since p is also a polynomial of degree n such that $p(A^\top) = 0$, the roots of p are also the eigenvalues of $A^\top$, thus A and $A^\top$ share eigenvalues.

(d) Show that if A is orthogonal, then $\frac{1}{\lambda}$ is an eigenvalue of A.

Solution: **Note**: This problem likely has a typo. Instead of orthogonal, it should probably be hermitian, however the standard definition of orthogonal does work in this context, it's just an odd choice to phrase it this way.

Solution: First, we show that if λ is an eigenvalue of A, and A^{-1} exists, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Let λ, v, be an eigenvalue with corresponding eigenvector. See that

$$\begin{aligned}Av=\lambda v &\implies A^{-1}Av = A^{-1}\lambda v\\ &\implies v=\lambda A^{-1}v\\ &\implies \frac{1}{\lambda}v = A^{-1}v\end{aligned}$$

Thus, we have that $\frac{1}{\lambda}$, is an eigenvalue of A^{-1}. Recall that since we are assuming that A is orthogonal, then $A^\top = A^{-1}$, so from part (c), $\frac{1}{\lambda}$ is an eigenvalueof A as desired provided that A orthogonal and λ is an eigenvalue of A.

2 Axler Problems

1. Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is normal. Prove that the minimal polynomial of T has no repeated zeros

Solution:
Note: This problem is implicitly assuming that V is a complex vector space. If we restrict this $\mathbb{R}$, then we can construct a normal matrix that is not diagonalizable. This is a good exercise to construct such a matrix. Our proof relies on diagonalizability, and so does the proof that Axler gives in his solutions manual. A problem like this on the prelim will probably assume $\mathbb{C}$ if phrased this way but it is much more ambiguous if the question said something to the effect of "prove or disprove".

Intuition: Recall that the powers of the terms in our minimal polynomials are the size of the largest block in our Jordan Canonical Form. This is not a directly stated theorem from the textbook but it is a fact that is consistently taken advantage of on previous prelim solutions as well as in most lectures that I have heard about. This concept is a consequence that we can make based on the proof of Theorem 8.60 in Axler based on the direct sums of the generalized eigenspace statements. However this is safe to assume as far as the prelims have been concerned up to know, but as always ask the proctor and/or committee for clarification on this if you are unsure.

So, since T is normal, there exists an orthonormal basis of V consisting of eigenvectors of T. So the matrix representation of T with respect to this basis is diagonal. It is also in Jordan form with all blocks of size 1, so the minimal polynomial will be of the form $(z-\lambda_1)\cdots(z-\lambda_k)$ where k is the number of unique eigenvalues, thus the minimal polynomial of T cannot have reapeated zeros as desired.

Solution: Since V is a complex vector space, and T is normal, we have an orthonormal basis of V consisting of eigenvectors of T. If we consider the matrix representation of T with respect to this basis, then we get a diagonal matrix of the form

$$\begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

Where for $k = 1, \dots, n$ each λ_k need not be unique. We also see that this matrix is also in Jordan form, with all block sizes equal 1. This tells us that for each unique eigenvalue, the associated term in the minimal polynomial will have degree 1 as the degree of each term is the largest block size. Thus our minimal polynomial will be of the form $p(z) = (z - \lambda_1)\cdots(z - \lambda_m)$ where m is the number of distinct eigenvalues. This means that p has no repeated zeros as desired.

Alternative Solution: There is another solution available via the solution to 8.C.12 in Axler's solutions. I don't follow the reason behind some of the steps, but it should be acceptable.

3 Other Problems

These are from the Berkely Problems pdf.

1. (7.5.16) Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B share a common eigenvector in $\mathbb{R}^n$.

Solution:

Intuition: We have two real symmetric matrices, so it is safe to assume that the real spectral theorem is going to be a good approach for us. IE that these matrices are diagonalizable and have eigenvalues. Let λ be an eigenvalue with associated eigenvector v.

See that

$$\begin{aligned} Av = \lambda v &\implies BAv = B(\lambda v) \\ &\implies ABv = \lambda Bv \end{aligned}$$

The second line comes from the fact that $AB = BA$.

Now, we have that Bv is an eigenvector associated with eigenvalue λ of A. This means that $Bv \in E(\lambda, A)$ ie Bv is in the eigenspace associated with λ of A. Since B is invariant in this subspace, and B is diagonalizable, there must exist some $u \in E(\lambda, A)$ such that for some eigenvalue of B denoted λ', $Bu = \lambda' u$. As if such a vector did not exist, then there would be a set of vectors in $\mathbb{R}^n$ that cannot be represented using eigenvectors of B. Thus u is an eigenvector of B, and since $u \in E(\lambda, A)$, we have that u is also an eigenvector of A associated with eigenvalue λ.

Note: This is a type of problem that I have seen variations of before. It's essentially testing how comfortable you are with invariant subspaces of a vector space and consequences of diagonalizability.

Solution: Since A and B are real symmetric matrices, we know that both of these matrices are diagonalizable. This means that there exists a basis of $\mathbb{R}^n$ that consists of eigenvectors of A and another basis consisting of eigenvectors of B. Let λ_A be an eigenvalue of A and let $v \in \mathbb{R}^n$ be an associated eigenvector. We see that from the definition of eigenvectors and how A and B are defined that

$$\begin{aligned} Av = \lambda_A v &\implies BAv = B(\lambda_A v) \\ &\implies ABv = \lambda_A Bv \end{aligned}$$

So, we have that if v is an eigenvector of A, then Bv is an eigenvector of A associated with the same eigenvalue. In other words, $v \in E(\lambda_A, A) \implies Bv \in E(\lambda_A, A)$. This gives us that B is invariant over the eigenspace of A associated with eigenvalue λ_A. From here, we will show that there exists some $u \in E(\lambda_A, A)$ such that $Bu = \lambda_B u$ for some eigenvalue of B denoted λ_B.

Since B is real symmetric, if we restrict the inputs to be only in $E(\lambda_A, A)$ it is still symmetric as any vector in $E(\lambda_A, A)$ is also in $\mathbb{R}^n$. Thus, B has an eigenvalue λ_B such that $Bu = \lambda_B u$ for some $u \in E(\lambda_A, A)$. Since $u \in E(\lambda_A, A)$, we also have that $Au = \lambda_A u$, thus u is an eigenvector of both A and B as desired.

Note 2: The above proof is essentially a more fleshed out version of an existing proof in the Berkeley PDF.