Bin 4 Problems

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This document holds problems that fit into the Bin 4 according to the prelim syllabus. Or are approached in a way most compatible with Bin 4

Contents

1 Prelim Problems

1. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$
   (a) (5 points) Prove T is invertible
   (b) (15 points) Find the minimal polynomial of T.
2. Answer the following
   (a) Is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ and $A^n = 0$? Give an example to show such a matrix exists (and explain why it satisfies both conditions), or disprove it.
   (b) Show that an $n \times n$ upper triangular matrix with $A^n \neq 0$ and $A^{n+1} = 0$ does not exist.
3. Let T be a linear map on a vector space V and $\dim V = n$
   (a) If for some vector v, the vectors $v, Tv, \ldots, T^{n-1}v$ are linearly independent, show that every eigenvalue of T has only one corresponding eigenvector up to a scalar multiplication
   (b) If T has n distinct eigenvalues and vector u is a sum of n eigenvectors, corresponding to the distinct eigenvalues, show that $u, Tu, \ldots, T^{n-1}u$ are linearly independent (and thus form a basis of V).
4. Let $A \in \mathcal{M}_n(\mathbb{C})$ and λ be an eigenvalue of A.
   (a) Show that λ^r is an eigenvalue of A^r for $r \in \mathbb{N}$.
   (b) Provide an example showing that the multiplicity of λ^r as an eigenvalue of A^r may be strictly higher than the multiplicity of λ as an eigenvalue of A
   (c) Show that $A^\top$ has the same eigenvalues as A.
   (d) Show that if A is orthogonal, then $\frac{1}{\lambda}$ is an eigenvalue of A.

2 Axler Problems

1. Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is normal. Prove that the minimal polynomial of T has no repeated zeros

3 Other Problems

These are from the Berkely Problems pdf.

1. (7.5.16) Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B share a common eigenvector in $\mathbb{R}^n$.