Bin 3 Problems

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This document holds problems that fit into the Bin 3 according to the prelim syllabus. Or are approached in a way most compatible with Bin 3

Contents

1 Prelim Problems

1. Let A be a Hermitian $n \times n$ complex matrix. Show that if $\langle Av, v\rangle \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T such that $A = T^*T$.

2. Let T be a positive operator on a complex inner product space V and S be an operator on V such that $ST = -TS$. Show that $ST = TS = 0$.

3.
   (a) Let T be an idempotent operator on an n-dimensional vector space V; that is $T^2 = T$, show that
      1. $V = \mathsf{range}\, T \oplus \operatorname{null} T$.
      2. $\mathsf{trace}\, T = \dim \mathsf{range}\, T$

   (b) Let $T_1, \ldots, T_m$ be idempotent operators on an n-dimensional vector space V. Show that if

   $$T_1 + \cdots + T_m = I$$

   Then

   $$V = \mathsf{range}\, T_1 \oplus \cdots \oplus \mathsf{range}\, T_m$$

   and

   $$T_i T_j = 0, \qquad i, j = 1, \ldots, m, i \neq j$$

4. Prove or give a counterexample to each of the following statements:
   (a) Let $T \in \mathcal{L}\left(R^3\right)$, and $\dim \operatorname{null} T \cap \mathsf{range}\, T \geq 1$. Then T is nilpotent.
   (b) Let $T \in \mathcal{L}\left(R^4\right)$, and $\dim \operatorname{null} T \cap \mathsf{range}\, T \geq 2$. Then T is nilpotent.

5. Let V be an n-dimensional vector space, and let $T_1, \ldots, T_{n+1} \in \mathcal{L}(V)$ such that

   $$T_i T_j = T_j T_i \qquad \text{for every } 1 \leq i \leq j \leq n+1 \text{ (the operators commute), and} \tag{2}$$
   $$T_1 \cdots T_{n+1} = 0 \tag{3}$$

   (a) Show that there exists some k such that $T_1 \cdots T_{k-1} T_{k+1} \cdots T_{n+1} = 0$ as follows:
   Show that for every k, we have
      1. $\mathsf{range}\, T_1 \cdots T_k \subset \mathsf{range}\, T_1 \cdots T_{k-1}$, and

2. $\mathsf{range}\, T_1 \cdots T_k \subset \operatorname{null} T_{k+1} \cdots T_n$

Then argue that for some k, we must have equality in (1.), and explain why this implies the desired statement

(b) Show that (2) is necessary for the previous conclusion by providing three operators (or matrices) $T_1, T_2, T_3 \in \mathcal{L}\left(\mathbb{R}^2\right)$ with $T_1 T_2 T_3 = 0$, but $T_1 T_2 \neq 0$, $T_1 T_3 \neq 0$, and $T_2 T_3 \neq 0$.