Bin 2 Problems

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This document holds problems that fit into the Bin 2 according to the prelim syllabus. Or are approached in a way most compatible with Bin 2

Contents

1 Prelim Problems

1. Let u be a unit vector in an $n-$ dimensional inner product space V over $\mathbb{R}$. Define $T \in \mathcal{L}(V)$ as:

$$T(x) = x - 2\langle x, u\rangle u, \quad x \in V$$

   Show that

   (a) T is an isometry

   (b) If $A = \mathcal{M}(T)$ is a matrix representation of T, then $\det A = -1$

   (c) If $S \in \mathcal{L}(V)$ is an isometry with 1 as an eigenvalue, and if the eigenspace of 1 is of dimension $n-1$, then there exists some $w \in V$ where w is a unit vector and for all $x \in V$:

$$S(x) = x - 2\langle x, w\rangle w$$

2. Let V be a finite dimensional real vector space with basis $e_1, \ldots, e_n$ (the standard basis of $\mathbb{R}^n$)

   (a) Let A be a positive definite bijective matrix in V (This means A is a matrix representation of some invertible linear operator in $\mathcal{L}(V)$). For any $v, w \in V$, expressed as coordinate vectors according to this basis (their standard representation if you were to write them down), define

$$\langle v, w\rangle := v^\top Aw.$$

   Show that this is an inner product.

   (b) Let $\langle \cdot, \cdot\rangle$ be an inner product in V. Define A to be a matrix such that $A_{ij} = \langle e_i, e_j\rangle$ is a positive bijective matrix such that $\langle v, w\rangle = v^\top Aw$.

3. V be a finite-dimensional inner product space over $\mathbb{C}$. Let T be a normal operator on V. Let $\lambda \in \mathbb{C}$ and let $v \in V$ be a unit vector (ie $\|v\| = 1$). Prove that T has an eigenvalue λ' such that

$$\|\lambda - \lambda'\| \leq \|Tv - \lambda v\|.$$

2 Axler Problems

1. Suppose $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are inner products on V over a field $\mathbb{F}$ such that for all $u, v \in V$, $\langle u, v \rangle_1 = 0 \iff \langle u, v \rangle_2 = 0$. Prove that there is a positive number c such that $\langle u, v \rangle_1 = c \langle u, v \rangle$ for all $u, v \in V$.

2. Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.