Bin 3 Problems

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This document holds problems that fit into the Bin 3 according to the prelim syllabus. Or are approached in a way most compatible with Bin 3

Contents

1 Prelim Problems

1. Let A be a Hermitian $n \times n$ complex matrix. Show that if $\langle Av, v \rangle \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T such that $A = T^*T$.

Solution:

<u>Intuition</u>: As is a pattern so far for most of these problems, we want to use the fact that A is diagonalizable. Since we are speaking in terms of matrices, we will use that flavor of diagonalizability. Since A is Hermitian, the Complex Spectral Theorem guarantees that we have an orthonormal list of vectors, denoted $v_1, \ldots, v_n$ with n not necessarily unique elements of $\mathbb{C}$ denoted $\lambda_1, \ldots, \lambda_n$ such that:

$$\begin{gathered} Av_1 = \lambda_1 v_1 \\ \vdots \\ Av_n = \lambda_n v_n \end{gathered}$$

However since A is hermitian (self-adjoint in Axler's terms) we know from Axler 7.13 that the eigenvalues, $\lambda_1, \ldots, \lambda_n$ are actually real! (This is needed to ensure uniquenss of the T we find, but this problem does require us to show it is unique)

Another way to view this is as

$$A = VDV^*$$

Where V is a matrix whose columns are the eigenvectors of A and D is a diagonal matrix with the eigenvalues of A on the diagonal. So if we can construct a second matrix $S = \sqrt{D}$, then we can define our desired matrix as

$$T = VSV^*$$

Then we would have

$$T^*T = VSV^*VSV^* = VS^2V^* = VDV^* = A$$

So, let's define S to be the matrix as:

$$S_{ij} = \begin{cases} \sqrt{\lambda_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Notice that when we multiply two diagonal matrices, we multiply their diagonals. So

$$S_{ij}^2 = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

which is the same as D! Now, define the matrix T as follows

$$T = VSV^*$$

Next, we see that

$$\begin{aligned} T^* &= (VSV^*)^* \\ &= (V^*)^* S^* V^* \\ &= VSV^* \\ &= T \end{aligned}$$

Finally, we have

$$\begin{aligned} T^*T &= T^2 \\ &= VSV^*VSV^* \\ &= VS^2V^* \\ &= VDV^* \\ &= A \end{aligned}$$

as desired.

Solution:

Since A is hermitian, A is orthogonally diagonalizable via the complex spectral theorem, and since A is Hermitian it also has only real eigenvalues. Denote the eigenvectors with associated eigenvalues as $v_1, \ldots, v_n$ and $\lambda_1, \ldots, \lambda_n$ respectively.

Define the matrix V such that

$$V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$

Also define the matrix D as

$$D_{ij} = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This means that the columns of V are the eigenvectors of A. From A being diagonalizable, we know that

$$A = VDV^*$$

Next, define the matrix S as

$$S_{ij} = \begin{cases} \sqrt{\lambda_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(Note the above square root is the real square root).

See that since S is diagonal, it is symmetric. In addition based on our construction of S, we have $S^2 = D$.

Finally, define the matrix T as:

$$T = VSV^*$$

Now, we will show that $T^*T = A$.

$$\begin{aligned} T^*T &= (VSV^*)^* VSV^* \\ &= VS^*V^*VSV^* \\ &= VS^2V^* \\ &= VDV^* \\ &= A. \end{aligned}$$

Thus, we have shown that there exists some matrix T such that $T^*T = A$ as desired.

2. Let T be a positive operator on a complex inner product space V and S be an operator on V such that $ST = -TS$. Show that $ST = TS = 0$.

Solution: **<u>Intuition</u>**: Since in this problem, we are not given much information about S, T aside from the fact that T is positive and S and T nearly commute. So we should look at eigenvalues and eigenvectors! Since T is positive, we know it has non-negative eigenvalues and is self adjoint from Axler 7.35. So from the Complex spectral theorem, we have that T is diagonalizable in an orthonormal basis consisting of eigenvectors of T. Let λ be an eigenvale of T with associated eigenvector v. This means $Tv = \lambda v$. However we also know that

$$\begin{aligned} \lambda v = Tv &\iff S\lambda v = STv \\ &\iff \lambda Sv = STv \\ &\iff \lambda Sv = -TSv \\ &\iff -\lambda Sv = TSv \end{aligned}$$

So we have two cases. The first case is that $Sv = 0$. If this is the case, then $STv = -TSv = 0$. However assume that $Sv \neq 0$. This means that Sv is an eigenvector of T associated with eigenvalue $-\lambda$. Since all eigenvalues of T are non-negative we have that both

$$\begin{aligned} \lambda &\geq 0 \\ -\lambda &\geq 0 \end{aligned}$$

This means that $\lambda = 0$. Then, $STv = -TSv = 0$.

Since we know that for all eigenvectors of T, we have that $TSv = STv = 0$, we know this also holds for a basis consisting of eigenvectors of T, so $TS = ST = 0$ as desired.

<u>Solution</u>: Since T is positive, by Axler 7.35 it is both self-adjoint and has non-negative eigenvalues. Since T is self adjoint, we know that there exists an orthonormal basis of V consisting of eigenvectors of T. Denote this basis $v_1, \dots, v_n$. Now, let (λ, v) be an arbitrary eigenpair of T. IE $Tv = \lambda v$. We see that

$$\begin{aligned} \lambda v = Tv &\iff S\lambda v = STv \\ &\iff \lambda Sv = STv \\ &\iff \lambda Sv = -TSv \\ &\iff TSv = -\lambda Sv \end{aligned}$$

From here, we have 2 cases for Sv, the first being that $Sv = \mathbf{0}$. If this is the case then $STv = TSv = \mathbf{0}$. Otherwise, $Sv \neq 0$. This means that Sv is an eigenvector of T associated with $-\lambda$. However, since T has non-negative eigenvalues we know

$$\begin{aligned} \lambda &\geq 0 \\ -\lambda &\geq 0 \end{aligned}$$

So, $\lambda = 0$. In this case we know $STv = TSv = 0$. Since we have that for all eigenvectors of T denoted v, $STv = TSv = 0$. We also know this is the case for $(v_1, \dots, v_n)$ (our basis of eigenvectors of T). Since for all $k = 1, \dots, n$, $STv_k = TSv_k = 0$, $ST = TS = 0$ as desired.

3.

(a) Let T be an idempotent operator on an n-dimensional vector space V; that is $T^2 = T$, show that

1. $V = \mathsf{range}\, T \oplus \operatorname{null} T$.
2. $\mathsf{trace}\, T = \dim \mathsf{range}\, T$

Solution:

Intuition: For this problem, there is a bit of a trick to it that would be a little hard to spot if you are not familiar with indepotent operators from a previous class. This is actually a fairly well known property of idempotent matrices/operators. The trick is to show that $V = \mathsf{range}\, T + \operatorname{null} T$. We can see this by looking at the following equation for any $v \in V$:

$$v = \underbrace{Tv}_{\text{This is in the range}} + \underbrace{(v - Tv)}_{\text{We will show this is in the nullspace}}$$

The second part of the above equation is in the nullspace because see that if we apply T to this vector, we have

$$\begin{aligned} T(v - Tv) &= Tv - T^2 v \\ &= Tv - Tv \qquad \text{This is because } T^2 = T \\ &= 0 \end{aligned}$$

So we have that any $v \in V$ can be represented as a sum of an element from the range and an element from the nullspace. Now all that is left is to show that this sum is a direct sum. From Axler 1.45, we need only show that $\mathsf{range}\, T \cap \operatorname{null} T = \{0\}$. So we will do just that

Let $v \in \mathsf{range}\, T \cap \operatorname{null} T$. Since $v \in \mathsf{range}\, T$, there exists some $u \in V$ such that $Tu = v$. Since $v \in \operatorname{null} T$, we have that $Tv = 0$. This gives us

$$\begin{aligned} 0 &= Tv \\ &= T(Tu) \\ &= T^2(u) \\ &= Tu \\ &= v \end{aligned}$$

So $v = 0$. Thus the sum of these spaces is a direct sum as required.

Next is to show that the trace of our operator is equal to the dimension of the range.

We will do this by showing T has 1 as an eigenvalue with multiplicity $\dim \mathsf{range}\, T$ and 0 as an eigenvalue with multiplicity $\dim \operatorname{null} T$, which will make up all n eigenvalues that we could have for our operator.

First let $\ell = \dim \mathsf{range}\, T$ and $k = \dim \operatorname{null} T$. Let $u_1, \dots, u_\ell$ be a basis of $\mathsf{range}\, T$ and $v_1, \dots, v_k$ be a basis of $\operatorname{null} T$.

Since for $i = 1, \dots, \ell$, u_i is in $\mathsf{range}\, T$, we know that there exists some $w_i \in V$ such that $Tw_i = u_i$. So we also have that

$$\begin{aligned} Tu_i &= T(Tw_i) \\ &= T^2 w_i \\ &= Tw_i \\ &= u_i \end{aligned}$$

So that means for each $i = 1, \ldots, \ell$, u_i is an eigenvector of T associated with eigenvalue 1. Clearly since $v_1, \ldots, v_k$ are a basis of the nullspace, we have that each v_j is an eigenvector associated with the eigenvalue 0. So, from Axler 10.9, we have that

$$\mathsf{trace}\, T = \sum_{i=1}^{\ell} 1 + \sum_{j=1}^{k} 0 = \ell = \dim \mathsf{range}\, T$$

as desired.

Solution:

In order to show that $V = \mathsf{range}\, T \oplus \operatorname{null} T$, we will first show that $V = \mathsf{range}\, T + \operatorname{null} T$.

Let $v \in V$. See that $v = Tv + (v - Tv)$.

Clearly, $Tv \in \mathsf{range}\, T$. Now we will show that $v - Tv \in \operatorname{null} T$.

$$\begin{aligned} T(v - Tv) &= Tv - T^2 v \\ &= Tv - Tv \qquad T^2 = T \text{ by assumption} \\ &= 0 \end{aligned}$$

Thus, v can be written as a sum of an element from $\mathsf{range}\, T$ and $\operatorname{null} T$. Since v was arbitrary, $v \in V$. Now we will show that this sum is actually a direct sum. We do this by using Axler 1.45 and show that $\mathsf{range}\, T \cap \operatorname{null} T = \{0\}$.

Let $v \in \mathsf{range}\, T \cap \operatorname{null} T$, since $v \in \mathsf{range}\, T$, there exists some $u \in V$ such that $Tu = v$. Since $v \in \operatorname{null} T$, we know that $Tv = 0$. Combining these facts, we see:

$$\begin{aligned} 0 &= Tv \\ &= T(Tu) \\ &= T^2 u \\ &= Tu \\ &= v \end{aligned}$$

So, $v = 0$. Thus this intersection is $\{0\}$ as desired. This means that our aforementioned sum is actually a direct sum.

Now, all that is left is to compute the trace. Our tactic is to use Axler 10.9 and show that T has eigenvalues 1 and 0 with multiplicty $\dim \mathsf{range}\, T$ and $\dim \operatorname{null} T$ respectively.

Let $\ell = \dim \mathsf{range}\, T$ and $k = \dim \operatorname{null} T$, and let $(u_1, \ldots, u_\ell)$ and $(v_1, \ldots, v_k)$ denote bases of $\mathsf{range}\, T$ and $\operatorname{null} T$ respectively. For $i = 1, \ldots, \ell$, since each $u_i \in \mathsf{range}\, T$, we know that there exists some $w_i \in V$ such that $Tw_i = u_i$. This means

$$\begin{aligned} Tu_i &= T(Tw_i) \\ &= T^2 w_i \\ &= Tw_i \\ &= u_i \end{aligned}$$

Thus, u_i is an eigenvector of T with associated eigenvalue 1.

Now, for $j = 1, \ldots, k$, we know $w_k \in \operatorname{null} T$, so it is an eigenvector of T with associated eigenvalue 0. Thus we found $\dim V$ linearly independent eigenvectors so we have found all possible eigenvalues. This means that we can appeal to Axler 10.9 to see that

$$\mathsf{trace}\, T = \ell * 1 + k * 0 = \ell = \dim V.$$

So, we have shown $\mathsf{trace}\, T = \dim V$ as desired.

(b) Let $T_1, \dots, T_m$ be idempotent operators on an n-dimensional vector space V. Show that if

$$T_1 + \cdots + T_m = I$$

Then

$$V = \mathsf{range}\, T_1 \oplus \cdots \oplus \mathsf{range}\, T_m$$

and

$$T_i T_j = 0, \qquad i, j = 1, \dots, m, i \neq j$$

Solution:

Intuition: For this part, we want to use the first part as it is a two part question where the second part seems to give us enough information to possibly use it. While it might look like the direct sum part may be helpful, it won't actually help us. All it could maybe do would be to give us that $\mathsf{range}\, T_2 \oplus \cdots \oplus T_m = \operatorname{null} T_1$. but remember that we can't "subtract" over the $\oplus$. (see Exercise 1.C.23)! So this isn't where it'll be helpful. We'll come back to this later after we show the direct sum works.

For the direct sum, we only know that each T_i is idempotent and that the sum is the identity operator. Let's use this fact.

Let $v \in V$. See that

$$\begin{aligned} v &= Iv \\ &= \left(\sum_{i=1}^{m} T_i \right) v \\ &= \sum_{i=1}^{m} T_i v \end{aligned}$$

So, $V = \sum_{i=1}^{m} \mathsf{range}\, T_i$. All that is left is to show this is a direct sum. Recall from Axler 3.78 that if we can show the dimensions of these operaters add up to be the dimension of V, then this sum is a direct sum.

This is where our previous part helps us out! As we can write I as the sum of our m operators, and the fact that the trace is equal to the dimension of the range as we showed above!

$$\begin{aligned} \dim V &= n \\ &= \mathsf{trace}\, I_n \\ &= \mathsf{trace} \sum_{i=1}^{m} T_i \\ &= \sum_{i=1}^{m} \mathsf{trace}\, T_i \\ &= \sum_{i=1}^{m} \dim \mathsf{range}\, T_i \end{aligned}$$

So, the sum of the ranges must be a direct sum as desired.

Next, we show for all $i \neq j = 1, \dots, m$, we have $T_i T_j = 0$. Let $v \in V$, and see for all $j = 1, \dots, m$, we have

$$
\begin{aligned}
T_j v &= I T_j v \\
&= \left(\sum_{i=1}^{m} T_i\right) T_j v \\
&= \sum_{i=1}^{m} T_i T_j v \\
&= T_j^2 v + \sum_{i=1, i\neq j}^{m} T_i T_j v \\
T_j v - T_j^2 v &= \sum_{i=1, i\neq j}^{m} T_i T_j v \\
T_j v - T_j v &= \sum_{i=1, i\neq j}^{m} T_i T_j v \\
0 &= \sum_{i=1, i\neq j}^{m} T_i T_j v
\end{aligned}
$$

See that $T_i T_j v \in \operatorname{range} T_i$. Since we have that the sum of these ranges is a direct sum, then because of Axler 1.44 we know that there is a unique way to write 0 making all of the vectors on the right hand side exactly the 0 vector. So, since v was arbitrary, we have for all $v \in V$, $i, j = 1, \ldots, m, i \neq j$:

$$T_i T_j v = 0$$

Since this is true for all vectors, we know that that the operator $T_i T_j = 0$ as we need!

<u>Solution</u>: We will first prove that $V = \operatorname{range} T_1 + \cdots + \operatorname{range} T_m$, then argue that it is actually a direct sum.

Let $v \in V$. Then see:

$$
\begin{aligned}
v &= I v \\
&= \left(\sum_{i=1}^{m} T_i\right) v \\
&= \sum_{i=1}^{m} T_i v
\end{aligned}
$$

So, $V = \sum_{i=1}^{m} \operatorname{range} T_i$.

Next, we will use Axler 3.78. So we will show $\dim V = \sum_{i=1}^{m} \operatorname{range} T_i$.

See that

$$
\begin{aligned}
\dim V &= n \\
&= \operatorname{trace} I_n \\
&= \operatorname{trace} \sum_{i=1}^{m} T_i \\
&= \sum_{i=1}^{m} \operatorname{trace} T_i && \text{the trace is linear} \\
&= \sum_{i=1}^{m} \dim \operatorname{range} T_i && \text{From the previous part}
\end{aligned}
$$

So, since the dimension of the sum of the ranges is equal to the dimension of our space, we have that our sum is a direct sum.

Next, we will show that for all $i, j \in \{1, \ldots, m\}$ such that $i \neq j$ we have

$$T_i T_j = 0$$

We will do this by showing $\operatorname{null} T_i T_j = V$.

Let $v \in V$. See:

$$\begin{aligned}
T_j v &= I T_j v \\
&= \left(\sum_{i=1}^{m} T_i \right) T_j v \\
&= \sum_{i=1}^{m} T_i T_j v \\
&= T_j^2 v + \sum_{i=1, i \neq j}^{m} T_i T_j v \\
T_j v - T_j^2 v &= \sum_{i=1, i \neq j}^{m} T_i T_j v \\
T_j v - T_j v &= \sum_{i=1, i \neq j}^{m} T_i T_j v \\
0 &= \sum_{i=1, i \neq j}^{m} T_i T_j v
\end{aligned}$$

See that $T_i T_j v \in \operatorname{range} T_i$. Since we have that the sum of these ranges is a direct sum, then because of Axler 1.44 we know that there is a unique way to write 0 making all of the vectors on the right hand side exactly the 0 vector. So, since v was arbitrary, we have for all $v \in V$, $i, j = 1, \ldots, m, i \neq j$:

$$T_i T_j v = 0$$

Telling us that $T_i T_j = V$, so $T_i T_j = 0$ as desired.

4. Prove or give a counterexample to each of the following statements:
   (a) Let $T \in \mathcal{L}\left(R^3\right)$, and $\dim \operatorname{null} T \cap \operatorname{range} T \geq 1$. Then T is nilpotent.
   (b) Let $T \in \mathcal{L}\left(R^4\right)$, and $\dim \operatorname{null} T \cap \operatorname{range} T \geq 2$. Then T is nilpotent.

Solution:

Intuition: This problem is a bit of trial and error, but the idea behind how we approach the problem is as follows.

For ease of notation, let $N = \operatorname{null} T$ and $R = \operatorname{range} T$. With intersections, we can only ever "shrink" in that we can lose elements when do an intersection but never gain them. S $\dim N \geq \dim N \cap R$ and $\dim R \geq \dim N \cap R$. We can use this to argue about the actual value of $\dim N \cap R$.

For part (a), we have $3 = \dim N + \dim R$, and that $\dim N \cap R \geq 1$. If we were to have that this dimension was 2, then by our inequalities above, we get that $3 \geq 2 + 2 = 4$, which is nonsense. So we would need this dimension to be exactly 1. Since this means that the range and nullspace must have different sizes, so they cannot be the same. So we can think about if an operator can have a range that maps one vector to itself, and a separate linearly independent vector mapping to the nullspace. This is actually possible! Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$, and define an operator T such that

$$\begin{aligned} Te_1 &= 0 \\ Te_2 &= e_1 \\ Te_3 &= e_3. \end{aligned}$$

See that from the first equation, $e_1 \in N$, and $e_1 \in R$. Thus, $\dim N \cap R \geq 1$ as we need in the problem statement, but for any $k \in \mathcal{N}$, we have that

$$T^k e_3 = e_3$$

So, T cannot be nilpotent. So the above would be a counter example

For part (b), however we see from a similar line of reasoning as above, $\dim N \cap R \not\geq 3$, so we have that $\dim N \cap R = 2$. Also from a similar argument, as before, we see that $\dim N = \dim R = \dim N \cap R = 2$. By definition both N, R are subsets of $N \cap R$, but all have the same dimension. So we have that $N = N \cap R = R$, so $N = R$. See that if $v \in \mathbb{R}^4$:

$$\begin{aligned} Tv \in R &\implies Tv \in N \\ &\implies T(Tv) = T^2 v = 0 \end{aligned}$$

Since v was arbitrary, this holds for all $v \in \mathbb{R}^4$, telling us that $T^2 = 0$ meaning that T is nilpotent.

Aside: We can see that if $\dim N \cap R = \dim V/2$, then we can show a similar result. Not really relevant for this problem, but still an interesting result to keep in mind!

Solution:

For ease of notation, for both parts, we will define $N = \operatorname{null} T$ and $R = \mathsf{range}\, T$.

(a) We will provide a counter example showing this statement is not true in general. Let e_1, e_2, e_3 denote the standard basis for $\mathbb{R}^3$. Define the operator T as follows

$$\begin{aligned} Te_1 &= 0 \\ Te_2 &= e_1 \\ Te_3 &= e_3. \end{aligned}$$

From the first equation, $e_1 \in N$, and from the second equation, $e_1 \in R$. Thus $e_1 \in N \cap R$, telling us that $\dim N \cap R \geq 1$ as this part requires. However see that for all $k \in \mathcal{N}$, we have

$$T^k e_3 = e_3$$

Which means that $T^{\dim \mathbb{R}^3} \neq 0$, so T is not nilpotent.

(b) We will prove this statement. We first show that $\dim N \cap R = 2$. Assume that $\dim N \cap R > 2$. From the fundamental theorem of linear maps, we have

$$4 = \dim \mathbb{R}^4 = \dim N + \dim R. \tag{1}$$

However since $N \cap R \in R$ and $N \cap R \in N$, we know that $\dim R \geq \dim N \cap R$ and $\dim N \geq \dim N \cap R$. Plugging this into equation (1) gives us that

$$\begin{aligned} 4 &= \dim N + \dim R \\ &\geq 2 * \dim N \cap R \\ &> 2 * 2 \\ &= 4 \end{aligned}$$

So we have that $4 > 4$, which is impossible, so we know that $\dim N \cap R = 2$. Next, we will argue that $N = N \cap R = R$. The first step is to argue that the dimensions of N and R are both 2.

WLOG we will do this by proving that $\dim N > 2$, and a similar argument will follow for R. Pluggin our known inequalities $\dim R \geq 2$ and $\dim N > 2$ into (1) , we get

$$\begin{aligned} 4 &= \dim N + \dim R \\ &\geq \dim N + 2 \\ &> 2 + 2 \\ &= 4. \end{aligned}$$

So, $4 > 4$. Meaning that the dimensions of these three spaces are the same. However, we know that $N \cap R \subset R$ and $N \cap R \subset N$, so we have this intersection is a subset of both R and N of the same dimension, so they must be the same. Giving us

$$N = N \cap R = R.$$

Finally, we will show that T is nilpotent by showing $T^2 v = 0$ for all $v \in V$.

Let $v \in V$. See:

$$\begin{aligned} Tv \in R &\implies Tv \in N \\ &\implies T(Tv) = T^2 v = 0 \end{aligned}$$

Since v was arbitrary, this holds for all $v \in \mathbb{R}^4$, telling us that $T^2 = 0$ meaning that T is nilpotent.

5. Let V be an n-dimensional vector space, and let $T_1, \ldots, T_{n+1} \in \mathcal{L}(V)$ such that

$$T_i T_j = T_j T_i \qquad \text{for every } 1 \leq i \leq j \leq n+1 \text{ (the operators commute), and} \tag{2}$$

$$T_1 \cdots T_{n+1} = 0 \tag{3}$$

Solution: Haven't talked about this one, so waiting to post the solution.

(a) Show that there exists some k such that $T_1 \cdots T_{k-1} T_{k+1} \cdots T_{n+1} = 0$ as follows:
Show that for every k, we have

1. $\mathsf{range}\, T_1 \cdots T_k \subset \mathsf{range}\, T_1 \cdots T_{k-1}$, and
2. $\mathsf{range}\, T_1 \cdots T_k \subset \operatorname{null} T_{k+1} \cdots T_n$

Then argue that for some k, we must have equality in (1.), and explain why this implies the desired statement

(b) Show that (2) is necessary for the previous conclusion by providing three operators (or matrices) $T_1, T_2, T_3 \in \mathcal{L}\left(\mathbb{R}^2\right)$ with $T_1 T_2 T_3 = 0$, but $T_1 T_2 \neq 0$, $T_1 T_3 \neq 0$, and $T_2 T_3 \neq 0$.