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$$\mathrm{Nul}\,(A) = \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} = \mathbf{0}_m\}$$

Null Space Example

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$$

Find $\operatorname{Nul}(A)$.

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Null Space is a Subspace.

Theorem

Let $A \in \mathbb{R}^{m \times n}$*, then* $\mathrm{Nul}\,(A)$ *is a subspace of* $\mathbb{R}^n$*.*

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   So, $\mathbf{u} + \mathbf{v} \in \mathrm{Nul}\,(A)$ meaning it is closed under addition

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$\square$

Column Space of a Matrix

Definition

Column Space: The column space of a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ is denoted as $\mathrm{Col}\,(A)$ and is the set of all linear combinations of columns of A.

$$\mathrm{Col}\,(A) = \mathrm{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_n) = \{\mathbf{b} \in \mathbb{R}^m \,|A\mathbf{x} = \mathbf{b} \text{ has a solution}\}$$

Finding a Basis of $\mathrm{Col}(A)$ Example

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$$

We want to know what an element of $\mathrm{Col}(A)$ looks like, so we solve $A\mathbf{x} = \mathbf{b}$ and determine what $\mathbf{b}$ has to look like!

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 2 & 3 & 6 & b_2 \\ 3 & 1 & 2 & b_3 \end{array}\right]$$

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Finding a Basis of $\mathrm{Col}\,(A)$ Example Continued

So, all the systems that we can solve have the form of

$$\mathrm{Col}\,(A) = \left\{ \mathbf{b} \in \mathbb{R}^3 \middle| \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ 7b_1 - 2b_2 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

forms a basis of $\mathrm{Col}\,(A)$

An Easier Way to Compute a Basis of $\mathrm{Col}\,(A)$

1. Reduce to RREF
2. The columns of A corresponding to pivot columns form a basis of $\mathrm{Col}\,(A)$.

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$$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the first two columns of A are a basis for $\mathrm{Col}\,(A)$!

Showing These are Both Bases

We will show that

$$\text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right)$$

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If we define $b_1 = c_1 + c_2, b_2 = 2c_1 + 3c_2,$ and $b_3 = 3c_1 + c_2$, then

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This is exactly what we said the systems we can solve look like!

The Column Space is a subspace of $\mathbb{R}^m$

We claim that for any $A \in \mathbb{R}^{m \times n}$ that $\mathrm{Col}\,(A)$ is a subspace of $\mathbb{R}^m$.

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2. Let $\mathbf{u}, \mathbf{v} \in \mathrm{Col}\,(A)$. This means there are some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{u}$ and $A\mathbf{y} = \mathbf{v}$. See that

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{u} + \mathbf{v}$$

So, $\mathbf{u} + \mathbf{v} \in \mathrm{Col}\,(A)$.

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$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{u} + \mathbf{v}$$
So, $\mathbf{u} + \mathbf{v} \in \mathrm{Col}(A)$.
3. Let $\mathbf{u} \in \mathrm{Col}(A), c \in \mathbb{R}$. Therefore, there is some $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{u}$. See that
$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{u}.$$
So, $c\mathbf{u} \in \mathrm{Col}(A)$!

$\mathrm{Col}\,(A)$ and $\mathrm{Nul}\,(A)$ Practice

For the following matrix $A \in \mathbb{R}^{4\times 3}$ find a basis for $\mathrm{Col}\,(A)$ and $\mathrm{Nul}\,(A)$.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \\ 1 & 5 & 7 \end{bmatrix}$$

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$\mathrm{Col}\,(A)$ and $\mathrm{Nul}\,(A)$ Practice Continued

So, we have that

$$\mathrm{Nul}\,(A) = \mathrm{Span}\left(\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)$$

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What do we notice about the dimension of these spaces?

Relating to Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then we define:

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Kernel: The kernel of T is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}_m$.

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This is just the null space of the matrix associated with T!

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$$\mathrm{Im}\,(T) = \mathrm{Range}\,(T)$$

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This is just the column space of the matrix associated with T!

Row Space of a Matrix

Let $A \in \mathbb{R}^{m \times n}$, then we can write it as

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

Where each $r_k^\top \in \mathbb{R}^n$ is a row of A.

Note: We are transposing the rows to make them column vectors!

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We define $\mathrm{Row}\,(A)$ to be all linear combinations of the rows of A. Or:

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Example

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$, then we saw that we can row reduce to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

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In addition, $\dim\left(\mathrm{Col}\,(A)\right) = \dim\left(\mathrm{Span}\,(\mathbf{a}_1, \mathbf{a}_2)\right) = 2$, so our definition is consistent!

Rank-Nullity Theorem

Theorem

Let $A \in \mathbb{R}^{m\times n}$, then we know that

$$\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = n$$

$$\operatorname{rank}(A) + \dim\left(\operatorname{Nul}\left(A^{\top}\right)\right) = m$$