What is a Vector Space?

A Vector Space is a set V that contains our *vectors*, a set F that contains our scalars with a vector addition operation and scalar multiplication operation where the following properties are true for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in F$.

1. V is closed under addition:
   $\mathbf{u} + \mathbf{v} \in V$
2. Vector addition is commutative:
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. Vector addition is associative:
   $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. Additive Identity:
   There exists some $\mathbf{0} \in V$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$
5. Additive Inverse: for each $\mathbf{u} \in V$, there is some $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. V is closed under scalar multiplication:
   $c\mathbf{v} \in V$.
7. Scalar multiplication distributes over vector addition:
   $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. Scalar multiplication distributes over scalar addition:
   $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
9. Scalar multiplication is associative:
   $c(d\mathbf{v}) = (cd)\mathbf{v}$
10. Multiplicative Identity:
    There exists some $1 \in F$ such that $1\mathbf{u} = \mathbf{u}$.

What are some examples?

$V = \mathbb{R}^n, F = \mathbb{R}$. See Slide 8 of Lecture slide 3 for properties 2-5 and 7-10.
For properties 1 and 2, we have the definitions of vector addition and scalar multiplication that guarantees this!

More Examples

$V = \mathcal{P}_2$ is the set of all polynomials with real coefficients of degree 2 or less.
$F = \mathbb{R}$, with the operations we'd expect.

1. $(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$
2. $f(x) + g(x) = g(x) + f(x)$
3. $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$
4. $\mathbf{0} = 0x^2 + 0x + 0$
5. $-f(x) = -a_2x^2 - a_1x - a_0$
6. $cf(x) = ca_2x^2 + ca_1x + ca_0$
7. $c(f(x) + g(x)) = cf(x) + cg(x)$
8. $(c + d)f(x) = cf(x) + df(x)$
9. $c(df(x)) = (cd)f(x)$
10. $1f(x) = f(x)$

Is this a Vector Space?

$V = \mathbb{R}^3, F = \mathbb{R}$ using standard scalar multiplication but $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_2 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

Uniqueness of $\mathbf{0}$

Theorem

If V is a vector space, then the $\mathbf{0}$ element is unique

Proof.

Let $\mathbf{w} \in V$ such that for every $\mathbf{u} \in V$ we have

$$\mathbf{w} + \mathbf{u} = \mathbf{u} + \mathbf{w} = \mathbf{u}$$

By taking $\mathbf{u} = \mathbf{0}$, we have:

$$\begin{aligned} \mathbf{0} + \mathbf{w} &= \mathbf{0} \\ \mathbf{w} + \mathbf{0} &= \mathbf{w} \end{aligned}$$

Thus, we see that $\mathbf{w} = \mathbf{0}$ $\square$

Uniqueness of Additive Inverse

Theorem

If V is a vector space, then for every $\mathbf{u} \in V$, we have that $-\mathbf{u}$ is unique.

Proof.

Let $\mathbf{u} \in V$, and suppose there are two additive identities IE that $-\mathbf{u}_1, -\mathbf{u}_2 \in V$ such that $\mathbf{u} + (-\mathbf{u}_1) = \mathbf{0} = \mathbf{u} + (-\mathbf{u}_2)$ See that

$$\mathbf{u} + (-\mathbf{u}_1) = 0 \to -\mathbf{u}_2 + (\mathbf{u} + (-\mathbf{u}_1)) = -\mathbf{u}_2 \to (-\mathbf{u}_2 + \mathbf{u}) + -\mathbf{u}_1 = -\mathbf{u}_2$$

$$\to -\mathbf{u}_1 = -\mathbf{u}_2$$

$\square$

Vector Space Practice

Work with your neighbors to determine if the following spaces are vector spaces

$V = \mathbb{R}^3, F = \mathbb{R}$ with the usual vector addition and $c\mathbf{u} = \begin{bmatrix} -cu_1 \\ -cu_2 \\ -cu_3 \end{bmatrix}$.

$V = \mathbb{R}^{3\times 3}, F = \mathbb{R}$ with the standard operations.

Vector Subspaces

Definition

A subspace of a vector space V is a subset H of V ($H \subseteq V$) that has the following properties (using the same vector addition, scalar multiplication, and F)

1. The $\mathbf{0}$ from V is in H.
2. H is closed under vector addition: for each $\mathbf{u}, \mathbf{v} \in H$, we have $\mathbf{u} + \mathbf{v} \in H$.
3. H is closed under scalar multiplication: for each $c \in F$ and $\mathbf{v} \in H$, we have $c\mathbf{v} \in H$.

Is it a Subspace?

Determine with your neighbors if each of the following sets are subspaces of $V = \mathbb{R}^3$.

$$H = \left\{ \mathbf{v} \in \mathbb{R}^3 \,\middle|\, \mathbf{v} = \begin{bmatrix} 3a+b \\ a+5 \\ 2a-5b \end{bmatrix} \text{ for } a, b \in \mathbb{R} \right\} \qquad H = \left\{ \mathbf{v} \in \mathbb{R}^3 \,\middle|\, \mathbf{v} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \text{ for } a, b \in \mathbb{R} \right\}$$

Spanning Sets and Subspaces

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ denote a set of p vectors in V. Then $\text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is a subspace of V.

1. $\text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$
2. Given any subspace H of V, a spanning set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in H such that $H = \text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$

Example

Determine if $H = \left\{ A \in \mathbb{R}^{3\times 3} \middle| A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, a+b+c=0 \right\}$ is a subspace of $\mathbb{R}^{3\times 3}$ and if so, give a spanning set for H.Show that $\mathbf{0} \in H$, Show that we are closed under "vector" addition, and show we are closed under scalar multiplication.

1. Set $a = b = c = 0$, clearly $a+b+c=0$ and then we have the 0 matrix!
2. Let $A, B \in H$. See that

$$A + B = \begin{bmatrix} a_1 + a_2 & 0 & 0 \\ 0 & b_1 + b_2 & 0 \\ 0 & 0 & c_1 + c_2 \end{bmatrix}$$

   and $(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = a_1 + b_1 + c_1 + a_2 + b_2 + c_2 = 0 + 0 = 0$, so $A + B \in H$.

Example continued

Determine if $H = \left\{ A \in \mathbb{R}^{3\times 3} \middle| A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, a+b+c=0 \right\}$ is a subspace of $\mathbb{R}^{3\times 3}$ and if so, give a spanning set for H.

3. let $x \in \mathbb{R}$. See that

$$xA = \begin{bmatrix} xa & 0 & 0 \\ 0 & xb & 0 \\ 0 & 0 & xc \end{bmatrix}$$

   And

$$xa + xb + xc = x(a+b+c) = x \cdot 0 = 0$$

So, H is a subspace of $\mathbb{R}^3$!

Example continued pt. 2

See that our "vectors" are 3×3 matrices, so our spanning set will have these kinds of matrices! Define

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

See that

$$\text{Span}\left(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right) = \left\{A \in \mathbb{R}^{3\times 3} \,|A = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3, \text{ for } a, b, c \in \mathbb{R}\right\}$$

And

Example continued pt. 3

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + b\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} + c\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2a - b - c & 0 & 0 \\ 0 & -a + 2b - c & 0 \\ 0 & 0 & -a - b + 2c \end{bmatrix}$$

Where

$$2a - b - c + (-a + 2b - c) + (-a - b + 2c) = 0$$

Basis of a Vector Space

Definition

Basis: A basis of a vector space V is a set of $v_1, \ldots, v_p \in V$ such that

1. $\operatorname{Span}(v_1, \ldots, v_p) = V$
2. $v_1, \ldots, v_p$ are linearly independent

Example

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is a basis of

$$H = \left\{ A \in \mathbb{R}^{3\times 3} \middle| A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, a + b + c = 0 \right\}$$

Length of Basis and Dimension of Vector Space

Theorem

All bases of a vector space V have the same number of elements

Definition

The dimension of a vector space, denoted $\dim(V)$ is the length of a basis of V.

Spanning and Independent List of Correct Size is a Basis

Theorem

Let V be a vector space with $n = \dim(V)$. Then, any linearly independent list of n vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ forms a basis of V.

Theorem

Let V be a vector space with $n = \dim(V)$. Then, any spanning of n vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ such that $Span(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$ is also a basis of V.