Diagonal Matrices

Recall that a diagonal matrix looks something like

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

And we can see that computing $D\mathbf{x}$ is pretty easy!

$$D\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_2 \\ 2x_3 \end{bmatrix}$$

So, our lives could be easier if we can find a diagonal matrix that a given matrix behaves like!

Similarity

Definition

Let $A, B \in \mathbb{R}^{n\times n}$. Then we say that A and B are similar if there is some $C \in \mathbb{R}^{n\times n}$ such that C is invertible and

$$A = CBC^{-1}$$

Or equivalently without using an inverse

$$AC = CB$$

Example

If

$$A = \begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Then A and B are similar as $AC = CB$.

Computing Large Matrix Powers

One application of similar matrices is that we can compute matrix powers really easy! Let $A, B, C \in \mathbb{R}^{n\times n}$ such that

$$A = CBC^{-1}$$

Then for any $k \geq 1$ we have that

$$A^k = \underbrace{A \cdots A}_{k \text{ times}} = CB^kC^{-1}$$

Which is really easy to compute if B is diagonal or some other nice structure!

Similarity Transformation

Let $A, B, C \in \mathbb{R}^{n \times n}$ such that

$$A = CBC^{-1}.$$

We sometimes call this a similarity transformation from A to B. We will see why this is important in the next few slides!

Similarity Transformation as a Change of Basis

Let's consider an invertible $C \in \mathbb{R}^{n\times n}$ with columns denoted $\mathbf{v}_1, \dots, \mathbf{v}_n$ as follows

$$C = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

Since C is invertible, $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent! So, this means

$$\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$$

forms a basis of $\mathbb{R}^n$, so we can talk about $\mathcal{B}$-Coordinates!

$\mathcal{B}$-Coordinates of $\mathbf{x}$

So, if we want to find the $\mathcal{B}$-Coordinates of some vector $\mathbf{x}$ we would get

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ where } \mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

This means that

$$C\,[\mathbf{x}]_{\mathcal{B}} = \mathbf{x} \leftrightarrow C^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

So C^{-1} takes a vector in the standard basis and converts it to coordinates in the $\mathcal{B}$ basis. Or, in otherwords, we're finding a basis under which the matrix A behaves "like" B does!

Putting it All Together for Similarity Transformations

Since C^{-1} takes a vector $\mathbf{x}$ and computes the $\mathcal{B}$-Coordinates of that vector and C returns it to the standard coordinates, we see that:

$$A\mathbf{x} = CBC^{-1}\mathbf{x} = C(B(C^{-1}\mathbf{x}))$$

performs the following actions

1. Computes the $\mathcal{B}$-Coordinates of $\mathbf{x}$
2. Transforms $[\mathbf{x}]_{\mathcal{B}}$ via B. IE $[\mathbf{y}]_{\mathcal{B}} = B\,[\mathbf{x}]_{\mathcal{B}}$
3. Returns $[\mathbf{y}]_{\mathcal{B}}$ to the standard coordinates

Similarity Transformations Preserve Eigenvalues!

The final properties that we will discuss are how eigenpairs behave under similarity transformations We first claim that the eigenvalues are preserved. See that

$$\begin{aligned} \det(A - \lambda I) &= \det\left(CBC^{-1} - \lambda CC^{-1}\right) = \det\left(C(B - \lambda I)C^{-1}\right) \\ &= \det(C)\det(B - \lambda I)\det\left(C^{-1}\right) = \det(B - \lambda I) \end{aligned}$$

So, any value λ that makes $\det(A - \lambda I) = 0$ will necessarily make $\det(B - \lambda I) = 0$, so the eigenvalues must be the same!

Similarity Transformations Also Transform Eigenvectors

We claim that if $(\lambda, \mathbf{v})$ is an eigenpair of A, then $(\lambda, C^{-1}\mathbf{v})$ is an eigenvector of B. See that

$$BC^{-1}\mathbf{v} = (C^{-1}C)BC^{-1}\mathbf{v} = C^{-1}A\mathbf{v} = \lambda C^{-1}\mathbf{v}$$

So $(\lambda, C^{-1}\mathbf{v})$ is an eigenpair of B.
Similarly if $(\lambda, \mathbf{v})$ is an eigenpair of B then $(\lambda, C\mathbf{v})$ is an eigenpair of A. See that

$$AC\mathbf{v} = (CBC^{-1})C\mathbf{v} = CB\mathbf{v} = \lambda C\mathbf{v}$$

So $(\lambda, C\mathbf{V})$ is an eigenpair of A!
This means we can think of eigenvectors of A and B as the same objects just with different coordinates!

Geometry of Similarity Transformations

See Section 5.3 of textbook. The images there are much better than what I will come up with

Some More Statements About Similarity

Some more properties that are nice to know are as follows

1. The only matrix similar to I_n is I_n itself
2. The only matrix similar to $0_{n\times n}$ is $0_{n\times n}$
3. Similarity has nothing to do with row equivalence