Eigenvalues and Eigenvectors[1]

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Note: Since the definitions of eigenvalues and eigenvectors depend on each other, we sometimes refer to the ordered pair

$$(\lambda, \mathbf{v})$$

as an eigenpair of A.

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Another Framing of Eigenvalues and Eigenvectors

From the definition of the eigenpair $(\lambda, \mathbf{v})$, we see that $\mathbf{v}$ is a non-trivial solution to the system

$$\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I_n)\mathbf{v}$$

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So, if λ is an eigenvalue of A, then the matrix $A - \lambda I$ has a non-trivial null space, and the eigenvectors will be the vectors of this null space!

Finding Eigenvectors for an Eigenvalue Example

Let's verify if $\lambda = 2$ is an eigenvalue of the following matrix, and if it is, find an eigenvector.

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 2 & 12 & 14 \\ 1 & 6 & 10 \end{bmatrix}$$

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Finding Eigenvectors for an Eigenvalue Practice

Determine if $\lambda = 1$ is an eigenvalue of the following matrix and if so, determine an eigenvector associated with $\lambda = 1$.

$$A = \begin{bmatrix} 2 & 2 & 9 \\ 2 & 8 & 30 \\ 1 & 4 & 18 \end{bmatrix}$$

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Eigenspace

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We define the Eigenspace of A associated with eigenvalue λ to be

$$E(A, \lambda) = \{\mathbf{v} \in \mathbb{R}^n \,|\, A\mathbf{v} = \lambda\mathbf{v}\}$$

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Since the eigenspace is really just a nullspace, we know how to find a basis of it!

Basis for an Eigenspace

A basis for an eigenspace of $E(A, \lambda)$ is just a basis for $\mathrm{Nul}\,(A - \lambda I_n)$. So, in order to find such a basis, we can

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1. Set up $A - \lambda I_n$.
2. Reduce to RREF
3. Write out a basis of this space as before

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Find a basis for the eigenspaces $E(A, 1)$ for

$$A = \begin{bmatrix} 7 & 0 & 6 \\ -3 & 4 & -6 \\ -3 & 0 & -2 \end{bmatrix}$$

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So, if this space has a non-trivial basis, then $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution!

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Meaning we can say that a matrix is invertible if and only if 0 is not an eigenvalue of A.

Linearly Independent Eigenvectors

Theorem

If $(\lambda_1, \mathbf{v}_1)$ *and* $(\lambda_2, \mathbf{v}_2)$ *are two eigenpairs of a matrix* A *such that* $\lambda_1 \neq \lambda_2$, *then* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent.*

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Proof.

We will prove this via a contradiction. IE assume $\lambda_1 \neq \lambda_2$ but $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

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meaning that $\lambda_1 = \lambda_2$, which contradicts our assumption that these eigenvalues are distinct. □