# Eigenpair Reminder

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We've discussed how to confirm a proposed $\lambda$ is actually an eigenvalue and to compute associated eigenvectors.
What about how to compute the eigenvalues themselves?

# Computing Eigenvalues

Recall that if $\lambda$ is an eigenvalue of a matrix $A$, then

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has a non-trivial solution. In other words, $A - \lambda I$ is not invertible!
So, we can figure out all $\lambda$ values such that

$$\det(A - \lambda I) = 0$$

# Characteristic Polynomial

**Definition**

Let $A \in \mathbb{R}^{n \times n}$. The characteristic polynomial of $A$ is the function $f(\lambda)$ given by

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Let $\lambda$ be an eigenvalue of $A$, then $A - \lambda I$ is not invertible, so $\det(A - \lambda I) = 0$, so $f(\lambda) = \det(A - \lambda I) = 0$.

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Let $\lambda$ be an eigenvalue of $A$, then $A - \lambda I$ is not invertible, so $\det(A - \lambda I) = 0$, so $f(\lambda) = \det(A - \lambda I) = 0$.
Let $\lambda$ be a root of $f(\lambda)$, then

$$0 = f(\lambda) = \det(A - \lambda I).$$

So, $A - \lambda I$ is not invertible, so $\lambda$ is an eigenvalue of $A$. $\square$

# Rational Root Theorem[1]

**Theorem**

*Let* $f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + c_n\lambda^n$ *be a polynomial with integer coefficients.*

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$$x = \frac{p}{q}$$

*Where* $p$ *is an integer factor of* $c_0$ *and* $q$ *is an integer factor of* $c_n$.

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### Example

Consider $f(x) = x^3 + x^2 - 10x + 8$. Then our possible roots are

$$\{\pm 1, \pm 2, \pm 4, \pm 8\}$$

And we plug in these values to see which one(s) are actual roots.

Note: We may not have rational roots depending on the polynomial itself (complex roots!)

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# Finding Eigenvalues Example

Let $A \in \mathbb{R}^{3\times 3}$ as given below. Then compute all the eigenvalues of $A$.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -3 \\ -4 & -4 & 5 \end{bmatrix}$$

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$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 1 & -1 \\ 1 & 4-\lambda & -3 \\ -4 & -4 & 5-\lambda \end{bmatrix}\right) = -\lambda^3 + 13\lambda^2 - 39\lambda + 27$$

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Our rational roots theorem states that the possible rational eigenvalues are:

$$\{\pm 1, \pm 3, \pm 9, \pm 27\}$$

So, we plug these in and find that $1, 3, 9$ are the eigenvalues.

# Finding Eigenvalues Practice

Let $A \in \mathbb{R}^{2\times 2}$ as given below. Then compute all the eigenvalues of $A$.

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$$\lambda = \pm 4$$

# Trace of a Matrix

### Definition

Let $A \in \mathbb{R}^{n \times n}$. Then we define the trace of $A$ as

$$\mathrm{Tr}\,(A) = a_{11} + \cdots + a_{nn}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

# Properties of the characteristic polynomial

Let $A \in \mathbb{R}^{n \times n}$. Then we know that

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \mathrm{Tr}\,(A)\,\lambda^{n-1} + \cdots + \det\,(A)$$

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then the characteristic polynomial has the form:

$$f(\lambda) = \lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

# What About Larger Problems? (IE $n \geq 5$)

It's great that we can solve these small problems by hand, but what about larger ones?

[2] More information can be found here:
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# What About Larger Problems? (IE $n \geq 5$)

It's great that we can solve these small problems by hand, but what about larger ones? Well, The Abel–Ruffini theorem[2] states that we cannot always solve for these eigenvalues when we have $n \geq 5$.

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However, we can still try to solve these problems! We will achieve this via some methods we learn later in the semester, and is how we currently compute eigenvalues in practice.

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