# Eigenvalues and Eigenvectors[1]

### Definition

Let $A \in \mathbb{R}^{n\times n}$, then we define:

1. An eigenvector of $A$ is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda$.
2. An eigenvalue of $A$ is a scalar $\lambda$ such that there is some $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

Note: Since the definitions of eigenvalues and eigenvectors depend on each other, we sometimes refer to the ordered pair

$$(\lambda, \mathbf{v})$$

as an eigenpair of $A$.

[1]Aside: The word "eigen" comes from German and roughly translates to either "own/self" or "characteristic".

# Another Framing of Eigenvalues and Eigenvectors

From the definition of the eigenpair $(\lambda, \mathbf{v})$, we see that $\mathbf{v}$ is a non-trivial solution to the system

$$\mathbf{0} = A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I_n)\mathbf{v}$$

So, if $\lambda$ is an eigenvalue of $A$, then the matrix $A - \lambda I$ has a non-trivial null space, and the eigenvectors will be the vectors of this null space!

# Finding Eigenvectors for an Eigenvalue Example

Let's verify if $\lambda = 2$ is an eigenvalue of the following matrix, and if it is, find an eigenvector.

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 2 & 12 & 14 \\ 1 & 6 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 6 \\ 2 & 12 & 14 \\ 1 & 6 & 10 \end{bmatrix} \xrightarrow{A = A - \lambda I} \begin{bmatrix} 1 & 4 & 6 \\ 2 & 10 & 14 \\ 1 & 6 & 8 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\lambda = 2$ is an eigenvalue! Let's find an eigenvector.

$$\xrightarrow{R_2 = \frac{R_2}{2}} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \text{ is an eigenvector of } A.$$

# Finding Eigenvectors for an Eigenvalue Practice

Determine if $\lambda = 1$ is an eigenvalue of the following matrix and if so, determine an eigenvector associated with $\lambda = 1$.

$$A = \begin{bmatrix} 2 & 2 & 9 \\ 2 & 8 & 30 \\ 1 & 4 & 18 \end{bmatrix}$$

$$\left(1, \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}\right)$$

is an eigenpair. See that

$$A\mathbf{v} = \begin{bmatrix} 2 & 2 & 9 \\ 2 & 8 & 30 \\ 1 & 4 & 18 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} = -\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 30 \\ 18 \end{bmatrix} = \begin{bmatrix} -2-8+9 \\ -2-32+30 \\ -1-16+18 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} = 1\mathbf{v}$$

# Eigenspace

**Definition**

We define the Eigenspace of $A$ associated with eigenvalue $\lambda$ to be

$$E(A, \lambda) = \{\mathbf{v} \in \mathbb{R}^n \,|\, A\mathbf{v} = \lambda \mathbf{v}\}$$

Or equivalently

$$E(A, \lambda) = \mathrm{Nul}(A - \lambda I_n)$$

Since the eigenspace is really just a nullspace, we know how to find a basis of it!

# Basis for an Eigenspace

A basis for an eigenspace of $E(A, \lambda)$ is just a basis for $\mathrm{Nul}\,(A - \lambda I_n)$. So, in order to find such a basis, we can

1. Set up $A - \lambda I_n$.
2. Reduce to RREF
3. Write out a basis of this space as before

# Basis for an Eigenspace Example

Find a basis for the eigenspaces $E(A, 1)$ for

$$A = \begin{bmatrix} 7 & 0 & 6 \\ -3 & 4 & -6 \\ -3 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 6 \\ -3 & 4 & -6 \\ -3 & 0 & -2 \end{bmatrix} \xrightarrow{A=A-\lambda I} \begin{bmatrix} 6 & 0 & 6 \\ -3 & 3 & -6 \\ -3 & 0 & -3 \end{bmatrix} \xrightarrow{R_1=\frac{R_1}{6}} \begin{bmatrix} 1 & 0 & 1 \\ -3 & 3 & -6 \\ -3 & 0 & -3 \end{bmatrix} \xrightarrow[R_3=R_3+3R_1]{R_2=R_2+3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2=\frac{R_2}{3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathrm{Nul}\,(A - I) = \mathrm{Span}\left( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

# Basis for an Eigenspace Practice

Find a basis for the eigenspaces $E(A, 4)$ for

$$A = \begin{bmatrix} 7 & 0 & 6 \\ -3 & 4 & -6 \\ -3 & 0 & -2 \end{bmatrix}$$

$$E(A, 4) = \text{Span}\left( \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

## The Eigenspace Associated With 0

What does $E(A, 0)$ look like?

$$E(A, 0) = \mathrm{Nul}\,(A - 0I) = \mathrm{Nul}\,(A)$$

So, if this space has a non-trivial basis, then $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution!
Meaning we can say that a matrix is invertible if and only if 0 is not an eigenvalue of $A$.

# Linearly Independent Eigenvectors

### Theorem

*If $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ are two eigenpairs of a matrix $A$ such that $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.*

### Proof.

We will prove this via a contradiction. IE assume $\lambda_1 \neq \lambda_2$ but $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. Then, we show that this leads to nonsense. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then there is some constant $c$ such that $\mathbf{v}_1 = c\mathbf{v}_2$. See that

$$\lambda_1 \mathbf{v}_1 = A\mathbf{v}_1 \; = cA\mathbf{v}_2 \; = \lambda_2 c\mathbf{v}_2 = \; \lambda_2 \mathbf{v}_1$$

So,

$$\lambda_1 \mathbf{v}_1 - \lambda_2 \mathbf{v}_1 = \mathbf{0}$$

meaning that $\lambda_1 = \lambda_2$, which contradicts our assumption that these eigenvalues are distinct. $\square$