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Example

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} .1 & .7 & .3 \\ .8 & .2 & .3 \\ .1 & .1 & .4 \end{bmatrix}$$

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Determine which of the following matrices are positive stochastic, stochastic, or neither.

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$$A = \begin{bmatrix} .1 & .2 & .7 \\ .2 & .3 & .5 \\ .4 & .5 & .1 \end{bmatrix}, B = \begin{bmatrix} .3 & .8 & .15 \\ .3 & .1 & .05 \\ .4 & .1 & .8 \end{bmatrix}, C = \begin{bmatrix} .2 & .8 & 0 \\ .7 & .1 & .2 \\ .5 & .1 & .8 \end{bmatrix}, D = \begin{bmatrix} .1 & .2 & .2 \\ .3 & .1 & .2 \\ .5 & .1 & .2 \end{bmatrix}, E = \begin{bmatrix} .2 & .8 & 0 \\ .3 & .1 & .2 \\ .5 & .1 & .8 \end{bmatrix}$$

Eigenvalues of Stochastic Matrices Part 1

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a stochastic matrix. Then we know that $\lambda = 1$ is an eigenvalue of A and if λ is an eigenvalue of A then $|\lambda| \leq 1$

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$$A^\top \mathbf{1}_n = \mathbf{1}_n$$

So, $(\lambda, \mathbf{1}_n)$ is an eigenpair of A. □

Eigenvalues of Stochastic Matrices Part 2

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Let $A \in \mathbb{R}^{n \times n}$ *be a stochastic matrix. Then we know that* $\lambda = 1$ *is an eigenvalue of* A *and if* λ *is an eigenvalue of* A *then* $|\lambda| \leq 1$

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Dividing by $|x_j|$ gives us that $|\lambda| \leq 1$ $\square$

Eigenvalues of Positive Stochastic Matrices

Theorem

If A is a positive stochastic matrix, then $\dim\left(E(A,1)\right) = 1$.

A (Very) Simple Weather Model

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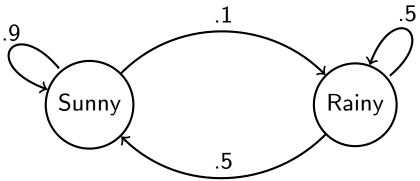

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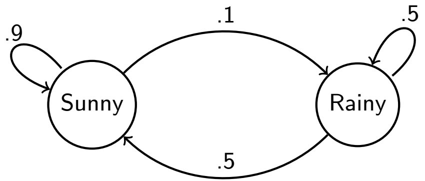

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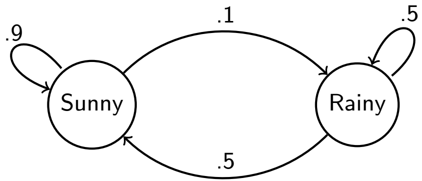

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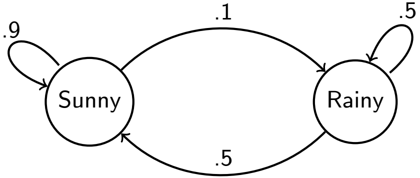

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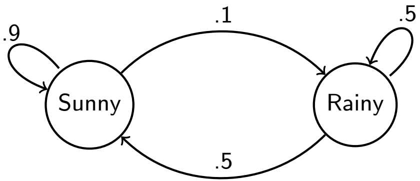

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3. P is also a stochastic matrix!

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This can be applied to many fields including: Epidemiology, Finance, Statistics, etc.

Markov Chains in Terms of Linear Algebra

A more formal definition with the tools that we have is as follows:
A Markov chain is a sequence of state vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbb{R}^n$ and a stochastic (transition) matrix P such that for $k = 0, 1, 2, \ldots$

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Where $\mathbf{x}_k = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ is a vector of probabilities we are in each state $1, \ldots n$.

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Note: We can say something very similar for non-diagonalizable matrices, but it's harder to phrase

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2. Divide $\mathbf{v}$ by the sum of its elements.
3. This new vector is the steady state vector!

Steady State Example

Let's find the steady state of a different stochastic matrix

$$P = \begin{bmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{bmatrix}$$

See Jupyter Notebook

A Practical Application: PageRank

The original algorithm that Google used in it's search engine was based on PageRank, which ranks websites based on the number of links to and from it.

The textbook discusses it in the end of chapter 5.6:
`https://textbooks.math.gatech.edu/ila/stochastic-matrices.html`.

A more comprehensive history can be found at the wikipedia page:
`https://en.wikipedia.org/wiki/PageRank#History`