What is a Vector Space?

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What are some examples?

$V = \mathbb{R}^n, F = \mathbb{R}.$

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$V = \mathbb{R}^n, F = \mathbb{R}$. See Slide 8 of Lecture slide 3 for properties 2-5 and 7-10.
For properties 1 and 2, we have the definitions of vector addition and scalar multiplication that guarantees this!

More Examples

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10. $1f(x) = f(x)$

Is this a Vector Space?

$V = \mathbb{R}^3, F = \mathbb{R}$ using standard scalar multiplication but $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_2 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

Uniqueness of $\mathbf{0}$

Theorem

If V is a vector space, then the $\mathbf{0}$ element is unique

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Proof.

Let $\mathbf{w} \in V$ such that for every $\mathbf{u} \in V$ we have

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Thus, we see that $\mathbf{w} = \mathbf{0}$ $\square$

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$\square$

Vector Space Practice

Work with your neighbors to determine if the following spaces are vector spaces

$V = \mathbb{R}^3, F = \mathbb{R}$ with the usual vector addition and $c\mathbf{u} = \begin{bmatrix} -cu_1 \\ -cu_2 \\ -cu_3 \end{bmatrix}$.

$V = \mathbb{R}^{3\times 3}, F = \mathbb{R}$ with the standard operations.

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3. H is closed under scalar multiplication: for each $c \in F$ and $\mathbf{v} \in H$, we have $c\mathbf{v} \in H$.

Is it a Subspace?

Determine with your neighbors if each of the following sets are subspaces of $V = \mathbb{R}^3$.

$$H = \left\{ \mathbf{v} \in \mathbb{R}^3 \middle| \mathbf{v} = \begin{bmatrix} 3a+b \\ a+5 \\ 2a-5b \end{bmatrix} \text{ for } a, b \in \mathbb{R} \right\}$$

$$H = \left\{ \mathbf{v} \in \mathbb{R}^3 \middle| \mathbf{v} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \text{ for } a, b \in \mathbb{R} \right\}$$

Spanning Sets and Subspaces

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ denote a set of p vectors in V. Then $\text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is a subspace of V.

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1. $\text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_p$
2. Given any subspace H of V, a spanning set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in H such that $H = \text{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_p)$

Example

Determine if $H = \left\{ A \in \mathbb{R}^{3\times 3} \middle| A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, a + b + c = 0 \right\}$ is a subspace of $\mathbb{R}^{3\times 3}$ and if so, give a spanning set for H.

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So, H is a subspace of $\mathbb{R}^3$!

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The dimension of a vector space, denoted $\dim(V)$ is the length of a basis of V.

Spanning and Independent List of Correct Size is a Basis

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Let V be a vector space with $n = \dim(V)$. Then, any linearly independent list of n vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ forms a basis of V.

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