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So, our lives could be easier if we can find a diagonal matrix that a given matrix behaves like!

Similarity

Definition

Let $A, B \in \mathbb{R}^{n\times n}$. Then we say that A and B are similar if there is some $C \in \mathbb{R}^{n\times n}$ such that C is invertible and

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Example

If

$$A = \begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Then A and B are similar as $AC = CB$.

Computing Large Matrix Powers

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Which is really easy to compute if B is diagonal or some other nice structure!

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We sometimes call this a similarity transformation from A to B. We will see why this is important in the next few slides!

Similarity Transformation as a Change of Basis

Let's consider an invertible $C \in \mathbb{R}^{n\times n}$ with columns denoted $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as follows

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Since C is invertible, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent! So, this means

$$\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$$

forms a basis of $\mathbb{R}^n$, so we can talk about $\mathcal{B}$-Coordinates!

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So, if we want to find the $\mathcal{B}$-Coordinates of some vector $\mathbf{x}$ we would get

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ where } \mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

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So C^{-1} takes a vector in the standard basis and converts it to coordinates in the $\mathcal{B}$ basis. Or, in otherwords, we're finding a basis under which the matrix A behaves "like" B does!

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$$\begin{aligned}\det(A-\lambda I) = \det\left(CBC^{-1} - \lambda CC^{-1}\right) &= \det\left(C(B-\lambda I)C^{-1}\right) \\ = \det(C)\det(B-\lambda I)\det\left(C^{-1}\right) &= \det(B-\lambda I)\end{aligned}$$

So, any value λ that makes $\det(A-\lambda I) = 0$ will necessarily make $\det(B-\lambda I) = 0$, so the eigenvalues must be the same!

Similarity Transformations Also Transform Eigenvectors

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So $(\lambda, C\mathbf{V})$ is an eigenpair of A!

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So $(\lambda, C\mathbf{V})$ is an eigenpair of A!
This means we can think of eigenvectors of A and B as the same objects just with different coordinates!

Geometry of Similarity Transformations

See Section 5.3 of textbook. The images there are much better than what I will come up with

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