Revisiting Vector Equations and Homogeneous Systems

If $A \in \mathbb{R}^{m\times n}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, then the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

which has the corresponding augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0} \end{array}\right]$$

- Homogeneous Linear Systems always have the trivial solution: $\mathbf{x} = \mathbf{0}$
- If a non-trivial solution exists, then:
  - At least one $x_i \neq 0$
  - At least one $\mathbf{a}_i$ can be written as a linear combination of the others!
- If there is only the trivial solution, then no columns can be written as a linear combination of the others!

Brief Proof (on board) Of the last bullet point.

Example

Determine whether the equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution.

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 1 & 6 & -10 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since x_3 is a free variable, we have non-trivial solutions

Python code to get this answer!

```
import sympy as sym
A = sym.Matrix([[2, -1, 6, 0],
                [1, 6, -10, 0],
                [0, 3, -6, 0]])
display(A.rref())
```

What does it mean?

We get

$$x_1 + 2x_3 = 0 \to x_1 = -2x_3$$
$$x_2 - 2x_3 = 0 \to x_2 = 2x_3$$

Where the solution set is

$$\mathbf{x} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Plugging in gives:

$$\mathbf{0} = A\mathbf{x} = -2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix}$$

Or equivalently

$$\begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}$$

So, one is a linear combination of the others!

Linear Independence

Definition

Linear Independence: We say that a set of $p > 1$ vectors in $\mathbb{R}^n$, $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent if we **cannot** write one as a linear combination of the others. Otherwise, the set is linearly dependent.

Example

Since

$$\begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix},$$

we know that the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} \right\}$ is linearly dependent!

Showing Linear Independence

Theorem

To show $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *is linearly independent, we can just show that the system*

$$\left[\begin{array}{ccc|c} \mathbf{v}_1 & \ldots & \mathbf{v}_p & \mathbf{0} \end{array}\right]$$

only has the trivial solution!

If we find another solution, then they are linearly dependent

Practice!

Determine if the following sets of vectors are linearly dependent or linearly independent

- $$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$$

  Since $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, they are linearly dependent

- $$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

  They are linearly independent! Perform Gaussian Elimination on the Augmented matrix to see this

Conceptual Practice!

For each of the following questions, either come up with an example or think of a reason it's false

1. Give an example of 1 vector in $\mathbb{R}^3$ so that the set is linearly independent
   Any vector will do!
2. Give an example of 2 vector in $\mathbb{R}^3$ so that the set is linearly independent
   e_1 and e_2 work!
3. Give an example of 3 vector in $\mathbb{R}^3$ so that the set is linearly independent
   This is the last one we can do!
4. Give an example of 4 vector in $\mathbb{R}^3$ so that the set is linearly independent
   This is not possible

But what about $\mathbf{0}$?

Theorem

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains the zero vector, $\mathbf{0}$, then the set is linearly dependent.

Proof.

Since we can reorder the list without changing the overall property, let $\mathbf{v}_1 = \mathbf{0}$.
See that

$$1\mathbf{0} + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

□

Number of Vectors and Their Dimension

Theorem

A set of p vectors, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.