Null Space of a Matrix

Definition

Null Space: The null space of a matrix $A \in \mathbb{R}^{m\times n}$ is defined to be

$$\mathrm{Nul}\,(A) = \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} = \mathbf{0}_m\}$$

Null Space Example

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$$

Find $\operatorname{Nul}(A)$. Solve $A\mathbf{x} = \mathbf{0}_3$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 3 & 6 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \mathbf{x} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix}, t \in \mathbb{R}$$

Null Space is a Subspace.

Theorem

Let $A \in \mathbb{R}^{m \times n}$, then $\mathrm{Nul}\,(A)$ is a subspace of $\mathbb{R}^n$.

Proof.

1. We know that $\mathbf{x} = \mathbf{0}$ is always a solution to $A\mathbf{x} = \mathbf{0}$
2. Let $\mathbf{u}, \mathbf{v} \in \mathrm{Nul}\,(A)$, then see that
$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
So, $\mathbf{u} + \mathbf{v} \in \mathrm{Nul}\,(A)$ meaning it is closed under addition
3. Let $\mathbf{u} \in \mathrm{Nul}\,(A)$, $c \in \mathbb{R}$. See that:
$$A(c\mathbf{u}) = cA(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$$
So, $c\mathbf{u} \in \mathrm{Nul}\,(A)$ meaning it is closed under multiplication.

$\square$

Column Space of a Matrix

Definition

Column Space: The column space of a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$ is denoted as $\mathrm{Col}\,(A)$ and is the set of all linear combinations of columns of A.

$$\mathrm{Col}\,(A) = \mathrm{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_n) = \{\mathbf{b} \in \mathbb{R}^m \,|A\mathbf{x} = \mathbf{b} \text{ has a solution}\}$$

Finding a Basis of $\mathrm{Col}\,(A)$ Example

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$$

We want to know what an element of $\mathrm{Col}\,(A)$ looks like, so we solve $A\mathbf{x} = \mathbf{b}$ and determine what $\mathbf{b}$ has to look like!

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 2 & 3 & 6 & b_2 \\ 3 & 1 & 2 & b_3 \end{array}\right] \xrightarrow{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 3 & 1 & 2 & b_3 \end{array}\right] \xrightarrow{R_3=R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & -2 & -4 & b_3 - 3b_1 \end{array}\right]$$

$$\xrightarrow{R_3=R_3+2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + 2b_2 - 7b_1 \end{array}\right] \xrightarrow{R_1=R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3b_1 - b_2 \\ 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + 2b_2 - 7b_1 \end{array}\right]$$

Finding a Basis of $\mathrm{Col}\,(A)$ Example Continued

So, all the systems that we can solve have the form of

$$\mathrm{Col}\,(A) = \left\{ \mathbf{b} \in \mathbb{R}^3 \,\middle|\, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ 7b_1 - 2b_2 \end{bmatrix} \right\}$$

Which can be written as

$$b_1 \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Meaning

$$\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

forms a basis of $\mathrm{Col}\,(A)$

An Easier Way to Compute a Basis of $\mathrm{Col}(A)$

1. Reduce to RREF
2. The columns of A corresponding to pivot columns form a basis of $\mathrm{Col}(A)$.

Example

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the first two columns of A are a basis for $\mathrm{Col}(A)$!

Showing These are Both Bases

We will show that

$$\text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right)$$

$$\text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right) = \left\{\mathbf{b} \in \mathbb{R}^3 \,\middle|\, \mathbf{b} = \begin{bmatrix}c_1 + c_2\\2c_1 + 3c_2\\3c_1 + c_2\end{bmatrix}\right\}$$

If we define $b_1 = c_1 + c_2, b_2 = 2c_1 + 3c_2,$ and $b_3 = 3c_1 + c_2$, then

$$7b_1 - 2b_2 = 7(c_1 + c_2) - 2(2c_1 + 3c_2) = 7c_1 + 7c_2 - 4c_1 - 6c_2 = 3c_1 + c_2 = b_3$$

This is exactly what we said the systems we can solve look like!

The Column Space is a subspace of $\mathbb{R}^m$

We claim that for any $A \in \mathbb{R}^{m \times n}$ that $\mathrm{Col}\,(A)$ is a subspace of $\mathbb{R}^m$.

1. $A\mathbf{0}_n = \mathbf{0}_m$, so $\mathbf{0} \in \mathrm{Col}\,(A)$.
2. Let $\mathbf{u}, \mathbf{v} \in \mathrm{Col}\,(A)$. This means there are some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{u}$ and $A\mathbf{y} = \mathbf{v}$. See that
$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{u} + \mathbf{v}$$
So, $\mathbf{u} + \mathbf{v} \in \mathrm{Col}\,(A)$.
3. Let $\mathbf{u} \in \mathrm{Col}\,(A)\,, c \in \mathbb{R}$. Therefore, there is some $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{u}$. See that
$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{u}.$$
So, $c\mathbf{u} \in \mathrm{Col}\,(A)$!

$\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$ Practice

For the following matrix $A \in \mathbb{R}^{4\times 3}$ find a basis for $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \\ 1 & 5 & 7 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \\ 1 & 5 & 7 \end{bmatrix} \xrightarrow{R_4 = R_4 - R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Col}\,(A)$ and $\mathrm{Nul}\,(A)$ Practice Continued

So, we have that

$$\mathrm{Nul}\,(A) = \text{Span}\left(\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$\mathrm{Col}\,(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 5 \end{bmatrix}\right)$$

What do we notice about the dimension of these spaces?

Relating to Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then we define:

Definition

Kernel: The kernel of T is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{0}_m$.
This is just the null space of the matrix associated with T!

Definition

Image or Range: The image or range of T denoted

$$\mathrm{Im}\,(T) = \mathrm{Range}\,(T)$$

is the set of all $\mathbf{b} \in \mathbb{R}^m$ such that there is some $\mathbf{x} \in \mathbb{R}^n$ where

$$T(\mathbf{x}) = \mathbf{b}$$

This is just the column space of the matrix associated with T!

Row Space of a Matrix

Let $A \in \mathbb{R}^{m\times n}$, then we can write it as

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

Where each $r_k^\top \in \mathbb{R}^n$ is a row of A.
Note: We are transposing the rows to make them column vectors!
We define $\mathrm{Row}\,(A)$ to be all linear combinations of the rows of A. Or:

$$\mathrm{Row}\,(A) = \left\{ \mathbf{b} \in \mathbb{R}^n \,\middle|\, \mathbf{b} = \sum_{k=1}^{m} c_k r_k^\top \right\} = \mathrm{Col}\left(A^\top\right)$$

Rank of a Matrix

Definition

Rank: The rank of a matrix is the number of linearly independent rows and columns. This is also the number of pivots and the dimension of $\mathrm{Col}\,(A)$!

Example

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \\ 3 & 1 & 2 \end{bmatrix}$, then we saw that we can row reduce to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Which, has 2 pivots, so $\mathrm{rank}\,(A) = 2$.

In addition, $\dim\left(\mathrm{Col}\,(A)\right) = \dim\left(\mathrm{Span}\,(\mathbf{a}_1, \mathbf{a}_2)\right) = 2$, so our definition is consistent!

Rank-Nullity Theorem

Theorem

Let $A \in \mathbb{R}^{m\times n}$, then we know that

$$\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = n$$

$$\operatorname{rank}(A) + \dim\left(\operatorname{Nul}\left(A^{\top}\right)\right) = m$$