Orthogonal Complement

Let V be a vector space with an inner product given by $\langle \cdot, \cdot \rangle$.

Definition

Orthogonal Complement: The orthogonal complement of a subspace of V (or equivalently $\mathbb{R}^n, \mathbb{C}^n$), W is given by

$$W^\perp = \{\mathbf{v} \in V \,|\, \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } w \in W\}$$

We read $W^\perp$ as "W perp" or "The orthogonal complement of W"

Note: This is the set of *all* vectors orthogonal to *all* vectors in W.

Example

Using the standard inner product and $V = \mathbb{R}^n$. Let $W = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right)$, then $W^\perp = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right)$

Computing Orthogonal Complements

Theorem

Let W be a subspace of V and A be a matrix such that $W = \mathrm{Col}\,(A)$. Then,

$$W^{\perp} = \mathrm{Nul}\left(A^{\top}\right)$$

Proving our Equality Part 1

Proof.

Let
$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}, W = \text{Span}\left(\mathbf{v}_1, \dots, \mathbf{v}_n\right)$$

We will first show that $W^{\perp} \subseteq \mathrm{Nul}\left(A^{\top}\right)$.

Let $\mathbf{x} \in W^{\perp}$. See that $A^{\top} = \begin{bmatrix} \mathbf{v}_1^{\top} \\ \vdots \\ \mathbf{v}_n^{\top} \end{bmatrix}$. Since $\mathbf{x} \in W^{\perp}$, we know that $\mathbf{v}_{\ell}^{\top}\mathbf{x} = 0$ for $\ell = 1, \dots, n$. So,

$$A^{\top}\mathbf{x} = \begin{bmatrix} \mathbf{v}_1^{\top} \\ \vdots \\ \mathbf{v}_n \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{v}_1^{\top}\mathbf{x} \\ \vdots \\ \mathbf{v}_n^{\top}\mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

So, $\mathbf{x} \in \mathrm{Nul}\left(A^{\top}\right)$ $\square$

Proving our Equality Part 2

Proof.

Let

$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}, W = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$$

We will now show that $\mathrm{Nul}(A^\top) \subseteq W^\perp$.

Let $\mathbf{x} \in \mathrm{Nul}(A^\top)$. This means $A^\top \mathbf{x} = \mathbf{0}$. From the previous slide, we have that

$$A^\top \mathbf{x} = \begin{bmatrix} \mathbf{v}_1^\top \\ \vdots \\ \mathbf{v}_n \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{v}_1^\top \mathbf{x} \\ \vdots \\ \mathbf{v}_n^\top \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

Now, let $\mathbf{w} \in W$. This means there exists some $c_1, \dots, c_n$ such that $\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$. See that

$$\mathbf{x}^\top \mathbf{w} = \mathbf{x}^\top(c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n) = c_1\mathbf{x}^\top\mathbf{v}_1 + \dots + c_n\mathbf{x}^\top\mathbf{v}_n = 0$$

So, $\mathbf{x} \in W^\perp$. Thus, our two spaces are the same! □

Algorithm for Computing Orthogonal Complements

In order to compute the orthogonal complement of a given space,W, we do the following

1. Determine a spanning set for our space If W is a span, then we just take the inside!
2. Write these vectors as rows of a matrix (Call it $A^\top$)
3. Compute $\mathrm{Nul}\left(A^\top\right)$
4. Write out a basis of this nullspace

Computing Orthogonal Set Example

Let's practice our algorithm!

$$W = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right)$$

So we construct and row reduce

$$A^\top = \begin{bmatrix} 2 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[R_2 = -R_2]{R_1 = \frac{R_1}{2}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

So, the null-space is given by the span of $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Properties of Orthogonal Complements

Let W be a subspace of our vector space V (V will be finite dimensional, meaning it has n basis vectors). Then we know

1. $W^\perp$ is also a subspace of V
2. $\left(W^\perp\right)^\perp = W$
3. $\dim(W) + \dim\left(W^\top\right) = n$.

Row Space of a Matrix

Definition

Row Space: The row space of a matrix $A \in \mathbb{R}^{m\times n}$, denoted $\mathrm{Row}\,(A)$ is the span of it's rows or equivalently:

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix} \qquad\qquad \mathrm{Row}\,(A) = \mathrm{Span}\,(\mathbf{a}_1, \dots, \mathbf{a}_m)$$

Fundamental Theorem of Linear Algebra

A fundamental theorem behind much of linear algebra is how our "fundamental" subspaces $(\mathrm{Col}(A), \mathrm{Row}(A), \mathrm{Nul}(A), \mathrm{Nul}(A^\top))$ relate to each other. It is summarized as

1. $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$
2. $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A)$
3. $\mathrm{Nul}(A)^\perp = \mathrm{Row}(A)$
4. $\mathrm{Nul}(A^\top)^\perp = \mathrm{Col}(A)$

Orthogonal (Orthonormal) Sets

Definition

Orthogonal Set: A set of *non-zero* vectors, $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, is an orthogonal set if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for all $i \neq j$. We instead say orthonormal if we also have $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = 1$ for all valid i.

Example

The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is orthogonal while $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is orthonormal

Orthogonal Sets are Linearly Independent

Theorem

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthogonal set. We also have that these vectors are linearly independent

Proof.

We will show the equation

$$c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m = \mathbf{0}$$

has only the trivial solution $c_1 = \cdots = c_m = 0$.
We apply both sides of our equality to our inner product with $\mathbf{u}_\ell$ for some $1 \leq \ell \leq m$, which gives us

$$\begin{aligned} 0 = \langle \mathbf{0}, \mathbf{u}_\ell \rangle &= \langle c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m, \mathbf{u}_\ell \rangle = c_1 \langle \mathbf{u}_1, \mathbf{u}_\ell \rangle + \cdots + c_m \langle \mathbf{u}_m, \mathbf{u}_\ell \rangle \\ &= c_\ell \langle \mathbf{u}_\ell, \mathbf{u}_\ell \rangle \end{aligned}$$

Since we know that $\mathbf{u}_\ell$ is non-zero, $\langle \mathbf{u}_\ell, \mathbf{u}_\ell \rangle \neq 0$, thus $c_\ell = 0$. Since ℓ was some arbitrary index, all must be 0. Therefore, we have a list of linearly independent vectors. $\square$