

# Inner Product

Let  $(V, \mathbb{F})$  be a vector space where  $V$  is the set our vectors come from and  $\mathbb{F}$  is the set our scalars come from (You can think of this as  $\mathbb{R}$  or  $\mathbb{C}$ )

## Definition

**Inner Product:** An **inner product** is a function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$  with the following properties for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , and  $a, b \in \mathbb{F}$ .

1.  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$  Note: If  $\mathbb{F} = \mathbb{R}$ , then we omit the conjugate!
2.  $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a \langle \mathbf{u}, \mathbf{w} \rangle + b \langle \mathbf{v}, \mathbf{w} \rangle$
3.  $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$
4.  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  if and only if  $\mathbf{v} = \mathbf{0}$ .

## Inner Product Example: $\mathbb{C}^n$ (standard) Part 1

Let  $V = \mathbb{C}^n$  and  $\mathbb{F} = \mathbb{C}$ . Then the following function is an inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \bar{\mathbf{y}}^\top \mathbf{x} = \sum_{k=1}^n x_k \bar{y}_k$$

Note that if we are in the real numbers, then we omit the conjugate of  $\mathbf{y}$ .

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Note: we sometimes abbreviate  $\bar{\mathbf{y}}^\top$  as  $\mathbf{y}^*$

## Inner Product Example: $\mathbb{C}^n$ (standard) Part 2

Property 1:  $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^n x_k \overline{y_k} = \sum_{k=1}^n \overline{y_k} x_k = \sum_{k=1}^n \overline{\overline{\overline{y_k} x_k}} = \overline{\sum_{k=1}^n y_k \overline{x_k}} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$$

Property 2:  $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$

$$\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = \sum_{k=1}^n (ax_k + by_k) \overline{z_k} = \sum_{k=1}^n ax_k \overline{z_k} + by_k \overline{z_k} = \sum_{k=1}^n ax_k \overline{z_k} + \sum_{k=1}^n by_k \overline{z_k} = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$$

## Inner Product Example: $\mathbb{C}^n$ (standard) Part 3

Property 3,4:  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$  and  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  if and only if  $\mathbf{x} = \mathbf{0}$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{k=1}^n x_k \overline{x_k} = \sum_{k=1}^n |x_k|^2 \geq 0$$

In addition, the only time  $\sum_{k=1}^n |x_k|^2 = 0$  is when all components are 0 or if  $\mathbf{x} = \mathbf{0}$

# Dot Product

## Definition

**Dot Product:** The **dot product** of two vectors in  $\mathbb{R}^n$  is a function given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^\top \mathbf{x}$$

## Theorem

*The **dot product** is an inner product.*

## Proof.

The 2 previous slides prove this. □

Note: For this course, we will only consider this inner product unless stated otherwise

# Norms

Let  $(V, \mathbb{F})$  be a vector space where  $V$  is the set our vectors come from and  $\mathbb{F}$  is the set our scalars come from (You can think of this as  $\mathbb{R}$  or  $\mathbb{C}$ )

## Definition

**Norm:** A **norm** is a function given by  $\|\cdot\| : V \rightarrow \mathbb{R}$  with the following properties for all  $\mathbf{x}, \mathbf{y} \in V$  and  $c \in \mathbb{F}$

1.  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
2.  $\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$
3.  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = \mathbf{0}$

# Induced Norms

## Theorem

Let  $(V, \mathbb{F})$  be a vector space with some inner product  $\langle \cdot, \cdot \rangle$ . Then the *induced norm* of this space is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

## Proof.

We can prove all 3 properties from the previous slide as consequences of us using the inner product. □

## Example of a Norm

If  $V = \mathbb{R}^n$  and  $\mathbb{F} = \mathbb{R}$ , then the induced norm is often called the “Euclidean Norm” and denoted as follows

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{k=1}^n x_k^2}$$

For this course, we will consider only this norm unless stated otherwise.



# Unit Vector

## Definition

**Unit Vector:** We say a vector is a **unit vector** if it has norm 1. In other words,  $\mathbf{x}$  is a unit vector if and only if

$$\|\mathbf{x}\| = 1$$

If we have any vector,  $\mathbf{x} \neq \mathbf{0}$ , then we can find a vector pointing in the same direction, but is also of *unit length* by doing:

$$\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

# Distance

For us, we can think of distance between vectors as “how large is the difference between two vectors”, or in other words, we say

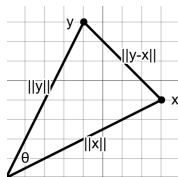
$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$$

Note: this idea is closely related to “Metric Spaces”<sup>1</sup>, which you will see in an analysis course.

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<sup>1</sup>[https://en.wikipedia.org/wiki/Metric\\_space#Definition](https://en.wikipedia.org/wiki/Metric_space#Definition)

# Angle Between Vectors



Using the law of cosines<sup>a</sup>, we have that

$$\|y - x\|^2 = \|x\|^2 + \|y\|^2 - 2 \|x\| \|y\| \cos(\theta)$$

$$(y - x)^T (y - x) = x^T x + y^T y - 2 \|x\| \|y\| \cos(\theta)$$

Which (assuming that  $x, y \neq 0$ ) can be solved for  $\theta$  to get

$$\theta = \cos^{-1} \left( \frac{x^T y}{\|x\| \|y\|} \right)$$

In higher dimensions and other vector spaces, this is how we define the angle between vectors

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<sup>a</sup>[https://en.wikipedia.org/wiki/Law\\_of\\_cosines](https://en.wikipedia.org/wiki/Law_of_cosines)

# Orthogonality

Let  $(V, \mathbb{F})$  be a vector space where  $V$  denotes the set our vectors come from,  $\mathbb{F}$  is the set our scalars come from, and we have some inner product  $\langle \cdot, \cdot \rangle$ .

## Definition

**Orthogonal Vectors:** We say that 2 vectors,  $(\mathbf{x}, \mathbf{y} \in V)$  are **orthogonal** (or **perpendicular**) if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

Note that assuming we have non-zero vectors, the angle between  $\mathbf{x}, \mathbf{y}$  would be  $90^\circ$ !

Since orthogonality is closely tied to our inner product, we will use our standard one for this course.

## Special Case for Orthogonality

### Theorem

*If  $V = \mathbb{R}^n$  (or equivalently  $\mathbb{C}^n$ ) with the usual inner product, then  $\mathbf{0}$  is orthogonal to every vector.*

### Proof.

Let  $\mathbf{x} \in \mathbb{R}^n$  and  $\langle \cdot, \cdot \rangle$  denote the standard inner product, then we have that

$$\langle \mathbf{0}, \mathbf{x} \rangle = \sum_{k=1}^n 0 \cdot x_k = 0$$

