Revisiting Vector Equations and Homogeneous Systems

If $A \in \mathbb{R}^{m \times n}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, then the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has the same solution set as the vector equation

$$x_1 \mathbf{a}_1 + \cdots + x_n \mathbf{a}_n = \mathbf{0}$$

which has the corresponding augmented matrix

$$\left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{array} \middle| \mathbf{0} \right]$$

▶ Homogeneous Linear Systems always have the trivial solution: $\mathbf{x} = \mathbf{0}$
▶ If a non-trivial solution exists, then:
  ▶ At least one $x_i \neq 0$
  ▶ At least one $\mathbf{a}_i$ can be written as a linear combination of the others!
▶ If there is only the trivial solution, then no columns can be written as a linear combination of the others!

Brief Proof (on board) Of the last bullet point.

Example

Determine whether the equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution.

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 1 & 6 & -10 & 0 \\ 0 & 3 & -6 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since x_3 is a free variable, we have non-trivial solutions

Python code to get this answer!

```python
import sympy as sym
A = sym.Matrix([[2, -1, 6, 0],
                [1, 6, -10, 0],
                [0, 3, -6, 0]])
display(A.rref())
```

What does it mean?

We get

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3$$
$$x_2 - 2x_3 = 0 \rightarrow x_2 = 2x_3$$

Where the solution set is

$$\mathbf{x} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Plugging in gives:

$$\mathbf{0} = A\mathbf{x} = -2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix}$$

Or equivalently

$$\begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}$$

So, one is a linear combination of the others!

Linear Independence

Definition

Linear Independence: We say that a set of $p > 1$ vectors in $\mathbb{R}^n$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if we **cannot** write one as a linear combination of the others. Otherwise, the set is linearly dependent.

Example

Since

$$\begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix},$$

we know that the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} \right\}$ is linearly dependent!

Showing Linear Independence

Theorem

To show $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, we can just show that the system

$$\begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_p & | & \mathbf{0} \end{bmatrix}$$

only has the trivial solution!

If we find another solution, then they are linearly dependent

Practice!

Determine if the following sets of vectors are linearly dependent or linearly independent

▶
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$$

▶
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

Conceptual Practice!

For each of the following questions, either come up with an example or think of a reason it's false

1. Give an example of 1 vector in $\mathbb{R}^3$ so that the set is linearly independent
2. Give an example of 2 vector in $\mathbb{R}^3$ so that the set is linearly independent
3. Give an example of 3 vector in $\mathbb{R}^3$ so that the set is linearly independent
4. Give an example of 4 vector in $\mathbb{R}^3$ so that the set is linearly independent

But what about $\mathbf{0}$?

> **Theorem**
>
> *If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains the zero vector, $\mathbf{0}$, then the set is linearly dependent.*

> **Proof.**
>
> Since we can reorder the list without changing the overall property, let $\mathbf{v}_1 = \mathbf{0}$.
> See that
>
> $$1\mathbf{0} + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$$
>
> $\square$

Number of Vectors and Their Dimension

Theorem

A set of p vectors, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.