Theorem Statement

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix $A \in \mathbb{R}^{n \times n}$.

1. *A is invertible*
2. *A has a pivot in every row*
3. *A has a pivot in every column*
4. *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution*
6. *The columns of A are linearly independent*
7. *The columns of A span all of $\mathbb{R}^n$.*
8. *T is invertible*
9. *T is injective*
10. *T is surjective*

We have the tools to prove all these, but really, it's just a summary of all the linear transformation and matrix algebra we've done so far!

One-sided inverse Implies the Other!

Theorem

Let $A \in \mathbb{R}^{n \times n}$. If there is some $B \in \mathbb{R}^{n \times n}$ such that

$$BA = I_n \text{ or } AB = I_n$$

Then, A is invertible and $A^{-1} = B$.

$AB = I_n$ case

Proof.

First, let $B \in \mathbb{R}^{n\times n}$ such that $AB = I_n$.
We will show that T is surjective by showing how we can get any $\mathbf{b} \in \mathbb{R}^n$.
Let $\mathbf{b} \in \mathbb{R}^n$. See that

$$\mathbf{b} = I_n\mathbf{b} = AB\mathbf{b} = A(B\mathbf{b}) = T(B\mathbf{b})$$

Since we have an input, $\mathbf{x} = B\mathbf{b}$ such that $T(\mathbf{x}) = \mathbf{b}$,
We know that T is surjective, so by above, A is invertible.
Also, see that $\square$

$BA = I_n$ case

Proof.

Let $B \in \mathbb{R}^{n \times n}$ such that $BA = I_n$.
We will show that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
Assume that $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. See that

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ BA\mathbf{x} &= B\mathbf{0} \\ I_n\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0} \end{aligned}$$

So, if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ must be $\mathbf{0}$!, which means A is invertible.
Next, we show $B = A^{-1}$

$$A^{-1} = I_nA^{-1} = BAA^{-1} = BI_n = B$$

□