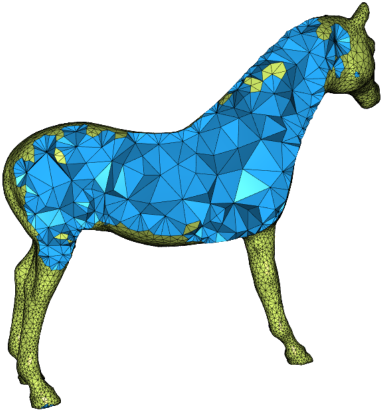
Applications of Linear Algebra to Computer Graphics



https://www.cg.tu-berlin.de/research/projects/harmonic-triangulations/

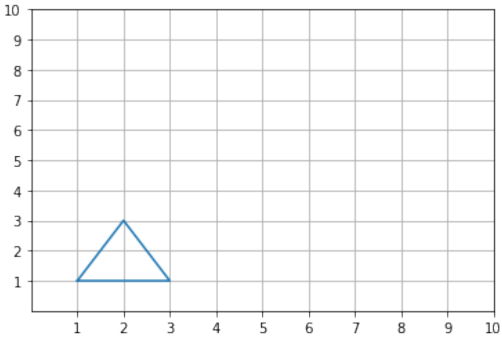
Image Credit: Star Wars low poly portraits, designed by Vladan Filipovic

Let's examine some ways to manipulate and display graphical images using matrices.

Scaling One Triangle

Consider one triangular piece. If we want to scale the triangle by a factor of three, then we apply the transformation

$$S : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}.$$



We have:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Vertical Shearing

If we would like to vertically shear the triangle by a factor of 2, we have

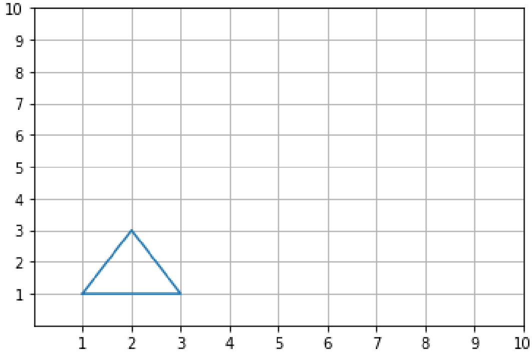
$$V : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

We have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$



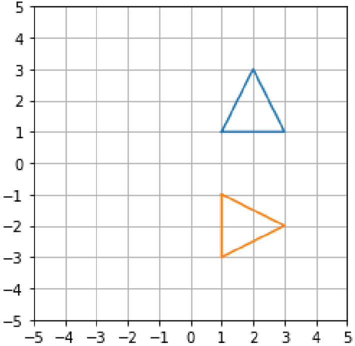
Rotation

If we would like to rotate the triangle counter-clockwise by an angle of θ, we have in general

$$R : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}.$$

For example a counterclockwise rotation by $\theta = \frac{3\pi}{2}$ would be given by

$$R : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$



We have:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Composing Maps

Imagine we want to apply three transformations to the triangle in the following order:

1. Scale the triangle by a factor of three.
2. Then vertically shear by a factor of 2 (in vertical direction only).
3. Finally rotate counter-clockwise by $\theta = 270^\circ = \frac{3\pi}{2}$ radians.

First apply the scaling matrix using matrix S:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Then apply the vertical shearing using V:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Finally, we rotate this result counter-clockwise by $\theta = \frac{3\pi}{2}$ using R:

$$\begin{bmatrix} \cos\left(\frac{3\pi}{2}\right) & -\sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{3\pi}{2}\right) & \cos\left(\frac{3\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

The image under the composition of these three transformations is $R\Big(V(S\mathbf{x})\Big) = (RVS)\mathbf{x}$

Composing Maps

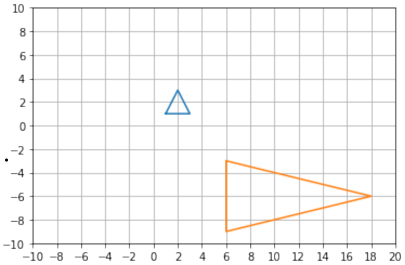
Imagine we want to apply three transformations to the triangle in the following order:

1. Scale the triangle by a factor of three.
2. Then vertically shear by a factor of 2 (in vertical direction only).
3. Finally rotate counter-clockwise by $\theta = 270^\circ = \frac{3\pi}{2}$ radians.

$$M : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto RVS\mathbf{x}.$$

where

$$RVS = \begin{bmatrix} \cos\left(\frac{3\pi}{2}\right) & -\sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{3\pi}{2}\right) & \cos\left(\frac{3\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -3 & 0 \end{bmatrix}.$$



Thus we have

$$\begin{bmatrix} 0 & 6 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Composing Linear Transformations

The composite of linear transformations A, B, and Cin the following order:

1. First apply transformation given be matrix A,
2. Then apply transformation given be matrix B, and
3. Finally apply transformation given be matrix C

Is equivalent to the linear transformation given by the product CBA.

Note we can extend this idea to compose any number of linear transformations.

Translations

To translate the vertex of the triangle at $(1, 1)$ to the left by 2 units and up by 3 units, we add vectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Doing the same for the other two vertices, we have

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

Homogeneous Coordinates

- Operations such as shearing, scaling, rotations are linear transformations.
  - These operations can be defined by matrix multiplication.
- A translation is not a linear transformation. We add vectors rather than multiply.
- How can we compose translations with the other linear transformations if translation cannot be represented as matrix multiplication?

One common way is to introduce homogeneous coordinates:

- Each point (x, y) in $\mathbb{R}^2$ can be identified with the point $(x, y, 1)$ in $\mathbb{R}^3$.
- We say (x, y) has homogeneous coordinate $(x, y, 1)$.

Homogeneous Coordinates

Now we can define translation by matrix multiplication. For example, if we want to shift the point $(3, 1)$ to the left by 2 units and up by the 3 units, then we have the product

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

In general, if we want to translate by h units in the horizontal direction and k units in the vertical direction, using homogeneous coordinates we have:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}.$$

Let's perform a composition of three different transformations to the triangle in the following order:

1. Scale the triangle by a factor of 3.
2. Rotate the triangle counter-clockwise by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

We have the following three matrices for the scaling, S, rotation, R, and translation B, respectively,

$$S = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying matrices we can now compose all three maps:

$$BRS = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Scale the triangle by a factor of 3.
2. Rotate the triangle counter-clockwise by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

Then we have each vertex (given in homogeneous coordinates) mapped as follows

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}.$$