

## Interpreting $A\mathbf{x} = \mathbf{b}$

The equation  $A\mathbf{x} = \mathbf{b}$  conceptually is:

- ▶ Finding some input  $\mathbf{x}$  to give us an output  $\mathbf{b}$
- ▶ Making sure  $A$  has enough information!

### Example

Some mappings we know from this class or previous are:

- ▶  $f : x \mapsto x^2$  maps from  $\mathbb{R} \rightarrow \mathbb{R}$ .
- ▶  $f : x \mapsto 7x$  maps from  $\mathbb{R} \rightarrow \mathbb{R}$ .
- ▶  $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$  maps from  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

- ▶  $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$  maps from  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ .
- ▶  $T : \mathbf{x} \mapsto A\mathbf{x}$  maps from  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ .

# Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

## Definition

**Transformation:** A transformation (or a function/mapping)  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is just a rule that assigns each vector  $\mathbf{x} \in \mathbb{R}^n$  to another vector  $T(\mathbf{x}) \in \mathbb{R}^m$ .

- ▶ All possible inputs,  $\mathbb{R}^n$  is the **domain** of  $T$ .
- ▶ The set where the mappings will live,  $\mathbb{R}^m$  is called the **codomain** of  $T$ .
- ▶ For each  $\mathbf{x}$  in the domain,  $T(\mathbf{x}) \in \mathbb{R}^m$  is called the **image** of  $\mathbf{x}$ .
- ▶ All possible images (or outputs) is called the **range** of  $T$ .

## Example

Consider

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix}$$

# Linear Transformations!

## Definition

**Linear Transformation:** A transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a **linear transformation** if and only if

- ▶ for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- ▶ for all  $\mathbf{v} \in \mathbb{R}^n$  and scalars  $c$ ,  $T(c\mathbf{v}) = cT(\mathbf{v})$

## Theorem

*Every matrix equation is a linear transformation, and vice versa (but only for  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ !)*

# Image of the Zero Vector

## Theorem

If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear map, then  $T(\mathbf{0}) = \mathbf{0}$ .

# Representing Transformations with Matrices

## Theorem

For all  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  there exists a unique  $A \in \mathbb{R}^{m \times n}$  such that

$$T(\mathbf{x}) = A\mathbf{x}$$

and is given by

$$A = [T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_n)]$$

## Example

Define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$$

Find the associated matrix  $A$  for this transformation. See that

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

## Another Example

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that

$$T \left( \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 8 \\ 16 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 8 \\ 8 \\ 20 \end{bmatrix}$$

Construct the  $A$  matrix associated with  $T$ .

## Another Example continued

Need to be able to find  $T(\mathbf{e}_1)$  and  $T(\mathbf{e}_2)$ !

So, we write  $\mathbf{e}_1$  and  $\mathbf{e}_2$  in terms of our given vectors. So we reduce:

$$\left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 4 & 4 & b_2 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -4 & b_2 - 4b_1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - \frac{b_2}{4} \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & \frac{b_2}{2} - b_1 \\ 0 & 1 & b_1 - \frac{b_2}{4} \end{array} \right]$$

So:

$$\mathbf{e}_1 = -1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \mathbf{e}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + -\frac{1}{4} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

## Another Example continued

$$T(\mathbf{e}_1) = T\left(-1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = -\begin{bmatrix} 6 \\ 8 \\ 16 \end{bmatrix} + \begin{bmatrix} 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T\left(\frac{1}{2} \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \frac{1}{2} T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) - \frac{1}{4} T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} 6 \\ 8 \\ 16 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

So:

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

## Now You Try!

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be such that

$$T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad T \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Compute the  $A$  associated with  $T$ .

## Properties of Linear Transformations

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. The mapping  $T$  is said to be:

- ▶ **Injective** if for each  $\mathbf{b} \in \mathbb{R}^m$ , we can find **at most one**  $\mathbf{x}$  such that  $T(\mathbf{x}) = \mathbf{b}$
- ▶ **Surjective** if for each  $\mathbf{b} \in \mathbb{R}^m$ , we can find **at least one**  $\mathbf{x}$  such that  $T(\mathbf{x}) = \mathbf{b}$

## Python Example

Consider  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ ,  $\mathbf{x} \mapsto A\mathbf{x}$  for

$$A = \begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

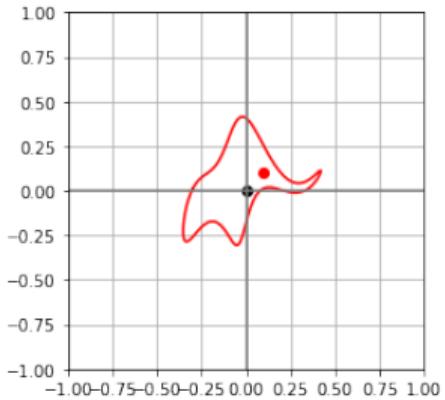
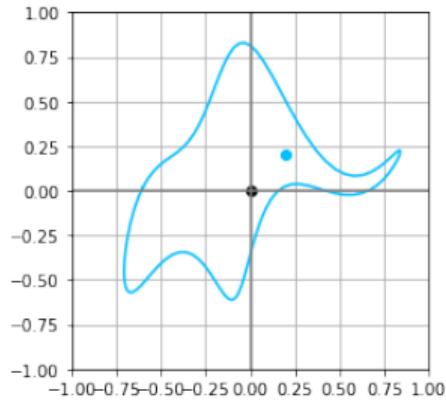
1. Find all  $\mathbf{x}$  such that  $T(\mathbf{x}) = \mathbf{0}$

$$2. \text{ Find all } \mathbf{x} \text{ (if any) such that } T(\mathbf{x}) = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix}$$

## Geometric Interpretation in $\mathbb{R}^2$

Consider the linear transformation given by  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \mathbf{x} \mapsto 0.5\mathbf{x}$ . Find the standard matrix  $A$  for this linear transformation.

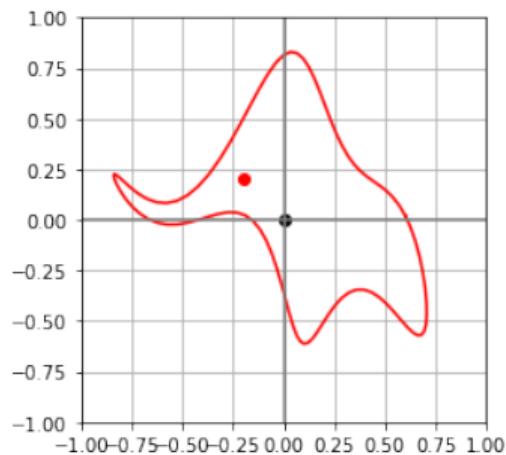
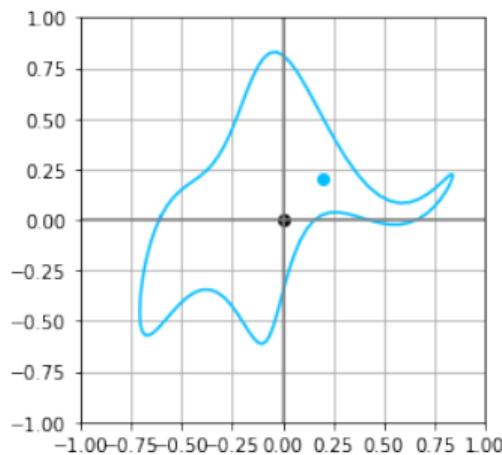
We have  $T(\mathbf{e}_1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$  and  $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$  giving the matrix  $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ .



## Geometric Interpretation in $\mathbb{R}^2$

Consider the linear transformation given by  $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

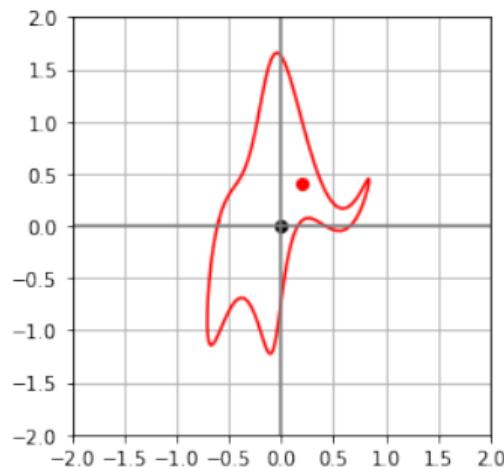
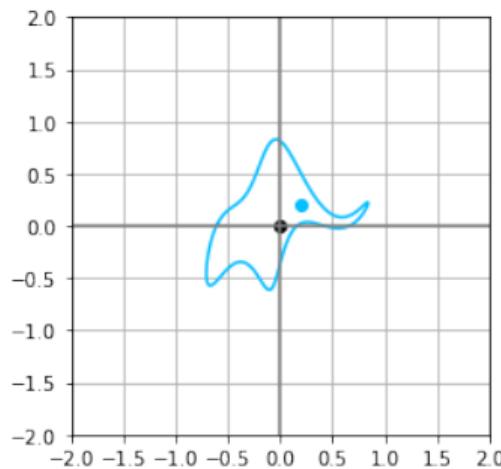
We have  $T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$  and  $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  giving the geometric interpretation seen below.



## Contractions and Expansions

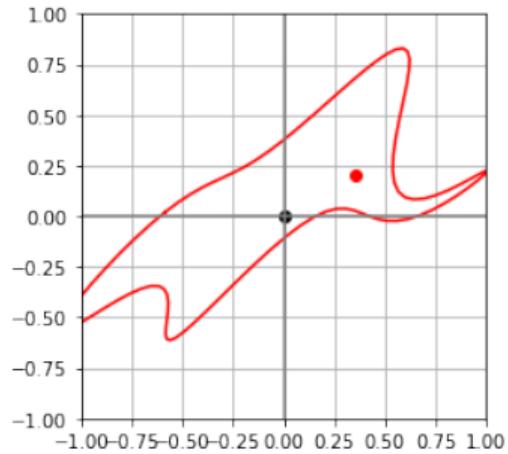
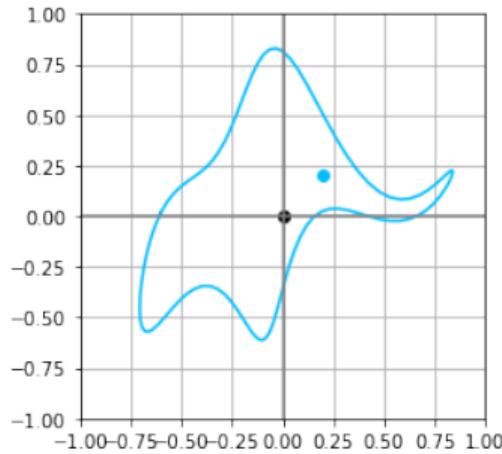
Consider the linear transformation given by  $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

We have  $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$  giving the geometric interpretation seen below.



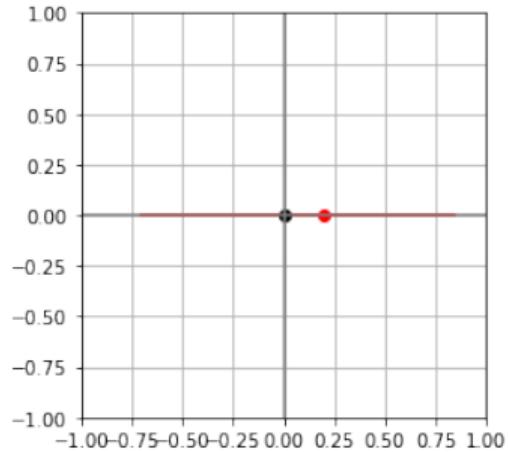
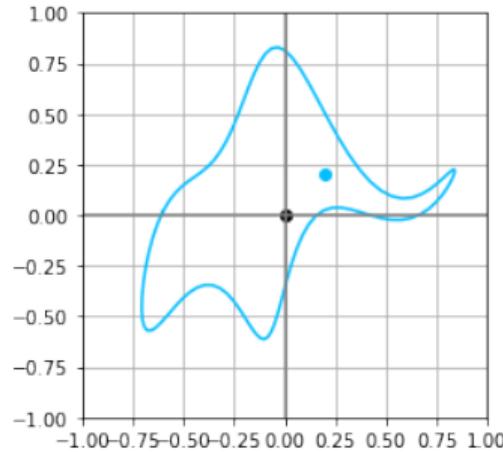
# Shear Transformations

Consider the linear transformation given by  $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0.75 \\ 0 & 1 \end{bmatrix}$



# Projections

Consider the linear transformation given by  $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



## Theorem

A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is injective if and only if  $T(\mathbf{x}) = \mathbf{0}$  has only the trivial solution

## Summary

A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with associated matrix  $A$  is **injective** if and only if

- ▶  $T(\mathbf{x}) = \mathbf{0}$  has only the trivial solution.
- ▶ The solution set to  $A\mathbf{x} = \mathbf{0}$  has no free variables.
- ▶ The matrix  $A$  has a pivot in every column.
- ▶ **The columns of  $A$  are linearly independent.**

A linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with associated matrix  $A$  is **surjective** if and only if

- ▶ For any  $\mathbf{b} \in \mathbb{R}^m$ , there exists at least one  $\mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{b}$ .
- ▶ **The columns of  $A$  span all of  $\mathbb{R}^m$ .**