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In fact, the determinant the unique function with these properties! But, proving this with the tools we have is difficult, so we will just take this for granted.

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1. Reduce to reduced row echelon form
2. Do operations in reverse following previous rules!

Practice!

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

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$$\det(A) = 1$$

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Practice!

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = -2$$

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Now you try!

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Special Types of Matrices

To make our later discussion easier, we define two new kinds of matrices

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$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & \dots & u_{2,n-1} & u_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & u_{n-1.n-1} & u_{n-1,n} \\ 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix}$$

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Definition

A matrix $U \in \mathbb{R}^{n \times n}$ is Upper Triangular if:

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Lower triangular

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Definition

A matrix $U \in \mathbb{R}^{n \times n}$ is Upper Triangular if:

$$u_{ij} = 0 \text{ for all } 1 \leq j < i \leq n$$

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$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & \dots & u_{2,n-1} & u_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & u_{n-1.n-1} & u_{n-1,n} \\ 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix}$$

Lower triangular

$$L = \begin{bmatrix} \ell_{11} & 0 & \dots & 0 & 0 \\ \ell_{12} & \ell_{22} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \ell_{1,n-1} & \ell_{1n} & \ddots & \ell_{n-1.n-1} & 0 \\ \ell_{2,n-1} & \ell_{2n} & \dots & \ell_{n-1,n} & \ell_{nn} \end{bmatrix}$$

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A matrix $U \in \mathbb{R}^{n\times n}$ is Upper Triangular if:

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Definition

A matrix $L \in \mathbb{R}^{n\times n}$ is Lower Triangular if:

$$\ell_{ij} = 0 \text{ for all } 1 \leq i < j \leq n$$

Determinant of a Matrix with a 0 row

A matrix with a 0 row will look something like below

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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So,

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We claim that in both cases, the determinant of A is the product of the elements on the diagonal

Less than n pivots

What does this look like?

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This idea extends to larger matrices too! Try to think about what that proof would look like!

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And:

n pivots

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And:

$$\det(A) = \det(B) \cdot a_{11} \cdot a_{22} \cdot a_{33} = a_{11} a_{22} a_{33}$$

General 2×2 formula

Theorem

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\det(A) = ad - bc$$

$a = 0$ case

Proof.

If $a = 0$, we need to have a pivot in A_{11}

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If $a = 0$, we need to have a pivot in A_{11}

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

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Proof.

If $a = 0$, we need to have a pivot in A_{11}

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = B$$

See that $\det(A) = -\det(B) = -bc$

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Proof.

If $a = 0$, we need to have a pivot in A_{11}

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = B$$

See that $\det(A) = -\det(B) = -bc = ad - bc$. $\square$

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If $a \neq 0$, we just need to eliminate c!

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If $a \neq 0$, we just need to eliminate c!

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{c}{a} R_1} \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix} = B$$

See that $\det(A) = \det(B) = a\left(d - \frac{bc}{a}\right)$

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Proof.

If $a \neq 0$, we just need to eliminate c!

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See that $\det(A) = \det(B) = a\left(d - \frac{bc}{a}\right) = ad - bc$. $\square$

General 3×3 formula

We could derive this formula, but it would be easier with Section 4.2, which we will not be covering in class.

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Theorem

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Then

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

More Practice

Determine if the determinant of the following systems is 0 or not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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3. $\det(AB) = \det(A)\det(B)$
4. $\det\left(A^\top\right) = \det(A)$