Diagonal Matrix Arithmetic

To motivate this section, let's look at how diagonal matrices multiply Let $D_1 = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, D_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, see that

$$D_1 D_2 = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \begin{bmatrix} 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \begin{bmatrix} 0 & 0 & z \end{bmatrix}$$

$$\begin{bmatrix} xa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & zc \end{bmatrix} = \begin{bmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{bmatrix}$$

Powers of Diagonal Matrices

From the previous slide, we see that for any diagonal $D = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$, we have

$$D^n = \begin{bmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \end{bmatrix}$$

Diagonalizability

Definition

Diagonalizable: We say that a matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if it is similar to a diagonal matrix.
Or equivalently, A is diagonalizable if there exists an invertible $C \in \mathbb{R}^{n\times n}$, and diagonal $D \in \mathbb{R}^{n\times n}$ such that

$$A = CDC^{-1}$$

An application is in computing matrix powers!

$$A^n = \left(CDC^{-1}\right)^n = \underbrace{CDC^{-1}CDC^{-1}\cdots CDC^{-1}CDC^{-1}}_{n \text{ times}} = CD^nC^{-1}$$

Diagonalization Theorem

Theorem

Let $A \in \mathbb{R}^{n \times n}$. Then, A is diagonalizable if and only if it has n linearly independent eigenvectors.

This means that if $A = CDC^{-1}$, then we have that

$$C = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

where $(\mathbf{v}_1, \lambda_1), \dots, (\mathbf{v}_n, \lambda_n)$ are each eigenpairs.

Remember that eigenvectors associated with distinct eigenvalues are linearly independent. So, if A has n distinct eigenvalues, then it is diagonalizable!

Diagonalization is not Unique 2×2 Example

Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$. We see that for $V_1, V_2, D \in \mathbb{R}^{2\times 2}$ as given below,

$$V_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, V_2 = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

We have

$$AV_1 = V_1 D \qquad AV_2 = V_2 D$$

But $V_1 \neq V_2$.

Diagonalization

In order to diagonalize a matrix, $A \in \mathbb{R}^{n \times n}$, we do the following

1. Compute all eigenvalues $\lambda_1, \ldots, \lambda_k$ (k may not be n)
2. Compute a basis for each $E(A, \lambda_\ell)$.
3. If the total number of basis vectors is less than n, then A is not diagonalizable, Otherwise, continue
4. Now $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (Eigenspace basis vectors!) form the columns of C, and their associated eigenvalues form the diagonal of D.

Diagonalization Example

Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ which has characteristic polynomial $-\lambda^3 + 8\lambda^2 - 13\lambda + 6$, which has roots $1, 6$. So we compute a basis for $E(A, 1)$ and $E(A, 6)$. We first do $E(A, 1)$.

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \xrightarrow{B=A-I} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Whose null space has a basis of $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Diagonalization Example Part 2

Now, we compute a basis for $E(A, 6)$. For

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \xrightarrow{B=A-6I} \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -1 \end{bmatrix} \xrightarrow{R_3=R_3-\frac{1}{2}R_1} \begin{bmatrix} -4 & 0 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2=-\frac{R_2}{5}]{R_1=-\frac{R_1}{4}} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whose null space has a basis of $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$.

Diagonalization Example Part 3

So we can say that A is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ with $C = \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ or

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Multiplicity of a Root Review

Recall that in the context of polynomials the multiplicity of a root is the number of times it is present in factored form.

Example

For the polynomial $x^3 - 3x + 2$, we can factor it into

$$(x-1)^2(x+2)$$

So, $x = 1$ is a root with multiplicity 2 and $x = -2$ is a root with multiplicity 1.

Eigenvalue Multiplicities

Definition

Let $A \in \mathbb{R}^{n \times n}$ with λ as an eigenvalue of A.
Algebraic Multiplicity: The algebraic multiplicity of λ is the multiplicity as a root of the characteristic polynomial of A.
Geometric Multiplicity: The geometric multiplicity of λ is the dimension of it's eigenspace (or $\dim(E(A, \lambda))$).

Let $A \in \mathbb{R}^{n \times n}$ and λ be an eigenvalue of A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda)$$

Variant of Diagonalizability Theorem

Theorem

Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:

1. *A is diagonalizable*
2. *The sum of the geometric multiplicities of all eigenvalues of A is equal to n.*

Finding Multiplicity Example

Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 2 \\ 2 & 3 & -1 \end{bmatrix}$. Find the algebraic and geometric multiplicities of 1.

The characteristic polynomial of A is

$$-\lambda^3 - 3\lambda^2 + 9\lambda - 5 = -(\lambda - 1)^2(\lambda + 5)$$

So the algebraic multiplicity of 1 is 2.
We now row reduce $A - I$

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & -4 & 2 \\ 2 & 3 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 3 & -2 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - \frac{1}{2}R_2} \begin{bmatrix} 2 & 3 & -2 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

There are 2 pivot variables, so $\dim(E(A, 1)) = 2$

Finding Multiplicities Practice

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has only 1 as an eigenvalue. Compute the algebraic and geometric multiplicities of 1.

Multiplicities for Similar Matrices

Theorem

Let $A, B \in \mathbb{R}^{n \times n}$ such that A and B are similar **and** *λ be an eigenvalue of both A, B. Then:*

1. *The algebraic multiplicity of λ is the same for A and B.*
2. *The geometric multiplicity of λ is the same for A and B.*