

Homogeneous Systems

Definition

Homogeneous: A system of equations is **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

Example

$$\begin{aligned}x_1 + x_2 &= 1 \\x_1 - 4x_2 &= 0\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 0 \\x_1 - 4x_2 &= 0\end{aligned}$$

Every homogeneous system has a **trivial solution** where $\mathbf{x} = \mathbf{0}$.

Nontrivial Solutions Existence

Recall that the system $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$ has:

- ▶ A unique solution if A has a pivot in every row
- ▶ Infinite number of solutions if A has a solution and at least one free variable

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the solution set has at least one free variable.

Solution Sets of Consistent Systems

Find the solution set to the homogeneous system write in vector form (from lecture 4):

$$x_1 + 3x_2 - 2x_3 = 0$$

$$-2x_1 - 6x_2 + 4x_3 = 0$$

More Practice!

Find the solution set to the **nonhomogeneous** system

$$x_1 + 3x_2 + 7x_3 = 4$$

$$x_2 + 3x_3 = 5$$

$$-2x_1 - 4x_2 - 8x_3 = 2$$

Comparing Homogeneous and Nonhomogeneous Solutions

$$\begin{aligned}x_1 + 3x_2 + 7x_3 &= 0 \\x_2 + 3x_3 &= 0 \\-2x_1 - 4x_2 - 8x_3 &= 0\end{aligned}$$

$$\begin{aligned}x_1 + 3x_2 + 7x_3 &= 4 \\x_2 + 3x_3 &= 5 \\-2x_1 - 4x_2 - 8x_3 &= 2\end{aligned}$$

Infinite Solutions from Homogeneous System

Theorem

Let $A\mathbf{x} = \mathbf{b}$ be a consistent system with solution $\mathbf{x} = \mathbf{p}$. Then there exists a solution set of vectors of the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}.$$

Where $t\mathbf{v}$ is a solution to the *homogeneous* system $A\mathbf{x} = \mathbf{0}$.

Proof.

We will show this by showing $A\mathbf{x} = A(\mathbf{p} + t\mathbf{v}) = \mathbf{b}$.

$$\begin{aligned} A\mathbf{x} &= A(\mathbf{p} + t\mathbf{v}) \\ &= A\mathbf{p} + A(t\mathbf{v}) \\ &= \mathbf{b} + \mathbf{0} \\ &= \mathbf{b} \end{aligned}$$

