

Vectors



Well not that kind of Vector

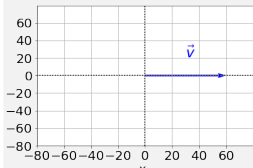
But the idea is the same!

Definition

Vector: A vector is an entity with **direction** and **length** (or magnitude!)

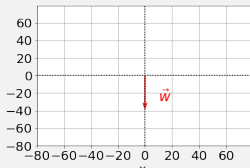
Example

A car traveling 60 miles per hour directly East.



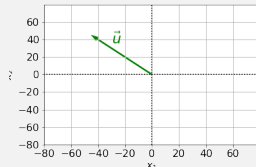
$$\mathbf{v} = \vec{v} = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

A car traveling 40 miles per hour directly South.



$$\mathbf{w} = \vec{w} = \begin{bmatrix} 0 \\ -40 \end{bmatrix}$$

A car traveling 64 miles per hour directly North West.



$$\mathbf{u} = \vec{u} = \begin{bmatrix} -\frac{64}{\sqrt{2}} \\ \frac{64}{\sqrt{2}} \end{bmatrix}$$

Vectors in \mathbb{R}^n

Definition

A **column vector** (which we will from now on refer to as a vector) is a matrix with only 1 column. We denote vectors by:

- ▶ a lowercase bold letter like \mathbf{v} or
- ▶ using an arrow above a lowercase letter like \vec{v}

If the vector \mathbf{v} has n different real numbers, then we say that $\mathbf{v} \in \mathbb{R}^n$, and

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Vector Equality

Definition

Vector Equality. We say that two vectors, $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ are **equal** if and only if

- ▶ They have the **same direction and length** or
- ▶ All entries of the two vectors are the same:

$$v_k = w_k \text{ for all } 1 \leq k \leq n$$

Vector Arithmetic (Addition)

Definition

We define adding two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ as $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$. Note: This means that we can only add vectors with the same dimensions together! And the result will have the same dimension!

Example

Adding the following vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ -5 \\ 10 \end{bmatrix}$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 + 4 \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 0 + 7 \\ -1 + -5 \\ 5 + 10 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 15 \end{bmatrix}$$

Vector Arithmetic (Scalar Multiplication)

Definition

A **scalar** is a quantity with only magnitude and no direction. Think about numbers like 2, $\sqrt{5}$, π , etc.

Definition

Scalar Product: For a vector $\mathbf{v} \in \mathbb{R}^n$, and a scalar $c \in \mathbb{R}$. We define the **scalar product** as $c\mathbf{v} = \begin{bmatrix} c \cdot v_1 \\ \vdots \\ c \cdot v_n \end{bmatrix}$

Note: This means that we can multiply any scalar by any vector, and the dimension of the output is the same as the vector input

Example

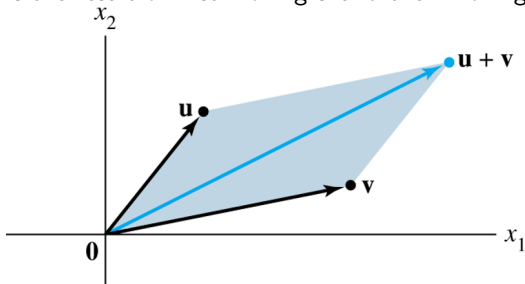
Computing the scalar product of the following

$$c = -3, \mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \quad c\mathbf{v} = \begin{bmatrix} -3 \cdot 2 \\ -3 \cdot -5 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

Geometric Descriptions of Vectors in \mathbb{R}^2

We can think of the vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ as a directed line segment (an arrow).

Consider two vectors \mathbf{u} and $\mathbf{v} \in \mathbb{R}^2$. The sum $\mathbf{u} + \mathbf{v}$ is the result of first moving \mathbf{u} and then moving \mathbf{v} .



Consider the vector $\mathbf{u} \in \mathbb{R}^2$ and nonzero scalar c . The scalar product $c\mathbf{u}$ gives a vector parallel to \mathbf{u} whose magnitude has been scaled by a factor of c .

- ▶ If $c > 0$, then $c\mathbf{u}$ has the same direction as \mathbf{u} .
- ▶ If $c < 0$, then $c\mathbf{u}$ has the opposite direction as \mathbf{u} .
- ▶ If $|c| < 1$, then $c\mathbf{u}$ is a compression of \mathbf{u} .
- ▶ If $|c| > 1$, then $c\mathbf{u}$ is a stretching of \mathbf{u} .

Parallelogram Rule for Addition: If \mathbf{u} and \mathbf{v} are in \mathbb{R}^2 , then the sum $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram spanned by \mathbf{u} and \mathbf{v} .

Properties of Vector Arithmetic

The **zero vector** is defined as the vector in \mathbb{R}^n with no magnitude, and it is denoted by $\mathbf{0}$ or $\vec{0}$.

As a result of this definition, it follows that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^n and all scalars c and d :

- | | |
|--|--|
| (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ | (vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$ |
| (iv) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | (viii) $1\mathbf{u} = \mathbf{u}$ |

Linear Combinations of Vectors

Definition

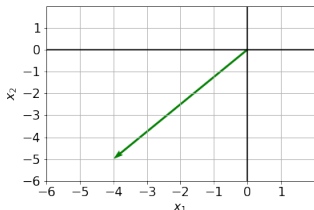
Linear Combination: A linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ with weights $c_1, \dots, c_p \in \mathbb{R}$ is the vector \mathbf{y} given by

$$\mathbf{y} = c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \sum_{k=1}^p c_k \mathbf{v}_k$$

Example

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix} + \begin{bmatrix} -3 \cdot -1 \\ -3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 + 3 \\ 2 - 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Giving Directions in a Grid



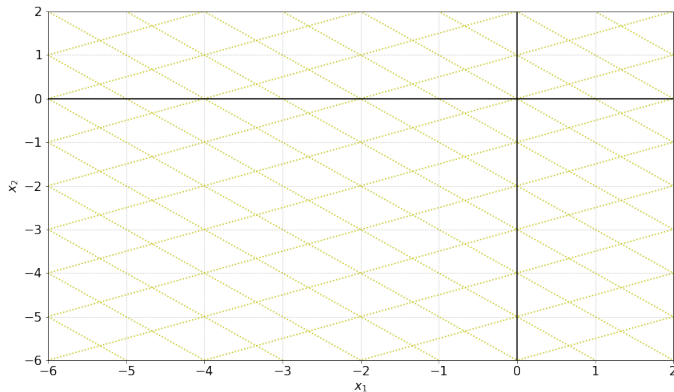
Example

Find constants x_1 and x_2 such that

$$\begin{bmatrix} -4 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the **standard column vectors**.

Giving Directions in a Different System



Example

Find constants x_1 and x_2 such that

$$\begin{bmatrix} -4 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving for the Weights

First, let's simplify our statement!

$$\begin{bmatrix} -4 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

Which becomes

$$-4 = 2x_1 - 1x_2$$

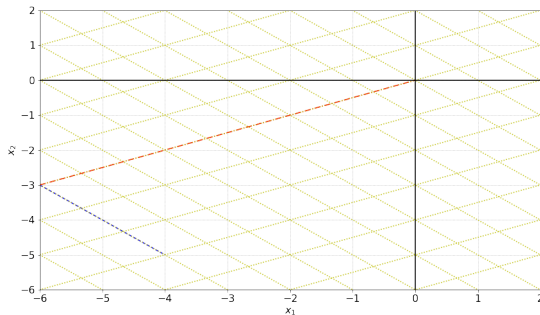
$$-5 = 1x_1 + 1x_2$$

Gaussian Elimination can solve this!

$$\left[\begin{array}{cc|c} 2 & -1 & -4 \\ 1 & 1 & -5 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \end{array} \right]$$

So, $(x_1, x_2) = (-3, -2)$. Or we can say $\mathbf{x} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

What Our Directions Look Like



$$\mathbf{x} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

What about an arbitrary vector $\mathbf{w} \in \mathbb{R}^2$? Can we find an \mathbf{x} such that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Span of a Set of Vectors

Definition

Span: Let $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$. Then we define $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ to be the subset of \mathbb{R}^n containing all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$. In other words

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \sum_{k=1}^p c_k \mathbf{v}_k \text{ for some } c_1, \dots, c_p \in \mathbb{R} \right\}$$

How do we test if $\mathbf{y} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$?

Solution

Solve the system

$$\left[\begin{array}{ccc|c} \mathbf{v}_1 & \cdots & \mathbf{v}_p & \mathbf{y} \end{array} \right]$$

Span Example

Example

Describe the span of the vectors $\mathbf{v}_1, \mathbf{v}_2$ given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Solution

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

So,

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \begin{bmatrix} c_1 \\ 2c_2 \\ c_1 + c_2 \end{bmatrix} \text{ for some } c_1, c_2 \in \mathbb{R} \right\}$$

Span of a Set of Vectors Practice

Describe the span of the following vectors

1. $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Span of a Set of Vectors Practice Solutions

Describe the span of the following vectors

1. $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

$$\text{Span}(\mathbf{v}_1) = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \begin{bmatrix} 0 \\ c_1 \\ c_1 \end{bmatrix} \text{ for some } c_1 \in \mathbb{R} \right\}$$

2. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \left\{ \mathbf{y} \in \mathbb{R}^2 \mid \mathbf{y} = \begin{bmatrix} 2c_1 - 4c_2 \\ c_1 - 2c_2 \end{bmatrix} \text{ for some } c_1, c_2 \in \mathbb{R} \right\}$$

3. $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\text{Span}(\mathbf{v}_1) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

4. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \\ -2c_1 - c_2 \end{bmatrix} \text{ for some } c_1, c_2 \in \mathbb{R} \right\}$$

More Span Practice

Determine if $y = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$ is in $\text{Span} \left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

Perform Gaussian Elimination on the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ -2 & -1 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 12 \end{array} \right]$$

No!

Systems with Infinite Solutions

Recall for the system,

$$1x_1 + 2x_2 + x_3 = -2$$

$$1x_1 + 3x_2 - 2x_3 = 1$$

Our answer is

$$\begin{cases} x_1 &= -8 - 7x_3 \\ x_2 &= 3 + 3x_3 \\ x_3 &\text{is free} \end{cases}$$

But writing this can be cumbersome!

Using Vectors!

We recall from earlier that we can write out our answer in terms of vectors!

$$\mathbf{x} = \begin{bmatrix} -8 - 7x_3 \\ 3 + 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -7x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$$

But what is x_3 ? It's just some real number, so we can use any letter we want. Let's use s .

$$\mathbf{x} = \begin{bmatrix} -8 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$$

Span of a Set of Vectors Conceptual Practice

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ be vectors in \mathbb{R}^n . Prove that if $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, then $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$