Interpreting $A\mathbf{x} = \mathbf{b}$

The equation $A\mathbf{x} = \mathbf{b}$ conceptually is:

- Finding some input $\mathbf{x}$ to give us an output $\mathbf{b}$
- Making sure A has enough information!

Example

Some mappings we know from this class or previous are:

- $f : x \mapsto x^2$ maps from $\mathbb{R} \to \mathbb{R}$.
- $f : x \mapsto 7x$ maps from $\mathbb{R} \to \mathbb{R}$.
- $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^2$.
- $f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ maps from $\mathbb{R}^2 \to \mathbb{R}^3$.
- $T : \mathbf{x} \mapsto A\mathbf{x}$ maps from $\mathbb{R}^n \to \mathbb{R}^m$.

Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

Transformation: A transformation (or a function/mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is just a rule that assigns each vector $\mathbf{x} \in \mathbb{R}^n$ to another vector $T(\mathbf{x}) \in \mathbb{R}^m$.

- All possible inputs, $\mathbb{R}^n$ is the domain of T.
- The set where the mappings will live, $\mathbb{R}^m$ is called the codomain of T.
- For each $\mathbf{x}$ in the domain, $T(\mathbf{x}) \in \mathbb{R}^m$ is called the image of $\mathbf{x}$.
- All possible images (or outputs) is called the range of T.

Example

Consider

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix}$$

Linear Transformations!

Definition

Linear Transformation: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if

- for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- for all $\mathbf{v} \in \mathbb{R}^n$ and scalars c, $T(c\mathbf{v}) = cT(\mathbf{v})$

Theorem

Every matrix equation is a linear transformation, and vice versa (but only for $T : \mathbb{R}^n \to \mathbb{R}^m$!)

Image of the Zero Vector

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear map, then* $T(\mathbf{0}) = \mathbf{0}$*.*

Proof.

Let $\mathbf{x} \in \mathbb{R}^n$ and $T(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m$. See that

$$T(\mathbf{0}) = T(\mathbf{x} - \mathbf{x}) = T(\mathbf{x}) - T(\mathbf{x}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

$\square$

Representing Transformations with Matrices

Theorem

For all $T : \mathbb{R}^n \to \mathbb{R}^m$ *there exists a unique* $A \in \mathbb{R}^{m \times n}$ *such that*

$$T(\mathbf{x}) = A\mathbf{x}$$

and is given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

Example

Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$$

Find the associated matrix A for this transformation. See that

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Another Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \\ 16 \end{bmatrix} \qquad \text{and } T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 8 \\ 20 \end{bmatrix}$$

Construct the A matrix associated with T.

Another Example continued

Need to be able to find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$!
So, we write $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of our given vectors. So we reduce:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 4 & 4 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -4 & b_2 - 4b_1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - \frac{b_2}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{b_2}{2} - b_1 \\ 0 & 1 & b_1 - \frac{b_2}{4} \end{array}\right]$$

So:

$$\mathbf{e}_1 = -1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \mathbf{e}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + -\frac{1}{4} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Another Example continued

$$T(\mathbf{e}_1) = T\left(-1\begin{bmatrix}1\\4\end{bmatrix} + 1\begin{bmatrix}2\\4\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\4\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = -\begin{bmatrix}6\\8\\16\end{bmatrix} + \begin{bmatrix}8\\8\\20\end{bmatrix} = \begin{bmatrix}2\\0\\4\end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T\left(\frac{1}{2}\begin{bmatrix}1\\4\end{bmatrix} - \frac{1}{4}\begin{bmatrix}2\\4\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\4\end{bmatrix}\right) - \frac{1}{4}T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}6\\8\\16\end{bmatrix} - \frac{1}{4}\begin{bmatrix}8\\8\\20\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

So:

$$A = \begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix} = \begin{bmatrix}2 & 1\\0 & 2\\4 & 3\end{bmatrix}$$

Now You Try!

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}4\\1\end{bmatrix}$$

Compute the A associated with T.

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 1 & 1 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -1 & b_2 - b_1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_1 - b_2 \end{array}\right]$$

So,

$$T(\mathbf{e}_1) = -T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, \qquad T(\mathbf{e}_2) = 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$$

And:

$$A = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$$

Properties of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation The mapping T is said to be:

- Injective if for each $\mathbf{b} \in \mathbb{R}^m$, we can find **at most one x** such that $T(\mathbf{x}) = \mathbf{b}$
- Surjective if for each $\mathbf{b} \in \mathbb{R}^m$, we can find **at least one x** such that $T(\mathbf{x}) = \mathbf{b}$

Python Example

Consider $T : \mathbb{R}^4 \to \mathbb{R}^4$, $\mathbf{x} \mapsto A\mathbf{x}$ for

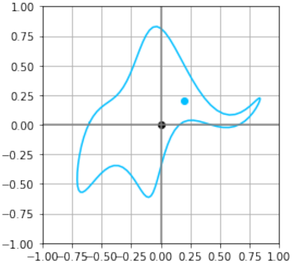
$$A = \begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$$

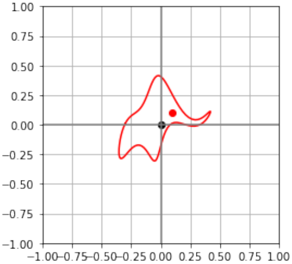
1. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$
2. Find all $\mathbf{x}$ (if any) such that $T(\mathbf{x}) = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix}$

Geometric Interpretation in $\mathbb{R}^2$

Consider the linear transformation given by $T : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto 0.5\mathbf{x}$. Find the standard matrix A for this linear transformation.

We have $T(\mathbf{e}_1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$ giving the matrix $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$.

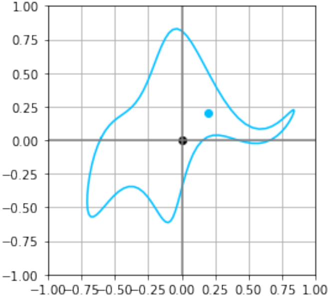


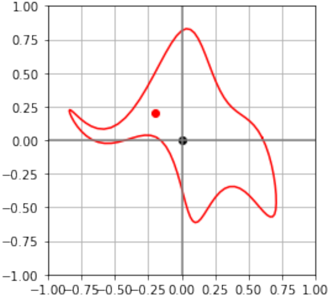


Geometric Interpretation in $\mathbb{R}^2$

Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

We have $T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ giving the geometric interpretation seen below.

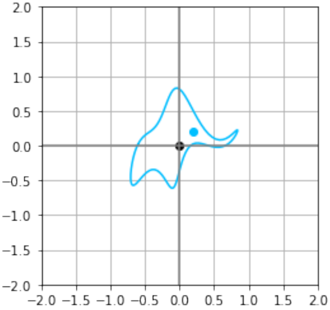


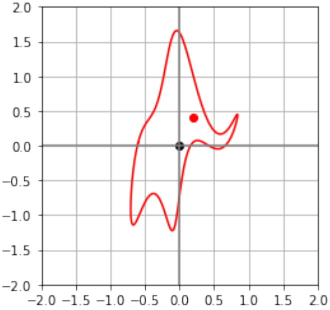


Contractions and Expansions

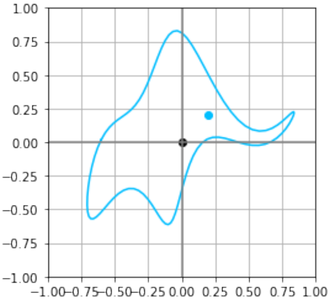
Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

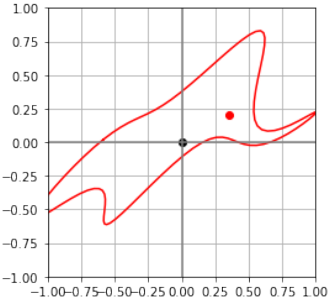
We have $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ giving the geometric interpretation seen below.





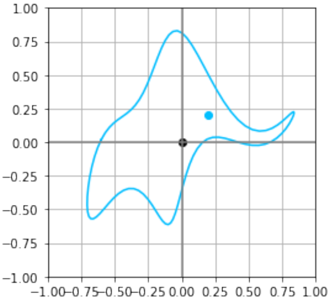
Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0.75 \\ 0 & 1 \end{bmatrix}$

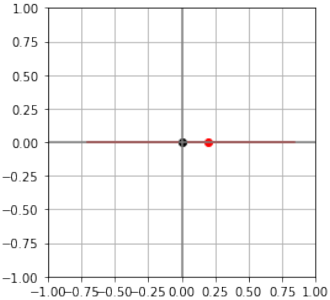




Projections

Consider the linear transformation given by $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$





Theorem

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution

Proof.

$\implies$ direction. Assume that T is injective and there is some $\mathbf{v} \neq \mathbf{0}$ such that $T(\mathbf{v}) = \mathbf{0}$. Well, this contradicts the fact that T is injective because $T(\mathbf{0}) = T(\mathbf{v}) = \mathbf{0}$.

$\impliedby$ direction. Assume T is not injective. This means there are some $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \neq \mathbf{v}$, and $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{b} \in \mathbb{R}^m$ and $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. See that

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Since $\mathbf{u} \neq \mathbf{v}$, we know that $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$. This contradicts the fact that $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. $\square$

Summary

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A is injective if and only if

- $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.
- The solution set to $A\mathbf{x} = \mathbf{0}$ has no free variables.
- The matrix A has a pivot in every column.
- The columns of A are linearly independent.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with associated matrix A is surjective if and only if

- For any $\mathbf{b} \in \mathbb{R}^m$, there exists at least one $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.
- The columns of A span all of $\mathbb{R}^m$.