Definition

Matrix-Vector Products Matrix-Vector Product: Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix where each column is given by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are column vectors in $\mathbb{R}^m$, and let $\mathbf{x} \in \mathbb{R}^n$ be a column vector. Then we define the matrix-vector product of A and x to be

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = \sum_{k=1}^{n} \mathbf{a}_k x_k$$

What does this mean about the relation between the columns and A and the number of elements in $\mathbf{x}$? $\mathbf{x}$ must have the same number of elements as A does rows!

Matrix-Vector Product Practice

Determine if each of the following matrix-vector products are defined. If they are, then compute the product.

a) $A = \underbrace{\begin{bmatrix} 2 & -4 & 3 & 1 \\ 6 & 2 & 1 & 9 \\ 1 & 0 & 2 & -1 \end{bmatrix}}_{4}$ and $\mathbf{x} = \left.\begin{bmatrix} -1 \\ -1 \\ 5 \\ 2 \end{bmatrix}\right\} 4$ b) $A = \underbrace{\begin{bmatrix} 2 & -4 & 3 & 1 \\ 6 & 2 & 1 & 9 \\ 1 & 0 & 2 & -1 \end{bmatrix}}_{4}$ and $\mathbf{x} = \left.\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}\right\} 3$

For (a):

$$A\mathbf{x} = \begin{bmatrix} 2 & -4 & 3 & 1 \\ 6 & 2 & 1 & 9 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 5 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} -1 \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} +5 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} +2 \begin{bmatrix} 1 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+4+15+2 \\ -6-2+5+18 \\ -1-0+10-2 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \\ 7 \end{bmatrix}$$

Matrix-Vector Product Properties

Let $A \in \mathbb{R}^{m \times n}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and c be a scalar. Then we know

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{v}) = cA\mathbf{v}$

Proof.

We will demonstrate that $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= A \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} = \mathbf{a}_1(u_1 + v_1) + \cdots + \mathbf{a}_n(u_n + v_n) \\ &= \mathbf{a}_1 u_1 + \mathbf{a}_1 v_1 + \cdots + \mathbf{a}_n u_n + \mathbf{a}_n v_n = \mathbf{a}_1 u_1 + \cdots + \mathbf{a}_n u_n + \mathbf{a}_1 v_1 + \cdots + \mathbf{a}_n v_n \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

□

Revisiting Span

Example

Determine if $\mathbf{b} = \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix}$ is in Span $\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

We solve either

$$\begin{aligned} 1x_1 + 0x_2 &= 6 \\ 0x_1 + 1x_2 &= -2 \\ -2x_1 + -1x_2 &= -10 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -2 \\ -2 & -1 & -10 \end{array}\right]$$

Or solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix}$$

Span with Matrix-Vector Products

If $A \in \mathbb{R}^{m\times n}$ has columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $b \in \mathbb{R}^m$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

Theorem

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of matrix A.

Span with General Vectors

We've answered questions about if a particular vector is in the span of others, but what about all vectors?

Example

Are all vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ in Span $\left(\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} \right)$?

Solution

Set up the augmented system and reduce!

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & b_1 \\ 1 & 0 & 8 & b_2 \\ -3 & 2 & 6 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 6 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ -3 & 2 & 6 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 6 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 5 & 24 & b_3 + 3b_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 6 & b_1 \\ 0 & -1 & 2 & b_2 - b_1 \\ 0 & 0 & 34 & b_3 + 3b_1 + 5(b_2 - b_1) \end{array}\right]$$

. See that regardless of b_1, b_2, b_3, we can get an answer!

Theorem

Let $A \in \mathbb{R}^{m \times n}$. The following 4 statements are equivalent.

1. *The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$*
2. *All vectors $\mathbf{b} \in \mathbb{R}^m$ can be written as linear combinations of columns of A.*
3. *The columns of A span all of $\mathbb{R}^m$.*
4. *A has a pivot in every row.*

Span Practice

For each of the following matrices, determine if their columns span all of $\mathbb{R}^3$.

1.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 1 & 4 \\ 6 & 1 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 7 \\ 1 & 1 & 4 \\ 6 & 1 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 3 & -2 & 7 \\ 6 & 1 & 19 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 4 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

NO!

2.

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 2 \\ 3 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -5 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

YES!