Uniqueness Representation Theorem

Theorem

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be a basis for a vector space* V*. Then, for every* $\mathbf{v} \in V$*, there is a unique set of* $c_1, \ldots, c_n$ *such that*

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

Proof of Uniqueness Representation Theorem

Proof.

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ forms a basis of V, then we know that

1. $\text{Span}\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right) = V$
2. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are all linearly independent.

Define $A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$, then since the columns of A span all of V, we know that we can solve $A\mathbf{c} = \mathbf{v}$ for every $\mathbf{v} \in V$. Since the columns of A are linearly independent, we know $\mathbf{c}$ is unique. Putting this all together means

$$\mathbf{v} = A\mathbf{c} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

$\square$

Basis as a Coordinate System

Definition

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then, for every $\mathbf{x} \in V$, we define the $\mathcal{B}$ coordinates of $\mathbf{x}$ to be the unique scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.

Definition

We define the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ to be:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Coordinate System Example

Let $V = \mathbb{R}^{2\times 2}$ and consider the basis given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

And label them as follows

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\left[\begin{bmatrix} 1 & 5 \\ -4 & 15 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \\ -4 \\ 15 \end{bmatrix}$$

Coordinate System Example pt. 2

Let $V = \mathbb{R}^3$, $F = \mathbb{R}$, and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ be a basis of V. Find

$$\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right]_{\mathcal{B}}$$

We want to solve the following augmented system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ -1 & 3 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 1 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 7 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{7} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right] \xrightarrow[R_1 = R_1 - 2R_3]{R_2 = R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right] \to \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} -\frac{2}{7} \\ \frac{2}{7} \\ \frac{1}{7} \end{bmatrix}$$

Coordinate System Practice

Let $V = \mathbb{R}^3$, $F = \mathbb{R}$, and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ be a basis of V. Find

$$\left[\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right]_{\mathcal{B}}$$

$$\left[\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

Coordinate Mapping

Definition

Coordinate Mapping: The mapping $C : \mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ where $\mathcal{B}$ is some basis of V, $n = \dim(\mathcal{B})$, and we call C the coordinate mapping.

Another way to phrase this is as follows:

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and define $A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$ to be a matrix whose columns are the basis vectors from $\mathcal{B}$.

Note: This may not be a matrix of the usual real numbers, could be functions, matrices, etc! Then for all $\mathbf{v} \in V$, define $\mathbf{x} = C(\mathbf{v})$, then $\mathbf{v} = A\mathbf{x}$.

We will now prove some properties of C!

Coordinate Mapping is an Injection

Remember that $A = \left[\mathbf{v}_1, \ldots, \mathbf{v}_n\right]$. We will show that the coordinate mapping is an injection first.

Proof.

Let $\mathbf{u}, \mathbf{v}$ be two vectors in V such that $C(\mathbf{u}) = C(\mathbf{v})$.
Since $C(\mathbf{u}) = C(\mathbf{v})$, there exists some $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{u} \qquad\qquad A\mathbf{x} = \mathbf{v}$$

Since A is a matrix whose columns are vectors, we can see that

$$\mathbf{u} - \mathbf{v} = A\mathbf{x} - A\mathbf{x} = A(\mathbf{x} - \mathbf{x}) = A(\mathbf{0}_n) = \mathbf{0}_V$$

So, $\mathbf{u} = \mathbf{v}$. □

Coordinate Mapping is a Surjection

Remember that $A = \left[\mathbf{v}_1, \ldots, \mathbf{v}_n\right]$. We will show that the coordinate mapping is a surjection. We want to show that we can get every $\mathbf{x} \in \mathbb{R}^n$ by choosing the correct $\mathbf{v} \in V$.

Proof.

Let $\mathbf{x} \in \mathbb{R}^n$. See that $A\mathbf{x} = \mathbf{v} \in V$. Then by definition of C, we have that $C(\mathbf{v}) = \mathbf{x}$. Thus, we can reach every element in $\mathbb{R}^n$.
Therefore, C is a surjection $\square$

Coordinate Mapping is a Bijection

Theorem

The coordinate mapping $C : V \to \mathbb{R}^n$ is a bijection.

Proof.

The last 2 slides proved that C is an injection and surjection. Therefore it is a bijection. $\square$

We also call bijections isomorphisms. This just means we can think about the vectors and their coordinate vectors interchangeably!

Coordinate Matrix

Definition

Coordinate Matrix: We define the coordinate matrix of a list of vectors: $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in the basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ as follows:

$$P_{\mathcal{B}} = \begin{bmatrix} [\mathbf{v}_1]_{\mathcal{B}} & \ldots & [\mathbf{v}_p]_{\mathcal{B}} \end{bmatrix}$$

Example

Consider $V = \mathbb{R}^{2\times 2}, F = \mathbb{R}$ with the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The Coordinate matrix for $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}$ is given by:

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 4 & 0 \\ 0 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

But Why do we Care?

In weird spaces, it's easier to check for pivot columns in $P_{\mathcal{B}}$ than it is to check if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent! Consider $V = \mathcal{P}_2(\mathbb{R}), F = \mathbb{R}$. (Note: $\mathcal{P}_2(\mathbb{R})$ is the set of all polynomial of degree 2 or less with real coefficients.)

Consider the "standard" basis of $\mathcal{B} = \left\{1, t, t^2\right\}$. Let's check if $1 - t + t^2, 3t + t^2, 2 - t$ are linearly independent!

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_3=R_3-R_1]{R_2=R_2+R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3=R_3-3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3=\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1=R_1-2R_3]{R_2=R_2+2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, these vectors are linearly independent!