Orthogonal Complement

Let V be a vector space with an inner product given by $\langle \cdot, \cdot \rangle$.

Definition

Orthogonal Complement: The orthogonal complement of a subspace of V (or equivalently $\mathbb{R}^n, \mathbb{C}^n$), W is given by

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Note: This is the set of *all* vectors orthogonal to *all* vectors in W.

Example

Using the standard inner product and $V = \mathbb{R}^n$. Let $W = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right)$, then $W^\perp = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right)$

Computing Orthogonal Complements

Theorem

Let W be a subspace of V and A be a matrix such that $W = \mathrm{Col}(A)$. Then,

$$W^{\perp} = \mathrm{Nul}\left(A^{\top}\right)$$

Proving our Equality Part 1

Proof.

Let

$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}, W = \text{Span}\left(\mathbf{v}_1, \dots, \mathbf{v}_n\right)$$

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So, $\mathbf{x} \in \mathrm{Nul}\left(A^\top\right)$ $\square$

Proving our Equality Part 2

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So, $\mathbf{x} \in W^\perp$. Thus, our two spaces are the same! □

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4. Write out a basis of this nullspace

Computing Orthogonal Set Example

Let's practice our algorithm!

$$W = \operatorname{Span}\left(\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right)$$

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So, the null-space is given by the span of $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

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3. $\dim(W) + \dim\left(W^{\top}\right) = n$.

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Fundamental Theorem of Linear Algebra

A fundamental theorem behind much of linear algebra is how our "fundamental" subspaces $(\mathrm{Col}\,(A)\,, \mathrm{Row}\,(A)\,, \mathrm{Nul}\,(A)\,, \mathrm{Nul}\left(A^\top\right))$ relate to each other. It is summarized as

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Example

The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is orthogonal while $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is orthonormal

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We will show the equation

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has only the trivial solution $c_1 = \cdots = c_m = 0$.
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has only the trivial solution $c_1 = \cdots = c_m = 0$.
We apply both sides of our equality to our inner product with $\mathbf{u}_\ell$ for some $1 \leq \ell \leq m$, which gives us

$$\begin{aligned} 0 = \langle \mathbf{0}, \mathbf{u}_\ell \rangle &= \langle c_1\mathbf{u}_1 + \cdots + c_m\mathbf{u}_m, \mathbf{u}_\ell \rangle = c_1 \langle \mathbf{u}_1, \mathbf{u}_\ell \rangle + \cdots + c_m \langle \mathbf{u}_m, \mathbf{u}_\ell \rangle \\ &= c_\ell \langle \mathbf{u}_\ell, \mathbf{u}_\ell \rangle \end{aligned}$$

Since we know that $\mathbf{u}_\ell$ is non-zero, $\langle \mathbf{u}_\ell, \mathbf{u}_\ell \rangle \neq 0$, thus $c_\ell = 0$. Since ℓ was some arbitrary index, all must be 0. Therefore, we have a list of linearly independent vectors. $\square$