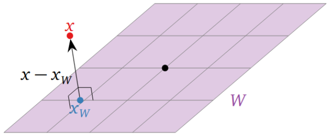
Orthogonal Projection



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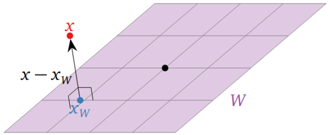

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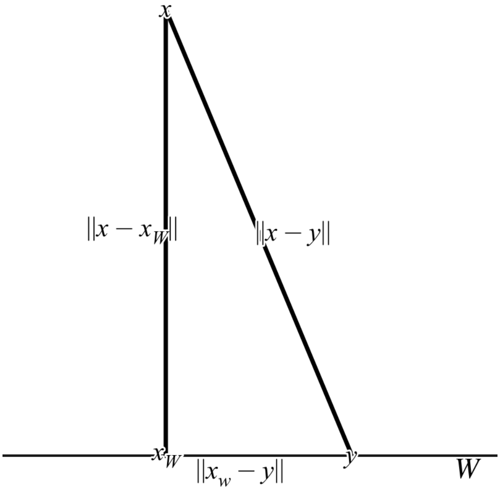
Definition

Orthogonal Projection: We call this vector $\mathbf{x}_W$ to be the orthogonal projection of $\mathbf{x}$ onto the space W.

Why call it Orthogonal? An $\mathbb{R}^2$ Figure



If we take any other point as $\mathbf{x}_W$, then we see that it would be further from $\mathbf{x}$.

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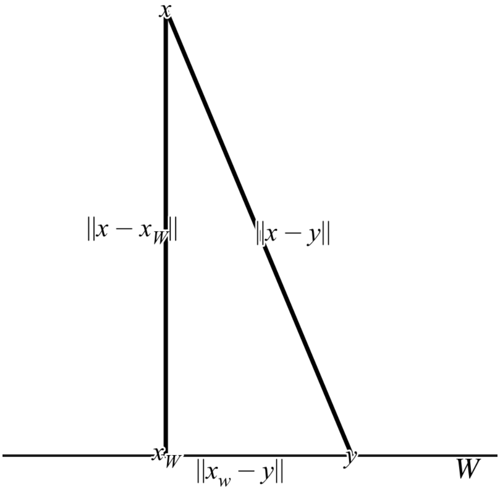

If we take any other point as $\mathbf{x}_W$, then we see that it would be further from $\mathbf{x}$. See that the vector $\mathbf{x} - \mathbf{x}_W$ is orthogonal to W!

Orthogonal Decomposition

Let's suppose we can compute this $\mathbf{x}_W$, and note something.

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This is called the orthogonal decomposition of $\mathbf{x}$. Where $\mathbf{x}_W$ is the orthogonal projection of $\mathbf{x}$ onto W and $\mathbf{x}_{W^\perp} = \mathbf{x} - \mathbf{x}_W$

Computing an Orthogonal Projection

Theorem

Let $A \in \mathbb{R}^{m \times n}$, $W = \mathrm{Col}\,(A)$, *and* $\mathbf{x} \in \mathbb{R}^m$. *Then the system of linear equations given by*

$$A^\top A\mathbf{c} = A^\top \mathbf{x}$$

is consistent and $\mathbf{x}_W = A\mathbf{c}$ *where* $\mathbf{c}$ *is some solution.*

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Note that if $n = 1$, then we have inner products instead of matrix multiplications!

Finding Orthogonal Projection Example

Let $W = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$. Find an orthogonal projection of $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

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Finding Orthogonal Projection Practice

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$$\mathbf{x}_W = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Orthogonal Projection as a Linear Transformation

Let's define this orthogonal projection to be the transformation T.

$$T : \mathbb{R}^n \to W \qquad T(\mathbf{x}) = \mathbf{x}_W$$

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Proof.

We will show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we have that $T(a\mathbf{x} + \mathbf{y}) = aT(\mathbf{x}) + T(\mathbf{y})$.

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$$A^\top A\mathbf{c}_z$$

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We will show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we have that $T(a\mathbf{x} + \mathbf{y}) = aT(\mathbf{x}) + T(\mathbf{y})$. For our convenience, we define $\mathbf{z} = a\mathbf{x} + \mathbf{y}$. Remember that $\mathbf{z}_W = A\mathbf{c}_z$ where $\mathbf{c}_z$ is a solution to $A^\top A\mathbf{c}_z = A^\top \mathbf{z}$, and similarly for $\mathbf{x}, \mathbf{y}$, so we need only show that $\mathbf{c}_z = a\mathbf{c}_x + \mathbf{c}_y$ is a solution to our system above.

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Orthogonal Projection as a Linear Transformation

Let's define this orthogonal projection to be the transformation T.

$$T : \mathbb{R}^n \to W \qquad T(\mathbf{x}) = \mathbf{x}_W$$

Theorem

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$\square$

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