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2. *A has a pivot in every row*
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6. *The columns of A are linearly independent*

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We have the tools to prove all these, but really, it's just a summary of all the linear transformation and matrix algebra we've done so far!

One-sided inverse Implies the Other!

Theorem

Let $A \in \mathbb{R}^{n \times n}$. If there is some $B \in \mathbb{R}^{n \times n}$ such that

$$BA = I_n \text{ or } AB = I_n$$

Then, A is invertible and $A^{-1} = B$.

$AB = I_n$ case

Proof.

First, let $B \in \mathbb{R}^{n \times n}$ such that $AB = I_n$.

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