Representing Matrices

- If $A \in \mathbb{R}^{m\times n}$, then it as m rows and n columns
- The j^{th} column vector is denoted $\mathbf{a}_j$
- There are n column vectors, where each $\mathbf{a}_j$ is in $\mathbb{R}^m$
- The entry in the i^{th} row and j^{th} column, is called a_{ij}.
- We **always** list the row index first then the column
- Note that in Python indexing starts at 0 while we use 1 here

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Special Kinds of Matrices

- Zero Matrix: A matrix with entries all equal to 0. Sometimes denoted $0_{m\times n}$
- Square Matrix: A matrix with the same number of rows and columns. (m=n)
- Diagonal Elements: Elements with the same row and column index. (a_{ii})
- Diagonal Matrix: A matrix with all elements NOT on the diagonal equal to 0
- Identity Matrix: A diagonal matrix with ones on the diagonal. If there are n rows and columns, then we use I_n
- Symmetric Matrix: A matrix satisfying $a_{ij} = a_{ji}$ for all valid i, j. (So, it must also be square!)

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 7 \end{bmatrix}$$

Matrix Arithmetic

- Let $A, B \in \mathbb{R}^{m \times n}$ (same size and shape!). The sum, $C = A + B$ is defined as
$$c_{ij} = a_{ij} + b_{ij}$$
- Let $A \in \mathbb{R}^{m \times n}$ and r be a scalar. The scalar multiple, $C = rA$ is defined as
$$c_{ij} = r \cdot a_{ij}$$
- Two matrices, $A \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$ are equal if
  1. $m_1 = m_2$
  2. $n_1 = n_2$ and
  3. $a_{ij} = b_{ij}$ for all i, j.

Matrix Addition & Scalar Multiplication Properties

For each of the following properties, let $A, B, C \in \mathbb{R}^{m\times n}$, and r, s be scalars

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0_{m\times n} = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $(r)sA = (rs)A$

The Transpose

Definition

Let $A \in \mathbb{R}^{m \times n}$. The transpose of A, denoted $A^\top \in \mathbb{R}^{n \times m}$ is the matrix with columns formed from rows of A. IE:

$$a_{ij}^\top = a_{ji}$$

for all valid i, j.

Example

Give the transpose of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix}$$

$$A^\top = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix}$$

Definition

Composition of Linear Transformations: Let $T : \mathbb{R}^m \to \mathbb{R}^p$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations defined by

$$T(\mathbf{y}) = A\mathbf{y} \text{ and } S(\mathbf{x}) = B\mathbf{x}$$

then we define the composition of T and S to be

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = AB\mathbf{x}$$

Note: AB is only defined with A has the same number of rows as B has columns

Matrix Multiplication

Definition

Matrix Product: Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$. We define the product of two matrices $C = AB$ to be the matrix such that for all $\mathbf{x} \in \mathbb{R}^p$ such that $C\mathbf{x} = A(B\mathbf{x})$.

There are 3 main ways to compute the matrix product, AB.

1. Column-wise
2. Component-wise
3. Sums of other matrices

Matrix Multiplication Method 1 (Column-wise)

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$. Then we define the matrix product as:

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p\end{bmatrix}$$

where $\mathbf{b}_k \in \mathbb{R}^n$ are the columns of B.

Example

Let $A = \begin{bmatrix}1 & -4\\ 2 & -5\\ 3 & -6\end{bmatrix}$ and $B = \begin{bmatrix}2 & 1\\ 1 & 4\end{bmatrix}$. Then

$$C = AB = \begin{bmatrix}\begin{bmatrix}1 & -4\\ 2 & -5\\ 3 & -6\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} & \begin{bmatrix}1 & -4\\ 2 & -5\\ 3 & -6\end{bmatrix}\begin{bmatrix}1\\ 4\end{bmatrix}\end{bmatrix} = \begin{bmatrix}-2 & -15\\ -1 & -18\\ 0 & -21\end{bmatrix}$$

Matrix Multiplication Method 2 (Component-wise)

Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$. Then we define the matrix product as the matrix $C = AB$ where for all i, j:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example

Let $A = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. Then

$$C = AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + -4 \cdot 1 & 1 \cdot 1 + -4 \cdot 4 \\ 2 \cdot 2 + -5 \cdot 1 & 2 \cdot 1 + -5 \cdot 4 \\ 3 \cdot 2 + -6 \cdot 1 & 3 \cdot 1 + -6 \cdot 4 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ -1 & -18 \\ 0 & -21 \end{bmatrix}$$

Matrix Multiplication Method 3 (Sums of other matrices)

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$. Then we define the matrix product as:

$$AB = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \mathbf{b}_1^\top \\ \vdots \\ \mathbf{b}_n^\top \end{bmatrix} = \mathbf{a}_1\mathbf{b}_1^\top + \cdots + \mathbf{a}_n\mathbf{b}_n^\top$$

where $\mathbf{a}_i \in \mathbb{R}^{m\times 1}$ are the columns of A and $\mathbf{b}_j^\top \in \mathbb{R}^{1\times p}$ are the rows of B (columns of $B^\top$!)

Method 3 Example

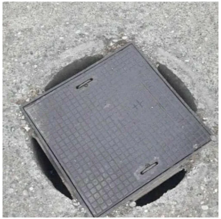
Example

Let $A = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. Then

$$C = AB = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} -4 & -16 \\ -5 & -20 \\ -6 & -24 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ -1 & -18 \\ 0 & -21 \end{bmatrix}$$

Is It The Correct Shape?

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$



1. For which matrices is addition with A defined?
2. For which matrices is addition with B defined?
3. For which matrices is addition with C defined?
4. For which matrices is multiplication with A defined?
5. For which matrices is multiplication with B defined?
6. For which matrices is multiplication with C defined?

Example

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Compute (if possible):

1. AF

This is defined!

$$AF = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 12 \\ -1 & -4 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 3 & 2 \\ 3 & 7 \end{bmatrix}$$

2. FA

Not defined!

Now You Try!

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 4 & 6 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1. BF

$$BF = \begin{bmatrix} 2 & 13 \\ 4 & 6 \end{bmatrix}$$

2. FB

$$FB = \begin{bmatrix} 4 & 7 \\ 8 & 4 \end{bmatrix}$$

Multiplication and Transpose Properties

Let $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times p}$, and r be a scalar.

1. $\left(A^\top\right)^\top = A$
2. $(A + B)^\top = A^\top + B^\top$
3. $(rA)^\top = rA^\top$
4. $(AB)^\top = B^\top A^\top$