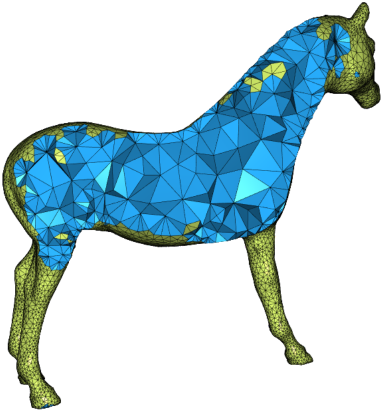
Applications of Linear Algebra to Computer Graphics



https://www.cg.tu-berlin.de/research/projects/harmonic-triangulations/

Image Credit: Star Wars low poly portraits, designed by Vladan Filipovic

Let's examine some ways to manipulate and display graphical images using matrices.

Scaling One Triangle

Consider one triangular piece. If we want to scale the triangle by a factor of three, then we apply the transformation

$$S : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}.$$

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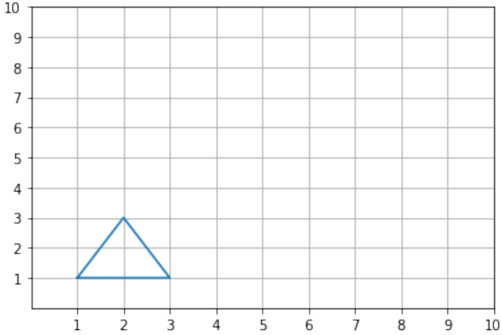

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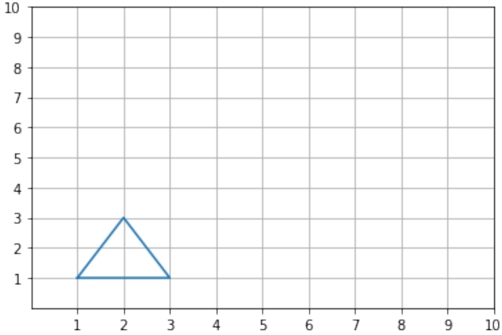

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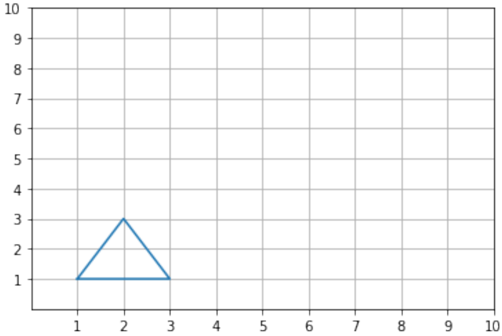

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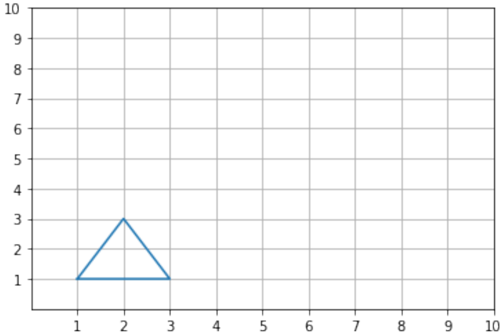

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Vertical Shearing

If we would like to vertically shear the triangle by a factor of 2, we have

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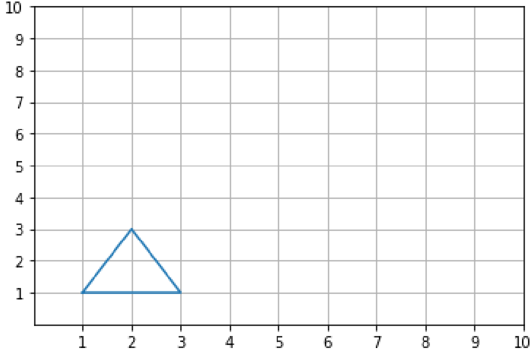

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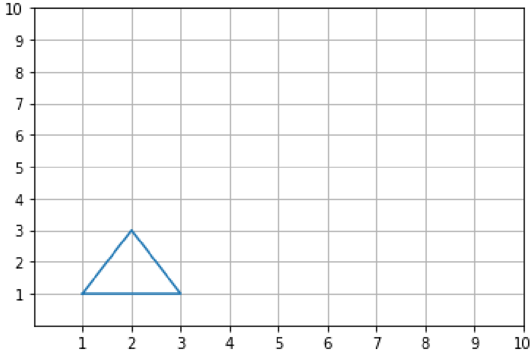

Rotation

If we would like to rotate the triangle counter-clockwise by an angle of θ, we have in general

$$R : \mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}.$$

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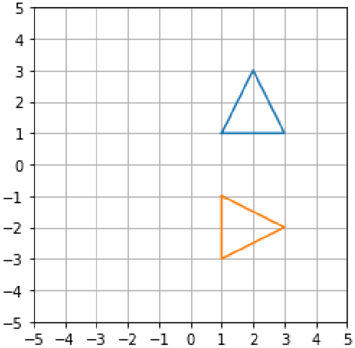

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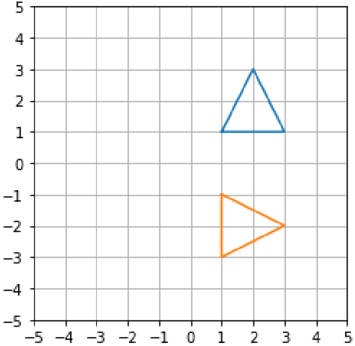

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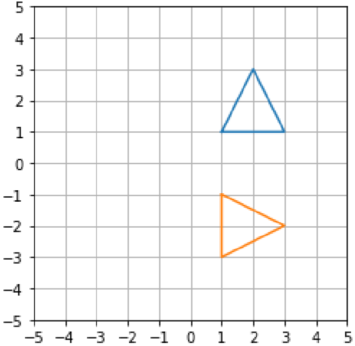

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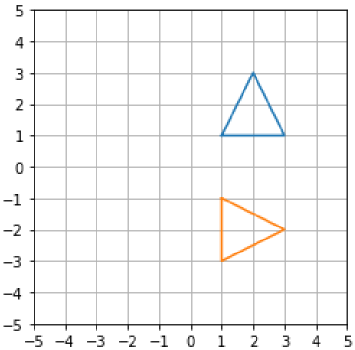

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The image under the composition of these three transformations is $R\Big(V(S\mathbf{x})\Big) = (RVS)\mathbf{x}$

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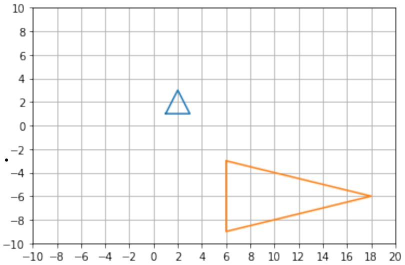

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Composing Linear Transformations

The composite of linear transformations A, B, and Cin the following order:

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Note we can extend this idea to compose any number of linear transformations.

Translations

To translate the vertex of the triangle at $(1, 1)$ to the left by 2 units and up by 3 units, we add vectors:

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- We say (x, y) has homogeneous coordinate $(x, y, 1)$.

Homogeneous Coordinates

Now we can define translation by matrix multiplication. For example, if we want to shift the point $(3, 1)$ to the left by 2 units and up by the 3 units, then we have the product

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1+3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

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2. Rotate the triangle counter-clockwise by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

We have the following three matrices for the scaling, S, rotation, R, and translation B, respectively,

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1. Scale the triangle by a factor of 3.
2. Rotate the triangle counter-clockwise by $\frac{\pi}{2}$.
3. Translate to the right by 5 units and down by 2 units.

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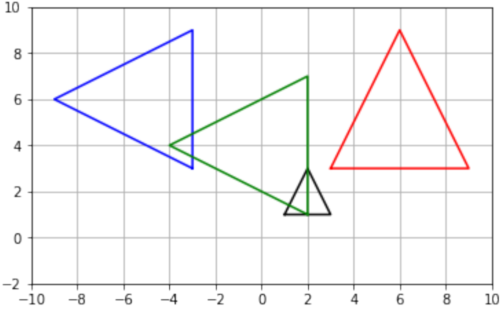
Multiplying matrices we can now compose all three maps:

$$BRS = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Scale the triangle by a factor of 3.



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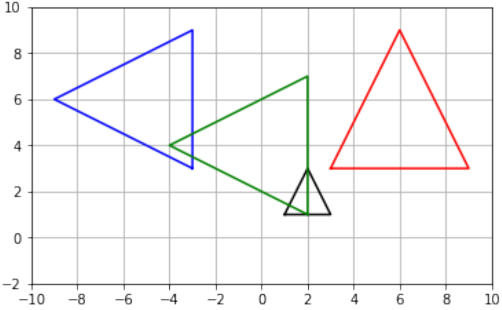

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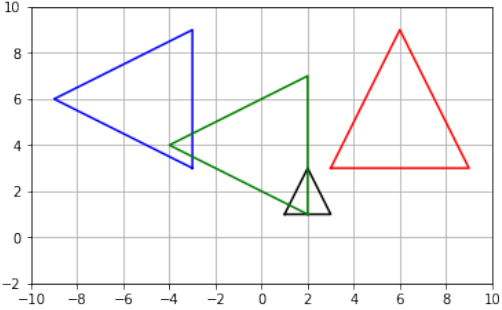

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Then we have each vertex (given in homogeneous coordinates) mapped as follows

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}.$$