Linear Equation Review

Definition

Linear Equations:
A linear equation of n variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where b and $a_1, \dots, a_n$ are constants

Practice

Determine which of the following equations are linear in x_1, x_2, x_3.

1. $x_1 + 4x_2 + x_1x_3 = 3$
2. $\pi x_1 - \frac{x_2}{e^2} = 4$
3. $\cos(4)x_1 + \sin(2)x_2 + x_3 = e\pi$

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Systems of Linear Equations

Definition

A *system of linear equations* is a collection of many linear equations using the same variables

Example

$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

A Linear System Solution Method

There are many ways that we learned to solve systems of linear equations in other math classes, the way we will discuss is called *Elimination*. We will use this method to solve

$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

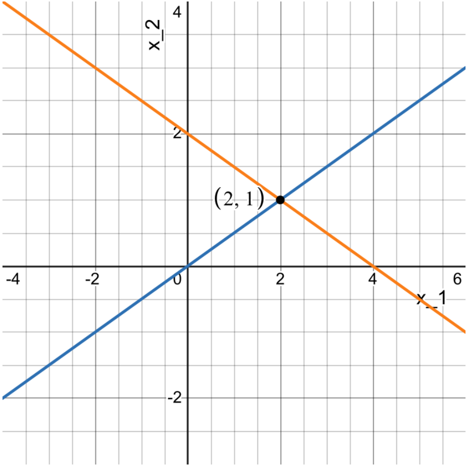
$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ 1x_1 - 2x_2 &= 0 \end{aligned} \rightarrow \begin{aligned} 2x_1 + 4x_2 &= 8 \\ 0x_1 - 4x_2 &= -4 \end{aligned} \rightarrow \begin{aligned} 2x_1 + 0x_2 &= 4 \\ 0x_1 - 4x_2 &= -4 \end{aligned} \rightarrow \begin{aligned} 1x_1 + 0x_2 &= 2 \\ 0x_1 + 1x_2 &= 1 \end{aligned}$$

So, our answer is $(x_1, x_2) = (2, 1)$

Remark

This method is called *Elimination* because we are *eliminating* variables until we only have one per equation!

Looking At Our Solution Visually

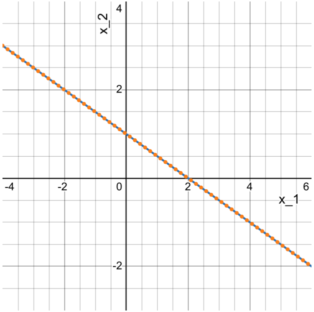


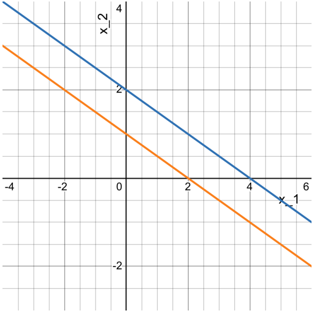
How Many Solutions Are There?

Let's consider two different linear systems

$$\begin{aligned} 2x_1 + 4x_2 &= 4 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

$$\begin{aligned} 2x_1 + 4x_2 &= 4 \\ x_1 + 2x_2 &= 4 \end{aligned}$$





Consistency

Definition

A linear system of equations is called *Consistent* if it has at least one solution, and it is called *Inconsistent* otherwise.

Example

From the previous slides, we would say that

$$\begin{aligned} 2x_1 + 4x_2 &= 4 \\ x_1 + 2x_2 &= 2 \end{aligned} \qquad\qquad \begin{aligned} 2x_1 + 4x_2 &= 8 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

are consistent while

$$\begin{aligned} 2x_1 + 4x_2 &= 4 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

is inconsistent.

Consistency Practice

Practice

Determine if the following systems are consistent or inconsistent

1. $$\begin{aligned} 2x_1 + 4x_2 &= 1 \\ 4x_1 + 8x_2 &= 2 \end{aligned}$$

2. $$\begin{aligned} 6x_1 + 3x_2 &= 10 \\ 12x_1 + 6x_2 &= 10 \end{aligned}$$

3. $$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 + 5x_2 &= 0 \end{aligned}$$

4. $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

Consistency Practice Answers

Practice

Determine if the following systems are consistent or inconsistent

1. $$\begin{aligned} 2x_1 + 4x_2 &= 1 \\ 4x_1 + 8x_2 &= 2 \end{aligned}$$

   This system is consistent. It has an infinite number of solutions!

2. $$\begin{aligned} 6x_1 + 3x_2 &= 10 \\ 12x_1 + 6x_2 &= 10 \end{aligned}$$

   This system is inconsistent. There cannot be a solution

3. $$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 + 5x_2 &= 0 \end{aligned}$$

   This system is consistent. It has exactly 1 solution!

4. $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 1 \end{aligned}$$

   This system is consistent. It has an infinite number of solutions!

Solution Set

Definition

A *solution set* is the set of all possible solutions to a system of linear equations.

Example

The solution set for the system

$$
\begin{aligned}
x + y &= 1 \\
x + y &= 1
\end{aligned}
$$

would be

$$\{(x, y) |\, y = 1 - x\}$$

Solution Set Practice

Practice

Determine the solution set for each of the following systems

1.
$$\begin{aligned} 2x_1 + 4x_2 &= 1 \\ 4x_1 + 8x_2 &= 2 \end{aligned}$$

2.
$$\begin{aligned} 6x_1 + 3x_2 &= 10 \\ 12x_1 + 6x_2 &= 10 \end{aligned}$$

3.
$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 + 5x_2 &= 0 \end{aligned}$$

Solution Set Practice Answers

Practice

Determine the solution set for each of the following systems

1.

$$\begin{aligned} 2x_1 + 4x_2 &= 1 \\ 4x_1 + 8x_2 &= 2 \end{aligned}$$

The solution set is $\{(x_1, x_2) |\, x_2 = \frac{1-2x_1}{4}\}$

2.

$$\begin{aligned} 6x_1 + 3x_2 &= 10 \\ 12x_1 + 6x_2 &= 10 \end{aligned}$$

The solution set is $\emptyset$

3.

$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 + 5x_2 &= 0 \end{aligned}$$

The solution set is $\{(0, 0)\}$.

Matrix Notation

The way we've been writing our Elimination steps is pretty wasteful. Luckily for us, we have a way of doing this in a more space efficient method: a matrix!

System of linear equations:

$$
\begin{aligned}
2x_1 + 5x_2 &= 1 \\
1x_1 + 2x_2 &= 7
\end{aligned}
$$

Coefficient Matrix:

$$
\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}
$$

Augmented Matrix:

$$
\left[\begin{array}{cc|c} 2 & 5 & 1 \\ 1 & 2 & 7 \end{array}\right]
$$

or

$$
\begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 7 \end{bmatrix}
$$

Some Properties of Matrices

How many rows and columns does the following augmented matrix have?

$$\begin{array}{c c} 3 & \left\{ \underbrace{\left[\begin{array}{cccc|c} 2 & 9 & 13 & 3 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 4 & 1 & 0 & 2 & 12 \end{array}\right]}_{5} \right. \end{array}$$

- This matrix has 3 rows. This means there were 3 equations in the original system
- This matrix has 5 columns. This means there were 4 variables in the original system
- We would say this is a 3×5 augmented matrix.

Note: The order is very important when we use the shorthand, rows always comes first and columns always go second.

Solving a Linear System With Matrices

Let's look at how we can use this augmented matrix by solving a system our old way and the new way simultaneously

$$2x_1 + 4x_2 = 8$$
$$1x_1 - 2x_2 = 0$$

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \tfrac{1}{2}R_1$$

$$2x_1 + 4x_2 = 8$$
$$0x_1 - 4x_2 = -4$$

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 0 & -4 & -4 \end{array}\right]$$

$$R_1 = R_1 + R_2$$

$$2x_1 + 0x_2 = 4$$
$$0x_1 - 4x_2 = -4$$

$$\left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & -4 & -4 \end{array}\right]$$

$$R_1 = \tfrac{1}{2}R_1,\ R_2 = -\tfrac{1}{4}R_2$$

$$1x_1 + 0x_2 = 2$$
$$0x_1 + 1x_2 = 1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

A Larger System!

Practice

Solve the following system of equations for x_1, x_2, x_3

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 7 \\ 5x_1 + x_2 + x_3 &= 11 \end{aligned}$$

A Larger System Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 7 \\ 5 & 1 & 1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 5 & 1 & 1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & -4 & -24 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -6 & -24 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

So, our solution is $(x_1, x_2, x_3) = (1, 2, 4)$

Elementary Row Operations

We have a name for the kinds of changes we are making to these matrices, we call them "Elementary Row Operations"

Definition

An operation on a row of a matrix called is an Elementary Row Operation if it is one of

1. Replacement: Replace one row by itself plus a multiple of another row
2. Interchange: Swap two rows
3. Scaling: Multiply an entire row by the same constant

Example

$$\left[\begin{array}{ccc|c} 1 & -8 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Equivalent Systems

Definition

Two matrices (augmented or not augmented) are Row Equivalent if we can transform them into each other using only Elementary Row Operations

Example

$$\left[\begin{array}{ccc|c} 1 & -8 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -3 \end{array}\right] \text{ and } \left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

are Row Equivalent augmented matrices.

Remark

If two augmented matrices are row equivalent, then the solutions to their respective systems are the same!

Equivalent Systems

Definition

Two linear systems are called Equivalent if they have the same solution.

Example

$$2x_1 + 4x_2 = 2 \qquad\qquad 4x_1 + 8x_2 = 4$$
$$1x_1 + 3x_2 = 5 \qquad\qquad 2x_1 + 6x_2 = 10$$

Are equivalent systems because the right one is just the left with both rows multiplied by 2.