

## Eigenpair Reminder

Remember that we defined an eigenpair of  $A$  as an ordered pair  $(\lambda, \mathbf{v})$  such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

We've discussed how to confirm a proposed  $\lambda$  is actually an eigenvalue and to compute associated eigenvectors.

What about how to compute the eigenvalues themselves?

# Computing Eigenvalues

Recall that if  $\lambda$  is an eigenvalue of a matrix  $A$ , then

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

has a non-trivial solution. In other words,  $A - \lambda I$  is not invertible!

So, we can figure out all  $\lambda$  values such that

$$\det(A - \lambda I) = 0$$

# Characteristic Polynomial

## Definition

Let  $A \in \mathbb{R}^{n \times n}$ . The **characteristic polynomial** of  $A$  is the function  $f(\lambda)$  given by

$$f(\lambda) = \det(A - \lambda I)$$

## Theorem

*The roots of  $f(\lambda)$  are exactly the eigenvalues of  $A$ .*

## Proof.

Let  $\lambda$  be an eigenvalue of  $A$ , then  $A - \lambda I$  is not invertible, so  $\det(A - \lambda I) = 0$ , so  $f(\lambda) = \det(A - \lambda I) = 0$ .

Let  $\lambda$  be a root of  $f(\lambda)$ , then

$$0 = f(\lambda) = \det(A - \lambda I).$$

So,  $A - \lambda I$  is not invertible, so  $\lambda$  is an eigenvalue of  $A$ . □

# Rational Root Theorem<sup>1</sup>

## Theorem

Let  $f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + c_n\lambda^n$  be a polynomial with integer coefficients. Then, all rational factors of  $f$  are of the form

$$x = \frac{p}{q}$$

Where  $p$  is an integer factor of  $c_0$  and  $q$  is an integer factor of  $c_n$ .

## Example

Consider  $f(x) = x^3 + x^2 - 10x + 8$ . Then our possible roots are

$$\{\pm 1, \pm 2, \pm 4, \pm 8\}$$

And we plug in these values to see which one(s) are actual roots.

Note: We may not have rational roots depending on the polynomial itself (complex roots!)

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<sup>1</sup>A proof can be found at [https://en.wikipedia.org/wiki/Rational\\_root\\_theorem](https://en.wikipedia.org/wiki/Rational_root_theorem)

## Finding Eigenvalues Example

Let  $A \in \mathbb{R}^{3 \times 3}$  as given below. Then compute all the eigenvalues of  $A$ .

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -3 \\ -4 & -4 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \left( \begin{bmatrix} 4 - \lambda & 1 & -1 \\ 1 & 4 - \lambda & -3 \\ -4 & -4 & 5 - \lambda \end{bmatrix} \right) = -\lambda^3 + 13\lambda^2 - 39\lambda + 27$$

Our rational roots theorem states that the possible rational eigenvalues are:

$$\{\pm 1, \pm 3, \pm 9, \pm 27\}$$

So, we plug these in and find that 1, 3, 9 are the eigenvalues.

## Finding Eigenvalues Practice

Let  $A \in \mathbb{R}^{2 \times 2}$  as given below. Then compute all the eigenvalues of  $A$ .

$$A = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$$

# Trace of a Matrix

## Definition

Let  $A \in \mathbb{R}^{n \times n}$ . Then we define the **trace** of  $A$  as

$$\text{Tr}(A) = a_{11} + \cdots + a_{nn}$$

where

$$A = \begin{bmatrix} \textcolor{red}{a}_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \textcolor{red}{a}_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \textcolor{red}{a}_{nn} \end{bmatrix}$$

# Properties of the characteristic polynomial

Let  $A \in \mathbb{R}^{n \times n}$ . Then we know that

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \text{Tr}(A) \lambda^{n-1} + \cdots + \det(A)$$

This means if  $A \in \mathbb{R}^{2 \times 2}$  of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then the characteristic polynomial has the form:

$$f(\lambda) = \lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$



## What About Larger Problems? (IE $n \geq 5$ )

It's great that we can solve these small problems by hand, but what about larger ones? Well, The Abel–Ruffini theorem<sup>2</sup> states that we cannot always solve for these eigenvalues when we have  $n \geq 5$ .

However, we can still try to solve these problems! We will achieve this via some methods we learn later in the semester, and is how we currently compute eigenvalues in practice.

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<sup>2</sup>More information can be found here:

[https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini\\_theorem](https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem)