The Inverse of a Map

Recall that for a function f, it's inverse function f^{-1} is a function that "undoes" what f did. Or in other words:

- $f^{-1}(f(x)) = x$ for all x in the domain of f.
- $f(f^{-1}(y)) = y$ for all x in the domain of f^{-1}.

Example

If $f(x) = 2x$, then $f^{-1}(x) = \frac{x}{2}$.
If $f(x) = \frac{x}{5} - 7$, then $f^{-1}(x) = (x+7) \cdot 5$.

The Inverse of a Matrix

Let $A \in \mathbb{R}^{n \times n}$. The inverse matrix if it exists, is denoted A^{-1} and is the unique matrix such that:

- $AA^{-1} = I_n$
- $A^{-1}A = I_n$

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

- What is A? Vertical expansion by factor of 5.
- How do we undo A? Vertical compression by factor of 5.

So:

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

Elementary Matrices

Definition

Elementary Matrices: An elementary matrix is one attainable by performing **one** row operation on an identity matrix.

Example

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We can also find E_2^{-1} and E_3^{-1}!

Example

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Demonstrating E_2^{-1}

Recall

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1}E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Are we done? No! Need to show $E_2E_2^{-1} = I_3$!

Showing $E_2E_2^{-1} = I_3$

Your turn!

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Inverse Properties

Let $A, B \in \mathbb{R}^{n\times n}$ such that both are invertible. Then we know that

- A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

More Inverse Properties

Let $A \in \mathbb{R}^{n \times n}$, then if A is invertible, so is $A^\top$ and

$$\left(A^\top\right)^{-1} = \left(A^{-1}\right)^\top$$

Product of Elementary Matrices

Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Write A as a product of elementary matrices.

1. $R_2 = R_2 - 2R_3$
2. Swap R_1 and R_2
3. $R_3 = 5R_3$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

How to Compute A^{-1} in General?

We can perform Gaussian Elimination on $\left[\begin{array}{c|c} A & I_n \end{array}\right]$!
If A is row equivalent to I_n (IE we reduce the left part to I_n), then

- $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ is row equivalent to $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$
- Otherwise, A^{-1} doesn't exist!

Example

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

How to Compute A^{-1} Practice 3×3

If possible, find the inverse of the given matrices.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ -4 & 0 & -2 \end{bmatrix}$$

A^{-1} Solution 1

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 1 & 0 \\ 0 & -3 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -\frac{R_2}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -3 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -\frac{3}{2} & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & \frac{9}{2} & -3 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -\frac{3}{2} & 1 \end{array}\right]$$

A^{-1} Solution 2

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ -4 & 0 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -4 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Can't find the inverse!

How to Compute A^{-1} Practice 2×2

If possible, find the inverse of the given matrices.

$$\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

A^{-1} solution 3

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - \frac{R_1}{3}} \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & 1 \end{array}\right]$$

Can't find the inverse!

A^{-1} solution 4

$$A = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - \frac{3R_1}{2}} \left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 0 & -5 & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow[R_2 = \frac{-R_2}{5}]{R_1 = \frac{-R_1}{2}} \left[\begin{array}{cc|cc} 1 & -2 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{10} & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{10} & -\frac{2}{5} \\ 0 & 1 & \frac{3}{10} & -\frac{1}{5} \end{array}\right]$$

Solving a System With A^{-1}

Let's say we know A^{-1}. How can we solve linear systems like $A\mathbf{x} = \mathbf{b}$?

$$A\mathbf{x} = \mathbf{b} \to A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \to \mathbf{x} = A^{-1}\mathbf{b}$$

This gives us our potential method as

1. Write our system as $A\mathbf{x} = \mathbf{b}$.
2. find A^{-1}
   2.1 If we cannot, then we must use GE as normal
   2.2 If we can, then continue
3. Multiply both sides of (1) by A^{-1}.
4. Solution is $\mathbf{x} = A^{-1}\mathbf{b}$

Note: If A is **NOT** square, then we cannot find an inverse!