Inner Product

Let $(V, \mathbb{F})$ be a vector space where V is the set our vectors come from and $\mathbb{F}$ is the set our scalars come from (You can think of this as $\mathbb{R}$ or $\mathbb{C}$)

Definition

Inner Product: An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ with the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $a, b \in \mathbb{F}$.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ Note: If $\mathbb{F} = \mathbb{R}$, then we omit the conjugate!
2. $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a \langle \mathbf{u}, \mathbf{w} \rangle + b \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$
4. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Inner Product Example: $\mathbb{C}^n$ (standard) Part 1

Let $V = \mathbb{C}^n$ and $\mathbb{F} = \mathbb{C}$. Then the following function is an inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\mathbf{y}}^\top \mathbf{x} = \sum_{k=1}^{n} x_k \overline{y_k}$$

Note that if we are in the real numbers, then we omit the conjugate of $\mathbf{y}$.

Note: we sometimes abbreviate $\overline{\mathbf{y}}^\top$ as $\mathbf{y}^*$

Inner Product Example: $\mathbb{C}^n$ (standard) Part 2

Property 1: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k \overline{y_k} = \sum_{k=1}^{n} \overline{y_k} x_k = \sum_{k=1}^{n} \overline{\overline{\overline{y_k} x_k}} = \overline{\sum_{k=1}^{n} y_k \overline{x_k}} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$$

Property 2: $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$

$$\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = \sum_{k=1}^{n} (ax_k + by_k)\overline{z_k} = \sum_{k=1}^{n} ax_k\overline{z_k} + by_k\overline{z_k} = \sum_{k=1}^{n} ax_k\overline{z_k} + \sum_{k=1}^{n} by_k\overline{z_k} = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$$

Inner Product Example: $\mathbb{C}^n$ (standard) Part 3

Property 3,4: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{k=1}^{n} x_k \overline{x_k} = \sum_{k=1}^{n} |x_k|^2 \geq 0$$

In addition, the only time $\sum_{k=1}^{n} |x_k|^2 = 0$ is when all components are 0 or if $\mathbf{x} = \mathbf{0}$

Dot Product

Definition

Dot Product: The dot product of two vectors in $\mathbb{R}^n$ is a function given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^\top \mathbf{x}$$

Theorem

The dot product is an inner product.

Proof.

The 2 previous slides prove this. $\square$

Note: For this course, we will only consider this inner product unless stated otherwise

Norms

Let $(V, \mathbb{F})$ be a vector space where V is the set our vectors come from and $\mathbb{F}$ is the set our scalars come from (You can think of this as $\mathbb{R}$ or $\mathbb{C}$)

Definition

Norm: A norm is a function given by $\|\cdot\| : V \to \mathbb{R}$ with the following properties for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in \mathbb{F}$

1. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
2. $\|c\mathbf{x}\| = |c| \, \|\mathbf{x}\|$
3. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

Induced Norms

Theorem

Let $(V, \mathbb{F})$ be a vector space with some inner product $\langle \cdot, \cdot \rangle$. Then the induced norm of this space is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Proof.

We can prove all 3 properties from the previous slide as consequences of us using the inner product. $\square$

Example of a Norm

If $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$, then the induced norm is often called the "Euclidean Norm" and denoted as follows

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{k=1}^{n} x_k^2}$$

For this course, we will consider only this norm unless stated otherwise.

Unit Vector

Definition

Unit Vector: We say a vector is a unit vector if it has norm 1. In other words, $\mathbf{x}$ is a unit vector if and only if

$$\|\mathbf{x}\| = 1$$

If we have any vector, $\mathbf{x} \neq \mathbf{0}$, then we can find a vector pointing in the same direction, but is also of *unit length* by doing:

$$\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

Distance

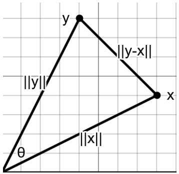
For us, we can think of distance between vectors as "how large is the difference between two vectors", or in other words, we say

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$$

Note: this idea is closely related to "Metric Spaces"[1], which you will see in an analysis course.

[1] https://en.wikipedia.org/wiki/Metric_space#Definition

Angle Between Vectors



Using the law of cosines[a], we have that

$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$$

$$(\mathbf{y}-\mathbf{x})^\top(\mathbf{y}-\mathbf{x}) = \mathbf{x}^\top\mathbf{x} + \mathbf{y}^\top\mathbf{y} - 2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$$

[a] https://en.wikipedia.org/wiki/Law_of_cosines

Which (assuming that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$) can be solved for θ to get

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}^\top\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right)$$

In higher dimensions and other vector spaces, this is how we define the angle between vectors

Orthogonality

Let $(V, \mathbb{F})$ be a vector space where V denotes the set our vectors come from, $\mathbb{F}$ is the set our scalars come from, and we have some inner product $\langle \cdot, \cdot \rangle$.

Definition

Orthogonal Vectors: We say that 2 vectors, $(\mathbf{x}, \mathbf{y} \in V)$ are orthogonal (or perpendicular) if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

Note that assuming we have non-zero vectors, the angle between $\mathbf{x}, \mathbf{y}$ would be 90°!

> Since orthogonality is closely tied to our inner product, we will use our standard one for this course.

Special Case for Orthogonality

Theorem

If $V = \mathbb{R}^n$ (or equivalently $\mathbb{C}^n$) with the usual inner product, then $\mathbf{0}$ is orthogonal to every vector.

Proof.

Let $\mathbf{x} \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denote the standard inner product, then we have that

$$\langle \mathbf{0}, \mathbf{x} \rangle = \sum_{k=1}^{n} 0 \cdot x_k = 0$$

$\square$