# A General Algorithm

Let's build a more systematic way of solving linear systems (that we can hopefully have a computer do for us!). We want an algorithm that we can:

- 1. Apply to any augmented matrix from a general system of equations
- 2. Transform to a form where we can find the solution(s) (if any exist!) by quickly looking
- 3. Using only Elementary Row Operations

$$\left[\begin{array}{ccc|c}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 4
\end{array}\right] \qquad \left[\begin{array}{ccc|c}
1 & 0 & -2 \\
0 & 1 & 8
\end{array}\right]$$

What are the similarities between these two augmented matrices?

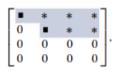
### Row Echelon Form

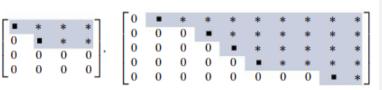
#### Definition

Row Echelon Form. We say that a matrix is in Row Echelon Form if and only if it satisfies the following requirements

- 1. All non-zero rows are above any rows of all zeros.
- 2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- 3. All entries in a column below a leading entry are zeros.

#### Example





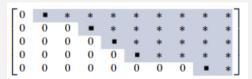
### Reduced Row Echelon Form

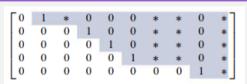
#### Definition

Reduced Row Echelon Form (RREF). We say that a matrix is in RREF if and only if it satisfies the requirements for Row Echelon Form and

- 4. The leading entry in each non-zero row is 1.
- 5. Each leading 1 is the only non-zero in its column.

#### Example





What kind of transformations can take us from the left to the right?

# Formalizing Our Algorithm

We've been doing this already actually! In order to better formalize it, let's break down our algorithm into 2 smaller steps.

### **Algorithm 1** RREF Transformation

**Input:**A matrix A with m rows and n columns of real numbers

Output: A transformed to RREF using Only Elementary Row Operations

- 1: Reduce A to Row Echelon Form
- 2: Reduce A to RREF

# Reducing to Row Echelon Form

Let's start with an example. Lets reduce the following augmented matrix to Row Echelon Form.

- 1. Start with the leftmost non-zero column. We call this the pivot column.
- 2. Pick some non-zero element. This is called the pivot. Move it to the top if necessary.
- Create zeros in all positions below the pivot.

$$\left[\begin{array}{ccc|c}
0 & 1 & 2 & 3 \\
2 & 8 & 1 & 2 \\
1 & 4 & 1 & 1
\end{array}\right]$$

$$\left[\begin{array}{ccc|c}
1 & 4 & 1 & 1 \\
2 & 8 & 1 & 2 \\
0 & 1 & 2 & 3
\end{array}\right]$$

$$\left[\begin{array}{ccc|c}
1 & 4 & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 2 & 3
\end{array}\right]$$

# Reducing to Row Echelon Form

4. Ignore the row containing the pivot and consider the remaining submatrix (highlighted in red). Then repeat the steps from before again!

$$\left[\begin{array}{ccc|c}
1 & 4 & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 2 & 3
\end{array}\right]$$

$$\left[\begin{array}{cc|cc|c} 1 & 4 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc|c} 1 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

# Reducing to RREF

Now that we are in Row Echelon, we're almost there! We have 2 more conditions to meet.

- 1. All our pivots are 1
- 2. All elements above our pivots are 0.

So, let's fix them!

# Reducing to RREF

- 5. Scale each row to make all pivots equal to 1.
- 6. Starting with the rightmost pivot column, zero out all elements above each non-zero pivot using Elementary Row Operations.

 $\left[\begin{array}{ccc|c}
1 & 4 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 0
\end{array}\right]$ 

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Then our answer is

$$(x_1, x_2, x_3) = (-11, 3, 0)$$

.

## Gaussian Elimination<sup>1</sup>

#### **Algorithm 2** Gaussian Elimination

**Input:**A matrix A with m rows and n columns of real numbers

#### Output: A transformed to RREF

- 1: Consider the leftmost column. Called the pivot column
- 2: Pick a non-zero entry as the pivot. Exchange rows if needed.
- 3: Use Elementary Row Operations to create zeros below the pivot
- 4: Ignore the row containing the pivot. Repeat the 3 steps above on the remaining submatrix until there are no more non-zero rows.
- 5: Scale each row so each non-zero pivot is 1.
- 6: Cancel out all elements above each pivot with Elementary Row Operations in each pivot column.

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# Existence and Uniqueness of RREF

- 1. Existence: Given any matrix, we can apply a series of elementary row operations to find an equivalent RREF.
- 2. The series of operations are not unique, but the result is unique!

$$\begin{bmatrix} 0 & 1 & | & -2 \\ -2 & -2 & | & -4 \\ -1 & 0 & | & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -2 & | & -4 \\ 0 & 1 & | & -2 \\ -1 & 0 & | & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -2 \\ -1 & 0 & | & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -2 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & | & -2 \\ -2 & -2 & | & -4 \\ -1 & 0 & | & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & | & -4 \\ -2 & -2 & | & -4 \\ 0 & 1 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 4 \\ -2 & -2 & | & -4 \\ 0 & 1 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -2 \\ -2 & -2 & | & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

#### Theorem

Every matrix is equivalent to one and only one reduced RREF matrix.

## Example

Find a solution to the system

$$x_1 + 2x_2 + x_3 = -2$$
  
 $x_1 + 3x_2 - 2x_3 = 1$ 

Using Gaussian Elimination, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 1 & 3 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & -3 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 7 & -8 \\ 0 & 1 & -3 & 3 \end{array}\right]$$

This gives the solution set:

$$x_1 +7x_3 = -8$$
  
 $x_2 -3x_3 = 3$ 

or in parametric form:

$$(-8-7x_3, 3+3x_3, x_3) = \begin{cases} x_1 & = -8-7x_3 \\ x_2 & = 3+3x_3 \\ x_3 & \text{is free} \end{cases}$$

### Parametric Form

#### Definition

Parametric Form: We say that a solution to a system of linear equations is in parametric form if it has a variable in the ordered pair.

#### Example

For example

$$(-8-7x_3,3+3x_3,x_3)$$

is in parametric form because  $x_3$  is still a variable and we don't know what it is! We will usually write either the variable with a subscript like above or using another variable. Like:

$$(-8-7s, 3+3s, s)$$

# Solutions of Linear Systems

We have

$$\begin{cases} x_1 &= -8 - 7x_3 \\ x_2 &= 3 + 3x_3 \\ x_3 & \text{is free} \end{cases}$$

- $\triangleright$  Variables  $x_1$  and  $x_2$  came from pivot columns, and we call these basic variables
- Variable  $x_3$  did not come from a pivot column, so we call it a free variable. This means it take on any value we want (IE we are free to choose it)
- If we have at least one free variable, then we have an infinite number of solutions.

## Another Example

Find a solution to the system

$$x_2 + 2x_3 = 0$$

$$x_1 + 2x_2 + 3x_3 = 1$$

$$x_1 + x_2 + x_3 = 2$$

Gaussian Elimination gives us

$$\begin{bmatrix} 0 & 1 & 2 & | & 0 \\ 1 & 2 & 3 & | & 1 \\ 1 & 1 & 1 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 1 & 2 & 3 & | & 1 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 2 & | & -1 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$$

This is an inconsistent system!

$$x_1 - x_3 = 0$$
  
 $x_2 + 2x_3 = 0$   
 $0 = 1$ 

# Summary

#### For a given system of linear equations:

- 1. Write the augmented matrix
- 2. Use Elementary Row Operations to reduce to Row Echelon Form
  - 2.1 If the rightmost column is a pivot column, then it is inconsistent, and no solution exists. We can now stop
  - 2.2 If the rightmost column is not a pivot column, then it is consistent, and we have a solution, so we need to continue
- 3. Reduce to RREF
- 4. Rewrite as a system of linear equations
- 5. Put all basic variables on the left and all free variables on the right.