Determinant Definition

Definition

The Determinant is a function given by

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

with the following properties:

1. Doing a row replacement does not change $\det(A)$
2. Scaling a row of A by a scaler c multiplies the determinant by c.
3. Swapping two rows of a matrix multiplies the determinant by -1.
4. The determinant of I_n is 1.

In fact, the determinant the unique function with these properties! But, proving this with the tools we have is difficult, so we will just take this for granted.

Method to Compute the Determinant

We can compute the determinant by doing:

1. Reduce to reduced row echelon form
2. Do operations in reverse following previous rules!

Practice!

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = -2$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \qquad \det(A) = -2$$

$$\xrightarrow{R_2 = \frac{-R_2}{2}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \det(A) = 1$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \det(A) = 1$$

Now you try!

Find the determinant of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Special Types of Matrices

To make our later discussion easier, we define two new kinds of matrices

Upper triangular

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & u_{1n} \\ 0 & u_{22} & \dots & u_{2,n-1} & u_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & u_{n-1.n-1} & u_{n-1,n} \\ 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix}$$

Definition

A matrix $U \in \mathbb{R}^{n \times n}$ is Upper Triangular if:

$$u_{ij} = 0 \text{ for all } 1 \leq j < i \leq n$$

Lower triangular

$$L = \begin{bmatrix} \ell_{11} & 0 & \dots & 0 & 0 \\ \ell_{12} & \ell_{22} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \ell_{1,n-1} & \ell_{1n} & \ddots & \ell_{n-1.n-1} & 0 \\ \ell_{2,n-1} & \ell_{2n} & \dots & \ell_{n-1,n} & \ell_{nn} \end{bmatrix}$$

Definition

A matrix $L \in \mathbb{R}^{n \times n}$ is Lower Triangular if:

$$\ell_{ij} = 0 \text{ for all } 1 \leq i < j \leq n$$

Determinant of a Matrix with a 0 row

A matrix with a 0 row will look something like below

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Recall that scaling a row of A by a scaler c multiplies the determinant by c.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{R_2 = -R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

So,

$$\det(A) = -\det(A) \to 2\det(A) = 0 \to \det(A) = 0.$$

Determinant of a Triangular Matrix

We will work with an upper triangular matrix, some good practice would be to do these arguments but for lower triangular matrices!
Remember that we have two cases to think about for a square matrix, $A \in \mathbb{R}^{n\times n}$

1. We have less than n pivots
2. We have n pivots

We claim that in both cases, the determinant of A is the product of the elements on the diagonal

Less than n pivots

What does this look like?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{a_{23}}{a_{33}} R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

There's a 0 row, so $\det(A) = 0$.
This idea extends to larger matrices too! Try to think about what that proof would look like!

n pivots

What does this look like?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{a_{23}}{a_{33}} R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \xrightarrow{R_1 = R_1 - \frac{a_{13}}{a_{33}} R_3} \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - \frac{a_{12}}{a_{22}} R_2} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \xrightarrow[R_3 = \frac{R_3}{a_{33}}]{R_1 = \frac{R_1}{a_{11}}, R_2 = \frac{R_2}{a_{22}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$$

And:

$$\det(A) = \det(B) \cdot a_{11} \cdot a_{22} \cdot a_{33} = a_{11} a_{22} a_{33}$$

General 2×2 formula

Theorem

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\det(A) = ad - bc$$

$a = 0$ case

Proof.

If $a = 0$, we need to have a pivot in A_{11}

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = B$$

See that $\det(A) = -\det(B) = -bc = ad - bc$. $\square$

$a \neq 0$ case

Proof.

If $a \neq 0$, we just need to eliminate c!

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_2 = R_2 - \frac{c}{a} R_1} \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix} = B$$

See that $\det(A) = \det(B) = a\left(d - \frac{bc}{a}\right) = ad - bc$. $\square$

General 3×3 formula

We could derive this formula, but it would be easier with Section 4.2, which we will not be covering in class.

Theorem

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Then

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

More Practice

Determine if the determinant of the following systems is 0 or not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant Properties

These are provable with what we have now, but we will take them for granted for now. I would think about how to demonstrate the second property though!
Let $A, B \in \mathbb{R}^{n \times n}$.

1. A is invertible if and only if $\det(A) \neq 0$.
2. If $\det(A) = 0$, then A has linearly dependent rows and columns
3. $\det(AB) = \det(A)\det(B)$
4. $\det\left(A^\top\right) = \det(A)$