Example 1 Part 1

Let

$$A = \begin{bmatrix} 4 & 14 & -4 \\ -12 & 28 & -8 \\ 12 & -13 & 18 \end{bmatrix}, \qquad B = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 10 \\ 0 & 0 & 20 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 2 & -2 \\ -3 & -1 & 0 \end{bmatrix}$$

We will demonstrate that A is similar to B using C. Compute AC, and CB.

$$AC = \begin{bmatrix} 4 & 14 & -4 \\ -12 & 28 & -8 \\ 12 & -13 & 18 \end{bmatrix} \begin{bmatrix} 2 & 2 & -3 \\ 0 & 2 & -2 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 40 & -40 \\ 0 & 40 & -20 \\ -30 & -20 & -10 \end{bmatrix}$$

Example 1 Part 2

$$AC = \begin{bmatrix} 20 & 40 & -40 \\ 0 & 40 & -20 \\ -30 & -20 & -10 \end{bmatrix}$$

Now we compute CB

$$CB = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 2 & -2 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 10 \\ 0 & 0 & 20 \end{bmatrix} = \begin{bmatrix} 20 & 40 & -40 \\ 0 & 40 & -20 \\ -30 & -20 & -10 \end{bmatrix}$$

Which is the same as AB.

Example 2 Part 1

Let A, B be given below, then compute a value of k such that A and B are similar.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & k \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

This is just asking us to find a k such that A has eigenvalues $1, 7$ and is diagonalizable! There are many ways to approach this. The easiest is probably to write out the characteristic polynomial we want

$$f(\lambda) = (\lambda - 1)(\lambda - 7) = \lambda^2 - 8\lambda + 7$$

Then finding the characteristic polynomial of A and setting them equal

$$\det\left(\begin{bmatrix} 5 - \lambda & 2 \\ 4 & k - \lambda \end{bmatrix}\right) = (5 - \lambda)(k - \lambda) - 8 = \lambda^2 - (5 + k)\lambda + 5k - 8$$

Example 2 Part 2

So, we will now set these polynomials equal to each other

$$\lambda^2 - 8\lambda + 7 = \lambda^2 - (5 + k)\lambda + 5k - 8$$

Remember that λ is our variable here, so we treat k as a constant. We are left with the following system to solve

$$\begin{aligned} -(5+k) &= -8 \\ 5k - 8 &= 7 \end{aligned}$$

We can solve this however we want, but the first equation gives us $k = 3$ and plugging this into the second gives us what we need, so k must be 3.

Example 2 Part 3

This means that we showed

$$\begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

have the same eigenvalues. Are we done? No! Just because two matrices have the same eigenvalues doesn't mean that they are similar. We need to compute the C that proves these matrices are similar.
Now we find the eigenvectors of A and then we would be done!