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3. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$
4. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

# Inner Product Example: $\mathbb{C}^n$ (standard) Part 1

Let $V = \mathbb{C}^n$ and $\mathbb{F} = \mathbb{C}$. Then the following function is an inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\mathbf{y}}^\top \mathbf{x} = \sum_{k=1}^{n} x_k \overline{y_k}$$

Note: we sometimes abbreviate $\overline{\mathbf{y}}^\top$ as $\mathbf{y}^*$

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Note that if we are in the real numbers, then we omit the conjugate of $\mathbf{y}$.

Note: we sometimes abbreviate $\overline{\mathbf{y}}^\top$ as $\mathbf{y}^*$

# Inner Product Example: $\mathbb{C}^n$ (standard) Part 2

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# Inner Product Example: $\mathbb{C}^n$ (standard) Part 3

Property 3,4: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

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In addition, the only time $\sum_{k=1}^{n} |x_k|^2 = 0$ is when all components are 0 or if $\mathbf{x} = \mathbf{0}$

# Dot Product

**Definition**

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Note: For this course, we will only consider this inner product unless stated otherwise

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# Induced Norms

**Theorem**

*Let $(V, \mathbb{F})$ be a vector space with some inner product $\langle \cdot, \cdot \rangle$. Then the induced norm of this space is*

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

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**Proof.**

We can prove all 3 properties from the previous slide as consequences of us using the inner product. □

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For this course, we will consider only this norm unless stated otherwise.

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If we have any vector, $\mathbf{x} \neq \mathbf{0}$, then we can find a vector pointing in the same direction, but is also of *unit length* by doing:

$$\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

# Distance

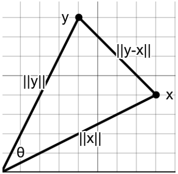
For us, we can think of distance between vectors as "how large is the difference between two vectors", or in other words, we say

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$$

Note: this idea is closely related to "Metric Spaces"[1], which you will see in an analysis course.

[1] `https://en.wikipedia.org/wiki/Metric_space#Definition`

# Angle Between Vectors



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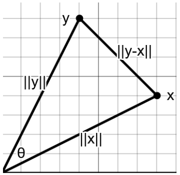


Using the law of cosines[a], we have that

$$\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$$

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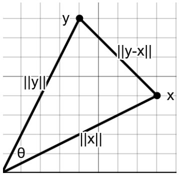


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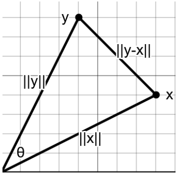


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Which (assuming that $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$) can be solved for $\theta$ to get

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}^\top\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right)$$

[a] https://en.wikipedia.org/wiki/Law_of_cosines

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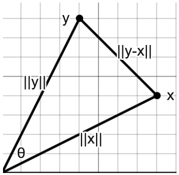


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In higher dimensions and other vector spaces, this is how we define the angle between vectors

[a] `https://en.wikipedia.org/wiki/Law_of_cosines`

# Orthogonality

Let $(V, \mathbb{F})$ be a vector space where $V$ denotes the set our vectors come from, $\mathbb{F}$ is the set our scalars come from, and we have some inner product $\langle \cdot, \cdot \rangle$.

**Definition**

Orthogonal Vectors: We say that 2 vectors, $(\mathbf{x}, \mathbf{y} \in V)$ are orthogonal (or perpendicular) if

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

# Orthogonality

Let $(V, \mathbb{F})$ be a vector space where $V$ denotes the set our vectors come from, $\mathbb{F}$ is the set our scalars come from, and we have some inner product $\langle \cdot, \cdot \rangle$.

**Definition**

**Orthogonal Vectors**: We say that 2 vectors, $(\mathbf{x}, \mathbf{y} \in V)$ are orthogonal (or perpendicular) if

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Note that assuming we have non-zero vectors, the angle between $\mathbf{x}, \mathbf{y}$ would be $90^\circ$!

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Since orthogonality is closely tied to our inner product, we will use our standard one for this course.

# Special Case for Orthogonality

### Theorem

*If $V = \mathbb{R}^n$ (or equivalently $\mathbb{C}^n$) with the usual inner product, then $\mathbf{0}$ is orthogonal to every vector.*

### Proof.

Let $\mathbf{x} \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denote the standard inner product, then we have that

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**Proof.**

Let $\mathbf{x} \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denote the standard inner product, then we have that

$$\langle \mathbf{0}, \mathbf{x} \rangle = \sum_{k=1}^{n} 0 \cdot x_k = 0$$

□