Revisiting Vector Equations and Homogeneous Systems

If $A \in \mathbb{R}^{m\times n}$ with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$, then the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

which has the corresponding augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{0} \end{array}\right]$$

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- If a non-trivial solution exists, then:
  - At least one $x_i \neq 0$
  - At least one $\mathbf{a}_i$ can be written as a linear combination of the others!
- If there is only the trivial solution, then no columns can be written as a linear combination of the others!

Brief Proof (on board) Of the last bullet point.

Example

Determine whether the equation

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ -10 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Python code to get this answer!

```
import sympy as sym
A = sym.Matrix([[2, -1, 6, 0],
                [1, 6, -10, 0],
                [0, 3, -6, 0]])
display(A.rref())
```

What does it mean?

We get

$$
\begin{aligned}
x_1 + 2x_3 &= 0 \\
x_2 - 2x_3 &= 0
\end{aligned}
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Plugging in gives:

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Or equivalently

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So, one is a linear combination of the others!

Linear Independence

Definition

Linear Independence: We say that a set of $p > 1$ vectors in $\mathbb{R}^n$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if we **cannot** write one as a linear combination of the others. Otherwise, the set is linearly dependent.

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If we find another solution, then they are linearly dependent

Practice!

Determine if the following sets of vectors are linearly dependent or linearly independent

- $$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$$

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- $$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

  They are linearly independent! Perform Gaussian Elimination on the Augmented matrix to see this

Conceptual Practice!

For each of the following questions, either come up with an example or think of a reason it's false

1. Give an example of 1 vector in $\mathbb{R}^3$ so that the set is linearly independent
2. Give an example of 2 vector in $\mathbb{R}^3$ so that the set is linearly independent
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   This is not possible

But what about $\mathbf{0}$?

Theorem

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains the zero vector, $\mathbf{0}$, then the set is linearly dependent.

Proof.

Since we can reorder the list without changing the overall property, let $\mathbf{v}_1 = \mathbf{0}$.

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See that

$$1\mathbf{0} + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

$\square$

Number of Vectors and Their Dimension

Theorem

A set of p vectors, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.