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Fundamental Theorem of Algebra

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where each $r_\ell \in \mathbb{C}$ and can be repeated!

Complex Eigenvalues of a Real Matrix Example

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Rotation-Scaling Matrices

Definition

We define a rotation-scaling matrix as a matrix of the form

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad a, b \in \mathbb{R} \qquad a \neq 0 \neq b$$

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We also have that the eigenvalues are $\lambda = a \pm bi$

Eigenvalues Relating to Rotation-Scaling Matrices

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Let $A \in \mathbb{R}^{2\times 2}$ with a complex eigenvalue $\lambda \notin \mathbb{R}$ and $\mathbf{v}$ be an eigenvector. Then $A = CBC^{-1}$ for

$$B = \begin{bmatrix} Re(\lambda) & Im(\lambda) \\ -Im(\lambda) & Re(\lambda) \end{bmatrix} \qquad C = \begin{bmatrix} Re(\mathbf{v}) & Im(\mathbf{v}) \end{bmatrix}$$

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Let A be given as below. Find the B and C in the previous theorem.

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Finally, we must compute an eigenvector associated with $\lambda = 1 + i$.

Rotation-Scaling Theorem Example 2×2 Part 2

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Meaning,

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So, $(\lambda, \mathbf{v})$ is an eigenpair of A!

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We can extend our Rotation-Scaling theorem to larger matrices! This is called the Block Diagonalization

Theorem

Let $A \in \mathbb{R}^{n \times n}$ suppose that for each eigenvalue (real or complex!) the algebraic and geometric multiplicities are equal, then $A = CBC^{-1}$ where B, C are as follows.

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In other words, if we are in $\mathbb{R}^{3 \times 3}$, and have $\lambda_1 \in \mathbb{C}, \lambda_2 \in \mathbb{R}$ as two eigenvalues $(\lambda_1 \notin \mathbb{C})$ of A, with $\mathbf{v}_1$ and $\mathbf{v}_2$ as their corresponding eigenvectors

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$$A = \begin{bmatrix} \mathrm{Re}(\mathbf{v}_1) & \mathrm{Im}(\mathbf{v}_1) & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\lambda_1) & \mathrm{Im}(\lambda_1) & 0 \\ -\mathrm{Im}(\lambda_1) & \mathrm{Re}(\lambda_1) & 0 \\ 0 & 0 & \lambda 2 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Example 3×3

Let A be given below:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

which has characteristic polynomial $p(\lambda) = \lambda^3 - 4\lambda^2 + 6\lambda - 4$. Find a matrix B such that

$$A = CBC^{-1}$$

where C is

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 1

First, we need to find the roots of $p(\lambda)$. By plugging in the values of $\pm 1, \pm 2, \pm 4$ we will find that $\lambda = 2$ is an eigenvalue.

Next we would divide out the factor $\lambda - 2$ to get $\lambda^2 - 2\lambda + 2$, which we use the quadratic formula to find that $\lambda = 1 \pm i$ are the other eigenvalues.

Now we just need to find an eigenvector associated with $\lambda = 1 + i$ and $\lambda = 2$. We do the real one first

$$A - 2I = \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 0 & -1 & 1-2 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 1

First, we need to find the roots of $p(\lambda)$. By plugging in the values of $\pm 1, \pm 2, \pm 4$ we will find that $\lambda = 2$ is an eigenvalue.

Next we would divide out the factor $\lambda - 2$ to get $\lambda^2 - 2\lambda + 2$, which we use the quadratic formula to find that $\lambda = 1 \pm i$ are the other eigenvalues.

Now we just need to find an eigenvector associated with $\lambda = 1 + i$ and $\lambda = 2$. We do the real one first

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Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 2

Now, we find an eigenvector for $\lambda = 1 + i$

$$A - (1+i)I = \begin{bmatrix} 1-(1+i) & 0 & -1 \\ 1 & 2-(1+i) & 1 \\ 0 & -1 & 1-(1+i) \end{bmatrix}$$

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So, an eigenvector is $\mathbf{v} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 3

Next, we see that

$$\mathrm{Re}\left(\mathbf{v}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathrm{Im}\left(\mathbf{v}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

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Which are the first 2 columns of C

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Which are the first 2 columns of C and the last column of C is $\mathbf{x}$.

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Which are the first 2 columns of C and the last column of C is $\mathbf{x}$. This means the block using the real and imaginary components must be in the first two columns of B and the third column corresponds to our real eigenvalue. In other words, since

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 3

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we have that

$$B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$