Review of Complex Numbers

Let $z \in \mathbb{C}$. Then we can write it as $z = a + bi$ for $a, b \in \mathbb{R}$.
We define some useful operations and properties of complex numbers. $z_1 = a + bi, z_2 = c + di$

1. $z_1 + z_2 = (a + c) + (b + d)i$
2. $z_1 - z_2 = (a - c) + (b - d)i$
3. $z_1 \cdot z_2 = (ac - bd) + (ad + cb)i$
4. $\overline{z}_1 = a - bi$
5. $\mathrm{Re}\,(z_1) = a$
6. $\mathrm{Im}\,(z_1) = b$
7. $|z_1| = \sqrt{a^2 + b^2} = \sqrt{z_1 \overline{z}_1}$

Division with Complex Numbers

What about division? We can determine what $\frac{1}{z}$ needs to look like if we know that $z = a + bi$ for $a, b \in \mathbb{R}$. See that

$$\frac{1}{z} = \frac{\overline{z}}{\overline{z}z} = \frac{a - bi}{a^2 + b^2}$$

Fundamental Theorem of Algebra

Recall that if we have an n^{th} degree polynomial with real (or complex) coefficients, then we have n roots counting multiplicity.
Or in other words if $p(x)$ is given by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

then we can factor it as

$$p(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

where each $r_\ell \in \mathbb{C}$ and can be repeated!

Complex Eigenvalues of a Real Matrix Example

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are going to be the roots of

$$p(\lambda) = \lambda^2 + 1$$

which has the roots $\lambda = \pm i$.

Complex Eigenvalues of a Real Matrix

Theorem

Let $A \in \mathbb{R}^{n \times n}$, then if $(\lambda, \mathbf{v})$ is an eigenpair of A then $(\overline{\lambda}, \overline{\mathbf{v}})$ is also an eigenpair! In other words, eigenpairs come in conjugate pairs.

Proof.

Let $(\lambda, \mathbf{v})$ be an eigenpair of $A \in \mathbb{R}^{n \times n}$

$$A\overline{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$$

$\square$

Rotation-Scaling Matrices

Definition

We define a rotation-scaling matrix as a matrix of the form

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \qquad a, b \in \mathbb{R} \qquad a \neq 0 \neq b$$

We can actually write A as follows

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

Where $r = \sqrt{a^2 + b^2} = \sqrt{\det(A)}$.
We also have that the eigenvalues are $\lambda = a \pm bi$

Eigenvalues Relating to Rotation-Scaling Matrices

Why do we care? Well, do we notice about A and how it relates to it's eigenvalues?

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \lambda = a \pm bi$$

We can write it as

$$A = \begin{bmatrix} \mathrm{Re}\,(\lambda) & \mathrm{Im}\,(\lambda) \\ -\mathrm{Im}\,(\lambda) & \mathrm{Re}\,(\lambda) \end{bmatrix}$$

Rotation-Scaling Theorem

Theorem

Let $A \in \mathbb{R}^{2\times 2}$ with a complex eigenvalue $\lambda \notin \mathbb{R}$ and $\mathbf{v}$ be an eigenvector. Then $A = CBC^{-1}$ for

$$B = \begin{bmatrix} Re(\lambda) & Im(\lambda) \\ -Im(\lambda) & Re(\lambda) \end{bmatrix} \qquad C = \begin{bmatrix} Re(\mathbf{v}) & Im(\mathbf{v}) \end{bmatrix}$$

Rotation-Scaling Theorem Example 2×2 Part 1

Let A be given as below. Find the B and C in the previous theorem.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 2$ The roots of this polynomial are exactly $1 \pm i$. However since we have conjugate pairs, we will consider only $\lambda = 1 + i$. This means that B is given by

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Finally, we must compute an eigenvector associated with $\lambda = 1 + i$.

Rotation-Scaling Theorem Example 2×2 Part 2

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \xrightarrow{B=A-(1+i)} \begin{bmatrix} 2-(1+i) & -1 \\ 2 & -(1+i) \end{bmatrix} \to \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \xrightarrow{R_2=R_2-\frac{2}{1-i}R_1} \begin{bmatrix} 1-i & -1 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1=\frac{R_1}{1-i}} \begin{bmatrix} 1 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$$

So, we have that

$$\mathbf{v} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} i$$

Meaning,

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

Rotation-Scaling Theorem Example 2×2 Part 3

Putting this together gives

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Is There an Easier Way?

The division was pretty tedious, so let's try to make an easier way. Remember that if λ is an eigenvalue of a matrix, then $\det(A - \lambda I) = 0$. So, if $A \in \mathbb{R}^{2\times 2}$, then the rows are multiples of each other! This means that

$$A - \lambda I = \begin{bmatrix} z & w \\ cz & cw \end{bmatrix}$$

For some $z, w, c \in \mathbb{C}$ See that if we define $\mathbf{v} = \begin{bmatrix} -w \\ z \end{bmatrix}$ then

$$(A - \lambda I)\mathbf{v} = \begin{bmatrix} z & w \\ cz & cw \end{bmatrix} \begin{bmatrix} -w \\ z \end{bmatrix} = \begin{bmatrix} -zw + wz \\ -czw + cwz \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $(\lambda, \mathbf{v})$ is an eigenpair of A!

A Special Similarity Transformation for Complex Eigenvalues

We can extend our Rotation-Scaling theorem to larger matrices! This is called the Block Diagonalization

Theorem

Let $A \in \mathbb{R}^{n\times n}$ suppose that for each eigenvalue (real or complex!) the algebraic and geometric multiplicities are equal, then $A = CBC^{-1}$ where B, C are as follows.

- *B is block diagonal with 1×1 blocks for real eigenvalues and 2×2 blocks for complex eigenvalues.*
- *The columns of C form a bases for the eigenspaces for the real eigenvectors or pairs $(Re(\mathbf{v}), Im(\mathbf{v}))$.*

In other words, if we are in $\mathbb{R}^{3\times 3}$, and have $\lambda_1 \in \mathbb{C}, \lambda_2 \in \mathbb{R}$ as two eigenvalues $(\lambda_1 \notin \mathbb{C})$ of A, with $\mathbf{v}_1$ and $\mathbf{v}_2$ as their corresponding eigenvectors, we get

$$A = \begin{bmatrix} \mathrm{Re}(\mathbf{v}_1) & \mathrm{Im}(\mathbf{v}_1) & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\lambda_1) & \mathrm{Im}(\lambda_1) & 0 \\ -\mathrm{Im}(\lambda_1) & \mathrm{Re}(\lambda_1) & 0 \\ 0 & 0 & \lambda 2 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Example 3×3

Let A be given below:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

which has characteristic polynomial $p(\lambda) = \lambda^3 - 4\lambda^2 + 6\lambda - 4$. Find a matrix B such that

$$A = CBC^{-1}$$

where C is

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 1

First, we need to find the roots of $p(\lambda)$. By plugging in the values of $\pm 1, \pm 2, \pm 4$ we will find that $\lambda = 2$ is an eigenvalue.

Next we would divide out the factor $\lambda - 2$ to get $\lambda^2 - 2\lambda + 2$, which we use the quadratic formula to find that $\lambda = 1 \pm i$ are the other eigenvalues.

Now we just need to find an eigenvector associated with $\lambda = 1 + i$ and $\lambda = 2$. We do the real one first

$$A - 2I = \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 0 & -1 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 = -R_2]{R_1 = -R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 2

Now, we find an eigenvector for $\lambda = 1 + i$

$$A - (1+i)I = \begin{bmatrix} 1-(1+i) & 0 & -1 \\ 1 & 2-(1+i) & 1 \\ 0 & -1 & 1-(1+i) \end{bmatrix} = \begin{bmatrix} -i & 0 & -1 \\ 1 & 1-i & 1 \\ 0 & -1 & -i \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1-i & 1 \\ -i & 0 & -1 \\ 0 & -1 & -i \end{bmatrix} \xrightarrow{R_2 = R_2 + iR_1} \begin{bmatrix} 1 & 1-i & 1 \\ 0 & 1+i & -1+i \\ 0 & -1 & -i \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 1-i & 1 \\ 0 & -1 & -i \\ 0 & 1+i & -1+i \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + (1-i)R_2} \begin{bmatrix} 1 & 1-i & 1 \\ 0 & -1 & -i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 + (1-i)R_2} \begin{bmatrix} 1 & 0 & -i \\ 0 & -1 & -i \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -R_2} \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

So, an eigenvector is $\mathbf{v} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$

Complex Eigenvalues of a Real Matrix Practice 3×3 Solution Part 3

Next, we see that

$$\mathrm{Re}\left(\mathbf{v}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathrm{Im}\left(\mathbf{v}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Which are the first 2 columns of C and the last column of C is $\mathbf{x}$. This means the block using the real and imaginary components must be in the first two columns of B and the third column corresponds to our real eigenvalue. In other words, since

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

we have that

$$B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$