Diagonal Matrix Arithmetic

To motivate this section, let's look at how diagonal matrices multiply Let $D_1 = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, D_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, see that

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Powers of Diagonal Matrices

From the previous slide, we see that for any diagonal $D = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$, we have

$$D^n = \begin{bmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \end{bmatrix}$$

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where $(\mathbf{v}_1, \lambda_1), \dots, (\mathbf{v}_n, \lambda_n)$ are each eigenpairs.

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Remember that eigenvectors associated with distinct eigenvalues are linearly independent. So, if A has n distinct eigenvalues, then it is diagonalizable!

Diagonalization is not Unique 2×2 Example

Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$. We see that for $V_1, V_2, D \in \mathbb{R}^{2\times 2}$ as given below,

$$V_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, V_2 = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

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But $V_1 \neq V_2$.

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4. Now $\mathbf{v}_1, \dots, \mathbf{v}_n$ (Eigenspace basis vectors!) form the columns of C, and their associated eigenvalues form the diagonal of D.

Diagonalization Example

Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ which has characteristic polynomial $-\lambda^3 + 8\lambda^2 - 13\lambda + 6$, which has roots $1, 6$. So we compute a basis for $E(A, 1)$ and $E(A, 6)$. We first do $E(A, 1)$.

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Whose null space has a basis of $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Diagonalization Example Part 2

Now, we compute a basis for $E(A, 6)$. For

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whose null space has a basis of $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$.

Diagonalization Example Part 3

So we can say that A is similar to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ with $C = \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

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$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Multiplicity of a Root Review

Recall that in the context of polynomials the multiplicity of a root is the number of times it is present in factored form.

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Example

For the polynomial $x^3 - 3x + 2$, we can factor it into

$$(x-1)^2(x+2)$$

So, $x = 1$ is a root with multiplicity 2 and $x = -2$ is a root with multiplicity 1.

Eigenvalue Multiplicities

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Let $A \in \mathbb{R}^{n\times n}$ and λ be an eigenvalue of A. Then

$$1 \le (\text{the geometric multiplicity of } \lambda) \le (\text{the algebraic multiplicity of } \lambda)$$

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2. *The sum of the geometric multiplicities of all eigenvalues of A is equal to n.*

Finding Multiplicity Example

Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 2 \\ 2 & 3 & -1 \end{bmatrix}$. Find the algebraic and geometric multiplicities of 1.

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There are 2 pivot variables, so $\dim\left(E(A, 1)\right) = 2$

Finding Multiplicities Practice

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has only 1 as an eigenvalue. Compute the algebraic and geometric multiplicities of 1.

Multiplicities for Similar Matrices

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