Homogeneous Systems

Definition

Homogeneous: A system of equations is homogeneous if it can be written in the form $A\mathbf{x} = \mathbf{0}$.

Example

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - 4x_2 &= 0 \end{aligned}$$

Not homogeneous!

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - 4x_2 &= 0 \end{aligned}$$

Yes it is homogeneous!

Every homogeneous system has a trivial solution where $\mathbf{x} = \mathbf{0}$.

Nontrivial Solutions Existence

Recall that the system $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$ has:

- A unique solution if A has a pivot in every row
- Infinite number of solutions if A has a solution and at least one free variable

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the solution set has at least one free variable.

Solution Sets of Consistent Systems

Find the solution set to the homogeneous system write in vector form (from lecture 4):

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= 0 \\ -2x_1 - 6x_2 + 4x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ -2 & -6 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us: $x_1 = -3x_2 + 2x_3$ or as a vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

More Practice!

Find the solution set to the nonhomogeneous system

$$\begin{aligned} x_1 + 3x_2 + 7x_3 &= 4 \\ x_2 + 3x_3 &= 5 \\ -2x_1 - 4x_2 - 8x_3 &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ -2 & -4 & -8 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 7 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -11 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 + 2x_3 \\ 5 - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v}(t \in \mathbb{R})$$

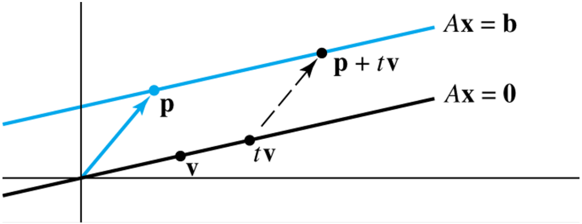
Comparing Homogeneous and Nonhomogeneous Solutions

$$
\begin{aligned}
x_1 + 3x_2 + 7x_3 &= 0 \\
x_2 + 3x_3 &= 0 \\
-2x_1 - 4x_2 - 8x_3 &= 0
\end{aligned}
$$

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = t\mathbf{v} (t \in \mathbb{R})$$

$$
\begin{aligned}
x_1 + 3x_2 + 7x_3 &= 4 \\
x_2 + 3x_3 &= 5 \\
-2x_1 - 4x_2 - 8x_3 &= 2
\end{aligned}
$$

$$\mathbf{x} = \begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \mathbf{p} + t\mathbf{v} (t \in \mathbb{R})$$



Infinite Solutions from Homogeneous System

Theorem

Let $A\mathbf{x} = \mathbf{b}$ be a consistent system with solution $\mathbf{x} = \mathbf{p}$. Then there exists a solution set of vectors of the form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}.$$

Where $t\mathbf{v}$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof.

We will show this by showing $A\mathbf{x} = A(\mathbf{p} + t\mathbf{v}) = b$.

$$\begin{aligned} A\mathbf{x} &= A(\mathbf{p} + t\mathbf{v}) \\ &= A\mathbf{p} + A(t\mathbf{v}) \\ &= \mathbf{b} + \mathbf{0} \\ &= \mathbf{b} \end{aligned}$$

□