Change of Coordinate Matrix in $\mathbb{R}^n$

Recall that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of a vector space V, then the matrix

$$P_{\mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{B}} & \ldots & [\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix}$$

Represents our basis vectors as an array of real numbers.
What does this mean for $V = \mathbb{R}^n$? Let's compute $P_{\mathcal{B}}\mathbf{x}$!

$$P_{\mathcal{B}}\mathbf{x} = x_1 [\mathbf{b}_1]_{\mathcal{B}} + \cdots + x_n [\mathbf{b}_n]_{\mathcal{B}}$$

This means if $\mathbf{x}$ is an array of coordinates for the $\mathcal{B}$ basis, then the output is a vector written in the standard basis!

Change of Coordinate Matrix in $\mathbb{R}^n$ Example

Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then, $P_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. So, if we say that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, and we want to find what it is in the standard basis we compute

$$\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 4 \end{bmatrix}$$

Change of Coordinate Matrix in $\mathbb{R}^n$ is a Linear Transformation

Theorem

If $V = \mathbb{R}^n$, then the Coordinate Matrix $P_{\mathcal{B}}$ is a Linear Transformation given by

$$P_{\mathcal{B}} : [\mathbf{x}]_{\mathcal{B}} \mapsto \mathbf{x},$$

which changes a vector from $\mathcal{B}$ coordinates to being written in the standard basis and

$$P_{\mathcal{B}}^{-1} : \mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$$

changes a vector from the standard basis back to $\mathcal{B}$ coordinates.

Changing Coordinate From One Basis to Another

In some data science applications, we may want to change problems from one basis to another.

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for a vector space V. Then, there exists a unique matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$

This is really just saying that we (1) change our vector from being written in $\mathcal{B}$ coordinates to being written in the standard basis , then (2) change our vector from the standard basis to $\mathcal{C}$ coordinates.

Is there a potentially better way?

A Better Way Without Explicitly Computing an Inverse

Yes!

We also have that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

Which we can solve in one of two ways

1) Solve

$$\left[\begin{array}{c|c} P_{\mathcal{C}} & P_{\mathcal{B}} \end{array}\right]$$

Which is solving the equation

$$P_{\mathcal{C}} X = P_{\mathcal{B}}$$

2) Solve

$$\left[\begin{array}{c|c} P_{\mathcal{C}} & \mathbf{y} \end{array}\right]$$

Which is solving the equation

$$P_{\mathcal{C}}\mathbf{x} = \mathbf{y}$$

then plugging in $\mathbf{y} = \mathbf{b}_1, \dots, \mathbf{b}_n$.

Either will give the right answer, it's just a matter of preference!

Change of Coordinate Example

Consider the following bases of $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.First, we try method 1.

$$\left[\begin{array}{c|c} P_{\mathcal{C}} & P_{\mathcal{B}} \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 2 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

So

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

Change of Coordinate Example

Consider the following bases of $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Next, we try method 2.

$$\left[\begin{array}{c|c} P_{\mathcal{C}} & \mathbf{y} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 1 & 1 & 0 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 0 & 1 & 0 & y_2 - y_1 \\ 0 & 1 & 1 & y_3 - y_1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 0 & 1 & 0 & y_2 - y_1 \\ 0 & 0 & 1 & y_3 - y_2 \end{array}\right]$$

So the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are given by

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \to \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

Change of Coordinate Practice

Consider the following bases of $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Matrix Vector Space Practice

Let V be the vector space of symmetric 2×2 real matrices. We have the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Consider the set $\mathcal{C} = \left\{ \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right\}$. Answer the following questions

1) Show that $\mathcal{C}$ is a basis for V.

Set up and show

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & -1 \\ 2 & 4 & 1 \end{bmatrix}$$

is invertible.

2) Write the matrix corresponding to

$$\left[\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \right]_{\mathcal{C}}$$

in $\mathcal{B}$ coordinates

$$\begin{bmatrix} -4 & 4 \\ 4 & 14 \end{bmatrix}$$