A General Algorithm

Let's build a more systematic way of solving linear systems (that we can hopefully have a computer do for us!). We want an algorithm that we can:

1. Apply to any augmented matrix from a general system of equations
2. Transform to a form where we can find the solution(s) (if any exist!) by quickly looking
3. Using only Elementary Row Operations

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 8 \end{array}\right]$$

What are the similarities between these two augmented matrices?

Row Echelon Form

Definition

Row Echelon Form. We say that a matrix is in Row Echelon Form if and only if it satisfies the following requirements

1. All non-zero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Example

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Reduced Row Echelon Form

Definition

Reduced Row Echelon Form (RREF). We say that a matrix is in RREF if and only if it satisfies the requirements for Row Echelon Form and

4. The leading entry in each non-zero row is 1.
5. Each leading 1 is the only non-zero in its column.

Example

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

What kind of transformations can take us from the left to the right?

Formalizing Our Algorithm

We've been doing this already actually! In order to better formalize it, let's break down our algorithm into 2 smaller steps.

Algorithm 1 RREF Transformation

Input:A matrix A with m rows and n columns of real numbers
Output:A transformed to RREF using Only Elementary Row Operations

1: Reduce A to Row Echelon Form
2: Reduce A to RREF

Reducing to Row Echelon Form

Let's start with an example. Lets reduce the following augmented matrix to Row Echelon Form.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 2 & 8 & 1 & 2 \\ 1 & 4 & 1 & 1 \end{array}\right]$$

1. Start with the leftmost non-zero column. We call this the pivot column.
2. Pick some non-zero element. This is called the pivot. Move it to the top if necessary.
3. Create zeros in all positions below the pivot.

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 2 & 8 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

Reducing to Row Echelon Form

4. Ignore the row containing the pivot and consider the remaining submatrix (highlighted in red). Then repeat the steps from before again!

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & \color{red}{0} & \color{red}{-1} & \color{red}{0} \\ 0 & \color{red}{1} & \color{red}{2} & \color{red}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Reducing to RREF

Now that we are in Row Echelon, we're almost there! We have 2 more conditions to meet.

1. All our pivots are 1
2. All elements above our pivots are 0.

So, let's fix them!

Reducing to RREF

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

5. Scale each row to make all pivots equal to 1.
6. Starting with the rightmost pivot column, zero out all elements above each non-zero pivot using Elementary Row Operations.

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Then our answer is

$$(x_1, x_2, x_3) = (-11, 3, 0)$$

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Gaussian Elimination[1]

Algorithm 2 Gaussian Elimination

Input: A matrix A with m rows and n columns of real numbers

Output: A transformed to RREF

1: Consider the leftmost column. Called the pivot column
2: Pick a non-zero entry as the pivot. Exchange rows if needed.
3: Use Elementary Row Operations to create zeros below the pivot
4: Ignore the row containing the pivot. Repeat the 3 steps above on the remaining submatrix until there are no more non-zero rows.
5: Scale each row so each non-zero pivot is 1.
6: Cancel out all elements above each pivot with Elementary Row Operations in each pivot column.

[1] https://en.wikipedia.org/wiki/Gaussian_elimination#History

Existence and Uniqueness of RREF

1. Existence: Given any matrix, we can apply a series of elementary row operations to find an equivalent RREF.
2. The series of operations are not unique, but the result is unique!

$$\left[\begin{array}{cc|c} 0 & 1 & -2 \\ -2 & -2 & -4 \\ -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} -2 & -2 & -4 \\ 0 & 1 & -2 \\ -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 1 & -2 \\ -2 & -2 & -4 \\ -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 0 & -4 \\ -2 & -2 & -4 \\ 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ -2 & -2 & -4 \\ 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Theorem

Every matrix is equivalent to one and only one reduced RREF matrix.

Example

Find a solution to the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -2 \\ x_1 + 3x_2 - 2x_3 &= 1 \end{aligned}$$

Using Gaussian Elimination, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 1 & 3 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & -3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 7 & -8 \\ 0 & 1 & -3 & 3 \end{array}\right]$$

This gives the solution set:

$$\begin{aligned} x_1 \quad + 7x_3 &= -8 \\ x_2 - 3x_3 &= 3 \end{aligned}$$

or in parametric form:

$$(-8 - 7x_3, 3 + 3x_3, x_3) = \begin{cases} x_1 & = -8 - 7x_3 \\ x_2 & = 3 + 3x_3 \\ x_3 & \text{is free} \end{cases}$$

Parametric Form

Definition

Parametric Form: We say that a solution to a system of linear equations is in parametric form if it has a variable in the ordered pair.

Example

For example

$$(-8 - 7x_3, 3 + 3x_3, x_3)$$

is in parametric form because x_3 is still a variable and we don't know what it is! We will usually write either the variable with a subscript like above or using another variable. Like:

$$(-8 - 7s, 3 + 3s, s)$$

Solutions of Linear Systems

We have

$$\begin{cases} x_1 & = -8 - 7x_3 \\ x_2 & = 3 + 3x_3 \\ x_3 & \text{is free} \end{cases}.$$

- Variables x_1 and x_2 came from pivot columns, and we call these basic variables
- Variable x_3 did not come from a pivot column, so we call it a free variable. This means it take on any value we want (IE we are free to choose it)
- If we have at least one free variable, then we have an infinite number of solutions.

Another Example

Find a solution to the system

$$\begin{aligned} x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

Gaussian Elimination gives us

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is an inconsistent system!

$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 1 \end{aligned}$$

Summary

For a given system of linear equations:

1. Write the augmented matrix
2. Use Elementary Row Operations to reduce to Row Echelon Form
    - 2.1 If the rightmost column is a pivot column, then it is inconsistent, and no solution exists. We can now stop
    - 2.2 If the rightmost column is not a pivot column, then it is consistent, and we have a solution, so we need to continue
3. Reduce to RREF
4. Rewrite as a system of linear equations
5. Put all basic variables on the left and all free variables on the right.