Eigenpair Reminder

Remember that we defined an eigenpair of A as an ordered pair $(\lambda, \mathbf{v})$ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

We've discussed how to confirm a proposed λ is actually an eigenvalue and to compute associated eigenvectors.
What about how to compute the eigenvalues themselves?

Computing Eigenvalues

Recall that if λ is an eigenvalue of a matrix A, then

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

has a non-trivial solution. In other words, $A - \lambda I$ is not invertible!
So, we can figure out all λ values such that

$$\det(A - \lambda I) = 0$$

Characteristic Polynomial

Definition

Let $A \in \mathbb{R}^{n \times n}$. The characteristic polynomial of A is the function $f(\lambda)$ given by

$$f(\lambda) = \det(A - \lambda I)$$

Theorem

The roots of $f(\lambda)$ are exactly the eigenvalues of A.

Proof.

Let λ be an eigenvalue of A, then $A - \lambda I$ is not invertible, so $\det(A - \lambda I) = 0$, so $f(\lambda) = \det(A - \lambda I) = 0$.
Let λ be a root of $f(\lambda)$, then

$$0 = f(\lambda) = \det(A - \lambda I).$$

So, $A - \lambda I$ is not invertible, so λ is an eigenvalue of A. $\square$

Rational Root Theorem[1]

Theorem

Let $f(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} + c_n\lambda^n$ be a polynomial with integer coefficients. Then, all rational factors of f are of the form

$$x = \frac{p}{q}$$

Where p is an integer factor of c_0 and q is an integer factor of c_n.

Example

Consider $f(x) = x^3 + x^2 - 10x + 8$. Then our possible roots are

$$\{\pm 1, \pm 2, \pm 4, \pm 8\}$$

And we plug in these values to see which one(s) are actual roots.

Note: We may not have rational roots depending on the polynomial itself (complex roots!)

[1] A proof can be found at `https://en.wikipedia.org/wiki/Rational_root_theorem`

Finding Eigenvalues Example

Let $A \in \mathbb{R}^{3\times 3}$ as given below. Then compute all the eigenvalues of A.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 4 & -3 \\ -4 & -4 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 1 & -1 \\ 1 & 4-\lambda & -3 \\ -4 & -4 & 5-\lambda \end{bmatrix}\right) = -\lambda^3 + 13\lambda^2 - 39\lambda + 27$$

Our rational roots theorem states that the possible rational eigenvalues are:

$$\{\pm 1, \pm 3, \pm 9, \pm 27\}$$

So, we plug these in and find that $1, 3, 9$ are the eigenvalues.

Finding Eigenvalues Practice

Let $A \in \mathbb{R}^{2\times 2}$ as given below. Then compute all the eigenvalues of A.

$$A = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$$

$$f(\lambda) = \lambda^2 - 16$$

which has roots

$$\lambda = \pm 4$$

Trace of a Matrix

Definition

Let $A \in \mathbb{R}^{n\times n}$. Then we define the trace of A as

$$\mathrm{Tr}\,(A) = a_{11} + \cdots + a_{nn}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Properties of the characteristic polynomial

Let $A \in \mathbb{R}^{n\times n}$. Then we know that

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \mathrm{Tr}\,(A)\,\lambda^{n-1} + \cdots + \det(A)$$

This means if $A \in \mathbb{R}^{2\times 2}$ of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then the characteristic polynomial has the form:

$$f(\lambda) = \lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

What About Larger Problems? (IE $n \geq 5$)

It's great that we can solve these small problems by hand, but what about larger ones? Well, The Abel–Ruffini theorem[2] states that we cannot always solve for these eigenvalues when we have $n \geq 5$.

However, we can still try to solve these problems! We will achieve this via some methods we learn later in the semester, and is how we currently compute eigenvalues in practice.

[2] More information can be found here: `https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem`