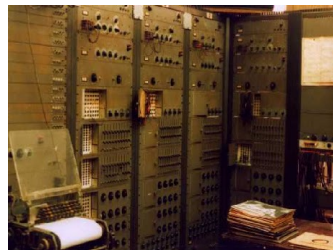


# Modeling the US Economy

- ▶ In 1949, economist Wassily Leontief worked with 250,000 data points produced by the United States BLS.
- ▶ Based on his research, he initially divided the US economy into 500 different sectors.
- ▶ One of the world's highest powered computers, Harvard's Mark II, could not process so much information.
- ▶ The model had to be further classified into a system of 42 equations of 42 unknowns.
- ▶ Such models are now known as **Leontief production models**.
- ▶ Mark II solved the system in 56 hours.



Based on *Linear Algebra and Its Applications* by D. Lay, et al.

Image Credit: <http://lasierrainformatica.blogspot.com/2013/06/el-harvard-mark-ii.html>

# Input-Output Model

- ▶ Instead of the 500 sectors Leontief identified, suppose the US economy is divided into 3 different sectors: Renewable Energy, Electricity, and Manufacturing.
- ▶ We can measure the total output for the year in each sector.
- ▶ We also know how each sector's output is divided among the other sectors.
- ▶ We can summarize this economy with an **input-output table**.

		Amount Purchased by		
Sector	Total Output (in Billions)	Renewables	Electricity	Manufacturing
Renewable Energy	40	10 (25%)	25 (62.5%)	5 (12.5%)
Electricity	100	7 (7%)	18 (18%)	75 (75%)
Manufacturing	125	20 (16%)	50 (40%)	55 (44%)

## Exchange Tables

We can convert the input-output table to an **exchange table** by giving the proportion of the output of each sector that is consumed by each of the other sectors.

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
0.625	0.18	0.4	Electricity
0.125	0.75	0.44	Manufacturing

- ▶ The total dollar amount of each sectors output is called the **price** of that sector.
- ▶ Thus,  $p_r$ ,  $p_e$ , and  $p_m$  denote total outputs of Renewable Energy, Electricity, and Manufacturing sectors, respectively.
- ▶ Leontief proved that there exists equilibrium prices that can be assigned to the total outputs for each sector in such a way that the income of each sector exactly balances its expenses
- ▶ How can we determine the equilibrium prices for each sector?

## Setting Up a System of Linear Equations

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
0.625	0.18	0.4	Electricity
0.125	0.75	0.44	Manufacturing

- ▶ Down each column is the proportion of that sector which is purchased by each of the other three sectors.
- ▶ Across each row we see for a given sector, what proportion of their inputs came from each sector's output.
- ▶ For the renewable energy sector, we therefore have the following linear equation:

output from renewables = (input from renewables) + (input from electricity) + (input from manufacturing)

$$p_r = 0.25p_r + 0.07p_e + 0.16p_m$$

$$0.75p_r - 0.07p_e - 0.16p_m = 0$$

# Setting Up a System of Linear Equations

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
0.625	0.18	0.4	Electricity
0.125	0.75	0.44	Manufacturing

Similarly deriving linear equations for the other sectors, we have the following system:

$$0.75p_r - 0.07p_e - 0.16p_m = 0$$

$$-0.625p_r + 0.82p_e - 0.4p_m = 0$$

$$-0.125p_r - 0.75p_e + 0.56p_m = 0$$

## Finding the Equilibrium for the Economy

We have the following augmented matrix associated to the system:

$$\left[ \begin{array}{ccc|c} 0.75 & -0.07 & -0.16 & 0 \\ -0.625 & 0.82 & -0.4 & 0 \\ -0.125 & -0.75 & 0.56 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -0.28 & 0 \\ 0 & 1 & -0.70 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

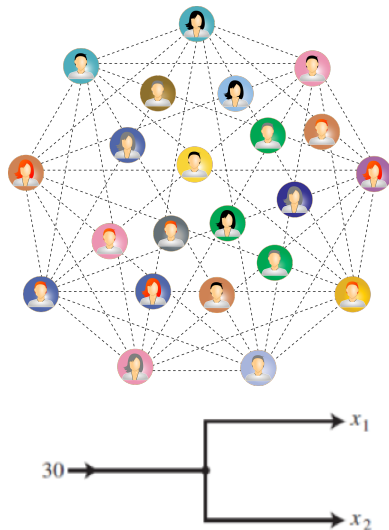
Notice  $p_m$  is a free variable, and we have equilibrium solution:

$$\mathbf{p} = \begin{bmatrix} p_r \\ p_e \\ p_m \end{bmatrix} = \begin{bmatrix} 0.28p_m \\ 0.7p_m \\ p_m \end{bmatrix} = p_m \begin{bmatrix} 0.28 \\ 0.7 \\ 1 \end{bmatrix}$$

- ▶ If this economy has  $p_m = 125$  billion dollars,
- ▶ Then if we want to ensure the economy is functioning at is equilibrium level (everything produced is used by other sectors):
  - ▶ Set  $p_r = (0.28)(125) = 35$  billion dollars, and
  - ▶ Set  $p_e = (0.7)(125) = 87.5$  billion dollars.

# Network Flow

- ▶ A **network** consists of a set of points, called **nodes** with lines, called **branches** connecting some or all of the nodes.
- ▶ The direction of the flow is indicated by each branch (are things flowing in or out of the node?).
- ▶ The flow amount (or rate) is either given or denoted by a variable.
- ▶ We assume the total flow into a network equals the total flow out of the network.
- ▶ The goal is to determine the flow in each branch when partial information is known.
- ▶ Network flows have applications to current flow through a circuit, flow of goods through supply chains, social networks, and **urban planning** to name a few.



# Traffic Flow in Baltimore

The network in the figure shows the flow of traffic (in vehicles per hour) over several one way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

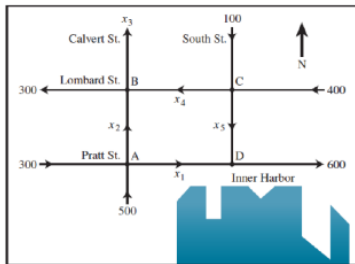


FIGURE 2 Baltimore streets.

Intersection	Flow in	Flow out
A	$300 + 500$	$= x_1 + x_2$
B	$x_2 + x_4$	$= 300 + x_3$
C	$100 + 400$	$= x_4 + x_5$
D	$x_1 + x_5$	$= 600$

$$\begin{array}{rclcl}
 x_1 & + & x_2 & & = & 800 \\
 & & x_2 & - & x_3 & + & x_4 & = & 300 \\
 & & & & x_4 & + & x_5 & = & 500 \\
 x_1 & & & & & + & x_5 & = & 600 \\
 & & x_3 & & & & & = & 400
 \end{array}$$



# Solving the System

We need to solve the following nonhomogeneous linear system of equations:

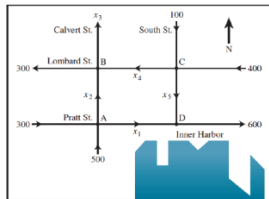


FIGURE 2 Baltimore streets.

$$\begin{array}{rclclcl} x_1 & + & x_2 & & & = & 800 \\ & & x_2 & - & x_3 & + & x_4 & = & 300 \\ & & & & x_4 & + & x_5 & = & 500 \\ x_1 & & & & & + & x_5 & = & 600 \\ & & x_3 & & & & & = & 400 \end{array}$$

We have an augmented matrix

$$\left[ \begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array} \right] \rightarrow \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_3 \\ x_5 \text{ is free} \end{cases}$$