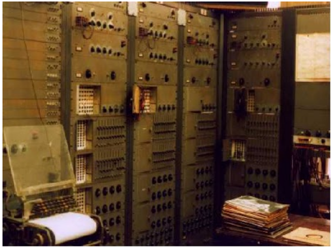
Modeling the US Economy

- In 1949, economist Wassily Leontief worked with 250,000 data points produced by the United States BLS.
- Based on his researched, he initially divided the US economy into 500 different sectors.
- One of the world's highest powered computers, Harvard's Mark II, could not process so much information.
- The model had to be further classified into a system of 42 equations of 42 unknowns.
- Such models are now know as Leontief production models.
- Mark II solved the system in 56 hours.



Based on *Linear Algebra and Its Applications* by D. Lay, et al.

Image Credit: http://lasierrainformatica.blogspot.com/2013/06/el-harvard-mark-ii.html

Input-Output Model

- Instead of the 500 sectors Leontief identified, suppose the US economy is divided into 3 different sectors: Renewable Energy, Electricity, and Manufacturing.
- We can measure the total output for the year in each sector.
- We also know how each sector's output is divided among the other sectors.
- We can summarize this economy with an input-output table.

		Amount Purchased by		
Sector	Total Output (in Billions)	Renewables	Electricity	Manufacturing
Renewable Energy	40	10 (25%)	25 (62.5%)	5 (12.5%)
Electricity	100	7 (7%)	18 (18%)	75 (75%)
Manufacturing	125	20 (16%)	50 (40%)	55 (44%)

Exchange Tables

We can convert the input-output table to an exchange table by giving the proportion of the output of each sector that is consumed by each of the other sectors.

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
0.625	0.18	0.4	Electricity
0.125	0.75	0.44	Manufacturing

- The total dollar amount of each sectors output is called the price of that sector.
- Thus, p_r, p_e, and p_m denote total outputs of Renewable Energy, Electricity, and Manufacturing sectors, respectively.
- Leontief proved that there exists equilibrium prices that can be assigned to the total outputs for each sector in such a way that the income of each sector exactly balances its expenses
- How can we determine the equilibrium prices for each sector?

Setting Up a System of Linear Equations

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
0.625	0.18	0.4	Electricity
0.125	0.75	0.44	Manufacturing

- Down each column is the proportion of that sector which is purchased by each of the other three sectors.
- Across each row we see for a given sector, what proportion of their inputs came from each sector's output.
- For the renewable energy sector, we therefore have the following linear equation:

output from renewables = (input from renewables) + (input from electricity) + (input from r

$$p_r = 0.25p_r + 0.07p_e + 0.16p_m$$

$$0.75p_r - 0.07p_e - 0.16p_m = 0$$

Setting Up a System of Linear Equations

Proportion of Output from:			
Renewable Energy	Electricity	Manufacturing	Purchased by
0.25	0.07	0.16	Renewable Energy
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0.125	0.75	0.44	Manufacturing

Similarly deriving linear equations for the other sectors, we have the following system:

$$\begin{aligned}
0.75p_r &- 0.07p_e - 0.16p_m = 0\\
-0.625p_r &+ 0.82p_e - 0.4p_m = 0\\
-0.125p_r &- -0.75p_e + 0.56p_m = 0
\end{aligned}$$

Finding the Equilibrium for the Economy

We have the following augmented matrix associated to the system:

$$\left[\begin{array}{ccc|c} 0.75 & -0.07 & -0.16 & 0 \\ -0.625 & 0.82 & -0.4 & 0 \\ -0.125 & -0.75 & 0.56 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -0.28 & 0 \\ 0 & 1 & -0.70 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

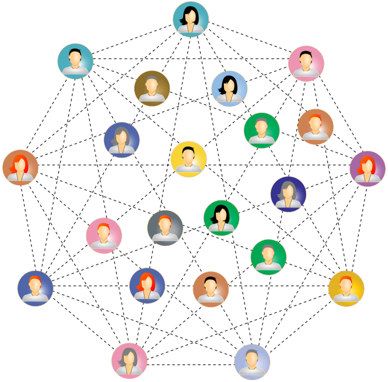
Notice p_m is a free variable, and we have equilibrium solution:

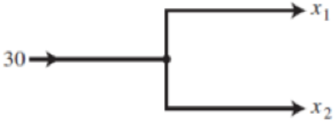
$$\mathbf{p} = \begin{bmatrix} p_r \\ p_e \\ p_m \end{bmatrix} = \begin{bmatrix} 0.28p_m \\ 0.7p_m \\ p_m \end{bmatrix} = p_m \begin{bmatrix} 0.28 \\ 0.7 \\ 1 \end{bmatrix}$$

- If this economy has $p_m = 125$ billion dollars,
- Then if we want to ensure the economy is functioning at is equilibrium level (everything produced is used by other sectors):
  - Set $p_r = (0.28)(125) = 35$ billion dollars, and
  - Set $p_e = (0.7)(125) = 87.5$ billion dollars.

Network Flow

- A network consists of a set of points, called nodes with lines, called branches connecting some or all of the nodes.
- The direction of the flow is indicated by each branch (are things flowing in or out of the node?).
- The flow amount (or rate) is either given or denoted by a variable.
- We assume the total flow into a network equals the total flow out of the network.
- The goal is to determine the flow in each branch when partial information is known.
- Network flows have applications to current flow through a circuit, flow of goods through supply chains, social networks, and urban planning to name a few.





Traffic Flow in Baltimore

The network in the figure shows the flow of traffic (in vehicles per hour) over several one way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

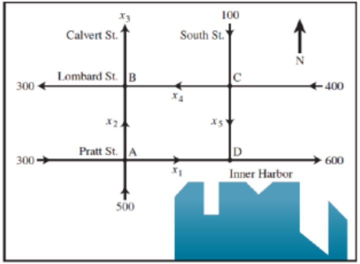


FIGURE 2 Baltimore streets.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

$$
\begin{array}{ccccccccccc}
x_1 & + & x_2 & & & & & & & = & 800 \\
 & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\
 & & & & & & x_4 & + & x_5 & = & 500 \\
x_1 & & & & & & & + & x_5 & = & 600 \\
 & & & & x_3 & & & & & = & 400
\end{array}
$$

Solving the System

We need to solve the following nonhomogeneous linear system of equations:

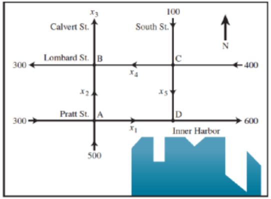


FIGURE 2 Baltimore streets.

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & & & & & = & 800 \\ & & x_2 & - & x_3 & + & x_4 & & & = & 300 \\ & & & & & & x_4 & + & x_5 & = & 500 \\ x_1 & & & & & & & + & x_5 & = & 600 \\ & & & & x_3 & & & & & = & 400 \end{array}$$

We have an augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_3 \\ x_5 \text{ is free} \end{cases}$$