Confidence intervals for a parameter with an upper or lower bound

Joshua N. Pritikin

Virginia Institute for Psychiatric and Behavioral Genetics
Virginia Commonwealth University

08 Dec 2016

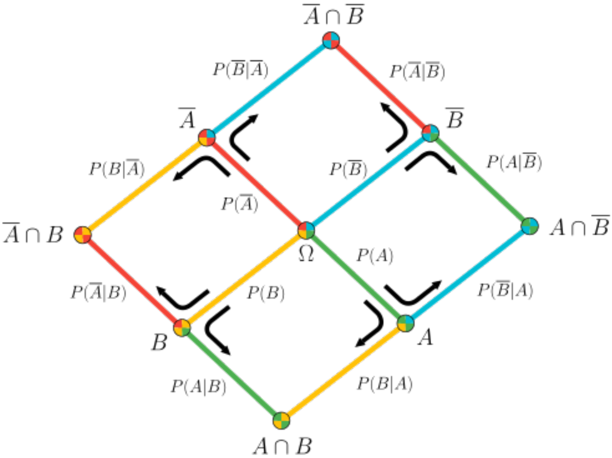
Acknowledgment

This research was aided by

- Mike Neale & Lance Rappaport
- Hao Wu (Boston College)
- OpenMx development team
- National Institute of Health R01-DA018673 (PI Neale)


In the
Beginning

Bayes' rule



$$P(A|B) \cdot P(B) = P(A \cap B) = P(B|A) \cdot P(A)$$

Maximum likelihood

Let θ be a column vector of model parameters.

$$\Pr(\theta|data) = \frac{\Pr(data|\theta)\Pr(\theta)}{\Pr(data)} \tag{1}$$

$$\Pr(\theta|data) \propto \Pr(data|\theta)\Pr(\theta) \tag{2}$$

$$\Pr(\theta|data) \propto \Pr(data|\theta) \tag{3}$$

Inferential test

Ingredients:

- False positive rate
- Test statistic
- Test distribution

Example:

- False positive rate: $\alpha = .05$
- Test statistic: t
- Test distribution: Student's t with $df = 10$

Inferential test

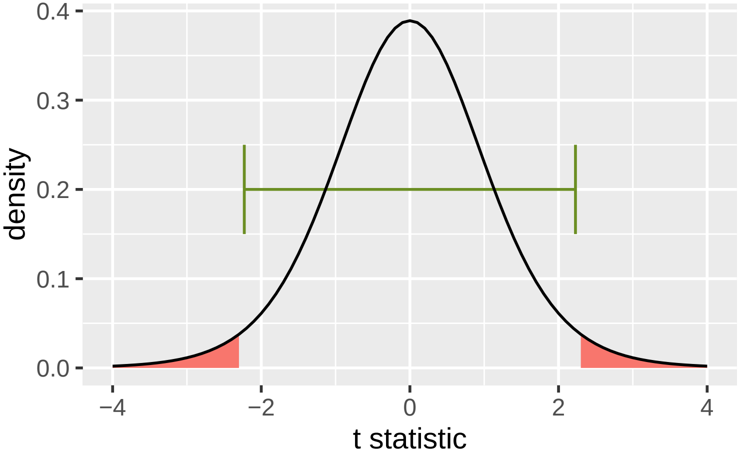
Ingredients:

- False positive rate
- Test statistic
- Test distribution

Example:

- False positive rate: $\alpha = .05$
- Test statistic: t
- Test distribution: Student's t with $df = 10$

Wald-based Confidence interval (CI)



t similar to z

$$t \approx z \equiv \frac{\theta - \mu}{\mathrm{SE}_\theta}$$

- Symmetric by assumption
- Accuracy depends on parameterization

Wilk's likelihood ratio test

For

- likelihood L
- parameter θ
- maximum likelihood point $\hat{\theta}$

and under certain regularity conditions[1],

$$(-2 \log L(\theta)) - (-2 \log L(\hat{\theta})) \xrightarrow{d} \chi^2_1.$$

[1] Steiger, Shapiro, and Browne (1985)

Asymmetric example

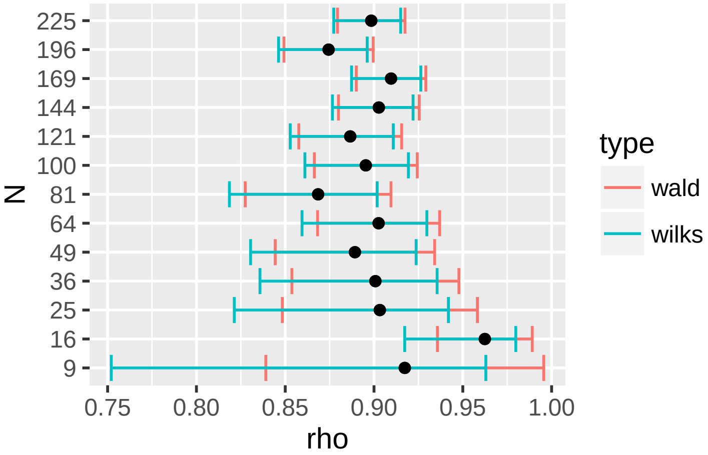
Estimate correlation ρ

$$\begin{bmatrix} 1.00 & \rho \\ \rho & 1.00 \end{bmatrix}$$

when data are generated with covariance matrix

$$\begin{bmatrix} 1.00 & 0.90 \\ 0.90 & 1.00 \end{bmatrix}$$

An asymmetric example



Two kinds of parameter boundaries

- **natural**
  a boundary beyond which the model distribution is invalid or degenerate
- **attainable**
  separates the interpretable part of a distribution from the uninterpretable part

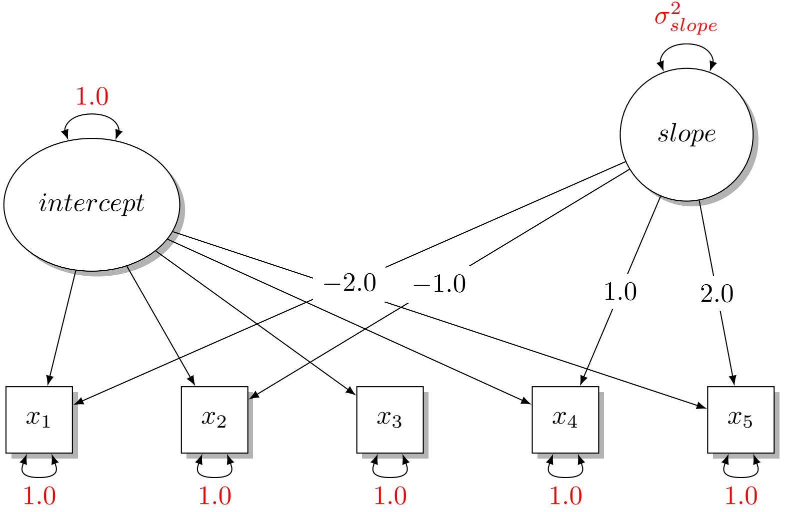
A natural boundary

Estimate correlation ρ

$$\begin{bmatrix} 1.00 & \rho \\ \rho & 1.00 \end{bmatrix}$$

ρ has natural boundaries at ± 1

Latent growth curve model

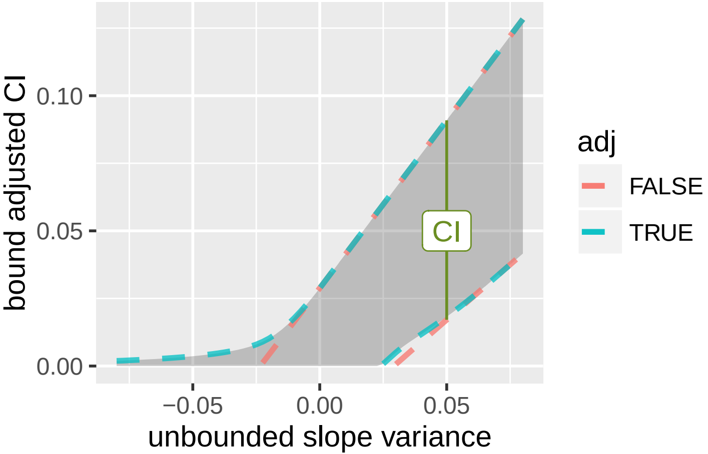


An attainable bound

- σ^2_{slope} has a natural lower bound near -0.1
- $\sigma^2_{slope} \leq 0$ is uninterpretable
- We place an attainable lower bound at 0

Wu and Neale (2012) described how to correct the CI for an attainable bound.

Adjusted CI


0.10
0.05
0.00
bound adjusted CI
−0.05
0.00
0.05
unbounded slope variance
CI
adj
FALSE
TRUE

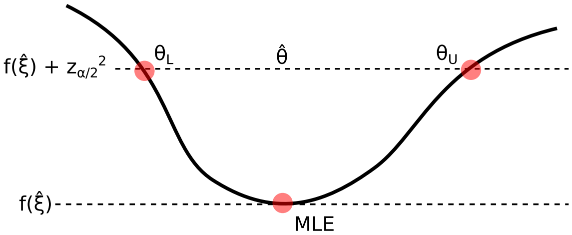
Some notation

- $\xi = (\theta, \zeta)'$
- $f(\xi)$
- $\hat{\xi}$
- z_a
- $\chi^2(1-a, 1) = z_a^2$

Estimation choices

- Geometry
  - Cone
  - Ring
- Optimization Form
  - Constrained
  - Unconstrained
- OpenMx Optimizer Engine
  - NPSOL
  - CSOLNP
  - SLSQP

Geometry



$$\arg\min_{\theta} f(\xi) \quad \text{subject to } f(\hat{\xi}) + z_{0.5\alpha}^2 = f(\xi) \tag{4}$$

$$\arg\min_{\theta} f(\xi) \quad \text{subject to } f(\hat{\xi}) + z_{0.5\alpha}^2 > f(\xi) \tag{5}$$

Optimization

Constrained (as on previous slide):

$$\arg\min_{\theta} f(\xi) \quad \text{subject to } f(\hat{\xi}) + z^2_{0.5\alpha} = f(\xi) \tag{6}$$

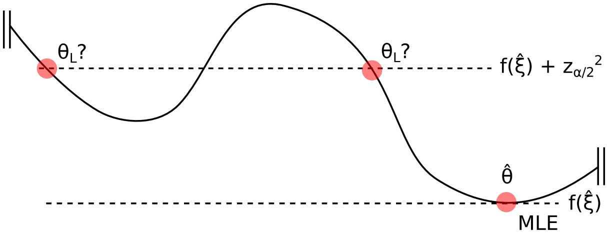
$$\arg\min_{\theta} f(\xi) \quad \text{subject to } f(\hat{\xi}) + z^2_{0.5\alpha} > f(\xi) \tag{7}$$

Unconstrained minimization:

$$\theta + \left[f(\hat{\xi}) + z^2_{0.5\alpha} - f(\xi) \right]^2 \tag{8}$$

$$\theta + \max \left[0, f(\hat{\xi}) + z^2_{0.5\alpha} - f(\xi) \right] \tag{9}$$

Identification



Compare with Wald SE

Diagnostics

```
summary(model, verbose=TRUE)
```

```
##   parameter value  side  fit residual vari  vars
## 1      vars 0.066 upper 1057     0.96 1.00 0.066
## 2      vars 0.005 lower 1056     1.05 0.98 0.005
##              method diagnostic statusCode
## 1 neale-miller-1997    success         OK
## 2     wu-neale-2012    success         OK
```

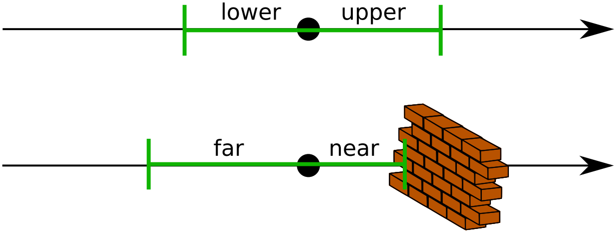
- `mxCheckIdentification`
- `mxTryHard`

Next: bound adjusted CIs

So far,

these slides apply to ordinary likelihood-based CI.

Change of reference

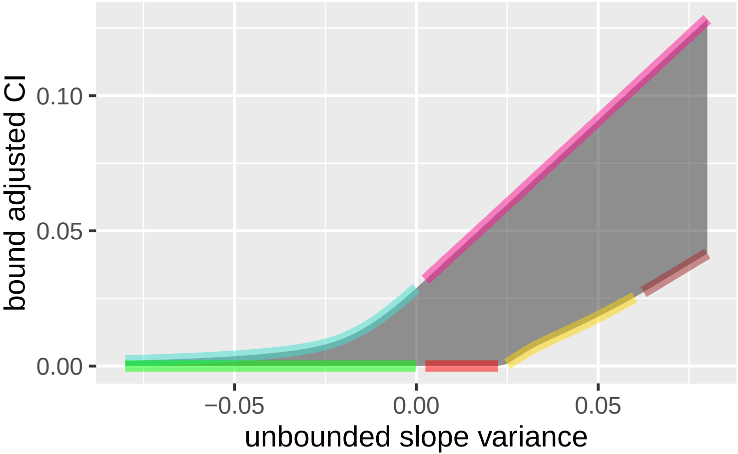


More notation

- $\xi = (\theta, \zeta)'$
- $f(\xi)$
- $\hat{\xi}$
- z_a
- $\chi^2(1-a, 1) = z_a^2$
- θ_b, θ_F, θ_N
- $\epsilon > 0$
- Φ

Algorithm tour by color

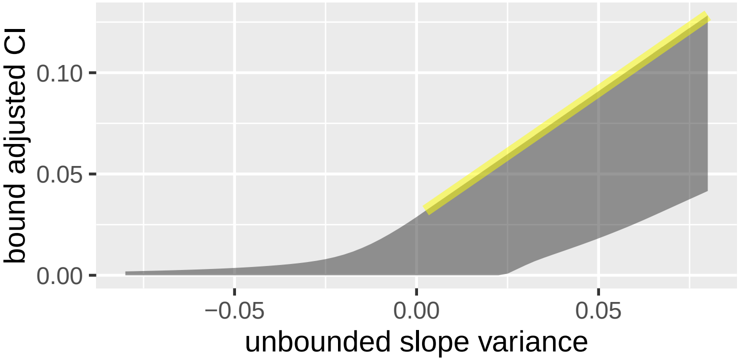
6 Cases



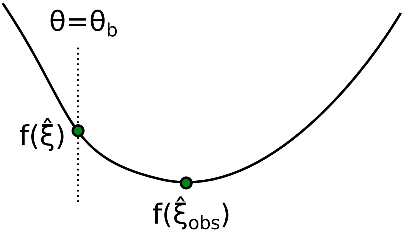
bound adjusted CI
0.10
0.05
0.00
−0.05
0.00
0.05
unbounded slope variance

Far limit θ_F

If $|\hat{\theta} - \theta_b| > \epsilon$ then the parameter bound θ_b is irrelevant.



Far limit θ_F



Far limit θ_F

$$\arg\min_{\pm\theta} f(\xi) \quad \text{subject to:} \tag{10}$$

$$d_1 < z_{0.5\alpha} \tag{11}$$

$$d_2 < z_{0.5\alpha} \tag{12}$$

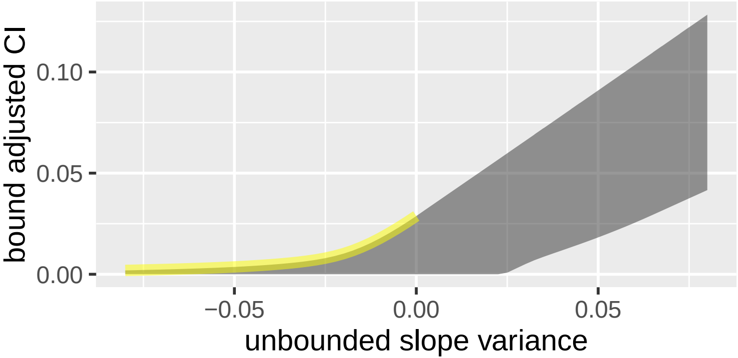
$$\log\left[\Phi(d_1) + \Phi(d_2)\right] > \log(\alpha) \tag{13}$$

where

$$d_1 \equiv \sqrt{\max\left[f(\xi) - f(\hat{\xi}), 0\right]} \tag{14}$$

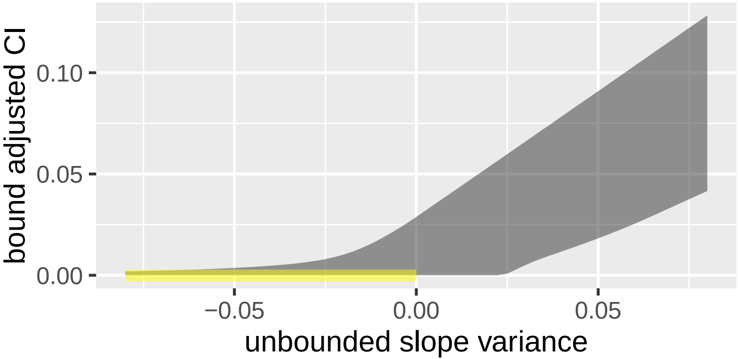
$$d_2 \equiv \sqrt{\max\left[f(\xi) - f(\hat{\xi}_{obs}), 0\right]} \tag{15}$$

Far limit θ_F

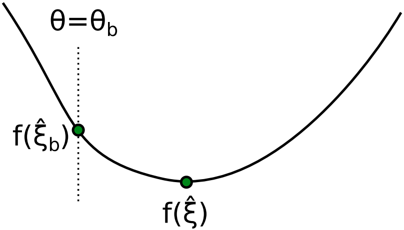


Near limit θ_N

If $|\hat{\theta} - \theta_b| < \epsilon$ then the parameter bound θ_b is irrelevant.



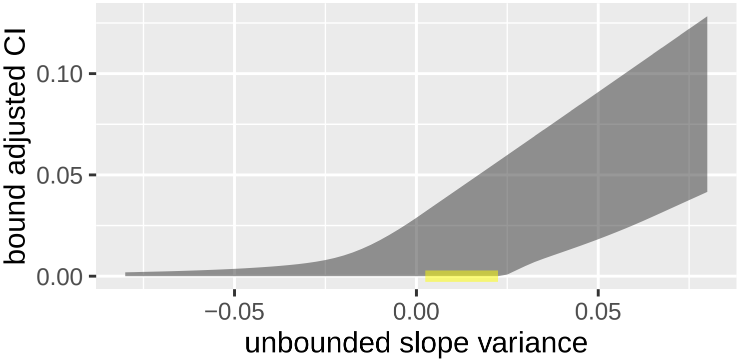
Near limit θ_N



Let $d_0 \equiv \sqrt{f(\hat{\xi}_b) - f(\hat{\xi})}$

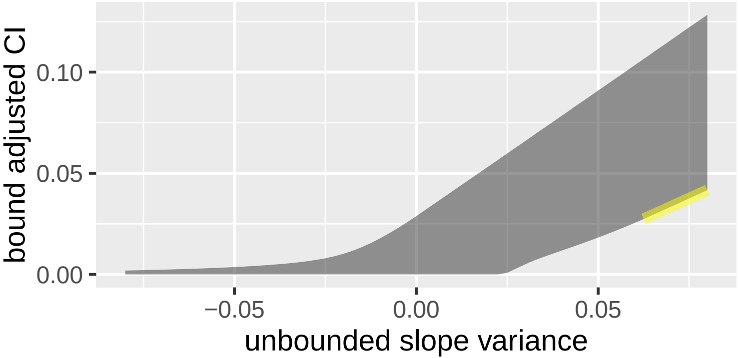
Near limit θ_N

If $d_0 < z_\alpha$ then set θ_N to θ_b.



Near limit θ_N

If $z_{0.5\alpha} \leq \frac{d_0}{2}$ then the attainable bound is irrelevant.



Near limit θ_N

Let $d_L \equiv \max(\frac{d_0}{2}, z_\alpha)$ and $d_U \equiv \min(d_0, z_{0.5\alpha})$.

$$\arg\min_{\pm\theta} f(\xi) \quad \text{subject to:} \tag{16}$$

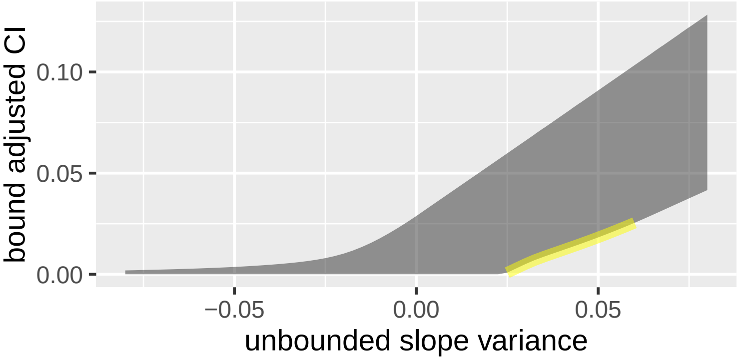
$$d_L < d < d_U \tag{17}$$

$$\log(\alpha) < \log\left(\Phi(d) + \Phi\left[\frac{d_0 - d}{2} + \frac{d^2}{2\max(d_0 - d, 10^{-3}d^2)}\right]\right) \tag{18}$$

where

$$d \equiv \sqrt{\max\left[f(\xi) - f(\hat{\xi}), 0\right]} \tag{19}$$

Near limit θ_N



OpenMx user interface

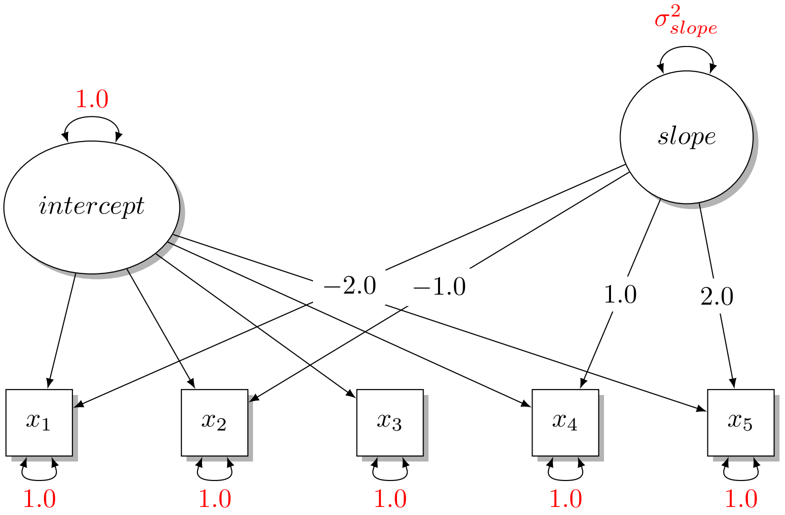
Simply set an attainable bound

```
mxBounds(vars, min=0.0, max=NA)
```

Easy.

Maximum likelihood resembles a rainbow

Latent growth curve model



Conditions

- 3 parameters
- False positive rate $\alpha = .05$
- $\sigma^2_{slope} \in \{0, 0.03, 0.06\}$
- 25k replications per *bound* $\times$ σ^2_{slope} condition

Simulation results

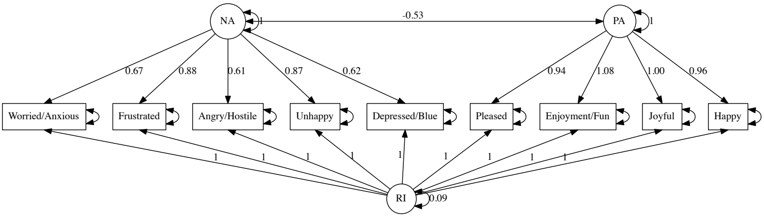
Percentage of replications that σ^2_{slope} was outside of the CI.

	σ^2_{slope}	lower	upper	total
unadjusted	0.0	2.364	2.840	5.204
	0.3	2.248	2.672	4.920
	0.6	2.244	2.696	4.940
adjusted	0.0	4.780	0.000	4.780
	0.3	2.248	2.672	4.920
	0.6	2.244	2.696	4.940

Total should be equal to false positive rate $\alpha = .05$.

Yeah!

Multilevel Confirmatory Factor Analysis



Results

Variable	Lower		Estimate	Upper	
	Estimate	Fit		Estimate	Fit
Unbounded					
σ^2_{RI}	0.07†	32470.3	0.09	0.17	32472.59
cor_{PA-NA}	$-17.23^\dagger$	32550.71	-0.53	-0.34	32472.59
Bounded					
σ^2_{RI}	0.01	32471.45	0.09	0.17	32472.59
cor_{PA-NA}	-0.73	32472.59	-0.53	$-0.34^{\dagger\dagger}$	32472.59

Note. † non-linear constraints were not satisfied during optimization. The optimizer might have converged to a satisfactory solution if given more time. †† At the upper bound of cor_{PA-NA}, σ^2_{RI} was at its lower bound of 0.

OpenMx is a free and open source extension to the R statistical environment.

Software and support available at
`http://openmx.ssri.psu.edu/`

Questions?

Steiger, J. H., Shapiro, A., & Browne, M. W. (1985). On the multivariate asymptotic distribution of sequential chi-square statistics. *Psychometrika*, *50*(3), 253–263.

Wu, H., & Neale, M. C. (2012). Adjusted confidence intervals for a bounded parameter. *Behavior genetics*, *42*(6), 886–898.