Ejercitación Unidad 4

1. $\Omega = \{1C, 1X, 2C, 2X, 3C, 3X, 4C, 4X, 5C, 5X, 6C, 6X\}$
$R_x = \{2, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6\}$

a. Obtenga la función de probabilidad $p(x)$

$$p(x) = \begin{cases} 0 & 7 < x < 0 \\ \dfrac{1}{12} & x = 1 \\ \dfrac{1}{12} & x = 7 \\ \dfrac{2}{12} & 2 \leq x \leq 6 \end{cases}$$

b. Obtenga la función de distribución acumulada $F_X(x)$

$F(x) = \sum_{i=1}^{7} p(x_i) = \dfrac{1}{12} + \dfrac{2}{12} + \dfrac{2}{12} + \dfrac{2}{12} + \dfrac{2}{12} + \dfrac{2}{12} + \dfrac{1}{12} = 1$

c. Halle $P(X > 3)$

$P(X > 3) = P(4) + P(5) + P(6) + P(7) = \boxed{\dfrac{7}{12}}$

d. Halle la probabilidad de que el puntaje obtenido sea un número impar.
C = {x es impar}
C = {1,3,5,7}

$P(x) = P(1) + P(3) + P(5) + P(7) = \boxed{\dfrac{6}{12}}$

e. Halle $E(X), V(X)$ y $E(2X + 3)$

$E(x) = \sum_{i=1}^{\infty} x_i p(x_i) = 1\dfrac{1}{12} + 2\dfrac{2}{12} + 3\dfrac{2}{12} + 4\dfrac{2}{12} + 5\dfrac{2}{12} + 6\dfrac{2}{12} + 7\dfrac{1}{12} = \boxed{4}$

$$\begin{aligned} V(x) &= E[X - E(X)]^2 \\ &= E(X^2) - [E(X)]^2 \\ &= 1\frac{1}{12} + 4\frac{2}{12} + 9\frac{2}{12} + 16\frac{2}{12} + 25\frac{2}{12} + 36\frac{2}{12} + 49\frac{1}{12} - 4^2 \\ &= \boxed{3{,}1667} \end{aligned}$$

$E(2X + 3) = 2E(X) + 3 = \boxed{11}$

2. **a.** Determinar el valor de la constante k

$\int_0^2 \int_0^2 k(x + y)dxdy = 1$

$$\begin{aligned}\int_0^2 k(x+y)dx &= \int_0^2 kxdx + \int_0^2 kydx\\ &= \tfrac{kx^2}{2}\Big|_0^2 + kxy\Big|_0^2\\ &= \frac{4k}{2} + 2ky\\ &= 2k(y+1)\end{aligned}$$

$$\begin{aligned}\int_0^2 2k(y+1)dy &= \int_0^2 2kydy + \int_0^2 2kdy\\ &= ky^2\Big|_0^2 + 2ky\Big|_0^2\\ &= 4k + 4k\\ &= 8k\end{aligned}$$

$$\int_0^2\int_0^2 k(x+y)dxdy = 8k = 1$$

$$\boxed{k = \frac{1}{8}}$$

b. Obtener las funciones de densidad marginales.

$$\begin{aligned}f_x(X) &= \int_0^2 \frac{1}{8}(x+y)dy\\ &= \int_0^2 \frac{1}{8}xdy + \int_0^2 \frac{1}{8}ydy\\ &= \frac{1}{8}xy\Big|_0^2 + \frac{1}{16}y^2\Big|_0^2 =\\ &= \frac{1}{4}x + \frac{1}{4}\\ &= \boxed{\frac{1}{4}(x+1)}\end{aligned}$$

$$\begin{aligned}
f_y(Y) &= \int_0^2 \frac{1}{8}(x+y)dx \\
&= \int_0^2 \frac{1}{8}xdx + \int_0^2 \frac{1}{8}ydx \\
&= \frac{1}{16}x^2\Big|_0^2 + \frac{1}{8}yx^2\Big|_0^2 \\
&= \frac{1}{4} + \frac{1}{4}y \\
&= \boxed{\frac{1}{4}(y+1)}
\end{aligned}$$

c. Calcular la $P(X < 1, Y \leq 1{,}5)$, $P(X \leq 1)$, $P(Y \leq 1{,}5 | X \leq 1)$

$$\begin{aligned}
F_x(x,y) &= \int_0^x \int_0^y f(x,y)dydx \\
&= \int_0^x \int_0^y \frac{1}{8}(x+y)dydx
\end{aligned}$$

$$\begin{aligned}
\int_0^y \frac{1}{8}xdy + \int_0^y \frac{1}{8}ydy &= \frac{1}{8}xy\Big|_0^y + \frac{1}{16}y^2\Big|_0^y \\
&= \frac{1}{8}xy + \frac{1}{16}y^2 \\
&= \frac{1}{16}y(2x+y)
\end{aligned}$$

$$\begin{aligned}
\int_0^x \frac{1}{16}y(2x+y)dx &= \int_0^x \frac{1}{8}yxdx + \int_0^x \frac{1}{16}y^2dx \\
&= \frac{1}{16}yx^2\Big|_0^x + \frac{1}{16}y^2\Big|_0^x \\
&= \frac{1}{16}xy(x+y)
\end{aligned}$$

$$\begin{aligned}
P(X < 1, Y \leq 1{,}5) &= \frac{1}{16}xy(x+y) \\
&= \boxed{0{,}234375}
\end{aligned}$$

$$\begin{aligned}
P(X \leq 1) &= \frac{1}{4}(x+1) \\
&= \boxed{0{,}5}
\end{aligned}$$

$$\begin{aligned}
P(Y \leq 1{,}5 | X \leq 1) &= \frac{f(x,y)}{f_x(X)} \\
&= \frac{\frac{1}{16} yx(x+y)}{\frac{1}{4}(x+1)} \\
&= \boxed{0{,}46835}
\end{aligned}$$

d. ¿Son independientes X e Y?

$$F_x(x,y) = \frac{1}{16} xy(x+y)$$

$$\begin{aligned}
F_x(x) \cdot F_y(y) &= \frac{1}{4}(x+1) \cdot \frac{1}{4}(y+1) \\
&= \frac{1}{16}(xy + x + y + 1)
\end{aligned}$$

$$\boxed{\begin{array}{l} F_x(x,y) \neq F_x(x) \cdot F_y(y) \\ \text{X e Y no son independientes} \end{array}}$$

3. $P(Y > X) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

$\int_0^y \int_0^y \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y} dy dx$

$$\int_0^y \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y} dy = \lambda_1 (e^{\lambda_2 y} - 1) e^{-\lambda_1 x - \lambda_2 y}$$

$$\int_0^y \lambda_1 (e^{\lambda_2 y} - 1) e^{-\lambda_1 x - \lambda_2 y} dx = (e^{\lambda_1 x} - 1)(e^{\lambda_2 y} - 1) e^{-\lambda_1 x - \lambda_2 y}$$