Hubbard dimer : exact solutions

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1 Introduction

Conventions :

$$|\uparrow\downarrow \quad \uparrow\downarrow\rangle = c^\dagger_{2\downarrow}c^\dagger_{2\uparrow}c^\dagger_{1\downarrow}c^\dagger_{2\uparrow}|0 \quad 0\rangle$$

$$c_i^\dagger \left|\ldots, n_i, \ldots\right\rangle_a = (1-n_i)\cdot(-1)^{\sum_{j>i} n_j} \left|\ldots, n_i+1, \ldots\right\rangle_a$$

$$c_i \left|\ldots, n_i, \ldots\right\rangle_a = n_i \cdot (-1)^{\sum_{j>i} n_j} \left|\ldots, n_i-1, \ldots\right\rangle_a$$

$$\hat{n}_{i\sigma} = c^\dagger_{i\sigma}c_{i\sigma}$$

$$\hat{n}_{i\sigma}\left|\ldots, n_{i\sigma}, \ldots\right\rangle_a = n_{i\sigma}\left|\ldots, n_{i\sigma}, \ldots\right\rangle_a$$

Hamiltonian :

$$H = -t \sum_{\substack{i,j=1,2 \\ i\neq j}} \sum_{\sigma} c^\dagger_{i\sigma}c_{j\sigma} + \frac{U}{2}\sum_{i=1,2}\sum_{\sigma\sigma'} c^\dagger_{i\sigma}c^\dagger_{i\sigma'}c_{i\sigma'}c_{i\sigma} + \epsilon_0 \sum_{\sigma,i=1,2} n_{i\sigma}$$

Useful relations :

$$\left\{c_{i\sigma}, c^\dagger_{j\sigma'}\right\} = \delta_{\sigma\sigma'}\delta_{ij} \quad \{c_{i\sigma}, c_{j\sigma'}\} = 0 \quad \left\{c^\dagger_{i\sigma}, c^\dagger_{j\sigma'}\right\} = 0$$

$$< \alpha|c_i|\beta > = < \beta|c_i^\dagger|\alpha >$$

2 General case : $U_1 \neq U_2$ and $\epsilon_1 \neq \epsilon_2$

2.1 Hamiltonian solutions

2.1.1 one electron solution

$$H = \begin{pmatrix} \epsilon_1 & 0 & -t & 0 \\ 0 & \epsilon_1 & 0 & -t \\ -t & 0 & \epsilon_2 & 0 \\ 0 & -t & 0 & \epsilon_2 \end{pmatrix}$$

E_i	$\lvert\uparrow 0\rangle$	$\lvert\downarrow 0\rangle$	$\lvert 0\uparrow\rangle$	$\lvert 0\downarrow\rangle$
$(\epsilon_1+\epsilon_2)/2-d$	0	$-(\epsilon_1-\epsilon_2-2d)/e$	0	$2t/e$
$(\epsilon_1+\epsilon_2)/2-d$	$-(\epsilon_1-\epsilon_2-2d)/e$	0	$2t/e$	0
$(\epsilon_1+\epsilon_2)/2+d$	0	$-(\epsilon_1-\epsilon_2+2d)/f$	0	$2t/f$
$(\epsilon_1+\epsilon_2)/2+d$	$-(\epsilon_1-\epsilon_2+2d)/f$	0	$2t/f$	0

with $d = \sqrt{(\epsilon_1-\epsilon_2)^2+4t^2}/2$, $e = \sqrt{(\epsilon_1-\epsilon_2-2d)^2+4t^2}$, and $f = \sqrt{(\epsilon_1-\epsilon_2+2d)^2+4t^2}$.

2.1.2 two electrons solution

$$H = \begin{pmatrix} \epsilon_1+\epsilon_2 & 0 & 0 & 0 & -t & -t \\ 0 & \epsilon_1+\epsilon_2 & 0 & 0 & t & t \\ 0 & 0 & \epsilon_1+\epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1+\epsilon_2 & 0 & 0 \\ -t & t & 0 & 0 & 2\epsilon_1+U_1 & 0 \\ -t & t & 0 & 0 & 0 & 2\epsilon_2+U_2 \end{pmatrix}$$

$$\begin{aligned}
A &= (18t^2+3U_1U_2-9(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_2+U_1-U_2))(U_1+U_2)-2(U_1^3+2U_2^3) \\
B &= -3(\epsilon_1-\epsilon_2)^2-3(U_1-U_2)(\epsilon_1-\epsilon_2)-U_1^2-U_2^2+U_1U_2-12t^2 \\
D &= \left(A+\sqrt{4B^3+A^2}\right)^{\frac{1}{3}} \\
K_0 &= \tfrac{U_1+U_2}{3}+\tfrac{2^{\frac{1}{3}}B}{3D}-\tfrac{D}{3\times 2^{\frac{1}{3}}} \\
K_+ &= \tfrac{U_1+U_2}{3}-\tfrac{(1+i\sqrt{3})B}{3\times 2^{\frac{2}{3}}D}+\tfrac{(1-i\sqrt{3})D}{6\times 2^{\frac{1}{3}}} \\
K_- &= \tfrac{U_1+U_2}{3}-\tfrac{(1-i\sqrt{3})B}{3\times 2^{\frac{2}{3}}D}+\tfrac{(1+i\sqrt{3})D}{6\times 2^{\frac{1}{3}}}
\end{aligned}$$

Despite K_i have imaginary parts, especially with D which is an imaginary number, all these quantites are real when evaluated numerically !

E_i	$\vert\uparrow\ \ \downarrow\rangle$	$\vert\downarrow\ \ \uparrow\rangle$	$\vert\uparrow\ \ \uparrow\rangle$	$\vert\downarrow\ \ \downarrow\rangle$	$\vert\uparrow\downarrow 0\rangle$	$\vert 0\uparrow\downarrow\rangle$
$\epsilon_1+\epsilon_2+K_0$	$-\frac{1}{C_0}\frac{t(U_1+U_2-2K_0)}{K_0(\epsilon_1-\epsilon_2+U_1-K_0)}$	$\frac{1}{C_0}\frac{t(U_1+U_2-2K_0)}{K_0(\epsilon_1-\epsilon_2+U_1-K_0)}$	0	0	$\frac{1}{C_0}\frac{\epsilon_2-\epsilon_1+U_2-K_0}{\epsilon_1-\epsilon_2+U_1-K_0}$	$\frac{1}{C_0}$
$\epsilon_1+\epsilon_2+K_+$	$-\frac{1}{C_+}\frac{t(U_1+U_2-2K_+)}{K_+(\epsilon_1-\epsilon_2+U_1-K_+)}$	$\frac{1}{C_+}\frac{t(U_1+U_2-2K_+)}{K_+(\epsilon_1-\epsilon_2+U_1-K_+)}$	0	0	$\frac{1}{C_+}\frac{\epsilon_2-\epsilon_1+U_2-K_+}{\epsilon_1-\epsilon_2+U_1-K_+}$	$\frac{1}{C_+}$
$\epsilon_1+\epsilon_2+K_-$	$-\frac{1}{C_-}\frac{t(U_1+U_2-2K_-)}{K_-(\epsilon_1-\epsilon_2+U_1-K_-)}$	$\frac{1}{C_-}\frac{t(U_1+U_2-2K_-)}{K_-(\epsilon_1-\epsilon_2+U_1-K_-)}$	0	0	$\frac{1}{C_-}\frac{\epsilon_2-\epsilon_1+U_2-K_-}{\epsilon_1-\epsilon_2+U_1-K_-}$	$\frac{1}{C_-}$
$\epsilon_1+\epsilon_2$	0	0	0	1	0	0
$\epsilon_1+\epsilon_2$	0	0	1	0	0	0
$\epsilon_1+\epsilon_2$	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0	0	0

By normalizing the vectors :

$$C_i = \sqrt{\left|-\frac{t(U_1+U_2-2K_i)}{K_i(\epsilon_1-\epsilon_2+U_1-K_i)}\right|^2 + \left|\frac{t(U_1+U_2-2K_i)}{K_i(\epsilon_1-\epsilon_2+U_1-K_i)}\right|^2 + \left|\frac{\epsilon_2-\epsilon_1+U_2-K_i}{\epsilon_1-\epsilon_2+U_1-K_i}\right|^2 + 1}$$

Numerically, we have these solution converging to the basic case solution !

2.1.3 three electrons solution

$$H = \begin{pmatrix} \epsilon_1+2\epsilon_2+U_2 & 0 & -t & 0 \\ 0 & \epsilon_1+2\epsilon_2+U_2 & 0 & -t \\ -t & 0 & 2\epsilon_1+\epsilon_2+U_1 & 0 \\ 0 & -t & 0 & 2\epsilon_1+\epsilon_2+U_1 \end{pmatrix}$$

E_i	$\vert\uparrow\uparrow\downarrow\rangle$	$\vert\downarrow\uparrow\downarrow\rangle$	$\vert\uparrow\downarrow\uparrow\rangle$	$\vert\uparrow\downarrow\downarrow\rangle$
$\frac{1}{2}(3\epsilon_1+3\epsilon_2+U_1+U_2-g)$	0	$\frac{m_+}{\sqrt{1+m_+^2}}$	0	$\frac{1}{\sqrt{1+m_+^2}}$
$\frac{1}{2}(3\epsilon_1+3\epsilon_2+U_1+U_2-g)$	$\frac{m_+}{\sqrt{1+m_+^2}}$	0	$\frac{1}{\sqrt{1+m_+^2}}$	0
$\frac{1}{2}(3\epsilon_1+3\epsilon_2+U_1+U_2+g)$	0	$\frac{m_-}{\sqrt{1+m_-^2}}$	0	$\frac{1}{\sqrt{1+m_-^2}}$
$\frac{1}{2}(3\epsilon_1+3\epsilon_2+U_1+U_2+g)$	$\frac{m_-}{\sqrt{1+m_-^2}}$	0	$\frac{1}{\sqrt{1+m_-^2}}$	0

with $g = \sqrt{((\epsilon_1-\epsilon_2)+(U_1-U_2))^2+4t^2}$, $m_+ = \frac{\epsilon_1-\epsilon_2+U_1-U_2+g}{2t}$, $m_- = \frac{\epsilon_1-\epsilon_2+U_1-U_2-g}{2t}$

2.1.4 other solutions

$$H|0\quad 0\rangle = 0|0\quad 0\rangle$$

$$H|\uparrow\downarrow\quad\uparrow\downarrow\rangle = (2\epsilon_1+2\epsilon_2+U_1+U_2)|\uparrow\downarrow\quad\uparrow\downarrow\rangle$$

2.2 One particle Green's function and density matrix

We chosed the spin-up configuration for $N = 1, 3$.

We used $\beta_{i,j}$ as coefficients for the eigenvector table for $N = 1$, we did the same with $\alpha_{i,j}$ and $\gamma_{i,j}$ for $N = 2$ and $N = 3$.

2.2.1 $N = 1$ electron

$$\begin{aligned}
G_{11\uparrow}(\omega) &= \frac{(\alpha_{0,1}\beta_{1,4} + \alpha_{0,3}\beta_{1,2})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,1}\beta_{1,4} + \alpha_{1,3}\beta_{1,2})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,1}\beta_{1,4} + \alpha_{2,3}\beta_{1,2})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\
&+ \frac{\beta_{2,3}^2}{\omega - (E_{\psi_4} - E_0) + i\eta} + \frac{\beta_{1,4}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,1}^2}{\omega - E_0 - i\eta}
\end{aligned}$$

$$\begin{aligned}
G_{22\uparrow}(\omega) &= \frac{(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,2}\beta_{1,2} - \alpha_{1,4}\beta_{1,4})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,2}\beta_{1,2} - \alpha_{2,4}\beta_{1,4})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\
&+ \frac{\beta_{2,1}^2}{\omega - (E_{\psi_4} - E_0) + i\eta} + \frac{\beta_{1,2}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,3}^2}{\omega - E_0 - i\eta}
\end{aligned}$$

$$\begin{aligned}
G_{12\uparrow}(\omega) = &- \frac{(\alpha_{0,1}\beta_{1,4} + \alpha_{0,3}\beta_{1,2})(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})}{\omega - (E_{\psi_0} - E_0) + i\eta} - \frac{(\alpha_{1,1}\beta_{1,4} + \alpha_{1,3}\beta_{1,2})(\alpha_{1,2}\beta_{1,2} - \alpha_{1,4}\beta_{1,4})}{\omega - (E_{\psi_1} - E_0) + i\eta} \\
&- \frac{(\alpha_{2,1}\beta_{1,4} + \alpha_{2,3}\beta_{1,2})(\alpha_{2,2}\beta_{1,2} - \alpha_{2,4}\beta_{1,4})}{\omega - (E_{\psi_2} - E_0) + i\eta} - \frac{\beta_{2,3}\beta_{2,1}}{\omega - (E_{\psi_4} - E_0) + i\eta} \\
&- \frac{\beta_{1,4}\beta_{1,2}/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,1}\beta_{2,3}}{\omega - E_0 - i\eta}
\end{aligned}$$

$$\begin{aligned}
G_{11\downarrow}(\omega) &= \frac{(\alpha_{0,2}\beta_{2,3} - \alpha_{0,3}\beta_{2,1})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,2}\beta_{2,3} - \alpha_{1,3}\beta_{2,1})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,2}\beta_{2,3} - \alpha_{2,3}\beta_{2,1})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\
&+ \frac{\beta_{1,4}^2}{\omega - (E_{\psi_3} - E_0) + i\eta} + \frac{\beta_{2,3}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,2}^2}{\omega - E_0 - i\eta}
\end{aligned}$$

$$\begin{aligned}
G_{22\downarrow}(\omega) &= \frac{(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,1}\beta_{2,1} + \alpha_{1,4}\beta_{2,3})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,1}\beta_{2,1} + \alpha_{2,4}\beta_{2,3})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\
&+ \frac{\beta_{1,2}^2}{\omega - (E_{\psi_3} - E_0) + i\eta} + \frac{\beta_{2,1}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,4}^2}{\omega - E_0 - i\eta}
\end{aligned}$$

$$G_{12\downarrow}(\omega) = -\frac{(\alpha_{0,2}\beta_{2,3} - \alpha_{0,3}\beta_{2,1})(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})}{\omega - (E_{\psi_0} - E_0) + i\eta} - \frac{(\alpha_{1,2}\beta_{2,3} - \alpha_{1,3}\beta_{2,1})(\alpha_{1,1}\beta_{2,1} + \alpha_{1,4}\beta_{2,3})}{\omega - (E_{\psi_1} - E_0) + i\eta}$$
$$- \frac{(\alpha_{2,2}\beta_{2,3} - \alpha_{2,3}\beta_{2,1})(\alpha_{2,1}\beta_{2,1} + \alpha_{2,4}\beta_{2,3})}{\omega - (E_{\psi_2} - E_0) + i\eta} - \frac{\beta_{1,4}\beta_{1,2}}{\omega - (E_{\psi_3} - E_0) + i\eta}$$
$$- \frac{\beta_{2,3}\beta_{2,1}/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,2}\beta_{1,4}}{\omega - E_0 - i\eta}$$

2.2.2 $N = 2$ electrons

$$G_{11\uparrow}(\omega) = \frac{(-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})^2}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{2,1} + \alpha_{0,2}\gamma_{2,3})^2}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})^2}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{4,1} + \alpha_{0,2}\gamma_{4,3})^2}{\omega - (E_{\phi_4} - E_0) + i\eta}$$
$$G_{22\uparrow}(\omega) = \frac{(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})^2}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(-\alpha_{0,1}\gamma_{2,1} + \alpha_{0,3}\gamma_{2,3})^2}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})^2}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(-\alpha_{0,1}\gamma_{4,1} + \alpha_{0,3}\gamma_{4,3})^2}{\omega - (E_{\phi_4} - E_0) + i\eta}$$
$$G_{12\uparrow}(\omega) = \frac{(-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{2,1} + \alpha_{0,2}\gamma_{2,3})(-\alpha_{0,1}\gamma_{2,1} + \alpha_{0,3}\gamma_{2,3})}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})(\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{4,1} + \alpha_{0,2}\gamma_{4,3})(-\alpha_{0,1}\gamma_{4,1} + \alpha_{0,3}\gamma_{4,3})}{\omega - (E_{\phi_4} - E_0) + i\eta}$$
$$G_{11\downarrow}(\omega) = \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})^2}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})^2}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,4}\gamma_{3,2} - \alpha_{0,1}\gamma_{3,4})^2}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})^2}{\omega + (E_{\chi_4} - E_0) - i\eta}$$
$$G_{22\downarrow}(\omega) = \frac{(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})^2}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})^2}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,2}\gamma_{3,2} + \alpha_{0,3}\gamma_{3,4})^2}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})^2}{\omega + (E_{\chi_4} - E_0) - i\eta}$$
$$G_{12\downarrow}(\omega) = \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,4}\gamma_{3,2} - \alpha_{0,1}\gamma_{3,4})(\alpha_{0,2}\gamma_{3,2} + \alpha_{0,3}\gamma_{3,4})}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})(\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})}{\omega + (E_{\chi_4} - E_0) - i\eta}$$

$$\gamma_{11\uparrow}(\omega) = (-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})^2 + (\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})^2$$

$$\gamma_{22\uparrow}(\omega) = (\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})^2 + (\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})^2$$

$$\gamma_{12\uparrow}(\omega) = (-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4}) + (\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})(\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})$$

$$\gamma_{11\downarrow}(\omega) = (\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})^2 + (\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})^2$$

$$\gamma_{22\downarrow}(\omega) = (\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})^2 + (\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})^2$$

$$\gamma_{12\downarrow}(\omega) = (\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3}) + (\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})(\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})$$

2.2.3 $N = 3$ electrons

$$\begin{aligned} G_{11\uparrow}(\omega) = &\frac{\gamma_{1,2}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{2,3} + \alpha_{0,4}\gamma_{2,1})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,2}\gamma_{2,3} + \alpha_{1,4}\gamma_{2,1})^2}{\omega + (E_{\psi_1} - E_0) - i\eta} \\ &+ \frac{(\alpha_{2,2}\gamma_{2,3} + \alpha_{2,4}\gamma_{2,1})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{1,4}^2}{\omega + (E_{\psi_3} - E_0) - i\eta} + \frac{\gamma_{2,3}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta} \end{aligned}$$

$$\begin{aligned} G_{22\uparrow}(\omega) = &\frac{\gamma_{1,4}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,3}\gamma_{2,3} - \alpha_{0,1}\gamma_{2,1})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,3}\gamma_{2,3} - \alpha_{1,1}\gamma_{2,1})^2}{\omega + (E_{\psi_1} - E_0) - i\eta} \\ &+ \frac{(\alpha_{2,3}\gamma_{2,3} - \alpha_{2,1}\gamma_{2,1})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{1,2}^2}{\omega + (E_{\psi_3} - E_0) - i\eta} + \frac{\gamma_{2,1}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta} \end{aligned}$$

$$\begin{aligned} G_{12\uparrow}(\omega) = &\frac{\gamma_{1,2}\gamma_{1,4}}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{2,3} + \alpha_{0,4}\gamma_{2,1})(\alpha_{0,3}\gamma_{2,3} - \alpha_{0,1}\gamma_{2,1})}{\omega + (E_{\psi_0} - E_0) - i\eta} \\ &+ \frac{(\alpha_{1,2}\gamma_{2,3} + \alpha_{1,4}\gamma_{2,1})(\alpha_{1,3}\gamma_{2,3} - \alpha_{1,1}\gamma_{2,1})}{\omega + (E_{\psi_1} - E_0) - i\eta} + \frac{(\alpha_{2,2}\gamma_{2,3} + \alpha_{2,4}\gamma_{2,1})(\alpha_{2,3}\gamma_{2,3} - \alpha_{2,1}\gamma_{2,1})}{\omega + (E_{\psi_2} - E_0) - i\eta} \\ &- \frac{\gamma_{1,4}\gamma_{1,2}}{\omega + (E_{\psi_3} - E_0) - i\eta} - \frac{\gamma_{2,3}\gamma_{2,1}/2}{\omega + (E_{\psi_5} - E_0) - i\eta} \end{aligned}$$

$$\begin{aligned} G_{11\downarrow}(\omega) = &\frac{\gamma_{2,1}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,4}\gamma_{1,2} - \alpha_{1,1}\gamma_{1,4})^2}{\omega + (E_{\psi_1} - E_0) - i\eta} \\ &+ \frac{(\alpha_{2,4}\gamma_{1,2} - \alpha_{2,1}\gamma_{1,4})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{2,3}^2}{\omega + (E_{\psi_4} - E_0) - i\eta} + \frac{\gamma_{1,4}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta} \end{aligned}$$

$$\begin{aligned} G_{22\downarrow}(\omega) = &\frac{\gamma_{2,3}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,2}\gamma_{1,2} + \alpha_{1,3}\gamma_{1,4})^2}{\omega + (E_{\psi_1} - E_0) - i\eta} \\ &+ \frac{(\alpha_{2,2}\gamma_{1,2} + \alpha_{2,3}\gamma_{1,4})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{2,1}^2}{\omega + (E_{\psi_4} - E_0) - i\eta} + \frac{\gamma_{1,2}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta} \end{aligned}$$

$$
\begin{aligned}
G_{12\downarrow}(\omega) &= \frac{\gamma_{2,1}\gamma_{2,3}}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})}{\omega + (E_{\psi_0} - E_0) - i\eta} \\
&+ \frac{(\alpha_{1,4}\gamma_{1,2} - \alpha_{1,1}\gamma_{1,4})(\alpha_{1,2}\gamma_{1,2} + \alpha_{1,3}\gamma_{1,4})}{\omega + (E_{\psi_1} - E_0) - i\eta} + \frac{(\alpha_{2,4}\gamma_{1,2} - \alpha_{2,1}\gamma_{1,4})(\alpha_{2,2}\gamma_{1,2} + \alpha_{2,3}\gamma_{1,4})}{\omega + (E_{\psi_2} - E_0) - i\eta} \\
&- \frac{\gamma_{2,3}\gamma_{2,1}}{\omega + (E_{\psi_4} - E_0) - i\eta} - \frac{\gamma_{1,4}\gamma_{1,2}/2}{\omega + (E_{\psi_5} - E_0) - i\eta}
\end{aligned}
$$

3 Basic case : $U_1 = U_2 = U$ and $\epsilon_1 = \epsilon_2 = \epsilon_0$

3.1 Hamiltonian solutions

3.1.1 one electron solution

$$H = \begin{pmatrix} \epsilon_0 & 0 & -t & 0 \\ 0 & \epsilon_0 & 0 & -t \\ -t & 0 & \epsilon_0 & 0 \\ 0 & -t & 0 & \epsilon_0 \end{pmatrix}$$

E_i	$\lvert\uparrow 0\rangle$	$\lvert\downarrow 0\rangle$	$\lvert 0\uparrow\rangle$	$\lvert 0\downarrow\rangle$
$\epsilon_0 - t$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
$\epsilon_0 - t$	$1/\sqrt{2}$	0	$1/\sqrt{2}$	0
$\epsilon_0 + t$	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$
$\epsilon_0 + t$	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	0

3.1.2 two electrons solution

$$H = \begin{pmatrix} 2\epsilon_0 & 0 & 0 & 0 & -t & -t \\ 0 & 2\epsilon_0 & 0 & 0 & t & t \\ 0 & 0 & 2\epsilon_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\epsilon_0 & 0 & 0 \\ -t & t & 0 & 0 & 2\epsilon_0 + U & 0 \\ -t & t & 0 & 0 & 0 & 2\epsilon_0 + U \end{pmatrix}$$

E_i	$\lvert\uparrow\downarrow\rangle$	$\lvert\downarrow\uparrow\rangle$	$\lvert\uparrow\uparrow\rangle$	$\lvert\downarrow\downarrow\rangle$	$\lvert\uparrow\downarrow 0\rangle$	$\lvert 0\uparrow\downarrow\rangle$
$2\epsilon_0 + (U-c)/2$	$\frac{4t}{a(c-U)}$	$-\frac{4t}{a(c-U)}$	0	0	1/a	1/a
$2\epsilon_0 + (U+c)/2$	$-\frac{4t}{b(c+U)}$	$\frac{4t}{b(c+U)}$	0	0	1/b	1/b
$2\epsilon_0 + U$	0	0	0	0	$-1/\sqrt{2}$	$1/\sqrt{2}$
$2\epsilon_0$	0	0	0	1	0	0
$2\epsilon_0$	0	0	1	0	0	0
$2\epsilon_0$	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0	0	0

With $a = \frac{\sqrt{2}}{(c-U)}\sqrt{16t^2 + (c-U)^2}$, $b = \frac{\sqrt{2}}{(c+U)}\sqrt{16t^2 + (c+U)^2}$, $c = \sqrt{16t^2 + U^2}$.

3.1.3 three electrons solution

$$H = \begin{pmatrix} 3\epsilon_0 + \mathrm{U} & 0 & -t & 0 \\ 0 & 3\epsilon_0 + \mathrm{U} & 0 & -t \\ -t & 0 & 3\epsilon_0 + \mathrm{U} & 0 \\ 0 & -t & 0 & 3\epsilon_0 + \mathrm{U} \end{pmatrix}$$

E_i	$\lvert\uparrow\uparrow\downarrow\rangle$	$\lvert\downarrow\uparrow\downarrow\rangle$	$\lvert\uparrow\downarrow\uparrow\rangle$	$\lvert\uparrow\downarrow\downarrow\rangle$
$3\epsilon_0 + \mathrm{U} - \mathrm{t}$	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
$3\epsilon_0 + \mathrm{U} - \mathrm{t}$	$1/\sqrt{2}$	0	$1/\sqrt{2}$	0
$3\epsilon_0 + \mathrm{U} + \mathrm{t}$	0	$-1/\sqrt{2}$	0	$1/\sqrt{2}$
$3\epsilon_0 + \mathrm{U} + \mathrm{t}$	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	0

3.2 One particle Green's function and density matrix

3.2.1 $N = 1$ electron

$$\begin{aligned} G_{ij\uparrow}(\omega) &= \frac{(-1)^{(i-j)}}{2}\left[\frac{1}{\omega - (\epsilon_0 + t) + i\eta} + \frac{(-1)^{(i-j)}}{\omega - (\epsilon_0 - t) - i\eta}\right] \\ G_{ij\downarrow}(\omega) &= \frac{(-1)^{(i-j)}}{4}\left[\frac{1}{\omega - (\epsilon_0 + t) + i\eta} + \frac{1}{\omega - (\epsilon_0 + t + U) + i\eta}\right] \\ &+ \frac{1}{2}\left[\frac{\frac{1}{a^2}\left(1 + \frac{4t}{(c-U)}\right)^2}{\omega - (\epsilon_0 + t - (c-U)/2) + i\eta} + \frac{\frac{1}{b^2}\left(1 - \frac{4t}{(c+U)}\right)^2}{\omega - (\epsilon_0 + t + (c+U)/2) + i\eta}\right] \end{aligned}$$

3.2.2 $N = 2$ electrons

$$\begin{aligned} G_d(\omega) &= \frac{1}{2a^2}\left[\frac{\left(1 + \frac{4t}{(c-U)}\right)^2}{\omega - (\epsilon_0 - t + (c+U)/2) + i\eta} + \frac{\left(1 - \frac{4t}{(c-U)}\right)^2}{\omega - (\epsilon_0 + t + (c+U)/2) + i\eta}\right] \\ &+ \frac{1}{2a^2}\left[\frac{\left(1 + \frac{4t}{(c-U)}\right)^2}{\omega - (\epsilon_0 + t - (c-U)/2) - i\eta} + \frac{\left(1 - \frac{4t}{(c-U)}\right)^2}{\omega - (\epsilon_0 - t - (c-U)/2) - i\eta}\right] \end{aligned}$$

$$G_a(\omega) = -\frac{1}{2a^2}\left[\frac{\left(1+\frac{4t}{(c-U)}\right)^2}{\omega-(\epsilon_0-t+(c+U)/2)+i\eta} - \frac{\left(1-\frac{4t}{(c-U)}\right)^2}{\omega-(\epsilon_0+t+(c+U)/2)+i\eta}\right]$$

$$+\frac{1}{2a^2}\left[\frac{\left(1+\frac{4t}{(c-U)}\right)^2}{\omega-(\epsilon_0+t-(c-U)/2)-i\eta} - \frac{\left(1-\frac{4t}{(c-U)}\right)^2}{\omega-(\epsilon_0-t-(c-U)/2)-i\eta}\right]$$

$$G^{N=2}(\omega) = \begin{pmatrix} G_{11\uparrow} & G_{12\uparrow} & 0 & 0 \\ G_{21\uparrow} & G_{22\uparrow} & 0 & 0 \\ 0 & 0 & G_{11\downarrow} & G_{12\downarrow} \\ 0 & 0 & G_{21\downarrow} & G_{22\downarrow} \end{pmatrix} = \begin{pmatrix} G_d & G_a & 0 & 0 \\ G_a & G_d & 0 & 0 \\ 0 & 0 & G_d & G_a \\ 0 & 0 & G_a & G_d \end{pmatrix}$$

$$\gamma_d(\omega) = \frac{1}{2a^2}\left(1+\frac{4t}{(c-U)}\right)^2 + \frac{1}{2a^2}\left(1-\frac{4t}{(c-U)}\right)^2$$

$$\gamma_a(\omega) = \frac{1}{2a^2}\left(1+\frac{4t}{(c-U)}\right)^2 - \frac{1}{2a^2}\left(1-\frac{4t}{(c-U)}\right)^2$$

3.2.3 $N = 3$ electrons

$$G_{ij\uparrow}(\omega) = \frac{(-1)^{(i-j)}}{4}\left[\frac{1}{\omega-(\epsilon_0-t)+i\eta} + \frac{1}{\omega-(\epsilon_0+U-t)+i\eta}\right]$$

$$+\frac{1}{2}\left[\frac{\frac{1}{a^2}\left(1-\frac{4t}{(c-U)}\right)^2}{\omega-(\epsilon_0+(U+c)/2-t)+i\eta} + \frac{\frac{1}{b^2}\left(1+\frac{4t}{(c+U)}\right)^2}{\omega-(\epsilon_0+(U-c)/2-t)+i\eta}\right]$$

$$G_{ij\downarrow}(\omega) = \frac{(-1)^{(i-j)}}{2}\left[\frac{1}{\omega-(\epsilon_0+U-t)-i\eta} + \frac{(-1)^{(i-j)}}{\omega-(\epsilon_0+U+t)-i\eta}\right]$$

4 Annexes

4.1 General case : calculus on eigenvectors

4.1.1 $N = 1$ electron : $|\chi_i\rangle$

For $i = 1, 3$: $|\chi_i\rangle = \beta_{i,2}|\downarrow \quad 0\rangle + \beta_{i,4}|0 \quad \downarrow\rangle$:

$$\begin{aligned}
c^\dagger_{1\uparrow}|\chi_i\rangle &= -\beta_{i,2}|\uparrow\downarrow \quad 0\rangle - \beta_{i,4}|\uparrow \quad \downarrow\rangle \\
c_{1\uparrow}|\chi_i\rangle &= 0 \\
c^\dagger_{2\uparrow}|\chi_i\rangle &= \beta_{i,2}|\downarrow \quad \uparrow\rangle - \beta_{i,4}|0 \quad \uparrow\downarrow\rangle \\
c_{2\uparrow}|\chi_i\rangle &= 0 \\
c^\dagger_{1\downarrow}|\chi_i\rangle &= -\beta_{i,4}|\downarrow \quad \downarrow\rangle \\
c_{1\downarrow}|\chi_i\rangle &= \beta_{i,2}|0 \quad 0\rangle \\
c^\dagger_{2\downarrow}|\chi_i\rangle &= \beta_{i,2}|\downarrow \quad \downarrow\rangle \\
c_{2\downarrow}|\chi_i\rangle &= \beta_{i,4}|0 \quad 0\rangle
\end{aligned}$$

For $i = 2, 4$: $|\chi_i\rangle = \beta_{i,1}|\uparrow \quad 0\rangle + \beta_{i,3}|0 \quad \uparrow\rangle$:

$$\begin{aligned}
c^\dagger_{1\uparrow}|\chi_i\rangle &= -\beta_{i,3}|\uparrow \quad \uparrow\rangle \\
c_{1\uparrow}|\chi_i\rangle &= \beta_{i,1}|0 \quad 0\rangle \\
c^\dagger_{2\uparrow}|\chi_i\rangle &= \beta_{i,1}|\uparrow \quad \uparrow\rangle \\
c_{2\uparrow}|\chi_i\rangle &= \beta_{i,3}|0 \quad 0\rangle \\
c^\dagger_{1\downarrow}|\chi_i\rangle &= \beta_{i,1}|\uparrow\downarrow \quad 0\rangle - \beta_{i,3}|\downarrow \quad \uparrow\rangle \\
c_{1\downarrow}|\chi_i\rangle &= 0 \\
c^\dagger_{2\downarrow}|\chi_i\rangle &= \beta_{i,1}|\uparrow \quad \downarrow\rangle + \beta_{i,3}|0 \quad \uparrow\downarrow\rangle \\
c_{2\downarrow}|\chi_i\rangle &= 0
\end{aligned}$$

4.1.2 $N = 2$ electrons : $|\psi_i\rangle$

For $i = 0, 1, 2$, $|\psi_i\rangle = \alpha_{i,1}|\uparrow \quad \downarrow\rangle + \alpha_{i,2}|\downarrow \quad \uparrow\rangle + \alpha_{i,3}|\uparrow\downarrow \quad 0\rangle + \alpha_{i,4}|0 \quad \uparrow\downarrow\rangle$:

$$\begin{aligned}
c^\dagger_{1\uparrow}|\psi_i\rangle &= \alpha_{i,2}|\uparrow\downarrow \quad \uparrow\rangle + \alpha_{i,4}|\uparrow \quad \uparrow\downarrow\rangle \\
c_{1\uparrow}|\psi_i\rangle &= -\alpha_{i,1}|0 \quad \downarrow\rangle - \alpha_{i,3}|\downarrow \quad 0\rangle \\
c^\dagger_{2\uparrow}|\psi_i\rangle &= -\alpha_{i,1}|\uparrow \quad \uparrow\downarrow\rangle + \alpha_{i,3}|\uparrow\downarrow \quad \uparrow\rangle \\
c_{2\uparrow}|\psi_i\rangle &= \alpha_{i,2}|\downarrow \quad 0\rangle - \alpha_{i,4}|0 \quad \downarrow\rangle \\
c^\dagger_{1\downarrow}|\psi_i\rangle &= -\alpha_{i,1}|\uparrow\downarrow \quad \downarrow\rangle + \alpha_{i,4}|\downarrow \quad \uparrow\downarrow\rangle \\
c_{1\downarrow}|\psi_i\rangle &= -\alpha_{i,2}|0 \quad \uparrow\rangle + \alpha_{i,3}|\uparrow \quad 0\rangle \\
c^\dagger_{2\downarrow}|\psi_i\rangle &= \alpha_{i,2}|\downarrow \quad \uparrow\downarrow\rangle + \alpha_{i,3}|\uparrow\downarrow \quad \downarrow\rangle \\
c_{2\downarrow}|\psi_i\rangle &= \alpha_{i,1}|\uparrow \quad 0\rangle + \alpha_{i,4}|0 \quad \uparrow\rangle
\end{aligned}$$

Pour $|\psi_3\rangle = |\downarrow \quad \downarrow\rangle$:

$$
\begin{aligned}
c^\dagger_{1\uparrow}|\psi_3\rangle &= |\uparrow\downarrow \quad \downarrow\rangle\\
c_{1\uparrow}|\psi_3\rangle &= 0\\
c^\dagger_{2\uparrow}|\psi_3\rangle &= -|\downarrow \quad \uparrow\downarrow\rangle\\
c_{2\uparrow}|\psi_3\rangle &= 0\\
c^\dagger_{1\downarrow}|\psi_3\rangle &= 0\\
c_{1\downarrow}|\psi_3\rangle &= -|0 \quad \downarrow\rangle\\
c^\dagger_{2\downarrow}|\psi_3\rangle &= 0\\
c_{2\downarrow}|\psi_3\rangle &= |\downarrow \quad 0\rangle
\end{aligned}
$$

Pour $|\psi_4\rangle = |\uparrow \quad \uparrow\rangle$:

$$
\begin{aligned}
c^\dagger_{1\uparrow}|\psi_4\rangle &= 0\\
c_{1\uparrow}|\psi_4\rangle &= -|0 \quad \uparrow\rangle\\
c^\dagger_{2\uparrow}|\psi_4\rangle &= 0\\
c_{2\uparrow}|\psi_4\rangle &= |\uparrow \quad 0\rangle\\
c^\dagger_{1\downarrow}|\psi_4\rangle &= -|\uparrow\downarrow \quad \uparrow\rangle\\
c_{1\downarrow}|\psi_4\rangle &= 0\\
c^\dagger_{2\downarrow}|\psi_4\rangle &= |\uparrow \quad \uparrow\downarrow\rangle\\
c_{2\downarrow}|\psi_4\rangle &= 0
\end{aligned}
$$

Pour $|\psi_5\rangle = \frac{1}{\sqrt{2}}|\uparrow \quad \downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow \quad \uparrow\rangle$:

$$
\begin{aligned}
c^\dagger_{1\uparrow}|\psi_5\rangle &= \tfrac{1}{\sqrt{2}}|\uparrow\downarrow \quad \uparrow\rangle\\
c_{1\uparrow}|\psi_5\rangle &= -\tfrac{1}{\sqrt{2}}|0 \quad \downarrow\rangle\\
c^\dagger_{2\uparrow}|\psi_5\rangle &= -\tfrac{1}{\sqrt{2}}|\uparrow \quad \uparrow\downarrow\rangle\\
c_{2\uparrow}|\psi_5\rangle &= \tfrac{1}{\sqrt{2}}|\downarrow \quad 0\rangle\\
c^\dagger_{1\downarrow}|\psi_5\rangle &= -\tfrac{1}{\sqrt{2}}|\uparrow\downarrow \quad \downarrow\rangle\\
c_{1\downarrow}|\psi_5\rangle &= -\tfrac{1}{\sqrt{2}}|0 \quad \uparrow\rangle\\
c^\dagger_{2\downarrow}|\psi_5\rangle &= \tfrac{1}{\sqrt{2}}|\downarrow \quad \uparrow\downarrow\rangle\\
c_{2\downarrow}|\psi_5\rangle &= \tfrac{1}{\sqrt{2}}|\uparrow \quad 0\rangle
\end{aligned}
$$

4.1.3 $N = 3$ electrons : $|\phi_i\rangle$

For $i = 1, 3$: $|\phi_i\rangle = \gamma_{i,2}|\downarrow \quad \uparrow\downarrow\rangle + \gamma_{i,4}|\uparrow\downarrow \quad \downarrow\rangle$:

$$
\begin{aligned}
&c^\dagger_{1\uparrow}|\phi_i\rangle = -\gamma_{i,2}|\uparrow\downarrow \quad \uparrow\downarrow\rangle\\
&c_{1\uparrow}|\phi_i\rangle = \gamma_{i,4}|\downarrow \quad \downarrow\rangle\\
&c^\dagger_{2\uparrow}|\phi_i\rangle = -\gamma_{i,4}|\uparrow\downarrow \quad \uparrow\downarrow\rangle\\
&c_{2\uparrow}|\phi_i\rangle = -\gamma_{i,2}|\downarrow \quad \downarrow\rangle\\
&c^\dagger_{1\downarrow}|\phi_i\rangle = 0\\
&c_{1\downarrow}|\phi_i\rangle = \gamma_{i,2}|0 \quad \uparrow\downarrow\rangle - \gamma_{i,4}|\uparrow \quad \downarrow\rangle\\
&c^\dagger_{2\downarrow}|\phi_i\rangle = 0\\
&c_{2\downarrow}|\phi_i\rangle = \gamma_{i,2}|\downarrow \quad \uparrow\rangle + \gamma_{i,4}|\uparrow\downarrow \quad 0\rangle
\end{aligned}
$$

For $i = 2, 4$: $|\phi_i\rangle = \gamma_{i,1}|\uparrow \quad \uparrow\downarrow\rangle + \gamma_{i,3}|\uparrow\downarrow \quad \uparrow\rangle$:

$$
\begin{aligned}
&c^\dagger_{1\uparrow}|\phi_i\rangle = 0\\
&c_{1\uparrow}|\phi_i\rangle = \gamma_{i,1}|0 \quad \uparrow\downarrow\rangle + \gamma_{i,3}|\downarrow \quad \uparrow\rangle\\
&c^\dagger_{2\uparrow}|\phi_i\rangle = 0\\
&c_{2\uparrow}|\phi_i\rangle = -\gamma_{i,1}|\uparrow \quad \downarrow\rangle + \gamma_{i,3}|\uparrow\downarrow \quad 0\rangle\\
&c^\dagger_{1\downarrow}|\phi_i\rangle = \gamma_{i,1}|\uparrow\downarrow \quad \uparrow\downarrow\rangle\\
&c_{1\downarrow}|\phi_i\rangle = -\gamma_{i,3}|\uparrow \quad \uparrow\rangle\\
&c^\dagger_{2\downarrow}|\phi_i\rangle = \gamma_{i,3}|\uparrow\downarrow \quad \uparrow\downarrow\rangle\\
&c_{2\downarrow}|\phi_i\rangle = \gamma_{i,1}|\uparrow \quad \uparrow\rangle
\end{aligned}
$$

4.2 General case : scalar products

4.2.1 With $|\chi_0\rangle = \beta_{1,2}|\downarrow \quad 0\rangle + \beta_{1,4}|0 \quad \downarrow\rangle + \beta_{2,1}|\uparrow \quad 0\rangle + \beta_{2,3}|0 \quad \uparrow\rangle$

$$\left\langle \chi_0 \left| c_{1\uparrow}^\dagger \right| 00 \right\rangle = \left\langle 00 \left| c_{1\uparrow} \right| \chi_0 \right\rangle = \beta_{2,1}$$
$$\left\langle \chi_0 \left| c_{2\uparrow}^\dagger \right| 00 \right\rangle = \left\langle 00 \left| c_{2\uparrow} \right| \chi_0 \right\rangle = \beta_{2,3}$$
$$\left\langle \chi_0 \left| c_{1\downarrow}^\dagger \right| 00 \right\rangle = \left\langle 00 \left| c_{1\downarrow} \right| \chi_0 \right\rangle = \beta_{1,2}$$
$$\left\langle \chi_0 \left| c_{2\downarrow}^\dagger \right| 00 \right\rangle = \left\langle 00 \left| c_{2\downarrow} \right| \chi_0 \right\rangle = \beta_{1,4}$$

$$\left\langle \psi_i \left| c_{1\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\uparrow} \right| \psi_i \right\rangle = -\alpha_{i,1}\beta_{1,4} - \alpha_{i,3}\beta_{1,2}$$
$$\left\langle \psi_i \left| c_{2\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\uparrow} \right| \psi_i \right\rangle = \alpha_{i,2}\beta_{1,2} - \alpha_{i,4}\beta_{1,4}$$
$$\left\langle \psi_i \left| c_{1\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\downarrow} \right| \psi_i \right\rangle = -\alpha_{i,2}\beta_{2,3} + \alpha_{i,3}\beta_{2,1}$$
$$\left\langle \psi_i \left| c_{2\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\downarrow} \right| \psi_i \right\rangle = \alpha_{i,1}\beta_{2,1} + \alpha_{i,4}\beta_{2,3}$$

$$\left\langle \psi_3 \left| c_{1\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\uparrow} \right| \psi_3 \right\rangle = 0$$
$$\left\langle \psi_3 \left| c_{2\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\uparrow} \right| \psi_3 \right\rangle = 0$$
$$\left\langle \psi_3 \left| c_{1\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\downarrow} \right| \psi_3 \right\rangle = -\beta_{1,4}$$
$$\left\langle \psi_3 \left| c_{2\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\downarrow} \right| \psi_3 \right\rangle = \beta_{1,2}$$

$$\left\langle \psi_4 \left| c_{1\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\uparrow} \right| \psi_4 \right\rangle = -\beta_{2,3}$$
$$\left\langle \psi_4 \left| c_{2\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\uparrow} \right| \psi_4 \right\rangle = \beta_{2,1}$$
$$\left\langle \psi_4 \left| c_{1\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\downarrow} \right| \psi_4 \right\rangle = 0$$
$$\left\langle \psi_4 \left| c_{2\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\downarrow} \right| \psi_4 \right\rangle = 0$$

$$\left\langle \psi_5 \left| c_{1\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\uparrow} \right| \psi_5 \right\rangle = -\frac{\beta_{1,4}}{\sqrt{2}}$$
$$\left\langle \psi_5 \left| c_{2\uparrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\uparrow} \right| \psi_5 \right\rangle = \frac{\beta_{1,2}}{\sqrt{2}}$$
$$\left\langle \psi_5 \left| c_{1\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{1\downarrow} \right| \psi_5 \right\rangle = -\frac{\beta_{2,3}}{\sqrt{2}}$$
$$\left\langle \psi_5 \left| c_{2\downarrow}^\dagger \right| \chi_0 \right\rangle = \left\langle \chi_0 \left| c_{2\downarrow} \right| \psi_5 \right\rangle = \frac{\beta_{2,1}}{\sqrt{2}}$$

4.2.2 With $|\psi_0\rangle = \alpha_{0,1}|\uparrow \quad \downarrow\rangle + \alpha_{0,2}|\downarrow \quad \uparrow\rangle + \alpha_{0,3}|\uparrow\downarrow \quad 0\rangle + \alpha_{0,4}|0 \quad \uparrow\downarrow\rangle$

For $i = 1,3 : |\chi_i\rangle = \beta_{i,2}|\downarrow \quad 0\rangle + \beta_{i,4}|0 \quad \downarrow\rangle$:

$$\begin{aligned}
\left\langle \psi_0 \left| c_{1\uparrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{1\uparrow} | \psi_0 \rangle = -\alpha_{0,3}\beta_{i,2} - \alpha_{0,1}\beta_{i,4} \\
\left\langle \psi_0 \left| c_{2\uparrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{2\uparrow} | \psi_0 \rangle = \alpha_{0,2}\beta_{i,2} - \alpha_{0,4}\beta_{i,4} \\
\left\langle \psi_0 \left| c_{1\downarrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{1\downarrow} | \psi_0 \rangle = 0 \\
\left\langle \psi_0 \left| c_{2\downarrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{2\downarrow} | \psi_0 \rangle = 0
\end{aligned}$$

For $i = 2,4 : |\chi_i\rangle = \beta_{i,1}|\uparrow \quad 0\rangle + \beta_{i,3}|0 \quad \uparrow\rangle$:

$$\begin{aligned}
\left\langle \psi_0 \left| c_{1\uparrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{1\uparrow} | \psi_0 \rangle = 0 \\
\left\langle \psi_0 \left| c_{2\uparrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{2\uparrow} | \psi_0 \rangle = 0 \\
\left\langle \psi_0 \left| c_{1\downarrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{1\downarrow} | \psi_0 \rangle = \alpha_{0,3}\beta_{i,1} - \alpha_{0,2}\beta_{i,3} \\
\left\langle \psi_0 \left| c_{2\downarrow}^\dagger \right| \chi_i \right\rangle &= \langle \chi_i | c_{2\downarrow} | \psi_0 \rangle = \alpha_{0,1}\beta_{i,1} + \alpha_{0,4}\beta_{i,3}
\end{aligned}$$

For $i = 1,3 : |\phi_i\rangle = \gamma_{i,2}|\downarrow \quad \uparrow\downarrow\rangle + \gamma_{i,4}|\uparrow\downarrow \quad \downarrow\rangle$:

$$\begin{aligned}
\langle \psi_0 | c_{1\uparrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{1\uparrow}^\dagger \right| \psi_0 \right\rangle = 0 \\
\langle \psi_0 | c_{2\uparrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{2\uparrow}^\dagger \right| \psi_0 \right\rangle = 0 \\
\langle \psi_0 | c_{1\downarrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{1\downarrow}^\dagger \right| \psi_0 \right\rangle = \gamma_{i,2}\alpha_{0,4} - \gamma_{i,4}\alpha_{0,1} \\
\langle \psi_0 | c_{2\downarrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{2\downarrow}^\dagger \right| \psi_0 \right\rangle = \gamma_{i,2}\alpha_{0,2} + \gamma_{i,4}\alpha_{0,3}
\end{aligned}$$

For $i = 2,4 : |\phi_i\rangle = \gamma_{i,1}|\uparrow \quad \uparrow\downarrow\rangle + \gamma_{i,3}|\uparrow\downarrow \quad \uparrow\rangle$:

$$\begin{aligned}
\langle \psi_0 | c_{1\uparrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{1\uparrow}^\dagger \right| \psi_0 \right\rangle = \gamma_{i,1}\alpha_{0,4} + \gamma_{i,3}\alpha_{0,2} \\
\langle \psi_0 | c_{2\uparrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{2\uparrow}^\dagger \right| \psi_0 \right\rangle = -\gamma_{i,1}\alpha_{0,1} + \gamma_{i,3}\alpha_{0,3} \\
\langle \psi_0 | c_{1\downarrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{1\downarrow}^\dagger \right| \psi_0 \right\rangle = 0 \\
\langle \psi_0 | c_{2\downarrow} | \phi_i \rangle &= \left\langle \phi_i \left| c_{2\downarrow}^\dagger \right| \psi_0 \right\rangle = 0
\end{aligned}$$

4.2.3 With $|\phi_0\rangle = \gamma_{2,1}|\uparrow \quad \uparrow\downarrow\rangle + \gamma_{1,2}|\downarrow \quad \uparrow\downarrow\rangle + \gamma_{2,3}|\uparrow\downarrow \quad \uparrow\rangle + \gamma_{1,4}|\uparrow\downarrow \quad \downarrow\rangle$

$$\begin{aligned}
\langle\uparrow\downarrow\uparrow\downarrow \left| c_{1\uparrow}^{\dagger} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\uparrow} \right| \uparrow\downarrow\uparrow\downarrow\rangle = -\gamma_{1,2} \\
\langle\uparrow\downarrow\uparrow\downarrow \left| c_{2\uparrow}^{\dagger} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\uparrow} \right| \uparrow\downarrow\uparrow\downarrow\rangle = -\gamma_{1,4} \\
\langle\uparrow\downarrow\uparrow\downarrow \left| c_{1\downarrow}^{\dagger} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\downarrow} \right| \uparrow\downarrow\uparrow\downarrow\rangle = \gamma_{2,1} \\
\langle\uparrow\downarrow\uparrow\downarrow \left| c_{2\downarrow}^{\dagger} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\downarrow} \right| \uparrow\downarrow\uparrow\downarrow\rangle = \gamma_{2,3}
\end{aligned}$$

$$\begin{aligned}
\langle\psi_i \left| c_{1\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\uparrow}^{\dagger} \right| \psi_i\rangle = \alpha_{i,2}\gamma_{2,3} + \alpha_{i,4}\gamma_{2,1} \\
\langle\psi_i \left| c_{2\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\uparrow}^{\dagger} \right| \psi_i\rangle = -\alpha_{i,1}\gamma_{2,1} + \alpha_{i,3}\gamma_{2,3} \\
\langle\psi_i \left| c_{1\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\downarrow}^{\dagger} \right| \psi_i\rangle = -\alpha_{i,1}\gamma_{1,4} + \alpha_{i,4}\gamma_{1,2} \\
\langle\psi_i \left| c_{2\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\downarrow}^{\dagger} \right| \psi_i\rangle = \alpha_{i,2}\gamma_{1,2} + \alpha_{i,3}\gamma_{1,4}
\end{aligned}$$

$$\begin{aligned}
\langle\psi_3 \left| c_{1\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\uparrow}^{\dagger} \right| \psi_3\rangle = \gamma_{1,4} \\
\langle\psi_3 \left| c_{2\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\uparrow}^{\dagger} \right| \psi_3\rangle = -\gamma_{1,2} \\
\langle\psi_3 \left| c_{1\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\downarrow}^{\dagger} \right| \psi_3\rangle = 0 \\
\langle\psi_3 \left| c_{2\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\downarrow}^{\dagger} \right| \psi_3\rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle\psi_4 \left| c_{1\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\uparrow}^{\dagger} \right| \psi_4\rangle = 0 \\
\langle\psi_4 \left| c_{2\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\uparrow}^{\dagger} \right| \psi_4\rangle = 0 \\
\langle\psi_4 \left| c_{1\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\downarrow}^{\dagger} \right| \psi_4\rangle = -\gamma_{2,3} \\
\langle\psi_4 \left| c_{2\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\downarrow}^{\dagger} \right| \psi_4\rangle = \gamma_{2,1}
\end{aligned}$$

$$\begin{aligned}
\langle\psi_5 \left| c_{1\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\uparrow}^{\dagger} \right| \psi_5\rangle = \tfrac{\gamma_{2,3}}{\sqrt{2}} \\
\langle\psi_5 \left| c_{2\uparrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\uparrow}^{\dagger} \right| \psi_5\rangle = -\tfrac{\gamma_{2,1}}{\sqrt{2}} \\
\langle\psi_5 \left| c_{1\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{1\downarrow}^{\dagger} \right| \psi_5\rangle = -\tfrac{\gamma_{1,4}}{\sqrt{2}} \\
\langle\psi_5 \left| c_{2\downarrow} \right| \phi_0\rangle &= \langle\phi_0 \left| c_{2\downarrow}^{\dagger} \right| \psi_5\rangle = \tfrac{\gamma_{1,2}}{\sqrt{2}}
\end{aligned}$$

4.3 General case : one particle Green's function

By symmetry, we have $G_{ij\sigma} = G_{ji\sigma}$ for all i, j, σ

4.3.1 $N = 1$ electron

$$G_{ij\sigma}(\omega) = \sum_{\psi} \frac{\left\langle \psi \left| c^{\dagger}_{i\sigma} \right| \chi_0 \right\rangle \langle \chi_0 \left| c_{j\sigma} \right| \psi \rangle}{\omega - (E_\psi - E_0) + i\eta} + \frac{\langle 0 \quad 0 \left| c_{i\sigma} \right| \chi_0 \rangle \left\langle \chi_0 \left| c^{\dagger}_{j\sigma} \right| 0 \quad 0 \right\rangle}{\omega - E_0 - i\eta}$$

With $E_0 = (\epsilon_1 + \epsilon_2)/2 - d$

$$\begin{aligned} G_{11\uparrow}(\omega) = & \frac{(\alpha_{0,1}\beta_{1,4} + \alpha_{0,3}\beta_{1,2})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,1}\beta_{1,4} + \alpha_{1,3}\beta_{1,2})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,1}\beta_{1,4} + \alpha_{2,3}\beta_{1,2})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\ & + \frac{\beta_{2,3}^2}{\omega - (E_{\psi_4} - E_0) + i\eta} + \frac{\beta_{1,4}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,1}^2}{\omega - E_0 - i\eta} \end{aligned}$$

$$\begin{aligned} G_{22\uparrow}(\omega) = & \frac{(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,2}\beta_{1,2} - \alpha_{1,4}\beta_{1,4})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,2}\beta_{1,2} - \alpha_{2,4}\beta_{1,4})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\ & + \frac{\beta_{2,1}^2}{\omega - (E_{\psi_4} - E_0) + i\eta} + \frac{\beta_{1,2}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,3}^2}{\omega - E_0 - i\eta} \end{aligned}$$

$$\begin{aligned} G_{12\uparrow}(\omega) = & -\frac{(\alpha_{0,1}\beta_{1,4} + \alpha_{0,3}\beta_{1,2})(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})}{\omega - (E_{\psi_0} - E_0) + i\eta} - \frac{(\alpha_{1,1}\beta_{1,4} + \alpha_{1,3}\beta_{1,2})(\alpha_{1,2}\beta_{1,2} - \alpha_{1,4}\beta_{1,4})}{\omega - (E_{\psi_1} - E_0) + i\eta} \\ & - \frac{(\alpha_{2,1}\beta_{1,4} + \alpha_{2,3}\beta_{1,2})(\alpha_{2,2}\beta_{1,2} - \alpha_{2,4}\beta_{1,4})}{\omega - (E_{\psi_2} - E_0) + i\eta} - \frac{\beta_{2,3}\beta_{2,1}}{\omega - (E_{\psi_4} - E_0) + i\eta} \\ & - \frac{\beta_{1,4}\beta_{1,2}/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{2,1}\beta_{2,3}}{\omega - E_0 - i\eta} \end{aligned}$$

$$\begin{aligned} G_{11\downarrow}(\omega) = & \frac{(\alpha_{0,2}\beta_{2,3} - \alpha_{0,3}\beta_{2,1})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,2}\beta_{2,3} - \alpha_{1,3}\beta_{2,1})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,2}\beta_{2,3} - \alpha_{2,3}\beta_{2,1})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\ & + \frac{\beta_{1,4}^2}{\omega - (E_{\psi_3} - E_0) + i\eta} + \frac{\beta_{2,3}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,2}^2}{\omega - E_0 - i\eta} \end{aligned}$$

$$\begin{aligned} G_{22\downarrow}(\omega) = & \frac{(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})^2}{\omega - (E_{\psi_0} - E_0) + i\eta} + \frac{(\alpha_{1,1}\beta_{2,1} + \alpha_{1,4}\beta_{2,3})^2}{\omega - (E_{\psi_1} - E_0) + i\eta} + \frac{(\alpha_{2,1}\beta_{2,1} + \alpha_{2,4}\beta_{2,3})^2}{\omega - (E_{\psi_2} - E_0) + i\eta} \\ & + \frac{\beta_{1,2}^2}{\omega - (E_{\psi_3} - E_0) + i\eta} + \frac{\beta_{2,1}^2/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,4}^2}{\omega - E_0 - i\eta} \end{aligned}$$

$$G_{12\downarrow}(\omega) = -\frac{(\alpha_{0,2}\beta_{2,3} - \alpha_{0,3}\beta_{2,1})(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})}{\omega - (E_{\psi_0} - E_0) + i\eta} - \frac{(\alpha_{1,2}\beta_{2,3} - \alpha_{1,3}\beta_{2,1})(\alpha_{1,1}\beta_{2,1} + \alpha_{1,4}\beta_{2,3})}{\omega - (E_{\psi_1} - E_0) + i\eta}$$
$$- \frac{(\alpha_{2,2}\beta_{2,3} - \alpha_{2,3}\beta_{2,1})(\alpha_{2,1}\beta_{2,1} + \alpha_{2,4}\beta_{2,3})}{\omega - (E_{\psi_2} - E_0) + i\eta} - \frac{\beta_{1,4}\beta_{1,2}}{\omega - (E_{\psi_3} - E_0) + i\eta}$$
$$- \frac{\beta_{2,3}\beta_{2,1}/2}{\omega - (E_{\psi_5} - E_0) - i\eta} + \frac{\beta_{1,2}\beta_{1,4}}{\omega - E_0 - i\eta}$$

4.3.2 $N = 2$ electrons

$$G_{ij\sigma}(\omega) = \sum_{\phi} \frac{\left\langle \phi \left| c_{i\sigma}^{\dagger} \right| \psi_0 \right\rangle \langle \psi_0 | c_{j\sigma} | \phi \rangle}{\omega - (E_{\phi} - E_0) + i\eta} + \sum_{\chi} \frac{\langle \chi | c_{i\sigma} | \psi_0 \rangle \left\langle \psi_0 \left| c_{j\sigma}^{\dagger} \right| \chi \right\rangle}{\omega + (E_{\chi} - E_0) - i\eta}$$

$$G_{11\uparrow}(\omega) = \frac{(-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})^2}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{2,1} + \alpha_{0,2}\gamma_{2,3})^2}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})^2}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{4,1} + \alpha_{0,2}\gamma_{4,3})^2}{\omega - (E_{\phi_4} - E_0) + i\eta}$$

$$G_{22\uparrow}(\omega) = \frac{(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})^2}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(-\alpha_{0,1}\gamma_{2,1} + \alpha_{0,3}\gamma_{2,3})^2}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})^2}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(-\alpha_{0,1}\gamma_{4,1} + \alpha_{0,3}\gamma_{4,3})^2}{\omega - (E_{\phi_4} - E_0) + i\eta}$$

$$G_{12\uparrow}(\omega) = \frac{(-\alpha_{0,3}\beta_{1,2} - \alpha_{0,1}\beta_{1,4})(\alpha_{0,2}\beta_{1,2} - \alpha_{0,4}\beta_{1,4})}{\omega + (E_{\chi_1} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{2,1} + \alpha_{0,2}\gamma_{2,3})(-\alpha_{0,1}\gamma_{2,1} + \alpha_{0,3}\gamma_{2,3})}{\omega - (E_{\phi_2} - E_0) + i\eta}$$
$$+ \frac{(\alpha_{0,3}\beta_{3,2} + \alpha_{0,1}\beta_{3,4})(\alpha_{0,2}\beta_{3,2} - \alpha_{0,4}\beta_{3,4})}{\omega + (E_{\chi_3} - E_0) - i\eta} + \frac{(\alpha_{0,4}\gamma_{4,1} + \alpha_{0,2}\gamma_{4,3})(-\alpha_{0,1}\gamma_{4,1} + \alpha_{0,3}\gamma_{4,3})}{\omega - (E_{\phi_4} - E_0) + i\eta}$$

$$G_{11\downarrow}(\omega) = \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})^2}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})^2}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,4}\gamma_{3,2} - \alpha_{0,1}\gamma_{3,4})^2}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})^2}{\omega + (E_{\chi_4} - E_0) - i\eta}$$

$$G_{22\downarrow}(\omega) = \frac{(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})^2}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})^2}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,2}\gamma_{3,2} + \alpha_{0,3}\gamma_{3,4})^2}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})^2}{\omega + (E_{\chi_4} - E_0) - i\eta}$$

$$G_{12\downarrow}(\omega) = \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})}{\omega - (E_{\phi_1} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{2,1} - \alpha_{0,2}\beta_{2,3})(\alpha_{0,1}\beta_{2,1} + \alpha_{0,4}\beta_{2,3})}{\omega + (E_{\chi_2} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{0,4}\gamma_{3,2} - \alpha_{0,1}\gamma_{3,4})(\alpha_{0,2}\gamma_{3,2} + \alpha_{0,3}\gamma_{3,4})}{\omega - (E_{\phi_3} - E_0) + i\eta} + \frac{(\alpha_{0,3}\beta_{4,1} - \alpha_{0,2}\beta_{4,3})(\alpha_{0,1}\beta_{4,1} + \alpha_{0,4}\beta_{4,3})}{\omega + (E_{\chi_4} - E_0) - i\eta}$$

4.3.3 $N = 3$ electrons

$$G_{ij\sigma}(\omega) = \frac{\left\langle \uparrow\downarrow\uparrow\downarrow \left| c_{i\sigma}^\dagger \right| \phi_0 \right\rangle \langle \phi_0 | c_{j\sigma} | \uparrow\downarrow\uparrow\downarrow \rangle}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \sum_\psi \frac{\langle \psi | c_{i\sigma} | \phi_0 \rangle \left\langle \phi_0 \left| c_{j\sigma}^\dagger \right| \psi \right\rangle}{\omega + (E_\psi - E_0) - i\eta}$$

With $E_0 = \frac{1}{2}(3\epsilon_1 + 3\epsilon_2 + U_1 + U_2 - g)$ and $E_{\uparrow\downarrow\uparrow\downarrow} = (2\epsilon_1 + 2\epsilon_2 + U_1 + U_2)$

$$G_{11\uparrow}(\omega) = \frac{\gamma_{1,2}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{2,3} + \alpha_{0,4}\gamma_{2,1})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,2}\gamma_{2,3} + \alpha_{1,4}\gamma_{2,1})^2}{\omega + (E_{\psi_1} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{2,2}\gamma_{2,3} + \alpha_{2,4}\gamma_{2,1})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{1,4}^2}{\omega + (E_{\psi_3} - E_0) - i\eta} + \frac{\gamma_{2,3}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta}$$

$$G_{22\uparrow}(\omega) = \frac{\gamma_{1,4}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,3}\gamma_{2,3} - \alpha_{0,1}\gamma_{2,1})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,3}\gamma_{2,3} - \alpha_{1,1}\gamma_{2,1})^2}{\omega + (E_{\psi_1} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{2,3}\gamma_{2,3} - \alpha_{2,1}\gamma_{2,1})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{1,2}^2}{\omega + (E_{\psi_3} - E_0) - i\eta} + \frac{\gamma_{2,1}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta}$$

$$G_{12\uparrow}(\omega) = \frac{\gamma_{1,2}\gamma_{1,4}}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{2,3} + \alpha_{0,4}\gamma_{2,1})(\alpha_{0,3}\gamma_{2,3} - \alpha_{0,1}\gamma_{2,1})}{\omega + (E_{\psi_0} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{1,2}\gamma_{2,3} + \alpha_{1,4}\gamma_{2,1})(\alpha_{1,3}\gamma_{2,3} - \alpha_{1,1}\gamma_{2,1})}{\omega + (E_{\psi_1} - E_0) - i\eta} + \frac{(\alpha_{2,2}\gamma_{2,3} + \alpha_{2,4}\gamma_{2,1})(\alpha_{2,3}\gamma_{2,3} - \alpha_{2,1}\gamma_{2,1})}{\omega + (E_{\psi_2} - E_0) - i\eta}$$
$$- \frac{\gamma_{1,4}\gamma_{1,2}}{\omega + (E_{\psi_3} - E_0) - i\eta} - \frac{\gamma_{2,3}\gamma_{2,1}/2}{\omega + (E_{\psi_5} - E_0) - i\eta}$$

$$G_{11\downarrow}(\omega) = \frac{\gamma_{2,1}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,4}\gamma_{1,2} - \alpha_{1,1}\gamma_{1,4})^2}{\omega + (E_{\psi_1} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{2,4}\gamma_{1,2} - \alpha_{2,1}\gamma_{1,4})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{2,3}^2}{\omega + (E_{\psi_4} - E_0) - i\eta} + \frac{\gamma_{1,4}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta}$$

$$G_{22\downarrow}(\omega) = \frac{\gamma_{2,3}^2}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})^2}{\omega + (E_{\psi_0} - E_0) - i\eta} + \frac{(\alpha_{1,2}\gamma_{1,2} + \alpha_{1,3}\gamma_{1,4})^2}{\omega + (E_{\psi_1} - E_0) - i\eta}$$
$$+ \frac{(\alpha_{2,2}\gamma_{1,2} + \alpha_{2,3}\gamma_{1,4})^2}{\omega + (E_{\psi_2} - E_0) - i\eta} + \frac{\gamma_{2,1}^2}{\omega + (E_{\psi_4} - E_0) - i\eta} + \frac{\gamma_{1,2}^2/2}{\omega + (E_{\psi_5} - E_0) - i\eta}$$

$$
\begin{aligned}
G_{12\downarrow}(\omega) &= \frac{\gamma_{2,1}\gamma_{2,3}}{\omega - (E_{\uparrow\downarrow\uparrow\downarrow} - E_0) + i\eta} + \frac{(\alpha_{0,4}\gamma_{1,2} - \alpha_{0,1}\gamma_{1,4})(\alpha_{0,2}\gamma_{1,2} + \alpha_{0,3}\gamma_{1,4})}{\omega + (E_{\psi_0} - E_0) - i\eta} \\
&+ \frac{(\alpha_{1,4}\gamma_{1,2} - \alpha_{1,1}\gamma_{1,4})(\alpha_{1,2}\gamma_{1,2} + \alpha_{1,3}\gamma_{1,4})}{\omega + (E_{\psi_1} - E_0) - i\eta} + \frac{(\alpha_{2,4}\gamma_{1,2} - \alpha_{2,1}\gamma_{1,4})(\alpha_{2,2}\gamma_{1,2} + \alpha_{2,3}\gamma_{1,4})}{\omega + (E_{\psi_2} - E_0) - i\eta} \\
&- \frac{\gamma_{2,3}\gamma_{2,1}}{\omega + (E_{\psi_4} - E_0) - i\eta} - \frac{\gamma_{1,4}\gamma_{1,2}/2}{\omega + (E_{\psi_5} - E_0) - i\eta}
\end{aligned}
$$