Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Steady State

Reversed Chain

Reversible Chains

Research Example: Aloha

Finite Absorbing Chains

Stochastic Network Modeling (SNM)

Llorenç Cerdà-Alabern
Universitat Politècnica de Catalunya
Departament d'Arquitectura de Computadors
llorenc@ac.upc.edu

Parts

I Introduction
II Discrete Time Markov Chains (DTMC)
III Continuous Time Markov Chains (CTMC)
IV Queuing Theory

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite Absorbing Chains

Part II

Discrete Time Markov Chains (DTMC)

Outline

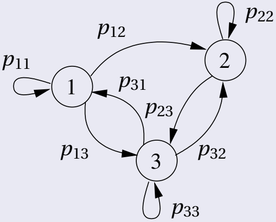
- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State
- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:
$\sum_{\forall j} p_{ij} = 1$

- Mathematically:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Properties of a DTMC

- The event $X(n) = i$ (at step n the system is in state i) must satisfy (memoryless property):

$$\boxed{\begin{aligned} P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \cdots) = \\ P(X(n) = j \mid X(n-1) = i) \end{aligned}}$$

- If $P\big(X(n) = j \mid X(n-1) = i\big) = P\big(X(1) = j \mid X(0) = i\big)$ for any n we have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$$

- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) =$$

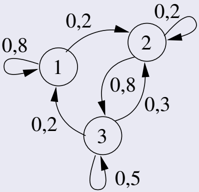
$$\sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = \boxed{1}$$

- $\mathbf{P}$ is a stochastic matrix, i.e. a matrix which rows sum 1.

Definition of a DTMC

Example

- Assume a terminal can be in 3 states:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



$$
\begin{array}{cc}
 & \text{to state} \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\mathbf{P} = & \begin{bmatrix} 0,8 & 0,2 & 0 \\ 0 & 0,2 & 0,8 \\ 0,2 & 0,3 & 0,5 \end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\quad \begin{array}{c} \text{from} \\ \text{state} \end{array}
$$

- The average transmission rate (throughput), v_a, is:

$$v_a = P(\text{the terminal is in state 3}) \times v$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time

Transient Solution

Classification of States

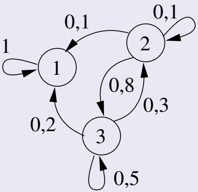
Steady State

Reversed Chain

Reversible Chains

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



$$\begin{array}{ccc} & \text{to state} & \\ 1 & 2 & 3 \end{array}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \begin{array}{c} \text{from} \\ \text{state} \\ \ \end{array}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

n-step transition probabilities

- Transition probabilities: $p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$
- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- We define the n-step transition probabilities:

$$p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$$

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are stochastic matrices: Their rows sum 1.

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

State Probabilities

- Define the probability of being in state i at step n:
$$\pi_i(n) = P(X(n) = i)$$
- In vector form (row vector)
$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$
- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the state probability at step n.

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n)=1), P(X(n)=2), \cdots).$$

- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1)=k)\, P\big(X(n)=i \,\big|\, X(n-1)=k\big) = \sum_k \pi_k(n-1)\, p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0)=k)\, P\big(X(n)=i \,\big|\, X(0)=k\big) = \sum_k \pi_k(0)\, p_{ki}(n)$$

- In matrix form:

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

State Probabilities

$$\boxed{\begin{aligned}\boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P}\\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n)\end{aligned}}$$

- Iterating

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P} = \boldsymbol{\pi}(n-2)\,\mathbf{P}\,\mathbf{P} = \boldsymbol{\pi}(n-3)\,\mathbf{P}\,\mathbf{P}\,\mathbf{P} = \cdots = \boldsymbol{\pi}(0)\,\mathbf{P}^n$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\; p_{kj}(n-r)}$$

- **Proof**:

$$\begin{aligned}
p_{ij}(n) &= P\big(X(n)=j \mid X(0)=i\big) = \sum_k P\big(X(n)=j, X(r)=k \mid X(0)=i\big) \\
&= \sum_k \frac{P\big(X(n)=j, X(r)=k, X(0)=i\big)}{P(X(0)=i)} \times \frac{P(X(r)=k, X(0)=i)}{P(X(r)=k, X(0)=i)} \\
&= \sum_k P\big(X(n)=j \mid X(r)=k, X(0)=i\big)\, P\big(X(r)=k \mid X(0)=i\big) \\
&= \sum_k P\big(X(n)=j \mid X(r)=k\big)\, P\big(X(r)=k \mid X(0)=i\big) \\
&= \sum_k p_{ik}(r)\, p_{kj}(n-r)
\end{aligned}$$

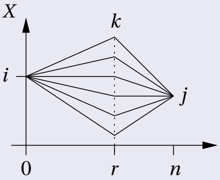
$\square$

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\; p_{kj}(n-r)}$$

- Graphical interpretation:



- In matrix form:

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Chapman-Kolmogorov Equations

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1)\,\mathbf{P}(n-1) = \mathbf{P}\,\mathbf{P}(n-1) = \mathbf{P}(n-1)\,\mathbf{P}$$

- Iterating:

$$\boxed{\mathbf{P}(n) = \mathbf{P}^n}$$

- Thus:

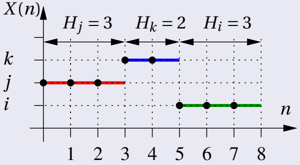
$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Sojourn or Holding Time

- Sojourn or holding time in state k: Is the RV H_k equal to the number of steps that the chain remains in state k before leaving to a different state:



- The Markov property implies:

$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\; n \geq 1}$$

- Which is a geometric distribution with mean:

$$\mathrm{E}[H_i] = \sum_{n=1}^{\infty} n P(H_i = n) = \frac{1}{1 - p_{ii}}.$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time

Transient Solution

Classification of States

Steady State

Reversed Chain

Reversible Chains

Sojourn or Holding Time

- NOTE: We allow that:

$$p_{ii} = 0 \Rightarrow H_i(n) = I(n=1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases} \text{, and}$$

$$p_{ii} = 1 \Rightarrow \mathrm{E}[H_i] = \infty \text{ (absorbing state).}$$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

Theorem

A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.

Proof.

- We have seen that a DTMC has a sojourn time

$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

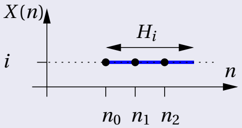
- Which is geometrically distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).

$\square$

Definition of a DTMC

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

The geometric distribution satisfies the Markov property (1)



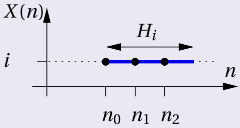
Proof

- Markov property:
  $P\big(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\big) = P\big(X(n_2) = i \mid X(n_1) = i\big)$
- Thus, the Markov property in terms of the sojourn time can be written as:

$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
State Transition Diagram
Properties of a DTMC
Transition Matrix
Absorbing Chains
n-step transition probabilities
State Probabilities
Chapman-Kolmogorov Equations
Sojourn or Holding Time
Transient Solution
Classification of States
Steady State
Reversed Chain
Reversible Chains

The geometric distribution satisfies the Markov property (2)



$$P\left(H_i > n_2 - n_0 \;\middle|\; H_i > n_1 - n_0\right) = P(H_i > n_2 - n_1)$$

- Since
$$P(H_i > k) = 1 - P(H_i \le k) = 1 - \sum_{n=1}^{k} p^{n-1}(1-p) = 1-(1-p)\frac{1-p^k}{1-p} = p^k$$
- We have:
$$P\left(H_i > n_2 - n_0 \;\middle|\; H_i > n_1 - n_0\right) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$
$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2-n_0}}{p^{n_1-n_0}} = p^{n_2-n_1} = P(H_i > n_2 - n_1) \quad \square$$

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Part II

Discrete Time Markov Chains (DTMC)

Outline

- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State
- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Transient Solution

- If we are interested in the transient evolution we shall study $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n$.
- If we can diagonalize **P**, we can obtain the transient evolution in close form.
- **P** can be diagonalized if **P** can be decomposed as:

$$\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$$

where **L** is some invertible matrix and Λ is the diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

with λ_l, $l = 1, \cdots N$ the eigenvalues of **P**.

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Eigenvalues

- The eigenvalues λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l}$ (or $\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* and *right* eigenvectors, respectively.

$$\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l} \Rightarrow \boldsymbol{l}(\mathbf{A} - \mathbf{I}\lambda_l) = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

$$\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r} \Rightarrow (\mathbf{A} - \mathbf{I}\lambda_l)\boldsymbol{r} = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

- Thus, λ_l solve the characteristic polynomial $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$.
- Note that, in general, *left* and *right* eigenvectors are different, but eigenvalues are the same (they solve the same characteristic polynomial).
- A matrix can be diagonalized if all eigenvalues are single (multiplicity = 1). If a matrix cannot be diagonalized it is called defective.

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Determinants

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{l} +a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{21}\, a_{32}\, a_{13} \\ -a_{31}\, a_{22}\, a_{13} - a_{12}\, a_{21}\, a_{33} - a_{23}\, a_{32}\, a_{11} \end{array}$$

- Cofactor Formula: expanding along a row i:

$$\det \mathbf{A} = \sum_{j=1}^{N} a_{ij}\, (-1)^{i+j} \det M_{ij},$$

where the minor matrices M_{ij} are obtained removing the row i and column j from $\mathbf{A}$. $(-1)^{i+j} \det M_{ij}$ is called the cofactor of a_{ij}.

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Properties of the determinants

$$\det \mathbf{A} = \prod \text{eigenvalues of } \mathbf{A}$$

$$\text{trace } \mathbf{A} = \sum \text{eigenvalues of } \mathbf{A}$$

where trace $\mathbf{A} = \sum$ elements of the diagonal of $\mathbf{A}$.

Transient Solution

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Transient Solution

- Assume a finite DTMC with N states. Then $\mathbf{P} = \mathbf{P}^{N\times N}$.
- Assume that $\mathbf{P}$ can be diagonalized: $\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{P}$.
- Since $\Lambda^n = \text{diag}(\lambda_1^n, \cdots \lambda_N^n)$, we have that

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n = \boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\Lambda^n\,\mathbf{L}) =$$
$$\boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\text{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L})$$

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Transient Solution

- But $\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L}$ are linear combinations of $\lambda_1^n, \cdots \lambda_N^n$. Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(n) = (\boldsymbol{\pi}(n))_i = \sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n}$$

where the unknown coefficients $a_i^{(l)}$ can be obtained solving the system of equations:

$$\sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n = (\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1$$

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

- We want the probability of being in state 2 in n steps starting from state 1: $\pi_2(n)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Solution

- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 2/5$.

$$\pi_2(n) = \lambda_1^n a + b\lambda_2^n = a + b(2/5)^n$$

- Imposing the boundary conditions $\pi_i(n) = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i$:

$$\pi_2(0) = a + b = (\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{P}^0)_2 = (\mathbf{P}^0)_{12} = 0$$

$$\pi_2(1) = a + b(2/5) = (\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{P}^1)_2 = (\mathbf{P})_{12} = 1/5$$

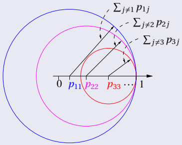
we have that $a = 1/3$, $b = -1/3$, thus:

$$\boxed{\begin{aligned}&\pi_2(n) = 1/3 - 1/3\,(2/5)^n, \quad n \ge 0\\ &\pi_1(n) = 1 - \pi_2(n) = 2/3 + 1/3\,(2/5)^n, \quad n \ge 0\end{aligned}}$$

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Eigenvalues of a Stochastic Matrix

- $\mathbf{P}$ has an eigenvalue equal to 1 ($\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 1$).
  Proof: $\mathbf{P}\,\mathbf{e} = \mathbf{e}$, where $\mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{P}$ add to 1). □
- All eigenvalues of $\mathbf{P}$ are $|\lambda_l| \leq 1$.
  Proof: Using Gerschgorin's theorem *The eigenvalues of a matrix $\mathbf{P}_{n\times n}$ lie within the union of the n circular disks with center p_{ii} and radius $\sum_{j\neq i}|p_{ij}|$* in $\mathbb{C}$. Since $\sum_j p_{ij} = 1$, the property is proved. □

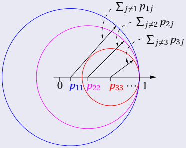


Gershgorin discs of a stochastic matrix

- The eigenvalue $\lambda = 1$ is single if $\mathbf{P}$ is irreducible (Perron-Frobenius theorem). $\mathbf{P}$ is irreducible if all states communicate: for some n, $p_{ij}(n) = (\mathbf{P}^n)_{ij} > 0$, $\forall\, i, j$.

Proof of Gerschgorin's theorem

Gerschgorin's theorem: *The eigenvalues of a matrix* $\mathbf{P}_{n\times n}$ *lie within the union of the n circular disks with center* p_{ii} *and radius* $\sum_{j\neq i}|p_{ij}|$ in $\mathbb{C}$.



Gershgorin discs
of a stochastic matrix

Proof: From $\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ we have

$$\sum_j p_{ij}\,x_j = \lambda\,x_i \quad \forall i \in \{1,\ldots,n\}.$$

We choose i such that $|x_i| = \max_j |x_j|$. Thus, $\sum_{j\neq i} p_{ij}\,x_j = \lambda\,x_i - p_{ii}\,x_i$, and

$$|\lambda - p_{ii}| = \left|\sum_{j\neq i} p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}\left|p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}|p_{ij}|$$

and the equation $|x-c| \le r$, $x,c\in\mathbb{C}, r\in\mathbb{R}$ is a disk of center c and radius r in $\mathbb{C}$. □

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Chain with a Defective Matrix

- What if $\mathbf{P}$ cannot be diagonalized? (defective matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{P}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$), and a possible eigenvalue $\lambda_1 = 0$ with multiplicity k_1. Then [1]:

$$\boxed{\pi_j(n) = \sum_{m=0}^{k_1-1} a_j^{(1,m)} I(n=m) + \sum_{l=2}^{L} \lambda_l^n \sum_{m=0}^{k_l-1} a_j^{(l,m)}\, n^m, \qquad 1 \leq j \leq N,\ n \geq 0}$$

$I(n=m)$ is the indicator func.: $I(n) = 1$ if $n = m$, $I(n) = 0$ if $n \neq m$.

[1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method.* Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html.

Transient Solution

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Transient Solution
Example
Eigenvalues of a Stochastic Matrix
Chain with a Defective Matrix
Example
Classification of States
Steady State
Reversed Chain
Reversible Chains
Research Example: Aloha
Finite

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

- We want the probability of being in state 1 in n steps starting from state 1: $\pi_1(n)$ with $\pi_1(0) = 1$.
- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1/4$ with multiplicity 2. We guess:

$$\pi_1(n) = a + 1/4^n(b + c\,n)$$

- Imposing $\pi_1(0) = 1$, $\pi_1(1) = 3/4$, $\pi_1(2) = (3/4)^2$, we have:

$$\boxed{\pi_1(n) = \frac{4}{9} + \frac{1}{4^n}\left(\frac{5}{9} + \frac{2}{3}n\right)}$$

Discrete Time Markov Chains (DTMC)

Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

Part II

Discrete Time Markov Chains (DTMC)

Outline

- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State
- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Objective

- Identify the different types of behavior that the chain can have.
- Introduce the concepts of first passage probability and mean recurrence time.

Classification of States

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

Irreducibility

- A state j is said to communicate with i, $i \leftrightarrow j$, if $p_{ij}(m_1) > 0$, $p_{ji}(m_2) > 0$ for some $m_1, m_2 \geq 0$.
- We define an irreducible closed set, ICS C_k as a set where all states communicate with each other, and have no transitions to other states out of the set:
  $i \leftrightarrow j, \forall i, j \in C_k$ and $p_{ij} = 0, \forall i \in C_k, j \notin C_k$
  (note that for $i \in C_k, j \notin C_k$ we have: $p_{ij}(2) = \sum_k p_{ik} p_{kj} = 0$, since $p_{ik} = 0$ if $k \notin C_k$, and $p_{kj} = 0$ if $k \in C_k$. Thus, $p_{ij}(n) = 0, \forall n$.)
- An absorbing state form an ICS of only one element. This state, i, must have $p_{ii} = 1$, $p_{ij} = 0 \; \forall j \neq i$.
- Transient states do not belong to any ICS.
- A MC is irreducible if all the states form a unique ICS.

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Irreducibility

- Assume a MC has M ICSs: By properly numbering the states, we can write $\mathbf{P}$ as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- Example, if $M = 3$:

$$\mathbf{P} = \left[\begin{array}{ccc|c} \mathbf{P}_1 & \multicolumn{3}{c}{\mathbf{0}} \\ \mathbf{0} & \mathbf{P}_2 & & \\ & & \mathbf{P}_3 & \\ \hline \multicolumn{3}{c|}{\text{at least one} > 0} & \mathbf{T} \end{array}\right] \Rightarrow \boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n = \boldsymbol{\pi}(0) \left[\begin{array}{ccc|c} \mathbf{P}_1^n & \multicolumn{3}{c}{\mathbf{0}} \\ \mathbf{0} & \mathbf{P}_2^n & & \\ & & \mathbf{P}_3^n & \\ \hline \multicolumn{3}{c|}{\text{at least one} > 0} & \mathbf{T}^n \end{array}\right]$$

- Note that the M sub-matrices are stochastic (their rows sum 1).

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

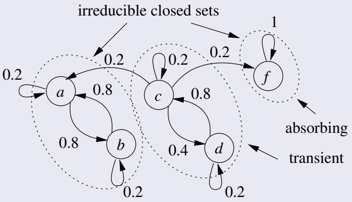
Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Example



$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccccc} a & b & f & c & d \end{array} \\ \begin{array}{c} a \\ b \\ f \\ c \\ d \end{array} & \begin{bmatrix} 0{,}2 & 0{,}8 & 0 & 0 & 0 \\ 0{,}8 & 0{,}2 & 0 & 0 & 0 \\ 0 & 0 & 1{,}0 & 0 & 0 \\ 0{,}2 & 0 & 0{,}2 & 0{,}2 & 0{,}4 \\ 0 & 0 & 0 & 0{,}8 & 0{,}2 \end{bmatrix} \end{array}$$

$$\mathbf{P}^{\infty} = \begin{array}{cc} & \begin{array}{ccccc} a & b & f & c & d \end{array} \\ \begin{array}{c} a \\ b \\ f \\ c \\ d \end{array} & \begin{bmatrix} 0{,}5 & 0{,}5 & 0 & 0 & 0 \\ 0{,}5 & 0{,}5 & 0 & 0 & 0 \\ 0 & 0 & 1{,}0 & 0 & 0 \\ 0{,}25 & 0{,}25 & 0{,}5 & 0 & 0 \\ 0{,}25 & 0{,}25 & 0{,}5 & 0 & 0 \end{bmatrix} \end{array}$$

- What is the meaning of the probabilities in $\mathbf{P}^{\infty}$? (recall that $(\mathbf{P}^n)_{ij} = p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$).

Example

$$\mathbf{P} = \begin{array}{|c|c|c|c|}\hline \mathbf{P}_1 & \multicolumn{3}{c|}{\mathbf{0}} \\ \hline & \mathbf{P}_2 & & \\ \mathbf{0} & & \mathbf{P}_3 & \\ \hline \multicolumn{3}{|c|}{\text{at least one} > 0} & \mathbf{T} \\ \hline \end{array}$$

Theorem *The multiplicity of the eigenvalue $\lambda = 1$ is equal to the number of irreducible closed sets.*

Proof The characteristic polynomial of $\mathbf{P}$ is equal to the product of the characteristic polynomials of the sub-matrices $\mathbf{P}_i$ and $\mathbf{T}$. Since $\mathbf{P}_i$ are irreducible stochastic, each will have a single eigenvalue equal to 1. For the transitorial states it must be $\lim_{n\to\infty} \mathbf{T}^n = \mathbf{0}$. Thus, all the eigenvalues of $\mathbf{T}$ must be $|\lambda| < 1$. □

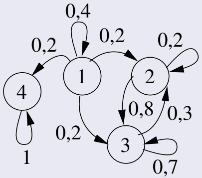
NOTE: in the closed form solution there is only one unknown associated with $\lambda = 1$, otherwise $\sum_{m=0}^{k_l-1} a_j^{(l,m)}\, n^m$ will diverge as $n \to \infty$ (i.e. $a_j^{(l,m)} = 0, m > 0$), and $a_j^{(l,0)} = \lim_{n\to\infty} \pi_j(n)$.

Classification of States

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

Transient and Recurrent

- Recurrent: States that, being visited, they are visited again with probability 1. They are visited an infinite number of times when $n \to \infty$.
- Transient: States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $n \to \infty$.
- Absorbing: A single (recurrent) state where the chain remains with probability $= 1$.



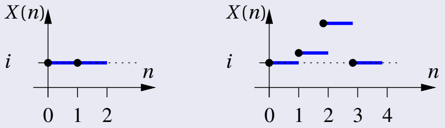
State 1 is transient
States 2 and 3 are recurrent
State 4 is absorbing

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

First Passage (Transition) Probabilities

- To derive a classification criteria, we shall study the distribution of the number of steps to go for the first time from a state i another state j. Definition:

$$\boxed{f_{ii}(n) = P\begin{pmatrix}\text{first transition into state } i \\ \text{in } n \text{ steps starting from } i\end{pmatrix}}$$



first transition in 1 step　　first transition in 3 steps

- Do not confuse with the n-step transition probability $p_{ii}(n)$, where the state i can be visited in the intermediate states.

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

- $f_{ii}(n)$ and $p_{ii}(n)$ satisfy:

$$\boxed{\begin{aligned} f_{ii}(1) &= p_{ii}(1) \\ p_{ii}(n) &= \sum_{l=1}^{n} f_{ii}(l)\, p_{ii}(n-l),\ n >= 1 \end{aligned}}$$

- The probability that the MC eventually enters state i starting from i is given by:

$$\boxed{f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n)}$$

- If $f_{ii} = 1$ we say i is a recurrent state.
- If $f_{ii} < 1$ we say i is a transient state.

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Generalization to Any State Pair

- Analogously to $f_{ii}(n)$, we define the probability of the first passage to state j starting from any state i in n steps: $f_{ij}(n)$.
- $f_{ij}(n)$ and $p_{ij}(n)$ satisfy:

$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l)\, p_{jj}(n-l),\ n \geq 1$$

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Recursive Equation for the First Passage Probabilities

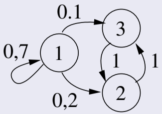
- Recall that the The probability that the MC eventually enters state j starting from i is given by: $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$
- f_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then we will reach j with probability f_{kj}. Thus:

$$\boxed{f_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}}$$

- If there are more than 1 absorbing states, we can compute the probability to reach them using this method (if there is only 1, say j, then $f_{ij} = 1, \forall i$).

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

Example: Recurrence Times Using the Definition



$$\mathbf{P} = \begin{bmatrix} 0,7 & 0,2 & 0,1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f_{21}(n) = f_{31}(n) = 0$$
$$f_{11}(n) = 0,7\, I(n=1)$$
$$f_{22}(n) = f_{33}(n) = I(n=2)$$
$$f_{23}(n) = f_{32}(n) = I(n=1)$$

$$f_{12}(n) = \begin{cases} 0,2, & n=1 \\ 0,7^{n-1}\,0,2 + 0,7^{n-2}\,0,1, & n>1 \end{cases}$$

$$f_{13}(n) = \begin{cases} 0,1, & n=1 \\ 0,7^{n-1}\,0,1 + 0,7^{n-2}\,0,2, & n>1 \end{cases}$$

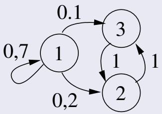
$$f_{11} = 0,7$$
$$f_{12} = f_{13} = 1 \qquad f_{22} = f_{23} = 1$$
$$f_{32} = f_{33} = 1 \qquad f_{21} = f_{31} = 0$$

- State 1 is transient. States 2 and 3 are recurrent.

Classification of States

Discrete Time Markov Chains (DTMC)
Definition of a DTMC
Transient Solution
Classification of States
Objective
Irreducibility
Example
Transient and Recurrent
First Passage (Transition) Probabilities
Relation between $f_{ii}(n)$ and $p_{ii}(n)$
Generalization to Any State Pair
Recursive Equation for the First Passage Probabilities
Example: Recurrence Times Using the Definition
Example: First Passage Probability Using

Example: First Passage Probability Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0,7 & 0,2 & 0,1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$f_{12} = p_{11} f_{12} + p_{12} + p_{13} f_{32}$$

- Clearly $f_{32} = 1$, thus:

$$f_{12} = 0,7 f_{12} + 0,2 + 0,1 \times 1 \Rightarrow f_{12} = 1$$

as before.

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Mean Recurrence Time

- If $f_{ii} = 1$ we say i is a recurrent state.
- If $f_{ii} < 1$ we say i is a transient state.
- When $f_{ii} = 1$, we define the mean recurrence time m_{ii} as:

$$\boxed{m_{ii} = \sum_{n=1}^{\infty} n f_{ii}(n)}$$

- m_{ii} is the average number of steps to eventually reach i starting from i. If $f_{ii} < 1$ (transient state) then we define $m_{ii} = \infty$.
- Classification of recurrent states ($f_{ii} = 1$):
  - If $m_{ii} = \infty$ the state is null recurrent: it takes an ∞ time to reach the state after leave it. Can only happen in chains with an infinite number of states.
  - If $m_{ii} < \infty$ the state is positive recurrent: the state is reached in a finite time after leave it.

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

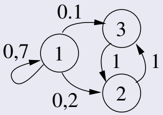
Example: First Passage Probability Using

Property of States

In finite MC:

1. States can be only of type positive recurrent or transient.
2. At least one state must be positive recurrent.
3. There are not null recurrent states.

- Example:



- State 1 is transient. States 2 and 3 are positive recurrent.

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Generalization to Any State Pair

- When $f_{ij} = 1$, the average number of steps to eventually reach j starting from i, m_{ij} is given by:

$$\boxed{m_{ij} = \sum_{n=1}^{\infty} n f_{ij}(n)}$$

- If state j can not be reached starting from state i with probability one (if $f_{ij} < 1$), then we define $m_{ij} = \infty$.

Classification of States

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

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Objective

Irreducibility

Example

Transient and Recurrent

First Passage (Transition) Probabilities

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

Generalization to Any State Pair

Recursive Equation for the First Passage Probabilities

Example: Recurrence Times Using the Definition

Example: First Passage Probability Using

Recursive Equation for the Mean Recurrence Time

- Recall that the mean recurrence time $m_{ij} = \sum_{n\geq 1} n f_{ij}(n)$ is the average number of steps to eventually reach j starting from i, i.e. it is the mean first passage time from state i to j.
- When $f_{ij} = 1$, m_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then it will take m_{kj} steps to reach j. Thus:

$$\boxed{m_{ij} = p_{ij} + \sum_{k\neq j} p_{ik}\,(1 + m_{kj}) = 1 + \sum_{k\neq j} p_{ik}\, m_{kj}}$$

since $\sum_j p_{ij} = 1$.

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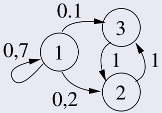
Example: Mean Recurrence Time Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$m_{12} = p_{12} + p_{11}\,(1 + m_{12}) + p_{13}\,(1 + m_{32}) = 1 + p_{11}\,m_{12} + p_{13}\,m_{32}$$

- Clearly $m_{32} = 1$, thus:

$$m_{12} = 1 + 0{,}7\,m_{12} + 0{,}1 \times 1 \Rightarrow m_{12} = 11/3.$$