Stochastic Network Modeling (SNM)

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Parts

Part II

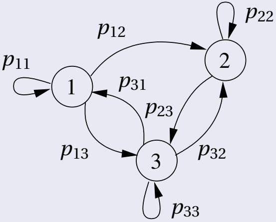
Discrete Time Markov Chains (DTMC)

Outline

Definition of a DTMC

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:
$\sum_{\forall j} p_{ij} = 1$

- Mathematically:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

Definition of a DTMC

Properties of a DTMC

- The event $X(n) = i$ (at step n the system is in state i) must satisfy (memoryless property):

$$\boxed{\begin{aligned} P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \cdots) = \\ P(X(n) = j \mid X(n-1) = i) \end{aligned}}$$

- If $P\big(X(n) = j \mid X(n-1) = i\big) = P\big(X(1) = j \mid X(0) = i\big)$ for any n we have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

Definition of a DTMC

Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

Definition of a DTMC

Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$$

- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) =$$

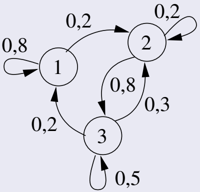
$$\sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = \boxed{1}$$

- **P** is a stochastic matrix, i.e. a matrix which rows sum 1.

Definition of a DTMC

Example

- Assume a terminal can be in 3 states:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



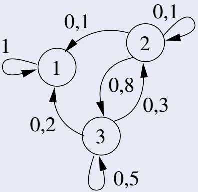
$$\mathbf{P} = \begin{array}{c c l} & \text{to state} & \\ & \begin{matrix} 1 & \ \ 2 & \ \ 3 \end{matrix} & \\ & \begin{bmatrix} 0{,}8 & 0{,}2 & 0 \\ 0 & 0{,}2 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \ \begin{matrix} \text{from} \\ \text{state} \\ \ \end{matrix} \end{array}$$

- The average transmission rate (throughput), v_a, is:

$$v_a = P(\text{the terminal is in state 3}) \times v$$

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



$$\begin{array}{cc} & \text{to state} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbf{P} = & \begin{bmatrix} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \begin{array}{c} \text{from} \\ \text{state} \end{array}$$

Definition of a DTMC

n-step transition probabilities

- Transition probabilities: $p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$
- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- We define the n-step transition probabilities:

$$p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$$

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are stochastic matrices: Their rows sum 1.

Definition of a DTMC

State Probabilities

- Define the probability of being in state i at step n:

$$\pi_i(n) = P(X(n) = i)$$

- In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$

- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the state probability at step n.

Definition of a DTMC

State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$

- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1) = k)\, P\big(X(n) = i \,\big|\, X(n-1) = k\big) = \sum_k \pi_k(n-1)\, p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0) = k)\, P\big(X(n) = i \,\big|\, X(0) = k\big) = \sum_k \pi_k(0)\, p_{ki}(n)$$

- In matrix form:

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

Definition of a DTMC

State Probabilities

$$\boxed{\begin{aligned}\boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n)\end{aligned}}$$

- Iterating

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P} = \boldsymbol{\pi}(n-2)\,\mathbf{P}\,\mathbf{P} = \boldsymbol{\pi}(n-3)\,\mathbf{P}\,\mathbf{P}\,\mathbf{P} = \cdots = \boldsymbol{\pi}(0)\,\mathbf{P}^n$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\; p_{kj}(n-r)}$$

- **Proof**:

$$p_{ij}(n) = P\big(X(n)=j \mid X(0)=i\big) = \sum_k P\big(X(n)=j, X(r)=k \mid X(0)=i\big)$$

$$= \sum_k \frac{P\big(X(n)=j, X(r)=k, X(0)=i\big)}{P(X(0)=i)} \times \frac{P(X(r)=k, X(0)=i)}{P(X(r)=k, X(0)=i)}$$

$$= \sum_k P\big(X(n)=j \mid X(r)=k, X(0)=i\big)\, P\big(X(r)=k \mid X(0)=i\big)$$

$$= \sum_k P\big(X(n)=j \mid X(r)=k\big)\, P\big(X(r)=k \mid X(0)=i\big)$$

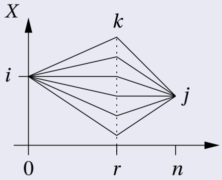
$$= \sum_k p_{ik}(r)\, p_{kj}(n-r)$$

□

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\, p_{kj}(n-r)}$$

- Graphical interpretation:



- In matrix form:

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1)\,\mathbf{P}(n-1) = \mathbf{P}\,\mathbf{P}(n-1) = \mathbf{P}(n-1)\,\mathbf{P}$$

- Iterating:

$$\boxed{\mathbf{P}(n) = \mathbf{P}^n}$$

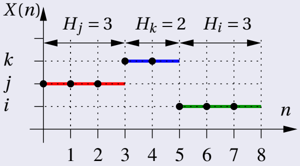
- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Sojourn or Holding Time

- Sojourn or holding time in state k: Is the RV H_k equal to the number of steps that the chain remains in state k before leaving to a different state:



- The Markov property implies:

$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

- Which is a geometric distribution with mean:

$$E[H_i] = \sum_{n=1}^{\infty} n P(H_i = n) = \frac{1}{1 - p_{ii}}.$$

Definition of a DTMC

Sojourn or Holding Time

- NOTE: We allow that:

$$p_{ii} = 0 \Rightarrow H_i(n) = I(n = 1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases} \text{, and}$$

$$p_{ii} = 1 \Rightarrow \mathrm{E}[H_i] = \infty \text{ (absorbing state).}$$

Definition of a DTMC

Theorem

A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.

Proof.

- We have seen that a DTMC has a sojourn time

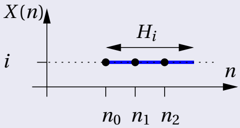
$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\, n \geq 1}$$

- Which is geometrically distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).

□

Definition of a DTMC

The geometric distribution satisfies the Markov property (1)



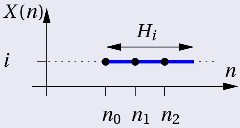
Proof

- Markov property:
  $P\big(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\big) = P\big(X(n_2) = i \mid X(n_1) = i\big)$
- Thus, the Markov property in terms of the sojourn time can be written as:

$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

Definition of a DTMC

The geometric distribution satisfies the Markov property (2)



$$P\big(H_i > n_2 - n_0 \;\big|\; H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

- Since

$$P(H_i > k) = 1 - P(H_i \le k) = 1 - \sum_{n=1}^{k} p^{n-1}(1-p) = 1-(1-p)\frac{1-p^k}{1-p} = p^k$$

- We have:

$$P\big(H_i > n_2 - n_0 \;\big|\; H_i > n_1 - n_0\big) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$

$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2 - n_0}}{p^{n_1 - n_0}} = p^{n_2 - n_1} = P(H_i > n_2 - n_1) \quad \square$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Transient Solution

- If we are interested in the transient evolution we shall study $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n$.
- If we can diagonalize $\mathbf{P}$, we can obtain the transient evolution in close form.
- $\mathbf{P}$ can be diagonalized if $\mathbf{P}$ can be decomposed as:

$$\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$$

where $\mathbf{L}$ is some invertible matrix and Λ is the diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{P}$.

Eigenvalues

- The eigenvalues λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l}$ (or $\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* and *right* eigenvectors, respectively.

$$\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l} \Rightarrow \boldsymbol{l}(\mathbf{A} - \mathbf{I}\lambda_l) = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

$$\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r} \Rightarrow (\mathbf{A} - \mathbf{I}\lambda_l)\boldsymbol{r} = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

- Thus, λ_l solve the characteristic polynomial $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$.
- Note that, in general, *left* and *right* eigenvectors are different, but eigenvalues are the same (they solve the same characteristic polynomial).
- A matrix can be diagonalized if all eigenvalues are single (multiplicity $= 1$). If a matrix cannot be diagonalized it is called defective.

Transient Solution

Determinants

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{l} +a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{21}\, a_{32}\, a_{13} \\ -a_{31}\, a_{22}\, a_{13} - a_{12}\, a_{21}\, a_{33} - a_{23}\, a_{32}\, a_{11} \end{array}$$

- Cofactor Formula: expanding along a row i:

  $$\det \mathbf{A} = \sum_{j=1}^{N} a_{ij}\, (-1)^{i+j} \det M_{ij},$$

  where the minor matrices M_{ij} are obtained removing the row i and column j from $\mathbf{A}$. $(-1)^{i+j} \det M_{ij}$ is called the cofactor of a_{ij}.

Transient Solution

Properties of the determinants

$$\det \mathbf{A} = \prod \text{eigenvalues of } \mathbf{A}$$

$$\text{trace } \mathbf{A} = \sum \text{eigenvalues of } \mathbf{A}$$

where trace $\mathbf{A} = \sum$ elements of the diagonal of $\mathbf{A}$.

Transient Solution

Transient Solution

- Assume a finite DTMC with N states. Then $\mathbf{P} = \mathbf{P}^{N \times N}$.
- Assume that $\mathbf{P}$ can be diagonalized: $\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{P}$.
- Since $\Lambda^n = \mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)$, we have that

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n = \boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\Lambda^n\,\mathbf{L}) =$$
$$\boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L})$$

Transient Solution

Transient Solution

- But $\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L}$ are linear combinations of $\lambda_1^n, \cdots \lambda_N^n$. Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(n) = (\boldsymbol{\pi}(n))_i = \sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n}$$

where the unknown coefficients $a_i^{(l)}$ can be obtained solving the system of equations:

$$\sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n = (\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1$$

Transient Solution

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

- We want the probability of being in state 2 in n steps starting from state 1: $\pi_2(n)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transient Solution

Solution

- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 2/5$.

$$\pi_2(n) = \lambda_1^n a + b\lambda_2^n = a + b(2/5)^n$$

- Imposing the boundary conditions $\pi_i(n) = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i$:

$$\pi_2(0) = a + b = (\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{P}^0)_2 = (\mathbf{P}^0)_{12} = 0$$

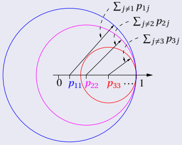
$$\pi_2(1) = a + b(2/5) = (\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{P}^1)_2 = (\mathbf{P})_{12} = 1/5$$

we have that $a = 1/3$, $b = -1/3$, thus:

$$\boxed{\begin{aligned} &\pi_2(n) = 1/3 - 1/3\,(2/5)^n, \quad n \ge 0 \\ &\pi_1(n) = 1 - \pi_2(n) = 2/3 + 1/3\,(2/5)^n, \quad n \ge 0 \end{aligned}}$$

Eigenvalues of a Stochastic Matrix

- $\mathbf{P}$ has an eigenvalue equal to 1 ($\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 1$).
  Proof: $\mathbf{P}\,\mathbf{e} = \mathbf{e}$, where $\mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{P}$ add to 1). □
- All eigenvalues of $\mathbf{P}$ are $|\lambda_l| \le 1$.
  Proof: Using Gerschgorin's theorem *The eigenvalues of a matrix $\mathbf{P}_{n\times n}$ lie within the union of the n circular disks with center p_{ii} and radius $\sum_{j\neq i}|p_{ij}|$ in $\mathbb{C}$.* Since $\sum_j p_{ij} = 1$, the property is proved. □



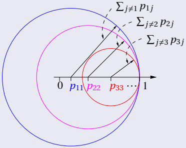
Gershgorin discs of a stochastic matrix

- The eigenvalue $\lambda = 1$ is single if $\mathbf{P}$ is irreducible (Perron-Frobenius theorem). $\mathbf{P}$ is irreducible if all states communicate: for some n, $p_{ij}(n) = (\mathbf{P}^n)_{ij} > 0,\ \forall\, i, j$.

Transient Solution

Proof of Gerschgorin's theorem

Gerschgorin's theorem: *The eigenvalues of a matrix* $\mathbf{P}_{n\times n}$ *lie within the union of the n circular disks with center* p_{ii} *and radius* $\sum_{j\neq i}|p_{ij}|$ in $\mathbb{C}$.



Gershgorin discs of a stochastic matrix

Proof: From $\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ we have

$$\sum_j p_{ij}\,x_j = \lambda\,x_i \quad \forall i \in \{1,\ldots,n\}.$$

We choose i such that $|x_i| = \max_j |x_j|$. Thus, $\sum_{j\neq i} p_{ij}\,x_j = \lambda\,x_i - p_{ii}\,x_i$, and

$$|\lambda - p_{ii}| = \left|\sum_{j\neq i} p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}\left|p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}|p_{ij}|$$

and the equation $|x - c| \le r$, $x, c \in \mathbb{C}, r \in \mathbb{R}$ is a disk of center c and radius r in $\mathbb{C}$. □

Transient Solution

Chain with a Defective Matrix

- What if $\mathbf{P}$ cannot be diagonalized? (defective matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{P}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$), and a possible eigenvalue $\lambda_1 = 0$ with multiplicity k_1. Then [1]:

$$\boxed{\pi_j(n) = \sum_{m=0}^{k_1-1} a_j^{(1,m)} I(n=m) + \sum_{l=2}^{L} \lambda_l^n \sum_{m=0}^{k_l-1} a_j^{(l,m)} n^m, \quad 1 \leq j \leq N,\ n \geq 0}$$

$I(n=m)$ is the indicator func.: $I(n) = 1$ if $n = m$, $I(n) = 0$ if $n \neq m$.

[1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method.* Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html.

Transient Solution

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

- We want the probability of being in state 1 in n steps starting from state 1: $\pi_1(n)$ with $\pi_1(0) = 1$.
- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1/4$ with multiplicity 2. We guess:

$$\pi_1(n) = a + 1/4^n (b + c\,n)$$

- Imposing $\pi_1(0) = 1$, $\pi_1(1) = 3/4$, $\pi_1(2) = (3/4)^2$, we have:

$$\boxed{\pi_1(n) = \frac{4}{9} + \frac{1}{4^n}\left(\frac{5}{9} + \frac{2}{3}n\right)}$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Objective

- Identify the different types of behavior that the chain can have.
- Introduce the concepts of first passage probability and mean recurrence time.

Classification of States

Irreducibility

- A state j is said to communicate with i, $i \leftrightarrow j$, if $p_{ij}(m_1) > 0$, $p_{ji}(m_2) > 0$ for some $m_1, m_2 \geq 0$.
- We define an irreducible closed set, ICS C_k as a set where all states communicate with each other, and have no transitions to other states out of the set:
  $i \leftrightarrow j, \forall i, j \in C_k$ and $p_{ij} = 0, \forall i \in C_k, j \notin C_k$
  (note that for $i \in C_k, j \notin C_k$ we have: $p_{ij}(2) = \sum_k p_{ik} p_{kj} = 0$, since $p_{ik} = 0$ if $k \notin C_k$, and $p_{kj} = 0$ if $k \in C_k$. Thus, $p_{ij}(n) = 0, \forall n$.)
- An absorbing state form an ICS of only one element. This state, i, must have $p_{ii} = 1$, $p_{ij} = 0\ \forall j \neq i$.
- Transient states do not belong to any ICS.
- A MC is irreducible if all the states form a unique ICS.

Classification of States

Irreducibility

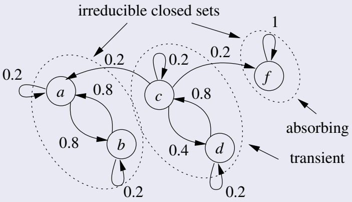
- Assume a MC has M ICSs: By properly numbering the states, we can write $\mathbf{P}$ as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- Example, if $M = 3$:

$$\mathbf{P} = \left[\begin{array}{ccc|c} \mathbf{P}_1 & \mathbf{0} & & \\ \mathbf{0} & \mathbf{P}_2 & & \\ & & \mathbf{P}_3 & \\ \hline \multicolumn{3}{c|}{\text{at least one} > 0} & \mathbf{T} \end{array}\right] \Rightarrow \boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n = \boldsymbol{\pi}(0) \left[\begin{array}{ccc|c} \mathbf{P}_1^n & \mathbf{0} & & \\ \mathbf{0} & \mathbf{P}_2^n & & \\ & & \mathbf{P}_3^n & \\ \hline \multicolumn{3}{c|}{\text{at least one} > 0} & \mathbf{T}^n \end{array}\right]$$

- Note that the M sub-matrices are stochastic (their rows sum 1).

Classification of States

Example



$$\mathbf{P} = \begin{array}{c} \\ a \\ b \\ f \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & f & c & d \end{array} \\ \left[\begin{array}{cc|c|cc} 0{,}2 & 0{,}8 & 0 & 0 & 0 \\ 0{,}8 & 0{,}2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1{,}0 & 0 & 0 \\ \hline 0{,}2 & 0 & 0{,}2 & 0{,}2 & 0{,}4 \\ 0 & 0 & 0 & 0{,}8 & 0{,}2 \end{array}\right] \end{array}$$

$$\mathbf{P}^{\infty} = \begin{array}{c} \\ a \\ b \\ f \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & f & c & d \end{array} \\ \left[\begin{array}{cc|c|cc} 0{,}5 & 0{,}5 & 0 & 0 & 0 \\ 0{,}5 & 0{,}5 & 0 & 0 & 0 \\ \hline 0 & 0 & 1{,}0 & 0 & 0 \\ \hline 0{,}25 & 0{,}25 & 0{,}5 & 0 & 0 \\ 0{,}25 & 0{,}25 & 0{,}5 & 0 & 0 \end{array}\right] \end{array}$$

- What is the meaning of the probabilities in $\mathbf{P}^{\infty}$? (recall that $(\mathbf{P}^n)_{ij} = p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$).

Example

$$\mathbf{P} = \begin{array}{|c|c|}\hline \begin{array}{c|c} \mathbf{P}_1 & \\ \hline & \mathbf{P}_2 \end{array} & \mathbf{0} \\ \mathbf{0} \quad \mathbf{P}_3 & \\ \hline \text{at least one} > 0 & \mathbf{T} \\ \hline \end{array}$$

Theorem *The multiplicity of the eigenvalue $\lambda = 1$ is equal to the number of irreducible closed sets.*

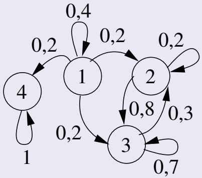
Proof The characteristic polynomial of $\mathbf{P}$ is equal to the product of the characteristic polynomials of the sub-matrices $\mathbf{P}_i$ and $\mathbf{T}$. Since $\mathbf{P}_i$ are irreducible stochastic, each will have a single eigenvalue equal to 1. For the transitorial states it must be $\lim_{n\to\infty} \mathbf{T}^n = \mathbf{0}$. Thus, all the eigenvalues of $\mathbf{T}$ must be $|\lambda| < 1$. □

NOTE: in the closed form solution there is only one unknown associated with $\lambda = 1$, otherwise $\sum_{m=0}^{k_l-1} a_j^{(l,m)} n^m$ will diverge as $n \to \infty$ (i.e. $a_j^{(l,m)} = 0, m > 0$), and $a_j^{(l,0)} = \lim_{n\to\infty} \pi_j(n)$.

Classification of States

Transient and Recurrent

- Recurrent: States that, being visited, they are visited again with probability 1. They are visited an infinite number of times when $n \to \infty$.
- Transient: States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $n \to \infty$.
- Absorbing: A single (recurrent) state where the chain remains with probability $= 1$.

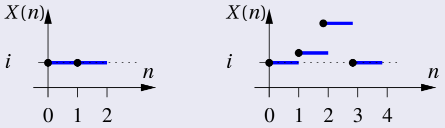


State 1 is transient
States 2 and 3 are recurrent
State 4 is absorbing

First Passage (Transition) Probabilities

- To derive a classification criteria, we shall study the distribution of the number of steps to go for the first time from a state i another state j. Definition:

$$\boxed{f_{ii}(n) = P\begin{pmatrix}\text{first transition into state } i \\ \text{in } n \text{ steps starting from } i\end{pmatrix}}$$



first transition in 1 step   first transition in 3 steps

- Do not confuse with the n-step transition probability $p_{ii}(n)$, where the state i can be visited in the intermediate states.

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

- $f_{ii}(n)$ and $p_{ii}(n)$ satisfy:

$$\boxed{\begin{aligned} f_{ii}(1) &= p_{ii}(1) \\ p_{ii}(n) &= \sum_{l=1}^{n} f_{ii}(l)\, p_{ii}(n-l),\ n >= 1 \end{aligned}}$$

- The probability that the MC eventually enters state i starting from i is given by:

$$\boxed{f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n)}$$

- If $f_{ii} = 1$ we say i is a recurrent state.
- If $f_{ii} < 1$ we say i is a transient state.

Generalization to Any State Pair

- Analogously to $f_{ii}(n)$, we define the probability of the first passage to state j starting from any state i in n steps: $f_{ij}(n)$.
- $f_{ij}(n)$ and $p_{ij}(n)$ satisfy:

$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l)\, p_{jj}(n-l),\ n \geq 1$$

Classification of States

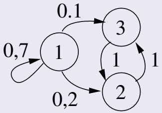
Recursive Equation for the First Passage Probabilities

- Recall that the The probability that the MC eventually enters state j starting from i is given by: $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$
- f_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then we will reach j with probability f_{kj}. Thus:

$$\boxed{f_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}}$$

- If there are more than 1 absorbing states, we can compute the probability to reach them using this method (if there is only 1, say j, then $f_{ij} = 1, \forall i$).

Example: Recurrence Times Using the Definition



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f_{21}(n) = f_{31}(n) = 0$$
$$f_{11}(n) = 0{,}7\, I(n=1)$$
$$f_{22}(n) = f_{33}(n) = I(n=2)$$
$$f_{23}(n) = f_{32}(n) = I(n=1)$$

$$f_{12}(n) = \begin{cases} 0{,}2, & n=1 \\ 0{,}7^{n-1}\, 0{,}2 + 0{,}7^{n-2}\, 0{,}1, & n>1 \end{cases}$$

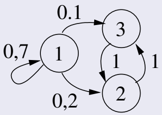
$$f_{13}(n) = \begin{cases} 0{,}1, & n=1 \\ 0{,}7^{n-1}\, 0{,}1 + 0{,}7^{n-2}\, 0{,}2, & n>1 \end{cases}$$

$$f_{11} = 0{,}7$$
$$f_{12} = f_{13} = 1 \qquad f_{22} = f_{23} = 1$$
$$f_{32} = f_{33} = 1 \qquad f_{21} = f_{31} = 0$$

- State 1 is transient. States 2 and 3 are recurrent.

Classification of States

Example: First Passage Probability Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$f_{12} = p_{11} f_{12} + p_{12} + p_{13} f_{32}$$

- Clearly $f_{32} = 1$, thus:

$$f_{12} = 0{,}7 f_{12} + 0{,}2 + 0{,}1 \times 1 \Rightarrow f_{12} = 1$$

as before.

Classification of States

Mean Recurrence Time

- If $f_{ii} = 1$ we say i is a recurrent state.
- If $f_{ii} < 1$ we say i is a transient state.
- When $f_{ii} = 1$, we define the mean recurrence time m_{ii} as:

$$\boxed{m_{ii} = \sum_{n=1}^{\infty} n f_{ii}(n)}$$

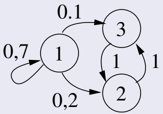
- m_{ii} is the average number of steps to eventually reach i starting from i. If $f_{ii} < 1$ (transient state) then we define $m_{ii} = \infty$.
- Classification of recurrent states ($f_{ii} = 1$):
  - If $m_{ii} = \infty$ the state is null recurrent: it takes an ∞ time to reach the state after leave it. Can only happen in chains with an infinite number of states.
  - If $m_{ii} < \infty$ the state is positive recurrent: the state is reached in a finite time after leave it.

Property of States

In finite MC:

1. States can be only of type positive recurrent or transient.
2. At least one state must be positive recurrent.
3. There are not null recurrent states.

- Example:



- State 1 is transient. States 2 and 3 are positive recurrent.

Classification of States

Generalization to Any State Pair

- When $f_{ij} = 1$, the average number of steps to eventually reach j starting from i, m_{ij} is given by:

$$\boxed{m_{ij} = \sum_{n=1}^{\infty} n f_{ij}(n)}$$

- If state j can not be reached starting from state i with probability one (if $f_{ij} < 1$), then we define $m_{ij} = \infty$.

Recursive Equation for the Mean Recurrence Time

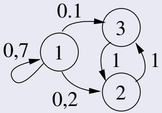
- Recall that the mean recurrence time $m_{ij} = \sum_{n \ge 1} n f_{ij}(n)$ is the average number of steps to eventually reach j starting from i, i.e. it is the mean first passage time from state i to j.
- When $f_{ij} = 1$, m_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then it will take m_{kj} steps to reach j. Thus:

$$\boxed{m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}\,(1 + m_{kj}) = 1 + \sum_{k \neq j} p_{ik}\, m_{kj}}$$

since $\sum_j p_{ij} = 1$.

Classification of States

Example: Mean Recurrence Time Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$m_{12} = p_{12} + p_{11}\,(1 + m_{12}) + p_{13}\,(1 + m_{32}) = 1 + p_{11}\,m_{12} + p_{13}\,m_{32}$$

- Clearly $m_{32} = 1$, thus:

$$m_{12} = 1 + 0{,}7\,m_{12} + 0{,}1 \times 1 \Rightarrow m_{12} = 11/3.$$

Classification of States

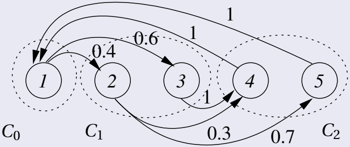
Periodic states

- A recurrent state j is periodic with period $d > 1$ if j can only be reached after leaving it with a multiple of d steps.
- If $d = 1$ the state is aperiodic.
- Any periodic irreducible chain can be partitioned in d cyclic classes $C_0, \cdots C_{d-1}$ such that at each step a transition occur from class C_i to $C_{(i+1) \bmod d}$.
- By properly numerating the states, the transition matrix can be written as (the sub-matrices $\mathbf{A}_i$ may not be square):

$$\mathbf{P} = \begin{array}{c} \\ C_0 \\ C_1 \\ \cdots \\ C_{d-1} \end{array} \begin{array}{c} \begin{array}{ccccc} C_0 & C_1 & C_2 & \cdots & C_{d-1} \end{array} \\ \left[\begin{array}{ccccc} 0 & \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{A}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{d-1} & 0 & 0 & \cdots & 0 \end{array} \right] \end{array}$$

Classification of States

Example



$$\mathbf{P} = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \end{bmatrix}, \mathbf{P}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \end{bmatrix}, \mathbf{P}^4 = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \cdots$$

- In periodic chains $\mathbf{P}^n$ does not converge.

Part II

Discrete Time Markov Chains (DTMC)

Outline

Steady State

Limiting Distribution

- Probability of being in state i at step n:

$$\pi_i(n) = P(X(n) = i) .$$

  In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots).$$

- The evolution of the chain depends on the initial distribution $\boldsymbol{\pi}(0)$.
- If we are interested in the transient evolution we shall study

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n.$$

- If we are interested the steady state we shall be interested in the limiting distribution (if the limit exists):

$$\boldsymbol{\pi}(\infty) = (\pi_1(\infty), \pi_2(\infty), \cdots)$$

Steady State

Limiting Distribution

Assume an irreducible chain with positive recurrent states.

- With infinite steps, we look for a probability converging to a value that depends only on the final state:

$$\pi_j(\infty) = \sum_i \pi_i(0) \lim_{n\to\infty} p_{ij}(n),\ \forall j \text{ and for any } \boldsymbol{\pi}(0),$$

  which implies:

$$\pi_j(\infty) = \lim_{n\to\infty} p_{ij}(n) \sum_i \pi_i(0) = p_{ij}(\infty),\ \forall j \Rightarrow$$

$$\mathbf{P}(\infty) = \lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \boldsymbol{\pi}(\infty) \\ \cdots \\ \boldsymbol{\pi}(\infty) \end{bmatrix}$$

- If this limit exists, we call $\mathbf{P}(\infty)$ the limiting matrix, and $\boldsymbol{\pi}(\infty)$ the limiting distribution.

Steady State

Example

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix} \qquad \mathbf{P}^2 = \begin{bmatrix} 0.764 & 0.168 & 0.068 \\ 0.760 & 0.170 & 0.070 \\ 0.752 & 0.174 & 0.074 \end{bmatrix}$$

$$\mathbf{P}^4 = \begin{bmatrix} 0{,}7628 & 0{,}1686 & 0{,}0686 \\ 0{,}7620 & 0{,}1690 & 0{,}0690 \\ 0{,}7604 & 0{,}1698 & 0{,}0698 \end{bmatrix} \qquad \mathbf{P}^8 = \begin{bmatrix} 0.762500 & 0.168750 & 0.068750 \\ 0.762499 & 0.168750 & 0.068750 \\ 0.762497 & 0.168752 & 0.068752 \end{bmatrix}$$

$\ldots$

$$\Rightarrow \boldsymbol{\pi}(\infty) = (0.76250, 0.16875, 0.06875)$$

Stationary distribution

- We have:

$$\pi_i(n) = P(X(n) = i) = \sum_k P(X(n-1) = k)\, P\big(X(n) = i \,\big|\, X(n-1) = k\big)$$

$$= \sum_k \pi_k(n-1)\, p_{ki}$$

- In matrix form: $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P}$
- If $\pi_i(n) = \pi_i(n-1) = \pi_i \;\forall i$, we call π_i the stationary probability of state i, and $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots)$, the stationary distribution of the chain.
- In matrix form (Global balance equations):

$$\boxed{\begin{array}{l} \boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P} \\ \boldsymbol{\pi}\,\mathbf{e} = 1, \mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}} \end{array}}$$

- Thus, the stationary distribution is the left-hand eigenvector corresponding to the unit eigenvalue of $\mathbf{P}$.
- $\boldsymbol{\pi}(n) = \boldsymbol{\pi} \Rightarrow \boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n)\,\mathbf{P} = \boldsymbol{\pi}\,\mathbf{P} = \boldsymbol{\pi} \Rightarrow \boxed{\boldsymbol{\pi}(k) = \boldsymbol{\pi},\, k \geq n}$

Steady State

Stationary distribution

- Do not confuse the limiting distribution $\boldsymbol{\pi}(\infty)$ and the stationary distribution $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$.
- $\boldsymbol{\pi}(\infty)$ and $\boldsymbol{\pi}$ may not be the same, e.g. in periodic chains $\boldsymbol{\pi}(\infty)$ does not exists ($\mathbf{P}$ does not converge), but we can compute the stationary distribution.
- Example: the periodic chain

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has the stationary distribution

$$\boldsymbol{\pi} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Numerical Solution

- Replace one equation method:

$$\boxed{\begin{aligned}&\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}\\ &\boldsymbol{\pi}\,\mathbf{e} = 1, \mathbf{e} = \begin{bmatrix}1 & 1 & \cdots\end{bmatrix}^{\mathrm{T}}\end{aligned}}$$

- We solve the equation $\boldsymbol{\pi}(\mathbf{I}-\mathbf{P}) = 0$ replacing the last equation by $\boldsymbol{\pi}\,\mathbf{e} = 1$:

$$\boxed{\boldsymbol{\pi}\begin{bmatrix} p_{11}-1 & p_{12} & \cdots p_{1n-1} & 1\\ p_{21} & p_{22}-1 & \cdots p_{2n-1} & 1\\ \vdots & \vdots & \vdots & \vdots\\ p_{n1} & p_{n2} & \cdots p_{nn-1} & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}}$$

Steady State

Numerical Solution

- Replace one equation method: $\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$
- With octave (matlab clone):

```
octave:1> P=[0.8,0.15,0.05;0.7,0.2,0.1;0.5,0.3,0.2];
octave:2> s=size(P,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [eye(s,s-1)-P(1:s,1:s-1), ones(s,1)]
ans =
0.762500   0.168750   0.068750
```

- With R

```
> P <- matrix(nc=3, byr=T, c(0.8,0.15,0.05,0.7,0.2,0.1,0.5,0.3,0.2))
> s <- nrow(P)
> solve(t(cbind(P[,1:(s-1)]-diag(nr=s,nc=s-1), rep(1,s))),
+ c(rep(0,s-1),1))
[1] 0.76250 0.16875 0.06875
```

NOTE: $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \Rightarrow \boldsymbol{\pi}^T = \mathbf{P}^T\boldsymbol{\pi}^T$. The transpose operator in R is `t()`.

Steady State

Ergodic Chains

Ergodic state positive recurrent and aperiodic state.
Ergodic chain if all states are ergodic.
Theorem: All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent, and aperiodic/periodic with the same period [1, chapter XV].
Consequences:

- Finite aperiodic and irreducible chains are ergodic (since all states are positive recurrent).
- Infinite aperiodic and irreducible chains can be:
  - Ergodic: all the states are positive recurrent (stable chains).
  - Non ergodic: all states are null recurrent or transient (unstable chains).

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition*. Wiley, 1968.

Theorems for ergodic chains

- Both stationary and limiting ditribution exist and are equal, $\boldsymbol{\pi} = \boldsymbol{\pi}(\infty)$.
- In stationary regime (when $\boldsymbol{\pi}(n)\,\mathbf{P} = \boldsymbol{\pi}(n)$), the mean number of steps the system remains in state j during k steps is given by

$$\boxed{k\pi_j}$$

thus, π_j is the average fraction of a step the chain remains in state j in stationary regime.
- In stationary regime the mean recurrence time (mean number of steps between two consecutive visits to state j) is given by

$$\boxed{m_{jj} = 1/\pi_j}$$

The last properties are also valid for periodic chains.

Theorems for ergodic chains (proofs)

- Both stationary and limiting ditribution exist and are equal, $\boldsymbol{\pi} = \boldsymbol{\pi}(\infty)$.
- Proof

  For an aperiodic irreducible chain with positive recurrent states:

$$\begin{cases} \boldsymbol{\pi}(\infty) & = \boldsymbol{\pi}(0)\,\mathbf{P}(\infty) \\ \mathbf{P}(\infty) & = \lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \boldsymbol{\pi}(\infty) \\ \cdots \\ \boldsymbol{\pi}(\infty) \end{bmatrix} \end{cases} \Rightarrow$$

$$\boldsymbol{\pi}(\infty)\,\mathbf{P} = (\boldsymbol{\pi}(0) \lim_{n\to\infty} \mathbf{P}^n)\,\mathbf{P} = \boldsymbol{\pi}(0)\,\mathbf{P}(\infty) = \boldsymbol{\pi}(\infty)$$

$$\Rightarrow \begin{cases} \boldsymbol{\pi}(\infty)\,\mathbf{P} = \boldsymbol{\pi}(\infty) \\ \boldsymbol{\pi}(\infty)\,\mathbf{e} = 1 \end{cases} \quad \boldsymbol{\pi}(\infty) \text{ satisfies the GBE} \Rightarrow \boldsymbol{\pi} = \boldsymbol{\pi}(\infty) \quad \square$$

Theorems for ergodic chains (proofs)

- In stationary regime (when $\boldsymbol{\pi}(n)\,\mathbf{P} = \boldsymbol{\pi}(n)$), the mean number of steps the system remains in state j during k steps is given by

$$\boxed{k\pi_j.}$$

- Proof

Assume the chain in stationary regime at time $t = 0$ ($\pi(0)\,\mathbf{P} = \boldsymbol{\pi}(0)$), and let $j(k)$ be the number of visits to j in k steps: $j(k) = \sum_{i=0}^{k-1} I(X(i) = j)$ ($I(A)$ is the indicator function: $I(A) = 1$ if A occurs, $I(A) = 0$ otherwise):

$$\mathrm{E}[j(k)] = \sum_{i=0}^{k-1} \mathrm{E}[I(X(i) = j)] = \sum_{i=0}^{k-1} P(X(i) = j) = k\pi_j \qquad \square$$

Theorems for ergodic chains (proofs)

- In stationary regime the mean recurrence time (mean number of steps between two consecutive visits to state j) is given by

$$\boxed{m_{jj} = 1/\pi_j}$$

- Proof

  Let $j(k)$ be the number of visits to j in k steps:

$$\pi_j = \lim_{k\to\infty} \frac{j(k)}{k} = \lim_{k\to\infty} \frac{1}{k/j(k)} = 1/m_{jj} \quad \square$$

Steady State

Global balance equations

- Why are they called Global balance equations?

$$\left.\begin{array}{l} \boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P} \Rightarrow \quad \pi_j = \sum_{i=0}^{\infty} \pi_i\, p_{ij} \\ \sum_{i=0}^{\infty} p_{ji} = 1 \Rightarrow \quad \pi_j \sum_{i=0}^{\infty} p_{ji} = \pi_j \end{array}\right\} \Rightarrow \sum_{i=0}^{\infty} \pi_i\, p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji}$$

$\sum_{i=0}^{\infty} \pi_i\, p_{ij}$ $\Rightarrow$ Frequency of transitions entering state j

$\pi_j \sum_{i=0}^{\infty} p_{ji}$ $\Rightarrow$ Frequency of transitions leaving state j

- In stationary regime, the frequency of transitions leaving state j is equal to the frequency of transitions entering state j.

Flux Balancing

- Define the flux F_{uv} from state u to v:

$$F_{uv} = \pi_u p_{uv}$$

- and the flux from set of states U to V:

$$F(U,V) = \sum_{u \in U} \sum_{v \in V} F_{uv}$$

- From the Global balance equations we have:

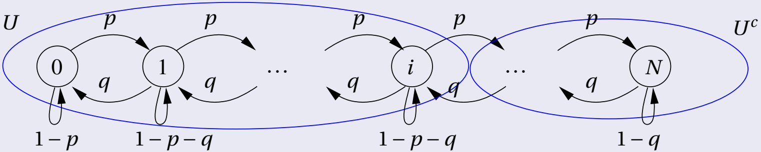
$$\sum_{i=0}^{\infty} \pi_i p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji} \Rightarrow \sum_{i \in U} F_{ij} + \sum_{i \notin U} F_{ij} = \sum_{i \in U} F_{ji} + \sum_{i \notin U} F_{ji}$$

- Adding for $j \in U$:

$$\sum_{j \in U} \sum_{i \in U} F_{ij} + \sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \in U} F_{ji} + \sum_{j \in U} \sum_{i \notin U} F_{ji} \Rightarrow \sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \notin U} F_{ji}$$

$$\Rightarrow \boxed{F(U,U^c) = F(U^c,U)}$$

Solution Using Flux Balancing



- Flux balancing $\Rightarrow p\pi_i = q\pi_{i+1}$
- Iterating: $\pi_1 = \rho\,\pi_0, \pi_2 = \rho\,\pi_1 = \rho\,\rho\,\pi_0, \cdots, \Rightarrow$

$$\boxed{\pi_i = \rho^i \pi_0,\ i = 1, \cdots, N} \quad \text{where: } \rho = \frac{p}{q},$$

- Normalizing: $\sum_{i=0}^{N} \pi_i = 1$

$$\boxed{\begin{array}{ll} \pi_0 = \dfrac{1-\rho}{1-\rho^{N+1}}, & p \neq q \\ \pi_0 = \dfrac{1}{N+1}, & p = q \end{array}}$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Reversed Chain

Definition

- Let $X(n)$ be an ergodic MC. The chain $X^r(n) = X(-n)$ is referred to as the time reversal chain of $X(n)$.
- Example, consider a possible sample path of $X(n)$:

$$\cdots (i_0, n_0), (i_1, n_1), (i_2, n_2), \cdots$$

The same path in the time reversal chain $X^r(n)$ would be:

$$\cdots (i_2, -n_2), (i_1, -n_1), (i_0, -n_0), \cdots$$

Properties

- Let p_{ij}, p^r_{ij} be the transition probabilities of $X(n)$ respectively $X^r(n)$, and π_i, π^r_i the stationary distributions of $X(n)$ respectively $X^r(n)$, then:

$$\boxed{\pi_i = \pi^r_i}$$

- Proof: the mean time in each state is the same for both chains. $\square$
- However, in general $p_{ij} \neq p^r_{ij}$. For example, $X(n)$ may be able to jump from state i to j, but not from j to $i \Rightarrow X^r(n)$ can jump from j to i, but not from i to j.
- But it must be $p_{ii} = p^r_{ii}$, since self-state transitions are the same in the direct and reversed chains.

Computation of p^r_{ij}

- The transition probabilities in the time reversal chain (p^r_{ji}) satisfy:

$$\boxed{\pi_i p_{ij} = \pi_j p^r_{ji}}$$

- Proof Assume the chain in steady state. We have:

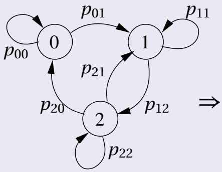
$$P\{X(n+1) = j, X(n) = i\} = P\{X^r(-n) = i, X^r(-n-1) = j\} =$$
$$P\{X^r(n+1) = i, X^r(n) = j\} \Rightarrow$$
$$P\{X(n) = i\}\, P\{X(n+1) = j \mid X(n) = i\} =$$
$$P\{X(n) = j\}\, P\{X^r(n+1) = i \mid X^r(n) = j\} \Rightarrow \pi_i p_{ij} = \pi_j p^r_{ji}. \quad \square$$

- We can compute p^r_{ji} using the reversed balance equations:
$\pi_i p_{ij} = \pi_j p^r_{ji} \Rightarrow \sum_{i\in U} \sum_{j\in V} \pi_i p_{ij} = \sum_{i\in U} \sum_{j\in V} \pi_j p^r_{ji} \Rightarrow$

$$\boxed{F(U,V) = F^r(V,U)}$$

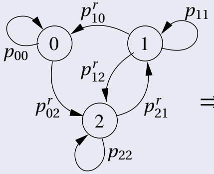
Reversed Chain

Example



$$\Rightarrow \begin{cases} \pi_0 = \dfrac{p_{12}\, p_{20}}{p_{12}\, p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\, p_{12}} \\ \pi_1 = \dfrac{p_{01}\,(p_{20} + p_{21})}{p_{12}\, p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\, p_{12}} \\ \pi_2 = \dfrac{p_{01}\, p_{12}}{p_{12}\, p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\, p_{12}} \end{cases}$$

- Time reversal chain:



$$\Rightarrow \begin{cases} \pi_0\, p_{01} = \pi_1\, p^r_{10} \\ \pi_1\, p_{12} = \pi_2\, p^r_{21} \\ \pi_2\, p_{21} = \pi_1\, p^r_{12} \\ \pi_2\, p_{20} = \pi_0\, p^r_{02} \end{cases} \Rightarrow \begin{cases} p^r_{10} = \dfrac{p_{12}\, p_{20}}{p_{20} + p_{21}} \\ p^r_{21} = p_{20} + p_{21} \\ p^r_{12} = \dfrac{p_{12}\, p_{21}}{p_{20} + p_{21}} \\ p^r_{02} = p_{01} \end{cases}$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Reversible Chains

Definition

- A chain is reversible if:

$$\boxed{p_{ij} = p^r_{ij}}$$

- This equality implies the reversibility balance equations:

$$\boxed{\pi_i p_{ij} = \pi^r_i p^r_{ij} \Rightarrow F(U,V) = F^r(U,V)}$$

- Using both reversed ($F^r(U,V) = F(V,U)$) and reversibility balance equations, the following relation holds for a reversible chain (detailed balance equations):

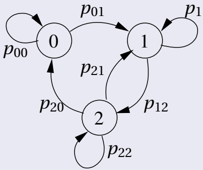
$$\boxed{F(U,V) = F(V,U)}$$

- NOTE: Compare with the global balance equations: $F(U,U^C) = F(U^C,U)$.

Notation

Definition of path

- Define a path as a possible sequence of transitions of the chain. For example, in the figure it could be $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$.



- We denote the sequence of states of one path l as:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m)$$

- For instance, if l is $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$, then $(l,1) = 0$, $(l,2) = 0$, $(l,3) = 1$, $(l,4) = 2$.
- We say that a path is closed if it starts and ends in the same state. For instance, a path stating and ending in state $(l,1)$:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m) \rightsquigarrow (l,1)$$

Reversible Chains

Kolmogorov Criteria

- Take a closed path l with $m \geq 0$ transitions, i.e.:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m) \rightsquigarrow (l,1),\ m \geq 0$$

- The chain is reversible iff for all l:

$$\boxed{\begin{aligned} p_{(l,1)(l,2)}\, p_{(l,2)(l,3)} \cdots p_{(l,m)(l,1)} &= \\ p_{(l,1)(l,m)}\, p_{(l,m)(l,m-1)} \cdots p_{(l,2)(l,1)} & \end{aligned}}$$

- Proof:
  - If the chain is reversible $\pi_i p_{ij} = \pi_j p_{ji}$ (detailed balance equations): $\Rightarrow \pi_{(l,k)}\, p_{(l,k)(l,k+1)} = \pi_{(l,k+1)}\, p_{(l,k+1)(l,k)}$
  - Multiplying for $k = 1, 2 \cdots m$ and simplifying we obtain the previous relation. $\square$

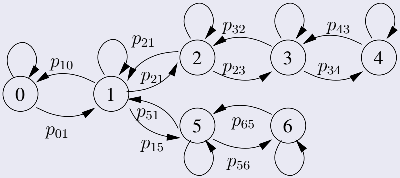
Reversible Chains

Kolmogorov Criteria. Corollary

- A reversible chain must satisfy:

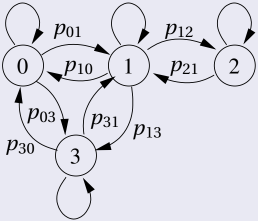
$$\boxed{\begin{aligned} p_{ij} > 0 &\Rightarrow p_{ji} > 0 \\ p_{ij} = 0 &\Rightarrow p_{ji} = 0 \end{aligned}}$$

- An ergodic tree chain is always reversible. We define a tree chain as chain having no cycles, i.e. the only possible transitions are bidirectional arcs between states, and self transitions.



Reversible Chains

Kolmogorov Criteria. Example



- The chain is reversible iif:

$$\boxed{p_{01}\,p_{13}\,p_{30} = p_{10}\,p_{03}\,p_{31}}$$

Product Form Solution

- Let $X(n)$ be a reversible MC with space state $S \Rightarrow$ the stationary probabilities of $X(n)$ can be computed as follows:
- Choose a state $s \in S$,
- For every other state $i \in S$, $i \neq s$ look for a possible path l_i from state s to state i:

$$s = (l_i,1) \rightsquigarrow (l_i,2) \rightsquigarrow \cdots \rightsquigarrow (l_i,m_{l_i}) = i,\; m_{l_i} \geq 1$$

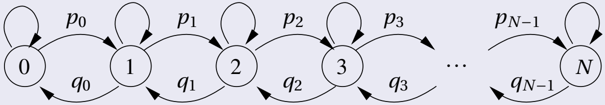
- The stationary probabilities are given by:

$$\boxed{\pi_i = \frac{\psi_i}{\sum_{j\in S}\psi_j},\; i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \prod_{k=1}^{m_{l_i}-1} \frac{p_{(l_i,k)(l_i,k+1)}}{p_{(l_i,k+1)(l_i,k)}}, & i \neq s \end{cases}}$$

- Proof Use the detailed balance equations $\pi_i p_{ij} = \pi_j p_{ji}$. □

Reversible Chains

Birth and Death Chains



- Birth and death chains are reversible.
- Applying the product form solution for reversible chains we obtain the general solution of birth death chains. Choosing $s = 0$:

$$\boxed{\pi_i = \frac{\psi_i}{\sum_{j=0}^{N} \psi_j},\ i \geq 0 \quad \text{where } \psi_i = \begin{cases} 1, & i = 0 \\ \prod_{k=0}^{i-1} \frac{p_k}{q_k}, & i = 1, \cdots, N \end{cases}}$$

Truncated Reversible Chain

- Consider a reversible MC X with a stationary distribution π_i.
- Suppose that we truncate the chain X and we obtain another irreducible chain X'.
- Then, X' is also reversible with stationary distribution:

$$\boxed{\pi'_i = \frac{\pi_i}{G}, \quad \sum_k \pi'_k = 1}$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Research Example: Aloha

Access Protocol (see the paper of Kleinrock and Lam [1])

- Pure Aloha:
  - Broadcast radio system.
  - Single hop system (all stations are in coverage).
  - Whenever a station has a frame ready, it is transmitted.
  - If two or more frames Tx overlap in time there is a collision, otherwise the frame is received correctly.
  - Colliding frames are reTx after a random time (backoff).
- Slotted Aloha:
  - Time is slotted.
  - Tx can only occur at the beginning of a slot.
  - Collisions occur when 2 or more stations Tx in the same slot.

[1] Leonard Kleinrock and Simon Lam. "Packet Switching in a Multiaccess Broadcast Channel: Performance Evaluation". In: *Communications, IEEE Transactions on* 23.4 (1975), pp. 410–423.

Research Example: Aloha

Analysis with finite population

Assumptions

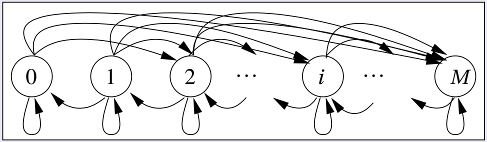
- Slotted Aloha.
- Acks are sent immediately after the reception of a frame, and are never lost.
- M nodes with a buffer of 1 frame.
- The nodes can be in 2 states:
  - Thinking: when the buffer is empty
  - Backlogged: when there is a frame in the buffer.
- A thinking node generate one frame in each slot with probability σ. When a frame collides, the frame is stored and the node becomes backlogged.
- A backlogged node ReTx the frame in each slot with probability ν.

Research Example: Aloha

Markov Chain

- The system state, $X(n)$, is the number of backlogged nodes:

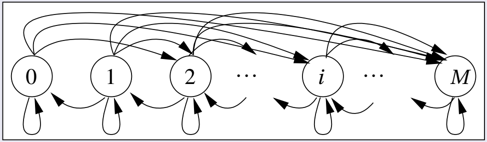
$$\boxed{p_{ij} = P\big(X(n) = j \text{ baklogged} \mid X(n-1) = i \text{ baklogged}\big)}$$



Research Example: Aloha

Transition probabilities

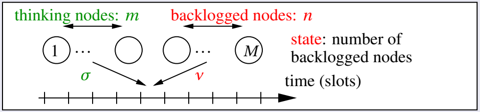
$$p_{ij} = P\big(X(n) = j \text{ baklogged} \mid X(n-1) = i \text{ baklogged}\big)$$



- 0 for $j < i - 1$.
- for $j = i - 1$: no thinking Tx and only 1 backlogged Tx.
- for $j = i$:
  1. no thinking Tx and none or more than 1 backlogged Tx,
  2. only 1 thinking Tx and no backlogged Tx.
- for $j = i + 1$: 1 thinking and 1 or more backlogged Tx.
- for $j > i + 1$: $j - i$ thinking Tx, regardless of backlogged Tx.

Research Example: Aloha

Transition probabilities



- Define the probabilities:
  - Arrivals: $Q_a(m,n)$, Probability of m thinking nodes Tx in a slot given that n nodes are backlogged:

$$Q_a(m,n) = P\left(\begin{array}{c} m \text{ think.} \\ \text{nodes Tx} \end{array} \,\middle|\, \begin{array}{c} n \text{ nodes are} \\ \text{backlogged} \end{array}\right) = \binom{M-n}{m} \sigma^m (1-\sigma)^{M-n-m}$$

  - Retransmissions: $Q_r(m,n)$, Probability of m backlogged nodes Tx in a slot given that n nodes are backlogged:

$$Q_r(m,n) = P\left(\begin{array}{c} m \text{ backl.} \\ \text{nodes Tx} \end{array} \,\middle|\, \begin{array}{c} n \text{ nodes are} \\ \text{backlogged} \end{array}\right) = \binom{n}{m} \nu^m (1-\nu)^{n-m}$$

Research Example: Aloha

Transition probabilities

- 0 for $j < i - 1$.
- for $j = i - 1$: no thinking Tx and only 1 backlogged Tx.
- for $j = i$:
  1. no thinking Tx and none or more than 1 backlogged Tx,
  2. only 1 thinking Tx and no backlogged Tx.
- for $j = i + 1$: 1 thinking and 1 or more backlogged Tx.
- for $j > i + 1$: $j - i$ thinking Tx, regardless of backlogged Tx.

$$p_{ij} = \begin{cases} 0, & j < i-1 \\ Q_a(0,i)\, Q_r(1,i), & j = i-1 \\ Q_a(0,i)\,(1 - Q_r(1,i)) + Q_a(1,i)\, Q_r(0,i), & j = i \\ Q_a(1,i)\,(1 - Q_r(0,i)), & j = i+1 \\ Q_a(j-i,i), & j > i+1 \end{cases}$$

Research Example: Aloha

Stationary distribution

- Solving the global balance equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}$$

$$\boldsymbol{\pi}\,\mathbf{e} = 1$$

- We obtain the probability of having i backlogged nodes:

$$\pi_i = P\left(i \text{ backlogged nodes}\right)$$

NOTE: there is no closed form solution of the chain. The matrix $\mathbf{P}$ must be constructed using the expression of p_{ij}, and solved numerically.

Research Example: Aloha

Throughput

- Define the probabilities:

$$P_{succ}(i) = P\big(\text{successful Tx} \mid i \text{ backlogged}\big)$$

- The normalized throughput, i.e. proportion of steps with a successful transmission) is:

$$S = \sum_{i=0}^{M} P\big(\text{successful Tx} \mid i \text{ backlogged}\big)\, P\big(i \text{ backlogged}\big) =$$

$$\sum_{i=0}^{M} P_{succ}(i)\, \pi_i$$

- For a slot to be successful: (i) 1 thinking Tx and no backlogged Tx, or (ii) no thinking Tx and 1 backlogged Tx:

$$P_{succ}(i) = Q_a(1,i)\, Q_r(0,i) + Q_a(0,i)\, Q_r(1,i)$$

Research Example: Aloha

Notes on the throughput

$$S = \sum_{i=0}^{M} P_{succ}(i)\,\pi_i$$

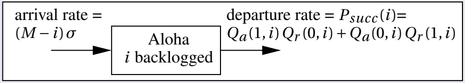
- For the special case $\sigma = \nu$ (thinking Tx with the same probability as backlogged): $P_{succ}(i) = M\sigma(1-\sigma)^{M-1}$, which does not depend on i, thus: $S = M\sigma(1-\sigma)^{M-1}$.
- The offered load (i.e. proportion of arrivals per slot) G is now: $G = M\sigma$, thus:

$$S = G\left(1-\frac{G}{M}\right)^{M-1} \Rightarrow \lim_{M\to\infty} S = Ge^{-G}$$

- We conclude that the infinite population model is the limit of the finite population if backlogged Tx with the same probability as thinking, and $M \to \infty$.

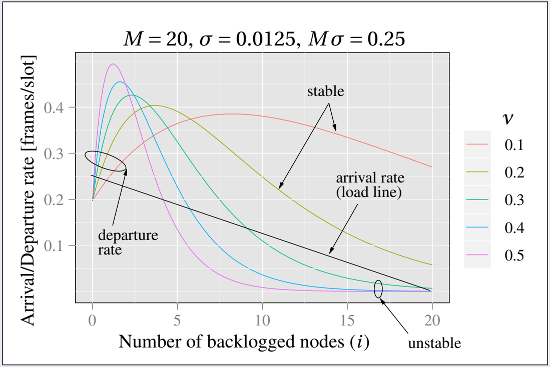
Research Example: Aloha

Dynamics



Note on the arrival rate (expected value of a binomial distribution):

$$\sum_{k=0}^{M-i} k \binom{M-i}{k} \sigma^k (1-\sigma)^{M-i-k} = (M-i)\,\sigma$$



Solving the chain:

$$S = \sum_{i=0}^{M} P_{succ}(i)\,\pi_i$$

ν	S
0.1	2.38e-01
0.2	2.42e-01
0.3	1.30e-02
0.4	4.98e-04
0.5	1.90e-05

Research Example: Aloha

Stabilizing Aloha

- The retransmission probabilities must adapt in accordance with the state of the system.
- Example: binary exponential backoff (ethernet). The retransmission rate at retransmission i is adapted as $\nu = 2^{-i}$. Thus, the higher are the number of retransmission trials i, the lower (exponentially) is the retransmission rate.

Part II

Discrete Time Markov Chains (DTMC)

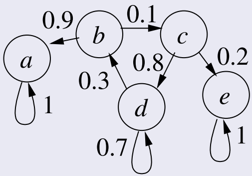
Outline

Finite Absorbing Chains

Canonical Form (from the book of Kemeny and Snell [1])

- Let $\mathbf{P}^{rxr}$ be the transition probability matrix of a chain with r states: s transient states and $r-s$ absorbing states. We can write $\mathbf{P}^{rxr}$ in the canonical form:

$$\boxed{\mathbf{P}^{rxr} = \begin{bmatrix} \mathbf{Q}^{s\times s} & \mathbf{R}^{s\times r-s} \\ \mathbf{0}^{r-s\times s} & \mathbf{I}^{r-s\times r-s} \end{bmatrix}}$$



$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccccc} b & c & d & a & e \end{array} \\ \begin{array}{c} b \\ c \\ d \\ a \\ e \end{array} & \left[\begin{array}{ccc|cc} 0 & 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

[1] John G Kemeny and James Laurie Snell. *Finite Markov Chains*. Springer-Verlag, 1976.

Finite Absorbing Chains

Results

- Define:

$$n_{ij} = \left\{ \begin{matrix} \text{number of steps in state } j \text{ before} \\ \text{absorption, starting from state } i \end{matrix} \right\},$$

$$t_i = \left\{ \begin{matrix} \text{number of steps in transient states before} \\ \text{absorption, starting from state } i \end{matrix} \right\},$$

$$b_{ij} = P\left(\text{probability to be absorbed } j \text{ starting } i\right)$$

- Then:

$$\boxed{\begin{array}{ll} \{\mathrm{E}[n_{ij}]\} = \mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}, & \{\mathrm{Var}[n_{ij}]\} = \mathbf{N}\,(2\mathbf{N}_{\text{diag}} - \mathbf{I}) - \mathbf{N}_{\text{sqr}} \\ \{\mathrm{E}[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\,\mathbf{e}, & \{\mathrm{Var}[t_i]\} = (2\,\mathbf{N} - \mathbf{I})\,\boldsymbol{\tau} - \boldsymbol{\tau}_{\text{sqr}} \\ \{b_{ij}\} = \mathbf{B} = \mathbf{N}\,\mathbf{R}. & \end{array}}$$

where $\{a_{ij}\}$ is a matrix with a_{ij} as element ij and $\mathbf{e}$ is a column vector of 1s. $\mathbf{N}$ is called the fundamental matrix.

Finite Absorbing Chains

Proof

- $\boxed{\{\mathrm{E}[n_{ij}]\} = \mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}}$

$$\mathrm{E}[n_{ij}] = \sum_{k \in A} p_{ik} \delta_{ij} + \sum_{k \in T} p_{ik} \mathrm{E}[n_{kj} + \delta_{ij}] = \delta_{ij} + \sum_{k \in T} p_{ik} \mathrm{E}[n_{kj}]$$

$$\Rightarrow \{\mathrm{E}[n_{ij}]\} = \mathbf{N} = \mathbf{I} + \mathbf{Q}\mathbf{N} \Rightarrow \mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1} \qquad \square$$

where A is the set of absorbing states and T is the set of transient states.

Notation: $\delta_{ij} = I(i = j) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$

Finite Absorbing Chains

Proof

- $\boxed{\{\mathrm{Var}[n_{ij}]\} = \mathbf{N}\,(2\,\mathbf{N}_{\text{diag}} - \mathbf{I}) - \mathbf{N}_{\text{sqr}}}$

$$\mathrm{Var}[n_{ij}] = \mathrm{E}[n_{ij}^2] - \mathrm{E}[n_{ij}]^2 \Rightarrow \{\mathrm{Var}[n_{ij}]\} = \{\mathrm{E}[n_{ij}^2]\} - \mathbf{N}_{\text{sqr}}$$

$$\mathrm{E}[n_{ij}^2] = \sum_{k\in A} p_{ik}\,\delta_{ij}^2 + \sum_{k\in T} p_{ik}\,\mathrm{E}[(n_{kj}+\delta_{ij})^2] =$$

$$\sum_{k\in A} p_{ik}\,\delta_{ij} + \sum_{k\in T} p_{ik}(\mathrm{E}[n_{kj}^2] + 2\,\mathrm{E}[n_{kj}]\,\delta_{ij} + \delta_{ij}) =$$

$$\delta_{ij} + \sum_{k\in T} (p_{ik}\mathrm{E}[n_{kj}^2] + 2\,p_{ik}\,\mathrm{E}[n_{kj}]\,\delta_{ij}) \Rightarrow$$

$$\{\mathrm{E}[n_{ij}^2]\} = \mathbf{I} + \mathbf{Q}\,\{\mathrm{E}[n_{ij}^2]\} + 2\,(\mathbf{Q}\,\mathbf{N})_{\text{diag}} =$$

$$(\mathbf{I}-\mathbf{Q})^{-1}(\mathbf{I} + 2(\mathbf{Q}\,\mathbf{N})_{\text{diag}}) =$$

$$\mathbf{N}(\mathbf{I} + 2(\mathbf{N}-\mathbf{I})_{\text{diag}}) = \mathbf{N}(2\,\mathbf{N}_{\text{diag}} - \mathbf{I}) \qquad \square$$

Finite Absorbing Chains

Proof

- $\boxed{\{\mathrm{E}[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\,\mathbf{e}}$

$$\mathrm{E}[t_i] = \sum_{k\in T} \mathrm{E}[n_{ij}]\} \Rightarrow \{E[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\,\mathbf{e} \quad \square$$

- $\boxed{\{\mathrm{Var}[t_i]\} = (2\,\mathbf{N} - \mathbf{I})\,\boldsymbol{\tau} - \boldsymbol{\tau}_{\mathrm{sqr}}}$

$$\mathrm{Var}[t_i] = \mathrm{E}[t_i^2] - \mathrm{E}[t_i]^2 \Rightarrow \{\mathrm{Var}[t_i]\} = \{\mathrm{E}[t_i^2]\} - \tau_{\mathrm{sqr}}$$

$$\mathrm{E}[t_i^2] = \sum_{k\in A} p_{ik} + \sum_{k\in T} p_{ik}\,\mathrm{E}[(t_k+1)^2] =$$

$$\sum_{k\in A} p_{ik} + \sum_{k\in T} p_{ik}\,(\mathrm{E}[t_k^2] + 2\,\mathrm{E}[t_k] + 1) =$$

$$1 + \sum_{k\in T} (p_{ik}\,\mathrm{E}[t_k^2] + 2\,p_{ik}\,\mathrm{E}[t_k]) \Rightarrow$$

$$\{E[t_i^2]\} = \mathbf{e} + \mathbf{Q}\,\{\mathrm{E}[t_i^2]\} + 2\,\mathbf{Q}\,\tau = (\mathbf{I}-\mathbf{Q})^{-1}(\mathbf{e} + 2\,\mathbf{Q}\,\tau) =$$

$$\mathbf{N}\,(\mathbf{e} + 2\,\mathbf{Q}\,\tau) = \tau + 2\,\mathbf{N}\,\mathbf{Q}\,\tau = \tau + 2\,(\mathbf{N}-\mathbf{I})\tau = (2\,\mathbf{N}-\mathbf{I})\tau \quad \square$$

Finite Absorbing Chains

Proof

- $\boxed{\{b_{ij}\} = \mathbf{B} = \mathbf{N}\,\mathbf{R}}$

$$b_{ij} = p_{ij} + \sum_{k\in T} p_{ik}\, b_{kj},\ j \in A \Rightarrow$$

$$\{b_{ij}\} = \mathbf{B} = \mathbf{R} + \mathbf{Q}\,\mathbf{B}$$

$$\Rightarrow \mathbf{B} = (\mathbf{I} - \mathbf{Q})^{-1}\,\mathbf{R} = \mathbf{N}\,\mathbf{R}. \qquad \square$$

Finite Absorbing Chains

Extension of the Results

- The previous results can be generalized to any group of states of $\mathbf{P}$:
- A set S is referred to as open if the chain can reach some state of S^c starting from any state of S. Let

$$\mathbf{Q} = \{p_{ij}, i \in S, j \in S\}$$

$$\mathbf{R} = \{p_{ij}, i \in S, j \in S^c\}$$

Let assume that the process starts from $i \in S$. Define:

$$n_{ij} = \left\{ \begin{matrix} \text{number of steps in state } j \text{ before} \\ \text{leaving } S \text{, starting from state } i \end{matrix} \right\},$$

$$\Rightarrow \boxed{\{\mathrm{E}[n_{ij}]\} = \mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}.}$$

- Similarly for the other results, e.g. $\boldsymbol{\tau} = \{\mathrm{E}[t_i]\} = \mathbf{N}\,\mathbf{e}$ and $\mathbf{B} = \{b_{ij}\} = \mathbf{N}\,\mathbf{R}$.

Finite Absorbing Chains

Computing the Inverse Using Cofactors

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^{\mathrm{T}}$$

where $\mathbf{C}^{\mathrm{T}}$ is the transposed cofactor matrix: $c_{ij} = (-1)^{i+j} \det \mathbf{M}_{ij}$, and $\mathbf{M}_{ij}$ are the minor matrices obtained removing the row i and column j from $\mathbf{A}$.

Computing the Inverse Using Gaussian Elimination

Do the transformation:

$$[\mathbf{A} \mid \mathbf{I}] \to [\mathbf{I} \mid \mathbf{A}^{-1}]$$

using the elementary row operations:

- Swapping two rows.
- Multiplying a row by a nonzero number.
- Adding a multiple of one row to another row.