The Dual Simplex Method

Combinatorial Problem Solving (CPS)

Javier Larrosa   Albert Oliveras   Enric Rodríguez-Carbonell

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Basic Idea

- **Abuse of terminology:**

  Henceforth sometimes by "optimal" we will mean
  "satisfying the optimality conditions"

  If not explicit, the context will disambiguate

- The algorithm as explained so far is known as primal simplex:

  starting with feasible basis,

  find optimal basis (= satisfying optimality conds.) while keeping feasibility

- There is an alternative algorithm known as dual simplex:

  starting with optimal basis (= satisfying optimality conds.),

  find feasible basis while keeping optimality

Basic Idea

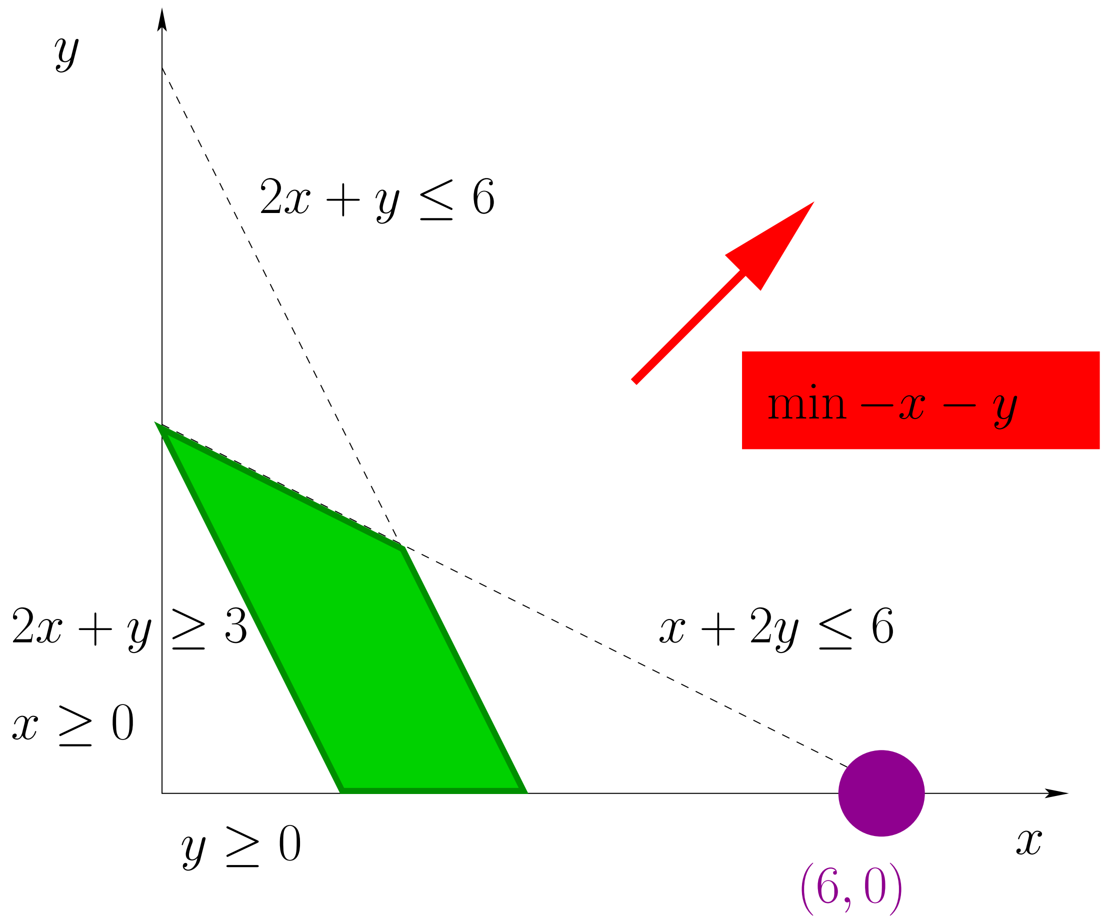
$$\begin{cases} \min \ -x - y \\ 2x + y \geq 3 \\ 2x + y \leq 6 \\ x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases} \implies \begin{cases} \min \ -x - y \\ 2x + y - s_1 = 3 \\ 2x + y + s_2 = 6 \\ x + 2y + s_3 = 6 \\ x, y, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\begin{cases} \min -6 \ +y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases}$$

Basis (x, s_1, s_2) is optimal
($=$ satisfies optimality conditions)

but is not feasible!

Basic Idea



Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase s_2 by making it non-basic: then it will be 0
- ... while preserving optimality ($=$ optimality conditions are satisfied)
  - If y replaces s_2 in the basis,
    then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$
  - If s_3 replaces s_2 in the basis,
    then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$

Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase s_2 by making it non-basic: then it will be 0
- ... while preserving optimality ($=$ optimality conditions are satisfied)
  - If y replaces s_2 in the basis,
    then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$
  - If s_3 replaces s_2 in the basis,
    then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$
  - To preserve optimality, y must replace s_2

Basic Idea

$$\begin{cases} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases} \Longrightarrow \begin{cases} \min\ -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\ x = 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\ y = 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\ s_1 = 3 - s_2 \end{cases}$$

Basic Idea

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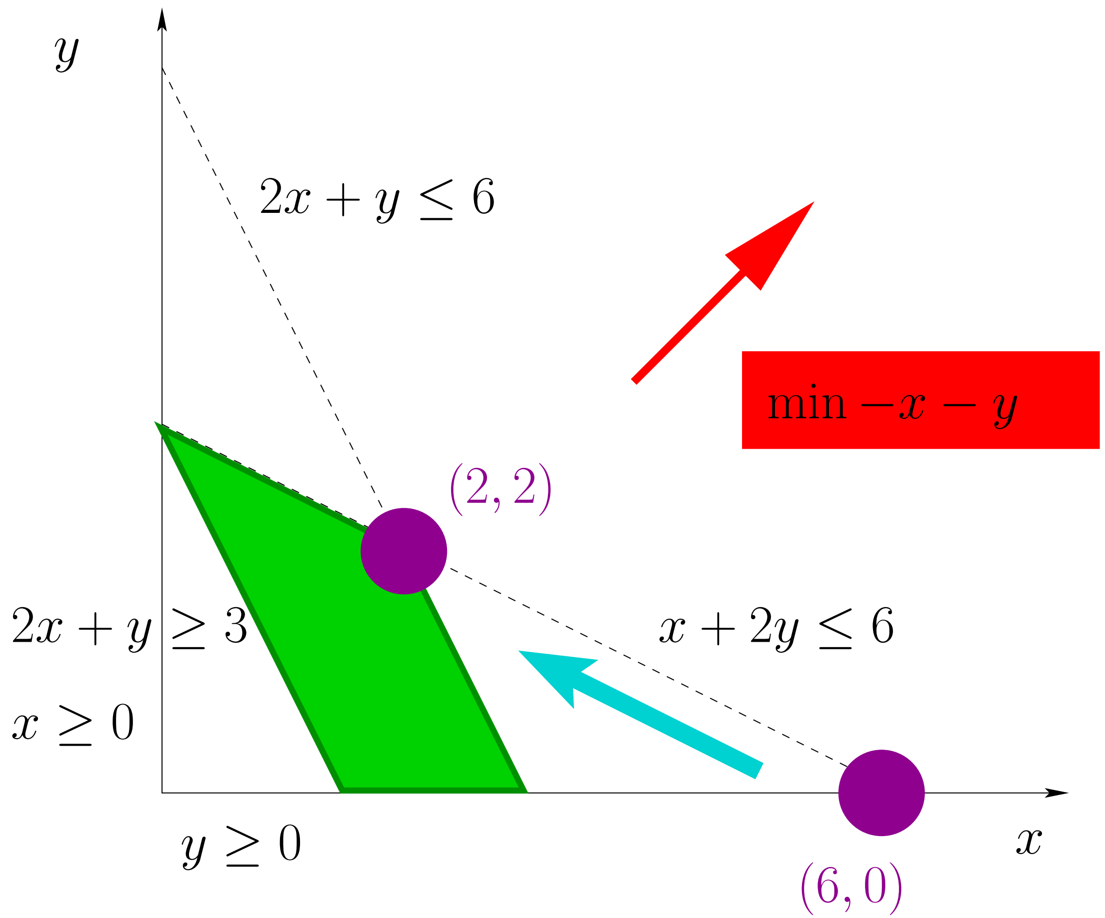
- Current basis is feasible and optimal!

Basic Idea



Outline of the Dual Simplex

1. `Initialization:` Pick an optimal basis.

2. `Dual Pricing:` If all basic values are ≥ 0,
   then return OPTIMAL.
   Else pick a basic variable with value < 0.

3. `Dual Ratio test:` Find non-basic variable for swapping that preserves optimality, i.e., non-negativity constraints on reduced costs.

   If it does not exist,
   then return INFEASIBLE.
   Else swap chosen non-basic variable with violating basic variable.

4. `Update:` Update the tableau and go to 2.

Duality

- To understand better how the dual simplex works: theory of duality
- We can get lower bounds on LP optimum value
  by adding constraints in a convenient way

$$\begin{cases} \min \ -x - y \\ 2x + y \geq 3 \\ 2x + y \leq 6 \\ x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} \min \ -x - y \\ 2x + y \geq 3 \\ -2x - y \geq -6 \\ -x - 2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{rcr} -x - 2y & \geq & -6 \\ y & \geq & 0 \\ \hline -x - y & \geq & -6 \end{array}$$

Duality

- In general we can get lower bounds on LP optimum value by linearly combining constraints with convenient multipliers

$$\begin{cases} \min \; -x-y \\ 2x+y \geq 3 \\ -2x-y \geq -6 \\ -x-2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{lrcl}
1 \cdot (& 2x+y & \geq & 3 \quad) \\
2 \cdot (& -2x-y & \geq & -6 \quad) \\
1 \cdot (& x & \geq & 0 \quad) \\
\hline
 & 2x+y & \geq & 3 \\
 & -4x-2y & \geq & -12 \\
 & x & \geq & 0 \\
\hline
 & -x-y & \geq & -9
\end{array}$$

- There may be different choices, each giving a different lower bound

Duality

- In general:

$$\begin{cases} \min \ -x-y \\ 2x+y \geq 3 \\ -2x-y \geq -6 \\ -x-2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{rrcl}
\mu_1 \cdot (& 2x+y & \geq & 3 \quad) \\
\mu_2 \cdot (& -2x-y & \geq & -6 \quad) \\
\mu_3 \cdot (& -x-2y & \geq & -6 \quad) \\
\mu_4 \cdot (& x & \geq & 0 \quad) \\
\mu_5 \cdot (& y & \geq & 0 \quad) \\
\hline
 & 2\mu_1 x + \mu_1 y & \geq & 3\mu_1 \\
 & -2\mu_2 x - \mu_2 y & \geq & -6\mu_2 \\
 & -\mu_3 x - 2\mu_3 y & \geq & -6\mu_3 \\
 & \mu_4 x & \geq & 0 \\
 & \mu_5 y & \geq & 0
\end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3 + \mu_4)\, x \ + \ (\mu_1 - \mu_2 - 2\mu_3 + \mu_5)\, y \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

- If $2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 = -1$, $\quad \mu_1 - \mu_2 - 2\mu_3 + \mu_5 = -1$,
  $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$, $\mu_5 \geq 0$,
  then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

- We can skip the multipliers of the non-negativity constraints

- We have:

$$\begin{cases} \min \ -x-y \\ 2x+y \geq 3 \\ -2x-y \geq -6 \\ -x-2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{lrcll} \mu_1 \cdot (& 2x+y & \geq & 3 &) \\ \mu_2 \cdot (& -2x-y & \geq & -6 &) \\ \mu_3 \cdot (& -x-2y & \geq & -6 &) \\ \hline & 2\mu_1 x + \mu_1 y & \geq & 3\mu_1 & \\ & -2\mu_2 x - \mu_2 y & \geq & -6\mu_2 & \\ & -\mu_3 x - 2\mu_3 y & \geq & -6\mu_3 & \end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3)\, x \ + \ (\mu_1 - \mu_2 - 2\mu_3)\, y \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

- Imagine $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$.
  In the coefficient of x we can "complete" $2\mu_1 - 2\mu_2 - \mu_3$ to reach -1 by adding a suitable multiple of $x \geq 0$ (the multiplier will be the slack)

- If $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$, $\mu_1 - \mu_2 - 2\mu_3 \leq -1$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

- Best possible lower bound with this “trick” can be found by solving

$$\begin{cases} \max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\ & 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\ & \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\ & \mu_1, \mu_2, \mu_3 \geq 0 \end{cases}$$

- How far will it be from the optimum?

Duality

- Best possible lower bound with this “trick” can be found by solving

$$
\begin{cases}
\max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\
& 2\mu_1 - 2\mu_2 - \mu_3 \le -1 \\
& \mu_1 - \mu_2 - 2\mu_3 \le -1 \\
& \mu_1, \mu_2, \mu_3 \ge 0
\end{cases}
$$

- How far will it be from the optimum?

- A best solution is given by $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})$

$$
\begin{array}{lrcl}
0 \cdot (& 2x + y & \ge & 3 \quad) \\
\frac{1}{3} \cdot (& -2x - y & \ge & -6 \quad) \\
\frac{1}{3} \cdot (& -x - 2y & \ge & -6 \quad) \\
\hline
& -x - y & \ge & -4
\end{array}
$$

Matches the optimum!

Dual Problem

- If we multiply constraints $Ax \geq b$ by multipliers y^T we get $y^T Ax \geq y^T b$
- Given an LP (called primal)

$$\begin{aligned} \min\ & c^T x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

its dual is the LP

$$\begin{aligned} \max\ & y^T b \\ & y^T A \leq c^T \\ & y^T \geq 0 \end{aligned} \qquad \text{or equivalently} \qquad \begin{aligned} \max\ & b^T y \\ & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

- Primal variables associated with columns of A
- Dual variables (multipliers) associated with rows of A
- Objective and right-hand side vectors swap their roles

Dual Problem

- **Prop.** The dual of the dual is the primal.

  Proof:

$$
\begin{array}{lcl}
\max\ b^T y & & -\min\ (-b)^T y \\
A^T y \le c & \Longrightarrow & -A^T y \ge -c \\
y \ge 0 & & y \ge 0
\end{array}
$$

$$
\begin{array}{lcl}
-\max\ -c^T x & & \min\ c^T x \\
(-A^T)^T x \le -b & \Longrightarrow & Ax \ge b \\
x \ge 0 & & x \ge 0
\end{array}
$$

- We say the primal and the dual form a primal-dual pair

Dual Problem

- **Prop.** $\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array}$ and $\begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$ form a primal-dual pair

Proof:

$$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min\ c^T x \\ Ax \geq b \\ -Ax \geq -b \\ x \geq 0 \end{array}$$

$$\begin{array}{l} \max\ b^T y_1 - b^T y_2 \\ A^T y_1 - A^T y_2 \leq c \\ y_1, y_2 \geq 0 \end{array} \quad \overset{y := y_1 - y_2}{\Longrightarrow} \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

Duality Theorems

- **Th. (Weak Duality)** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \; c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \; b^T y \\ A^T y \leq c \end{array}$$

If x is feasible solution to P and y is feasible solution to D then $b^T y \leq c^T x$

Proof:

$c - A^T y \geq 0$, i.e., $c^T - y^T A \geq 0$, and $x \geq 0$ imply $c^T x - y^T A x \geq 0$.

So $c^T x \geq y^T A x$, and

$$c^T x \geq y^T A x = y^T b = b^T y$$

Duality Theorems

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- Feasible solutions to P give upper bounds on D
- Will the two optimum values be always equal?

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- Feasible solutions to P give upper bounds on D
- Will the two optimum values be always equal?
- **Th. (Strong Duality)** Let (P, D) be a primal-dual pair

$$(P)\quad \begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

If any of P or D has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.

Duality Theorems

- Proof (Th. of Strong Duality):

  By symmetry it is sufficient to prove only one direction.
  Wlog. let us assume P is feasible with finite optimum.

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  By symmetry it is sufficient to prove only one direction.
  Wlog. let us assume P is feasible with finite optimum.

  After executing the Simplex algorithm to P we find
  B optimal feasible basis. Then:

  - $c_{\mathcal{B}}^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
  - $c_{\mathcal{B}}^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

  So $c_{\mathcal{B}}^T B^{-1} A \leq c^T$.
  Hence $\pi^T := c_{\mathcal{B}}^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$

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  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$

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  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$

  By the theorem of weak duality, π is optimum for D

- If B is an optimal feasible basis for P,
  then simplex multipliers $\pi^T := c_{\mathcal{B}}^T B^{-1}$ are optimal feasible solution for D
- We can solve the dual by applying the simplex algorithm on the primal
- We can solve the primal by applying the simplex algorithm on the dual

Duality Theorems

- **Prop.** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \; c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \; b^T y \\ A^T y \leq c \end{array}$$

(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

Proof:

Let us prove (1) by contradiction.

If y were a feasible solution to D,
by the weak duality theorem, objective of P would be lower bounded!

(2) is proved by duality.

Duality Theorems

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- And the converse?
  Does infeasibility of one imply unboundedess of the other?

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- And the converse?
  Does infeasibility of one imply unboundedess of the other?

$$\begin{array}{lrl} \min & 3x_1 + 5x_2 & \\ & x_1 + 2x_2 & = 3 \\ & 2x_1 + 4x_2 & = 1 \\ & x_1, x_2 \text{ free} & \end{array} \qquad \begin{array}{lrl} \max & 3y_1 + y_2 & \\ & y_1 + 2y_2 & = 3 \\ & 2y_1 + 4y_2 & = 5 \\ & y_1, y_2 \text{ free} & \end{array}$$

Duality Theorems

$$\boxed{\begin{array}{lcl} \text{Primal unbounded} & \Longrightarrow & \text{Dual infeasible} \\ \text{Dual unbounded} & \Longrightarrow & \text{Primal infeasible} \\ \text{Primal infeasible} & \Longrightarrow & \text{Dual} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \\ \text{Dual infeasible} & \Longrightarrow & \text{Primal} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \end{array}}$$

Karush Kuhn Tucker Opt. Conds.

- Consider a primal-dual pair of the form

$$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \text{and} \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array} \iff \begin{array}{l} \max\ b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}$$

- Karush-Kuhn-Tucker (KKT) optimality conditions are

  - $Ax = b$
  - $A^T y + w = c$
  - $x, w \geq 0$
  - $x^T w = 0$ (complementary slackness)

- They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions $(x, (y, w))$

- Used, e.g., as a test of quality in LP solvers

Karush Kuhn Tucker Opt. Conds.

$$(P) \begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad (D) \begin{array}{l} \max\ b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}$$

(KKT)

- $Ax = b$
- $A^T y + w = c$
- $x, w \geq 0$
- $x^T w = 0$

- **Th.** $(x, (y, w))$ is solution to KKT iff x optimal solution to P and (y, w) optimal solution to D

  Proof:

  $\Rightarrow$ By $0 = x^T w = x^T (c - A^T y) = c^T x - b^T y$, and Weak Duality

  $\Leftarrow$ x is feasible solution to P, (y, w) is feasible solution to D.
  By Strong Duality $x^T w = x^T (c - A^T y) = c^T x - b^T y = 0$
  as both solutions are optimal

~~Relating Bases~~

- Consider a primal-dual pair of the form

$$
(P)\ \begin{array}{l} \min\ z = c^T x \\ Ax = b \\ x \geq 0 \end{array}
\qquad\qquad
(D)\ \begin{array}{l} \max\ Z = b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}
$$

- Let us denote by a_1, ..., a_n the columns of A, i.e., $A = (a_1, \dots, a_n)$
- Let B be a basis of P. Let us see how we can get a basis of D.

  Assume that the basic variables are the first m: $B = (a_1, \dots, a_m)$.

  Then $R = (a_{m+1}, \dots, a_n)$.

  If slacks w are split into $w_{\mathcal{B}}^T = (w_1, \dots, w_m)$, $w_{\mathcal{R}}^T = (w_{m+1}, \dots, w_n)$, then

$$
A^T y + w = \begin{pmatrix} a_1^T y \\ \dots \\ a_m^T y \\ \hline a_{m+1}^T y \\ \dots \\ a_n^T y \end{pmatrix} + \begin{pmatrix} w_1 \\ \dots \\ w_m \\ \hline w_{m+1} \\ \dots \\ w_n \end{pmatrix} = \begin{pmatrix} B^T y + w_{\mathcal{B}} \\ \hline R^T y + w_{\mathcal{R}} \end{pmatrix}
$$

~~Relating Bases~~

- Hence we have

$$A^T y + w = \left(\begin{array}{c} B^T y + w_{\mathcal{B}} \\ \hline R^T y + w_{\mathcal{R}} \end{array} \right)$$

- Then the matrix of the system in the dual problem D is

$$\left(\begin{array}{c|c|c} B^T & I & 0 \\ \hline R^T & 0 & I \end{array} \right) \left(\begin{array}{c} y \\ w_{\mathcal{B}} \\ w_{\mathcal{R}} \end{array} \right)$$

- Now let us consider the submatrix of vars y and vars $w_{\mathcal{R}}$:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

- Note $\hat{B}$ is a square $n \times n$ matrix

Relating Bases

- Dual variables $\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ determine a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array}\right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)$$

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- In the next slides we answer the following questions:

  1. If basis $\hat{B}$ of the dual D is feasible,
     what can we say about basis B of the primal P?

  2. If basis $\hat{B}$ of the dual D is optimal (satisfies the optimality conds.),
     what can we say about basis B of the primal P?

  3. If we apply the simplex algorithm to the dual D using basis $\hat{B}$,
     how does that translate into the primal P and its basis B?

~~Relating Bases~~

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     how does that translate into the primal P and its basis B?

- Recall that each variable w_j in D is associated to a variable x_j in P.
- Note that w_j is $\hat{\mathcal{B}}$-basic iff x_j is not $\mathcal{B}$-basic

~~Dual Feasibility — Primal Optimality~~

- ~~If $\cancel{\hat{B}}$ is feasible for dual D, what about B in primal P?~~

$$\cancel{\hat{B}}^{-1}c = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)\left(\begin{array}{c} c_{\mathcal{B}} \\ \hline c_{\mathcal{R}} \end{array}\right) = \left(\begin{array}{c} B^{-T}c_{\mathcal{B}} \\ \hline -R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \end{array}\right)$$

- There is no restriction on the sign of $y_1, \ldots, y_m$
- Variables w_j have to be non-negative. But

$$-R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c_{\mathcal{R}}^T - c_{\mathcal{B}}^T B^{-1} R \geq 0$$

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- $\hat{B}$ is dual feasible iff $d_j \geq 0$ for all $j \in \mathcal{R}$

- Dual feasibility is primal optimality!

~~Dual Optimality = Primal Feasibility~~

- If $\hat{B}$ satisfies the optimality conds. for dual D, what about B in primal P?
- Non $\hat{\mathcal{B}}$-basic vars: $w_{\mathcal{B}}$ with costs (0)
- $\hat{\mathcal{B}}$-basic vars: $(y \mid w_{\mathcal{R}})$ with costs $(b^T \mid 0)$
- Matrix of non $\hat{\mathcal{B}}$-basic vars: $\left(\frac{I}{0}\right)$
- Optimality condition: $0 \geq$ reduced costs (maximization!)

$$0 \geq (\ 0\ ) - (\ b^T \mid 0\ ) \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right) \left(\frac{I}{0}\right) =$$

$$(\ 0\ ) - (\ b^T B^{-T} \mid 0\ ) \left(\frac{I}{0}\right) = -b^T B^{-T} = -(B^{-1} b)^T$$

Dual Optimality − Primal Feasibility

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Dual Optimality — Primal Feasibility

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- In the dual problem,
  for all $1 \leq p \leq m$, var w_{k_p} cannot improve objective function iff $\beta_p \geq 0$
- Dual optimality is primal feasibility!

Improving a Non-Optimal Solution

- Next we apply the simplex algorithm to basis $\hat{B}$ in the dual problem D and translate it to the primal problem P

- Let p (where $1 \leq p \leq m$) be such that $\beta_p < 0$.

  I.e., the reduced cost of non-basic dual variable w_{k_p} is positive.

  So by giving w_{k_p} a larger value we can improve the dual objective value.

  If w_{k_p} takes value $t \geq 0$:

$$\begin{pmatrix} y(t) \\ w_{\mathcal{R}}(t) \end{pmatrix} = \hat{B}^{-1}c - \hat{B}^{-1}te_p =$$

$$= \begin{pmatrix} B^{-T}c_{\mathcal{B}} \\ d_{\mathcal{R}} \end{pmatrix} - \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)\begin{pmatrix} te_p \\ \hline 0 \end{pmatrix} = \begin{pmatrix} B^{-T}c_{\mathcal{B}} - tB^{-T}e_p \\ \hline d_{\mathcal{R}} + tR^T B^{-T} e_p \end{pmatrix}$$

- Dual objective value improvement is

$$\Delta Z = b^T y(t) - b^T y(0) = -tb^T B^{-T} e_p = -t\beta^T e_p = -t\beta_p = t(-\beta_p)$$

~~Improving a Non-Optimal Solution~~

- Of all basic dual variables, only $w_{\mathcal{R}}$ variables need to be ≥ 0
- For $j \in \mathcal{R}$

$$w_j(t) = d_j + t a_j^T B^{-T} e_p = d_j + t e_p^T B^{-1} a_j = d_j + t e_p^T \alpha_j = d_j + t \alpha_j^p$$

where α_j^p is the p-th component of α_j. Hence:

$$w_j(t) \geq 0 \iff d_j + t\alpha_j^p \geq 0$$

  - If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$
  - If $\alpha_j^p < 0$ we need $\frac{d_j}{-\alpha_j^p} \geq t$

- Best improvement achieved with

$$\Theta_D := \min\{\frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0\}$$

- Variable w_q is blocking when $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j \in \mathcal{R}$ such that $\alpha_j^p < 0$):

   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.

   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and w_q is blocking:

   When setting $w_{k_p} = \Theta_D$,
   non-negativity constraints of basic vars of dual are respected

   In particular, $w_q(\Theta_D) = d_q + \Theta_D \alpha_q^p = d_q + (\frac{d_q}{-\alpha_q^p})\alpha_q^p = 0$

   We can make a basis change:

   - In dual: w_{k_p} enters $\hat{\mathcal{B}}$ and w_q leaves
   - In primal: x_{k_p} leaves $\mathcal{B}$ and x_q enters

Update

- We do not actually need to form the dual LP:
  it is enough to have a representation of the primal LP

- New basic indices: $\bar{\mathcal{B}} = (k_1, \ldots, k_{p-1}, q, k_{p+1} \ldots, k_m)$

- New dual objective value: $\bar{Z} = Z - \Theta_D \beta_p$

- New dual basic sol: $\bar{y} = y - \Theta_D \rho_p$
  $\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$

- New primal basic sol: $\bar{\beta}_p = \Theta_P$, $\bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$
  where $\Theta_P = \frac{\beta_p}{\alpha_q^p}$

- New basis inverse: $\bar{B}^{-1} = E B^{-1}$
  where $E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m)$ and
  $\eta^T = \left(\left(\frac{-\alpha_q^1}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^{p-1}}{\alpha_q^p} \right), \frac{1}{\alpha_q^p} \left(\frac{-\alpha_q^{p+1}}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^m}{\alpha_q^p} \right) \right)^T$

Algorithmic Description

1. `Initialization:` Find an initial dual feasible basis $\mathcal{B}$
   Compute B^{-1}, $\beta = B^{-1}b$,
   $y^T = c_{\mathcal{B}}^T B^{-1}$, $d_{\mathcal{R}}^T = c_{\mathcal{R}}^T - y^T R$, $Z = b^T y$

2. `Dual Pricing:`
   If for all $i \in \mathcal{B}, \beta_i \geq 0$ then return OPTIMAL
   Else let p be such that $\beta_p < 0$.
   Compute $\rho_p^T = e_p^T B^{-1}$ and $\alpha_j^p = \rho_p^T a_j$ for $j \in \mathcal{R}$

3. `Dual Ratio test:` Compute $\mathcal{J} = \{j \mid j \in \mathcal{R}, \alpha_j^p < 0\}$.
   If $\mathcal{J} = \emptyset$ then return INFEASIBLE
   Else compute $\Theta_D = \min_{j \in \mathcal{J}} (\frac{d_j}{-\alpha_j^p})$ and q st. $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Algorithmic Description

4. `Update:`

$$\bar{\mathcal{B}} = \mathcal{B} - \{k_p\} \cup \{q\}$$
$$\bar{Z} = Z - \Theta_D \beta_p$$

<u>`Dual solution`</u>

$$\bar{y} = y - \Theta_D \rho_p$$
$\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$

<u>`Primal solution`</u>

Compute $\alpha_q = B^{-1} a_q$ and $\Theta_P = \frac{\beta_p}{\alpha_q^p}$

$\bar{\beta}_p = \Theta_P$, $\quad \bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$

$$\bar{B}^{-1} = E B^{-1}$$

Go to 2.

Primal vs. Dual Simplex

PRIMAL	DUAL
Can handle bounds efficiently	Can handle bounds efficiently (not explained here)
Many years of research and implementation	Developments in the 90's made it an alternative
There are classes of LP's for which it is the best	Nowadays on average it gives better performance
Not suitable for solving LP's with integer variables	**Suitable for solving LP's with integer variables**