Stochastic Network Modeling (SNM)

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Parts

I. Introduction
II. Discrete Time Markov Chains (DTMC)
III. Continuous Time Markov Chains (CTMC)
IV. Queuing Theory

Part II

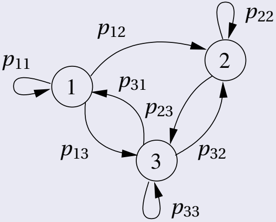
Discrete Time Markov Chains (DTMC)

Outline

Definition of a DTMC

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:
$\sum_{\forall j} p_{ij} = 1$

- Mathematically:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

Definition of a DTMC

Properties of a DTMC

- The event $X(n) = i$ (at step n the system is in state i) must satisfy (memoryless property):

$$\boxed{\begin{aligned} P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \cdots) = \\ P(X(n) = j \mid X(n-1) = i) \end{aligned}}$$

- If $P\big(X(n) = j \mid X(n-1) = i\big) = P\big(X(1) = j \mid X(0) = i\big)$ for any n we have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

Definition of a DTMC

Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

Definition of a DTMC

Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$$

- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) =$$

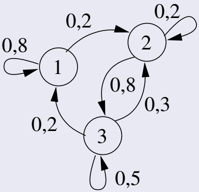
$$\sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = \boxed{1}$$

- $\mathbf{P}$ is a stochastic matrix, i.e. a matrix which rows sum 1.

Definition of a DTMC

Example

- Assume a terminal can be in 3 states:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



$$\mathbf{P} = \begin{array}{c} \text{to state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0{,}8 & 0{,}2 & 0 \\ 0 & 0{,}2 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \\ \\ \text{from} \\ \text{state} \\ \\ \end{array}$$

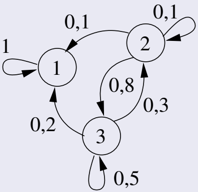
- The average transmission rate (throughput), v_a, is:

$$v_a = P(\text{the terminal is in state 3}) \times v$$

Definition of a DTMC

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



$$\begin{array}{cccc} & \text{to state} & & \\ & \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} & & \\ \mathbf{P} = & \begin{bmatrix} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{array}{c} \text{from} \\ \text{state} \\ \end{array} \end{array}$$

Definition of a DTMC

n-step transition probabilities

- Transition probabilities: $p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$
- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- We define the n-step transition probabilities:

$$p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$$

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are stochastic matrices: Their rows sum 1.

Definition of a DTMC

State Probabilities

- Define the probability of being in state i at step n:
$$\pi_i(n) = P(X(n) = i)$$
- In vector form (row vector)
$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$
- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the state probability at step n.

Definition of a DTMC

State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$

- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1) = k)\ P\big(X(n) = i \mid X(n-1) = k\big) = \sum_k \pi_k(n-1)\, p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0) = k)\ P\big(X(n) = i \mid X(0) = k\big) = \sum_k \pi_k(0)\, p_{ki}(n)$$

- In matrix form:

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.