Stochastic Network Modeling (SNM)

Llorenç Cerdà-Alabern
Universitat Politècnica de Catalunya
Departament d'Arquitectura de Computadors
llorenc@ac.upc.edu

Parts

Part III

Continuous Time Markov Chains (CTMC)

Outline

Definition of a CTMC

Properties of a continuous time MC

- The states must be a numerable set.
- Let $X(t)$ be the event {at time t the system is in state i}, then it must hold the memoryless property:

$$\boxed{\begin{aligned} &P\big(X(t_n) = i \mid X(t_1) = j, X(t_2) = k, \ldots\big) = \\ &\qquad P\big(X(t_n) = i \mid X(t_1) = j\big) \text{ for any } t_n > t_1 > t_2 > t_3 \ldots \end{aligned}}$$

Transition Matrix

- Transition probabilities:

$$p_{ij}(t_1, t_2) = P\big(X(t_2) = j \mid X(t_1) = i\big)$$

- For an homogeneous chain:

$$p_{ij}(t) = P\big(X(t_1 + t) = j \mid X(t_1) = i\big) = \\ = P\big(X(t) = j \mid X(0) = i\big), \ \forall t_1$$

- In matrix form (transition probability matrix):

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \ t \geq 0$$

- Notes:
  - Compare with the n-step prob. matrix of a DTMC: $\mathbf{P}(n)$.
  - $\mathbf{P}(t)$ must be a stochastic matrix (all rows add to 1).

Transition Matrix

$$p_{ij}(t) = P\big(X(t) = j \mid X(0) = i\big),\; t \geq 0$$

- We look for an equivalent 1-step prob. matrix **P** of DTMCs.
- For consistency: $\lim_{t \to 0} p_{ij}(t) = \delta_{ij}$. In matrix form:

$$\lim_{t \to 0} \mathbf{P}(t) = \mathbf{I}.$$

- And assume that the following transition rates exist:

$$q_{ij} = \lim_{t \to 0} \frac{p_{ij}(t)}{t},\; i \neq j; \quad q_{ii} = \lim_{t \to 0} \frac{p_{ii}(t) - 1}{t}$$

- In matrix form: $\mathbf{Q} = \lim_{t \to 0} \frac{\mathbf{P}(t) - \mathbf{I}}{t}$
- Note that $\sum_j p_{ij}(t) = 1 \Rightarrow p_{ii}(t) = 1 - \sum_{j \neq i} p_{ij}(t)$, thus:

$$q_{ii} = \lim_{t \to 0} \frac{p_{ii}(t) - 1}{t} = \lim_{t \to 0} \frac{-\sum_{j \neq i} p_{ij}(t)}{t} = -\sum_{j \neq i} q_{ij}$$

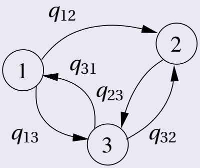
Transition Matrix

- The matrix $\mathbf{Q}$ is called the transition rate or infinitesimal generator of the chain.
- Since $q_{ii} = -\sum_{j \neq i} q_{ij}$, all the rows of $\mathbf{Q}$ add to 0.
- The rate q_{ij}, $i \neq j$ measures "how fast" the chain moves from state i to j: the higher is q_{ij}, the faster it moves from i to j.
- For $q_{ii} = -\sum_{j \neq i} q_{ij}$, the higher $-q_{ii}$ is, the faster the chain leaves state i.
- If $q_{ij} = 0, \forall j \Rightarrow q_{ii} = 0$, then i is an absorbing state: the chain "moves with rate 0 from i to other states", i.e. never leaves i.

Definition of a CTMC

State Transition Diagram

- A continuous MC is characterized by the transition rate or infinitesimal generator: the Q-matrix.
- The state transition diagram is now represented as:



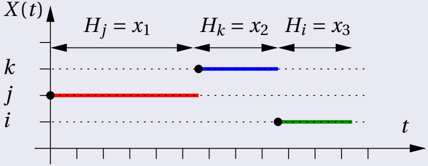
$$\Rightarrow \mathbf{Q} = \begin{bmatrix} -(q_{12}+q_{13}) & q_{12} & q_{13} \\ 0 & -q_{23} & q_{23} \\ q_{31} & q_{32} & -(q_{31}+q_{32}) \end{bmatrix}$$

- Note that now we have transition rates ($0 \leq q_{ij} < \infty$, $i \neq j$) and not probabilities.
- The rates q_{ii} are not written in the diagram, no self transitions.

Definition of a CTMC

Sojourn Time

- Sojourn or holding time: Is the RV H_k equal to the sojourn time in state k:

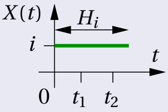


- The Markov property implies that the sojourn time is exponentially distributed with parameter q_{ii}:

$$\boxed{P(H_i \le x) = 1 - \mathrm{e}^{q_{ii}x} \Rightarrow P(H_i > x) = \mathrm{e}^{q_{ii}x},\ q_{ii} = -\sum_{j \neq i} q_{ij},\ x \ge 0}$$

Definition of a CTMC

The exponential distribution satisfies the Markov property



- Markov property (memoryless):

$$P\left(X(t_2) = i \mid X(t_1) = i, X(0) = i\right) =$$

$$P\left(X(t_2) = i \mid X(t_1) = i\right),\ t_2 > t_1 > 0$$

- In terms of the sojourn time:

$$P\left(H_i > t_2 \mid H_i > t_1\right) = P(H_i > t_2 - t_1)$$

- But:

$$P\left(H_i > t_2 \mid H_i > t_1\right) =$$

$$\frac{P(H_i > t_2, H_i > t_1)}{P(H_i > t_1)} = \frac{P(H_i > t_2)}{P(H_i > t_1)} = \frac{e^{q_{ii}\,t_2}}{e^{q_{ii}\,t_1}} = e^{q_{ii}\,(t_2 - t_1)} =$$

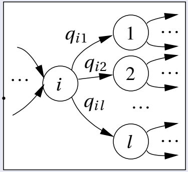
$$P(H_i > t_2 - t_1) \quad \square$$

- The exponential distribution is the only one satisfying the memoryless property.

Definition of a CTMC

Exponential Jumps Description of a CTMC

Assume a process built as follows:



- Upon reaching a state i
  1. the process can jump to a state $j \in \{1, 2, \cdots l\}$.
  2. A set of independent exponential RVs, $\{H_{i1}, H_{i2}, \cdots H_{il}\}$, with parameters $\{q_{i1}, q_{i1}, \cdots q_{il}\}$ are triggered. That is, $P(H_{ij} \leq t) = 1 - \mathrm{e}^{-q_{ij}t}$, $t \geq 0$.
- If $\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij} \Rightarrow$ the process jumps to the state j. In other words, a transition occurs to state j if the RV H_{ij} is the minimum of $\{H_{i1}, H_{i2}, \cdots H_{il}\}$.

$\boxed{\text{Theorem: This process is a CTMC with transition rates } q_{ij}.}$

Exponential Jumps Description of a CTMC

$$P(H_{ij} \le t) = 1 - e^{-q_{ij}\,t}.$$

Theorem: This process is a CTMC with transition rates q_{ij}.

Proof:

- The RV $H_i = \min\{H_{i1}, H_{i2}, \cdots H_{il}\}$ (sojourn time in state i) is exponentially distributed with parameter $q_i = \sum_j q_{ij}$: $P(H_i \le t) = 1 - e^{-q_i\,t}$.
- $P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) = q_{ij} / \sum_j q_{ij}$. Thus, the transition rate to state j is:

$$\lim_{t\to 0} \frac{p_{ij}(t)}{t} = \lim_{t\to 0} \frac{P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) \times P(H_i \le t)}{t} =$$

$$\frac{q_{ij}}{\sum_j q_{ij}} \left.\frac{\partial P(H_i \le t)}{\partial t}\right|_{t=0} = \frac{q_{ij}}{\sum_j q_{ij}} \sum_j q_{ij} = q_{ij} \qquad \square$$

Definition of a CTMC

Example: Pure Aloha System

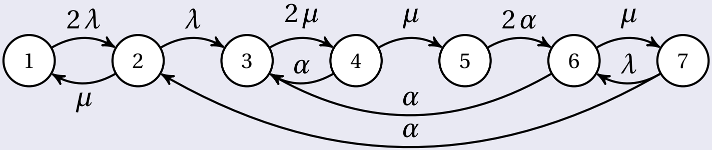
- Consider a Pure Aloha System with 2 nodes:
  - Nodes in thinking state Tx a packet in a time exponentially distributed with rate λ.
  - Transmission time is exponentially distributed with rate μ.
  - If two transmissions overlap, the packet is lost and stations become backlogged (after the packet transmission) until the packet is successfully transmitted.
  - Nodes in backlogged state Tx a packet in a time exponentially distributed with rate α.

Questions

- Build the state transition diagram.

Definition of a CTMC

Example: Pure Aloha System



State	Condition
1	T,T
2	X,T
3	C,C
4	B,C
5	B,B
6	X,B
7	T,B

Legend	
T	Thinking
X	Transmitting
C	Collided transmission
B	Backlogged

Part III

Continuous Time Markov Chains (CTMC)

Outline

Chapman-Kolmogorov Equations

- Chapman-Kolmogorov: $p_{ij}(t) = \sum_k p_{ik}(t-\alpha) p_{kj}(\alpha), 0 \le \alpha \le t$
- Thus:
$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_k \left\{ \frac{p_{ik}(t+\Delta t-\alpha) - p_{ik}(t-\alpha)}{\Delta t} p_{kj}(\alpha) \right\}$$
- Taking the limit
$$\alpha \to t, \Delta t \to 0 \Rightarrow \begin{cases} p_{ik}(t-\alpha) \to 0, & i \ne k \\ p_{ik}(t-\alpha) \to 1, & i = k \end{cases}$$
and using:
$$\begin{cases} q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t}, & i \ne j \\ q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)-1}{t}, & i = j \end{cases}$$
we have:
$$\boxed{\frac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik} p_{kj}(t), \quad t \ge 0, \forall i,j}$$

Chapman-Kolmogorov Equations (cont)

- we have: $\dfrac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik}\, p_{kj}(t),\ t \geq 0,\ \forall i,j$
- In matrix form: $\boxed{\dfrac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{Q}\,\mathbf{P}(t),\ t \geq 0}$ known as the master equations of a CTMC.
- The solution of the previous matrix differential equation is the exponential matrix:

$$\boxed{\mathbf{P}(t) = e^{\mathbf{Q}\,t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \mathbf{I} + \mathbf{Q}\,t + \frac{\mathbf{Q}^2\,t^2}{2!} + \frac{\mathbf{Q}^3\,t^3}{3!} + \cdots,\ t \geq 0}$$

- Due to rounding errors, the previous series is difficult to compute numerically (the powers of $\mathbf{Q}$ have positive and negative entries).

Transient Solution

State Probabilities

- Define the probability of being in state i at time t:

$$\boxed{\pi_i(t) = P(X(t) = i)}$$

- In vector form (row vector)

$$\boldsymbol{\pi}(t) = (\pi_1(t), \pi_2(t), \cdots).$$

- Clearly:

$$\pi_i(t) = \sum_k P(X(0) = k)\, P\big(X(t) = i \,\big|\, X(0) = k\big) = \sum_k \pi_k(0)\, p_{ki}(t)$$

- In matrix form:

$$\boxed{\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t},\ t \geq 0}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

- NOTE: Compare with DTMC

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n,\ n \geq 0}$$

Transient Solution

Transient Solution

- If we are interested in the transient evolution we shall study $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,e^{\mathbf{Q}\,t}$, $t \geq 0$.
- Assume a finite CTMC with N states (infinitesimal generator $\mathbf{Q}^{N\times N}$).
- Assume that $\mathbf{Q}$ can be diagonalized: $\mathbf{Q} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{Q}$.
- NOTE: the eigenvalues λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l\,\boldsymbol{l}$ (or $\mathbf{A}\,\boldsymbol{r} = \lambda_l\,\boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* and *right* eigenvectors, respectively. Thus, solve the characteristic polynomial $\det(\lambda\,\mathbf{I} - \mathbf{A}) = 0$.

Transient Solution

Transient Solution

... assume that $\mathbf{Q}$ can be diagonalized: $\mathbf{Q} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$

- Since

$$\mathbf{P}(t) = e^{\mathbf{Q}\,t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \sum_{i=0}^{\infty} \frac{(\mathbf{L}^{-1}\,\Lambda\,\mathbf{L}\,t)^i}{i!} =$$

$$\mathbf{L}^{-1}\,\mathrm{diag}\left(\sum_{i=0}^{\infty} \frac{(\lambda_1\,t)^i}{i!}, \cdots\right)\mathbf{L} = \mathbf{L}^{-1}\,\mathrm{diag}(e^{\lambda_1\,t}, \cdots)\,\mathbf{L}$$

  we have that

$$\boxed{\begin{aligned}&\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,e^{\mathbf{Q}\,t} = \\ &\quad \boldsymbol{\pi}(0)\,\mathbf{L}^{-1}\,\mathrm{diag}(e^{\lambda_1\,t}, e^{\lambda_2\,t}, \cdots e^{\lambda_L\,t})\,\mathbf{L}\end{aligned}}$$

Transient Solution

Transient Solution

... we have that $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{L}^{-1}\,\mathrm{diag}(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \cdots \mathrm{e}^{\lambda_L t})\,\mathbf{L}$

- Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(t) = (\boldsymbol{\pi}(t))_i = \sum_{l=1}^{N} a_i^{(l)}\,\mathrm{e}^{\lambda_l t},\ t \ge 0}$$

where the unknown coefficients $a_i^{(l)}$ can be obtained solving the system of equations:

$$\boxed{\left.\frac{\partial^n \pi_i(t)}{\partial t^n}\right|_{t=0} = (\boldsymbol{\pi}(0)\,\mathbf{Q}^n)_i,\ n = 0, \cdots N-1}$$

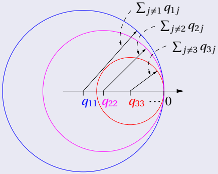
NOTE: Compare with $(\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1$

Transient Solution

Eigenvalues of an Infinitesimal Generator

- $\mathbf{Q}$ has an eigenvalue equal to 0 ($\mathbf{Q}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 0$, $\boldsymbol{x} \neq \mathbf{0}$).
  Proof: $\mathbf{Q}\,\mathbf{e} = \mathbf{0}$, where $\mathbf{e} = (1, 1, \cdots)^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{Q}$ add to 0). □
- The eigenvalue $\lambda = 0$ is single if $\mathbf{Q}$ is irreducible (Perron-Frobenius theorem). $\mathbf{Q}$ is irreducible if all states communicate: for $t > 0$, $p_{ij}(t) > 0$, $\forall i, j$.
- All eigenvalues of $\mathbf{Q}$ are $\lambda_l \leq 0$.

Proof: Using Gerschgorin's theorem and the fact that the rows of $\mathbf{Q}$ add to 0. □



Gershgorin discs of a rate matrix

Transient Solution

Example

- Assume a CTMC with

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 1/2 & -1/2 \end{bmatrix}$$

- We want the probability of being in state 2 at time t starting from state 1: $\pi_2(t)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transient Solution

Solution

- It can be easily found that the eigenvalues of $\mathbf{Q}$ are $\lambda_1 = 0$ and $\lambda_2 = -3/2$.

$$\pi_2(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t} = a + b e^{-(3/2)\, t}$$

- Imposing the boundary conditions:

$$\pi_2(0) = a + b = (\boldsymbol{\pi}(0)\,\mathbf{Q}^0)_2 = (\boldsymbol{\pi}(0)\,\mathbf{I})_2 = (\boldsymbol{\pi}(0))_2 = 0$$

$$\left.\frac{\partial \pi_2(t)}{\partial t}\right|_{t=0} = b(-3/2) = (\boldsymbol{\pi}(0)\,\mathbf{Q})_2 = \mathbf{Q}_{12} = 1$$

we have that $a = 2/3$, $b = -2/3$, thus:

$$\pi_2(t) = 2/3\,(1 - e^{-(3/2)\, t}), \quad t \geq 0$$

Transient Solution

Chain with a Defective Matrix

- What if $\mathbf{Q}$ cannot be diagonalized? (defective matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{Q}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$). Then [1]:

$$\boxed{\pi_j(t) = \sum_{l=1}^{L} \mathrm{e}^{\lambda_l t} \sum_{m=0}^{k_l-1} a_j^{(l,m)}\, t^m}$$

where $a_j^{(l,m)}$ are constants. So, exponentials associated with eigenvalues λ_l of multiplicity $k_l > 1$ are multiplied by polynomials in t of degree $k_l - 1$.

[1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method*. Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: `https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html`.

Transient Solution

Example

- Assume the CTMC:

$$\textcircled{1} \xrightarrow{1} \textcircled{2} \xrightarrow{1} \textcircled{3}$$

$$\pi(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- We have $\lambda_1 = 0$ and $\lambda_2 = -1$ with multiplicity 2. Thus:

$$\pi_3(t) = a + e^{-t}(b + c\,t)$$

- We have that $a = 1$, because state 3 is absorbing. Imposing $\pi_3(0) = 0$ and $\pi_3'(0) = 0$, we have $b = c = -1$, and

$$\boxed{\pi_3(t) = 1 - e^{-t}(1 + t),\ t \ge 0}$$

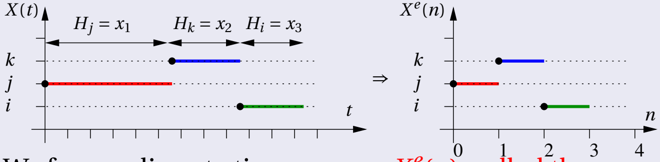
Part III

Continuous Time Markov Chains (CTMC)

Outline

Embedded MC of a CTMC

Definition



- We form a discrete time process $X^e(n)$, called the *Embedded MC (EMC)*, by looking a CTMC at the transition time instants.

  Theorem: This process is a DTMC with transition probabilities:

$$p^e_{ij} = \begin{cases} 0, & i = j \\ \dfrac{q_{ij}}{\sum_{j \neq i} q_{ij}}, & i \neq j \end{cases}$$

- NOTE: If i is absorbing ($q_{ii} = 0$), we define $p^e_{ii} = 1$.

Proof

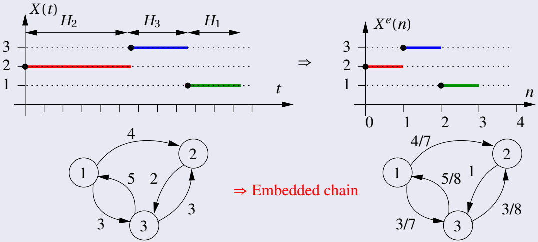
Theorem: This process is a DTMC with transition probabilities:

$$p^e_{ij} = \begin{cases} 0, & i = j \\ \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}, & i \neq j \end{cases}$$

- The EMC satisfies the memoryless property.
- Since we look the system only upon transition to a different state: $p^e_{ii} = 0$. NOTE: it might be $p^e_{ii} \neq 0$ if we look at transitions that end up in the same state.
- The probability that there is a transition from state i to j in the CTMC is the probability that the exponentially distributed RV with parameter q_{ij} is the minimum from the independent exponentially distributed RVs with parameters $\{q_{ik}\}_{k \neq i}$. This probability is $q_{ij} / \sum_{k \neq i} q_{ik}$. $\square$

Embedded MC of a CTMC

Example



$$\mathbf{Q} = \begin{bmatrix} -7 & 4 & 3 \\ 0 & -2 & 2 \\ 5 & 3 & -8 \end{bmatrix} \Rightarrow \quad \mathbf{P}_e = \begin{bmatrix} 0 & 4/7 & 3/7 \\ 0 & 0 & 1 \\ 5/8 & 3/8 & 0 \end{bmatrix}$$

- Each transition in the CTMC is a transition in the EMC.
- One step in i in the EMC is a sojourn time H_i in the CTMC.

Part III

Continuous Time Markov Chains (CTMC)

Outline

Classification of States

Irreducibility

- A state j is said to communicate with i, $i \leftrightarrow j$, if $p_{ij}(t_1) > 0$, $p_{ji}(t_2) > 0$ for some $t_1 \geq 0$, $t_2 \geq 0$.
- We define an irreducible closed set, ICS C_k as a set where all states communicate with each other, and have no transitions to other states out of the set: $i \leftrightarrow j, \forall i, j \in C_k$ and $q_{ij} = 0, \forall i \in C_k, j \notin C_k$
- An absorbing state form an ICS of only one element. This state, i, must have $q_{ij} = 0 \; \forall i, j$.
- Transient states do not belong to any ICS.
- A MC is irreducible if all the states form a unique ICS.

Classification of States

Irreducibility

- Assume a MC has M ICSs: By properly numbering the states, we can write $\mathbf{P}$ as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- Example, if $M = 3$:

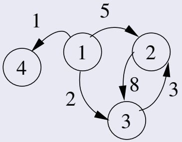
$$\mathbf{Q} = \begin{array}{|c|c|c|c|}\hline \mathbf{Q}_1 & & \mathbf{0} & \\ \hline \mathbf{0} & \mathbf{Q}_2 & & \\ & & \mathbf{Q}_3 & \\ \hline \multicolumn{3}{|c|}{\text{at least one} > 0} & \mathbf{T} \\ \hline \end{array} \Rightarrow \boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\, e^{\mathbf{Q}\,t} = \boldsymbol{\pi}(0) \begin{array}{|c|c|c|c|}\hline e^{\mathbf{Q}_1 t} & & \mathbf{0} & \\ \hline \mathbf{0} & e^{\mathbf{Q}_2 t} & & \\ & & e^{\mathbf{Q}_3 t} & \\ \hline \multicolumn{3}{|c|}{\text{at least one} > 0} & e^{\mathbf{T}t} \\ \hline \end{array}$$

- Note that the M sub-matrices are infinitesimal generators (their rows add to 0).

Classification of States

Transient and Recurrent

- Recurrent: States that, being visited, have a probability > 0 of being visited again. They are visited an infinite number of times when $t \to \infty$.
- Transient: States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $t \to \infty$.
- Absorbing: A single (recurrent) state where the chain remains with probability $= 1$.



State 1 is transient
States 2 and 3 are recurrent
State 4 is absorbing

Classification of States

Transient and Recurrent

- To derive a classification criteria, we shall study the embedded MC (EMC), and proceed as in DTMC: Let $f^e_{ij}(n)$ the first passage prob. of the EMC, and $f^e_{ij} = \sum_{n=1}^{\infty} f^e_{ij}(n)$.
- If $f^e_{ii} = 1$ we say i is a recurrent state.
- If $f^e_{ii} < 1$ we say i is a transient state.
- When $f^e_{ij} = 1$, we define the mean recurrence time of the EMC $m^e_{ij} = \sum_{n=1}^{\infty} n f^e_{ij}(n)$. NOTE: in steps, not time units.
- If $m^e_{ii} = \infty$ the state is null recurrent.
- If $m^e_{ii} < \infty$ the state is positive recurrent.
- NOTEs: (i) Even if the EMC is periodic, there are not periodic CTMC (it has no sense). (ii) f^e_{ij} and m^e_{ij} can be computed using the recursive equations for DTMC.

Classification of States

Mean recurrence time of the CTMC

- If the chain is in i at a time t, it takes an expected time to leave i equal to $1/(-q_{ii}) = 1/\sum_{j\neq i} q_{ij}$ (sojourn time exponentially distributed with rate $q_i = -q_{ii} = \sum_{j\neq i} q_{ij}$).
- Thus, if the chain is in state i, it takes a mean time to enter state j (mean first passage time):

$$\boxed{m_{ij} = \frac{1}{q_i} + \sum_{k\neq i} p^e_{ik}\, m_{kj}}$$

- Since: $p^e_{ij} = \begin{cases} 0, & i = j \\ \dfrac{q_{ij}}{\sum_{j\neq i} q_{ij}} = \dfrac{q_{ij}}{q_i}, & i \neq j \end{cases}$ we have:

$$\boxed{m_{ij} = \frac{1}{q_i} + \sum_{k\neq i} p^e_{ik}\, m_{kj} = \frac{1}{q_i} + \sum_{k\neq i} \frac{q_{ik}}{q_i}\, m_{kj} \text{ [time units]}}$$

Part III

Continuous Time Markov Chains (CTMC)

Outline

Limiting Distribution

- The probability to be in state i at time t is: $\pi_i(t) = P(X(t) = i) = \sum_k \pi_k(0)\, p_{ki}(t),\ t \geq 0$
- In matrix form: $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,e^{\mathbf{Q}\,t},\ t \geq 0$
- Assume that the following limit exists:

$$\boldsymbol{\pi}(\infty) = \lim_{t\to\infty} \boldsymbol{\pi}(t) = \lim_{t\to\infty} \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0) \lim_{t\to\infty} e^{\mathbf{Q}\,t}$$

- for any $\boldsymbol{\pi}(0)$, which implies

$$\lim_{t\to\infty} e^{\mathbf{Q}\,t} = \mathbf{P}(\infty) = \begin{bmatrix}\boldsymbol{\pi}(\infty) & \cdots & \boldsymbol{\pi}(\infty)\end{bmatrix}^{\mathrm{T}}$$

- If this limit exists, we call $\mathbf{P}(\infty)$ the limiting matrix, and $\boldsymbol{\pi}(\infty)$ the limiting distribution.
- $\mathbf{P}(\infty) = \begin{bmatrix}\boldsymbol{\pi}(\infty) & \cdots & \boldsymbol{\pi}(\infty)\end{bmatrix}^{\mathrm{T}}$ does not exist if the CTMC has more than one irreducible closed set (each ICS will converge to a diagonal block, and $\boldsymbol{\pi}(\infty)$ will depend on $\boldsymbol{\pi}(0)$).

Stationary Distribution

- We have: $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t}$, $t \geq 0$.
- In steady state the probabilities do not change. We look for a probability vector $\boldsymbol{\pi} = \boldsymbol{\pi}(t_1)$ satisfying: $\boldsymbol{\pi}(t_1)\,\mathrm{e}^{\mathbf{Q}\,t} = \boldsymbol{\pi}(t_1)$. In other words, for $t \geq t_1$ the probability vector reach the steady state $\boldsymbol{\pi}$, and do not change anymore. Thus:

$$\boxed{\boldsymbol{\pi}\,\frac{\partial \mathrm{e}^{\mathbf{Q}\,t}}{\partial t} = \boldsymbol{\pi}\,\mathbf{Q}\,\mathrm{e}^{\mathbf{Q}\,t} = \mathbf{0}}$$

- and we obtain that the stationary distribution $\boldsymbol{\pi}$ can be computed with the Global balance equations:

$$\boxed{\begin{array}{l}\boldsymbol{\pi}\,\mathbf{Q} = \mathbf{0} \\ \boldsymbol{\pi}\,\mathbf{e} = 1,\ \mathbf{e}^{\mathrm{T}} = (1, 1, \cdots)\end{array}}$$

- NOTE: Compare with DTMC $\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}$, $\boldsymbol{\pi}\,\mathbf{e} = 1$.

Numerical Solution

- Replace one equation method:

$$\boxed{\begin{array}{l} \boldsymbol{\pi}\,\mathbf{Q} = \mathbf{0} \\ \boldsymbol{\pi}\,\mathbf{e} = 1, \mathbf{e}^{\mathrm{T}} = (1, 1, \cdots) \end{array}}$$

- We solve the equation $\boldsymbol{\pi}\,\mathbf{Q} = 0$ replacing the last equation by $\boldsymbol{\pi}\,\mathbf{e} = 1$:

$$\boxed{\boldsymbol{\pi} \begin{bmatrix} q_{11} & q_{12} & \cdots q_{1n-1} & 1 \\ q_{21} & q_{22} & \cdots q_{2n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ q_{n1} & q_{n2} & \cdots q_{nn-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}$$

Numerical Solution

- Replace one equation method: $\mathbf{Q} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\quad \begin{matrix} \boldsymbol{\pi}\,\mathbf{Q} = 0 \\ \boldsymbol{\pi}\,\mathbf{e} = 1 \end{matrix}$
- Solving with octave (matlab clone):

```
octave:1> Q=[-2,1,1;1,-2,1;1,1,-2];
octave:2> s=size(Q,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [Q(1:s,1:s-1), ones(s,1)]
ans =
  0.33333  0.33333  0.33333
```

- With R

```
> Q <- matrix(nc=3, byr=T, c(-2,1,1,1,-2,1,1,1,-2))
> s <- nrow(Q)
> solve(t(cbind(Q[,1:(s-1)], rep(1,s))), c(rep(0,s-1),1))
[1] 0.3333333 0.3333333 0.3333333
```

Steady State

Global balance equations

- Why are they called Global balance equations?

$$\left.\begin{array}{rr} \boldsymbol{\pi}\,\mathbf{Q} = \mathbf{0} \Rightarrow & \displaystyle\sum_{i=0}^{\infty} \pi_i q_{ij} = 0 \\ \displaystyle\sum_{i=0}^{\infty} q_{ji} = 0 \Rightarrow & \pi_j \displaystyle\sum_{i=0}^{\infty} q_{ji} = 0 \end{array}\right\} \Rightarrow \pi_j \sum_{i=0}^{\infty} q_{ji} = \sum_{i=0}^{\infty} \pi_i q_{ij}$$

$\sum_{i=0}^{\infty} \pi_i q_{ij}$ $\Rightarrow$Frequency of transitions entering state j

$\pi_j \sum_{i=0}^{\infty} q_{ji}$ $\Rightarrow$Frequency of transitions leaving state j

- In stationary regime, the frequency of transitions leaving state j is equal to the frequency of transitions entering state j.

Solving using flux balancing

- Define the flux F_{uv} from state u to v:

$$F_{uv} = \pi_u q_{uv}$$

- and the flux from set of states U to V:

$$F(U,V) = \sum_{u \in U} \sum_{v \in V} F_{uv}$$

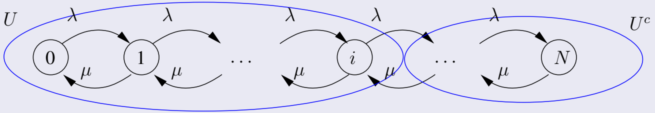
- From the Global balance equations, and reasoning exactly as in DTMC:

$$\boxed{F(U,U^c) = F(U^c,U)}$$

- NOTE: Same equation as in DTMC, changing p_{ij} by q_{ij}.

Steady State

Example: Birth-dead Process



- Flux balancing $\Rightarrow \lambda \pi_i = \mu \pi_{i+1}$
- Iterating:

$$\boxed{\pi_i = \pi_0 \rho^i,\ i = 0, 1, \cdots N-1,\ \rho = \frac{\lambda}{\mu}}$$

- Normalizing:

$$\boxed{\pi_0 = \frac{1-\rho}{1-\rho^N}}$$

Steady State

Ergodic Chains

- Ergodic state: positive recurrent ($f_{ii}^e = 1$, $m_{ii}^e < \infty$).
- Ergodic chain if all states are ergodic.
- Theorem: All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent (see [1, chapter XV]).
- Consequences:
  - Finite irreducible chains are ergodic (since all states are positive recurrent).
  - Infinite irreducible chains can be:
    - Ergodic: all the states are positive recurrent (stable chains).
    - Non ergodic: all states are null recurrent or transient (unstable chains).

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition*. Wiley, 1968.

Steady State

Theorems for ergodic chains

- $\boldsymbol{\pi} = \boldsymbol{\pi}(\infty)$. Proof: $\boldsymbol{\pi}(\infty)$ satisfies the GBE.
- In stationary regime (when $\boldsymbol{\pi} = \boldsymbol{\pi}\, e^{\mathbf{Q}\,t}$), the mean number of time the system remains in state j during T time units is given by

$$\boxed{T\pi_j}$$

  thus, π_j is the fraction of time the chain remains in state j. The proof is analogous to DTMC.
- NOTE: The relation of DTMC between mean recurrence time and stationary probabilities does not hold for CTMC. I.e., the mean number of time units between two consecutive visits to state j, m_{jj}, cannot be computed as $1/\pi_j$. It must be computed with the recursive equations (slide 35).

Reversible Chains

- Let $X(t)$ be an ergodic MC. The chain $X^r(t) = X(-t)$ is referred to as the time reversal chain of $X(t)$.
- The same results obtained for DTMC reversed chains apply to CTMC, changing p_{ij} by q_{ij}:
  - The reversed chain transition rates q^r_{ij}, given by:

$$\boxed{\pi_i q_{ij} = \pi_j q^r_{ji}}$$

    satisfy the reversed balance equations: $F(U,V) = F^r(V,U)$
- A chain is reversible if:

$$\boxed{q_{ij} = q^r_{ij}}$$

- Reversible chains satisfy the detailed balance equations:

$$\boxed{F(U,V) = F(V,U), \forall (V,U),\ V \cap U = \emptyset}$$

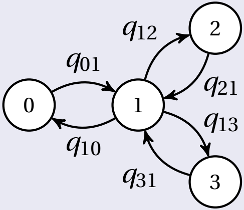
Reversible Chains

- The same results obtained for DTMC Reversible Chains apply to CTMC: Kolmogorov Criteria and Product Form Solution for the stationary distribution (changing p_{ij} by q_{ij}).
- E.g. the stationary probabilities are given by:
  - Choose a state $s \in S$,
  - For every other state $i \in S$, $i \neq s$ look for a possible path l_i from state s to state i:

$$s = (l_i,1) \rightsquigarrow (l_i,2) \rightsquigarrow \cdots \rightsquigarrow (l_i,m_{l_i}) = i,\ m_{l_i} \geq 1$$

$$\boxed{\pi_i = \frac{\psi_i}{\sum_{j\in S}\psi_j},\ i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \displaystyle\prod_{k=1}^{m_{l_i}-1} \frac{q_{(l_i,k)(l_i,k+1)}}{q_{(l_i,k+1)(l_i,k)}}, & i \neq s \end{cases}}$$

Reversible Chains, Example



- An ergodic tree is always reversible, thus

$$\pi_0 = \frac{1}{G}, \pi_1 = \frac{1}{G}\frac{q_{01}}{q_{10}}, \pi_2 = \frac{1}{G}\frac{q_{01}}{q_{10}}\frac{q_{12}}{q_{21}}, \pi_3 = \frac{1}{G}\frac{q_{01}}{q_{10}}\frac{q_{13}}{q_{31}}.$$

- Normalizing:

$$G = 1 + \frac{q_{01}}{q_{10}} + \frac{q_{01}}{q_{10}}\frac{q_{12}}{q_{21}} + \frac{q_{01}}{q_{10}}\frac{q_{13}}{q_{31}}$$

Part III

Continuous Time Markov Chains (CTMC)

Outline

Semi-Markov Process

Introduction

- Define the continuous RV H_i equal to the sojourn time in state i.
- In a semi-Markov process we leave the H_i distribution to be arbitrary. If H_i is exponentially distributed, we have a CTMC.
- NOTE: If H_i is not exponentially distributed, considering only the current state does not satisfy the Markov property (memoryless) since the evolution of the process depends on the current state and the sojourn time in the state: (i, t_i).
- If we consider (i, t_i) as the state, the state would satisfy the Markov property, but we would have a Markov process (since t_i is not a discrete RV).

Embedded MC (EMC) of a semi-Markov process

- Embedded MC (EMC) of the process: We only look at the state transition instants.
  - The EMC is a DTMC with transition probabilities p^e_{ij}.
  - The time step is variable.
  - There are not self transitions ($p^e_{ii} = 0$), unless we look at some memoryless event that produce a self transition.
- Theorem: let π^e_i and π_i be the stationary distribution of the EMC and the semi-Markov process respectively. Let $\mathrm{E}[H_i]$ be the mean sojourn time in state i, then:

$$\boxed{\pi_i = \frac{\pi^e_i \mathrm{E}[H_i]}{\sum_j \pi^e_j \mathrm{E}[H_j]}}$$

NOTE: By *stationary distribution* for the semi-Markov process we mean to the long-run proportion of time that the process is in each state.

Embedded MC (EMC) of a semi-Markov process

Proof:

- For n steps of the EMC, define:
  - $f_i(n)$: proportion of time the process is in state i.
  - $N_i(n)$: number of visits to state i.
  - $H_i(l)$: sojourn time in state i in the visit number l.

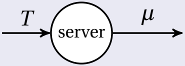
$$f_i(n) = \frac{\sum_{l=1}^{N_i(n)} H_i(l)}{\sum_j \sum_{l=1}^{N_j(n)} H_j(l)} = \frac{\frac{N_i(n)}{n} \sum_{l=1}^{N_i(n)} \frac{H_i(l)}{N_i(n)}}{\sum_j \frac{N_j(n)}{n} \sum_{l=1}^{N_j(n)} \frac{H_j(l)}{N_j(n)}} \Rightarrow \boxed{\pi_i = \frac{\pi_i^e \, \mathrm{E}[H_i]}{\sum_j \pi_j^e \, \mathrm{E}[H_j]}}$$

- since:

$$\boxed{\lim_{n\to\infty} f_i(n) = \pi_i}, \quad \boxed{\lim_{n\to\infty} \sum_{l=1}^{N_i(n)} \frac{H_i(l)}{N_i(n)} = \mathrm{E}[H_i]}, \quad \boxed{\lim_{n\to\infty} \frac{N_i(n)}{n} = \pi_i^e}.$$

Embedded MC (EMC) of a semi-Markov process, Example

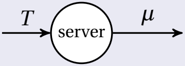
Suppose the system:



- Packets arrive deterministically every T time units.
- Upon a packet arrival it goes immediately into service if the server is empty, and it is lost if the server is busy.
- Services are exponentially distributed with rate μ.

Semi-Markov Process

Embedded MC (EMC) of a semi-Markov process, Example



Define a semi-Markov process with states $\begin{cases} 0 & \text{server empty,} \\ 1 & \text{server busy.} \end{cases}$

1. Derive the EMC and stationary distribution of the EMC and continuous time process.
2. Compute the throughput and loss probability.

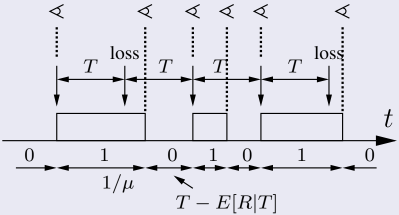
Hint: The distribution of an event R exponentially distributted with rate μ, given that occurs in an interval $t \in [0, T]$, is

$$F_R(t|T) = \frac{P(R \le t, R \le T)}{P(R \le T)} = \frac{P(R \le t)_{t \in [0,T]}}{P(R \le T)} = \frac{1 - e^{-\mu t}}{1 - e^{-\mu T}}, \ t \in [0, T], \ \text{and}$$

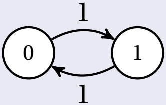
$$E[R|T] = \int_0^T (1 - F_R(t|T))\, dt = \frac{1 - \alpha - \alpha \mu T}{\mu (1 - \alpha)}, \text{where } \alpha = e^{-\mu T}.$$

Semi-Markov Process

Embedded MC (EMC) of a semi-Markov process, time diagram



EMC:



Embedded MC (EMC) of a semi-Markov process, Solution

EMC:

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \end{tikzcd}$$

- $\pi_0^e = \pi_1^e = 1/2$,
- $\mathrm{E}[H_0] = T - \mathrm{E}[R|T] = \dfrac{T\mu - (1-\alpha)}{\mu(1-\alpha)}, \mathrm{E}[H_1] = \dfrac{1}{\mu}$.

And the continuous time process:

$$G = \pi_0^e \mathrm{E}[H_0] + \pi_1^e \mathrm{E}[H_1] = \frac{T}{2(\alpha-1)} \left[\frac{\text{time units}}{\text{step}}\right]$$

$$\pi_0 = \frac{\pi_0^e \mathrm{E}[H_0]}{G} = \frac{1-\alpha}{\mu T}, \pi_1 = \frac{\pi_1^e \mathrm{E}[H_1]}{G} = \frac{\mu T - (1-\alpha)}{\mu T}$$

Throughput: $S = \mu \pi_1 = \dfrac{1-\alpha}{T}$ (check $S = \dfrac{1}{\mathrm{E}[H_0] + \mathrm{E}[H_1]}$)

Loss probability: $S = \dfrac{1}{T}(1-p_L), p_L = 1 - ST = \alpha = e^{-\mu T}$.