Radomized Algorithms Problemes (24-28) Fall 2019.
Deadline Nov. 26th

24.- Suppose X and Y are independent Poisson rv with parameters 1 and 2 respectively. Find:

(a) $\mathbf{Pr}\left[X = 1 \text{and} Y = 2\right]$

(b) $\mathbf{Pr}\left[\frac{X+Y}{2} \geq 1\right]$

(c) $\mathbf{Pr}\left[X = 1 \mid \frac{X+Y}{2} \geq 2\right]$

25.- Let X be a Poisson rv. with parameter λ. Find:

(a) $\mathbf{E}\left[3X + 5\right]$.

(b) $\mathbf{Var}\left[3X + 5\right]$.

(c) $\mathbf{E}\left[\frac{1}{1+X}\right]$.

26.- (MU5.7) Suppose that n balls are thrown independently and u.a.r. into n bins.

(a) Find the conditional probability that bin 1 has one ball given that exactly one ball fell into the first three bins.

(b) Find the conditional expectation of the number of balls in bin 1 under the condition that bin 2 received no balls.

(c) Write an expression for the probability that bin 1 receives more balls that bin 2.

27.- (Number of collisions) (*) Send n balls into m bins, let Z be a random variable counting the number of collisions.

(a) Compute $\mu = \mathbf{E}\left[X\right]$

(b) Use Chebyshev to show that $\mathbf{Pr}\left[|Z - \mu| \geq c\sqrt{\mu}\right] \leq 1/c^2$, for constant $c > 0$.

(c) Assume $n = 2\sqrt{n}$ then $\mu \leq 1$. Use Chernoff bounds plus the union bound to bound the probability that no bin has more than 1 ball.

28.- (*) (Coupon collector) Recall that we showed the number of balls we need to throw before every one of the m bin has at least one ball, is $\mu = \mathbf{E}[Y] \leq cm \ln m$.

(a) Use Chebyshev inequality to show $\mathbf{Pr}[|Y - \mu| \geq c\mu] \leq \frac{\pi^2}{6c^2 H_m}$, where $H_m = \sum_{i=1}^{m} \frac{1}{i} \sim \lg n$. *(Use the know fact that $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is the Euler-Riemann function $\zeta(2)$ and it is known that $\zeta(2) = \pi^2/6$.)*

(b) If we throw $n = m \ln m + cm$ balls, use Chernoff plus union-bound, to choose a value for c such that no bin is empty with probability $> 1 - \delta$, for $0 < \delta \leq 1$.

(c) How do you compare those two bounds with the bound produced in the class?