Solution

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Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

Stochastic Network Modeling (SNM). Autumn 2019.

Second assessment, Continuous Time Markov Chains. 25/11/2019.

Problem 1

Assume a *Carrier Sense Multiple Access with Collision Avoidance* (CSMA/CA) MAC protocol with 2 nodes. This protocol is similar to Aloha, but nodes listen before transmitting (CSMA). Initially both nodes are in thinking state, and enter this state after a successful packet transmission. If the medium is idle the packet is transmitted, otherwise the node enters in backlogged state, and CSMA is tried again after a backoff time (CA). Assume that:

- The system is in steady state.
- Nodes in thinking state run CSMA in a time exponentially distributed with rate $\lambda = 1/4$.
- Transmission time is exponentially distributed with rate $\mu = 1$.
- Nodes in backlogged state run CSMA in a backoff time exponentially distributed with rate $\alpha = 3/4$.
- **Packets are transmitted only twice**. That is, nodes discard the packets and move into thinking state when they find the medium busy for the second time, as shown in figure 1.

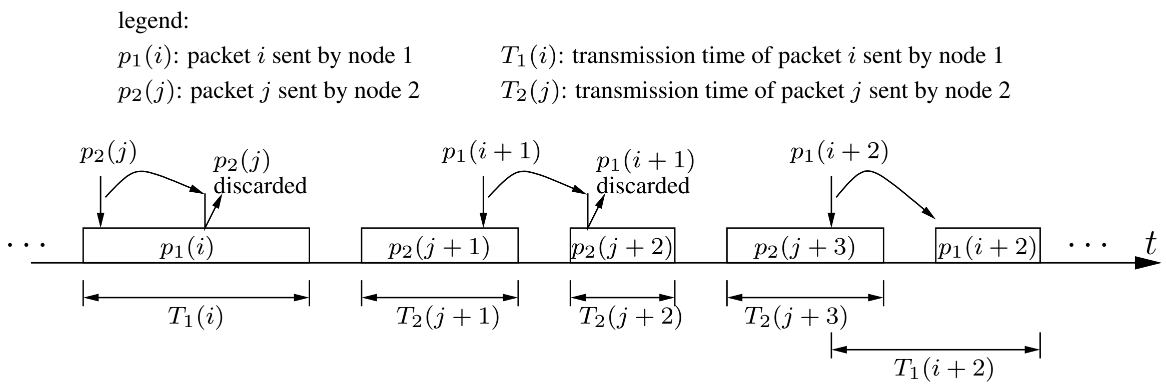


Figure 1: Time diagram of the system.

1.A (1.5 points) Draw the transition diagram of a CTMC that allows computing the throughput. Define clearly the meaning of each state and the value of the transition rates.

1.B (1.5 points) Compute the stationary distribution.

1.C (1.5 points) Compute the overall throughput S (expected number of successful packets transmitted per time unit).

1.D (1.5 points) Compute the probability that a new arriving packet is eventually discarded.

1.E (1 point) Define the events

$$A = \text{when a tinking node starts a new transmission finds the medium idle}$$
$$B = \text{when a tinking node starts a new transmission finds the medium busy}$$

Compute $P(A)$ and $P(B)$.

1.F (1 point) Let T be the random variable equal to the transmission time of non discarded packets (see figure 1). Compute $P(T \le t \mid A)$, where A is the event defined in the previous item.

1.G (1 point) Let T be the random variable equal to the transmission time of a node (see figure 1). Compute $P(T \le t \mid B)$, where B is the event defined previously. Hint: use an absorbing CTMC.

1.H (1 point) Let T be the random variable equal to the transmission time of a node (see figure 1). Compute $P(T \le t)$ and E$[T]$ using the results of the previous items.

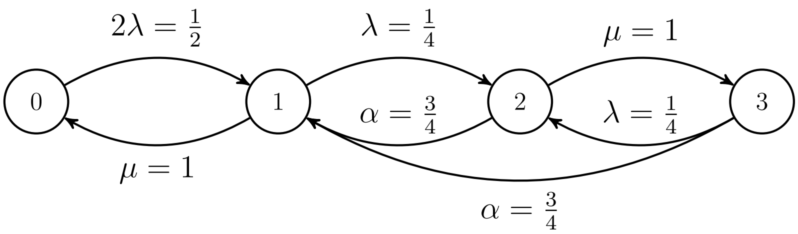
Solution

Problem 1

1.A Define the states:

0 2 nodes thinking
1 1 node transmitting and 1 node thinking
2 1 node transmitting and 1 node backlogged
3 1 node thinking and 1 node backlogged

we have the chain:



1.B Using flux balancing:

$$\pi_0 \frac{1}{2} = \pi_1$$
$$\pi_1 \frac{1}{4} = \pi_2 \frac{3}{4} + \pi_3 \frac{3}{4}$$
$$\pi_2 = \pi_3$$

which yields $\pi_0 = \frac{12}{20}$, $\pi_1 = \frac{6}{20}$, $\pi_2 = \pi_3 = \frac{1}{20}$.

1.C

$$S = \mu\,(\pi_1 + \pi_2) = \frac{7}{20}$$

1.D We have a loss rate equal to

$$L = \alpha\,\pi_2 = \frac{3}{80},$$

and a new packet arrivals rate of

$$G = 2\,\lambda\,\pi_0 + \lambda\,\pi_1 + \lambda\,\pi_3 = \frac{31}{80}.$$

Thus, the proportion of new packet arrivals eventually discarded is:

$$p_L = \frac{\text{number of lost packets}}{\text{new packets arrivals}} = \frac{L}{G} = \frac{3}{31}$$

Check: $S = G\,(1 - p_L) = \frac{31}{80}(1 - \frac{3}{31}) = \frac{7}{20}$, as expected.

1.E We have

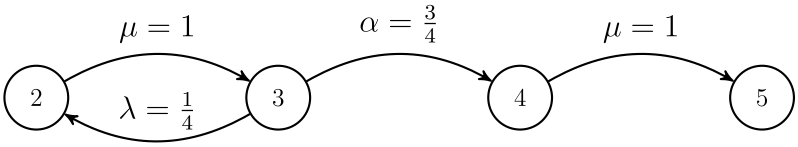
$$P(A) = \frac{\text{new packet arrivals that find the medium idle}}{\text{new packet arrivals}} = \frac{2\,\lambda\,\pi_0 + \lambda\,\pi_3}{G} = \frac{25}{31}$$
$$P(B) = 1 - P(A) = \frac{6}{31}$$

1.F Clearly,
$$P(T \le t \mid A) = 1 - \mathrm{e}^{-\mu t} = 1 - \mathrm{e}^{-t},\ t \ge 0$$

1.G Let n_1 be the tagged node. Define the absorbing chain with states:

2 n_2 transmitting and n_1 backlogged
3 n_2 thinking and n_1 backlogged
4 n_2 transmitting
5 end of n_2 transmission

we have the chain:



with rate matrix:
$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1/4 & -1 & 3/4 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and characteristic polynomial
$$\det(\lambda\, I - Q) = \lambda\,(\lambda+1)\,[(\lambda+1)^2 - 1/4] = \lambda\,(\lambda+1)\,(\lambda^2 + 2\,\lambda + 3/4)$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -1/2$, $\lambda_4 = -3/2$. Thus, we guess
$$\pi_5(t) = 1 + a\,\mathrm{e}^{-t} + b\,\mathrm{e}^{-t/2} + c\,\mathrm{e}^{-3\,t/2},\ t \ge 0.$$

Imposing the boundary conditions:
$$\begin{aligned} \pi_5(0) &= 1 + a + b + c = 0 \\ \pi_5'(0) &= -a - b/2 - 3\,c/2 = 0 \\ \pi_5''(0) &= a + b/4 + 9\,c/4 = 0 \end{aligned}$$

we get $a = 3$, $b = -3$, $c = -1$ and
$$P(T \le t \mid B) = \pi_5(t) = 1 + 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-t/2} - \mathrm{e}^{-3\,t/2},\ t \ge 0.$$

1.H We have
$$\begin{aligned} P(T \le t) &= P(T \le t \mid A)\,P(A) + P(T \le t \mid B)\,P(B) \\ &(1 - \mathrm{e}^{-t})\,\frac{25}{31} + (1 + 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-t/2} - \mathrm{e}^{-3\,t/2})\,\frac{6}{31} = \\ &1 - \frac{7}{31}\,\mathrm{e}^{-t} - \frac{18}{31}\,\mathrm{e}^{-t/2} - \frac{6}{31}\mathrm{e}^{-3\,t/2},\ t \ge 0, \end{aligned}$$

and
$$\mathrm{E}[T] = \int_0^\infty (1 - P(T \le t))\,dt = \frac{7}{31} + \frac{18}{31}\,\frac{1}{1/2} + \frac{6}{31}\,\frac{1}{3/2} = \frac{47}{31}$$