The Dual Simplex Method

Combinatorial Problem Solving (CPS)

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Basic Idea

- **Abuse of terminology:**

  Henceforth sometimes by "optimal" we will mean "satisfying the optimality conditions"

  If not explicit, the context will disambiguate

- The algorithm as explained so far is known as primal simplex:

  starting with feasible basis,

  find optimal basis (= satisfying optimality conds.) while keeping feasibility

- There is an alternative algorithm known as dual simplex:

  starting with optimal basis (= satisfying optimality conds.),

  find feasible basis while keeping optimality

Basic Idea

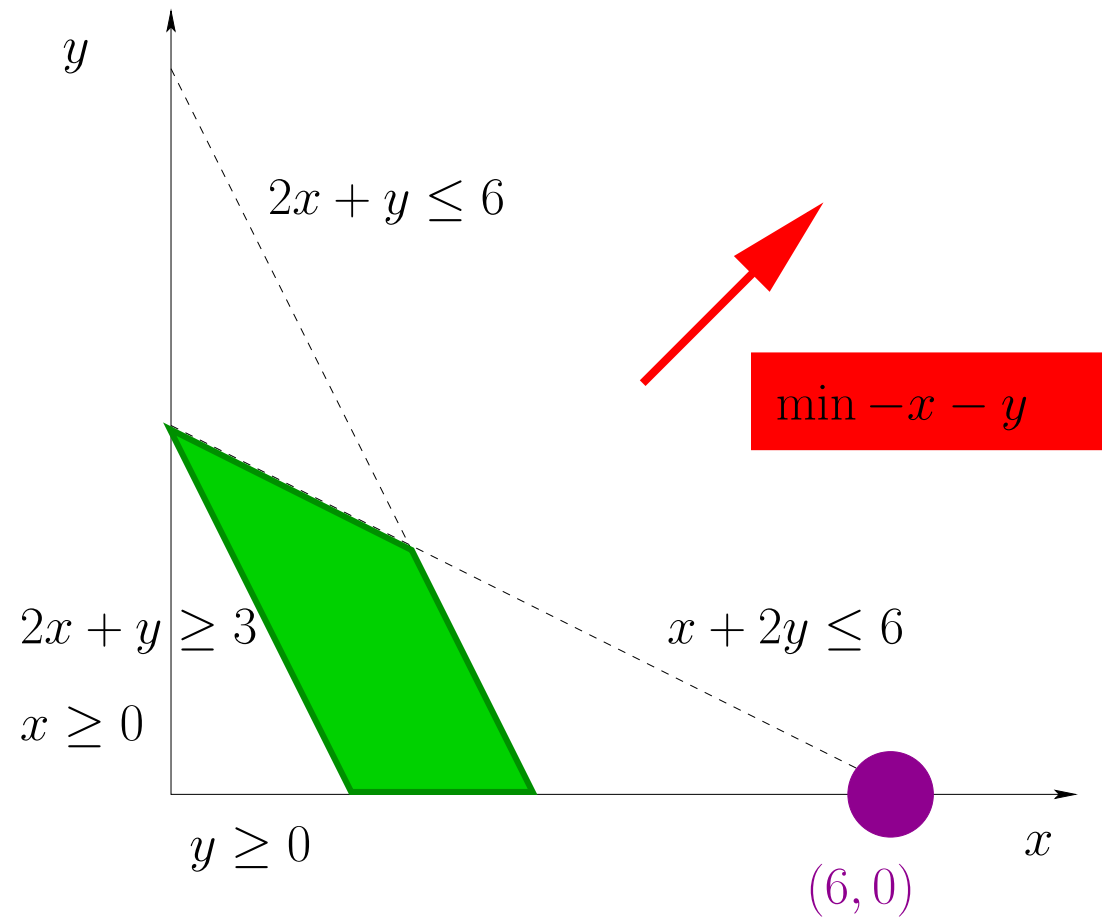$$\begin{cases} \min \ -x - y \\ 2x + y \geq 3 \\ 2x + y \leq 6 \\ x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases} \implies \begin{cases} \min \ -x - y \\ 2x + y - s_1 = 3 \\ 2x + y + s_2 = 6 \\ x + 2y + s_3 = 6 \\ x, y, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\begin{cases} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases}$$

Basis (x, s_1, s_2) is optimal
($=$ satisfies optimality conditions)

but is not feasible!

Basic Idea

Basic Idea

- Let us make a violating basic variable non-negative ...

  - Increase s_2 by making it non-basic: then it will be 0

- ... while preserving optimality ($=$ optimality conditions are satisfied)

  - If y replaces s_2 in the basis,
    then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$

  - If s_3 replaces s_2 in the basis,
    then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$

Basic Idea

- Let us make a violating basic variable non-negative …

  - Increase s_2 by making it non-basic: then it will be 0

- … while preserving optimality ($=$ optimality conditions are satisfied)

  - If y replaces s_2 in the basis,
    then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$

  - If s_3 replaces s_2 in the basis,
    then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$

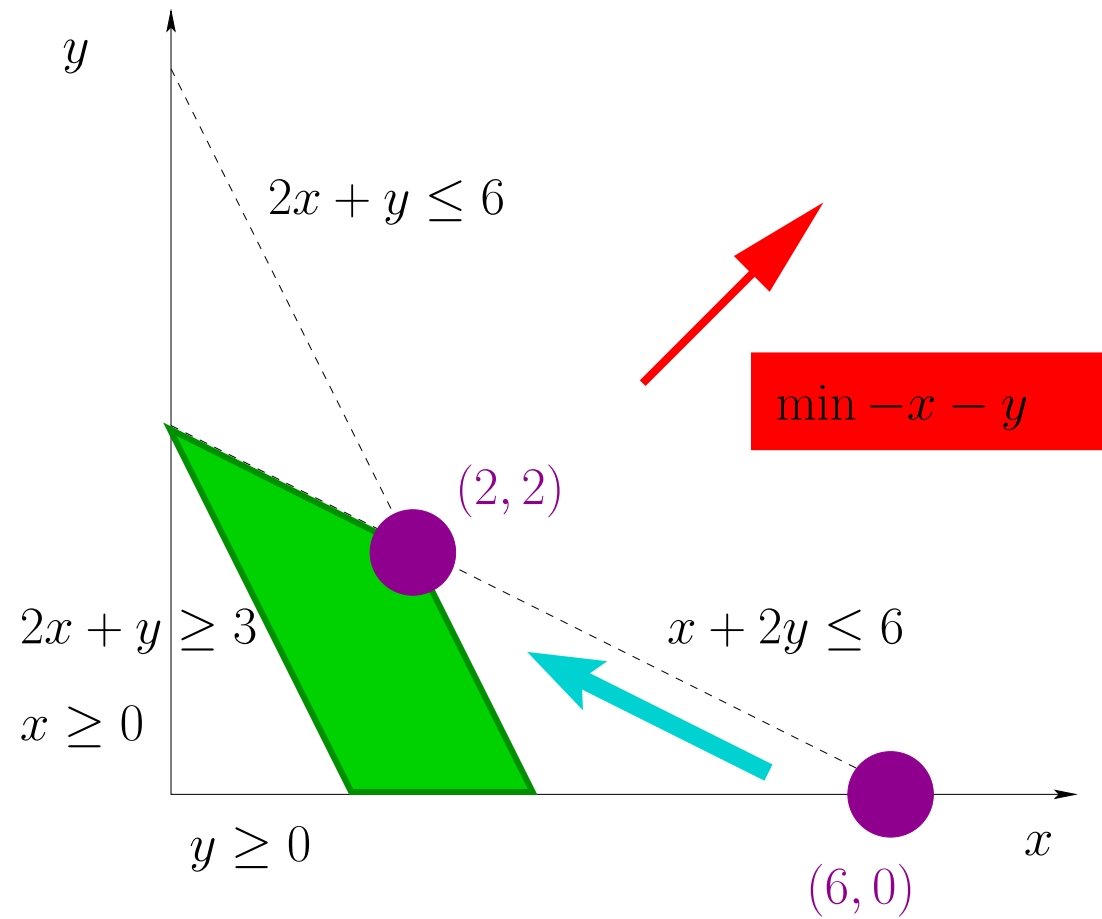 - To preserve optimality, y must replace s_2

Basic Idea

$$\begin{cases} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases} \implies \begin{cases} \min \ -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\ x = 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\ y = 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\ s_1 = 3 - s_2 \end{cases}$$

Basic Idea

$$\begin{cases} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases} \implies \begin{cases} \min \ -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\ x = 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\ y = 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\ s_1 = 3 - s_2 \end{cases}$$

■ Current basis is feasible and optimal!

Basic Idea

Outline of the Dual Simplex

1. `Initialization:` Pick an optimal basis.

2. `Dual Pricing:` If all basic values are ≥ 0,
    then return OPTIMAL.
    Else pick a basic variable with value < 0.

3. `Dual Ratio test:` Find non-basic variable for swapping that
    preserves optimality, i.e., non-negativity constraints on reduced costs.

    If it does not exist,
    then return INFEASIBLE.
    Else swap chosen non-basic variable with violating basic variable.

4. `Update:` Update the tableau and go to 2.

Duality

■ To understand better how the dual simplex works: theory of duality

■ We can get lower bounds on LP optimum value
by adding constraints in a convenient way

$$
\begin{cases}
\min \ -x - y \\
2x + y \geq 3 \\
2x + y \leq 6 \\
x + 2y \leq 6 \\
x \geq 0 \\
y \geq 0
\end{cases}
\Rightarrow
\begin{cases}
\min \ -x - y \\
2x + y \geq 3 \\
-2x - y \geq -6 \\
-x - 2y \geq -6 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

$$
\begin{array}{rcl}
-x - 2y & \geq & -6 \\
y & \geq & 0 \\
\hline
-x - y & \geq & -6
\end{array}
$$

Duality

■ In general we can get lower bounds on LP optimum value by linearly combining constraints with convenient multipliers

$$
\begin{cases}
\min \ -x - y \\
2x + y \geq 3 \\
-2x - y \geq -6 \\
-x - 2y \geq -6 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

$$
\begin{array}{rrcrr}
1 \cdot (& 2x + y & \geq & 3 &) \\
2 \cdot (& -2x - y & \geq & -6 &) \\
1 \cdot (& x & \geq & 0 &)
\end{array}
$$

$$
\begin{array}{rcr}
2x + y & \geq & 3 \\
-4x - 2y & \geq & -12 \\
x & \geq & 0
\end{array}
$$

$$
-x - y \ \geq \ -9
$$

■ There may be different choices, each giving a different lower bound

Duality

$$\mu_1 \cdot (\quad\quad 2x + y \quad \geq \quad 3 \quad\quad)$$
$$\mu_2 \cdot (\quad\quad -2x - y \quad \geq \quad -6 \quad\quad)$$
$$\mu_3 \cdot (\quad\quad -x - 2y \quad \geq \quad -6 \quad\quad)$$
$$\mu_4 \cdot (\quad\quad x \quad \geq \quad 0 \quad\quad)$$
$$\mu_5 \cdot (\quad\quad y \quad \geq \quad 0 \quad\quad)$$

- In general:

$$
\begin{cases}
\min \ -x - y \\
2x + y \geq 3 \\
-2x - y \geq -6 \\
-x - 2y \geq -6 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

$$2\mu_1 x + \mu_1 y \quad \geq \quad 3\mu_1$$
$$-2\mu_2 x - \mu_2 y \quad \geq \quad -6\mu_2$$
$$-\mu_3 x - 2\mu_3 y \quad \geq \quad -6\mu_3$$
$$\mu_4 x \quad \geq \quad 0$$
$$\mu_5 y \quad \geq \quad 0$$

$$\left(2\mu_1 - 2\mu_2 - \mu_3 + \mu_4\right) x \ + \ \left(\mu_1 - \mu_2 - 2\mu_3 + \mu_5\right) y \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

- If $2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 = -1$, $\quad \mu_1 - \mu_2 - 2\mu_3 + \mu_5 = -1$,
  $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$, $\mu_5 \geq 0$,
  then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

■ We can skip the multipliers of the non-negativity constraints

■ We have:

$$\begin{cases} \min\ -x-y \\ 2x+y \geq 3 \\ -2x-y \geq -6 \\ -x-2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{llll} \mu_1 \cdot (& 2x+y & \geq & 3 &) \\ \mu_2 \cdot (& -2x-y & \geq & -6 &) \\ \mu_3 \cdot (& -x-2y & \geq & -6 &) \end{array}$$

$$\begin{array}{rcl} 2\mu_1 x + \mu_1 y & \geq & 3\mu_1 \\ -2\mu_2 x - \mu_2 y & \geq & -6\mu_2 \\ -\mu_3 x - 2\mu_3 y & \geq & -6\mu_3 \end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3)\, x \ + \ (\mu_1 - \mu_2 - 2\mu_3)\, y \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

■ Imagine $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$.
In the coefficient of x we can "complete" $2\mu_1 - 2\mu_2 - \mu_3$ to reach -1 by adding a suitable multiple of $x \geq 0$ (the multiplier will be the slack)

■ If $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$, $\mu_1 - \mu_2 - 2\mu_3 \leq -1$,
$\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

■ Best possible lower bound with this "trick" can be found by solving

$$
\begin{cases}
\max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\
& 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\
& \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\
& \mu_1, \mu_2, \mu_3 \geq 0
\end{cases}
$$

■ How far will it be from the optimum?

Duality

■ Best possible lower bound with this "trick" can be found by solving

$$\left\{ \begin{array}{ll} \max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\ & 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\ & \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\ & \mu_1, \mu_2, \mu_3 \geq 0 \end{array} \right.$$

■ How far will it be from the optimum?

■ A best solution is given by $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})$

$$\begin{array}{rcccc} 0 \cdot (& 2x + y & \geq & 3 &) \\ \frac{1}{3} \cdot (& -2x - y & \geq & -6 &) \\ \frac{1}{3} \cdot (& -x - 2y & \geq & -6 &) \end{array}$$

Matches the optimum!

$$\begin{array}{ccc} -x - y & \geq & -4 \end{array}$$

Dual Problem

- If we multiply constraints $Ax \geq b$ by multipliers y^T we get $y^T Ax \geq y^T b$
- Given an LP (called primal)

$$\min \ c^T x$$
$$Ax \geq b$$
$$x \geq 0$$

its dual is the LP

$$\max \ y^T b \qquad\qquad \max \ b^T y$$
$$y^T A \leq c^T \qquad \text{or equivalently} \qquad A^T y \leq c$$
$$y^T \geq 0 \qquad\qquad\qquad\qquad\qquad y \geq 0$$

- Primal variables associated with columns of A

- Dual variables (multipliers) associated with rows of A

- Objective and right-hand side vectors swap their roles

Dual Problem

■ **Prop.** The dual of the dual is the primal.

Proof:

$$\begin{array}{l} \max\ b^T y \\ A^T y \le c \\ y \ge 0 \end{array} \qquad \Longrightarrow \qquad \begin{array}{l} -\min\ (-b)^T y \\ -A^T y \ge -c \\ y \ge 0 \end{array}$$

$$\begin{array}{l} -\max\ -c^T x \\ (-A^T)^T x \le -b \\ x \ge 0 \end{array} \qquad \Longrightarrow \qquad \begin{array}{l} \min\ c^T x \\ Ax \ge b \\ x \ge 0 \end{array}$$

■ We say the primal and the dual form a primal-dual pair

Dual Problem

■ **Prop.** $\begin{aligned}\min\ &c^T x\\ &Ax = b\\ &x \geq 0\end{aligned}$ and $\begin{aligned}\max\ &b^T y\\ &A^T y \leq c\end{aligned}$ form a primal-dual pair

Proof:

$$\begin{aligned}\min\ &c^T x\\ &Ax = b\\ &x \geq 0\end{aligned} \implies \begin{aligned}\min\ &c^T x\\ &Ax \geq b\\ &-Ax \geq -b\\ &x \geq 0\end{aligned}$$

$$\begin{aligned}\max\ &b^T y_1 - b^T y_2\\ &A^T y_1 - A^T y_2 \leq c\\ &y_1, y_2 \geq 0\end{aligned} \xRightarrow{y := y_1 - y_2} \begin{aligned}\max\ &b^T y\\ &A^T y \leq c\end{aligned}$$

Duality Theorems

- **Th. (Weak Duality)** Let (P, D) be a primal-dual pair

$$
(P) \quad \begin{array}{l} \min \ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \ b^T y \\ A^T y \leq c \end{array}
$$

If x is feasible solution to P and y is feasible solution to D then $b^T y \leq c^T x$

Proof:

$c - A^T y \geq 0$, i.e., $c^T - y^T A \geq 0$, and $x \geq 0$ imply $c^T x - y^T A x \geq 0$.

So $c^T x \geq y^T A x$, and

$$
c^T x \geq y^T A x = y^T b = b^T y
$$

Duality Theorems

- Feasible solutions to D give lower bounds on P

- Feasible solutions to P give upper bounds on D

- Will the two optimum values be always equal?

Duality Theorems

■ Feasible solutions to D give lower bounds on P

■ Feasible solutions to P give upper bounds on D

■ Will the two optimum values be always equal?

■ **Th. (Strong Duality)** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \ b^T y \\ A^T y \leq c \end{array}$$

If any of P or D has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.

Duality Theorems

■ Proof (Th. of Strong Duality):

By symmetry it is sufficient to prove only one direction.
Wlog. let us assume P is feasible with finite optimum.

Duality Theorems

- **Proof (Th. of Strong Duality):**

  By symmetry it is sufficient to prove only one direction.
  Wlog. let us assume P is feasible with finite optimum.

  After executing the Simplex algorithm to P we find
  B optimal feasible basis. Then:

  - $c_{\mathcal{B}}^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
  - $c_{\mathcal{B}}^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

  So $c_{\mathcal{B}}^T B^{-1} A \leq c^T$.
  Hence $\pi^T := c_{\mathcal{B}}^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$

Duality Theorems

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  Hence $\pi^T := c_{\mathcal{B}}^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$
  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$
  By the theorem of weak duality, π is optimum for D

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  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$
  By the theorem of weak duality, π is optimum for D

- If B is an optimal feasible basis for P,
  then simplex multipliers $\pi^T := c_{\mathcal{B}}^T B^{-1}$ are optimal feasible solution for D
- We can solve the dual by applying the simplex algorithm on the primal
- We can solve the primal by applying the simplex algorithm on the dual

Duality Theorems

- **Prop.** Let (P, D) be a primal-dual pair

$$
(P) \quad \begin{aligned} \min \ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{and} \qquad (D) \quad \begin{aligned} \max \ & b^T y \\ & A^T y \leq c \end{aligned}
$$

(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

Proof:

Let us prove (1) by contradiction.

If y were a feasible solution to D,
by the weak duality theorem, objective of P would be lower bounded!

(2) is proved by duality.

Duality Theorems

■ **Prop.** Let (P, D) be a primal-dual pair

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(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

■ And the converse?
  Does infeasibility of one imply unboundedess of the other?

Duality Theorems

- **Prop.** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \ b^T y \\ A^T y \leq c \end{array}$$

(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

- And the converse?
  Does infeasibility of one imply unboundedess of the other?

$$\begin{array}{ll} \min & 3x_1 + 5x_2 \\ & x_1 + 2x_2 = 3 \\ & 2x_1 + 4x_2 = 1 \\ & x_1, x_2 \ \text{free} \end{array} \qquad\qquad \begin{array}{ll} \max & 3y_1 + y_2 \\ & y_1 + 2y_2 = 3 \\ & 2y_1 + 4y_2 = 5 \\ & y_1, y_2 \ \text{free} \end{array}$$

Duality Theorems

$$
\begin{array}{lcl}
\text{Primal unbounded} & \Longrightarrow & \text{Dual infeasible} \\
\text{Dual unbounded} & \Longrightarrow & \text{Primal infeasible} \\
\text{Primal infeasible} & \Longrightarrow & \text{Dual} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \\
\text{Dual infeasible} & \Longrightarrow & \text{Primal} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases}
\end{array}
$$

Karush Kuhn Tucker Opt. Conds.

■ Consider a primal-dual pair of the form

$$\min \ c^T x$$
$$Ax = b \quad \text{and}$$
$$x \geq 0$$

$$\max \ b^T y$$
$$A^T y \leq c \quad \Longleftrightarrow$$

$$\max \ b^T y$$
$$A^T y + w = c$$
$$w \geq 0$$

■ Karush-Kuhn-Tucker (KKT) optimality conditions are

- $Ax = b$
- $A^T y + w = c$

- $x, w \geq 0$
- $x^T w = 0$ (complementary slackness)

■ They are necessary and sufficient conditions
for optimality of the pair of primal-dual solutions $(x, (y, w))$

■ Used, e.g., as a test of quality in LP solvers

Karush Kuhn Tucker Opt. Conds.

$$(KKT)$$

- $Ax = b$
- $A^T y + w = c$
- $x, w \geq 0$
- $x^T w = 0$

$$\begin{array}{ll} & \min \ c^T x \\ (P) & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} & \max \ b^T y \\ (D) & A^T y + w = c \\ & w \geq 0 \end{array}$$

■ **Th.** $(x, (y, w))$ is solution to KKT iff
x optimal solution to P and (y, w) optimal solution to D

Proof:

$\Rightarrow$ By $0 = x^T w = x^T (c - A^T y) = c^T x - b^T y$, and Weak Duality

$\Leftarrow x$ is feasible solution to P, (y, w) is feasible solution to D.
By Strong Duality $x^T w = x^T (c - A^T y) = c^T x - b^T y = 0$
as both solutions are optimal

Relating Bases

- Consider a primal-dual pair of the form

$$
\begin{array}{ll}
\min\ z = c^T x & \max\ Z = b^T y \\
(P)\ \ Ax = b & (D)\ \ A^T y + w = c \\
\ \ \ \ x \geq 0 & \ \ \ \ w \geq 0
\end{array}
$$

- Let us denote by a_1, ..., a_n the columns of A, i.e., $A = (a_1, \ldots, a_n)$
- Let B be a basis of P. Let us see how we can get a basis of D.

  Assume that the basic variables are the first m: $B = (a_1, \ldots, a_m)$.

  Then $R = (a_{m+1}, \ldots, a_n)$.

  If slacks w are split into $w_{\mathcal{B}}^T = (w_1, \ldots, w_m)$, $w_{\mathcal{R}}^T = (w_{m+1}, \ldots, w_n)$, then

$$
A^T y + w = \begin{pmatrix} a_1^T y \\ \cdots \\ a_m^T y \\ \hline a_{m+1}^T y \\ \cdots \\ a_n^T y \end{pmatrix} + \begin{pmatrix} w_1 \\ \cdots \\ w_m \\ \hline w_{m+1} \\ \cdots \\ w_n \end{pmatrix} = \begin{pmatrix} B^T y + w_{\mathcal{B}} \\ \hline R^T y + w_{\mathcal{R}} \end{pmatrix}
$$

Relating Bases

■ Hence we have

$$A^T y + w = \left(\frac{B^T y + w_{\mathcal{B}}}{R^T y + w_{\mathcal{R}}} \right)$$

■ Then the matrix of the system in the dual problem D is

$$\left(\begin{array}{c|c|c} B^T & I & 0 \\ \hline R^T & 0 & I \end{array} \right) \left(\begin{array}{c} y \\ w_{\mathcal{B}} \\ w_{\mathcal{R}} \end{array} \right)$$

■ Now let us consider the submatrix of vars y and vars $w_{\mathcal{R}}$:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

■ Note $\hat{B}$ is a square $n \times n$ matrix

Relating Bases

■ Dual variables $\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ determine a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right)$$

Relating Bases

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■ In the next slides we answer the following questions:

1. If basis $\hat{B}$ of the dual D is feasible,
   what can we say about basis B of the primal P?

2. If basis $\hat{B}$ of the dual D is optimal (satisfies the optimality conds.),
   what can we say about basis B of the primal P?

3. If we apply the simplex algorithm to the dual D using basis $\hat{B}$,
   how does that translate into the primal P and its basis B?

Relating Bases

- Dual variables $\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ determine a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right)$$

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     what can we say about basis B of the primal P?

  3. If we apply the simplex algorithm to the dual D using basis $\hat{B}$,
     how does that translate into the primal P and its basis B?

- Recall that each variable w_j in D is associated to a variable x_j in P.
- Note that w_j is $\hat{\mathcal{B}}$-basic iff x_j is not $\mathcal{B}$-basic

Dual Feasibility $=$ Primal Optimality

■ If $\hat{B}$ is feasible for dual D, what about B in primal P?

$$\hat{B}^{-1}c = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right) \left(\begin{array}{c} c_{\mathcal{B}} \\ c_{\mathcal{R}} \end{array} \right) = \left(\begin{array}{c} B^{-T}c_{\mathcal{B}} \\ \hline -R^T B^{-T}c_{\mathcal{B}} + c_{\mathcal{R}} \end{array} \right)$$

■ There is no restriction on the sign of $y_1, ..., y_m$

■ Variables w_j have to be non-negative. But

$$-R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c_{\mathcal{R}}^T - c_{\mathcal{B}}^T B^{-1} R \geq 0$$

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- $\hat{B}$ is dual feasible iff $d_j \geq 0$ for all $j \in \mathcal{R}$

- Dual feasibility is primal optimality!

Dual Optimality $=$ Primal Feasibility

■ If $\hat{B}$ satisfies the optimality conds. for dual D, what about B in primal P?

■ Non $\hat{\mathcal{B}}$-basic vars: $w_{\mathcal{B}}$ with costs (0)

■ $\hat{\mathcal{B}}$-basic vars: $(y \mid w_{\mathcal{R}})$ with costs $(b^T \mid 0)$

■ Matrix of non $\hat{\mathcal{B}}$-basic vars: $\left(\dfrac{I}{0} \right)$

■ Optimality condition: $0 \geq$ reduced costs (maximization!)

$$0 \geq (\, 0 \,) - (\, b^T \mid 0 \,) \left(\frac{B^{-T} \mid 0}{-R^T B^{-T} \mid I} \right) \left(\frac{I}{0} \right) =$$

$$(\, 0 \,) - (\, b^T B^{-T} \mid 0 \,) \left(\frac{I}{0} \right) = -b^T B^{-T} = -(B^{-1}b)^T$$

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- In the dual problem,
  for all $1 \leq p \leq m$, var w_{k_p} cannot improve objective function iff $\beta_p \geq 0$

- Dual optimality is primal feasibility!

Improving a Non-Optimal Solution

■ Next we apply the simplex algorithm to basis $\hat{B}$ in the dual problem D and translate it to the primal problem P

■ Let p (where $1 \le p \le m$) be such that $\beta_p < 0$.

I.e., the reduced cost of non-basic dual variable w_{k_p} is positive.

So by giving w_{k_p} a larger value we can improve the dual objective value.

If w_{k_p} takes value $t \ge 0$:

$$\begin{pmatrix} y(t) \\ w_{\mathcal{R}}(t) \end{pmatrix} = \hat{B}^{-1}c - \hat{B}^{-1}te_p =$$

$$= \begin{pmatrix} B^{-T}c_{\mathcal{B}} \\ d_{\mathcal{R}} \end{pmatrix} - \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right) \begin{pmatrix} te_p \\ 0 \end{pmatrix} = \begin{pmatrix} B^{-T}c_{\mathcal{B}} - tB^{-T}e_p \\ d_{\mathcal{R}} + tR^T B^{-T}e_p \end{pmatrix}$$

■ Dual objective value improvement is

$$\Delta Z = b^T y(t) - b^T y(0) = -tb^T B^{-T}e_p = -t\beta^T e_p = -t\beta_p = t(-\beta_p)$$

Improving a Non-Optimal Solution

- Of all basic dual variables, only $w_{\mathcal{R}}$ variables need to be ≥ 0

- For $j \in \mathcal{R}$

$$w_j(t) = d_j + ta_j^T B^{-T} e_p = d_j + te_p^T B^{-1} a_j = d_j + te_p^T \alpha_j = d_j + t\alpha_j^p$$

where α_j^p is the p-th component of α_j. Hence:

$$w_j(t) \geq 0 \iff d_j + t\alpha_j^p \geq 0$$

- ◆ If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$

- ◆ If $\alpha_j^p < 0$ we need $\dfrac{d_j}{-\alpha_j^p} \geq t$

- Best improvement achieved with

$$\Theta_D := \min\{\frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0\}$$

- Variable w_q is blocking when $\Theta_D = \dfrac{d_q}{-\alpha_q^p}$

Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j \in \mathcal{R}$ such that $\alpha_j^p < 0$):

   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.
   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and w_q is blocking:

   When setting $w_{k_p} = \Theta_D$,
   non-negativity constraints of basic vars of dual are respected

   In particular, $w_q(\Theta_D) = d_q + \Theta_D \alpha_q^p = d_q + (\frac{d_q}{-\alpha_q^p})\alpha_q^p = 0$

   We can make a basis change:

   - In dual: w_{k_p} enters $\hat{\mathcal{B}}$ and w_q leaves
   - In primal: x_{k_p} leaves $\mathcal{B}$ and x_q enters

Update

- We do not actually need to form the dual LP:
  it is enough to have a representation of the primal LP

- New basic indices: $\bar{\mathcal{B}} = (k_1, \ldots, k_{p-1}, q, k_{p+1} \ldots, k_m)$

- New dual objective value: $\bar{Z} = Z - \Theta_D \beta_p$

- New dual basic sol: $\bar{y} = y - \Theta_D \rho_p$
  $\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$

- New primal basic sol: $\bar{\beta}_p = \Theta_P$, $\quad \bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$
  where $\Theta_P = \frac{\beta_p}{\alpha_q^p}$

- New basis inverse: $\bar{B}^{-1} = E B^{-1}$
  where $E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m)$ and
  $\eta^T = \left(\left(\frac{-\alpha_q^1}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^{p-1}}{\alpha_q^p} \right), \frac{1}{\alpha_q^p} \left(\frac{-\alpha_q^{p+1}}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^m}{\alpha_q^p} \right) \right)^T$

Algorithmic Description

1. `Initialization:` Find an initial dual feasible basis $\mathcal{B}$
   Compute B^{-1}, $\beta = B^{-1}b$,
   $y^T = c_\mathcal{B}^T B^{-1}$, $d_\mathcal{R}^T = c_\mathcal{R}^T - y^T R$, $Z = b^T y$

2. `Dual Pricing:`
   If for all $i \in \mathcal{B}, \beta_i \geq 0$ then return OPTIMAL
   Else let p be such that $\beta_p < 0$.
      Compute $\rho_p^T = e_p^T B^{-1}$ and $\alpha_j^p = \rho_p^T a_j$ for $j \in \mathcal{R}$

3. `Dual Ratio test:` Compute $\mathcal{J} = \{j \mid j \in \mathcal{R}, \alpha_j^p < 0\}$.
   If $\mathcal{J} = \emptyset$ then return INFEASIBLE
   Else compute $\Theta_D = \min_{j \in \mathcal{J}}(\frac{d_j}{-\alpha_j^p})$ and q st. $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Algorithmic Description

4. Update:

$$\bar{\mathcal{B}} = \mathcal{B} - \{k_p\} \cup \{q\}$$
$$\bar{Z} = Z - \Theta_D \beta_p$$

<u>Dual solution</u>

$$\bar{y} = y - \Theta_D \rho_p$$
$$\bar{d}_j = d_j + \Theta_D \alpha_j^p \text{ if } j \in \mathcal{R}, \ \bar{d}_{k_p} = \Theta_D$$

<u>Primal solution</u>

Compute $\alpha_q = B^{-1} a_q$ and $\Theta_P = \dfrac{\beta_p}{\alpha_q^p}$

$$\bar{\beta}_p = \Theta_P, \quad \bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i \ \text{ if } \ i \neq p$$

$$\bar{B}^{-1} = E B^{-1}$$

Go to 2.

Primal vs. Dual Simplex

	PRIMAL		DUAL

PRIMAL

- Can handle bounds efficiently

- Many years of research and implementation

- There are classes of LP's for which it is the best

- **Not suitable** for solving LP's with **integer** variables

DUAL

- Can handle bounds efficiently (not explained here)

- Developments in the 90's made it an alternative

- Nowadays on average it gives better performance

- **Suitable** for solving LP's with **integer** variables