UC BERKELEY, CS 174: COMBINATORICS AND DISCRETE PROBABILITY (FALL 2010)

Solutions to Problem Set 1

1. We flip a fair coin ten times. Find the probability of the following events.

   (a) The number of heads and the number of tails are equal.
   There are 10 flips of which we choose 5 heads, and there are total of 2^{10} ways to flip the coin. Therefore, the probability is

   $$\frac{\binom{10}{5}}{2^{10}} = \frac{63}{256}$$

   (b) There are more heads than tails.
   Let X_i be the number of heads. Then

   $$\begin{aligned}\mathbb{P}[\text{more heads than tails}] &= \sum_{i=6}^{10} \mathbb{P}[X_i] \\ &= \frac{1}{2^{10}} \sum_{i=6}^{10} \binom{10}{i} \\ &= \frac{386}{1024}\end{aligned}$$

   (c) The ith flip and the $(11-i)$th flip are the same for $i = 1, 2, 3, 4, 5$.
   There are 2^5 choices of the first five flips, which are repeated, according to the pattern, in the next five flips. Therefore the probability is

   $$2^5/2^{10} = 1/2^5.$$

   (d) We flip at least four consecutive heads.
   Clearly $\mathbb{P}[\text{flip} \geq 4 \text{ consecutive heads}] = 1 - \mathbb{P}[\text{flip} < 4 \text{ consecutive heads}]$. Notice that there are four sequences that do not lead to four consecutive heads:

   $$\begin{aligned}\mathbb{P}[T] &= 1/2 \\ \mathbb{P}[HT] &= 1/2^2 \\ \mathbb{P}[HHT] &= 1/2^3 \\ \mathbb{P}[HHHT] &= 1/2^4\end{aligned}$$

   Therefore we can set up a recursion for k flips where P_k is the probability of not observing four consecutive heads in k flips. Notice that $P_0 = P_1 = P_2 = P_3 = 1$, in order to allow sequences ending in heads. So, we have

   $$\begin{aligned}P_k &= 1/2P_{k-1} + 1/4P_{k-2} + 1/8P_{k-3} + 1/16P_{k-4} \\ P_{10} &= 0.245\end{aligned}$$

2. **(The inclusion-exclusion principle)** Let E and F be events. Prove: $\mathbb{P}[E \cup F] \geq \mathbb{P}[E] + \mathbb{P}[F] - 1$.

$$\begin{aligned}
\mathbb{P}[E \cup F] &= \mathbb{P}[E] + \mathbb{P}[F] - \mathbb{P}[E \cap F] \qquad \text{(by inclusion-exclusion)} \\
&\geq \mathbb{P}[E] + \mathbb{P}[F] - 1 \qquad (\text{by } \mathbb{P}[E \cup F] \leq 1)
\end{aligned}$$

3. We are playing a tournament in which we stop as soon as one of us wins n games. We are evenly matched, so that each of us wins any game with probability $1/2$ independently of other games. What is the probability that the loser has won exactly k games when the match is over?

   Let E_k be the event that the loser has won k games, where $0 \leq k \leq n-1$. There are a total of $n+k$ games. The last game is won by the winner, therefore the loser wins k games out of $n+k-1$ games. The total number of ways for any wins to occur, excluding the last game, is 2^{n+k-1}.

$$\mathbb{P}[E_k] = \frac{1}{2^{n+k-1}} \binom{n+k-1}{k}$$

4. Assume that each child born is equally likely to be a boy or a girl. If a family has two children, what is the probability that they will both be girls given that:

   (a) The elder is a girl?
   Let B and G be the events that a child is a boy or a girl, respectively. The question is asking

$$\begin{aligned}
\mathbb{P}[G, G | G, B \text{ or } G, G] &= \frac{\mathbb{P}[G, G \cap G, B \text{ or } G, G]}{\mathbb{P}[G, B \text{ or } G, G]} \\
&= \frac{(1/2)(1/2)}{(1/2)} \\
&= 1/2
\end{aligned}$$

   (b) At least one is a girl?
   Using the same notation as above

$$\begin{aligned}
\mathbb{P}[G, G | \text{not } B, B] &= \frac{\mathbb{P}[G, G \cup \text{not } B, B]}{\mathbb{P}[\text{not } B, B]} \\
&= \frac{1/4}{1 - 1/4} \\
&= 1/3
\end{aligned}$$

5. **(Conditional Probability)** If two fair dice are tossed what is the conditional probability that the first die is six given that the sum is seven?

   Let D_1 and D_2 be random variables for the two dice rolls.

$$\begin{aligned}
\mathbb{P}[D_1 = 6 | D_1 + D_2 = 7] &= \frac{\mathbb{P}[D_1 = 6 \cap D_1 + D_2 = 7]}{\mathbb{P}[D_1 + D_2 = 7]} \\
&= \frac{\mathbb{P}[D_1 = 6 \text{ and } D_2 = 1]}{\mathbb{P}[D_1 + D_2 = 7]} \\
&= \frac{1/36}{6/36} \\
&= 1/6
\end{aligned}$$

6. **(Bayes Rule)** Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is heads then a ball from urn 1 is selected, while if the outcome is tails a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails? Explain your computation.

   Let H and T represent the events that the coin lands heads and tails, respectively. Let W be the event that a white ball is chosen.

$$\begin{aligned}\mathbb{P}[T|W] &= \frac{\mathbb{P}[W|T]\mathbb{P}[T]}{\mathbb{P}[W|T]\mathbb{P}[T]+\mathbb{P}[W|H]\mathbb{P}[H]} \\ &= \frac{(3/15)(1/2)}{(3/15)(1/2)+(5/12)(1/2)} \\ &= 12/37\end{aligned}$$

7. **(Induction)** Consider the following balls-and-bins game. We start with one black ball and one white ball in the bin. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same color. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$. Hint: use mathematical induction.

   Let W_n be a random variable representing the number of white balls at round n.
   Base case. Let $n = 3$. Then $\mathbb{P}[W_2 = 1] = 1/2$ and $\mathbb{P}[W_2 = 2] = 1/2$, since there is equal probability of drawing a black or white ball in the first round.
   Inductive Step. Assume that $\mathbb{P}[W_n = i] = 1/(n-1)\ \forall 1 \le i \le n-1$. Then for the $(n+1)$-th ball, we draw a ball X_{n+1} at random taking two values, white, w, and black, b, which is drawn from a bin with n balls which contains W_n white balls $n - W_n$ black balls. For $1 < i < n$,

$$\begin{aligned}\mathbb{P}[W_{n+1}=i] &= \mathbb{P}[W_{n+1}=i-1|W_n=i-1]\mathbb{P}[W_n=i-1]+\mathbb{P}[W_{n+1}=i|W_n=i]\mathbb{P}[W_n=i] \\ &= \mathbb{P}[W_{n+1}=i-1|W_n=i-1,X_{n+1}=w]\mathbb{P}[X_{n+1}|W_n=i-1]\mathbb{P}[W_n=i-1] \\ &\quad +\mathbb{P}[W_{n+1}=i|W_n=i,X_{n+1}=b]\mathbb{P}[X_{n+1}=b|W_n=i]\mathbb{P}[W_n=i] \\ &= \mathbb{P}[X_{n+1}|W_n=i-1]\mathbb{P}[W_n=i-1]+\mathbb{P}[X_{n+1}=b|W_n=i]\mathbb{P}[W_n=i] \\ &= \left(\frac{i-1}{n}\right)\left(\frac{1}{n-1}\right)+\left(\frac{n-i}{n}\right)\left(\frac{1}{n-1}\right) \\ &= 1/n.\end{aligned}$$

   The end cases are similar, except there are more zero-probability events. For $i = 1$, we have

$$\begin{aligned}\mathbb{P}[W_{n+1}=1] &= \mathbb{P}[X_{n+1}=b|W_n=i]\mathbb{P}[W_n=i] \\ &= \left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) \\ &= 1/n.\end{aligned}$$

And, for $i = n$, we have

$$\begin{aligned}
\mathbb{P}[W_{n+1} = n] &= \mathbb{P}[X_{n+1} | W_n = i-1]\mathbb{P}[W_n = i-1] \\
&= \left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) \\
&= 1/n.
\end{aligned}$$

8. **(Randomized Min-Cut, MU 1.25)** There may be several different min-cut sets in an n-vertex graph. Using the analysis of the randomized min-cut algorithm, argue that there can be at most $n(n-1)/2$ distinct min-cut sets.

   Thm 1.8 says that the algorithm outputs any particular min-cut set with probability $\geq \frac{2}{n(n-1)}$. Consider the set of all min-cuts, each min-cut is a disjoint event, so their probabilities sum to at most one. Therefore, $n(n-1)/2 \geq c$, where c is the number of cut-sets.