

Games with pure equilibria

Fall 2020

1 Best response dynamics

2 Potential games

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Best response dynamics

Consider a strategic game $\Gamma = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, \dots, n\}$

- PNE are defined as the fix point among mutually best responses.

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- PNE are defined as the fix point among mutually best responses.
- It seems natural to consider variants of the process of local changes to try to get a PNE.
- Consider the algorithm:
 - choose $s \in A_1 \times \dots \times A_n$
 - while s is not a NE do
 - choose $i \in \{1, \dots, n\}$ such that $s_i \notin BR(s_{-i})$
 - Set s_i to be an action in $BR(s_{-i})$
- The process looks similar to local search algorithms. Is there any difference?

Best response graph

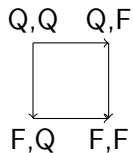
- The **Nash dynamics** or **Best Response graph** has
 - $V = A_1 \times \dots \times A_n$
 - An edge $(s, (s_{-i}, s'_i))$ for $i \in N$, $s_i \notin BR(s_{-i})$ and $s'_i \in BR(s_{-i})$.
- Performing local search on the best response graph
 - Does it produce a PNE?
 - If so, how much time?
 - Let's look to some examples.

Prisoner's dilemma

	Quiet	Fink
Quiet	2,2	0,3
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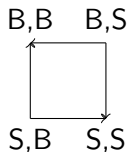


Bach and Stravinsky

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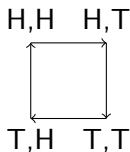


Matching Pennies

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Other games

- sending from s to t ?
- congestion games?

In those games we cannot get the best response graph in polynomial time.

However we can perform a local improvement step in polynomial time.

Although, even assuring convergence, it might take exponential time to reach a NE.

Best response graph: properties

- A NE is a **sink** (a node with out-degree 0) in the best response graph.
- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE.

Best response graph: properties

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- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE. Furthermore, best response dynamics converges to a PNE, maybe with a lot of time.

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- 2 Potential games**
- 3 Congestion games
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Potential games

(Monderer and Shapley 96)

- Consider a strategic game $\Gamma = (N, A_1, \dots, A_n, u_1, \dots, u_n)$.
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Potential games

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- Consider a strategic game $\Gamma = (N, A_1, \dots, A_n, u_1, \dots, u_n)$.
Let $S = A_1 \times \dots \times A_n$.

- A function $\Phi : S \rightarrow \mathbb{R}$ is an **exact potential function** for Γ if

$$\forall i \in N \forall s \in S \forall s'_i \in A_i \quad u_i(s) - u_i(s_{-i}, s'_i) = \Phi(s) - \Phi(s_{-i}, s'_i)$$

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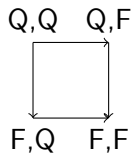
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- Γ is a **potential game** if it admits a potential function.

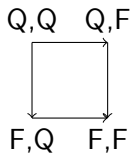
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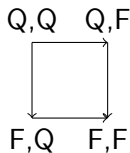
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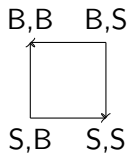


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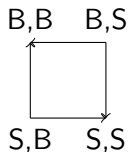
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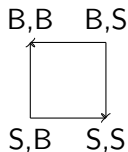
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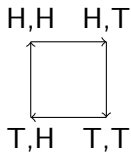
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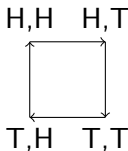
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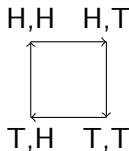
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The property on Φ cannot hold along a cycle in the best response graph.

Potential games

Theorem

A strategic game is a potential game iff the best response graph is acyclic

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- Let G be the best response graph of Γ .
- The existence of a potential function Φ and the fact that, for each pair of connected strategy profiles in G , at least one player improves, implies the non existence of cycles in G .
- If G is acyclic, a topological sort of the graph provides a potential function for Γ .



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We have a way to show that a game has a PNE by showing that it is a potential game.

1 Best response dynamics

2 Potential games

3 **Congestion games**

4 References

Congestion games

Congestion games

A congestion game

- is defined on a finite set E of resources.
- There is a delay function d mapping $E \times \mathbb{N}$ to the integers.
- Player's actions are subsets of E (all or some).
- The **cost** functions are the following:

$$c_i(a_1, \dots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

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- A **singleton congestion game** has $A_i = \{\{r\} \mid r \in E\}$.

An example of a congestion game

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- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.

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- We have a factory with two end production lines, each having a cutting and a packing unit. Orders are cut down and then packed.
- We have 3 orders that have to be send to one of the end production lines.
- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.
- The packing machine on the first line takes 2 additional hours to pack a single order, 3 hours to pack 2 and 7 hours to pack 3. The packing machine on the second line takes instead 0, 2 and 9 hours respectively.

An example of a congestion game

- We have 4 resources C_1, C_2, P_1, P_2 and 3 players $N = \{1, 2, 3\}$
- $A_i = \{\{C_1, P_1\}, \{C_2, P_2\}\}, i = 1, 2, 3$
- Delay functions are defined by the processing times.

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C_1	1	2	4
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Does this game have a PNE?

Rosenthal's theorem

Theorem (Rosenthal 73)

Every congestion game is a potential game,

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Every congestion game is a potential game,

- For a strategy profile $s = (a_1, \dots, a_n)$, define

$$\Phi(s) = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k)$$

where $r(s) = \cup_{i \in N} a_i$.

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Let us show that Φ is a potential function.

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$$\Phi(s) - \Phi(s') = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e' \in r(s')} \sum_{k=1}^{f(s',e')} d(e', k)$$

Cost difference

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$$\begin{aligned} c_i(s) - c_i(s_{-i}, s'_i) &= \left(\sum_{e \in a_i} d(e, f(s, e)) \right) - \left(\sum_{e' \in a'_i} d(e, f(s', e')) \right) \\ &= \sum_{e \in a_i, e \notin a'_i} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a'_i} d(e, f(s', e')) \end{aligned}$$

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 - $e \in a_i$ and $e \notin a'_i$: $f(s, e) = f(s', e) + 1$
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$$\begin{aligned}
 \Phi(s) - \Phi(s') &= \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e' \in r(s')} \sum_{k=1}^{f(s',e')} d(e', k) \\
 &= \sum_{e \in a_i, e \notin a'_i} \left[\sum_{k=1}^{f(s',e)+1} d(e, k) - \sum_{k=1}^{f(s',e)} d(e, k) \right] \\
 &\quad + \sum_{e \notin a_i, e \in a'_i} \left[\sum_{k=1}^{f(s,e)} d(e, k) - \sum_{k=1}^{f(s,e)+1} d(e, k) \right]
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 &= \sum_{e \in a_i, e \notin a'_i} d(e, f(s', e) + 1) - \sum_{e \notin a_i, e \in a'_i} d(e, f(s, e) + 1) \\
 &= \sum_{e \in a_i, e \notin a'_i} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a'_i} d(e, f(s', e)) \\
 &= c_i(s) - c_i(s_{-i}, s'_i)
 \end{aligned}$$

Network congestion games

- A **network congestion game** is a congestion game defined by a directed graph G and a collection of pairs of vertices (s_i, t_i) .
 - The set of resources are the arcs in G .
 - The actions, for player i , are the $s_i \rightarrow t_i$ paths on G .
- A network congestion game is **symmetric** when $s_i = s$ and $t_i = t$, for $i \in N$.

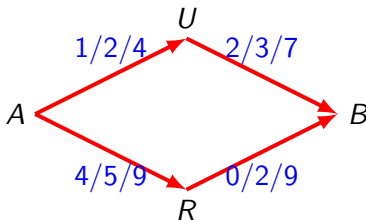
An example of a network congestion game

An example of a network congestion game

- There are three players.
- and a network (with a delay function on arcs)

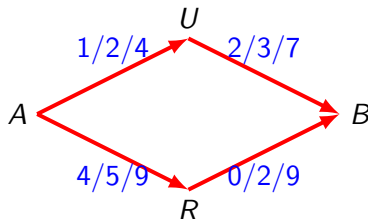
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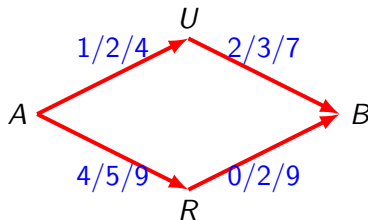
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- Strategy profiles: paths from A to B .
- A NE?

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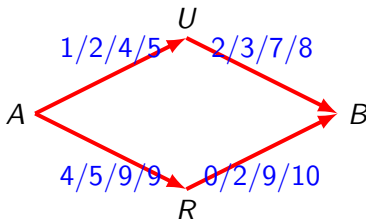
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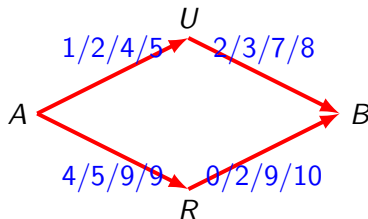
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- Player's objective: send w_i units from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B .
- A NE?

Results on convergence time

Theorem (Fabrikant, Papadimitriou, Talwar (STOC 04))

There exist network congestion games with an initial strategy profile from which all better response sequences have exponential length.

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Theorem (leong, McGrew, Nudelman, Shoham, Sun (AAAI 05))

In singleton congestion games all best response sequences have length at most $n^2 m$.

Complexity classification?

Optimization problem

An **optimization problem** is a structure $\Pi = (I, \text{sol}, m, \text{goal})$, where

- C is the input set to Π ;
- $\text{sol}(x)$ is the set of feasible solutions for an input x .
- m is an integer measure defined over pairs (x, y) , $x \in I$ and $y \in \text{sol}(x)$.
- goal is the optimization criterium MAX or MIN.

An optimization problem is a function problem whose goal, with respect to an instance x , is to find an optimum solution, that is, a feasible solution y such that

$$y = \text{goal}\{(m(x, y') \mid y' \in \text{sol}(x))\}.$$

Example: Given a graph and two vertices, obtain a path joining them with minimum length.

PLS

- A **local search problem** is an optimization problem with
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- finding **initial feasible solution** $s \in \text{sol}(x)$,

PLS

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A **PLS reduction** from (Π_1, \mathcal{N}_1) to (Π_2, \mathcal{N}_2) is

- a polynomial time computable function $f : I_{\Pi_1} \rightarrow I_{\Pi_2}$ and
- a polynomial time computable function $g : \text{sol}(f(x)) \rightarrow \text{sol}(x)$, for $x \in I_{\Pi_1}$ such that
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A PLS problem (Π, \mathcal{N}) is **PLS-complete** if every problem in PLS is PLS-reducible to (Π, \mathcal{N}) .

PLS complete problems

- **MAX-SAT** (maximum satisfiability) problem
 - Given a Boolean formula in conjunctive normal form with a positive integer weight for each clause.
 - A solution is an assignment of the value 0 or 1 to all variables.
 - Its weight, to be maximized, is the sum of the weights of all satisfied clauses.
 - As neighborhood consider the **Flip-neighborhood**, where two assignments are neighbors if one can be obtained from the other by flipping the value of a single variable.

PLS complete problems

- **MaxCut** problem.
 - Given a graph $G = (V, E)$ with non-negative edge weights.
 - A feasible solution is a partition of V into two sets A and B .
 - The objective is to maximize the weight of the edges between the two sets A and B .
 - In the **Flip-neighborhood** two solutions are neighbors if one can be obtained from the other by moving a single vertex from one set to the other.

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- We provide a reduction from MaxCut under the Flip-neighborhood.

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 - Solutions (A, B) of MaxCut corresponds to strategy S_v^a for $v \in A$ and S_v^b for $v \in B$.

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 - Furthermore, the local optima of the MaxCut instance coincide with the Nash equilibria of the congestion game.
- We have a PLS-reduction from MaxCut.

- 1 Best response dynamics
- 2 Potential games
- 3 Congestion games
- 4 References**

Reference

B. Vöcking, Congestion Games: Optimization in Competition