

Stochastic Network Modeling (SNM)

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Parts

- I Introduction
- II Discrete Time Markov Chains (DTMC)
- III Continuous Time Markov Chains (CTMC)
- IV Queuing Theory

Part III

Continuous Time Markov Chains (CTMC)

Outline

- Definition of a CTMC
- Transient Solution
- Embedded MC of a CTMC
- Classification of States
- Steady State
- Semi-Markov Process
- Finite Absorbing Chains

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Properties of a continuous time MC

- The states must be a numerable set.
- Let $X(t)$ be the event {at time t the system is in state i }, then it must hold the **memoryless property**:

$$P(X(t_n) = i \mid X(t_1) = j, X(t_2) = k, \dots) =$$

$$P(X(t_n) = i \mid X(t_1) = j) \text{ for any } t_n > t_1 > t_2 > t_3 \dots$$



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Transition Matrix

- **Transition probabilities:**

$$p_{ij}(t_1, t_2) = P(X(t_2) = j \mid X(t_1) = i)$$

- For an **homogeneous chain**:

$$\begin{aligned} p_{ij}(t) &= P(X(t_1 + t) = j \mid X(t_1) = i) = \\ &= P(X(t) = j \mid X(0) = i), \forall t_1 \end{aligned}$$

- In matrix form (**transition probability matrix**):

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, t \geq 0$$

- Notes:

- Compare with the n-step prob. matrix of a DTMC: $\mathbf{P}(n)$.
- $\mathbf{P}(t)$ must be a **stochastic matrix** (all rows add to 1).

Transition Matrix

$$p_{ij}(t) = P(X(t) = j \mid X(0) = i), t \geq 0$$

- We look for an equivalent 1-step prob. matrix **P** of DTMCs.
- For consistency: $\lim_{t \rightarrow 0} p_{ij}(t) = \delta_{ij}$. In matrix form:

$$\lim_{t \rightarrow 0} \mathbf{P}(t) = \mathbf{I}.$$

- And assume that the following **transition rates** exist:

$$q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t}, i \neq j; \quad q_{ii} = \lim_{t \rightarrow 0} \frac{p_{ii}(t) - 1}{t}$$

- In matrix form: $\mathbf{Q} = \lim_{t \rightarrow 0} \frac{\mathbf{P}(t) - \mathbf{I}}{t}$
- Note that $\sum_j p_{ij}(t) = 1 \Rightarrow p_{ii}(t) = 1 - \sum_{j \neq i} p_{ij}(t)$, thus:

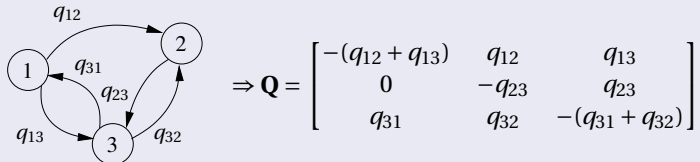
$$q_{ii} = \lim_{t \rightarrow 0} \frac{p_{ii}(t) - 1}{t} = \lim_{t \rightarrow 0} \frac{-\sum_{j \neq i} p_{ij}(t)}{t} = -\sum_{j \neq i} q_{ij}$$

Transition Matrix

- The matrix **Q** is called the **transition rate or infinitesimal generator** of the chain.
- Since $q_{ii} = -\sum_{j \neq i} q_{ij}$, **all the rows of Q add to 0**.
- The rate q_{ij} , $i \neq j$ measures “how fast” the chain moves from state i to j : the higher is q_{ij} , the faster it moves from i to j .
- For $q_{ii} = -\sum_{j \neq i} q_{ij}$, the higher $-q_{ii}$ is, the faster the chain leaves state i .
- If $q_{ij} = 0, \forall j \Rightarrow q_{ii} = 0$, then i is an **absorbing state**: the chain “moves with rate 0 from i to other states”, i.e. never leaves i .

State Transition Diagram

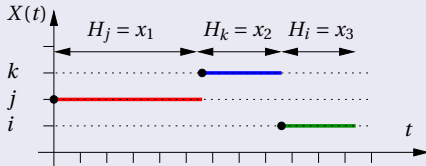
- A continuous MC is characterized by the transition rate or infinitesimal generator: the Q-matrix.
- The state transition diagram is now represented as:



- Note that now we have **transition rates** ($0 \leq q_{ij} < \infty, i \neq j$) **and not probabilities**.
- The **rates q_{ii} are not written** in the diagram, **no self transitions**.

Sojourn Time

- Sojourn or holding time: Is the RV H_k equal to the sojourn time in state k :



- The Markov property implies that **the sojourn time is exponentially distributed with parameter q_{ii}** :

$$P(H_i \leq x) = 1 - e^{q_{ii}x} \Rightarrow P(H_i > x) = e^{q_{ii}x}, \quad q_{ii} = -\sum_{j \neq i} q_{ij}, \quad x \geq 0$$

The exponential distribution satisfies the Markov property

- Markov property (**memoryless**):

$$P(X(t_2) = i \mid X(t_1) = i, X(0) = i) =$$

$$P(X(t_2) = i \mid X(t_1) = i), t_2 > t_1 > 0$$

- In terms of the sojourn time:

$$P(H_i > t_2 \mid H_i > t_1) = P(H_i > t_2 - t_1)$$

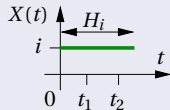
- But:

$$P(H_i > t_2 \mid H_i > t_1) =$$

$$\frac{P(H_i > t_2, H_i > t_1)}{P(H_i > t_1)} = \frac{P(H_i > t_2)}{P(H_i > t_1)} = \frac{e^{q_{ii} t_2}}{e^{q_{ii} t_1}} = e^{q_{ii} (t_2 - t_1)} =$$

$$P(H_i > t_2 - t_1) \quad \square$$

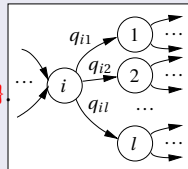
- The **exponential distribution is the only one satisfying the memoryless property.**



Exponential Jumps Description of a CTMC

Assume a process built as follows:

- Upon reaching a state i
 - the process can jump to a state $j \in \{1, 2, \dots, l\}$.
 - A set of **independent exponential RVs**, $\{H_{i1}, H_{i2}, \dots, H_{il}\}$, with parameters $\{q_{i1}, q_{i2}, \dots, q_{il}\}$ are triggered. That is, $P(H_{ij} \leq t) = 1 - e^{-q_{ij}t}$, $t \geq 0$.
- If $\min\{H_{i1}, H_{i2}, \dots, H_{il}\} = H_{ij} \Rightarrow$ the process jumps to the state j . In other words, a transition occurs to state j if the RV H_{ij} is the minimum of $\{H_{i1}, H_{i2}, \dots, H_{il}\}$.



Theorem: This process is a CTMC with transition rates q_{ij} .

Exponential Jumps Description of a CTMC

$$P(H_{ij} \leq t) = 1 - e^{-q_{ij}t}.$$

Theorem: This process is a CTMC with transition rates q_{ij} .

Proof:

- The RV $H_i = \min\{H_{i1}, H_{i2}, \dots, H_{il}\}$ (sojourn time in state i) is **exponentially distributed** with parameter $q_i = \sum_j q_{ij}$:
 $P(H_i \leq t) = 1 - e^{-q_i t}$.
- $P(\min\{H_{i1}, H_{i2}, \dots, H_{il}\} = H_{ij}) = q_{ij} / \sum_j q_{ij}$. Thus, the **transition rate to state j** is:

$$\lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t} = \lim_{t \rightarrow 0} \frac{P(\min\{H_{i1}, H_{i2}, \dots, H_{il}\} = H_{ij}) \times P(H_i \leq t)}{t} =$$

$$\frac{q_{ij}}{\sum_j q_{ij}} \left. \frac{\partial P(H_i \leq t)}{\partial t} \right|_{t=0} = \frac{q_{ij}}{\sum_j q_{ij}} \sum_j q_{ij} = q_{ij} \quad \square$$



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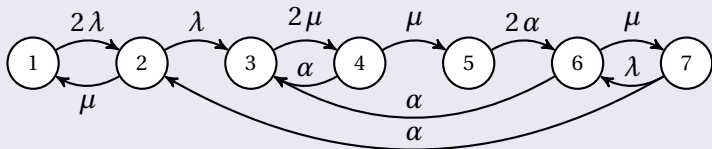
Example: Pure Aloha System

- Consider a **Pure Aloha System** with **2 nodes**:
 - Nodes in **thinking state** Tx a packet in a time exponentially distributed with rate λ .
 - Transmission time** is exponentially distributed with rate μ .
 - If two transmissions overlap, the packet is lost and stations become backlogged (after the packet transmission) until the packet is successfully transmitted.
 - Nodes in **backlogged state** Tx a packet in a time exponentially distributed with rate α .

Questions

- Build the state **transition diagram**.

Example: Pure Aloha System



State	Condition	Legend
1	T, T	T Thinking
2	X, T	X Transmitting
3	C, C	C Collided transmission
4	B, C	B Backlogged
5	B, B	
6	X, B	
7	T, B	



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Chapman-Kolmogorov Equations

- **Chapman-Kolmogorov:** $p_{ij}(t) = \sum_k p_{ik}(t - \alpha) p_{kj}(\alpha), 0 \leq \alpha \leq t$

- Thus:

$$\frac{p_{ij}(t + \Delta t) - p_{ij}(t)}{\Delta t} = \sum_k \left\{ \frac{p_{ik}(t + \Delta t - \alpha) - p_{ik}(t - \alpha)}{\Delta t} p_{kj}(\alpha) \right\}$$

- Taking the limit

$$\alpha \rightarrow t, \Delta t \rightarrow 0 \Rightarrow \begin{cases} p_{ik}(t - \alpha) \rightarrow 0, & i \neq k \\ p_{ik}(t - \alpha) \rightarrow 1, & i = k \end{cases}$$

and using:

we have:

$$\begin{cases} q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t}, & i \neq j \\ q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t) - 1}{t}, & i = j \end{cases}$$

$$\frac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik} p_{kj}(t), \quad t \geq 0, \forall i, j$$

Chapman-Kolmogorov Equations (cont)

- we have: $\frac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik} p_{kj}(t), t \geq 0, \forall i, j$

- In matrix form: $\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{Q} \mathbf{P}(t), t \geq 0$ known as the master equations of a CTMC.

- The solution of the previous matrix differential equation is the **exponential matrix**:

$$\mathbf{P}(t) = e^{\mathbf{Q}t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}t)^i}{i!} = \mathbf{I} + \mathbf{Q}t + \frac{\mathbf{Q}^2 t^2}{2!} + \frac{\mathbf{Q}^3 t^3}{3!} + \dots, t \geq 0$$

- Due to rounding errors, the previous series is difficult to compute numerically (the powers of \mathbf{Q} have positive and negative entries).



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State Probabilities

- Define the probability of being in state i at time t :

$$\pi_i(t) = P(X(t) = i)$$

- In vector form (row vector)

$$\boldsymbol{\pi}(t) = (\pi_1(t), \pi_2(t), \dots).$$

- Clearly:

$$\pi_i(t) = \sum_k P(X(0) = k) P(X(t) = i \mid X(0) = k) = \sum_k \pi_k(0) p_{ki}(t)$$

- In matrix form:

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \mathbf{P}(t) = \boldsymbol{\pi}(0) \mathbf{e}^{\mathbf{Q}t}, t \geq 0$$

where $\boldsymbol{\pi}(0)$ is the **initial distribution**.

- NOTE:** Compare with **DTMC**

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n, n \geq 0$$



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Transient Solution

- If we are interested in the **transient evolution** we shall study $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \mathbf{P}(t) = \boldsymbol{\pi}(0) e^{\mathbf{Q}t}$, $t \geq 0$.
- Assume a **finite CTMC** with N states (infinitesimal generator $\mathbf{Q}^{N \times N}$).
- Assume that \mathbf{Q} can be **diagonalized**: $\mathbf{Q} = \mathbf{L}^{-1} \boldsymbol{\Lambda} \mathbf{L}$, where $\boldsymbol{\Lambda}$ is the diagonal matrix $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_N)$, with λ_l , $l = 1, \dots, N$ the eigenvalues of \mathbf{Q} .
- **NOTE**: the **eigenvalues** λ_l of a matrix \mathbf{A} are scalars that satisfy: $\mathbf{l}\mathbf{A} = \lambda_l \mathbf{l}$ (or $\mathbf{A}\mathbf{r} = \lambda_l \mathbf{r}$) for some row vectors \mathbf{l} (column vectors \mathbf{r}), referred to as **left and right eigenvectors**, respectively. Thus, solve the **characteristic polynomial** $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$.

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... assume that \mathbf{Q} can be **diagonalized**: $\mathbf{Q} = \mathbf{L}^{-1} \mathbf{\Lambda} \mathbf{L}$

- Since

$$\mathbf{P}(t) = e^{\mathbf{Q}t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}t)^i}{i!} = \sum_{i=0}^{\infty} \frac{(\mathbf{L}^{-1} \mathbf{\Lambda} \mathbf{L}t)^i}{i!} =$$

$$\mathbf{L}^{-1} \text{diag} \left(\sum_{i=0}^{\infty} \frac{(\lambda_1 t)^i}{i!}, \dots \right) \mathbf{L} = \mathbf{L}^{-1} \text{diag}(e^{\lambda_1 t}, \dots) \mathbf{L}$$

we have that

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) e^{\mathbf{Q}t} =$$

$$\boldsymbol{\pi}(0) \mathbf{L}^{-1} \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots e^{\lambda_L t}) \mathbf{L}$$



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... we have that $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \mathbf{L}^{-1} \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_L t}) \mathbf{L}$

- Thus, the probability of being in state i is given by:

$$\pi_i(t) = (\boldsymbol{\pi}(t))_i = \sum_{l=1}^N a_i^{(l)} e^{\lambda_l t}, t \geq 0$$

where the **unknown coefficients** $a_i^{(l)}$ can be obtained solving the system of equations:

$$\left. \frac{\partial^n \pi_i(t)}{\partial t^n} \right|_{t=0} = (\boldsymbol{\pi}(0) \mathbf{Q}^n)_i, n = 0, \dots, N-1$$

NOTE: Compare with $(\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0) \mathbf{P}^n)_i, n = 0, \dots, N-1$

Transient Solution

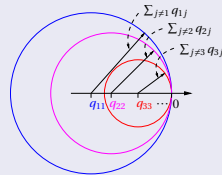
Eigenvalues of an Infinitesimal Generator

- \mathbf{Q} has an eigenvalue equal to 0 ($\mathbf{Q}\mathbf{x} = \lambda\mathbf{x}$, for $\lambda = 0$, $\mathbf{x} \neq \mathbf{0}$).

Proof: $\mathbf{Q}\mathbf{e} = \mathbf{0}$, where $\mathbf{e} = (1, 1, \dots)^T$ is a column vector of 1 (all rows of \mathbf{Q} add to 0). □

- The eigenvalue $\lambda = 0$ is single if \mathbf{Q} is irreducible (Perron-Frobenius theorem). \mathbf{Q} is irreducible if all states communicate: for $t > 0$, $p_{ij}(t) > 0$, $\forall i, j$.
- All eigenvalues of \mathbf{Q} are $\lambda_l \leq 0$.

Proof: Using Gerschgorin's theorem and the fact that the rows of \mathbf{Q} add to 0. □



Gerschgorin discs
of a rate matrix



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- Assume a CTMC with

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 1/2 & -1/2 \end{bmatrix}$$

- We want the probability of being in state 2 at time t starting from state 1: $\pi_2(t)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

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Solution

- It can be easily found that the **eigenvalues** of \mathbf{Q} are $\lambda_1 = 0$ and $\lambda_2 = -3/2$.

$$\pi_2(t) = ae^{\lambda_1 t} + be^{\lambda_2 t} = a + be^{-(3/2)t}$$

- Imposing the **boundary conditions**:

$$\pi_2(0) = a + b = (\boldsymbol{\pi}(0) \mathbf{Q}^0)_2 = (\boldsymbol{\pi}(0) \mathbf{I})_2 = (\boldsymbol{\pi}(0))_2 = 0$$

$$\left. \frac{\partial \pi_2(t)}{\partial t} \right|_{t=0} = b(-3/2) = (\boldsymbol{\pi}(0) \mathbf{Q})_2 = \mathbf{Q}_{12} = 1$$

we have that $a = 2/3$, $b = -2/3$, thus:

$$\pi_2(t) = 2/3 (1 - e^{-(3/2)t}), \quad t \geq 0$$

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Chain with a Defective Matrix

- What if \mathbf{Q} cannot be diagonalized? (**defective** matrix).
- Let λ_l , $l = 1, \dots, L$ be the eigenvalues of $\mathbf{Q}^{N \times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$). Then [1]:

$$\pi_j(t) = \sum_{l=1}^L e^{\lambda_l t} \sum_{m=0}^{k_l-1} a_j^{(l,m)} t^m$$

where $a_j^{(l,m)}$ are constants. So, exponentials associated with eigenvalues λ_l of multiplicity $k_l > 1$ are multiplied by polynomials in t of degree $k_l - 1$.

- [1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method*. Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html.

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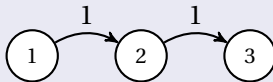
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Example

- Assume the CTMC:



$$\pi(0) = [1 \quad 0 \quad 0]$$

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- We have $\lambda_1 = 0$ and $\lambda_2 = -1$ with multiplicity 2. Thus:

$$\pi_3(t) = a + e^{-t}(b + ct)$$

- We have that $a = 1$, because state 3 is absorbing. Imposing $\pi_3(0) = 0$ and $\pi'_3(0) = 0$, we have $b = c = -1$, and

$$\pi_3(t) = 1 - e^{-t}(1 + t), \quad t \geq 0$$



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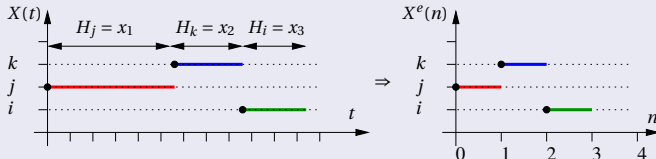
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Definition



- We form a discrete time process $X^e(n)$, called the **Embedded MC (EMC)**, by looking at a CTMC at the transition time instants.

Theorem: This process is a **DTMC** with transition probabilities:

$$p_{ij}^e = \begin{cases} 0, & i = j \\ \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}, & i \neq j \end{cases}$$

- NOTE:** If i is **absorbing** ($q_{ii} = 0$), we define $p_{ii}^e = 1$.

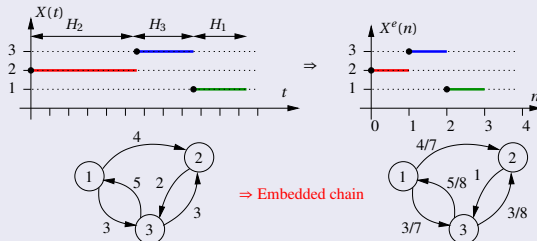
Proof

Theorem: This process is a DTMC with transition probabilities:

$$p_{ij}^e = \begin{cases} 0, & i = j \\ \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}, & i \neq j \end{cases}$$

- The EMC satisfies the **memoryless** property.
- Since we look the system only upon transition to a different state: $p_{ii}^e = 0$. NOTE: it might be $p_{ii}^e \neq 0$ if we look at transitions that end up in the same state.
- The probability that there is a transition from state i to j in the CTMC is the probability that the exponentially distributed RV with parameter q_{ij} is the **minimum from the independent exponentially distributed RVs** with parameters $\{q_{ik}\}_{k \neq i}$. This probability is $q_{ij} / \sum_{k \neq i} q_{ik}$. \square

Example



$$\mathbf{Q} = \begin{bmatrix} -7 & 4 & 3 \\ 0 & -2 & 2 \\ 5 & 3 & -8 \end{bmatrix} \Rightarrow \mathbf{P}_e = \begin{bmatrix} 0 & 4/7 & 3/7 \\ 0 & 0 & 1 \\ 5/8 & 3/8 & 0 \end{bmatrix}$$

- Each **transition** in the CTMC is a transition in the EMC.
- One step in ***i*** in the EMC is a **sojourn time H_i** in the CTMC.



Part III

Continuous Time Markov Chains (CTMC)

Outline

- Definition of a CTMC
- Steady State
- Transient Solution
- Semi-Markov Process
- Embedded MC of a CTMC
- Finite Absorbing Chains
- Classification of States

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Irreducibility

- A state j is said to **communicate** with i , $i \leftrightarrow j$, if $p_{ij}(t_1) > 0$, $p_{ji}(t_2) > 0$ for some $t_1 \geq 0$, $t_2 \geq 0$.
- We define an **irreducible closed set, ICS** C_k as a set where all states communicate with each other, and have no transitions to other states out of the set:
 $i \leftrightarrow j, \forall i, j \in C_k$ and $q_{ij} = 0, \forall i \in C_k, j \notin C_k$
- An **absorbing state** form an ICS of only one element. This state, i , must have $q_{ij} = 0 \forall i, j$.
- **Transient states** do not belong to any ICS.
- A MC is **irreducible** if all the states form a unique ICS.

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Irreducibility

- Assume a MC has ***M* ICSs**: By properly numbering the states, we can write \mathbf{P} as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- Example**, if $M = 3$:

$$\mathbf{Q} = \begin{array}{|c|c|c|} \hline \mathbf{Q}_1 & & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_2 & \\ \hline & & \mathbf{Q}_3 \\ \hline \text{at least} & & \\ \text{one} > 0 & & \mathbf{T} \\ \hline \end{array}$$

$$\Rightarrow \boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) e^{\mathbf{Q}t} = \boldsymbol{\pi}(0)$$

$$\begin{array}{|c|c|c|} \hline e^{\mathbf{Q}_1 t} & & \mathbf{0} \\ \hline \mathbf{0} & e^{\mathbf{Q}_2 t} & \\ \hline & & e^{\mathbf{Q}_3 t} \\ \hline \text{at least} & & \\ \text{one} > 0 & & e^{\mathbf{T}t} \\ \hline \end{array}$$

- Note that **the M sub-matrices are infinitesimal generators** (their rows add to 0).



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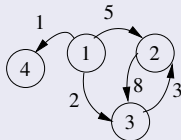
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Transient and Recurrent

- **Recurrent:** States that, being visited, have a probability > 0 of being visited again. They are visited an infinite number of times when $t \rightarrow \infty$.
- **Transient:** States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $t \rightarrow \infty$.
- **Absorbing:** A single (recurrent) state where the chain remains with probability $= 1$.



State 1 is **transient**
States 2 and 3 are **recurrent**
State 4 is **absorbing**



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Transient and Recurrent

- To derive a classification criteria, we shall study the **embedded MC (EMC)**, and proceed as in DTMC: Let $f_{ij}^e(n)$ the first passage prob. of the EMC, and $f_{ij}^e = \sum_{n=1}^{\infty} f_{ij}^e(n)$.
- If $f_{ii}^e = 1$ we say i is a **recurrent state**.
- If $f_{ii}^e < 1$ we say i is a **transient state**.
- When $f_{ii}^e = 1$, we define the **mean recurrence time of the EMC** $m_{ii}^e = \sum_{n=1}^{\infty} n f_{ii}^e(n)$. **NOTE:** in **steps**, not time units.
- If $m_{ii}^e = \infty$ the state is **null recurrent**.
- If $m_{ii}^e < \infty$ the state is **positive recurrent**.
- **NOTES:** (i) Even if the EMC is periodic, **there are not periodic CTMC** (it has no sense). (ii) f_{ij}^e and m_{ij}^e can be computed using the **recursive equations** for DTMC.

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Mean recurrence time of the CTMC

- If the chain is in i at a time t , it takes an **expected time to leave i** equal to $1/(-q_{ii}) = 1/\sum_{j \neq i} q_{ij}$ (**sojourn time exponentially distributed** with rate $q_i = -q_{ii} = \sum_{j \neq i} q_{ij}$).
- Thus, if the chain is **in state i** , it takes a **mean time to enter state j** (**mean first passage time**):

$$m_{ij} = \frac{1}{q_i} + \sum_{k \neq i} p_{ik}^e m_{kj}$$

- Since: $p_{ij}^e = \begin{cases} 0, & i = j \\ \frac{q_{ij}}{\sum_{j \neq i} q_{ij}} = \frac{q_{ij}}{q_i}, & i \neq j \end{cases}$ we have:

$$m_{ij} = \frac{1}{q_i} + \sum_{k \neq i} p_{ik}^e m_{kj} = \frac{1}{q_i} + \sum_{k \neq i} \frac{q_{ik}}{q_i} m_{kj} \text{ [time units]}$$



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Limiting Distribution

- The probability to be in state i at time t is:
 $\pi_i(t) = P(X(t) = i) = \sum_k \pi_k(0) p_{ki}(t), t \geq 0$
- In matrix form: $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \mathbf{P}(t) = \boldsymbol{\pi}(0) \mathbf{e}^{\mathbf{Q}t}, t \geq 0$
- Assume that the following limit exists:

$$\boldsymbol{\pi}(\infty) = \lim_{t \rightarrow \infty} \boldsymbol{\pi}(t) = \lim_{t \rightarrow \infty} \boldsymbol{\pi}(0) \mathbf{P}(t) = \boldsymbol{\pi}(0) \lim_{t \rightarrow \infty} \mathbf{e}^{\mathbf{Q}t}$$

- for any $\boldsymbol{\pi}(0)$, which implies

$$\lim_{t \rightarrow \infty} \mathbf{e}^{\mathbf{Q}t} = \mathbf{P}(\infty) = [\boldsymbol{\pi}(\infty) \quad \cdots \quad \boldsymbol{\pi}(\infty)]^T$$

- If this limit exists, we call $\mathbf{P}(\infty)$ the **limiting matrix**, and $\boldsymbol{\pi}(\infty)$ the **limiting distribution**.
- $\mathbf{P}(\infty) = [\boldsymbol{\pi}(\infty) \quad \cdots \quad \boldsymbol{\pi}(\infty)]^T$ does not exist if the CTMC has more than one irreducible closed set (each ICS will converge to a diagonal block, and $\boldsymbol{\pi}(\infty)$ will depend on $\boldsymbol{\pi}(0)$).

Stationary Distribution

- We have: $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \mathbf{e}^{\mathbf{Q}t}$, $t \geq 0$.
- In steady state the probabilities do not change. We look for a probability vector $\boldsymbol{\pi} = \boldsymbol{\pi}(t_1)$ satisfying: $\boldsymbol{\pi}(t_1) \mathbf{e}^{\mathbf{Q}t} = \boldsymbol{\pi}(t_1)$. In other words, for $t \geq t_1$ the probability vector reach the steady state $\boldsymbol{\pi}$, and do not change anymore. Thus:

$$\boldsymbol{\pi} \frac{\partial \mathbf{e}^{\mathbf{Q}t}}{\partial t} = \boldsymbol{\pi} \mathbf{Q} \mathbf{e}^{\mathbf{Q}t} = \mathbf{0}$$

- and we obtain that the **stationary distribution $\boldsymbol{\pi}$ can be computed with the Global balance equations:**

$$\boldsymbol{\pi} \mathbf{Q} = \mathbf{0}$$

$$\boldsymbol{\pi} \mathbf{e} = 1, \mathbf{e}^T = (1, 1, \dots)$$

- NOTE:** Compare with DTMC $\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}$, $\boldsymbol{\pi} \mathbf{e} = 1$.

Numerical Solution

- Replace one equation method:

$$\pi \mathbf{Q} = \mathbf{0}$$

$$\pi \mathbf{e} = 1, \mathbf{e}^T = (1, 1, \dots)$$

- We solve the equation $\pi \mathbf{Q} = \mathbf{0}$ replacing the last equation by $\pi \mathbf{e} = 1$:

$$\pi \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n-1} & 1 \\ q_{21} & q_{22} & \cdots & q_{2n-1} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn-1} & 1 \end{bmatrix} = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

Numerical Solution

- **Replace one equation method:** $\mathbf{Q} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\pi \mathbf{Q} = 0$
 $\pi \mathbf{e} = 1$
- Solving with **octave** (matlab clone):

```
octave:1> Q=[-2,1,1;1,-2,1;1,1,-2];
octave:2> s=size(Q,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [Q(1:s,1:s-1), ones(s,1)]
ans =
    0.33333    0.33333    0.33333
```

- With **R**

```
> Q <- matrix(nc=3, byr=T, c(-2,1,1,1,-2,1,1,1,-2))
> s <- nrow(Q)
> solve(t(cbind(Q[,1:(s-1)], rep(1,s))), c(rep(0,s-1),1))
[1] 0.3333333 0.3333333 0.3333333
```


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Global balance equations

- Why are they called Global balance equations?

$$\left. \begin{aligned} \pi \mathbf{Q} = \mathbf{0} &\Rightarrow \sum_{i=0}^{\infty} \pi_i q_{ij} = 0 \\ \sum_{i=0}^{\infty} q_{ji} = 0 &\Rightarrow \pi_j \sum_{i=0}^{\infty} q_{ji} = 0 \end{aligned} \right\} \Rightarrow \pi_j \sum_{i=0}^{\infty} q_{ji} = \sum_{i=0}^{\infty} \pi_i q_{ij}$$

$$\sum_{i=0}^{\infty} \pi_i q_{ij} \Rightarrow \text{Frequency of transitions entering state } j$$

$$\pi_j \sum_{i=0}^{\infty} q_{ji} \Rightarrow \text{Frequency of transitions leaving state } j$$

- In **stationary regime**, the frequency of transitions leaving state j is equal to the frequency of transitions entering state j .



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Solving using flux balancing

- Define the **flux** F_{uv} from state u to v :

$$F_{uv} = \pi_u q_{uv}$$

- and the flux from set of states U to V :

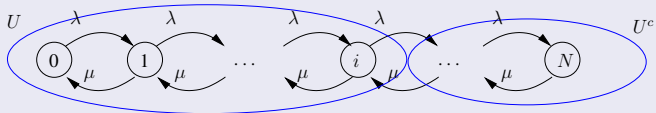
$$F(U, V) = \sum_{u \in U} \sum_{v \in V} F_{uv}$$

- From the Global balance equations, and reasoning exactly as in DTMC:

$$F(U, U^c) = F(U^c, U)$$

- NOTE:** Same equation as in DTMC, changing p_{ij} by q_{ij} .

Example: Birth-dead Process



- Flux balancing $\Rightarrow \lambda \pi_i = \mu \pi_{i+1}$
- Iterating:

$$\pi_i = \pi_0 \rho^i, i = 0, 1, \dots, N-1, \rho = \frac{\lambda}{\mu}$$

- Normalizing:

$$\pi_0 = \frac{1 - \rho}{1 - \rho^N}$$



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Ergodic Chains

- **Ergodic state**: positive recurrent ($f_{ii}^e = 1, m_{ii}^e < \infty$).
- **Ergodic chain** if all states are ergodic.
- **Theorem**: All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent (see [1, chapter XV]).
- **Consequences**:
 - **Finite irreducible** chains are **ergodic** (since all states are positive recurrent).
 - **Infinite irreducible** chains can be:
 - **Ergodic**: all the states are positive recurrent (stable chains).
 - **Non ergodic**: all states are null recurrent or transient (unstable chains).

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition*. Wiley, 1968.



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Theorems for ergodic chains

- $\pi = \pi(\infty)$. Proof: $\pi(\infty)$ satisfies the GBE.
- In stationary regime (when $\pi = \pi e^{Q^t}$), the **mean number of time the system remains in state j** during T time units is given by

$$T\pi_j$$

thus, π_j is the fraction of time the chain remains in state j .
The proof is analogous to DTMC.

- **NOTE:** The relation of DTMC between **mean recurrence time** and stationary probabilities does not hold for CTMC. I.e., the mean number of time units between two consecutive visits to state j , **m_{jj} , cannot be computed as $1/\pi_j$** . It must be computed with the **recursive equations** (slide 35).

Reversible Chains

- Let $X(t)$ be an **ergodic** MC. The chain $X^r(t) = X(-t)$ is referred to as the **time reversal chain** of $X(t)$.
- The same results obtained for DTMC reversed chains apply to CTMC, changing p_{ij} by q_{ij} :

- The reversed chain transition rates q_{ij}^r , given by:

$$\pi_i q_{ij} = \pi_j q_{ji}^r$$

satisfy the **reversed balance equations**: $F(U, V) = F^r(V, U)$

- A chain is **reversible** if:

$$q_{ij} = q_{ij}^r$$

- Reversible chains satisfy the **detailed balance equations**:

$$F(U, V) = F(V, U), \forall (V, U), V \cap U = \emptyset$$

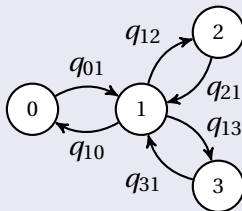
Reversible Chains

- The same results obtained for DTMC Reversible Chains apply to CTMC: **Kolmogorov Criteria** and **Product Form Solution** for the stationary distribution (changing p_{ij} by q_{ij}).
- E.g. the **stationary probabilities** are given by:
 - Choose a state $s \in S$,
 - For every other state $i \in S$, $i \neq s$ look for a possible path l_i from state s to state i :

$$s = (l_i, 1) \rightsquigarrow (l_i, 2) \rightsquigarrow \dots \rightsquigarrow (l_i, m_{l_i}) = i, m_{l_i} \geq 1$$

$$\pi_i = \frac{\psi_i}{\sum_{j \in S} \psi_j}, i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \prod_{k=1}^{m_{l_i}-1} \frac{q_{(l_i, k)(l_i, k+1)}}{q_{(l_i, k+1)(l_i, k)}}, & i \neq s \end{cases}$$

Reversible Chains, Example



- An ergodic tree is always reversible, thus

$$\pi_0 = \frac{1}{G}, \pi_1 = \frac{1}{G} \frac{q_{01}}{q_{10}}, \pi_2 = \frac{1}{G} \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}}, \pi_3 = \frac{1}{G} \frac{q_{01}}{q_{10}} \frac{q_{13}}{q_{31}}.$$

- Normalizing:

$$G = 1 + \frac{q_{01}}{q_{10}} + \frac{q_{01}}{q_{10}} \frac{q_{12}}{q_{21}} + \frac{q_{01}}{q_{10}} \frac{q_{13}}{q_{31}}$$



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Introduction

- Define the continuous RV H_i equal to the **sojourn time** in state i .
- In a **semi-Markov process** we leave the H_i distribution to be **arbitrary**. If H_i is exponentially distributed, we have a CTMC.
- **NOTE: If H_i is not exponentially distributed**, considering only the current state does not satisfy the Markov property (memoryless) since **the evolution of the process depends on the current state and the sojourn time in the state: (i, t_i)** .
- If we consider (i, t_i) as the state, the state would satisfy the Markov property, but we would have a **Markov process** (since t_i is not a discrete RV).

Embedded MC (EMC) of a semi-Markov process

- **Embedded MC (EMC)** of the process: **We only look at the state transition instants.**
 - **The EMC is a DTMC** with transition probabilities p_{ij}^e .
 - The **time step is variable**.
 - There are not self transitions ($p_{ii}^e = 0$), unless we look at some memoryless event that produce a self transition.
- **Theorem:** let π_i^e and π_i be the stationary distribution of the EMC and the semi-Markov process respectively. Let $E[H_i]$ be the mean sojourn time in state i , then:

$$\pi_i = \frac{\pi_i^e E[H_i]}{\sum_j \pi_j^e E[H_j]}$$

NOTE: By *stationary distribution* for the semi-Markov process we mean to the long-run proportion of time that the process is in each state.

Embedded MC (EMC) of a semi-Markov process

Proof:

- For n steps of the EMC, define:
 - $f_i(n)$: proportion of time the process is in state i .
 - $N_i(n)$: number of visits to state i .
 - $H_i(l)$: sojourn time in state i in the visit number l .

$$f_i(n) = \frac{\sum_{l=1}^{N_i(n)} H_i(l)}{\sum_j \sum_{l=1}^{N_j(n)} H_j(l)} = \frac{\frac{N_i(n)}{n} \sum_{l=1}^{N_i(n)} \frac{H_i(l)}{N_i(n)}}{\sum_j \frac{N_j(n)}{n} \sum_{l=1}^{N_j(n)} \frac{H_j(l)}{N_j(n)}} \Rightarrow \pi_i = \frac{\pi_i^e E[H_i]}{\sum_j \pi_j^e E[H_j]}$$

- since:

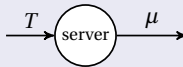
$$\lim_{n \rightarrow \infty} f_i(n) = \pi_i$$

$$\lim_{n \rightarrow \infty} \sum_{l=1}^{N_i(n)} \frac{H_j(l)}{N_i(n)} = E[H_i]$$

$$\lim_{n \rightarrow \infty} \frac{N_i(n)}{n} = \pi_i^e$$

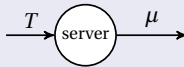
Embedded MC (EMC) of a semi-Markov process, Example

Suppose the system:



- Packets arrive deterministically every T time units.
- Upon a packet arrival it goes immediately into service if the server is empty, and it is lost if the server is busy.
- Services are exponentially distributed with rate μ .

Embedded MC (EMC) of a semi-Markov process, Example



Define a semi-Markov process with states $\begin{cases} \textcircled{0} & \text{server empty,} \\ \textcircled{1} & \text{server busy.} \end{cases}$

- ① Derive the EMC and stationary distribution of the EMC and continuous time process.
- ② Compute the throughput and loss probability.

Hint: The distribution of an event R exponentially distributed with rate μ , given that occurs in an interval $t \in [0, T]$, is

$$F_R(t|T) = \frac{P(R \leq t, R \leq T)}{P(R \leq T)} = \frac{P(R \leq t)_{t \in [0, T]}}{P(R \leq T)} = \frac{1 - e^{-\mu t}}{1 - e^{-\mu T}}, \quad t \in [0, T], \text{ and}$$

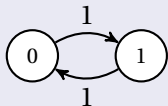
$$E[R|T] = \int_0^T (1 - F_R(t|T)) dt = \frac{1 - \alpha - \alpha \mu T}{\mu(1 - \alpha)}, \text{ where } \alpha = e^{-\mu T}.$$



Semi-Markov Process

Embedded MC (EMC) of a semi-Markov process, Solution

EMC:



- $\pi_0^e = \pi_1^e = 1/2$,
- $E[H_0] = T - E[R|T] = \frac{T\mu - (1 - \alpha)}{\mu(1 - \alpha)}$, $E[H_1] = \frac{1}{\mu}$.

And the continuous time process:

$$G = \pi_0^e E[H_0] + \pi_1^e E[H_1] = \frac{T}{2(1 - \alpha)} \left[\frac{\text{time units}}{\text{step}} \right]$$

$$\pi_0 = \frac{\pi_0^e E[H_0]}{G} = \frac{\mu T - (1 - \alpha)}{\mu T}, \pi_1 = \frac{\pi_1^e E[H_1]}{G} = \frac{1 - \alpha}{\mu T}$$

$$\text{Throughput: } S = \mu \pi_1 = \frac{1 - \alpha}{T} \quad (\text{check } S = \frac{1}{E[H_0] + E[H_1]})$$

$$\text{Loss probability: } S = \frac{1}{T} (1 - p_L), p_L = 1 - ST = \alpha = e^{-\mu T}.$$

Embedded MC of a CTMC

- Assume that the semi-Markov process is a **CTMC** (sojourn times are exponentially distributed).
- The transition probabilities p_{ij}^e of the EMC, and the stationary distribution π_i of the CTMC are given by:

$$\begin{cases} p_{ij}^e = q_{ij} / q_i, & i \neq j \\ p_{ij}^e = 0, & i = j \end{cases} \quad \pi_i = \frac{\pi_i^e / q_i}{\sum_k \pi_k^e / q_k}$$

where: $q_i = \sum_{k \neq i} q_{ik} = -q_{ii}$.

Embedded MC of a CTMC

Proof:

- The equations for p_{ij}^e was proven in a previous section.
- The distribution of the **sojourn time in state i in the CTMC** is the distribution of the **minimum of independent exponentially distributed RV** with parameters $\{q_{ik}\}_{k \neq i}$. This distribution is **exponentially distributed with parameter $\sum_{k \neq i} q_{ik}$** . Thus $E[H_i] = 1 / \sum_{k \neq i} q_{ik} = 1 / q_i$, $q_i = \sum_{k \neq i} q_{ik}$.
- Substituting:

$$\pi_i = \frac{\pi_i^e E[H_i]}{\sum_j \pi_j^e E[H_j]} = \frac{\pi_i^e / q_i}{\sum_j \pi_j^e / q_j}$$

where: $q_i = \sum_{k \neq i} q_{ik} = -q_{ii}$. \square



Part III

Continuous Time Markov Chains (CTMC)

Outline

- Definition of a CTMC
- Transient Solution
- Embedded MC of a CTMC
- Classification of States
- Steady State
- Semi-Markov Process
- **Finite Absorbing Chains**

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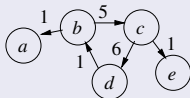
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Canonical Form

- Let \mathbf{Q}^{rxr} be the transition probability matrix of a chain with a set S of **s transient** states and a set A of **r - s absorbing** states. We can write \mathbf{Q}^{rxr} in the **canonical** form:

$$\mathbf{Q}^{rxr} = \begin{bmatrix} \mathbf{T}^{s \times s} & \mathbf{R}^{s \times r-s} \\ \mathbf{0}^{r-s \times s} & \mathbf{0}^{r-s \times r-s} \end{bmatrix}$$



$$\mathbf{P} = \begin{array}{c} \begin{matrix} & b & c & d & | & a & e \end{matrix} \\ \begin{matrix} b \\ c \\ d \\ a \\ e \end{matrix} \left[\begin{array}{ccc|cc} -6 & 5 & 0 & 1 & 0 \\ 0 & -7 & 6 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

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- Define:

$$n_{ij} = \left\{ \begin{array}{l} \text{time in state } j \text{ before absorption,} \\ \text{starting from state } i \end{array} \right\},$$

$$t_i = \left\{ \begin{array}{l} \text{time in transient states before} \\ \text{absorption, starting from state } i \end{array} \right\},$$

$$b_{ij} = P(\text{probability to be absorbed } j \text{ starting } i).$$

- Then:

$$\{E[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}$$

$$\{E[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\mathbf{e}$$

$$\{b_{ij}\} = \mathbf{B} = \mathbf{N}\mathbf{R}$$

where $\{a_{ij}\}$ is a matrix with a_{ij} as element ij and \mathbf{e} is a column vector of 1s. \mathbf{N} is called the **fundamental matrix**.

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Proof

Let **S** be the set of **transient** states and **A** the set of **absorbing** states.

- $\{E[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}$

$$E[n_{ij}] = \frac{\delta_{ij}}{-q_{ii}} + \sum_{\substack{k \neq i \\ k \in S}} p_{ik}^e E[n_{kj}] = \frac{\delta_{ij}}{-q_{ii}} + \sum_{\substack{k \neq i \\ k \in S}} \frac{q_{ik}}{-q_{ii}} E[n_{kj}] \Rightarrow$$

$$- \sum_{k \in S} q_{ik} E[n_{kj}] = \delta_{ij} \Rightarrow -\mathbf{T} \{E[n_{ij}]\} = \mathbf{I} \Rightarrow \{E[n_{ij}]\} = -\mathbf{T}^{-1} = \mathbf{N}$$

- $\{E[t_i]\} = \boldsymbol{\tau} = \mathbf{N} \mathbf{e}$

$$E[t_i] = \sum_{k \in S} E[n_{ik}] \Rightarrow \{E[t_i]\} = \boldsymbol{\tau} = \mathbf{N} \mathbf{e}$$

- $\{b_{ij}\} = \mathbf{B} = \mathbf{N} \mathbf{R}$

$$b_{ij} = p_{ij}^e + \sum_{\substack{k \neq i \\ k \in S}} p_{ik}^e b_{kj} = \frac{q_{ij}}{-q_{ii}} + \sum_{\substack{k \neq i \\ k \in S}} \frac{q_{ik}}{-q_{ii}} b_{kj}, i \in S, j \in A \Rightarrow$$

$$- \sum_{k \in S} q_{ik} b_{kj} = q_{ij} \Rightarrow -\mathbf{T} \{b_{ij}\} = \mathbf{R} \Rightarrow \{b_{ij}\} = \mathbf{B} = -\mathbf{T}^{-1} \mathbf{R} = \mathbf{N} \mathbf{R}.$$



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- The previous results can be generalized to any group of states of \mathbf{Q} :
- A set S is referred to as **open** if the chain can reach some state of S^c starting from any state of S . Let

$$\mathbf{T} = \{q_{ij}, i \in S, j \in S\}$$

$$\mathbf{R} = \{q_{ij}, i \in S, j \in S^c\}$$

Let assume that the process starts from $i \in S$. Define:

$$n_{ij} = \left\{ \begin{array}{l} \text{number of steps in state } j \text{ before} \\ \text{leaving } S, \text{ starting from state } i \end{array} \right\},$$

$$\Rightarrow \{E[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}.$$

- Similarly for the other results, e.g. $\boldsymbol{\tau} = \{E[t_i]\} = \mathbf{N}\mathbf{e}$ and $\mathbf{B} = \{b_{ij}\} = \mathbf{N}\mathbf{R}$.