Dimension and codimension of simple games

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Abstract

This paper studies the complexity of computing a representation of a simple game as the intersection (union) of weighted majority games, as well as, the dimension or the codimension. We also present some examples with linear dimension and exponential codimension with respect to the number of players.

Keywords: Simple games, Dimension, Codimension, Computational complexity

1 Introduction and preliminaries

We consider the so-called *simple games* and the computational complexity of representing them as unions or intersections of weighted majority games.

Simple games and its dimension, as well as, weighted majority games, were defined by Taylor and Zwicker [10]. Later, Freixas and Marciniak [3] introduced a new concept, the codimension of simple games.

A *simple game* is a tuple $\Gamma = (N, \mathcal{W})$, where N is a finite set of *players* and $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotonic family of subsets of N. Furthermore, *its dual* $\Gamma^* = (N, \mathcal{W}^*)$ is the game such that $\mathcal{W}^* = \{S \subseteq N : N \setminus S \notin \mathcal{W}\}$. Γ is said to be *self-dual* if $\Gamma = \Gamma^*$. Note that $(\Gamma^*)^* = \Gamma$. Given two simple games $\Gamma_1 = (N_1, \mathcal{W}_1)$ and $\Gamma_2 = (N_2, \mathcal{W}_2)$, they are *equivalent* if $N_1 = N_2$ and $\mathcal{W}_1 = \mathcal{W}_2$. The subsets of N are called *coalitions*, the set N is the *grand coalition* and each $X \in \mathcal{W}$ is a *winning coalition*. The complement of the family of winning coalitions is the family of *losing coalitions* $\mathcal{L}$, i.e., $\mathcal{L} = \mathcal{P}(N) \setminus \mathcal{W}$. Any of those set families determine uniquely the game Γ and constitute one of the usual forms of representation for simple games [10], although the size of the representation is not, in general, polynomial in the number of players [8].

A simple game Γ is a *weighted majority game* (WMG) if it admits a representation by means of $n+1$ nonnegative real numbers $[q; w_1, \ldots, w_n]$ such that $S \in \mathcal{W} \iff w(S) \geq q$ where, for each coalition $S \subseteq N$, $w(S) = \sum_{i \in S} w_i$. The number q is called the quota and w_i the weight of the player i. It is well known that any WMG admits a representation with integer numbers. The *dimension* of a simple game Γ is the least k such that there exists WMGs $\Gamma_1, \ldots, \Gamma_k$ such that $\Gamma = \Gamma_1 \cap \ldots \cap \Gamma_k$. On the other hand, the *codimension* of a simple game Γ is the least k such that there exists WMGs $\Gamma_1, \ldots, \Gamma_k$ such that $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_k$.

There are many theoretical results and examples about dimension and codimension [9,7,3,6,4,10,5] including computational complexity results [1]. We present some results that will be used later on.

Lemma 1.1 *The dimension of a simple game v is bounded above by $\left|\mathcal{L}^M\right|$ and the co-dimension is bounded above by $|\mathcal{W}^m|$.*

Lemma 1.2 *Let Γ be a simple game. Γ is the intersection of t weighted games if and only if Γ^* is the union of t weighted games. Furthermore a representation, as union (intersection), of Γ^* can be obtained from a representation, as intersection (union), of Γ in polynomial time. Moreover, dim(Γ)=codim(Γ^*), and if Γ is self-dual then dim(Γ) = codim(Γ).*

Note that the converse statement of the last sentence is not true in general as there are weighted games which are not self-dual.

2 Computational complexity of related problems

First, we present a simple game with $2n$ players, dimension n and codimension 2^{n-1}. Other examples of high dimensional games can be found in [10,9].

Example 2.1 Given a positive integer n, Freixas and Marcinicak (Theorem 2 of [3]) define a simple game with $2n$ players and dimension n. Let $\Gamma = (N, \mathcal{W})$ be a simple game defined by $N = \{1, 2, \ldots, 2n\}$ and $S \in \mathcal{W}$ iff $S \cap \{2i-1, 2i\} \neq \emptyset$, $i \in \{1, \ldots, n\}$, then Γ has dimension n.

As S is a winning coalition in Γ^* iff $N \setminus S$ is a losing coalition in Γ, $\Gamma^* = (N, \mathcal{W}_1^* \cup \ldots \cup \mathcal{W}_n^*)$, where $\mathcal{W}_i^* = \{S \subseteq N : \{2i-1, 2i\} \subseteq S\}$, $i \in \{1, \ldots, n\}$. As Γ^* is a composition of n unanimity games, Γ^* has dimension 2^{n-1} [4] and Γ has codimension 2^{n-1} (by Lemma 1.2).

Proposition 2.2 *Given a simple game Γ as union (intersection) of weighted games, computing a representation of Γ as intersection (union) of weighted games requires exponential time.*

The complexity of several problems about representations of simple games as intersections of WMGs were analyzed in [1]. We provide here a new reduction from the NP-hard *Subset Sum Problem (*SSP*)*. Our reduction differs in the fact that, for the game $\Gamma(I, d)$ associated to an instance I, we know both the dimension and the codimension.

Lemma 2.3 *Let $d > 1$. When I is a yes instance of* SSP *then $dim(\Gamma(I, d)) = d$ and $codim(\Gamma(I, d)) = 2^d$, otherwise, $dim(\Gamma(I, d)) = codim(\Gamma(I, d)) = 1$.*

Combining lemmas 2.3 and 1.2 we can prove the following results.

Proposition 2.4 *Let d_1 and d_2 be two integers with $1 \leq d_2 < d_1$. Then the problem of deciding whether the union of d_1 given WMGs can also be represented as the union of d_2 WMGs is NP-hard.*

Proposition 2.5 *Let d_1 and d_2 be two integers with $1 \leq d_1, d_2$. Then the problem of deciding whether the intersection (union) of d_1 given WMGs can also be represented as the union (intersection) of d_2 WMGs is NP-hard.*

As a consequence of the previous results, given a simple game Γ as union or intersection of WMGs, to compute $dim(\Gamma)$, $codim(\Gamma)$ or deciding whether Γ is weighted are NP-hard problems. Recall that two game representations are said to be *equivalent* whenever the represented games have the same set of winning coalitions. We can extend several results on equivalence problems from [2] to games given as unions of WMG, in particular we have.

Proposition 2.6 *Checking whether a given union of* **WMGs** *is equivalent to a given union of* **WMGs** *is co-NP-complete, even if all weights are equal to 0 or 1.*

It remains open to exhaustively classify the dimension and codimension of all complete simple game up to n players. Some bounds about dimension are given by Freixas and Puente [4] and Olsen *et al.* [9]. As well as to find complete simple games with *small* dimension (codimension), but with *large* codimension (dimension), and to construct analytical examples with specific dimension and codimension. It is also interesting to find real simple games with *large* dimension or codimension as the example given by Kurz and Napel [7].

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