Universitat Politècnica de Catalunya. Departament d'Arquitectura de Computadors.
Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

Stochastic Network Modeling (SNM). Autumn 2018.

Second assessment, Continuous Time Markov Chains. 22/11/2018.

Problem 1

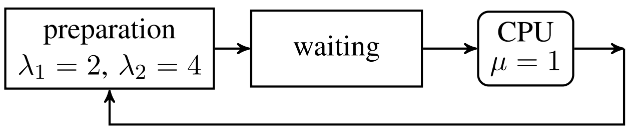
Consider a system with 2 processes p_1 and p_2 that can be in three stages (see the figure):

(a) Preparation: where p_1 and p_2 stay a time exponentially distributed with parameters $\lambda_1 = 2$ and $\lambda_2 = 4$, respectively.

(b) After preparation the processes go into the CPU, or wait if busy.

(c) In the CPU each process remains a time exponentially distributed with parameter $\mu = 1$.

(d) After the CPU the processes return to preparation.



1.A (2.5 points) Consider the embedded Markov chain obtained observing the system when the processes enter and leave the CPU. Formulate the Markov chain and compute the stationary distribution of the embedded chain. Indicate clearly the meaning of each state.

1.B (1.5 points) Obtain the stationary distribution of the continuous time process of the previous chain.

1.C (1 points) Use the previous results to compute the expected number of p_1 and p_2 processes dispatched by the CPU per time unit (S_1 and S_2, respectively).

Problem 2

Suppose now that the time in the CPU is deterministic, equal to 1 time unit.

2.A (2.5 points) Consider the embedded Markov chain obtained observing the system when the processes enter and leave the CPU. Formulate the Markov chain and compute the stationary distribution of the embedded chain. To make numerical calculus easily, you can round numbers to two decimal places.

2.B (1.5 points) Obtain the stationary distribution of the continuous time process of the previous chain.

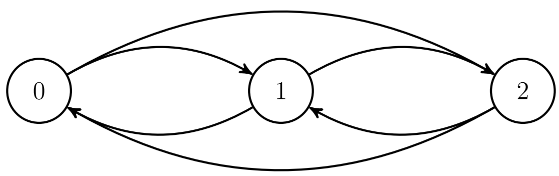
2.C (1 points) Use the previous results to compute the expected number of p_1 and p_2 processes dispatched by the CPU per time unit (S_1 and S_2, respectively).

Solution

Problem 1

1.A The states we can observe are the following:

0 CPU idle
1 p_1 starts a CPU cycle
2 p_2 starts a CPU cycle



with transition probabilities:

$$
\begin{aligned}
p_{01} &= \frac{\lambda_1}{\lambda_1+\lambda_2} = 1/3 & p_{02} &= 1 - p_{01} = 2/3 \\
p_{10} &= \frac{\mu}{\lambda_2+\mu} = 1/5 & p_{12} &= 1 - p_{10} = 4/5 \\
p_{20} &= \frac{\mu}{\lambda_1+\mu} = 1/3 & p_{21} &= 1 - p_{20} = 2/3
\end{aligned}
$$

and stationary distribution

$$\pi^e = \begin{bmatrix} 3/14 & 5/14 & 6/14 \end{bmatrix} \approx \begin{bmatrix} 0.214 & 0.357 & 0.429 \end{bmatrix}.$$

1.B The sojourn times are:

$$
\begin{aligned}
\mathrm{E}[H_0] &= \frac{1}{\lambda_1+\lambda_2} = 1/6 \\
\mathrm{E}[H_1] &= \frac{1}{\mu} = 1 \\
\mathrm{E}[H_2] &= \frac{1}{\mu} = 1.
\end{aligned}
$$

The time unit per step:

$$T = \sum \pi^e \, \mathrm{E}[H_i] = 23/28 \approx 0.821,$$

and the stationary distribution of the continuous time process:

$$
\begin{aligned}
\pi_0 &= \frac{\pi_0^e \, \mathrm{E}[H_0]}{T} = 1/23 \approx 0.043 \\
\pi_1 &= \frac{\pi_1^e \, \mathrm{E}[H_1]}{T} = 10/23 \approx 0.435 \\
\pi_2 &= \frac{\pi_2^e \, \mathrm{E}[H_2]}{T} = 12/23 \approx 0.522
\end{aligned}
$$

1.C

$$
\begin{aligned}
S_1 &= \pi_1 \, \mu \approx 0.435 \\
S_2 &= \pi_2 \, \mu \approx 0.522
\end{aligned}
$$

Problem 2

2.A The states and the chain are as in probmem 1, with transition probabilities:

$$
\begin{aligned}
p_{01} &= \frac{\lambda_1}{\lambda_1+\lambda_2} = 1/3 & p_{02} &= 1 - p_{01} = 2/3 \\
p_{10} &= \mathrm{e}^{-\lambda_2} = \mathrm{e}^{-4} = \sigma^2 & p_{12} &= 1 - p_{10} = 1 - \sigma^2 \\
p_{20} &= \mathrm{e}^{-\lambda_1} = \mathrm{e}^{-2} = \sigma & p_{21} &= 1 - p_{20} = 1 - \sigma
\end{aligned}
$$

and stationary distribution

$$
\begin{aligned}
\pi_0^e &= \frac{3\,\sigma + 3\,\sigma^2 - 3\,\sigma^3}{6 + \sigma + 2\,\sigma^2 - 3\,\sigma^3} \approx 0.074 \\
\pi_1^e &= \frac{3 - 2\,\sigma}{6 + \sigma + 2\,\sigma^2 - 3\,\sigma^3} \approx 0.443 \\
\pi_2^e &= \frac{3 - \sigma^2}{6 + \sigma + 2\,\sigma^2 - 3\,\sigma^3} \approx 0.484.
\end{aligned}
$$

2.B The sojourn times are:

$$
\begin{aligned}
\mathrm{E}[H_0] &= \frac{1}{\lambda_1+\lambda_2} = 1/6\\
\mathrm{E}[H_1] &= 1\\
\mathrm{E}[H_2] &= 1.
\end{aligned}
$$

The time unit per step:

$$T = \sum \pi^e \,\mathrm{E}[H_i] = \frac{1}{2}\,\frac{12-3\,\sigma-\sigma^2-\sigma^3}{6+\sigma+2\,\sigma^2-3\,\sigma^3} \approx 0.939,$$

and the stationary distribution of the continuous time process:

$$
\begin{aligned}
\pi_0 &= \frac{\pi_0^e\,\mathrm{E}[H_0]}{T} = \frac{\sigma+\sigma^2-\sigma^3}{12-3\,\sigma-\sigma^2-\sigma^3} \approx 0.013\\
\pi_1 &= \frac{\pi_1^e\,\mathrm{E}[H_1]}{T} = \frac{2\,(3-2\,\sigma)}{12-3\,\sigma-\sigma^2-\sigma^3} \approx 0.472\\
\pi_2 &= \frac{\pi_2^e\,\mathrm{E}[H_2]}{T} = \frac{2\,(3-\sigma^2)}{12-3\,\sigma-\sigma^2-\sigma^3} \approx 0.515
\end{aligned}
$$

2.C

$$
\begin{aligned}
S_1 &= \pi_1\,1 \approx 0.472\\
S_2 &= \pi_2\,1 \approx 0.515.
\end{aligned}
$$

Rounding to two decimal places we have:

$$
\begin{aligned}
p_{01} &= \frac{\lambda_1}{\lambda_1+\lambda_2} = 1/3 \approx 0.33 & p_{02} &= 1-p_{01} = 2/3 \approx 0.67\\
p_{10} &= \mathrm{e}^{-\lambda_2} = \mathrm{e}^{-4} \approx 0.02 & p_{12} &= 1-p_{10} \approx 0.98\\
p_{20} &= \mathrm{e}^{-\lambda_1} = \mathrm{e}^{-2} \approx 0.14 & p_{21} &= 1-p_{20} \approx 0.86
\end{aligned}
$$

and stationary distribution

$$\pi^e \approx \begin{bmatrix}0.07 & 0.45 & 0.47\end{bmatrix}.$$

The time unit per step:

$$T = \sum \pi^e \,\mathrm{E}[H_i] \approx 0.93,$$

and the stationary distribution of the continuous time process:

$$\pi \approx \begin{bmatrix}0.01 & 0.48 & 0.51\end{bmatrix}.$$