Algorithmic Game Theory
Homework 6 - Cooperative Games

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1 Problem 10

1.1 Question a

1.1.1 Monotone

In order to probe if this game is *monotone* we need to probe that $v(C) \leq v(D)$ for any C, D such that $C \subseteq D$. Since we have a case function we are going to analyze the different parts.

Let have a graph $G_x = (V, X)$. Since by definition the players are the Edges, we can add player to X, so we are adding edges Y to the graph G_x, such that $X' = X \cup Y$. Lets call this new graph $G'_x = (V', X')$.

- If G'_x is connected then $v(X') = 2|X'| - \text{diam}(G'_x)$. Since we are adding more edges $|X'| > |X|$, and the diameter is at least greater or equal $diam(G'_x) \geq diam(G_x)$ because it is connected and by definition of diameter if we are adding edges the greatest length of the shortest path cannot be smaller with more edges.
- If G'_x is not connected also $\frac{|X'|}{2} > \frac{|X|}{2}$.

Therefore for any $G_x \subseteq G'_x$, $v(X) \leq v(X')$, so it is **Monotone**.

1.1.2 Superadditive

A game is Superadditive if $v(C \cup D) \geq v(C) + v(D)$ for any 2 disjoint coalitions C and D.

Lets analyize each case. Lets take if $G_{X \cup Y}$ is connected, then

$$v(X \cup Y) = 2(|X| + |Y|) - diam(G_{X \cup Y}) \tag{1a}$$
$$= 2|X| + 2|Y| - diam(G_{X \cup Y}) \tag{1b}$$

And $2|X| + 2|Y| - diam(G_{X \cup Y}) \geq 2|X| - diam(G_X) + 2|Y| - diam(G_Y)$ because $diam(G_{X \cup Y}) \leq diam(G_X) + diam(G_Y)$ since it is included in one of the 2.

Lets take if $G_{X \cup Y}$ is not connected, then

$$v(X \cup Y) = \frac{|X| + |Y|}{2} \tag{2a}$$
$$= \frac{|X|}{2} + \frac{|Y|}{2} \tag{2b}$$
$$= v(X) + v(Y) \tag{2c}$$

Therefore it is superadditive.

1.1.3 Supermodular

A game is Supermodular if $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$.

By the previous item we have probe that it is superadditive, so if we sum $v(C \cap D)$ to the union, we are still going to have something greater or equal of the sum of the individual valuation functions, therefore it is also supermodular.

1.2 Question b

Yes, a graph of only 1 edge because. Let suppose that that graph is $A = (Z, Y)$ where $Z = z, v$ and $Y = z, v$. Lets assume that we take $X = Y$, then

- $v(X) = 2|X| - diam(A_X) = 2 \times 1 - 1 = 1$
- Also we have that $x(X) = 1$ because we have 1 edge.
- Therefore the core is not empty.

2 Problem 16

In order to probe if it is $NP-$ Complete we need to first show that it is $NP-$ hard.

Lets call X the votes that are not in M.

Lets consider every *pairwise* competition and we could see that every *pairwise* is determined without M except for the pairwise a and b

- d wins to a and b in first row $2K+1$
- a and b wins to c
- c wins to d

If there is someone who wins between a and b pairwise, that one either a or b will tie with d. So, taking into consideration the previous statement d wins **Copeland** manipulation election $\iff a \wedge b$ tie in their *pairwise*. But, after the votes in X a and b are tied. In order to maintain this tie we need that the **combined weights** $k_i \in M$ such that $\{b \mid b\ P\ a\}$ (b is preferred over a) is the same as **combined weights** $k_i \in M$ such that $\{a \mid a\ P\ b\}$ (a is preferred over b).

Since there is a $PARTITION$ instance $(k1, \ldots, k_n)$ with $\sum_{i=1}^{n} = 2K$, so M can be balance and preserved the tie.

Therefore since $PARTITION$ is $NP-$ Complete, *Copeland-CWM* is also $NP-$ Complete.