

Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Queuing Theory

## Stochastic Network Modeling (SNM)

Llorenç Cerdà-Alabern Universitat Politècnica de Catalunya Departament d'Arquitectura de Computadors llorenc@ac.upc.edu

#### **Parts**

- Introduction
- Discrete Time Markov Chains (DTMC)
- Continuous Time Markov Chains (CTMC)
- Queuing Theory



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Queuing Theory

Part IV

Queuing Theory

Outline

- Introduction
- Kendal Notation
- Little Theorem
- PASTA Theorem
- The M/M/1 Queue
- M/G/1 Oueue

- M/G/1/K Queue
- M/G/1 Busy Period
- M/G/1 Delays
- Oueues in Tandem
- Networks of Oueues
- Matrix Geometric Method



### Introduction

Queuing Theory

Introduction

Little Theore

PASTA Theorem

The M/M/ Oueue

M/G/1 Queue

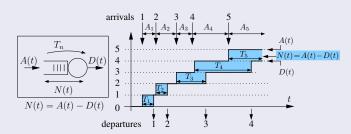
mr or i Queuc

M/G/1 Bus

M/G/1 Delay

Queues in

Tandem



- Queueing theory is the mathematical study of waiting lines, or queues.
- Common notation:
  - A(t): number of arrivals [0, t].
  - $A_n$ : interarrival time between customers n and n+1.
  - $T_n$ : time in the system (response time) for customer n.
  - N(t): number in the system at time t.



### Kendal Notation

Queuing Theory

Kendal

Notation

**Kendal Notation** 

A/S/k[/c/p]

- A: arrival process,
- S: service process,
- k: number of servers.
- c: maximum number in the system (number of servers + queue size). Note: some authors use the queue size.
- p: population. If "c" or "p" are missing, they are assumed to be infinite.

arrivals departures



### **Kendal Notation**

Queuing Theory

Introduction

Kendal

Notation

PASTA Theorem

The M/M/1 Queue

M/G/1 Queue

M/G/1 Bus

M/G/1 Delays

Tandem

### Common arrivals/service processes

- G: general (non specific process is assumed),
- M: Markovian (exponentially or geometrically distributed),
- D: deterministic.
- P: Poisson (discrete RV, *N*, equal to the number of arrivals exponentially dist. in a time *t*):

$$P_p(N=n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n \ge 0, t \ge 0.$$

• Er: Erlang (continuous RV equal to the time *t* that last *n* arrivals exponentially dist.):

$$f_e(t) = \lambda P_p(N = n - 1, t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, t \ge 0, n \ge 1$$

#### Examples

- M/M/1: M. arr. / M. serv. / 1 server,  $\infty$  queue and population.
- M/G/1: M. arr. / Gen. serv. / 1 server, ∞ queue and population.



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems Stochastic Network Modeling (SNM)

Queuing Theory

Part IV

ma o a a cuo.

Little Theorem

Application to the waiting line and the

Mean number in the

Mean number in th Server

PASTA Theorem

Queue

M/G/1 Queue

M/G/1/K Queue

M/G/1 Bus Period

Period

# **Queuing Theory**

#### Outline

- Introduction
- Kendal Notation
- Little Theorem
- PASTA Theorem
- The M/M/1 Queue
- M/G/1 Oueue

- M/G/1/K Queue
- M/G/1 Busy Period
- M/G/1 Delays
- Queues in Tandem
- Networks of Queues
- Matrix Geometric Method

M/G/1 Dela



Queuing Theory

Introduction

Little Theorem

Graphical proof
Application to the
waiting line and the
server

Mean number in the Server

PASTA Theorem

M/G/1 Oue

M/G/1/K Queue

M/G/1 Busy Period

Period

#### Little Theorem

- Define the stochastic processes:
  - A(t): number of arrivals [0, t].
  - $T_n$ : time in the system (response time) for customer n.
  - N(t): number in the system at time t.
- And the mean values:
  - Mean number of customers in the system:

$$N = \lim_{t \to \infty} \frac{1}{t} \int_0^t N(s) \, \mathrm{d}s$$

- Arrival rate:  $\lambda = \lim_{t \to \infty} A(t)/t$
- Mean time in the system:  $T = \lim_{t \to \infty} (\sum_n T_n) / A(t)$
- The following relation follows:

$$N = \lambda T$$

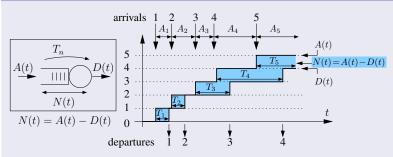
Mnemonic: NAT (Number = Arrivals x Time).



Queuing Theory

Graphical proof

### Graphical proof



From the graph we have:

$$\frac{1}{t} \int_0^t N(s) \, ds = \frac{1}{t} \sum_{i=1}^{A(t)} T_i = \frac{A(t)}{t} \frac{\sum_{i=1}^{A(t)} T_i}{A(t)}$$

Taking the limit  $t \to \infty$ :  $N = \lambda T$ 



Queuing Theory

Introduction

Little Theorem

Graphical proof

Application to the waiting line and the server

Mean number in the

PASTA Theorem

The M/M/1

M/G/1 Queu

M/G/1/K Queue

M/G/1 Busy Period

M/C/1 Deleve

### Application to the waiting line and the server

 We can apply the Little theorem to the waiting line and the server:

Waiting time in the queue of customer n:  $W_n$  Service time:  $S_n$   $A(t) \qquad D(t)$ 

N(t)

Time in the system:  $T_n = W_n + S_n$  Expected value:

$$T = W + S$$
  
where  
 $T = E[T_n], W = E[W_n],$ 

 $S = E[S_n]$ 

- Mean number of customers in the queue:  $N_O = \lambda W$ .
- Mean number of customers in the server:  $N_S = \rho = \lambda S$ .



Queuing Theory

Introduction

Little Theore

Graphical proof

Application to the waiting line and the server

Mean number in the Server

Theorem

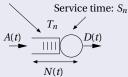
M/G/1 Ouer

M/G/1/K Queue

M/G/1 Busy Period

#### Mean number in the Server

Waiting time in the queue of customer n:  $W_n$ 



Time in the system:

$$T_n = W_n + S_n$$
  
Expected value:

$$T = W + S$$
  
where  
 $T = E[T_n], W = E[W_n],$   
 $S = E[S_n]$ 

• In a single server queue (even if not Markovian):

$$\rho = N_S = \mathbb{E}[N_S(t)] = \lambda \, \mathbb{E}[S]$$
  
$$\mathbb{E}[N_S(t)] = 0 \times \pi_0 + 1 \times (1 - \pi_0) = 1 - \pi_0 \Rightarrow \pi_0 = 1 - \rho$$

•  $\rho = N_S = \lambda E[S] = 1 - \pi_0$  is the proportion of time the system is busy, in other words, is the server utilization or load.



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems Stochastic Network Modeling (SNM)

Queuing Theory

Part IV

**Queuing Theory** 

PASTA

Theorem

### Outline

- PASTA Theorem



### PASTA Theorem

Queuing Theory

Introductio:

ivotation

Little Theorei

PASTA Theorem

Example of P/

Queue

M/G/1 Queue

M/G/1/K Queue

T CHOU

M/G/1 Delays

Queues in Tandem

### PASTA Theorem: Poisson Arrivals See Time Averages

- The mean time the chain is in state i is  $\pi_i \Rightarrow$  using PASTA, the probability that a Markovian arrival see the system in state i is  $\pi_i$  (proof: see [1]).
- The equivalent theorem in discrete time is the arrival theorem, RASTA: Random Arrivals See Time Averages: the probability that a random arrival see the system in state i is  $\pi_i$ .
- [1] Ronald W Wolff. "Poisson arrivals see time averages". In: *Operations Research* 30.2 (1982), pp. 223–231.



Queuing Theory

Introduction

Example of PASTA

Little Theorem

PASTA Theorem

Example of PASTA

Queue

M/G/1 Queue

M/G/1/K Queue

M/C/1 Dolor

O.....

Queues in Tandem Assume that a system can have, at most, *N* customers (e.g *N* – 1 in the queue and 1 in service).

• Assume that an arrival is lost when the system is full.

• By PASTA the proportion of Poisson arrivals that see the system full, and are lost, is equal to the proportion of time the system has N in the system,  $\pi_N$ .

• Thus, the loss probability is  $\pi_N$ .



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Queuing Theory

Part IV

**Queuing Theory** 

Outline

The M/M/1 Queue

The M/M/1 Queue



Queuing Theory

Introduction

Notation

Little Theorer

PASTA Theoren

The M/M/1

Queue Q-matrix

Distribution Properties Stability

Stability
Example: Loss
probability in a
telephone switchi

M/G/1 Queu

M/G/1/K Queue

#### The M/M/1 Queue

$$A(t) \xrightarrow{N(t)} T_n = W_n + S_n$$

• Markovian arrivals with rate  $\lambda \Rightarrow$  the interarrival time is exponentially distributed with mean  $1/\lambda$ :

$$P\{A_n \le x\} = 1 - e^{-\lambda x}, x \ge 0$$

 $\Rightarrow$  A(t) is a Poisson process:

$$P(A(t) = i) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}, i \ge 0, t \ge 0$$

• Markovian Services with rate  $\mu \Rightarrow$  service time exponentially distributed with mean  $1/\mu$ :

$$P\{S_n \le x\} = 1 - e^{-\mu x}, x \ge 0$$



Queuing Theory

Introduction

Little Theorem

PASTA

Theorem

Queue O-matrix

Q-matrix

Stationary Distribution

Stability
Example: Loss

M/G/1 Queu

M/G/1/K Queue

#### Q-matrix

• The process  $N(t) = \{\text{number in the system at time } t \ge 0\}$  is a CTMC.

OBSERVATION: for a non Markovian service, the process N(t) would not be a MC! State transition diagram:

• Q-matrix:

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\mu + \lambda) & \lambda & 0 & \cdots \\ 0 & \mu & -(\mu + \lambda) & \lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$



Queuing Theory

Introduction

Little Tileoreii

PASTA Theoren

The M/N

Queue

Stationary Distribution

Stability
Example: Loss
probability in a

M/G/1 Que

M/G/1/k Queue

### Stationary Distribution

• Solving the M/M/1 queue using flux balancing (or the general solution of a reversible chain):

$$\pi_i = (1 - \rho) \rho^i, i = 0, \cdots, \infty$$

where  $\rho = \frac{\lambda}{\mu}$ 



Queuing Theory

Introduction

Notation

Little Theorei

PASTA Theoren

The M/M/

Queue

Stationar

Properties

Example: Loss probability in a telephone switching

M/G/10

M/G/1/K Queue

### **Properties**

• Mean customers in the system:

$$N = \lim_{t \to \infty} \frac{1}{t} \int_0^t N(s) \, ds = \sum_{i=0}^{\infty} i \pi_i = \sum_{i=0}^{\infty} i (1 - \rho) \, \rho^i = \frac{\rho}{1 - \rho}$$

• Mean time in the system (response time):

Little: 
$$N = \lambda T \Rightarrow T = \frac{N}{\lambda} = \frac{\rho}{\lambda (1 - \rho)} = \frac{1}{\mu - \lambda}$$

- Mean time in the queue:  $W = T \frac{1}{\mu} = \frac{\rho}{\mu \lambda}$
- Mean Number in the queue:  $N_Q = \lambda W = \frac{\rho^2}{1-\rho}$
- Mean number in the server:  $N_s = N N_Q = \rho$ NOTE:  $\pi_0 = 1 - \rho$



Queuing Theory

Introductio

introductio

Little Theore

PASTA Theorem

The M/N

Queue

Q-matrix

Distrib

Properti Stability

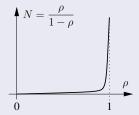
Example: Loss probability in a telephone switchin center

M/G/1 Que

M/G/1/K Queue

### Stability

- *N* and *T* are proportional to  $1/(1-\rho) \Rightarrow$  when  $\rho \to 1 \Rightarrow N, T \to \infty$ .
- The process N(t) is positive recurrent, null recurrent or transient according to whether  $\rho = \lambda/\mu$  is below, equal or greater than 1, respectively.





Queuing Theory

Introduction

Notation

Little Theorem

PASTA Theorem

The M/N

Queue

Q-matrix

Distribu

Stability
Example: Loss

probability in a telephone switching center

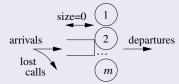
Wi/G/T Que

M/G/1/I Queue

M/G/1 Rusy

### Example: Loss probability in a telephone switching center

• Hypothesis: Switching center with m circuits and "lost call", infinite population, Markovian arrivals with rate  $\lambda$  and exponentially distributed call duration with mean  $1/\mu \Rightarrow M/M/m/m$  queue.





Queuing Theory

Introduction

Little Theorem

PASTA Theoren

The M/N

Queue

Q-matrix

Distribu

Stability

Example: Loss probability in a telephone switching center

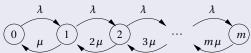
M/G/1 Que

M/G/1/I Queue

G/1 Busv

### Example: Loss probability in a telephone switching center

• Since the minimum of i independent and identically exponentially distributed RV with parameter service time is exponentially distributed with parameter  $i\mu$ :





Queuing Theory

Example: Loss probability in a telephone switching

#### Example: Loss probability in a telephone switching center

- Stationary Distribution of the queue M/M/m/m:
- Solving using the general solution of a reversible chain:

Define 
$$\rho_k = \frac{\lambda}{(k+1)\mu}$$
,  $k = 0, \dots, m-1$ 

$$\pi_0 = \frac{1}{G}, \, \pi_i = \frac{1}{G} \prod_{k=0}^{i-1} \rho_k = \frac{1}{G} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}, \, 0 < i \le m \Rightarrow$$

$$\pi_i = \frac{1}{G} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}, 0 \le i \le m. \ G = \sum_{k=0}^m \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}.$$

• Using PASTA Theorem (Poisson Arrivals See Time Average): the loss call probability is the probability that the queue is in state m:  $\pi_m$ , "Erlang B Formula".



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Queuing Theory

Part IV

**Queuing Theory** 

M/G/1 Oueue

#### Outline

- M/G/1 Oueue



Queuing Theory

Introduction

. . . .

Little Theorer

PASTA Theorem

The M/M/ Queue

M/G/1 Queue

Matrix
Properties of the stationary distribution  $(\pi = \pi P, \pi e = 1)$ Proof of the Level

M/G/1 Oueue

M/G/1 Busy

#### M/G/1 Queue

- The process  $N(t) = \{\text{number in the system at time } t \ge 0\}$  in general it is not a MC (it is so only if G is Markovian).
- We can build a semi-Markov process observing the system at departure times  $t_n$  (note that  $t_n$  are also the service completion times). Define the discrete time process:

 $X(n) = \{\text{number in the system at time } t_n \ge 0, n = 0, 1, \dots \}$ 

- Theorem: The process X(n) is a DTMC.
- Proof: X(n) only depends on the number of arrivals in non overlapping intervals. Since arrivals are Markovian, this is a memoryless process.
- NOTE: Looking at departure times the chain may have self transitions (in contrast to observing at transition times): we can have the same number in the system after a departure.



Queuing Theory

Introductio

Notation

Little Theorem

PASTA Theorem

The M/M/I Queue

M/G/1 Ouer

Transition Probability
Matrix
Properties of the stationary distribution  $(\pi = \pi P, \pi e = 1)$ 

Proof of the Level
Crossing Law Theore

M/G/1 Bus

M/G/1 Busy Period

#### **Transition Probability Matrix**

- Let  $f_S(x)$ ,  $x \ge 0$  be the service time density function.
- Define the RV V = {number of arrivals during a service time}, and the probabilities: v<sub>i</sub> = P{V = i}.
- Conditioning on the service duration:

$$v_i = \int_{x=0}^{\infty} P\{i \text{ arrivals in time } x \mid S = x\} f_S(x) dx \Rightarrow$$

$$v_i = \int_{x=0}^{\infty} \frac{(\lambda x)^i}{i!} e^{-\lambda x} f_S(x) dx$$



Queuing Theory

Introduction

. . . . .

Little Theorei

PASTA Theoren

The M/M/ Oueue

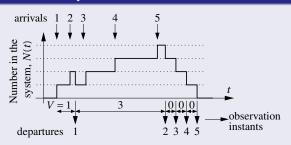
M/G/1 Queue

Transition Probability Matrix Properties of the stationary distribution  $(\pi = \pi P, \pi e = 1)$ Proof of the Level

M/G/1

M/G/1 Busy Period

### **Transition Probability Matrix**



•  $v_i = P\{\text{number of arrivals during a service time} = i\} \Rightarrow$ 

$$p_{ij} = \begin{cases} 0, & j < i-1 \quad (N(t) \text{ can only be decreased by 1}) \\ v_j, & i = 0, j \ge 0 \quad (i = 0 \to \text{the queue was empty}) \\ v_{j-i+1}, & i > 0, j \ge i-1 \quad (i > 0 \to \text{the queue was busy}) \end{cases}$$



Queuing Theory

Introduction

Little Theorer

PASTA

The M/M/

M/G/1 Queue

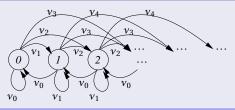
Transition Probability Matrix Properties of the

 $(\pi = \pi P, \pi e = 1)$ Proof of the Level Crossing Law Theorem

M/G/1/

M/G/1 Busy Period

### **Transition Probability Matrix**



$$p_{ij} = \begin{cases} 0, & j < i-1 \\ v_j, & i = 0, j \ge 0 \\ v_{j-i+1}, & i > 0, j \ge i-1 \end{cases} \Rightarrow \mathbf{P} = \begin{bmatrix} v_0 & v_1 & v_2 & v_3 & \cdots \\ v_0 & v_1 & v_2 & v_3 & \cdots \\ 0 & v_0 & v_1 & v_2 & \cdots \\ 0 & 0 & v_0 & v_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

• Stationary distribution:  $\pi = \pi P$ ,  $\pi e = 1$ .



Queuing Theory

Properties of the

### <u>Properties of the stationary distribution ( $\pi = \pi P$ , $\pi e = 1$ )</u>

• Using the "Level Crossing Law" theorem: a queue with unitary arrivals and departures satisfies:

> $P\{\text{an arriving customer finds } i \text{ in the system}\} =$  $P\{a \text{ departing customer leaves } i \text{ in the system}\} \Rightarrow$

 $\pi_i = P\{\text{an arriving customer find } i \text{ in the system}\}\$ 

Using PASTA:

 $\pi_i = P\{\text{there are } i \text{ customers in the } \}$ system at an arbitrary time}

So, in an M/G/1 the stationary distribution of the EMC obtained observing the departures, is the stationary distribution of the continuous time process.



Queuing Theory

#### r...............................

introductio

Little Theorei

PASTA Theorem

The M/M/I Oueue

### M/G/1 Queu

Hansilion Probability
Matrix
Properties of the stationary distribution  $(\pi = \pi P, \pi e = 1)$ Proof of the Level
Crossing Law Theorem

M/G/1/

M/G/1 Busy

#### Proof of the Level Crossing Law Theorem

- Define:
  - $A_i(t) = \{\text{number of arrivals finding } i \text{ in the system at } t \ge 0\}$
  - D<sub>i</sub>(t) ={number of departures leaving i in the system at t ≥ 0}
  - $P\{\text{a customer finds } i \text{ in the system}\} = \lim_{t\to\infty} A_i(t)/A(t)$
  - $P\{\text{a customer leave } i \text{ in the system}\} = \lim_{t\to\infty} D_i(t)/D(t)$
- An arriving customer that finds i in the system produce a transition  $i \rightarrow i+1$ . A customer leaving i in the system produce a transition  $i+1 \rightarrow i$ .
- Since arrivals and departures are unitary, the number of transitions  $i \to i+1$  and  $i+1 \to i$  can only differ in 1:  $|A_i(t) D_i(t)| \le 1$ . Note that N(t) = A(t) D(t).
- For a stable queue:  $A(t) D(t) < \infty$



Queuing Theory

Proof of the Level Crossing Law Theorem

#### Proof of the Level Crossing Law Theorem

- We have:
  - $A_i(t) = \{\text{number of arrivals finding } i \text{ customer in the system}\}$
  - $D_i(t) = \{\text{number of departures leaving } i \text{ customers in the } i \text{ cu$ system}
  - $P\{\text{a customer finds } i \text{ in the system}\} = \lim_{t\to\infty} A_i(t)/A(t)$
  - $P\{a \text{ customer leave } i \text{ in the system}\} = \lim_{t\to\infty} D_i(t)/D(t)$
  - $|A_i(t) D_i(t)| \le 1$ ,  $N(t) = A(t) D(t) < \infty$ .
  - $\lim_{t\to\infty} A(t) = \infty$ ,  $\lim_{t\to\infty} D(t) = \infty$ .
- Thus:

$$\lim_{t \to \infty} \left\{ \frac{A_i(t)}{A(t)} - \frac{D_i(t)}{D(t)} \right\} = \lim_{t \to \infty} \left\{ \frac{A_i(t)}{A(t)} - \frac{D_i(t)}{A(t)} - \left( \frac{D_i(t)}{D(t)} - \frac{D_i(t)}{A(t)} \right) \right\} = \lim_{t \to \infty} \left\{ \frac{A_i(t) - D_i(t)}{A(t)} - \frac{D_i(t)}{D(t)} \frac{A(t) - D(t)}{A(t)} \right\} = 0$$



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Queuing Theory

M/G/1/K

Queue

Part IV

**Queuing Theory** 

#### Outline

- M/G/1/K Queue



Queuing Theory

Introduction

-----

Little Theorem

Theoren

The M/M/I

M/G/1 Oueur

..., G, I Quouc

Problem Formulation

Stationary

Distribution

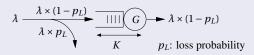
Loss Probability

Loss Probability

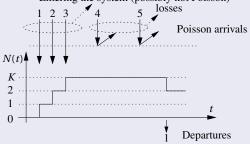
M/G/1 Dela

M/G/1 Delay

### **Problem Formulation**



Entering the system (possibly not Poisson)





Queuing Theory

#### **Stationary Distribution**

- Using the general solution of an M/G/1/K we obtain the stationary distribution of the number in the system left by a departing customer:  $\pi_i^d$ .
- By the Level Crossing Law this is the stationary distribution of the number in the system found by the successful arrivals:

$$\pi_i^s = \pi_i^d, i = 0, 1, \dots K - 1.$$

and

$$\pi_i^s = P(a \text{ customer entering the system finds } i)$$

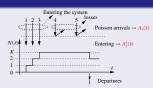
 NOTE: a departing customer cannot leave the system full (nor an arrival can enter the system when it is full).



Queuing Theory

Loss Probability

### Loss Probability



#### Define:

- $A_i^a(t)$ : Number of arrivals (lost or not) finding i in the system.
- $A_i^s(t)$ : Number of successful arrivals finding i in the system.
- $\pi_i^a$ ,  $\pi_i^s$  the stationary distribution of the embedded Markov chains  $A_i^a(t)$ ,  $A_i^s(t)$ . By PASTA  $\pi_i^a$  is also the stationary distribution of the continuous time process. Thus,

 $\pi_i^s = P(\text{a customer entering the system finds } i), i = 0, 1, \dots K - 1 \Rightarrow$ 

$$\pi_i^s = \lim_{t \to \infty} \frac{A_i^s(t)}{\sum_{k=0}^{K-1} A_k^s(t)} \frac{\sum_{k=0}^K A_k^a(t)}{\sum_{k=0}^K A_k^a(t)} = \frac{\pi_i^a}{\sum_{k=0}^{K-1} \pi_i^a} = \frac{\pi_i^a}{1 - \pi_K^a} = \frac{\pi_i^a}{1 - p_L}, \Rightarrow$$

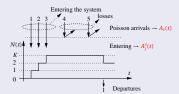
$$\pi_i^a = \pi_i^s (1 - p_L) = \pi_i^d (1 - p_L), i = 0, 1, \dots \frac{K - 1}{I}$$



Queuing Theory

Loss Probability

### Loss Probability



- Applying Little:  $\rho_S = E[N_S] = 1 \pi_0 = \lambda (1 p_L) E[S] = \rho (1 p_L)$ . Where  $\rho = \lambda E[S]$  and  $\pi_0$  is the proportion of time the server is empty.
- Using PASTA:  $\pi_0 = \pi_0^a$  (Poisson arrivals). Using  $\pi_i^a = \pi_i^d (1 p_I)$ :

$$\begin{vmatrix} 1 - \pi_0 = 1 - \pi_0^a = 1 - \pi_0^d (1 - p_L) \\ 1 - \pi_0 = \rho (1 - p_L) \end{vmatrix} \Rightarrow \boxed{ p_L = \frac{\rho + \pi_0^d - 1}{\rho + \pi_0^d}, \rho = \lambda \operatorname{E}[S] }$$

• Where  $\pi_0^d$  is computed using the general solution of an M/G/1/K.