Stochastic Network Modeling (SNM)

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Parts

I Introduction
II Discrete Time Markov Chains (DTMC)
III Continuous Time Markov Chains (CTMC)
IV Queuing Theory

Part II

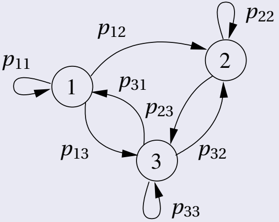
Discrete Time Markov Chains (DTMC)

Outline

Definition of a DTMC

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:
$$\sum_{\forall j} p_{ij} = 1$$

- Mathematically:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

Definition of a DTMC

Properties of a DTMC

- The event $X(n)=i$ (at step n the system is in state i) must satisfy (memoryless property):

$$\boxed{\begin{aligned}P(X(n)=j \mid X(n-1)=i, X(n-2)=k, \cdots) =\\ P(X(n)=j \mid X(n-1)=i)\end{aligned}}$$

- If $P\big(X(n)=j \mid X(n-1)=i\big) = P\big(X(1)=j \mid X(0)=i\big)$ for any n we have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$\boxed{p_{ij} = P\big(X(n)=j \mid X(n-1)=i\big)}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

Definition of a DTMC

Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

Definition of a DTMC

Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$$

- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) =$$

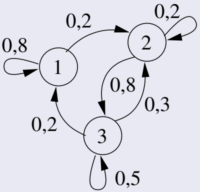
$$\sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = \boxed{1}$$

- $\mathbf{P}$ is a stochastic matrix, i.e. a matrix which rows sum 1.

Definition of a DTMC

Example

- Assume a terminal can be in 3 states:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



$$
\begin{array}{c}
\text{to state} \\
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\mathbf{P} = & \left[\begin{array}{ccc} 0{,}8 & 0{,}2 & 0 \\ 0 & 0{,}2 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{array}\right]
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\quad \begin{array}{c} \text{from} \\ \text{state} \end{array}
\end{array}
$$

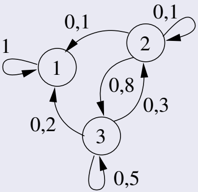
- The average transmission rate (throughput), v_a, is:

$$v_a = P(\text{the terminal is in state 3}) \times v$$

Definition of a DTMC

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



to state

$$\mathbf{P} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\mathbf{P} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

from state

Definition of a DTMC

n-step transition probabilities

- Transition probabilities: $p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$
- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- We define the n-step transition probabilities:

$$p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$$

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are stochastic matrices: Their rows sum 1.

Definition of a DTMC

State Probabilities

- Define the probability of being in state i at step n:
$$\pi_i(n) = P(X(n) = i)$$
- In vector form (row vector)
$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$
- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the state probability at step n.

Definition of a DTMC

State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n)=1), P(X(n)=2), \cdots).$$

- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1)=k)\, P\big(X(n)=i \,\big|\, X(n-1)=k\big) = \sum_k \pi_k(n-1)\, p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0)=k)\, P\big(X(n)=i \,\big|\, X(0)=k\big) = \sum_k \pi_k(0)\, p_{ki}(n)$$

- In matrix form:

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

Definition of a DTMC

State Probabilities

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

- Iterating

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P} = \boldsymbol{\pi}(n-2)\,\mathbf{P}\,\mathbf{P} = \boldsymbol{\pi}(n-3)\,\mathbf{P}\,\mathbf{P}\,\mathbf{P} = \cdots = \boldsymbol{\pi}(0)\,\mathbf{P}^n$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\ p_{kj}(n-r)}$$

- **Proof**:

$$\begin{aligned}
p_{ij}(n) &= P\big(X(n)=j \mid X(0)=i\big) = \sum_k P\big(X(n)=j, X(r)=k \mid X(0)=i\big) \\
&= \sum_k \frac{P\big(X(n)=j, X(r)=k, X(0)=i\big)}{P(X(0)=i)} \times \frac{P(X(r)=k, X(0)=i)}{P(X(r)=k, X(0)=i)} \\
&= \sum_k P\big(X(n)=j \mid X(r)=k, X(0)=i\big)\ P\big(X(r)=k \mid X(0)=i\big) \\
&= \sum_k P\big(X(n)=j \mid X(r)=k\big)\ P\big(X(r)=k \mid X(0)=i\big) \\
&= \sum_k p_{ik}(r)\ p_{kj}(n-r)
\end{aligned}$$

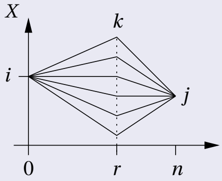
$\square$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\; p_{kj}(n-r)}$$

- Graphical interpretation:



- In matrix form:

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1)\,\mathbf{P}(n-1) = \mathbf{P}\,\mathbf{P}(n-1) = \mathbf{P}(n-1)\,\mathbf{P}$$

- Iterating:

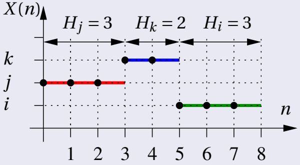
$$\boxed{\mathbf{P}(n) = \mathbf{P}^n}$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Sojourn or Holding Time

- Sojourn or holding time in state k: Is the RV H_k equal to the number of steps that the chain remains in state k before leaving to a different state:



- The Markov property implies:

$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

- Which is a geometric distribution with mean:

$$\mathrm{E}[H_i] = \sum_{n=1}^{\infty} n P(H_i = n) = \frac{1}{1 - p_{ii}}.$$

Definition of a DTMC

Sojourn or Holding Time

- NOTE: We allow that:

$$p_{ii} = 0 \Rightarrow H_i(n) = I(n = 1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases} \text{, and}$$

$$p_{ii} = 1 \Rightarrow \mathrm{E}[H_i] = \infty \text{ (absorbing state).}$$

Definition of a DTMC

Theorem

A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.

Proof.

- We have seen that a DTMC has a sojourn time

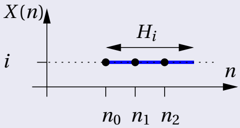
$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

- Which is geometrically distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).

□

Definition of a DTMC

The geometric distribution satisfies the Markov property (1)



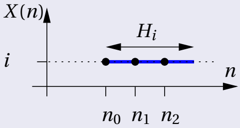
Proof

- Markov property:
  $P\big(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\big) = P\big(X(n_2) = i \mid X(n_1) = i\big)$
- Thus, the Markov property in terms of the sojourn time can be written as:

$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

Definition of a DTMC

The geometric distribution satisfies the Markov property (2)



$$P\left(H_i > n_2 - n_0 \;\middle|\; H_i > n_1 - n_0\right) = P(H_i > n_2 - n_1)$$

- Since

$$P(H_i > k) = 1 - P(H_i \le k) = 1 - \sum_{n=1}^{k} p^{n-1}(1-p) = 1-(1-p)\frac{1-p^k}{1-p} = p^k$$

- We have:

$$P\left(H_i > n_2 - n_0 \;\middle|\; H_i > n_1 - n_0\right) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$

$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2 - n_0}}{p^{n_1 - n_0}} = p^{n_2 - n_1} = P(H_i > n_2 - n_1) \quad \square$$

Part II

Discrete Time Markov Chains (DTMC)

Outline

Transient Solution

Transient Solution

- If we are interested in the transient evolution we shall study $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n$.
- If we can diagonalize **P**, we can obtain the transient evolution in close form.
- **P** can be diagonalized if **P** can be decomposed as:

$$\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$$

where **L** is some invertible matrix and Λ is the diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \cdots \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

with λ_l, $l = 1, \cdots N$ the eigenvalues of **P**.

Transient Solution

Eigenvalues

- The eigenvalues λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l}$ (or $\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* and *right* eigenvectors, respectively.

$$\boldsymbol{l}\mathbf{A} = \lambda_l \boldsymbol{l} \Rightarrow \boldsymbol{l}(\mathbf{A} - \mathbf{I}\lambda_l) = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

$$\mathbf{A}\boldsymbol{r} = \lambda_l \boldsymbol{r} \Rightarrow (\mathbf{A} - \mathbf{I}\lambda_l)\boldsymbol{r} = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

- Thus, λ_l solve the characteristic polynomial $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$.
- Note that, in general, *left* and *right* eigenvectors are different, but eigenvalues are the same (they solve the same characteristic polynomial).
- A matrix can be diagonalized if all eigenvalues are single (multiplicity $= 1$). If a matrix cannot be diagonalized it is called defective.

Transient Solution

Determinants

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{l} +a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{21}\, a_{32}\, a_{13} \\ -a_{31}\, a_{22}\, a_{13} - a_{12}\, a_{21}\, a_{33} - a_{23}\, a_{32}\, a_{11} \end{array}$$

- Cofactor Formula: expanding along a row i:

$$\det \mathbf{A} = \sum_{j=1}^{N} a_{ij}\, (-1)^{i+j} \det M_{ij},$$

where the minor matrices M_{ij} are obtained removing the row i and column j from $\mathbf{A}$. $(-1)^{i+j} \det M_{ij}$ is called the cofactor of a_{ij}.

Transient Solution

Properties of the determinants

$$\det \mathbf{A} = \prod \text{eigenvalues of } \mathbf{A}$$

$$\text{trace } \mathbf{A} = \sum \text{eigenvalues of } \mathbf{A}$$

where trace $\mathbf{A} = \sum$ elements of the diagonal of $\mathbf{A}$.

Transient Solution

Transient Solution

- Assume a finite DTMC with N states. Then $\mathbf{P} = \mathbf{P}^{N \times N}$.
- Assume that $\mathbf{P}$ can be diagonalized: $\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{P}$.
- Since $\Lambda^n = \mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)$, we have that

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n = \boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\Lambda^n\,\mathbf{L}) = \boldsymbol{\pi}(0)\,(\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L})$$

Transient Solution

Transient Solution

- But $\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L}$ are linear combinations of $\lambda_1^n, \cdots \lambda_N^n$. Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(n) = (\boldsymbol{\pi}(n))_i = \sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n}$$

where the unknown coefficients $a_i^{(l)}$ can be obtained solving the system of equations:

$$\sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n = (\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1$$

Transient Solution

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

- We want the probability of being in state 2 in n steps starting from state 1: $\pi_2(n)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transient Solution

Solution

- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 2/5$.

$$\pi_2(n) = \lambda_1^n a + b\lambda_2^n = a + b(2/5)^n$$

- Imposing the boundary conditions $\pi_i(n) = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i$:

$$\pi_2(0) = a + b = (\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{P}^0)_2 = (\mathbf{P}^0)_{12} = 0$$

$$\pi_2(1) = a + b(2/5) = (\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{P}^1)_2 = (\mathbf{P})_{12} = 1/5$$

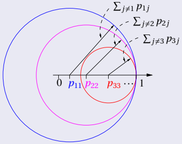
we have that $a = 1/3$, $b = -1/3$, thus:

$$\boxed{\begin{aligned}\pi_2(n) &= 1/3 - 1/3\,(2/5)^n, \quad n \geq 0\\ \pi_1(n) &= 1 - \pi_2(n) = 2/3 + 1/3\,(2/5)^n, \quad n \geq 0\end{aligned}}$$

Transient Solution

Eigenvalues of a Stochastic Matrix

- $\mathbf{P}$ has an eigenvalue equal to 1 ($\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 1$).
  Proof: $\mathbf{P}\,\mathbf{e} = \mathbf{e}$, where $\mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{P}$ add to 1). $\square$
- All eigenvalues of $\mathbf{P}$ are $|\lambda_l| \leq 1$.
  Proof: Using Gerschgorin's theorem *The eigenvalues of a matrix* $\mathbf{P}_{n\times n}$ *lie within the union of the n circular disks with center* p_{ii} *and radius* $\sum_{j\neq i}|p_{ij}|$ in $\mathbb{C}$. Since $\sum_j p_{ij} = 1$, the property is proved. $\square$

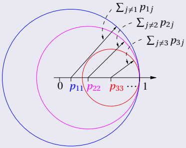


Gershgorin discs of a stochastic matrix

- The eigenvalue $\lambda = 1$ is single if $\mathbf{P}$ is irreducible (Perron-Frobenius theorem). $\mathbf{P}$ is irreducible if all states communicate: for some n, $p_{ij}(n) = (\mathbf{P}^n)_{ij} > 0$, $\forall\, i, j$.

Transient Solution

Proof of Gerschgorin's theorem



Gershgorin discs of a stochastic matrix

Gerschgorin's theorem: *The eigenvalues of a matrix* $\mathbf{P}_{n\times n}$ *lie within the union of the n circular disks with center* p_{ii} *and radius* $\sum_{j\neq i}|p_{ij}|$ in $\mathbb{C}$.

Proof: From $\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ we have

$$\sum_j p_{ij}\,x_j = \lambda\,x_i \quad \forall\, i \in \{1,\ldots,n\}.$$

We choose i such that $|x_i| = \max_j |x_j|$. Thus, $\sum_{j\neq i} p_{ij}\,x_j = \lambda\,x_i - p_{ii}\,x_i$, and

$$|\lambda - p_{ii}| = \left|\sum_{j\neq i} p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}\left|p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}|p_{ij}|$$

and the equation $|x - c| \le r$, $x, c \in \mathbb{C}, r \in \mathbb{R}$ is a disk of center c and radius r in $\mathbb{C}$. □

Transient Solution

Chain with a Defective Matrix

- What if $\mathbf{P}$ cannot be diagonalized? (defective matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{P}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$), and a possible eigenvalue $\lambda_1 = 0$ with multiplicity k_1. Then [1]:

$$\boxed{\pi_j(n) = \sum_{m=0}^{k_1-1} a_j^{(1,m)} I(n=m) + \sum_{l=2}^{L} \lambda_l^n \sum_{m=0}^{k_l-1} a_j^{(l,m)} n^m, \qquad 1 \leq j \leq N,\ n \geq 0}$$

$I(n=m)$ is the indicator func.: $I(n) = 1$ if $n = m$, $I(n) = 0$ if $n \neq m$.

[1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method*. Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html.

Transient Solution

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

- We want the probability of being in state 1 in n steps starting from state 1: $\pi_1(n)$ with $\pi_1(0) = 1$.
- It can be easily found that the eigenvalues of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1/4$ with multiplicity 2. We guess:

$$\pi_1(n) = a + 1/4^n(b + c\,n)$$

- Imposing $\pi_1(0) = 1$, $\pi_1(1) = 3/4$, $\pi_1(2) = (3/4)^2$, we have:

$$\boxed{\pi_1(n) = \frac{4}{9} + \frac{1}{4^n}\left(\frac{5}{9} + \frac{2}{3}n\right)}$$