Radomized Algorithms Problemes (29-34) Fall 2019. (Sol to all exercises)

29.- Necessitem que $\pi^* = \pi^* P$ i a més $\sum_{i=0}^{3} \pi_i^* = 1$, el que dóna

$$\begin{aligned}
\frac{1}{10}\pi_1^* + \frac{1}{10}\pi_2^* + \frac{9}{10}\pi_3^* &= \pi_0^* \\
\frac{3}{10}\pi_0^* + \frac{1}{10}\pi_1^* + \frac{7}{10}\pi_2^* + \frac{1}{10}\pi_3^* &= \pi_1^* \\
\frac{1}{10}\pi_0^* + \frac{7}{10}\pi_1^* + \frac{1}{10}\pi_2^* &= \pi_2^* \\
\frac{3}{5}\pi_0^* + \frac{1}{10}\pi_1^* + \frac{1}{10}\pi_2^* &= \pi_3^* \\
\pi_0^* + \pi_1^* + \pi_2^* + \pi_3^* &= 1
\end{aligned}$$

que dóna $\pi_0^* = \frac{19}{81} = 0.2346$, $\pi_1^* = \frac{395}{1296} = 0.3048$, $\pi_2^* = \frac{341}{1296} = 0.2631$, $\pi_3^* = \frac{16}{81} = 0.1975$.

Amb distribució inicial $\pi(0) = (1, 0, 0, 0)$ hem de calcular $P_{0,3}^{32}$ = la darrere posició de $\pi(0)P = (0.2345, 0.3047, 0.2631, 0.1975)$, que és $P_{0,3}^{32} = 0.1975$.

(b) $(1/4, 1/4, 1/4, 1/4)P^{128} = (0.2354, 0.3047, 0.2631, 0.1975)$, per tant $P_{i,3}^{128} = 0.1975$.

(c) Volem $\max_i |P_{0,i}^t - \pi_i^*| \leq 0.01$ que és 13

30.- Done in class, see slides.

31.- (a) It is irreducible as the graph is strongly connected, i.e. there is a path from every state to any other state. It is aperiodic, as the values for t s.t. $P_{1,1}^t > 0$ are 1,2,3,4,... and their gcd is 1.

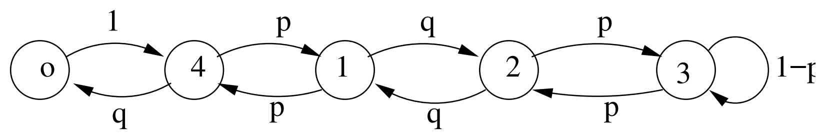
(b) It is Irreducible, as the graph is strongly connected. It is aperiodic, as the values for t s.t. $P_{1,1}^t > 0$ are 1,2,3,4,... and their gcd is 1.

(c) It is Irreducible, because there is a path from every state to any other state. It is periodic as the values for t for $p_{1,1}^t > 0$ are 2,4,6,... and their gcd is 2.

(d) It is NOT irreducible, as 2 is an absorbing state, so starting from it we do not go anywhere else. It is periodic, with period , because for each state i $p_{i,i} > 0$.

(e) It is irreducible as the graph is strongly connected. It is aperiodic as the values for t s.t. $P_{1,1}^t > 0$ are 1,2,3,4,... and their gcd is 1.

32.- * Define a Markov chain taking values in the set $S = \{i | i = 0, 1, 2, 3, 4\}$, where i represents the number of umbrellas in the place where I am currently at (home or office). If $i = 1$ and it rains then I take the umbrella, move to the other place, where there are already 3 umbrellas, and, including the one I bring, I have next 4 umbrellas. Thus, $p_{1,4} = p$, as $p =$ probability of rain. If $i = 1$ but it does not rain then I do not take the umbrella, I go to the other place and find 3 umbrellas. Thus, $p_{1,3} = 1-p = q$. We have the following Markov chain:



Let us find the stationary distribution. From picture: $\pi[1] = \pi[2] = \pi[3] = \pi[4]$ $\pi[0] = \pi[4]q$, $\pi[0] + \pi[1] + \pi[2] + \pi[3] + \pi[4] = 1$ Expressing all probabilities in terms of $\pi[4]$ and inserting in this last equation, we get $q\pi[4] + 4\pi[4] = 1 \Rightarrow \pi[4] = \frac{1}{q+4} = \pi[1] = \pi[2] = \pi[3]$, $\pi[4] = \frac{q}{q+4}$. I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is $\pi[0]$, the chance it rains is p so the probability I get wet is $\pi[0]p = \frac{qp}{q+4}$, which for $p = 0.6$ is 0.0545 i.e. $< 6\%$.

33.- * Formulate a new Markov chain with n^2 states of the form $(i, j) \in [1, n]^2$. Each node (i, j) in the new chain is connected to $N(i)N(j)$ neighbors, where $N(i)$ denotes the number of neighbors of state i in the old Markov chain. Hence the number of edges in the new chain comes to

$$2|B| = \sum_i \sum_j N((i,j)) = \sum_i \sum_j N(i)N(j) = (\sum_i N(i))(\sum_j N(j)) = m^2.$$

We have seing that if an edge exists between nodes $u = (i_1, j_1)$ and $v = (i_2, j_2)$, then $h_{u,v} \leq 2|E| = 4m^2$. In order to obtain the $O(m^2 n)$ upper bound, we need to show that for any node (i, j), there exists a path of length $O(n)$ connecting it to some node of the form (v, v). In fact, we show that there exists a length $O(n)$ path between (i, j) and (i, i). Since the graph is undirected, the cat can always go back to node i in two steps. At the same time, because the graph is connected, theres a path of length $k < n$ from j to i. If k is even, then the mouse will run into the cat. If k is odd, then the mouse will get to node i when the cat is away. But since the chain is non-bipartite, there must be a path of odd length from i back to itself; let the mouse follow this path, and it will run into the cat on the next return to i. Thus the total length of this path from (i, j) to (i, i) is at most $3n$. Each edge on this path requires at most $4m^2$ steps, thus the desired upper bound on the time to collision is $O(m^2 n)$ steps.

34.- * See MU book page 159–160

35.- * See MU book page 161–163

36.- (Lollipop graph RW)

1. We need the expected time it takes to travel the stick part of the lollipop from v to u ($h_{v,u}$), and the expected cover time of the clique part of the lollipop starting from node). Say there are k nodes in the stick part (excluding u), and k nodes in the ball part of the graph (including u), so that the total number of nodes is $n = 2k$. $h_{v,u}$ is just the time it takes to reach the kth. node on a chain starting from 0, i.e., t_0 in the chain for 2-SAT. So $h_{v,u} = k^2$. On the other hand c_u is upper bounded by the expected time it takes to travel to each of the nodes in the clique and return to u, so $c_u \leq \sum_{w \in K_k} h_{u,w} + h_{w,u}$. Let w and x denote nodes in the clique other than u, and let $i \in \{1, 2, \ldots, k\}$ denote the nodes on the stick, with 1 being the neighbor of u and k the of v. Then $h_{u,w}$ can be written in terms of the following system of equations

$$
\begin{aligned}
h_{u,w} =& \frac{1}{k} \cdot 0 + \frac{k-2}{k} h_{x,w} + \frac{1}{k} h_{1,w} + 1 \\
h_{x,w} =& \frac{1}{k} \cdot 0 + \frac{k-3}{k-1} h_{x,w} + \frac{1}{k-1} h_{u,w} + 1 \\
h_{1,w} =& \frac{1}{2} h_{u,w} + \frac{1}{2} h_{2,w} + 1 \\
h_{2,w} =& \frac{1}{2} h_{1,w} + \frac{1}{2} h_{3,w} + 1 \\
& \ldots \\
h_{k-1,w} =& \frac{1}{2} h_{k-2,w} + \frac{1}{2} h_{k,w} + 1 \\
h_{k,w} =& \frac{1}{2} h_{k-1,w} + 1
\end{aligned}
$$

   We obtain $h_{k-i,w} = h_{k-i-1,w} + (2i+1)$, and hence $h_{1,w} = h_{u,w} + 2k - 1$. Solving the equations, we get $h_{u,w} = \frac{k^2+9k-2}{2k}$.

   Using the same proof that in the cover lemma, $\frac{2|E|}{d(u)|=h_{u,u}=\frac{1}{d(u)}\sum_w \in N(u)}(1+ h_{w,u})$, and $\sum_{w \in N(u)} h_{w,u} = 2|E| - k = 2(k(k-1)+k) - k = 2k^2 - k$. So we have $c_u \leq \sum_{w \in K_k} h_{u,w} + h_{w,u} \leq (k-1)\frac{k^2+9k-2}{2k} + 2k^2 - k$.

   Combining everything, we get

$$
k^2 = h_{v,u} \leq \text{cover time starting from } v \leq h_{v,u} + c_u = O(k^2) = \Theta(n^2).
$$

2. Using the same node naming convention as before, we can write down the

following system of equations

$$\begin{aligned} h_{u,v} =& \frac{k-1}{k} h_{w,v} + \frac{1}{k} h_{1,v} + 1 \\ h_{w,v} =& \frac{k-2}{k-1} h_{w,v} + \frac{1}{k-1} h_{u,v} + 1 \\ h_{i,v} =& \frac{1}{2} h_{i-1,v} + \frac{1}{2} h_{i+1,v} + 1, \end{aligned}$$

so we get $h_{w,v} = h_{u,v} + k - 1$ and $h_{k-i,v} = \frac{i}{i+1} h_{k-i-1,v} + i$, hence $h_{1,v} = k - 1 k h_{u,v} + (k-1)$. Fromthis we get $h_{u,v} = k^3$.