Pure Nash Equilibria
complexity versus succinctness

Fall 2020

1. Complexity framework
2. Complexity analysis
3. Other succinct representations
4. Concluding remarks

Natural problems related to PNE

Is Nash (IsN)
Given a game Γ and a strategy profile a, decide whether a is a Nash equilibrium of Γ.

Exists Pure Nash (EPN)
Given a strategic game Γ, decide whether Γ has a Pure Nash equilibrium.

Pure Nash with Guarantees (PNGrant)
Given a strategic game Γ and a value v, decide whether there is a pure Nash equilibrium in which the first player gets payoff at least v.

How to represent a game?

- We are interested in fixing the representation of a game as an input to a program.
- It is natural to consider different levels of succinctness.
- In the most generic model some components of the game have to be represented by a TM, for example the utility functions.

TMs in game representations

- All the TMs appearing in the description of games are deterministic.

TMs in game representations

- All the TMs appearing in the description of games are deterministic.
- The TMs will work for a limited number of time steps (t) given as part of the input in unary ($\langle M, 1^t \rangle$).

TMs in game representations

- All the TMs appearing in the description of games are deterministic.
- The TMs will work for a limited number of time steps (t) given as part of the input in unary ($\langle M, 1^t \rangle$).
- Convention: there is a pre-fixed interpretation of the contents of the output tape of a TM so that, both when the machine stops or when the machine is stopped (after at most t steps), it always computes a rational value.

TMs in game representations

- All the TMs appearing in the description of games are deterministic.
- The TMs will work for a limited number of time steps (t) given as part of the input in unary ($\langle M, 1^t \rangle$).
- Convention: there is a pre-fixed interpretation of the contents of the output tape of a TM so that, both when the machine stops or when the machine is stopped (after at most t steps), it always computes a rational value.

We only consider rational valued utility functions
The convention guarantees a correct and unique game definition from its description

Explicit form

Strategic games in explicit form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, A_1, \ldots, A_n, T\rangle.$$

- *It has n players,*
- *For each player i, A_i is given explicitly by listing its elements.*
- *T is a table with an entry for each strategy profile s and player i.*
- *So, $u_i(s) = T(s, i)$.*

General form

Strategic games in general form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle.$$

- *It has n players,*
- *For each player i, A_i is given explicitly by listing its elements.*
- *The description of their pay-off is given by $\langle M, 1^t \rangle$.*
- *So, for each strategy profile s and player i, $u_i(s) = M(s, i)$ stopping after t steps.*

Implicit form

Strategic games in implicit form.

- *A game is given by a tuple*

$$\Gamma = \langle 1^n, 1^m, M, 1^t \rangle.$$

- *It has n players,*
- *For each player i, $A_i = \Sigma^m$*
- *The description of their pay-off is given by $\langle M, 1^t \rangle$.*
- *So, for each strategy profile s and player i, $u_i(s) = M(s, i)$ stopping after t steps.*

Forms of representation

Strategic games in explicit form. *A game is described by a tuple* $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$.

Strategic games in general form. *A game is described by a tuple* $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$.

Strategic games in implicit form. *A game is described by a tuple* $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$.

What is the most suitable level of succinctness?

- Prisoners' dilemma?

What is the most suitable level of succinctness?

- Prisoners' dilemma?
  Explicit

What is the most suitable level of succinctness?

- Prisoners' dilemma?
  Explicit
- Sending from s to t?

What is the most suitable level of succinctness?

- Prisoners' dilemma?
  Explicit
- Sending from s to t?
  General

What is the most suitable level of succinctness?

- Prisoners' dilemma?
  Explicit
- Sending from s to t?
  General
- Congestion games?

What is the most suitable level of succinctness?

- Prisoners' dilemma?
  Explicit
- Sending from s to t?
  General
- Congestion games?
  Implicit

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \; \forall a'_i \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a'_i)$$

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a'_i \in A_i \ u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combinations

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a'_i \in A_i \ u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a_i' \in A_i \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a_i' \in A_i \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, M, 1^t \rangle$ the cost is

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a'_i \in A_i \ u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N\ \forall a_i' \in A_i\ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T\rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t\rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t\rangle$ the cost is

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification?

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a_i' \in A_i \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification?
  The condition $u_i(s) \geq u_i(s_{-i}, a_i')$ can be checked in polynomial time given Γ, s, and a_i.

Solving the IsPN

IsPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification?
  The condition $u_i(s) \geq u_i(s_{-i}, a_i')$ can be checked in polynomial time given Γ, s, and a_i.
  Thus the problem is in

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \; \forall a'_i \in A_i \; u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification?
  The condition $u_i(s) \geq u_i(s_{-i}, a'_i)$ can be checked in polynomial time given Γ, s, and a_i.
  Thus the problem is in coNP.

Solving the IsPN

ISPN Given a game Γ and a strategy profile s, is s is a PNE?.

$$\forall i \in N \ \forall a'_i \in A_i \ u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combinations

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification?
  The condition $u_i(s) \geq u_i(s_{-i}, a'_i)$ can be checked in polynomial time given Γ, s, and a_i.
  Thus the problem is in coNP.
  Is this classification tight?

IsPN implicit form: Hardness

A coNP-complete problem?

IsPN implicit form: Hardness

A coNP-complete problem?

SAT: Given a boolean formula F in CNF form, determine whether F is satisfiable.

Is an NP-complete problem.

IsPN implicit form: Hardness

A coNP-complete problem?

SAT: Given a boolean formula F in CNF form, determine whether F is satisfiable.

Is an NP-complete problem. So, its complement is coNP-complete.

IsPN implicit form: Hardness

A coNP-complete problem?

SAT: Given a boolean formula F in CNF form, determine whether F is satisfiable.

Is an NP-complete problem. So, its complement is coNP-complete.

We have to associate F to a game Γ and a strategy profile s , i.e a reduction function f, where $f(F) = \langle \Gamma, s \rangle$, so that:

- F is not satisfiable iff s is a PNE of Γ
- and $\langle \Gamma, s \rangle$ can be obtained from F in polynomial time

IsPN implicit form: Hardness

Given a CNF formula F on n variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0,1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

IsPN implicit form: Hardness

Given a CNF formula F on n variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0,1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

Consider the strategy $a_1 = 0^{n+1}$.

IsPN implicit form: Hardness

Given a CNF formula F on n variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0,1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

Consider the strategy $a_1 = 0^{n+1}$.

a_1 is a PNE iff F is unsatisfiable

IsPN implicit form: Hardness

Given a CNF formula F on n variables consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, for any $x \in \{0,1\}^n$
- $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

Consider the strategy $a_1 = 0^{n+1}$.

a_1 is a PNE iff F is unsatisfiable

Thus $\Gamma(F), 0^{n+1}$ verify the first requirement.

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$
- M:

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$
- M: There is a TM M' that given a CNF formula F and a truth assignment x computes $F(x)$ in linear time $O(|F|)$.

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$
- M: There is a TM M' that given a CNF formula F and a truth assignment x computes $F(x)$ in linear time $O(|F|)$. M on input ax outputs 0 if $a = 0$, otherwise transfer the control to M' after writing in the input tape F and x.
- $t = (n + |F|)^2$.

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$
- M: There is a TM M' that given a CNF formula F and a truth assignment x computes $F(x)$ in linear time $O(|F|)$. M on input ax outputs 0 if $a = 0$, otherwise transfer the control to M' after writing in the input tape F and x.
- $t = (n + |F|)^2$.

The time required to obtain $\langle 1^n, 1^m, M, 1^t \rangle$, given F, is

IsPN implicit form: Hardness

Given a boolean formula F on n variables, consider the game $\Gamma(F)$ which:

- Has one player and $A_1 = \{0,1\}^{n+1}$
- $u_1(0x) = 0$, $u_1(1x) = F(x)$, for any $x \in \{0,1\}^n$

An implicit form representation of $\Gamma(F)$ as $\langle 1^n, 1^m, M, 1^t \rangle$?

- $n = 1$, $m = n + 1$
- M: There is a TM M' that given a CNF formula F and a truth assignment x computes $F(x)$ in linear time $O(|F|)$. M on input ax outputs 0 if $a = 0$, otherwise transfer the control to M' after writing in the input tape F and x.
- $t = (n + |F|)^2$.

The time required to obtain $\langle 1^n, 1^m, M, 1^t \rangle$, given F, is polynomial in $|F|$.

IsPN implicit form

Theorem

The IsPN problem for strategic games in implicit form is coNP-complete.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \ \forall i \in N \ \forall a_i' \in A_i \ \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \ \forall i \in N \ \forall a_i' \in A_i \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \ \forall i \in N \ \forall a_i' \in A_i \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential. Even we can easily show that it is in NP.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s\ \forall i \in N\ \forall a_i' \in A_i\ \ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential.
  Even we can easily show that it is in NP.
  In the case that n is constant, in P.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a'_i \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential.
  Even we can easily show that it is in NP.
  In the case that n is constant, in P.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a_i' \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential.
  Even we can easily show that it is in NP.
  In the case that n is constant, in P.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s\ \forall i \in N\ \forall a_i' \in A_i\ u_i(s) \geq u_i(s_{-i}, a_i')$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, T\rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \dots, A_n, M, 1^t\rangle$ the cost is exponential.
  Even we can easily show that it is in NP.
  In the case that n is constant, in P.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t\rangle$ the cost is exponential.
  A better classification?

Solving the EPN

EPN Given a game Γ does it have a PNE?.

$$\exists s \; \forall i \in N \; \forall a'_i \in A_i \;\; u_i(s) \geq u_i(s_{-i}, a'_i)$$

Algorithm: Brute force, try all combination

- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, T \rangle$ the cost is polynomial.
- Given $\Gamma = \langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ the cost is exponential.
  Even we can easily show that it is in NP.
  In the case that n is constant, in P.
- Given $\Gamma = \langle 1^n, 1^m, M, 1^t \rangle$ the cost is exponential.
  A better classification? in Σ_2^p.

EPN: general form

Theorem

The EPN problem for strategic games in general form is NP-complete.

We provide a reduction from SAT. Let F be a CNF formula.

- $F \to \Gamma(F) = \langle 1^n, \{0,1\} \ldots \{0,1\}, M^F, 1^{(n+|F|)^2} \rangle$ where
- n is the number of variables in F and
- M^F is a TM that on input (a, i), evaluates F on assignment a and afterwards it implements the utility function of the i-th player. According to the following definition:

EPN: general form

$$u_1(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \end{cases}$$

$$u_2(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 4 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 0, \\ 3 & \text{if } F(a) = 0 \wedge a_1 = 0 \wedge a_2 = 1, \\ 2 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 1, \\ 1 & \text{if } F(a) = 0 \wedge a_1 = 1 \wedge a_2 = 0. \end{cases}$$

And, for any $j > 2$

$$u_j(a) = \begin{cases} 5 & \text{if } F(a) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Reduction correctness

We have that

- Given a description of F, $\Gamma(F)$ is computable in polynomial time.

Reduction correctness

We have that

- Given a description of F, $\Gamma(F)$ is computable in polynomial time.
  Similar arguments as before.

Reduction correctness

We have that

- Given a description of F, $\Gamma(F)$ is computable in polynomial time.
  Similar arguments as before.
- F is satisfiable iff $\Gamma(F)$ has a PNE?

Reduction trick

Look at the two player strategic game that can be played by the first and second players:

	0	1
0	1,4	4,3
1	2,1	3,2

PNE?

Reduction trick

Look at the two player strategic game that can be played by the first and second players:

	0	1
0	1,4	4,3
1	2,1	3,2

PNE?

None

Reduction correctness

- F is a yes instance of SAT.

Reduction correctness

- F is a yes instance of SAT.
  There is a satisfying assignmet x. So $u_i(x) = 5$, for any i.
  Such a strategy profile is a PNE.

Reduction correctness

- F is a yes instance of SAT.
  There is a satisfying assignmet x. So $u_i(x) = 5$, for any i.
  Such a strategy profile is a PNE.
- F is a no instance of SAT.

Reduction correctness

- F is a yes instance of SAT.
  There is a satisfying assignmet x. So $u_i(x) = 5$, for any i.
  Such a strategy profile is a PNE.
- F is a no instance of SAT.
  For any strategy profile the payoff of players $j > 2$ is always 1.
  So they cannot change strategy and improve payoff.
  However, players 1 and 2 are engaged in a game with no PNE so one of them can change strategy and increase its payoff.
  Therefore $\Gamma(F)$ has no PNE

Σ_2^p definition and a complete problem

Let $L \subseteq \Sigma^*$ be a language.
$L \in \Sigma_2^p$ if and only if there is a polynomially time decidable relation R and a polynomial p such that

$$L = \{x \mid \exists z : |z| \leq p(|x|)\, \forall y : |y| \leq p(|x|)\, \langle x, y, z \rangle \in R\}.$$

Σ_2^p definition and a complete problem

Let $L \subseteq \Sigma^*$ be a language.
$L \in \Sigma_2^p$ if and only if there is a polynomially time decidable relation R and a polynomial p such that

$$L = \{x \mid \exists z : |z| \leq p(|x|) \, \forall y : |y| \leq p(|x|) \, \langle x, y, z \rangle \in R\}.$$

Q2SAT
Given $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$ where F is a Boolean formula over the boolean variables $\alpha_1, \ldots, \alpha_{n_1}, \beta_1, \ldots, \beta_{n_2}$, decide whether Φ is valid.

Q2SAT is Σ_2^p-complete.

EPN: implicit form

Theorem

The EPN problem for strategic games in implicit form is Σ_2^p-complete.

Lets provide a reduction from $\mathrm{Q2SAT}$.

EPN implicit form:reduction

For each $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:

EPN implicit form:reduction

For each $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$ and
$A_1 = \{0,1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$.

EPN implicit form:reduction

For each $\Phi = \exists\alpha_1, \ldots, \alpha_{n_1}\forall\beta_1, \ldots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$ and $A_1 = \{0, 1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$.
- Player 2, the *universal player*, assigns truth values to the boolean variables $\beta_1, \ldots, \beta_{n_2}$ and $A_2 = \{0, 1\}^{n_2}$ and $a_2 = (\beta_1, \ldots, \beta_{n_2}) \in A_2$.

EPN implicit form:reduction

For each $\Phi = \exists \alpha_1, \ldots, \alpha_{n_1} \forall \beta_1, \ldots \beta_{n_2} F$
we define a game $\Gamma(\Phi)$ as follows.
There are four players:

- Player 1, the *existential player*, assigns truth values to the boolean variables $\alpha_1, \ldots, \alpha_{n_1}$ and $A_1 = \{0,1\}^{n_1}$ and $a_1 = (\alpha_1, \ldots \alpha_{n_1}) \in A_1$.
- Player 2, the *universal player*, assigns truth values to the boolean variables $\beta_1, \ldots, \beta_{n_2}$ and $A_2 = \{0,1\}^{n_2}$ and $a_2 = (\beta_1, \ldots, \beta_{n_2}) \in A_2$.
- Players 3 and 4 avoid entering into a Nash equilibrium when the actions played by players 1 and 2 do not satisfy F. Their set of actions are $A_3 = A_4 = \{0,1\}$.

Let us denote by $F(a_1, a_2)$ the truth value of F under the assignment given by a_1 and a_2.

$$u_1(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_2(a_1, a_2, a_3, a_4) = \begin{cases} 1 & \text{if } F(a_1, a_2) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_3(a_1,a_2,a_3,a_4) = \begin{cases} 5 & \text{if } F(a_1,a_2)=1, \\ 4 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=1, \\ 3 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=1, \\ 2 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=0, \\ 1 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=0. \end{cases}$$

$$u_4(a_1,a_2,a_3,a_4) = \begin{cases} 5 & \text{if } F(a_1,a_2)=1, \\ 3 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=1, \\ 2 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=1, \\ 1 & \text{if } F(a_1,a_2)=0 \wedge a_3=1 \wedge a_4=0, \\ 4 & \text{if } F(a_1,a_2)=0 \wedge a_3=0 \wedge a_4=0. \end{cases}$$

EPN implicit form:reduction correcteness

- Let us assume that $\Phi = \exists \alpha_1, \ldots, \alpha_n \forall \beta_1, \ldots, \beta_m F$, where F is a Boolean formula over the boolean variables $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$, is true.
- Then there exists $\alpha \in \{0,1\}^n$ such that for all $\beta \in \{0,1\}^m$, $F(\alpha, \beta) = 1$.
- This means that if player 1 plays action α, for each $\beta \in \{0,1\}^m$, $a_3, a_4 \in \{0,1\}$, no player has incentive to change strategy.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha,\beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha, \beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.
- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha, \beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.
- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.
- Case 2: $F(\alpha, \beta) = 1$, since Φ is not valid, there exists $\beta' \in \{0,1\}^m$ such that $F(\alpha, \beta') = 0$. Therefore player 2 has an incentive to change strategy β by β'.

EPN implicit form:reduction correcteness

- Let us assume that Φ is not valid.
- It means that for any $\alpha \in \{0,1\}^n$ there exists $\beta \in \{0,1\}^m$ such that $F(\alpha, \beta) = 0$.
- Let (α, β, a, b) be a strategy profile. We have two cases.
- Case 1: $F(\alpha, \beta) = 0$, in this case players 3 an 4 engage in a no PNE game.
- Case 2: $F(\alpha, \beta) = 1$, since Φ is not valid, there exists $\beta' \in \{0,1\}^m$ such that $F(\alpha, \beta') = 0$. Therefore player 2 has an incentive to change strategy β by β'.
- Therefore, the strategy profile is not a PNE.

PNGrant problem

PNGrant Given a strategic game Γ and a value v, decide whether there is a PNE s so the $u_1(s) \geq v$.

Theorem

The PNGrant problem
can be solved in polynomial time for strategic games given in explicit form but it
is NP-complete for strategic games given in general form
is Σ_2^p-complete for strategic games given in implicit form.

PNGrant problem

PNGrant Given a strategic game Γ and a value v, decide whether there is a PNE s so the $u_1(s) \geq v$.

Theorem

The PNGrant problem
can be solved in polynomial time for strategic games given in explicit form but it
is NP-complete for strategic games given in general form
is Σ_2^p-complete for strategic games given in implicit form.

Membership follows from the same arguments.

PNGrant problem

PNGrant Given a strategic game Γ and a value v, decide whether there is a PNE s so the $u_1(s) \geq v$.

Theorem

The PNGrant problem
can be solved in polynomial time for strategic games given in explicit form but it
is NP-complete for strategic games given in general form
is Σ_2^p-complete for strategic games given in implicit form.

Membership follows from the same arguments.
In all the reduction the utility for the first player in all PNE is constant, this provides the value of v in each reduction.

(Boolean) Circuit games

[Schoenebeck and Vadhan, EC 2006 - ACM TCT 2012]

- In a circuit game, players still control disjoint sets of variables, but each player's payoff is given by a single boolean circuit.
- The boolean circuit computes a rational value as the quotient of two integers
- Boolean circuit games are the special case of circuit games where each player controls a single boolean variable.

(Boolean) Circuit games

[Schoenebeck and Vadhan, EC 2006 - ACM TCT 2012]

- In a circuit game, players still control disjoint sets of variables, but each player's payoff is given by a single boolean circuit.
- The boolean circuit computes a rational value as the quotient of two integers
- Boolean circuit games are the special case of circuit games where each player controls a single boolean variable.

TMs can be simulated by circuits and viceversa

(Boolean) Circuit games

[Schoenebeck and Vadhan, EC 2006 - ACM TCT 2012]

- In a circuit game, players still control disjoint sets of variables, but each player's payoff is given by a single boolean circuit.
- The boolean circuit computes a rational value as the quotient of two integers
- Boolean circuit games are the special case of circuit games where each player controls a single boolean variable.

TMs can be simulated by circuits and viceversa

- Circuit games are equivalent to implicit form games
- Boolean circuit games are a subset of general form games.

(Boolean) weighted formula games

[Mavronicolas, Monien, Wagner, WINE 2007]

- In a formula game, players still control disjoint sets of variables, but each player's payoff is given by a weighted combination of boolean formulas.
- Boolean formula games are the special case of formula games where each player controls a single boolean variable.

(Boolean) weighted formula games

[Mavronicolas, Monien, Wagner, WINE 2007]

- In a formula game, players still control disjoint sets of variables, but each player's payoff is given by a weighted combination of boolean formulas.
- Boolean formula games are the special case of formula games where each player controls a single boolean variable.
- Formulas can be casted as circuits but not viceversa as the size might grow exponentially.

(Boolean) weighted formula games

[Mavronicolas, Monien, Wagner, WINE 2007]

- In a formula game, players still control disjoint sets of variables, but each player's payoff is given by a weighted combination of boolean formulas.
- Boolean formula games are the special case of formula games where each player controls a single boolean variable.

- Formulas can be casted as circuits but not viceversa as the size might grow exponentially.
- Nevertheless the utility functions of the provided reductions can be easily described in this way.
So the problems are equivalent from the complexity point of view.

Graphical games

[Gottlob, Greco and Scarcello, JAIR 2005]

- Graphical games are a representation of multiplayer games meant to capture and exploit locality or sparsity of direct influences.
- They are most appropriate for large population games in which the payoffs of each player are determined by the actions of only a small subpopulation.
- Players' relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.

Graphical games

[Gottlob, Greco and Scarcello, JAIR 2005]

- Graphical games are a representation of multiplayer games meant to capture and exploit locality or sparsity of direct influences.
- They are most appropriate for large population games in which the payoffs of each player are determined by the actions of only a small subpopulation.
- Players' relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.
- Provide a complementary framework to analyze complexity based on the graph parameters:

Graphical games

[Gottlob, Greco and Scarcello, JAIR 2005]

- Graphical games are a representation of multiplayer games meant to capture and exploit locality or sparsity of direct influences.
- They are most appropriate for large population games in which the payoffs of each player are determined by the actions of only a small subpopulation.
- Players' relationship is described by a graph and the payoff of a player depends only on the actions of its neighbors.
- Provide a complementary framework to analyze complexity based on the graph parameters: bounded degree, bounded treewidth, . . .

1. Complexity framework
2. Complexity analysis
3. Other succinct representations
4. Concluding remarks

Conclusions

- We have analyzed some ways of describing strategic games with polynomial time computable utilities
- We have concentrated on the study of two computational problems.
- As expected complexity increases with succinctness.
- There are many other
    - game classes
    - and problems of interest

  with similar behavior.

References

Contents taken from a subset of the results in

- C. Alvarez, J. Gabarró, M. Serna
  Equilibria problems on games: Complexity versus succinctness
  J. of Comp. and Sys. Sci. 77:1172-1197, 2011

References

Further suggested reading (among many others)

- G. Gottlob, G. Greco, F. Scarcello
  Pure Nash equilibria: Hard and easy games
  J. Artificial Intelligence Res. 24:357–406, 2005
- J. Gabarro, A. Garcia, M. Serna
  The complexity of game isomorphism
  Theor. Comput. Sci. 412(48): 6675-6695, 2011.

References

- M. Mavronicolas, B. Monien, K. Wagner
  Weighted boolean formula games
  in: X. Deng, F. Graham (Eds.), WINE 2007,
  Lecture Notes in Comput. Sci., 4858:469–481, 2007.
- G.R. Schoenebeck, S. Vadhan
  The Computational Complexity of Nash Equilibria in Concisely Represented Games
  ACM Transactions on Computational Theory, 4(2) article 4, 2012