Stochastic Network Modeling (SNM)

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Parts

- I Introduction
- II Discrete Time Markov Chains (DTMC)
- III Continuous Time Markov Chains (CTMC)
- IV Queuing Theory

Part III

Continuous Time Markov Chains (CTMC)

Outline

Definition of a CTMC

Properties of a continuous time MC

- The states must be a numerable set.
- Let $X(t)$ be the event {at time t the system is in state i}, then it must hold the memoryless property:

$$\boxed{\begin{aligned} & P\big(X(t_n) = i \mid X(t_1) = j, X(t_2) = k, \dots\big) = \\ & \qquad P\big(X(t_n) = i \mid X(t_1) = j\big) \text{ for any } t_n > t_1 > t_2 > t_3 \dots \end{aligned}}$$

Definition of a CTMC

Transition Matrix

- Transition probabilities:

$$p_{ij}(t_1, t_2) = P\big(X(t_2) = j \mid X(t_1) = i\big)$$

- For an homogeneous chain:

$$\begin{aligned} p_{ij}(t) &= P\big(X(t_1 + t) = j \mid X(t_1) = i\big) = \\ &= P\big(X(t) = j \mid X(0) = i\big), \ \forall t_1 \end{aligned}$$

- In matrix form (transition probability matrix):

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \ t \geq 0$$

- Notes:
  - Compare with the n-step prob. matrix of a DTMC: $\mathbf{P}(n)$.
  - $\mathbf{P}(t)$ must be a stochastic matrix (all rows add to 1).

Definition of a CTMC

Transition Matrix

$$p_{ij}(t) = P\big(X(t) = j \mid X(0) = i\big),\ t \geq 0$$

- We look for an equivalent 1-step prob. matrix **P** of DTMCs.
- For consistency: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$. In matrix form:

$$\lim_{t\to 0} \mathbf{P}(t) = \mathbf{I}.$$

- And assume that the following transition rates exist:

$$q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t},\ i \neq j; \quad q_{ii} = \lim_{t\to 0} \frac{p_{ii}(t) - 1}{t}$$

- In matrix form: $\mathbf{Q} = \lim_{t\to 0} \frac{\mathbf{P}(t) - \mathbf{I}}{t}$
- Note that $\sum_j p_{ij}(t) = 1 \Rightarrow p_{ii}(t) = 1 - \sum_{j\neq i} p_{ij}(t)$, thus:

$$q_{ii} = \lim_{t\to 0} \frac{p_{ii}(t) - 1}{t} = \lim_{t\to 0} \frac{-\sum_{j\neq i} p_{ij}(t)}{t} = -\sum_{j\neq i} q_{ij}$$

Definition of a CTMC

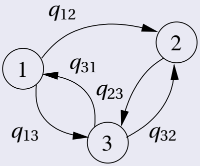
Transition Matrix

- The matrix $\mathbf{Q}$ is called the transition rate or infinitesimal generator of the chain.
- Since $q_{ii} = -\sum_{j \neq i} q_{ij}$, all the rows of $\mathbf{Q}$ add to 0.
- The rate q_{ij}, $i \neq j$ measures "how fast" the chain moves from state i to j: the higher is q_{ij}, the faster it moves from i to j.
- For $q_{ii} = -\sum_{j \neq i} q_{ij}$, the higher $-q_{ii}$ is, the faster the chain leaves state i.
- If $q_{ij} = 0, \forall j \Rightarrow q_{ii} = 0$, then i is an absorbing state: the chain "moves with rate 0 from i to other states", i.e. never leaves i.

Definition of a CTMC

State Transition Diagram

- A continuous MC is characterized by the transition rate or infinitesimal generator: the Q-matrix.
- The state transition diagram is now represented as:



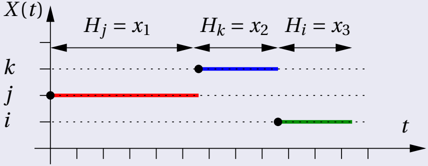
$$\Rightarrow \mathbf{Q} = \begin{bmatrix} -(q_{12}+q_{13}) & q_{12} & q_{13} \\ 0 & -q_{23} & q_{23} \\ q_{31} & q_{32} & -(q_{31}+q_{32}) \end{bmatrix}$$

- Note that now we have transition rates ($0 \le q_{ij} < \infty$, $i \neq j$) and not probabilities.
- The rates q_{ii} are not written in the diagram, no self transitions.

Definition of a CTMC

Sojourn Time

- Sojourn or holding time: Is the RV H_k equal to the sojourn time in state k:

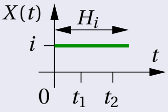


- The Markov property implies that the sojourn time is exponentially distributed with parameter q_{ii}:

$$\boxed{P(H_i \le x) = 1 - e^{q_{ii}x} \Rightarrow P(H_i > x) = e^{q_{ii}x},\ q_{ii} = -\sum_{j \ne i} q_{ij},\ x \ge 0}$$

Definition of a CTMC

The exponential distribution satisfies the Markov property



- Markov property (memoryless):

$$P\big(X(t_2) = i \mid X(t_1) = i, X(0) = i\big) = P\big(X(t_2) = i \mid X(t_1) = i\big),\ t_2 > t_1 > 0$$

- In terms of the sojourn time:
$$P\big(H_i > t_2 \mid H_i > t_1\big) = P(H_i > t_2 - t_1)$$

- But:

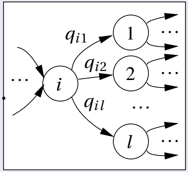
$$P\big(H_i > t_2 \mid H_i > t_1\big) = \frac{P(H_i > t_2, H_i > t_1)}{P(H_i > t_1)} = \frac{P(H_i > t_2)}{P(H_i > t_1)} = \frac{e^{q_{ii}\,t_2}}{e^{q_{ii}\,t_1}} = e^{q_{ii}\,(t_2 - t_1)} = P(H_i > t_2 - t_1) \quad \square$$

- The exponential distribution is the only one satisfying the memoryless property.

Definition of a CTMC

Exponential Jumps Description of a CTMC

Assume a process built as follows:



- Upon reaching a state i
  1. the process can jump to a state $j \in \{1, 2, \cdots l\}$.
  2. A set of independent exponential RVs, $\{H_{i1}, H_{i2}, \cdots H_{il}\}$, with parameters $\{q_{i1}, q_{i1}, \cdots q_{il}\}$ are triggered. That is, $P(H_{ij} \le t) = 1 - e^{-q_{ij} t}$, $t \ge 0$.
- If $\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij} \Rightarrow$ the process jumps to the state j. In other words, a transition occurs to state j if the RV H_{ij} is the minimum of $\{H_{i1}, H_{i2}, \cdots H_{il}\}$.

Theorem: This process is a CTMC with transition rates q_{ij}.

Exponential Jumps Description of a CTMC

$$P(H_{ij} \le t) = 1 - e^{-q_{ij}\,t}.$$

Theorem: This process is a CTMC with transition rates q_{ij}.

Proof:

- The RV $H_i = \min\{H_{i1}, H_{i2}, \cdots H_{il}\}$ (sojourn time in state i) is exponentially distributed with parameter $q_i = \sum_j q_{ij}$: $P(H_i \le t) = 1 - e^{-q_i\,t}$.
- $P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) = q_{ij} / \sum_j q_{ij}$. Thus, the transition rate to state j is:

$$\lim_{t\to 0} \frac{p_{ij}(t)}{t} = \lim_{t\to 0} \frac{P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) \times P(H_i \le t)}{t} =$$

$$\frac{q_{ij}}{\sum_j q_{ij}} \left.\frac{\partial P(H_i \le t)}{\partial t}\right|_{t=0} = \frac{q_{ij}}{\sum_j q_{ij}} \sum_j q_{ij} = q_{ij} \qquad \square$$

Definition of a CTMC

Example: Pure Aloha System

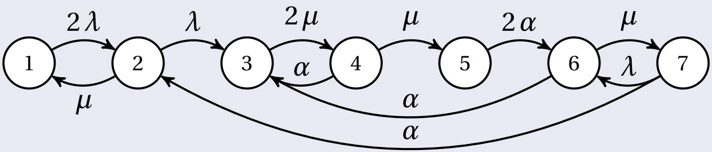
- Consider a Pure Aloha System with 2 nodes:
  - Nodes in thinking state Tx a packet in a time exponentially distributed with rate λ.
  - Transmission time is exponentially distributed with rate μ.
  - If two transmissions overlap, the packet is lost and stations become backlogged (after the packet transmission) until the packet is successfully transmitted.
  - Nodes in backlogged state Tx a packet in a time exponentially distributed with rate α.

Questions

- Build the state transition diagram.

Definition of a CTMC

Example: Pure Aloha System



State	Condition
1	T,T
2	X,T
3	C,C
4	B,C
5	B,B
6	X,B
7	T,B

Legend	
T	Thinking
X	Transmitting
C	Collided transmission
B	Backlogged

Part III

Continuous Time Markov Chains (CTMC)

Outline

Transient Solution

Chapman-Kolmogorov Equations

- Chapman-Kolmogorov: $p_{ij}(t) = \sum_k p_{ik}(t-\alpha) p_{kj}(\alpha), 0 \le \alpha \le t$
- Thus:
$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_k \left\{ \frac{p_{ik}(t+\Delta t-\alpha) - p_{ik}(t-\alpha)}{\Delta t} p_{kj}(\alpha) \right\}$$
- Taking the limit
$$\alpha \to t, \Delta t \to 0 \Rightarrow \begin{cases} p_{ik}(t-\alpha) \to 0, & i \ne k \\ p_{ik}(t-\alpha) \to 1, & i = k \end{cases}$$

and using:

$$\begin{cases} q_{ij} = \lim_{t \to 0} \frac{p_{ij}(t)}{t}, & i \ne j \\ q_{ij} = \lim_{t \to 0} \frac{p_{ij}(t) - 1}{t}, & i = j \end{cases}$$

we have:

$$\boxed{\frac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik} p_{kj}(t), \quad t \ge 0, \forall i,j}$$

Chapman-Kolmogorov Equations (cont)

- we have: $\dfrac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik}\, p_{kj}(t),\ t \geq 0,\ \forall i,j$
- In matrix form: $\boxed{\dfrac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{Q}\,\mathbf{P}(t),\ t \geq 0}$
- The solution of the previous matrix differential equation is the exponential matrix:

$$\boxed{\mathbf{P}(t) = e^{\mathbf{Q}\,t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \mathbf{I} + \mathbf{Q}\,t + \frac{\mathbf{Q}^2\,t^2}{2!} + \frac{\mathbf{Q}^3\,t^3}{3!} + \cdots,\ t \geq 0}$$

- Due to rounding errors, the previous series is difficult to compute numerically (the powers of **Q** have positive and negative entries).

Transient Solution

State Probabilities

- Define the probability of being in state i at time t:

$$\boxed{\pi_i(t) = P(X(t) = i)}$$

- In vector form (row vector)

$$\boldsymbol{\pi}(t) = (\pi_1(t), \pi_2(t), \cdots).$$

- Clearly:

$$\pi_i(t) = \sum_k P(X(0) = k)\, P\big(X(t) = i \,\big|\, X(0) = k\big) = \sum_k \pi_k(0)\, p_{ki}(t)$$

- In matrix form:

$$\boxed{\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t},\ t \ge 0}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

- NOTE: Compare with DTMC

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n,\ n \ge 0$$

Transient Solution

Transient Solution

- If we are interested in the transient evolution we shall study $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,e^{\mathbf{Q}\,t}$, $t \geq 0$.
- Assume a finite CTMC with N states (infinitesimal generator $\mathbf{Q}^{N\times N}$).
- Assume that $\mathbf{Q}$ can be diagonalized: $\mathbf{Q} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{Q}$.
- NOTE: the eigenvalues λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l\,\boldsymbol{l}$ (or $\mathbf{A}\,\boldsymbol{r} = \lambda_l\,\boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* and *right* eigenvectors, respectively. Thus, solve the characteristic polynomial $\det(\lambda\,\mathbf{I} - \mathbf{A}) = 0$.

Transient Solution

Transient Solution

... assume that $\mathbf{Q}$ can be diagonalized: $\mathbf{Q} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$

- Since

$$\mathbf{P}(t) = e^{\mathbf{Q}\,t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \sum_{i=0}^{\infty} \frac{(\mathbf{L}^{-1}\,\Lambda\,\mathbf{L}\,t)^i}{i!} =$$

$$\mathbf{L}^{-1}\,\mathrm{diag}\left(\sum_{i=0}^{\infty} \frac{(\lambda_1\,t)^i}{i!}, \cdots\right)\mathbf{L} = \mathbf{L}^{-1}\,\mathrm{diag}(e^{\lambda_1\,t}, \cdots)\,\mathbf{L}$$

we have that

$$\boxed{\begin{aligned}&\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,e^{\mathbf{Q}\,t} = \\ &\quad \boldsymbol{\pi}(0)\,\mathbf{L}^{-1}\,\mathrm{diag}(e^{\lambda_1\,t}, e^{\lambda_2\,t}, \cdots e^{\lambda_L\,t})\,\mathbf{L}\end{aligned}}$$

Transient Solution

Transient Solution

... we have that $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{L}^{-1}\,\mathrm{diag}(\mathrm{e}^{\lambda_1 t}, \mathrm{e}^{\lambda_2 t}, \cdots \mathrm{e}^{\lambda_L t})\,\mathbf{L}$

- Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(t) = (\boldsymbol{\pi}(t))_i = \sum_{l=1}^{N} a_i^{(l)}\,\mathrm{e}^{\lambda_l t},\ t \geq 0}$$

where the unknown coefficients $a_i^{(l)}$ can be obtained solving the system of equations:

$$\boxed{\left.\frac{\partial^n \pi_i(t)}{\partial t^n}\right|_{t=0} = (\boldsymbol{\pi}(0)\,\mathbf{Q}^n)_i,\ n = 0, \cdots N-1}$$

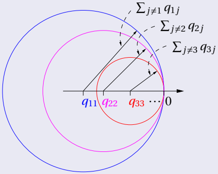
NOTE: Compare with $(\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1$

Transient Solution

Eigenvalues of an Infinitesimal Generator

- $\mathbf{Q}$ has an eigenvalue equal to 0 ($\mathbf{Q}\boldsymbol{x} = \lambda\boldsymbol{x}$, for $\lambda = 0$, $\boldsymbol{x} \neq \mathbf{0}$).
  Proof: $\mathbf{Q}\,\mathbf{e} = \mathbf{0}$, where $\mathbf{e} = (1, 1, \cdots)^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{Q}$ add to 0). $\square$
- The eigenvalue $\lambda = 0$ is single if $\mathbf{Q}$ is irreducible (Perron-Frobenius theorem). $\mathbf{Q}$ is irreducible if all states communicate: for $t > 0$, $p_{ij}(t) > 0$, $\forall i, j$.
- All eigenvalues of $\mathbf{Q}$ are $\lambda_l \leq 0$.

Proof: Using Gerschgorin's theorem and the fact that the rows of $\mathbf{Q}$ add to 0. $\square$



Gershgorin discs of a rate matrix

Transient Solution

Example

- Assume a CTMC with

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 1/2 & -1/2 \end{bmatrix}$$

- We want the probability of being in state 2 at time t starting from state 1: $\pi_2(t)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Transient Solution

Solution

- It can be easily found that the eigenvalues of $\mathbf{Q}$ are $\lambda_1 = 0$ and $\lambda_2 = -3/2$.

$$\pi_2(t) = a e^{\lambda_1 t} + b e^{\lambda_2 t} = a + b e^{-(3/2)\,t}$$

- Imposing the boundary conditions:

$$\pi_2(0) = a + b = (\boldsymbol{\pi}(0)\,\mathbf{Q}^0)_2 = (\boldsymbol{\pi}(0)\,\mathbf{I})_2 = (\boldsymbol{\pi}(0))_2 = 0$$

$$\left.\frac{\partial \pi_2(t)}{\partial t}\right|_{t=0} = b(-3/2) = (\boldsymbol{\pi}(0)\,\mathbf{Q})_2 = \mathbf{Q}_{12} = 1$$

we have that $a = 2/3$, $b = -2/3$, thus:

$$\pi_2(t) = 2/3\,(1 - e^{-(3/2)\,t}), \quad t \geq 0$$

Transient Solution

Chain with a Defective Matrix

- What if $\mathbf{Q}$ cannot be diagonalized? (defective matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{Q}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$). Then [1]:

$$\boxed{\pi_j(t) = \sum_{l=1}^{L} \mathrm{e}^{\lambda_l t} \sum_{m=0}^{k_l-1} a_j^{(l,m)}\, t^m}$$

where $a_j^{(l,m)}$ are constants. So, exponentials associated with eigenvalues λ_l of multiplicity $k_l > 1$ are multiplied by polynomials in t of degree $k_l - 1$.

[1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method*. Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: `https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html`.

Transient Solution

Example

- Assume the CTMC:

$$1 \xrightarrow{1} 2 \xrightarrow{1} 3$$

$$\pi(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- We have $\lambda_1 = 0$ and $\lambda_2 = -1$ with multiplicity 2. Thus:

$$\pi_3(t) = a + \mathrm{e}^{-t}(b + c\,t)$$

- We have that $a = 1$, because state 3 is absorbing. Imposing $\pi_3(0) = 0$ and $\pi'_3(0) = 0$, we have $b = c = -1$, and

$$\boxed{\pi_3(t) = 1 - \mathrm{e}^{-t}(1 + t),\ t \geq 0}$$