

Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification of States

Steady State

Reversed Chain

Reversible Chains

Research Example: Aloha

Finite Absorbing

Stochastic Network Modeling (SNM)

Llorenç Cerdà-Alabern Universitat Politècnica de Catalunya Departament d'Arquitectura de Computadors llorenc@ac.upc.edu

Parts

- Introduction
- ① Discrete Time Markov Chains (DTMC)
- Continuous Time Markov Chains (CTMC)
- Queuing Theory



Master in Innovation and Research in Informatics (MIRI) Computer Networks and Distributed Systems

Stochastic Network Modeling (SNM)

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

Transient Solution

Classification

Steady State

Reversed Chair

Reversible Chains

Research Example: Aloha

Finite Absorbing

Part II

Discrete Time Markov Chains (DTMC)

Outline

- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State

- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains



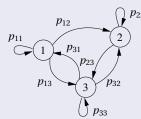
Definition of a DTMC

Discrete Time Markov Chains (DTMC)

State Transition Diagram

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:

$$\sum_{\forall j} p_{ij} = 1$$

Mathematically:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Properties of a DTMC

Properties of a DTMC

• The event X(n) = i (at step n the system is in state i) must satisfy (memoryless property):

$$P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \dots) =$$

 $P(X(n) = j \mid X(n-1) = i)$

- If $P(X(n) = j \mid X(n-1) = i) = P(X(1) = j \mid X(0) = i)$ for any nwe have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

 The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Transition Matrix

Transition Matrix

Transition probabilities:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

In matrix form:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Transition Matrix

Transition Matrix

We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

 For the model to be consistent, the probability to move from *i* to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P(X(n) = j \mid X(n-1) = i) =$$

$$\sum_{\forall j} \frac{P\big(X(n-1)=i \bigm| X(n)=j\big) P\big(X(n)=j\big)}{P(X(n-1)=i)} = \frac{P(X(n-1)=i)}{P(X(n-1)=i)} = \boxed{1}$$

• P is a stochastic matrix, i.e. a matrix which rows sum 1.



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of DTMC

Diagram

Properties of a DT

Transition Matrix

Absorbing Chai

State Probabilities

Chapman-Kolmogoro Equations

Transier

Classificat

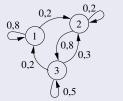
Steady Sta

Reversed Chair

Reversib

Example

- Assume a terminal can be in 3 states:
 - State 1: Idle.
 - State 2: Active without sending data.
 - State 3: Active and sending data at a rate v bps.



$$\mathbf{P} = \begin{bmatrix} \mathbf{to} \text{ state} \\ 1 & 2 & 3 \\ 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & \text{from} \\ 2 & \text{state} \\ 3 \end{bmatrix}$$

• The average transmission rate (throughput), v_a , is:

 $v_a = P$ (the terminal is in state 3) × v



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of DTMC

Diagram

Properties of a DTM

Transition Matrix

Absorbing Chains

State Probabilities
Chapman-Kolmogorov
Equations

Sojourn or Holding Time

Solution Classificati

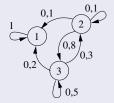
Steady Stat

Reversed Chai

Revers

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state *i* is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



$$\mathbf{P} = \begin{bmatrix} \mathbf{to} \ \mathbf{state} \\ 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0,1 & 0,1 & 0,8 \\ 0,2 & 0,3 & 0,5 \end{bmatrix} \begin{bmatrix} 1 & \text{from} \\ 2 & \text{state} \\ 3 \end{bmatrix}$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of

State Transition

Diagram

Properties of a DTM

Transition Matrix

n-step transition

State Probabilities

Equations

Transient

Classification of States

teady Stat

Reversed Chair

n-step transition probabilities

- Transition probabilities: $p_{ij} = P(X(n) = j \mid X(n-1) = i)$
- In matrix form:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

• We define the **n-step** transition probabilities:

$$p_{ij}(n) = P(X(n) = j \mid X(0) = i)$$

$$\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

• **P** and P(n) are stochastic matrices: Their rows sum 1.

Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

State Transition
Diagram
Properties of a DT

Absorbing Chains n-step transition

State Probabilities

Equations
Sojourn or Holding
Time

Transient Solution

Classificatio of States

Reversed Chai

Reversed Chai

Revers

State Probabilities

• Define the probability of being in state *i* at step *n*:

$$\pi_i(n) = P(X(n) = i)$$

In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$

• Thus, the vector $\pi(n)$ is the distribution of the random variable X(n), and it is called the state probability at step n.



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

State Probabilities

State Probabilities

State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$

• Law of total prob. $P(A) = \sum_{n} P(A \cap B_n) = \sum_{n} P(A|B_n)P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1) = k) \ P\big(X(n) = i \ \big| \ X(n-1) = k\big) = \sum_k \pi_k(n-1) \ p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0) = k) \ P\big(X(n) = i \ \big| \ X(0) = k\big) = \sum_k \pi_k(0) \ p_{ki}(n)$$

In matrix form:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P}$$

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \, \mathbf{P}(n)$$

where $\pi(0)$ is the initial distribution.



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

State Probabilities

State Probabilities

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) \mathbf{P}$$
$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}(n)$$

Iterating

$$\pi(n) = \pi(n-1) \mathbf{P} = \pi(n-2) \mathbf{P} \mathbf{P} = \pi(n-3) \mathbf{P} \mathbf{P} \mathbf{P} = \dots = \pi(0) \mathbf{P}^n$$

Thus:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \, \mathbf{P}(n) = \boldsymbol{\pi}(0) \, \mathbf{P}^n$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Chapman-Kolmogorov

Equations

Chapman-Kolmogorov Equations

$$p_{ij}(n) = \sum_{k} p_{ik}(r) \ p_{kj}(n-r)$$

Proof:

$$p_{ij}(n) = P(X(n) = j \mid X(0) = i) = \sum_{k} P(X(n) = j, X(r) = k \mid X(0) = i)$$

$$= \sum_{k} \frac{P(X(n) = j, X(r) = k, X(0) = i)}{P(X(0) = i)} \times \frac{P(X(r) = k, X(0) = i)}{P(X(r) = k, X(0) = i)}$$

$$= \sum_{k} P(X(n) = j \mid X(r) = k, X(0) = i) P(X(r) = k \mid X(0) = i)$$

$$= \sum_{k} P(X(n) = j \mid X(r) = k) P(X(r) = k \mid X(0) = i)$$

$$= \sum_{k} P(X(n) = j \mid X(r) = k) P(X(r) = k \mid X(0) = i)$$

$$= \sum_{k} P(X(n) = j \mid X(r) = k) P(X(r) = k \mid X(0) = i)$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

State Transition

Diagram

Properties of a

..

n-step transiti

State Probabilities

Chapman-Kolmogorov

Equations
Sojourn or Holding

Transient Solution

of States

Steady Stat

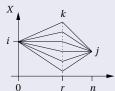
Reversed Chair

Revers

Chapman-Kolmogorov Equations

$$p_{ij}(n) = \sum_{k} p_{ik}(r) \ p_{kj}(n-r)$$

Graphical interpretation:



In matrix form:

$$\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

State Transition

Diagram

Transition Mat

Absorbing Cha

n-step transitie

probabilities

State Probabilities

Chapman-Kolmogorov

Equations Sojourn or Holding

Sojourn or Holding Time

Classification

Classification of States

Reversed Chai

Chains

Chapman-Kolmogorov Equations

$$\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)$$

• Particularly:

$$P(n) = P(1)P(n-1) = PP(n-1) = P(n-1)P$$

Iterating:

$$\mathbf{P}(n) = \mathbf{P}^n$$

• Thus:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \, \mathbf{P}(n) = \boldsymbol{\pi}(0) \, \mathbf{P}^n$$



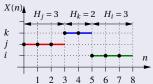
Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Sojourn or Holding

Sojourn or Holding Time

• Sojourn or holding time in state k: Is the RV H_k equal to the number of steps that the chain remains in state *k* before leaving to a different state:



The Markov property implies:

$$H_i(n) = P(H_i = n) = p_{ii}^{n-1} (1 - p_{ii}), n \ge 1$$

• Which is a geometric distribution with mean:

$$E[H_i] = \sum_{n=1}^{\infty} nP(H_i = n) = \frac{1}{1 - p_{ii}}.$$



Definition of a DTMC

Sojourn or Holding Time NOTE: We allow that:

Discrete Time Markov Chains (DTMC)

Sojourn or Holding

 $p_{ii} = 0 \Rightarrow H_i(n) = I(n = 1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$, and

 $p_{ii} = 1 \Rightarrow E[H_i] = \infty$ (absorbing state).



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Sojourn or Holding

Theorem

A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.

Proof.

We have seen that a DTMC has a sojourn time

$$H_i(n) = P(H_i = n) = p_{ii}^{n-1} (1 - p_{ii}), n \ge 1$$

- Which is geometrically distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a DTMC

State Transition

Diagram

Properties of a

11411311101111111

Thospioling Citi

n-step transitio

State Probabilities

Sojourn or Holding

Transient Solution

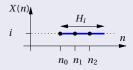
Classification of States

of States

Reversed Chai

Reversit

The geometric distribution satisfies the Markov property (1)



Proof

Markov property:

$$P\big(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\big) = P\big(X(n_2) = i \mid X(n_1) = i\big)$$

 Thus, the Markov property in terms of the sojourn time can be written as:

$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = P(H_i > n_2 - n_1)$$



Definition of a DTMC

Discrete Time Markov Chains (DTMC)

Definition of a

State Transitio

Diagram

Troperties or a

Absorbing Chai

n-ston transitio

n-step transition

State Probabilities

Sojourn or Holding

Transient

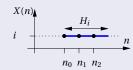
Classification

of States

Reversed Chai

Reversi

The geometric distribution satisfies the Markov property (2)



$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = P(H_i > n_2 - n_1)$$

Since

$$P(H_i > k) = 1 - P(H_i \le k) = 1 - \sum_{n=1}^k p^{n-1} (1-p) = 1 - (1-p) \frac{1-p^k}{1-p} = p^k$$

• We have:

$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$

$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2 - n_0}}{p^{n_1 - n_0}} = p^{n_2 - n_1} = P(H_i > n_2 - n_1) \quad \Box$$