Stochastic Network Modeling (SNM)

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Syllabus

Parts

1. Introduction
2. Discrete Time Markov Chains (DTMC)
3. Continuous Time Markov Chains (CTMC)
4. Queuing Theory

Evaluation

- NF = 0.1 * NP + 0.30 * max(EF, C) + 0.60 * EF
  where:
  - NF = final mark
  - EF = final theory exam
  - NP = Problems delivered by the students
  - C = average assessments mark: C = 0.5*C1 + 0.5*C2

Part I

Introduction

Outline

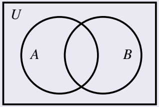
- Probability
- Stochastic Process (SP)

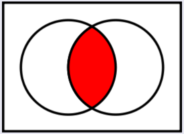
Ingredients of Probability

- Random experiment, e.g. toss a die.
- Outcome, ω, e.g. tossing a die can be $\omega = 2$, choosing a fruit can be ω = orange.
- Sample space or Universal set, U, set of all possible outcomes. E.g. tossing a die $U = \{1,2,3,4,5,6\}$.
- Event, A, any subset of U (e.g. tossing a die $A = \{1,2,3\}$). We say the event A occurs if the outcome of the experiment $\omega \in A$. U is the sure event, and we represent by the empty set $\emptyset$ an impossible outcome.

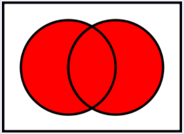
Venn Diagrams

Graphical representation of events

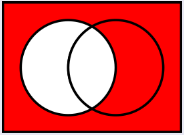




Intersection $A \cap B$

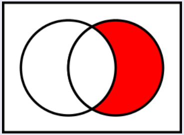


Union $A \cup B$



Complement of A in U

$A^c = U \setminus A$



Complement of A in B

(B minus A) $A^c \cap B = B \setminus A$

source: wikipedia

Random Variable

- For simplicity it is defined a random variable (RV), X as a function that assigns a real number to each outcome in the sample space U, i.e.:
$$X : U \to \mathbb{R}$$
- We will represent the experiment by a RV, X, and the possible outcomes by its values. $X = x_i$ is the outcome $X(\omega_i) = x_i$.
- Using RVs the sample space is mapped in a subset of $\mathbb{R}$. So, in terms of X, U is a set of points of $\mathbb{R}$. The same for any event.
- Normally the definition of X comes naturally from the experiment, e.g. tossing a die: $X = \{\text{number in the toss}\}$.
- RVs can be discrete (e.g. tossing a die) or continuous (e.g. waiting time of a packet in a queue).

Probability Measure of Discrete RV

- If the sample space U of the RV X is finite (discrete RV), $U = \{x_1, \cdots x_n\}$, a probability measure is an assignment of numbers $P(x_i)$, referred to as probabilities, to each outcome x_i such that:

$$0 \le P(x_i) \le 1$$

$$P(A) = \sum_{x_i \in A} P(x_i)$$

$$P(U) = 1$$

E.g. tossing a fair die,

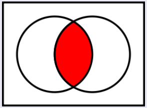
$$P(x_i) = 1/6$$

$$P(X \in \{2,4,6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Conditional Probability and Bayes Formula

- Given the the sample space U and the events $A, B \in U$ with $P(B) > 0$ the probability of A conditioned by B is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Intersection $A \cap B$

NOTE: It's common to use commas to denote set intersection, and write $P(A \cap B)$ as $P(A, B)$.

- Bayes Formula

$$P(A|B)\,P(B) = P(B|A)\,P(A) \Rightarrow P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Law of total probability

- Let B_i a partition of the sample space U ($\cup_i B_i = U$, $B_i \cap B_j = \emptyset, \forall i \neq j$), then
$$P(A) = \sum_i P(A|B_i)\, P(B_i)$$
- For conditional probabilities:
$$P(A|C) = \sum_i P(A|C \cap B_i)\, P(B_i|C)$$
- If C is independent of any of the B_i
$$P(A|C) = \sum_i P(A|C \cap B_i)\, P(B_i)$$

Probability Measure of Continuous RV

- If the sample space of the RV X is continuous (continuous RV), the events are intervals of $\mathbb{R}$. The probability measure is defined by means of the cumulative distribution function, CDF:

$$F(x) = P(X \in (-\infty, x]) = P(X \leq x)$$

- X is called absolutely continuous[a] if there exists the probability density function, PDF, such that for any interval $I = \{x \mid a \leq x \leq b\}$:

$$\int_a^b f(x)\, dx = P(X \in I) = F(b) - F(a)$$

[a] Some special distributions, called singular, do not have a PDF. One example is the Cantor distribution (see Wikipedia).

Probability

Expected value

- Given the discrete $N \in \mathbb{Z}$, respectively continuous $X \in \mathbb{R}$ RV, the expected value is:

$$\mathrm{E}[N] = \sum_{k=-\infty}^{\infty} k\, P(N = k)$$

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Variance

- The amount of dispersion of a RV X with expected value $\mu = \mathrm{E}[X]$ is measured by the Variance:

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}[(X-\mu)^2] = \mathrm{E}[X^2] - \mu^2$$

- Often it is used the standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$.

Indicator Function

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$\mathrm{E}[I(A)] = 0 \times P(I(A) = 0) + 1 \times P(I(A) = 1) = P(A)$$

Expected value of non negative RVs

- For non negative RVs, $N \geq 0$ discrete and $X \geq 0$ continous:

$$\mathrm{E}[N] = \sum_{k=0}^{\infty} k\, P(N = k) = \sum_{k=0}^{\infty} P(N > k)$$

$$\mathrm{E}[X] = \int_0^{\infty} x f(x)\, dx = \int_0^{\infty} P(X > x)\, dx = \int_0^{\infty} (1 - F(x))\, dx$$

Proof

$$N = \sum_{k=0}^{N-1} 1 = \sum_{k=0}^{\infty} I(N > k)$$

$$X = \int_0^{X} dx = \int_0^{\infty} I(X > x)\, dx$$

and take expectations.

Wald's Equation

- **Definition**: An positive integer RV $N > 0$ is a **stopping time** of a sequence $X_1, X_2, \cdots$ if the event $N = n$ is independent of $X_{n+1}, X_{n+2}, \cdots$.
  E.g. toss a die until you get 6. Let N be the number of tosses. N does not depend on the values obtained after getting 6.
- **Wald's Equation** If $X_1, X_2, \cdots$ are independent and identically distributed and N is a stopping time:

$$\boxed{\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}[X]\,\mathrm{E}[N]}$$

Wald's Equation

- **Wald's Equation** If $X_1, X_2, \cdots$ are independent and identically distributed and N is a stopping time:

$$\boxed{\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}[X]\,\mathrm{E}[N]}$$

- **Proof**

$$\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}\left[\sum_{n=1}^{\infty} X_n I(n \le N)\right] = \sum_{n=1}^{\infty} \mathrm{E}[X_n]\,\mathrm{E}[I(n \le N)] =$$

$$\mathrm{E}[X] \sum_{n=1}^{\infty} P(n \le N) = \mathrm{E}[X] \sum_{n=0}^{\infty} P(N > n) = \mathrm{E}[X]\, E[N]$$

$\square$

Probability in $\mathbb{R}^k$
Aka multivariate random variable

If we have a set of k RV $\boldsymbol{X} = (X_1, \cdots X_k)$ taking values in $\mathbb{R}^k$ ($\boldsymbol{X} \in \mathbb{R}^k$), we define the joint distribution:

- Discrete RV

$$P(\boldsymbol{x}) = P(x_1, \cdots x_k) = P(X_1 = x_1, \cdots X_k = x_k)$$

- Continuos RV:
  - cumulative distribution function, CDF:

$$F(\boldsymbol{x}) = F(x_1, \cdots x_k) = P(X_1 \in (-\infty, x_1], \cdots X_k \in (-\infty, x_k])$$

  - with joint density function $f(\boldsymbol{x}) = f(x_1, \cdots x_k)$ (if exists):

$$F(\boldsymbol{x}) = F(x_1, \cdots x_k) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(x_1, \cdots x_k)\, dx_k \cdots dx_1$$

$$f(\boldsymbol{x}) = f(x_1, \cdots x_k) = \frac{\partial^k F(x_1, \cdots x_k)}{\partial x_1 \cdots \partial x_k}$$

Marginal distributions

Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X} \in \mathbb{R}^k, \boldsymbol{X}_1 \in \mathbb{R}^r, \boldsymbol{X}_2 \in \mathbb{R}^{k-r}$, $1 \le r < k$:

- Discrete RV

$$P(\boldsymbol{x}_2) = \sum_{x_1} \cdots \sum_{x_r} P(\boldsymbol{x}_1, \boldsymbol{x}_2)$$

- Continuos RV

$$f(\boldsymbol{x}_2) = \int_{x_1} \cdots \int_{x_r} f(\boldsymbol{x}_1, \boldsymbol{x}_2)\, dx_1 \cdots dx_r$$

Independent RV

- Discrete RV

$$P(\boldsymbol{x}) = P(x_1, \cdots x_k) =$$
$$P(X_1 = x_1, \cdots X_k = x_k) =$$
$$P(X_1 = x_1) \cdots P(X_k = x_k)$$

- Continuos RV

$$F(\boldsymbol{x}) = F(x_1, \cdots x_k) = F_{X_1}(x_1) \cdots F_{X_k}(x_k)$$
$$f(\boldsymbol{x}) = f(x_1, \cdots x_k) = f_{X_1}(x_1) \cdots f_{X_k}(x_k)$$

Conditional Distribution

- Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X} \in \mathbb{R}^k, \boldsymbol{X}_1 \in \mathbb{R}^r, \boldsymbol{X}_2 \in \mathbb{R}^{k-r}$, the r-dimensional distribuion of $\boldsymbol{X}_1$ conditioned by $\boldsymbol{X}_2 = \boldsymbol{x}_2$, $P(\{\boldsymbol{X}_2 = \boldsymbol{x}_2\}) > 0$ is:

$$F(\boldsymbol{X}_1|\boldsymbol{X}_2) = P(\boldsymbol{X}_1 \le \boldsymbol{x}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \frac{P(\boldsymbol{X}_1 \le \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2)}{P(\boldsymbol{X}_2 = \boldsymbol{x}_2)}.$$

If $\boldsymbol{X}$ is discrete with probability $P(\boldsymbol{x}_1, \boldsymbol{x}_2)$ or absolutely continuous with density $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$:

$$P(\boldsymbol{x}_1|\boldsymbol{x}_2) = \frac{P(\boldsymbol{x}_1, \boldsymbol{x}_2)}{P(\boldsymbol{x}_2)}$$

$$f(\boldsymbol{x}_1|\boldsymbol{x}_2) = \frac{f(\boldsymbol{x}_1, \boldsymbol{x}_2)}{f(\boldsymbol{x}_2)}$$

Probability

Bayes formula

$$f(\boldsymbol{x}_1|\boldsymbol{x}_2)\,f(\boldsymbol{x}_2) = f(\boldsymbol{x}_2|\boldsymbol{x}_1)\,f(\boldsymbol{x}_1) \Rightarrow f(\boldsymbol{x}_1|\boldsymbol{x}_2) = \frac{f(\boldsymbol{x}_2|\boldsymbol{x}_1)\,f(\boldsymbol{x}_1)}{f(\boldsymbol{x}_2)}$$

in which f is the density or probability, accordingly.

Law of total probability
Aka composition of marginals and conditionals

Using the previous formulas we can compute ($\boldsymbol{X}$ can be a mixture of discrete and continuous RV):

- If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are discrete RV: $P(\boldsymbol{x}_2) = \sum_{\boldsymbol{x}_1} P(\boldsymbol{x}_2|\boldsymbol{x}_1)\, P(\boldsymbol{x}_1)$
- If $\boldsymbol{x}_1$ is discrete and $\boldsymbol{x}_2$ is cont.: $f(\boldsymbol{x}_2) = \sum_{\boldsymbol{x}_1} f(\boldsymbol{x}_2|\boldsymbol{x}_1)\, P(\boldsymbol{x}_1)$
- If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are cont.: $f(\boldsymbol{x}_2) = \int_{\boldsymbol{x}_1} f(\boldsymbol{x}_2|\boldsymbol{x}_1)\, f(\boldsymbol{x}_1)\, d\boldsymbol{x}_1$
- If $\boldsymbol{x}_1$ is cont. and $\boldsymbol{x}_2$ is discrete: $P(\boldsymbol{x}_2) = \int_{\boldsymbol{x}_1} P(\boldsymbol{x}_2|\boldsymbol{x}_1)\, f(\boldsymbol{x}_1)\, d\boldsymbol{x}_1$

Conditional expected value

- Given $X \in \mathbb{R}$, $\boldsymbol{Y} \in \mathbb{R}^k$ with density $f(x,\boldsymbol{y})$:

$$\mathrm{E}[X \mid \boldsymbol{Y} = \boldsymbol{y}] = \int_{\mathbb{R}} x f(x|\boldsymbol{y})\, dx$$

$$\mathrm{E}[X] = \int_{\mathbb{R}^k} \mathrm{E}[X \mid \boldsymbol{Y} = \boldsymbol{y}]\, f(\boldsymbol{y})\, d\boldsymbol{y}$$

where the marginal $f(\boldsymbol{y}) = \int_{x=-\infty}^{\infty} f(x,\boldsymbol{y})\, dx$ and the conditional $f(x|\boldsymbol{y}) = f(x,\boldsymbol{y}) / f(\boldsymbol{y})$.
Thus, the law of total probability also applies to expected value, and it is known as law of total expectation.

Part I

Introduction

Outline

Stochastic Process (SP)

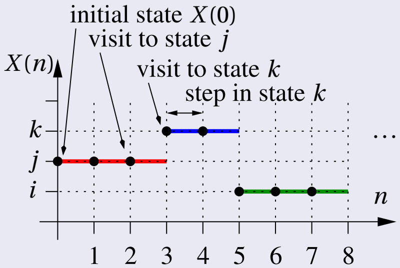
Introduction

- Sequence of RVs $\{X(t)\}_{t\geq 0}$.
- $X(t)$ is the state at time t.
- The state $X(t)$ can be continuous or discrete.
- The index can be continuous or discrete. We shall use n for the index, and refer to it as steps when it is discrete, and t and refer to it as time when it is continuous.
- We call a possible sequence of states of the SP the sample function (or sample path) of the SP.

Stochastic Process (SP)

Sample Path

- Possible evolution (sample path) of a discrete state, discrete time SP $\{X(n)\}_{n \geq 0}$:



- To characterize the stochastic process we would need the distribution and joint probabilities of the $\{X(n)\}_{n \geq 0}$ RVs:

$$P\big(X(n) = i, X(n-1) = k, \cdots X(0) = j\big)$$

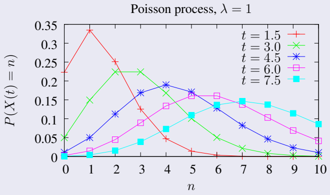
Stochastic Process (SP)

Example: Poisson Process

- It is a discrete state continuous time SP.
- It counts the number of events ocurred in a time interval.
- Often used to build models of other stochastic processes.
- Definition: The number of "events" in any interval of length t, $X(t)$, is Poisson distributed with mean λt, i.e.

$$\boxed{\begin{gathered} P(X(t+s) - X(s) = n) = P(X(t) - X(0) = n) = \\ P(X(t) = n) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!} \end{gathered}}$$

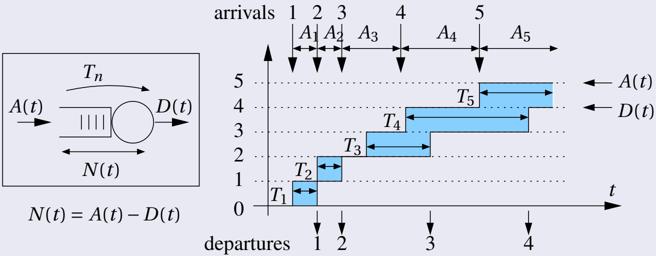
where we assume $X(0) = 0$.



Stochastic Process (SP)

Example: Queue with Poisson Arrivals

- The queue arrivals, $A(t)$, are modeled as a Poisson process with mean λt. Each event model an arrival.

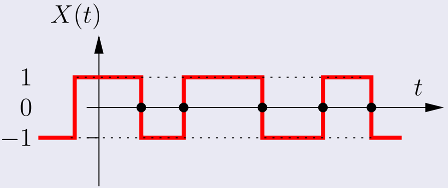


Stochastic Process (SP)

Example: Telegraph signal [1]

- The signal is modeled as a Poisson process with mean λt such that $X(0) = 1$ or $X(0) = -1$ with equal probability of 1/2 and:

$$X(t) = \begin{cases} 1 & \text{if the number of events in } (0, t] \text{ is even} \\ -1 & \text{if the number of events in } (0, t] \text{ is odd} \end{cases}$$



[1] Athanasios Papoulis and S Unnikrishna Pillai. *Probability, Random Variables and Stochastic Processes*. McGraw-Hill Education, 2002.

Stochastic Process (SP)

Analysis of Stochastic Processes

- Signal Theory: Normally interested in the spectral analysis of the signal. The basic tool is the Fourier transform of the autocorrelation function of the process (energy spectral density). We will not do this analysis.

$R(t) = \mathrm{E}[X(\tau)\, X(\tau - t)]$ autocorrelation

$$F(f) = \mathscr{F}[R(t)] = \int_{-\infty}^{\infty} R(t)\, \mathrm{e}^{-j2\pi f t}\, \mathrm{d}t$$ Fourier transform (energy spectral density)

- Computer Networks: Normally interested in probabilistic models using Markov Chains and Queueing Theory.