Solution

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Stochastic Network Modeling (SNM). Autumn 2019.

First assessment, Discrete Time Markov Chains. 4/11/2019.

Problem 1

Assume a slotted Aloha system with 2 nodes, n_1, n_2. Both nodes transmit with probability $\sigma = 1/3$ when they are thinking. When they are backlogged n_1 transmits deterministically after every 1 slot, and n_2 continues transmitting with probability $\sigma = 1/3$, as shows figure 1.

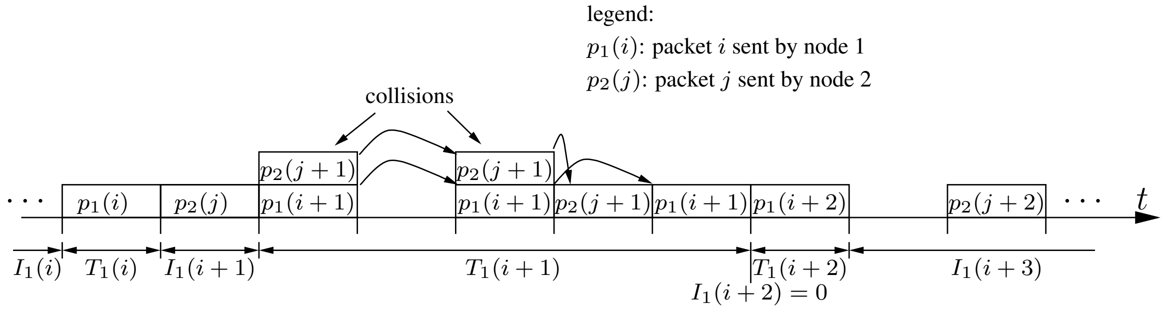


Figure 1: Time diagram of the system.

1.A (1.5 points) Draw the transition diagram and derive the one step transition probabilities of a DTMC that allows computing the throughput of each node.

1.B (1 points) Compute the stationary distribution.

1.C (1.5 point) Compute the throughput S_1, S_2, of each node (expected number of successful packets transmitted per slot).

1.D (1.5 point) Let $N_i \geq 1$, $i = 1{,}2$ be the random variable equal to number transmissions per packet of node n_i. That is, if a new packet is successfully transmitted in the first trial, $N_i = 1$. If it collides in the first trial and it is successfully transmitted in the second trial, then $N_i = 2$, and so on. Compute E$[N_i]$, $i = 1{,}2$.

1.E (1 points) Let $I_1 \geq 0$ be the random variable equal to the number of slots that node n_1 is in thinking state between transmissions (idle time). That is, if n_1 transmits a new packet immediately after a n_1 successful transmission, then $I_1 = 0$, and so on (see figure 1). Compute the distribution of I_1, $P(I_1 = n)$, and its expected value E$[I_1]$.

1.F (1.5 points) Let $T_1 \geq 1$ be the random variable equal to the transmission time of node n_1 (time that follows every idle time). That is, if n_1 successfully transmits a new packet in the first trial $T_1 = 1$. If it collides in the first trial and it is successfully transmitted in the second trial, then $N_1 = 3$. If it collides twice (as $p_1(i+1)$ in figure 1), then $N_1 = 5$, and so on (see figure 1). Compute the distribution of T_1, $P(T_1 = n)$, and its expected value E$[T_1]$.

1.G (1 point) Say what relation there is between the throughput of n_1, E$[I_1]$ and E$[T_1]$. Check it with the values obtained in the previous items.

1.H (1 point) Let A be the event $A = \{$both nodes are in thinking state$\}$ and $T_2 \geq 1$ be the random variable equal to the transmission time of node n_2. Compute E$[T_2 \mid A]$, that is, the expected value of T_2 given that the transmission of n_2 occurs when both nodes are in thinking state. Use an absorbing DTMC, and describe clearly the meaning of each state.

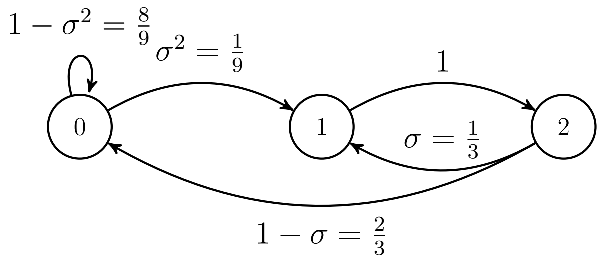
Solution

Problem 1

1.A Node n_2 transmits always with the same probability, therefore, we only need to remember the state of node n_1. Define the states:

(0) n_1 thinking
(1) n_1 backlogged slot 1
(2) n_1 backlogged slot 2

we have the chain:



1.B Using flow balancing:

$$\pi_0\,\frac{1}{9}=\pi_2\,\frac{3}{3}$$
$$\pi_1=\pi_2$$

which yields $\pi_0=\frac{6}{8}$, $\pi_1=\pi_2=\frac{1}{8}$.

1.C

$$S_1=\pi_0\,\sigma\,(1-\sigma)+\pi_2\,(1-\sigma)=1/4$$
$$S_2=\pi_0\,\sigma\,(1-\sigma)+\pi_1\,\sigma=5/24.$$

1.D Offered load of nodes n_1 and n_2:

$$G_1=\pi_0\,\sigma+\pi_2=9/24$$
$$G_2=\sigma=1/3.$$

Thus, we have:

$$\mathrm{E}[N_1]=G_1/S_1=3/2,$$
$$\mathrm{E}[N_2]=G_2/S_2=8/5.$$

1.E

$$\mathrm{E}[I_1]=\sum_{n=0}^{\infty}n\,(1-\sigma)^n\,\sigma=\frac{1-\sigma}{\sigma}=2.$$

1.F

$$P(T_1=n)=\begin{cases}1-\sigma, & n=1\\ 0, & n=2\\ \sigma\,(1-\sigma), & n=3\\ 0, & n=4\\ \sigma^2\,(1-\sigma), & n=5\\ \dots & \\ \sigma^{k-1}\,(1-\sigma), & n=2\,k-1\end{cases}$$

thus,

$$P(T_1 = n) = \sigma^{k-1}\,(1-\sigma),\, n = 2\,k-1,\, k \geq 1.$$

and

$$\mathrm{E}[T_1] = \sum_{k=1}^{\infty}(2\,k-1)\,\sigma^{k-1}\,(1-\sigma) = 2\,(1-\sigma)\sum_{k=1}^{\infty} k\,\sigma^{k-1} - (1-\sigma)\sum_{k=1}^{\infty}\sigma^k =$$
$$2\,(1-\sigma)\frac{1}{(1-\sigma)^2} - (1-\sigma)\,\frac{1}{1-\sigma} = \frac{1+\sigma}{1-\sigma} = 2.$$

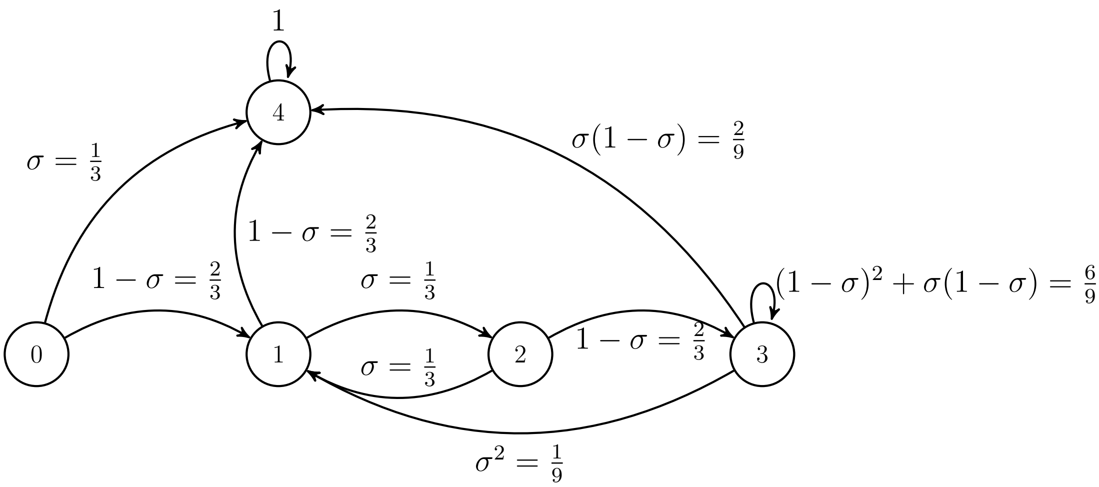
1.G It must be:

$$S_1 = \frac{1}{\mathrm{E}[I_1] + \mathrm{E}[T_1]} = \frac{1}{2+2} = \frac{1}{4}.$$

1.H Define the states:

0 both nodes thinking (event A)
1 n_1 backlogged slot 1
2 n_1 backlogged slot 2
3 n_1 thinking, n_2 backlogged
4 absorbing state n_1 successfully transmits the packet

We have the chain:



We have $\mathrm{E}[T_2 \mid A] = m_{04}$ (first passage time), where:

$$\begin{aligned}
m_{04} &= 1 + \frac{1}{3}\,m_{14}\\
m_{14} &= 1 + \frac{2}{3}\,m_{24}\\
m_{24} &= 1 + \frac{1}{3}\,m_{14} + \frac{2}{3}\,m_{34}\\
m_{34} &= 1 + \frac{1}{9}\,m_{14} + \frac{6}{9}\,m_{34}.
\end{aligned}$$

Solving the system we get $m_{04} = 44/17$.

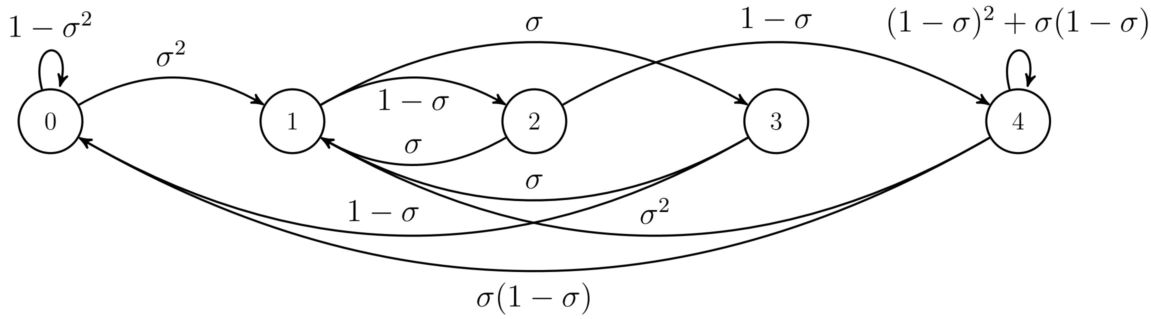
NOTE: Defining the event $B = \{n_1$ backlogged in slot 2 and n_2 thinking$\}$ we have:

$$\mathrm{E}[T_2] = \mathrm{E}[T_2 \mid A]\,P(A) + \mathrm{E}[T_2 \mid B]\,P(B) \tag{1}$$

where $\mathrm{E}[T_2 \mid B] = 1 + m_{14} = 1 + 81/17 = 98/17$ in the chain above. In order to compute $P(A)$ and $P(B)$ we can consider the chain with states:

0 both nodes thinking (event A)
1 n_1 backlogged slot 1, n_2 backlogged
2 n_1 backlogged slot 2, n_2 backlogged
3 n_1 backlogged slot 2, n_2 thinking (event B)
4 n_1 thinking, n_2 backlogged

and transition diagram:



Solving the chain we get the stationary distribution:

$$\begin{aligned}
\pi_0 &= 14/24 \\
\pi_1 &= \ \ 3/24 \\
\pi_2 &= \ \ 2/24 \\
\pi_3 &= \ \ 1/24 \\
\pi_4 &= \ \ 4/24
\end{aligned}$$

and

$$\begin{aligned}
P(A) =& \frac{\pi_0}{\pi_0 + \pi_3} = \frac{14}{15} \\
P(B) =& \frac{\pi_3}{\pi_0 + \pi_3} = \frac{1}{15}.
\end{aligned}$$

Substituting into (1) we get

$$\mathrm{E}[T_2] = \frac{44}{17}\frac{14}{15} + \frac{98}{17}\frac{1}{15} = \frac{14}{5}.$$

Since $\mathrm{E}[I_2] = \mathrm{E}[I_1] = 2$, we have:

$$S_2 = \frac{1}{\mathrm{E}[I_2] + \mathrm{E}[T_2]} = \frac{1}{2 + 14/5} = \frac{5}{24}$$

as expected.