Algorithmic Game Theory
Homework 3

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1 Problem 15

Singleton Congestion Game. *In singleton congestion games with n players accessing to a set of m resources, all best response sequences have length at most n^2m*

Proof. Having the set of delay functions $V = \{d(e, f(k, e)) | e \in E \wedge 1 \leq k \leq n\}$, we sort this set in increasing order and we define a new delay function such as each value is the index of the ordered previous set V.

$$d'(e, f(k, e)) = \text{index of } d(e, f(k, e)) \text{ of sorted set } V$$

Having this, it is trivial to observed that if some the strategy selected (s_{-i}, s'_i) for some player i, reduce the cost in $d'_e(k)$ also it is reduced on $d_e(k)$. As we are analyzing for a Singleton Congestion game we know that $s_{-i} = r$ and $s'_i = r'$ by definition, therefore $c(s_{-i}) = d(r, f(s_{-i}, r))$ and $c(s'_i) = d(r', f(s'_i, r'))$. Since for player i, (s_{-i}, s'_i) is an improvement, we know that $d(r', f(s'_i, r')) < d(r, f(s_{-i}, r))$, and also $d'(r', f(s'_i, r')) < d'(r, f(s_{-i}, r))$.

We also observe that $d'(e, f(k, e)) \leq nm, \forall\, e \in E, k \in N$, since $|V| \leq nm$.

Taking **Rossenthal's** potential function delay can be upper bound as:

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{f(s,e)} d'(e, f(k, e)) \leq \sum_{e \in E} \sum_{k=1}^{f(s,e)} nm \leq n^2m \tag{1a}$$

n^2 is taking because every player uses exactly one resource, therefore $\sum_{e \in E} f(k,e) = n$. ■

2 Problem 17

It can be seen that $PoA = PoS = \frac{k}{k} = 1$ and this is because of the following:

- In order to maximize their utility each player is going to perform the task until k select as strategy perform the task.
- As we have analyzed this problem before there is a $NE \iff \sum_{i=1}^{n} x_i \neq k-1$, this indicates that $PoA = \frac{k}{k}$, because when reaching the equilibrium non player has incentive to change its strategy.
- The same happen with $PoS = \frac{k}{k}$
- Therefore it is a stabilized game