Algorithmic Game Theory
Homework 1 - Solutions

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1 Problem 6

Given the following definition of the problem:

Let $G = (V, E)$ be the graph representing with V the number of players as vertices and E the set of edges such as $\forall_{i,j \in V}\ (i, j) \in E$ if it is a bad pairing.

1.1 Formal Characterization

$N = V$.

$\forall_{i \in N}, A_i = \{1, 0\}$ 1 if it is in Group 1, 0 if it is in Group 2.

$$u_i(v_2, \ldots, v_n) = \begin{cases} 1 & \exists\ j \neq i, (i, j) \in E \text{ and } j \text{ is in the other group of } i \\ 0 & otherwise \end{cases}$$

$$BR_i(v_{-i}) = \begin{cases} \{1\} & \forall_{j \neq i}, (i, j) \in E \implies \sum_j 1 - v_j > \sum_j v_j \\ \{0\} & \forall_{j \neq i}, (i, j) \in E \implies \sum_j v_j > \sum_j 1 - v_j \\ \{0, 1\} & otherwise \end{cases}$$

NPE Analysis 1. *$v = (v_1, \ldots, v_n)$ is a $NPE \iff \forall_{i,j}, (i, j) \in E\ \sum_{i=1}^{n} 1 - v_i = \sum_{i=1}^{n} v_j$*

Proof.

Part 1. $\Longleftarrow\ \forall_{i,j},(i,j) \in E\ \sum_{i=1}^{n} 1 - v_i = \sum_{i=1}^{n} v_j \implies$

$$\implies \forall_{i,j},(i,j) \in E\ \sum_{j\neq i} 1 - v_i > \sum_{j\neq i} v_i \implies \{1\} \in BR_i \tag{1a}$$

$$\implies \forall_{i,j},(i,j) \in E\ \sum_{j\neq i} 1 - v_i < \sum_{j\neq i} v_i \implies \{0\} \in BR_i \tag{1b}$$

Part 2. $\implies$
$\forall_{i,j},(i,j) \in E\ \sum_{i=1}^{n} 1 - v_i \neq \sum_{i=1}^{n} v_j$ *if this holds, then*

$$\exists j, v_j = 1 \iff (i,j) \in E\ \wedge \sum_{j\neq i} 1 - v_i < \sum_{j\neq i} v_i + v_j \tag{2a}$$

$$, v_j = 0 \iff (i,j) \in E\ \wedge \sum_{j\neq i} (1 - v_i) + v_j > \sum_{j\neq i} v_i \tag{2b}$$

$$\tag{2c}$$

But if this is true i changes to other group.
Therefore $\forall_{i,j},(i,j) \in E\ \sum_{i=1}^{n} 1 - v_i = \sum_{i=1}^{n} v_j$

■

1.2 Complexity Analysis

As we can see by the definition of the problem, the complexity is polinomial in the size of V by the size of E, because we only need to sum over the vertices that have an edge over other vertex. Also this problem characterization matches the ***Strategy General Form***, which we have seen in the theory that it is polynomial. Therefore the complexity is $O(|V||E|)$ worst case.