Efficiency of Nash Equilibria

Fall 2020

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- The players' goals can be different from those of the society.
- Fixing a social goal an optimal situation can be defined.
- How good/bad are NE with respect to this goal?

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- To perform such an analysis for strategic games we have first to define a global function to optimize, this function is usually called the social cost or social utility.
- Society is interested in minimizing the social cost or maximizing the social utility.

Social cost

Consider a n-player game $\Gamma = (A_1, \ldots, A_n, c_1, \ldots, c_n)$. Let

- $A = A_1 \times \cdots \times A_n$,
- $PNE(\Gamma)$ be the set of PNE of Γ,
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Social cost

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- $PNE(\Gamma)$ be the set of PNE of Γ,
- $NE(\Gamma)$ be the set of NE of Γ,
- $\mathcal{C} : A \to \mathbb{R}$ be a social cost function.

  $\mathcal{C}$ can be extended to mixed strategy profiles by computing the average under the joint product distribution.

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- Game specific cost/utility defined by the model motivating the game.

Price of Anarchy/Stability

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The Price of anarchy of Γ is defined as

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For social utility functions the terms are inverted in the definition.

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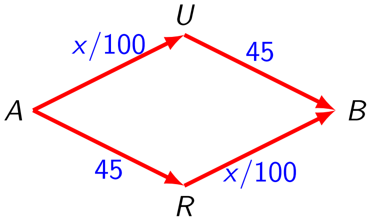
- PoA measures the worst decentralized equilibrium scenario giving the maximum system degradation.
- PoS measures the best decentralized equilibrium scenario giving the best possible degradation.

Network

- 4000 drivers drive from A to B on



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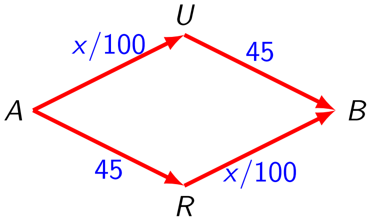

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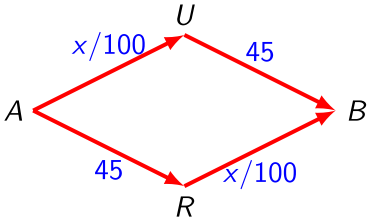

- Set the social cost to be the maximum travel time.
- Optimal social cost is reached when half of the drivers take $A - U - B$ and the other half $A - R - B$ with social cost 65.

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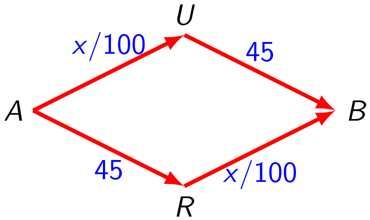

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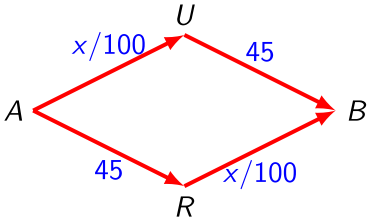

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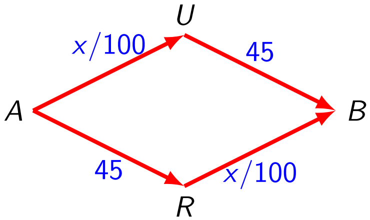

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- In the NE half of the drivers take $A - U - B$ and the other half $A - R - B$.
- $PoA = PoS = 65/65 = 1$

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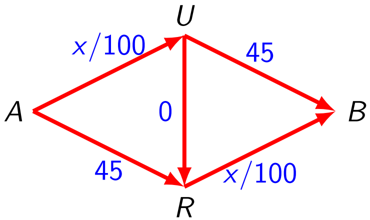

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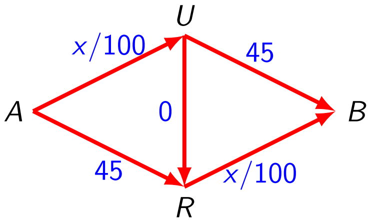

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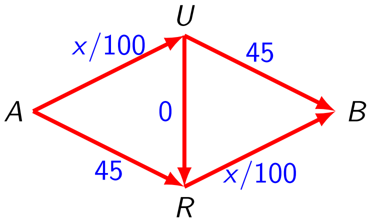

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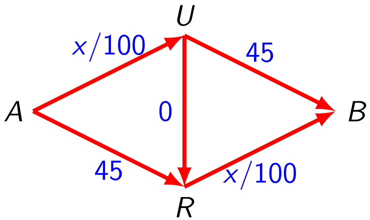

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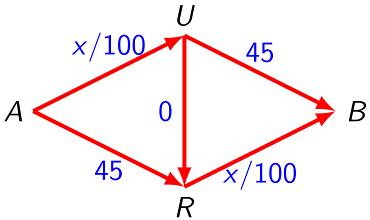

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- In the NE all drivers take $A - U - R - B$ with social cost 80.

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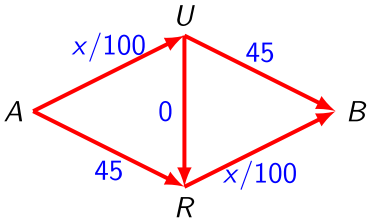

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- In the NE all drivers take $A - U - R - B$ with social cost 80.
- $PoA = PoS = 80/65 = 16/13$

Load Balancing game

- There are m servers and n jobs. Job i has load p_i.
- The game has n players, corresponding to the n jobs.
- Each player has to decide the server that will process its job. $A_i = \{1, \ldots, m\}$
- The response time of server j is proportional to its load

$$L_j(s) = \sum_{i | s_i = j} p_i.$$

- Each job wants to be assigned to the server that minimizes its response time:

$$c_i(s) = L_{s_i}(s).$$

Load Balancing game: PNE?

Consider the best response dynamic

- Start with an arbitrary state.
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- Start with an arbitrary state.
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- How to prove that such a process converges to a PNE?
- Seek for an adequate kind of potential function.

Load Balancing game: PNE?

BR-inspired-algorithm

- Order the servers with decreasing load (i.e., the decreasing response time):
$$L_1 \geq L_2 \geq \cdots \geq L_m.$$
- Job i moves from server j to k, $L_k + p_i < L_j$.
- We must have $L_1 \geq \cdots \geq L_j \geq \cdots \geq L_k \geq \cdots \geq L_m$.
- Thus, $L_j - p_i < L_j$ and $L_k + p_i < L_j$.

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- Thus, $L_j - p_i < L_j$ and $L_k + p_i < L_j$.
- Reorder the servers by decreasing load and repeat the process until no job can move.

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- Does the algorithm converge?
- There are a finite number of (possibly exponential) assignments of jobs to servers.
- At each step the sorted load sequence decreases!
- So BR-inspired-algorithm terminates (although it can be rather slow).
- The load balancing game has a PNE.

Load Balancing game: Social cost

- The natural social cost is the total finish time i.e., the maximum of the server's loads

$$c(s) = \max_{j=1}^{m} L_j.$$

- How bad/good is a PNE?

Load Balancing game: PoS

- Let s be an assignment with optimal cost.
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Load Balancing game: PoS

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- not necessarily, no player in the worst server can improve, however other players can get a better benefit.
- However, starting from an optimal solution the BR-inspired-algorithm terminates on a PNE with the same maximum load.
- Therefore, $PoS(\Gamma) = 1$.

Load Balancing game: PoA

Theorem

The max load of a Nash equilibrium s is within twice the max load of an optimum assignment, i.e.,.

$$C(s) \leq 2 \min_{s'} C(s').$$

Which will give $PoA(\Gamma) \leq 2$.

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- We get

$$C(s) = L_j \leq (\textstyle\sum_k L_k)/m + p_i \leq (\textstyle\sum_\ell p_\ell)/m + p_i \leq C(s') + C(s').$$

Congestion games

Congestion games

A congestion game $(E, N, (d_e)_{e \in E})$

- is defined on a finite set E of resources and
- has n players and,
- for each resource e, a delay function d_e mapping $\mathbb{N}$ to the integers.
- The actions for each player are subsets of E.
- The cost functions are the following:

$$c_i(a_1, \dots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f_e(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

Weighted congestion games

Weighted congestion games

A weighted congestion game $(E, N, (d_e)_{e\in E}, (w_i)_{i\in N})$

- is defined on a finite set E of resources and
- has n players. Player i has an associated positive integer weight w_i.
- Each resource e has a delay function d_e mapping $\mathbb{N}$ to the integers.
- The actions for each player are subsets of E.
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \left(\sum_{e\in a_i} d(e, f(a_1, \ldots, a_n, e))\right)$$

being $f_e(a_1, \ldots, a_n, e) = \sum_{i|e\in a_i} w_i$.

Network weighted congestion games

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A network weighted congestion game
$(N, G = (V, E), (d_e)_{e \in E}, (w_i)_{i \in N}, (s_i)_{i \in N}, (t_i)_{i \in N})$

- Is defined on a directed graph $G = (V, E)$, the resources are the arcs (E)
- The game has n players, player i has an associated positive integer weight w_i and two vertices $s_i, t_i \in V$.
- For each arc e a delay function d_e mapping $\mathbb{N}$ to the integers.
- The action set for player i is the set of $(s_i - t_i)$-paths in G.
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \ldots, a_n, e)) \right)$$

being $f(a_1, \ldots, a_n, e) = \sum_{i | e \in a_i} w_i$.

PNE in weighted congestion games

- There are weighted network congestion games without PNE

PNE in weighted congestion games

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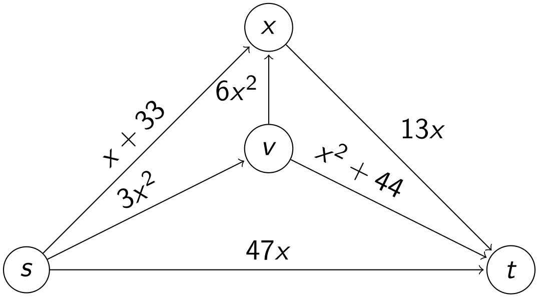
- Consider the following network with 2 players having weights $w_1 = 1$ and $w_2 = 2$.



Not always PNE in weighted congestion games

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s_{-i}	BR_1	BR_2
$P_1 : s \to t$	P_4	P_2
$P_2 : s \to v \to t$	P_4	P_4
$P_3 : s \to w \to t$	P_1	P_2
$P_3 : s \to v \to w \to t$	P_1	P_3

Not always PNE in weighted congestion games

s_{-i}	BR_1	BR_2
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$P_3 : s \to v \to w \to t$	P_1	P_3

Therefore the game has no PNE

PoA for affine congestion games

Consider unweighted congestion games such that the delay functions are affine functions, i.e., for each resource e,

$$d_e(x) = a_e x + b_e,$$

for some $a_e, b_e > 0$.

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$$d_e(x) = a_e x + b_e,$$

for some $a_e, b_e > 0$.

Let C be the usual social cost:

$$C(s) = \sum_{e \in E} d_e(f_e(s))$$

Smoothness

A game is called (λ, μ)-smooth, for $\lambda > 0$ and $\mu \leq 1$ if, for every pair of strategy profiles s and s', we have

$$\sum_{i \in N} c_i(s_{-i}, s_i') \leq \lambda C(s') + \mu C(s).$$

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Smoothness directly gives a bound for the PoA:

Theorem

In a (λ, μ)-smooth game, the PoA for PNE is at most $\frac{\lambda}{1-\mu}$.

Proof of smoothness bound on PoA

Let s be the worst PNE and s^* be an optimum solution.

$$\begin{aligned} C(s) = \sum_{i \in N} c_i(s) \le \sum_{i \in N} c_i(s_{-i}, s_i^*) \\ \le \lambda C(s^*) + \mu C(s) \end{aligned}$$

Substracting $\mu C(s)$ on both sides gives

$$(1 - \mu) C(s) \le \lambda C(s^*).$$

Theorem

Every congestion game with affine delay functions is $(5/3, 1/3)$*-smooth. Thus, PoA* $\leq 5/2$.

Theorem

Every congestion game with affine delay functions is $(5/3, 1/3)$-smooth. Thus, PoA $\leq 5/2$.

The proof uses a technical lemma:

Lemma (Christodoulou, Koutsoupias, 2005)

For all integers y, z we have

$$y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2.$$

Proof of smoothness for affine functions

Recall that $d_e(x) = a_e x + b_e$. Note that using the Lemma

$$a_e y(z+1)+b_e y \leq a_e(\frac{5}{3}y^2+\frac{1}{3}z^2)+b_e y = \frac{5}{3}(a_e y^2+b_e y)+\frac{1}{3}(a_e z^2+b_e z).$$

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Taking $y = f_e(s^*)$ and $z = f_e(s)$ we get

$$(a_e(f_e(s)+1)+b_e)f_e(s^*) \leq \frac{5}{3}(a_e f_e(s^*)+b_e)f_e(s^*))+\frac{1}{3}(a_e f_e(s)+b_e)f_e(s)).$$

Proof of smoothness for affine functions

Recall that $d_e(x) = a_e x + b_e$. Note that using the Lemma

$$a_e y(z+1) + b_e y \leq a_e(\frac{5}{3}y^2 + \frac{1}{3}z^2) + b_e y = \frac{5}{3}(a_e y^2 + b_e y) + \frac{1}{3}(a_e z^2 + b_e z).$$

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Summing up all the inequalities

$$\sum_{e\in E}(a_e(f_e(s)+1)+b_e)f_e(s^*) \leq \frac{5}{3}C(s^*) + \frac{1}{3}C(s).$$

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$$\sum_{e \in E} (a_e(f_e(s)+1) + b_e) f_e(s^*) \leq \frac{5}{3} C(s^*) + \frac{1}{3} C(s).$$

But,

$$\sum_{i \in N} c_i(s_{-i}, s_i^*) \leq \sum_{e \in E} (a_e(f_e(s)+1) + b_e) f_e(s^*)$$

as there are at most $f_e(s^*)$ players that might move to resource r. Each of them by unilaterally deviating incur a delay of $(a_e(f_e(s)+1) + b_e$.

Proof of smoothness for affine functions

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Each of them by unilaterally deviating incur a delay of $(a_e(f_e(s) + 1) + b_e$.
This gives the $(5/3, 1/3)$-smoothness.

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