Solution

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Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

Stochastic Network Modeling (SNM). Autumn 2020.

First assessment, Discrete Time Markov Chains. 10/11/2020.

Problem 1

A machine can be either working or in repair states. In both states the machine stays at leas 1 day. If it is working there is a probability $\alpha = 1/3$ that it fails at the end of the day and goes into repair next day. If it is in repair there is a probability $\beta = 1/2$ that it is repaired at the end of the day, and will be working next day. The machine can be at most 3 consecutive days in repair state. If it is not repaired at the end of 3 consecutive days, it is replaced next day, i.e. the beginning of day 4, by a new one that starts in working state. Note that the machine can be repaired in the third day, and no replacement is done in this case. Every day the machine is working it produces a profit of $G = 500$ €. Every day the machine is in repair it produces a cost of $C(n) = 200\,n$ €, where n is the number of consecutive repair days. Every new machine has a cost of $M = 1500$ €.

1.1 (1 point) Formulate a DTMC that allows computing the expected benefit.

1.2 (1 point) Derive the transition probabilities and one step transition probability matrix.

1.3 (1 point) Compute the stationary distribution using the flow balancing method.

1.4 (1 point) Compute the expected number of new machines bought per day, in stationary regime.

1.5 (1 point) Compute the expected benefit in one year, in stationary regime.

1.6 (1 point) Let R be the random variable equal to the time since a machine enters in repair state until a machine is in working state. Compute the distribution of R (i.e. $P(R \leq n)$, $n \geq 0$) using a DTMC.

1.7 (1 point) Compute the expected value of R using its distribution.

1.8 (1 point) Compute the expected value of R using the mean recurrence time of the DTMC formulated in item 1.6.

1.9 (1 point) Suppose that a machine enters in repair. Compute the probability that there will be a machine working again because it is replaced by a new machine.

1.10 (1 point) Suppose that the machine is never replaced. That is, when it is not working, we wait until it is repaired. Compute the expected benefit in a year, in stationary regime. Hint: Let B be the benefit. The expected value can be computed as $\mathrm{E}[B] = \sum_i \mathrm{E}[B \mid X = i]\,\pi_i$.

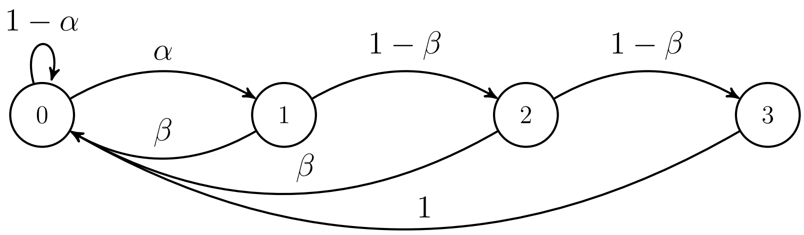
Solution

Problem 1

1.1 Define the states:

(0) working
(1) day 1 in repair
(2) day 2 in repair
(3) day 3 in repair

we have the chain:



1.2 Transition probabilities are shown in the figure above.

1.3 Using flow balancing we get

$$\boldsymbol{\pi} = \begin{bmatrix} 12/19 & 4/19 & 2/19 & 1/19 \end{bmatrix}$$

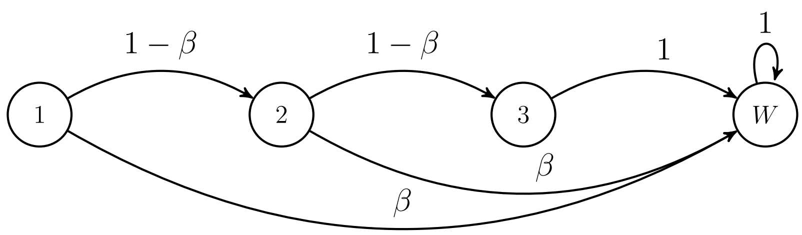
1.4

$$S = \pi_3 \, (1 - \beta) = 1/38 \text{ [New machines/step]}$$

1.5

$$\mathrm{E}[B] = 365 \left(G \, \pi_0 - \sum_{i=1}^{3} C(i) \, \pi_i - M \, S \right) \approx 58.592{,}1 \text{ €}$$

1.6



$$\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1-\beta & 0 & \beta \\ 0 & 0 & 1-\beta & \beta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = 0$ with multiplicity 3. We guess the solution:

$$\pi_W(n) = 1 + a \, I(n=0) + b \, I(n=1) + b \, I(n=2), \; n \geq 0$$

where $I()$ is the indicator function. Imposing the boundary conditions

$$\begin{aligned}
\pi_W(0) &= a + 1 = 0 \\
\pi_W(1) &= b + 1 = \beta \\
\pi_W(2) &= c + 1 = \beta + \beta\,(1-\beta)
\end{aligned}$$

we get

$$P(R \le n) = \pi_W(n) = 1 - I(n=0) - (1-\beta)\,I(n=1) - (1-\beta\,(2-\beta))\,I(n=2),\; n \ge 0$$

1.7

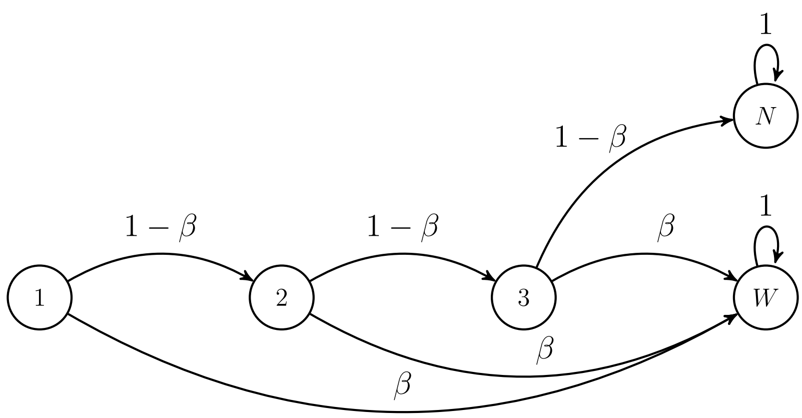
$$\sum_{n=0}^{\infty} n\,P(R=n) = \sum_{n=0}^{\infty}(1 - P(R \le n)) =$$
$$\sum_{n=0}^{\infty}\left(I(n=0) + (1-\beta)\,I(n=1) + (1-\beta\,(2-\beta))\,I(n=2)\right) =$$
$$1 + (1-\beta) + 1 - \beta\,(2-\beta) = 3 - \beta\,(3-\beta) = 7/4$$

1.8

$$\begin{aligned}
m_{1W} &= \beta + (1-\beta)\,(1 + m_{2W}) \\
m_{2W} &= \beta + (1-\beta)\,(1 + m_{3W}) \\
m_{3W} &= 1
\end{aligned}$$

substituting we get $m_{1W} = 3 - \beta\,(3-\beta)$, as expected.

1.9



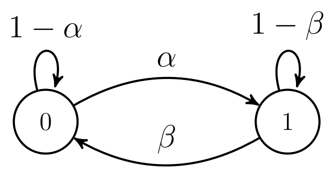
where state N is buy a new machine. Clearly the requested probability is

$$f_{1N} = (1-\beta)^3 = 1/8$$

1.10 Define the states:

(0) working
(1) in repair

we have the chain:



with stationary distribution

$$\pi_0 = \frac{\alpha}{\alpha + \beta}$$
$$\pi_1 = \frac{\beta}{\alpha + \beta}$$

Thus:

$$\mathrm{E}[B] = 365\,(\pi_0\,G - \pi_1\,\mathrm{E}[C])$$

where

$$\mathrm{E}[C] = \sum_{n=1}^{\infty} C(n)\,(1-\beta)^{n-1}\,\beta = \sum_{n=1}^{\infty} 200\,n\,(1-\beta)^{n-1}\,\beta = \frac{200}{\beta}$$

substituting we get $\mathrm{E}[B] = 51.100{,}0$ €. Thus, it is worth changing the machine if not repaired at day 3.