Algorithmic Game Theory

Exercises Cooperative Game theory

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Problem 1

Consider a cooperative game with $N = 3$ players $\{a, b, c\}$ and the following characteristic function v:

S	$\emptyset$	a	b	c	a,b	a,c	b,c	a,b,c
v	0	1	0	1	4	3	5	8

Which imputations belong to the core?

We know that an **outcome** of a game $\Gamma = (N, v)$ is a pair (P, x), where

- P is a **coalition structure**.
- x is a **payoff vector** which distributes the value of each coalition in P.

An outcome is called an **imputation** if it satisfies individual rationality, that is: $X_i \geq v(i)$ for all $i \in N$.

To end with, the **core** of a game Γ is the set of all stable outcomes (outcomes that no coalition wants to deviate from).

To find the elements in the core, first we analyze some characteristics of this function v:

- Normalized: $v(\emptyset) = 0$
- Non-negative: $v(C) \geq 0,$ for any coalition $C \subseteq N$
- Monotone: $v(C) \leq v(D),$ for any C, D such that $C \subseteq D$
- Superadditive: $v(C \cup D) \geq v(C) + v(D)$ where $C \cap D = \emptyset$

As the game is superadditive, players can always merge without losing profit, hence we can assume $P = (N, \emptyset)$. Therefore, players must form the **grand coalition**. In this case, we identify outcomes with payoff vectors for the grand coalition.

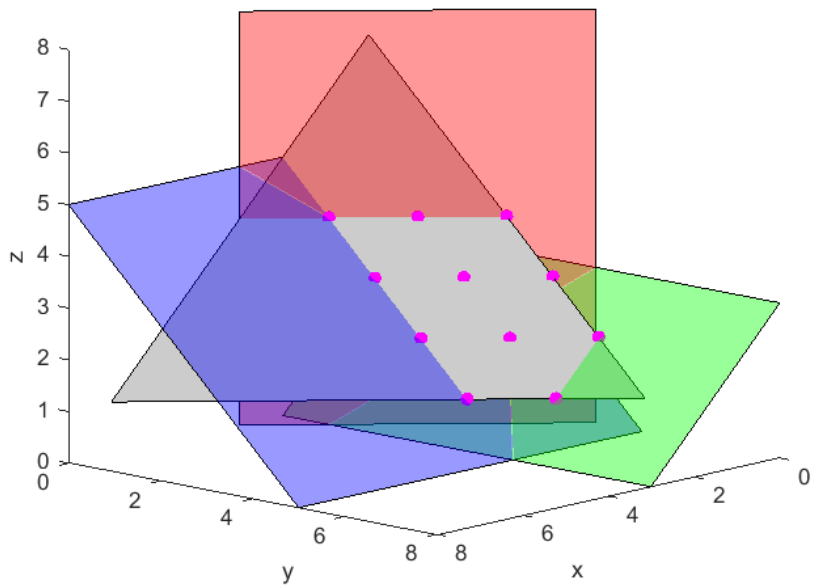

Figure 1: Solution representation

In this case, we can depict the solution visually. We can have points in the plane with three coordinates (x, y, z) representing the payoff vector for players a, b and c respectively. As the value for the grand coalition is 8, then $x+y+z=8$ with $x, z \geq 1$ and $y \geq 0$ (to assure individual rationality). We consider the equilateral triangle (gray region) that verify the previous restrictions, corresponding to the vertex coordinates: $(1, 6, 1)$,$(1, 0, 7)$ and $(7, 0, 1)$.

In addition, we know the following:

- $x + y >= 4$. Otherwise, players a and b would prefer to form their own coalition (red plane).
- $x + z >= 3$. Otherwise, players a and c would prefer to form their own coalition (green plane).
- $y + z >= 5$. Otherwise, players b and c would prefer to form their own coalition (blue plane).

Therefore, we can add the following constraints (where the "limits" are represented by planes) to see which points satisfy them. We can observe that, if we consider only integer payoffs, the solutions (magenta dots) are:

$(1, 3, 4), (1, 4, 3), (1, 5, 2), (2, 2, 4), (2, 3, 3), (2, 4, 2),$
$(2, 5, 1), (3, 1, 4), (3, 2, 3), (3, 3, 2), (3, 4, 1)$

Otherwise, the solutions are the region in the triangle delimited by the planes.