Stochastic Network Modeling (SNM)

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Parts

I. Introduction
II. Discrete Time Markov Chains (DTMC)
III. Continuous Time Markov Chains (CTMC)
IV. Queuing Theory

Part II

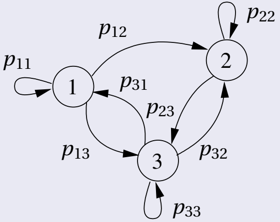
Discrete Time Markov Chains (DTMC)

Outline

Definition of a DTMC

State Transition Diagram

- We are interested in a process that evolve in stages.
- For the model to be tractable, it is convenient to represent the SP by giving all possible states (there may be ∞), and the possible transitions between them:



For the model to be consistent:
$$\sum_{\forall j} p_{ij} = 1$$

- Mathematically:

$$\boxed{p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)}$$

Definition of a DTMC

Properties of a DTMC

- The event $X(n) = i$ (at step n the system is in state i) must satisfy (memoryless property):

$$\boxed{\begin{aligned} P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \cdots) &= \\ P(X(n) = j \mid X(n-1) = i) \end{aligned}}$$

- If $P\big(X(n) = j \mid X(n-1) = i\big) = P\big(X(1) = j \mid X(0) = i\big)$ for any n we have an homogeneous DTMC. We shall only consider homogeneous DTMC.
- We call one-step transition probabilities to:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

Definition of a DTMC

Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

Definition of a DTMC

Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$$

- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) =$$

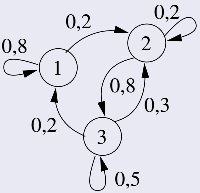
$$\sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = \boxed{1}$$

- $\mathbf{P}$ is a stochastic matrix, i.e. a matrix which rows sum 1.

Definition of a DTMC

Example

- Assume a terminal can be in 3 states:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



$$\mathbf{P} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \text{to state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0{,}8 & 0{,}2 & 0 \\ 0 & 0{,}2 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \\ \\ \text{from} \\ \text{state} \\ \\ \end{array}$$

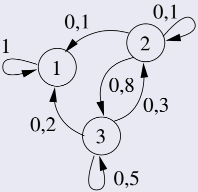
- The average transmission rate (throughput), v_a, is:

$$v_a = P(\text{the terminal is in state 3}) \times v$$

Definition of a DTMC

Absorbing Chains

- It is possible to have chains with absorbing states.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



to state

$$\mathbf{P} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

from state

Definition of a DTMC

n-step transition probabilities

- Transition probabilities: $p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)$
- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- We define the n-step transition probabilities:

$$p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$$

$$\boxed{\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are stochastic matrices: Their rows sum 1.

Definition of a DTMC

State Probabilities

- Define the probability of being in state i at step n:
$$\pi_i(n) = P(X(n) = i)$$
- In vector form (row vector)
$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n) = 1), P(X(n) = 2), \cdots).$$
- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the state probability at step n.

Definition of a DTMC

State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P(X(n)=1), P(X(n)=2), \cdots).$$

- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)$:

$$\pi_i(n) = \sum_k P(X(n-1)=k)\, P\big(X(n)=i \,\big|\, X(n-1)=k\big) = \sum_k \pi_k(n-1)\, p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0)=k)\, P\big(X(n)=i \,\big|\, X(0)=k\big) = \sum_k \pi_k(0)\, p_{ki}(n)$$

- In matrix form:

$$\boxed{\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned}}$$

where $\boldsymbol{\pi}(0)$ is the initial distribution.

Definition of a DTMC

State Probabilities

$$\boxed{\begin{aligned}\boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P}\\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n)\end{aligned}}$$

- Iterating

  $$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P} = \boldsymbol{\pi}(n-2)\,\mathbf{P}\,\mathbf{P} = \boldsymbol{\pi}(n-3)\,\mathbf{P}\,\mathbf{P}\,\mathbf{P} = \cdots = \boldsymbol{\pi}(0)\,\mathbf{P}^n$$

- Thus:

  $$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\; p_{kj}(n-r)}$$

- **Proof**:

$$p_{ij}(n) = P\big(X(n)=j \mid X(0)=i\big) = \sum_k P\big(X(n)=j, X(r)=k \mid X(0)=i\big)$$

$$= \sum_k \frac{P\big(X(n)=j, X(r)=k, X(0)=i\big)}{P(X(0)=i)} \times \frac{P(X(r)=k, X(0)=i)}{P(X(r)=k, X(0)=i)}$$

$$= \sum_k P\big(X(n)=j \mid X(r)=k, X(0)=i\big)\; P\big(X(r)=k \mid X(0)=i\big)$$

$$= \sum_k P\big(X(n)=j \mid X(r)=k\big)\; P\big(X(r)=k \mid X(0)=i\big)$$

$$= \sum_k p_{ik}(r)\, p_{kj}(n-r)$$

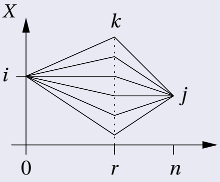
$\square$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{p_{ij}(n) = \sum_k p_{ik}(r)\, p_{kj}(n-r)}$$

- Graphical interpretation:



- In matrix form:

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

Definition of a DTMC

Chapman-Kolmogorov Equations

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)}$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1)\,\mathbf{P}(n-1) = \mathbf{P}\,\mathbf{P}(n-1) = \mathbf{P}(n-1)\,\mathbf{P}$$

- Iterating:

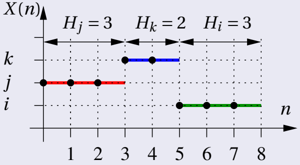
$$\boxed{\mathbf{P}(n) = \mathbf{P}^n}$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n}$$

Sojourn or Holding Time

- Sojourn or holding time in state k: Is the RV H_k equal to the number of steps that the chain remains in state k before leaving to a different state:



- The Markov property implies:

$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

- Which is a geometric distribution with mean:

$$\mathrm{E}[H_i] = \sum_{n=1}^{\infty} n P(H_i = n) = \frac{1}{1 - p_{ii}}.$$

Definition of a DTMC

Sojourn or Holding Time

- NOTE: We allow that:

$$p_{ii} = 0 \Rightarrow H_i(n) = I(n = 1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases}, \text{ and}$$

$$p_{ii} = 1 \Rightarrow \mathrm{E}[H_i] = \infty \text{ (absorbing state).}$$

Definition of a DTMC

Theorem

A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.

Proof.

- We have seen that a DTMC has a sojourn time

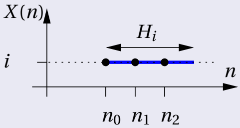
$$\boxed{H_i(n) = P(H_i = n) = p_{ii}^{n-1}(1 - p_{ii}),\ n \geq 1}$$

- Which is geometrically distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).

□

Definition of a DTMC

The geometric distribution satisfies the Markov property (1)



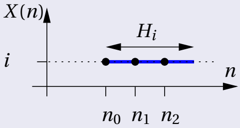
Proof

- Markov property:
  $P\big(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\big) = P\big(X(n_2) = i \mid X(n_1) = i\big)$
- Thus, the Markov property in terms of the sojourn time can be written as:

$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

Definition of a DTMC

The geometric distribution satisfies the Markov property (2)



$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = P(H_i > n_2 - n_1)$$

- Since

$$P(H_i > k) = 1 - P(H_i \le k) = 1 - \sum_{n=1}^{k} p^{n-1}(1-p) = 1-(1-p)\frac{1-p^k}{1-p} = p^k$$

- We have:

$$P\big(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\big) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$

$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2-n_0}}{p^{n_1-n_0}} = p^{n_2-n_1} = P(H_i > n_2 - n_1) \quad \square$$