Solution

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Problem 1

Consider a system where a task can be in 3 stages (see figure 1): preparation, CPU and DISK. In each stage during a time exponentially distributed with parameters λ, μ_1 and μ_2, respectively. Upon finishing the CPU, the task may go to the DISK stage, finish and go to preparation stage, or repeat a CPU stage with probabilities α_1, α_2, $1-\alpha_1-\alpha_2$, respectively.

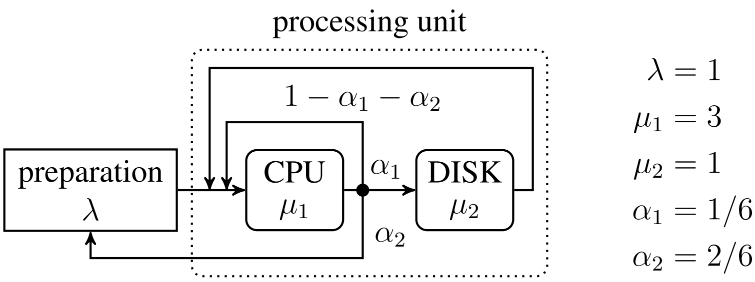


Figure 1: System.

Compute the following. For the Markov chains used to solve the questions say clearly the states and their meaning.

1.A (2 points) Throughput, that is, number of tasks that finish the processing unit per time unit.

1.B (2 points) Average time that the task is in the processing unit.

1.C (2 points) Let T be the random variable equal to the time that a task is in the processing unit. Compute the distribution, $P(T \leq t)$.

1.D (2 points) Assume that the CPU consumes 1 W while a task is running (in the CPU), and 0 W otherwise. The DISK consumes 2 W while a task is running (in the DISK), and 0 W otherwise. Compute the expected energy (in Watts$\times$ time unit) consumed by the task each time it visits the processing unit.

1.E (2 points) Average time that the tasks that visit the DISK are in the CPU before visiting the DISK for the first time.

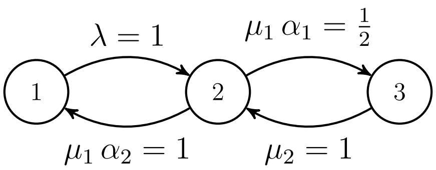
Hint: When the task enters the CPU the system is equivalent to trigger 3 exponentials with rates $\mu_1\,\alpha_1$, $\mu_1\,\alpha_2$ and $\mu_1\,(1-\alpha_1-\alpha_2)$. Upon expiring the first of them, the task moves to the DISK, preparation or CPU, respectively.

Solution

Problem 1

1.A Consider a chain with states (see figure 3):

① task in preparation phase
② task running in CPU
③ task running in DISK



$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 1 & -1 \end{bmatrix}$$

Figure 2: CTMC of the system.

The chain is reversible. Thus, the stationary distribution is given by:

$$\pi_1 = \frac{1}{G}$$
$$\pi_2 = \frac{1}{G}\,\frac{1}{1}$$
$$\pi_3 = \frac{1}{G}\,\frac{1}{1}\,\frac{1/2}{1}$$

which yields $\pi_1 = 2/5$, $\pi_2 = 2/5$, $\pi_3 = 1/5$.

The throughput is the number of tasks leaving the preparation stage per time unit (which must be equal to the tasks leaving the processing unit per time unit). This is the proportion of time the system is in state 1, times the arrival rate in state 1. That is, $S = \lambda\,\pi_1 = 2/5$ tasks/t.u.

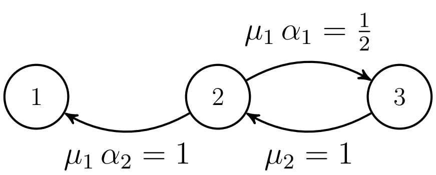
Check: the number of tasks leaving the processing unit per time unit is $\pi_1 \times \mu_1 \times \alpha_2 = 2/5 \times 3 \times 2/6 = 2/5$ tasks/t.u. as expected.

1.B The average time that the task is in the processing unit is the mean recurrent time m_{21}:

$$m_{21} = \frac{1}{3/2} + \frac{1/2}{3/2}\,m_{31}$$
$$m_{31} = 1 + m_{21}$$

which gives $m_{21} = 3/2$ t.u.
Check: it must be $S = 2/5 = 1/(1/\lambda + m_{21}) = 1/(1 + m_{21})$, which yields $m_{21} = 3/2$ t.u. as expected.

1.C Consider the absorbing chain:



$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 1 & -1 \end{bmatrix}$$

Figure 3: CTMC of the system.

with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -1/2$, $\lambda_2 = -2$. We have: $\pi_1(t) = 1 + a\,\mathrm{e}^{-t/2} + b\,\mathrm{e}^{-2t}$. Imposing $\pi_1(0) = 0$ and $\pi_1'(0) = 1$ we get

$$P(T \leq t) = \pi_1(t) = 1 - \frac{2}{3}\,\mathrm{e}^{-t/2} - \frac{1}{3}\,\mathrm{e}^{-2t}.$$

Check:

$$\mathrm{E}[T] = \int_0^\infty (1 - \pi_1(t))\,dt = \frac{2}{3}\frac{1}{1/2} + \frac{1}{3}\frac{1}{2} = \frac{3}{2} = m_{21}$$

as expected.

1.D Assume the absorbing chain of item 1.C. Let n_{ij} be the expected time in state j stating from state i until absorbtion. We have

$$\begin{aligned} n_{22} &= \frac{1}{3/2} + \frac{1/2}{3/2} n_{32} \\ n_{32} &= n_{22} \end{aligned}$$

which gives $n_{22} = 1$ t.u. and

$$\begin{aligned} n_{23} &= \frac{1/2}{3/2} n_{33} \\ n_{33} &= 1 + n_{23} \end{aligned}$$

which gives $n_{23} = 1/2$ t.u. Thus, the expected energy is

$$E = 1\text{ W} \times 1\text{ t.u.} + 2\text{ W} \times 1/2\text{ t.u.} = 2\text{ W t.u.}$$

1.E In the following 3 different reasoning are provided:

i. Consider the embedded discrete chain obtained looking at the task upon entering the CPU until it moves to the preparation or DISK stages:

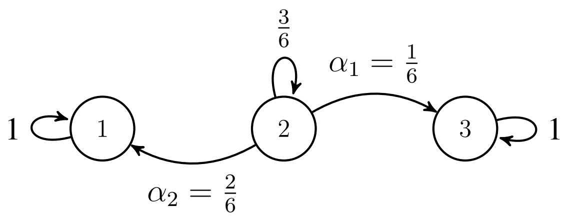
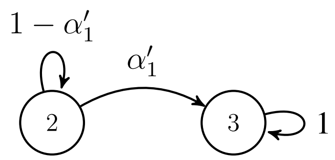


Figure 4: Unconditioned chain.

If we know that the task visits the DISK, we have to consider the chain:



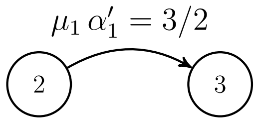
where

$$\alpha_1' = P(X(n+1) = 3 \mid X(n) = 2,\, X(\infty) = 3)$$

in the chain of figure 4. We have:

$$\begin{aligned} \alpha_1' = P(X(n+1) = 3 \mid X(n) = 2,\, X(\infty) = 3) = \\ \frac{P(X(n+1) = 3, X(n) = 2,\, X(\infty) = 3)}{P(X(n) = 2,\, X(\infty) = 3)} = \\ \frac{P(X(n+1) = 3, X(n) = 2)}{P(X(n) = 2,\, X(\infty) = 3)} = \frac{\alpha_1}{\alpha_1/(\alpha_1 + \alpha_2)} = \alpha_1 + \alpha_2 = \frac{1}{2} \end{aligned}$$

since $P(X(n) = 2,\, X(\infty) = 3) = f_{23} = \alpha_1 + (1 - \alpha_1 - \alpha_2)\, f_{23}$, so $f_{23} = \alpha_1/(\alpha_1 + \alpha_2)$. The chain in continuous time is:



Thus, the requested time is $1/(\mu_1\, \alpha_1') = 2/3$ t.u.

ii. An equivalent reasoning, using basic probability:

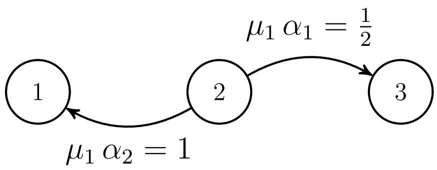
$$\alpha_1' = P(\text{move from CPU to DISK} \mid \text{end in DISK}) = \frac{P(\text{move from CPU to DISK} \cap \text{end in DISK})}{P(\text{end in DISK})} = \frac{P(\text{move from CPU to DISK})}{P(\text{end in DISK})} = \frac{\alpha_1}{P(\text{end in DISK})}$$

but, we move to DISK in n steps if we move $n-1$ times to CPU and then we jump to DISK:

$$P(\text{end in DISK}) = \sum_{n=1}^{\infty} (1 - \alpha_1 - \alpha_2)^{n-1}\, \alpha_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

So, we get $\alpha_1' = \frac{\alpha_1}{\alpha_1/(\alpha_1+\alpha_2)} = \alpha_1 + \alpha_2$, as expected.

iii. Regardless whether the task moves to the preparation or DISK for the first time, the average time in the CPU will be the sojourn time in the CPU until it moves out of the CPU for the first time. In other words, the sojourn time in state 2 of the following absorbing CTMC:



which is

$$\mathrm{E}[H_2] = \frac{1}{\mu_1\, (\alpha_1 + \alpha_2)} = 2/3 \text{ t.u.}$$

as expected.