

Simple Games

Fall 2020

- 1 Simple Games
- 2 Problems on simple games
- 3 IsWeighted

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- A **simple game** is a cooperative game (N, v) such that $v : \mathcal{C}_N \rightarrow \{0, 1\}$ and it is monotone.
- A simple game can be described by a pair (N, \mathcal{W}) :
 - N is a set of players,
 - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of **winning coalitions**, those coalitions X with $v(X) = 1$.
 - $\mathcal{L} = \mathcal{C}_N \setminus \mathcal{W}$ is the set of **losing coalitions** those coalitions X with $v(X) = 0$.

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- Members of $N = \{1, \dots, n\}$ are called **players** or **voters**.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N :

- *winning coalitions* \mathcal{W} .
- *losing coalitions* \mathcal{L} .
- *minimal winning coalitions* \mathcal{W}^m
$$\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$$
- *maximal losing coalitions* \mathcal{L}^M
$$\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq Z\}$$

This provides us with many representation forms for simple games.

Weighted voting games

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A simple game for which there exists a **quota** q and it is possible to assign to each $i \in N$ a **weight** w_i , so that

$$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$

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$$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$

- WVG can be represented by a tuple of integers

$(q; w_1, \dots, w_n)$.

as **any weighted game admits such an integer realization**,

[Carreras and Freixas, Math. Soc.Sci., 1996]

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Both are simple games

- A simple game Γ is a **vector weighted voting game** if there are WVGs $\Gamma_1, \dots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \dots \cap \Gamma_k$.

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- Assume it is given by $(q; w_1, w_2, w_3, w_4)$.
- We have $w_1 + w_2 \geq q$ and $w_3 + w_4 \geq q$.
- Thus $\max\{w_1, w_2\} \geq q/2$ and $\max\{w_3, w_4\} \geq q/2$,
- So, $\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q$ which cannot be.

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 - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
Any set that is not contained in C wins!
 - The intersection of the above games describes Γ .
A winning coalition cannot be a subset of any losing coalition.
- The **dimension** of a simple games is the minimum number of WVGs that allows its representation as VWVG

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A representation as WVGs

The game Γ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$
- This gives four WVG, according to the previous construction

$$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0].$$

Input representations

- Simple Games
 (N, \mathcal{W}) : extensive winning, (N, \mathcal{W}^m) : minimal winning
 (N, \mathcal{L}) : extensive losing, (N, \mathcal{L}^M) maximal losing
 (N, C) : monotone circuit winning
 (N, F) : monotone formula winning,
- Weighted voting games: $(q; w_1, \dots, w_n)$
- Vector weighted voting games:
 $(q_1; w_1^1, \dots, w_n^1), \dots, (q_k; w_1^k, \dots, w_n^k)$

All numbers are integers

The core of simple games

The core of simple games

- It is standard to assume that the grand coalition forms, even if the simple game is not superadditive.
- A player is a **veto player** if $v(C) = 0$, for any $C \subseteq N \setminus \{i\}$.
- Ex: Consider the unanimity game (N, v) where $v(C) = 0$, if $C \neq N$ and $v(N) = 1$.

The game indeed is a simple game and can be described in (minimal) winning form by $(N, \{N\})$.

In the unanimity game all players are veto players.

The core of simple games

Theorem

A simple game has non-empty core iff it has a veto player.

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A simple game has non-empty core iff it has a veto player.

- If Γ has a veto player i .
 - Consider the payoff $x_i = 1$ and $x_j = 0$, for $j \neq i$
 - For C with $i \in C$, $v(C) = 1$ and $x(C) = 1$.
 - For C with $i \notin C$, $v(C) = 0$ and $x(C) = 0$.
 - Thus, x is in the core.
- If Γ does not have a veto player and non-empty core.
 - Consider a payoff x that is in the core.
 - $x(N) = v(N) = 1$, so there exists i with $x_i > 0$.
 - So, $x(N \setminus \{i\}) < 1$. But, $v(N \setminus \{i\}) = 1$ as i is not a veto player.
 - Thus, x is not in the core.

Is the core empty?

- Determining if the core is empty or not can be done by checking for every player whether it is a veto player or not.
- For this it is enough to check whether $v(N \setminus \{i\}) = 0$.
- For reasonable v , polynomial time computable, this can be done in poly time

Shapley value and Banzhaf index

- Player i is **pivotal** for coalition C if $v(C) = 1$ and $v(C \setminus \{i\}) = 0$.
- The sum counts those the terms for which the player is pivotal.

$$\varphi_i(\Gamma) = \frac{1}{n!} |\{i \text{ is pivotal for } S_\pi(i)\}|$$

- $\varphi_i(\Gamma)$ is the probability that the arrival of player i turns a losing coalition into a winning one.
- The Banzhaf value gives the probability of this fact over **random** coalitions.
Players in $N \setminus \{i\}$ select to be or not in the coalition tossing a fair coin.

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Problems on simple games

In general we state a property P , for simple games, and consider the associated decision problem which has the form:

Name: IsP

Input: A simple game/WVG/VWVG Γ

Question: Does Γ satisfy property P ?

Four properties

A simple game (N, \mathcal{W}) is

- **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- **proper** if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a **weighted voting game**.
- a **vector weighted voting game**.

IsStrong: Simple Games

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Theorem

The ISSTRONG problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.

- First observe that, given a family of subsets F , we can check, for any set in F , whether its complement is not in F in polynomial time.
- Therefore, the ISSTRONG problem, when the input is given in explicit losing form is polynomial time solvable.

IsStrong: Simple Games losing forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

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- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \wedge N \setminus S \subseteq L_2$$

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- which is equivalent to there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.

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- This can be checked in polynomial time, given \mathcal{L}^M .



IsStrong: explicit winning forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given (N, \mathcal{W}) , for $i \in N$ consider the family $\mathcal{W}_i = \{X \setminus \{i\} \mid X \in \mathcal{W}\}$ and $R = \cup_{i \in N} \mathcal{W}_i$.
- All the coalitions in $R \setminus \mathcal{W}$ are losing coalitions.
- Furthermore for a coalition $X \in \mathcal{L}^M$ and $i \notin X$, $X \cup \{i\} \in \mathcal{W}$.
- Therefore, $\mathcal{L}^M \subseteq R \setminus \mathcal{W}$ and $(R \setminus \mathcal{W})^M = \mathcal{L}^M$.
- Then, we compute \mathcal{L}^M from \mathcal{W} in polynomial time and then use the algorithm for the maximal losing form.

IsStrong: minimal winning forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The ISSTRONG problem is coNP-complete when the input game is given in explicit minimal winning form.

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The ISSTRONG problem is coNP-complete when the input game is given in explicit minimal winning form.

- The property can be expressed as

$$\forall S [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and \mathcal{W}^m .
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete **set splitting** problem.
- An instance of the **set splitting problem** is a collection C of subsets of a finite set N . The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

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- We have to decide whether $P \subseteq N$ exists such that

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We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m) .

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- Now assume that $P \subseteq N$ satisfies

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- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m) .

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- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m) .
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m .
- Therefore, (N, C) has a set splitting iff (N, C^m) is not strong.

IsProper: winning forms

Γ is **proper** if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The ISPROPER problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.

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The ISPROPER problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.

- As before, given a family of subsets F , we can check, for any set in F , whether its complement is not in F in polynomial time.

Taking into account the definitions, the ISPROPER problem is polynomial time solvable for the explicit forms

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- equivalent to there are two minimal winning coalitions W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$.
- Which can be checked in polynomial time when \mathcal{W}^m is given.

IsProper: maximal losing form

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Theorem

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- A game is *not proper* iff

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- Therefore ISPROPER belongs to coNP.

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- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

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- Which can be obtained in polynomial time.
- Besides, Γ is strong iff Γ' is proper.

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Input: n integer values, x_1, \dots, x_n

Question: Is there $S \subseteq \{1, \dots, n\}$ for which

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i.$$

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Observe that, for any instance of the PARTITION problem in which the sum of the n input numbers is odd, the answer must be NO.

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Theorem

The ISSTRONG and the ISPROPER problems, when the input is described by an integer realization of a weighted game $(q; w)$, are coNP-complete.

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Theorem

The ISSTRONG and the ISPROPER problems, when the input is described by an integer realization of a weighted game $(q; w)$, are coNP-complete.

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation $(2; 1, 1, 1)$ is both proper and strong.

Hardness

We transform an instance $x = (x_1, \dots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \dots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

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$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \dots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

- Function f can be computed in polynomial time provided q does.
- Independently of q , when $x_1 + \dots + x_n$ is *odd*, x is a NO input for partition, but $f(x)$ is a YES instance of ISSTRONG or ISPROPER.

IsStrong

Assume that $x_1 + \dots + x_n$ is even.

Let $s = (x_1 + \dots + x_n)/2$ and $N = \{1, \dots, n\}$.

Set $q(x) = s + 1$.

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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.

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- If S and $N \setminus S$ are losing coalitions in $f(x)$.
If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning.
Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of x .

IsProper

Assume that $x_1 + \dots + x_n$ is even.

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Set $q(x) = s$.

IsProper

Assume that $x_1 + \dots + x_n$ is even.

Let $s = (x_1 + \dots + x_n)/2$ and $N = \{1, \dots, n\}$.

Set $q(x) = s$.

- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.

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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.
- When $f(x)$ is not proper

$$\exists S \subseteq N : \sum_{i \in S} x_i \geq s \wedge \sum_{i \notin S} x_i \geq s,$$

and thus $\sum_{i \in S} x_i = s$.

- 1 Simple Games
- 2 Problems on simple games
- 3 IsWeighted**

Explicit forms

Lemma

The ISWEIGHTED problem can be solved in polynomial time when the input game is given in explicit winning or losing form.

Explicit forms

Lemma

The ISWEIGHTED problem can be solved in polynomial time when the input game is given in explicit winning or losing form.

We can obtain \mathcal{W}^m and \mathcal{L}^M in polynomial time.

Once this is done we write, in polynomial time, the LP

$$\begin{array}{ll} \min q & \\ \text{subject to} & w(S) \geq q \quad \text{if } S \in \mathcal{W}^m \\ & w(S) < q \quad \text{if } S \in \mathcal{L}^M \\ & 0 \leq w_i \quad \text{for all } 1 \leq i \leq n \\ & 0 \leq q \end{array}$$

IsWeighted: Minimal and Maximal

Lemma

The ISWEIGHTED problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

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- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the i 'th coordinate equal to 1 if and only if $i \in C$.

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Lemma

The ISWEIGHTED problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the i 'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\bigwedge_{i \in S} x_i)$$

IsWeighted: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

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- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

IsWeighted: Maximal losing

- we can prove a similar result given (N, L^M) .
- The **dual** of game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where

$$S \in \mathcal{W}^d \text{ iff } N \setminus S \notin \mathcal{W}.$$
- Observe that Γ is weighted iff Γ^d is weighted.
- We can compute a monotone CNF formula describing the losing coalitions of Γ . Negating this formula we get a DNF on negated variables. Replacing \bar{x}_i by y_i we get a DNF describing \mathcal{W}^d .
- As the formula can be computed in polynomial time the result follows.

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