Epidemic models (part II)
Search in networks

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Recap: Homogeneous SIS model

Equations of dynamics

$$\frac{ds}{dt} = \gamma x - \beta\langle k\rangle sx \qquad \frac{dx}{dt} = \beta\langle k\rangle sx - \gamma x$$

Solution

$$x(t) = x_0 \frac{(\beta\langle k\rangle - \gamma)e^{(\beta\langle k\rangle - \gamma)t}}{\beta\langle k\rangle - \gamma + \beta\langle k\rangle x_0 e^{(\beta\langle k\rangle - \gamma)t}}$$

Observations

- Same behavior as in the non-networked model
- Epidemic threshold at $\beta\langle k\rangle - \gamma = 1$
  - Equivalent to $\frac{\beta}{\gamma} \leqslant \frac{1}{\langle k\rangle}$, same as SIR

The scale-free model of epidemics for SIS I

From [Pastor-Satorras and Vespignani, 2001]

Instead of assuming homogeneous mixing, have a different equation for all nodes of same degree k:

$$\frac{dx_k}{dt} = \beta k(1 - x_k)\Theta(\beta) - \gamma x_k$$

where

- $(1 - x_k)$ is the probability that a node of degree k is not infected
- $\Theta(\beta)$ is the probability that a neighbor is infected
- $\beta k\Theta(\beta)$ is the probability of contagion of a k-degree node from an infected neighbor

The scale-free model of epidemics for SIS II

From [Pastor-Satorras and Vespignani, 2001]

Imposing stationarity ($\frac{dx_k}{dt} = 0$, for all k), we obtain

$$x_k = \frac{k\beta\Theta(\beta)}{\gamma + k\beta\Theta(\beta)}$$

and so nodes with higher degree are more susceptible to being infected. W.l.o.g. may assume $\gamma = 1$.

The probability that any edge points to an s-degree node is proportional to $sP(s)$, and by def. $\sum_s sP(s) = \langle k \rangle$. Therefore

$$\Theta(\beta) = \sum_k \frac{kP(k)x_k}{\sum_s sP(s)} = \frac{1}{\langle k \rangle} \sum_k kP(k)x_k$$

The scale-free model of epidemics for SIS III

From [Pastor-Satorras and Vespignani, 2001]

Plug in the expression for x_k to obtain

$$\Theta(\beta) = \frac{1}{\langle k \rangle} \sum_k kP(k) \frac{k\beta\Theta(\beta)}{1 + k\beta\Theta(\beta)}$$

A non-zero stationary prevalence ($x_k \neq 0$) is obtained when both sides of previous eq., taken as funct. of Θ, cross in $0 < \Theta \leqslant 1$. This corresponds to

$$\frac{d}{d\Theta} \left(\frac{1}{\langle k \rangle} \sum_k kP(k) \frac{k\beta\Theta}{1 + k\beta\Theta} \right) \Bigg|_{\Theta=0} \geqslant 1$$

The critical epidemic threshold β_c is the value β which yields equality above. This is given by

$$\frac{1}{\langle k \rangle} \sum_k kP(k) \beta_c k = \frac{\langle k^2 \rangle}{\langle k \rangle} \beta_c = 1$$

The scale-free model of epidemics for SIS IV

From [Pastor-Satorras and Vespignani, 2001]

Hence, $\beta = \dfrac{\langle k \rangle}{\langle k^2 \rangle}$

This implies that in scale-free networks, for which $\langle k^2 \rangle \to \infty$, we have $\beta_c = 0$.

So, there is no epidemic threshold for (infinite) scale-free networks.

In practice, the epidemic threshold in scale-free networks is going to be very small. As a consequence, viruses can spread and proliferate at any rate. However, this spreading rate is in general exponentially small.

(In the scale-free model of [Barabasi and Albert, 1999] we have $P(k) = 2m^2/k^3$ and so we obtain in this case (w.l.o.g. $\gamma = 1$)

$$\Theta(\beta) = \frac{e^{-\frac{1}{m\beta}}}{\beta m} \qquad \text{and} \qquad x \approx 2e^{-\frac{1}{m\beta}} \qquad)$$

SEARCH

Milgram's small-world experiment

[Milgram, 1967, Travers and Milgram, 1969]

Instructions

Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is "closest" to the target

Outcome

20% of initiated chains reached target average chain length = 6.5

Small-world experiment revisited
[Dodds et al., 2003]

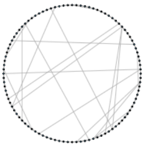
We report on a global social-search experiment in which more than **60,000 e-mail users** *attempted to reach one of* **18 target persons** *in* **13 countries** *by forwarding messages to acquaintances. We find that successful social search is conducted primarily through intermediate to weak strength ties, does not require highly connected "hubs" to succeed, and, in contrast to unsuccessful social search, disproportionately relies on professional relationships. By accounting for the attrition of message chains, we estimate that social searches can reach their targets in a median of* **five to seven steps**, *depending on the separation of source and target, although small variations in chain lengths and participation rates generate large differences in target reachability. We conclude that although global social networks are, in principle, searchable, actual success depends sensitively on individual incentives.*

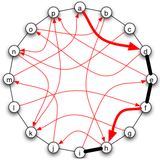
Reflections on Milgram's experiment

1. Short paths exist between random pairs of people ("six degrees of separation")
   - Explained by models with small diameter, e.g. Watts-Strogatz model
2. With little "local" information, people are **able to find them**
   - [Kleinberg, 2000b]: *The success of Milgram's experiment suggests a source of latent navigational "cues" embedded in the underlying social network, by which a message could implicitly be guided quickly from source to target. It is natural to ask what properties a social network must possess in order for it to exhibit such cues, and enable its members to find short chains through it.*

Reproducing Milgram's result: Kleinberg's model
[Kleinberg, 2000b, Kleinberg, 2000a]

Variation on Watts-Strogatz small-world model[1]





- n nodes arranged on a ring
- each node connects to immediately adjacent nodes
  - mimics "local" information
- each node has an additional long-range **shortcut**
  - $Prob(\text{shortcut from } u \text{ to } v) \propto d(u,v)^{-\alpha}$
  - α is a parameter, the "clustering exponent"
    - if $\alpha = 0$, like WS model
    - if $\alpha > 0$, preference for *closer* nodes

[1] Originally defined on a 2D grid, here explained with 1D ring for simplicity.

Myopic search

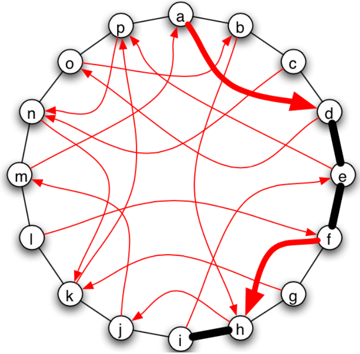
Goal: to show that networks are navigatable: for any i, j, there are shortest paths and to find them one do not need to know all the network, but only use *local information*.

Given a source node s and a destination node t, the decentralized algorithm works as follows:

1. Each node has a coordinate and knows its position on the ring, included the positions of s and t ("geographical" information)
2. Each node knows its neighbors and its shortcut ("local" information)
3. Each node forwards the "message" *greedily*, each time moving as close to t as possible

Myopic search

Example



- Source $s = a$; destination $t = i$
- Myopic search selects path $a - d - e - f - h - i$ (length 5)
- Shortest path is $a - b - h - i$ (length 3)

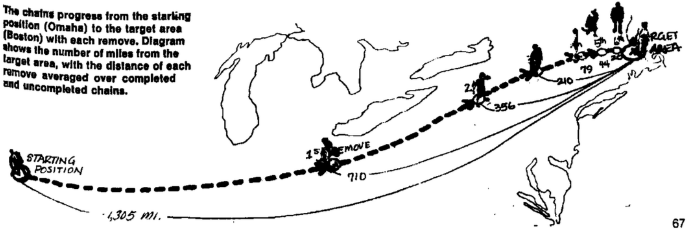
For $\lim_{n\to\infty}$, the expected number of steps needed to reach target $E[X]$, is:

$$E[X] = \begin{cases} \Omega(n^{1-\alpha}) & \alpha < 1 \\ O(\log^2 n) & \alpha = 1 \\ \Omega(n^{\alpha-1}) & \alpha > 1 \end{cases}$$

Fast myopic search with $\alpha = 1$

Intuition

From [Milgram, 1967]:



"Track how long it takes to for the message to reduce its distance by factors of 2"

- X_j is the nr. of steps taken in *phase j*
- Phase j: portion of the search in which message is at distance between 2^j and 2^{j+1}
- Will show that $E[X_j] = O(\log(n))$ for each j

$$E[X] = E[X_1] + E[X_2] + .. + E[X_{\log n}]$$

Normalizing constant for $\alpha = 1$

What is the probability distribution, exactly?

$$P[\text{shortcut from } u \text{ to } v] \propto \frac{1}{d(u,v)}$$

Need to figure out normalizing constant $Z = \sum_v \frac{1}{d(u,v)}$ for the distribution of shortcuts for node u.

Fix arbitrary node u. Then, there are 2 nodes at distance 1, 2 at distance 2, and in general 2 at each distance up to $n/2$:

$$\begin{aligned} Z &= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n/2}\right) \\ &\leqslant 2(1 + \ln(n/2)) \\ &\leqslant 2(1 + \log_2(n/2)) \\ &= 2\log_2(n) \end{aligned}$$

Halving the distance to destination is quick for $\alpha = 1$ I

- Assume we are at node v in phase j, somewhere at distance d from destination for $2^j \leqslant d \leqslant 2^{j+1}$
- If current node v has shortcut to node at distance at most $d/2$, then we are done (since current node takes shortcut and search leaves phase j and goes into phase $j-1$ or better)
- There are $d+1$ nodes at distance $d/2$ from destination; these nodes are at distance at most $d + d/2 = 3d/2$ from v
- Probability of v having a shortcut to any one of these $d+1$ nodes (call it w) is $Prob[v \text{ shortcuts to } w] = \frac{1}{Z}\frac{1}{d(v,w)} \geqslant \frac{1}{2\log(n)}\frac{1}{3d/2} = \frac{1}{3d\log(n)}$
- The probability that v shortcuts to *any* one of them is at least $\frac{1}{3\log(n)}$ (since there are $d+1$ of them)

Halving the distance to destination is quick for $\alpha = 1$ II

- After each step, the probability of leaving phase j is at least $\frac{1}{3\log(n)}$ so the probability of staying in phase j for i steps is at least $(1 - \frac{1}{3\log(n)})^{i-1}$ and so $P[X_j \geqslant i] \leqslant (1 - \frac{1}{3\log(n)})^{i-1}$
- Now,

$$
\begin{aligned}
E[X_j] &= \sum_{k \geqslant 0} k \times P[X_j = k] \\
&= 1P[X_j = 1] + 2P[X_j = 2] + 3P[X_j = 3] + .. \\
&= P[X_j \geqslant 1] + P[X_j \geqslant 2] + P[X_j \geqslant 3] + .. \\
&\leqslant 1 + \left(1 - \frac{1}{3\log(n)}\right)^1 + \left(1 - \frac{1}{3\log(n)}\right)^2 + .. \\
&= 3\log(n)
\end{aligned}
$$

where the last step is due to the geometric series $\frac{1}{1-x} = \sum_{n \geqslant 0} x^n$ Q.E.D.

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