The Simplex Method

Combinatorial Problem Solving (CPS)

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April 3, 2020

Global Idea

- The Fundamental Theorem of Linear Programming ensures
  it is sufficient to explore basic feasible solutions
  to find the optimum of a feasible and bounded LP

- The simplex method moves from one basic feasible solution
  to another that does not worsen the objective function while

  - optimality or
  - unboundedness

  are not detected

Bases and Tableaux

- Given a basis B, its tableau is the system of equations

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

which expresses values of basic variables in terms of non-basic variables

$$\begin{array}{l} \min -x - 2y \\ x + y + s_1 = 3 \\ x + s_2 = 2 \\ y + s_3 = 2 \\ x, y, s_1, s_2, s_3 \geq 0 \end{array}$$

$$\mathcal{B} = \{x, y, s_2\}$$

$$\begin{cases} x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

Basic Solution in a Tableau

- The basic solution can be easily obtained from the tableau by looking at independent terms

$$\begin{cases} x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

Note that by definition of basic solution,
the values for non-basic variables are null

Detecting Optimality (1)

- Tableaux can be extended with
  the expression of the cost function in terms of the non-basic variables

$$\begin{cases} \min -x - 2y \Longrightarrow \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

- Value of objective function at basic solution
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- Coefficients of non-basic variables in objective function after substitution
  are called reduced costs
- By convention, reduced costs of basic variables are 0

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- Value of objective function at basic solution
  can be easily found by looking at independent term
- Coefficients of non-basic variables in objective function after substitution are called reduced costs
- By convention, reduced costs of basic variables are 0
- Sufficient condition for optimality: all reduced costs are ≥ 0
  The cost of any other feasible solution can't improve on the basic solution
  So the basic solution is optimal!

Detecting Optimality (2)

- If reduced costs ≥ 0:
  sufficient condition for optimality but not necessary

- In the example, both bases are optimal
  but in one we cannot detect optimality!

$$\begin{array}{l} \min x + 2y \\ x + y = 0 \\ x, y \geq 0 \end{array} \qquad \begin{array}{c} \mathcal{B} = \{x\} \\ \\ \left\{ \begin{array}{l} \min \;\; y \\ x = -y \end{array} \right. \end{array} \qquad \begin{array}{c} \mathcal{B} = \{y\} \\ \\ \left\{ \begin{array}{l} \min \;\; -x \\ y = -x \end{array} \right. \end{array}$$

Improving the Basic Solution

- What to do when the tableau does not satisfy the optimality condition?

$$\begin{aligned} &\min -x - 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

$$\mathcal{B} = (s_1, s_2, s_3)$$

$$\begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$$

- E.g. variable y has a negative reduced cost
- If we can get a new solution where $y > 0$ and the rest of non-basic variables does not worsen the objective value, we will get a better solution
- In general, to improve the objective value: increase the value of a non-basic variable with negative reduced cost while the rest of non-basic variables are frozen to 0

  E.g. increase y while keeping $x = 0$

Improving the Basic Solution

- Let us increase value of variable y
while satisfying non-negativity constraints on basic variables

$$\left\{ \begin{array}{ll} s_1 = 3 - x - y & \text{Limits new value to } \leq 3 \\ s_2 = 2 - x & \text{Does not limit new value} \\ s_3 = 2 - y & \text{Limits new value to } \leq 2 \end{array} \right.$$

- Best possible new value for y is $\min(3, 2) = 2$

- The bound due to s_3 is tight, i.e.,
the constraint $s_3 \geq 0$ limits the new value for y

Improving the Basic Solution

- The new solution does not seem to be basic... but in fact it is.
We only need to change the basis.

- When increasing the value of the improving non-basic variable, all basic variables for which the bound is tight become 0

$$y = 2 \rightarrow s_3 = 0$$

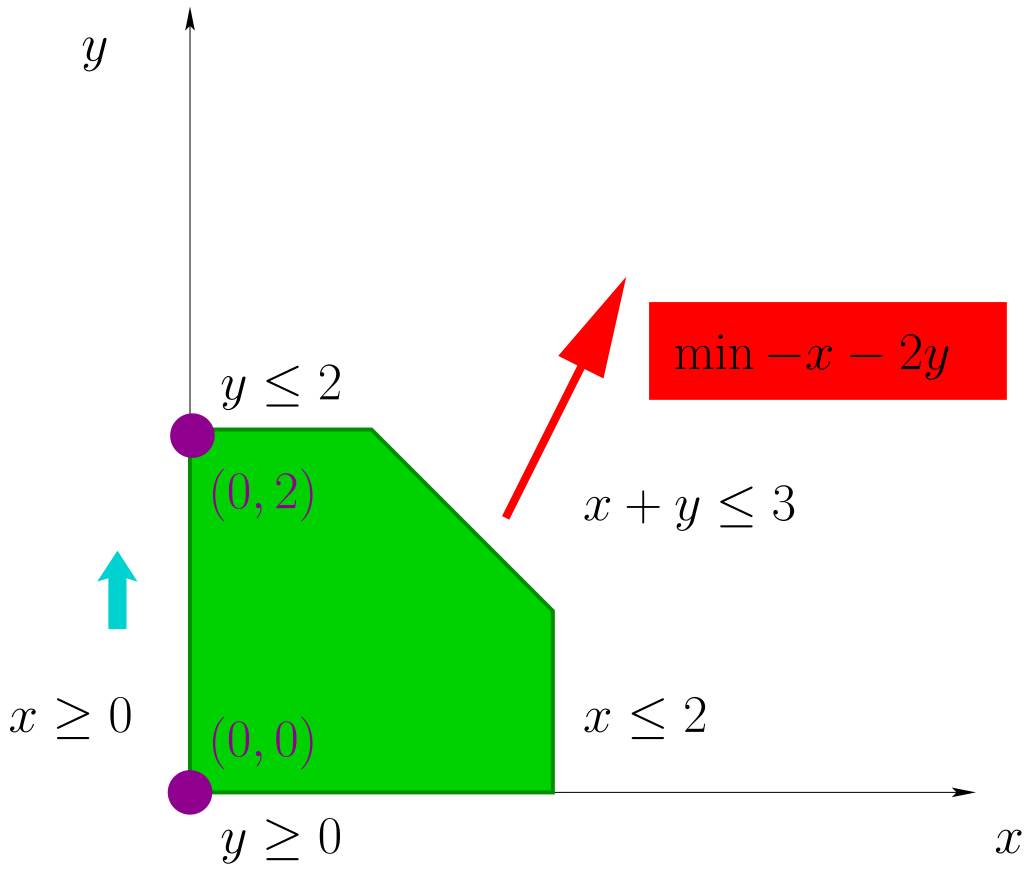
- Choose a tight basic variable, here s_3, to be exchanged with the improving non-basic variable, here y

- We can get the tableau of the new basis by solving the non-basic variable in terms of the basic one and substituting:

$$s_3 = 2 - y \quad \Rightarrow \quad y = 2 - s_3$$

$$\begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \min \ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases}$$

Improving the Basic Solution



Improving the Basic Solution

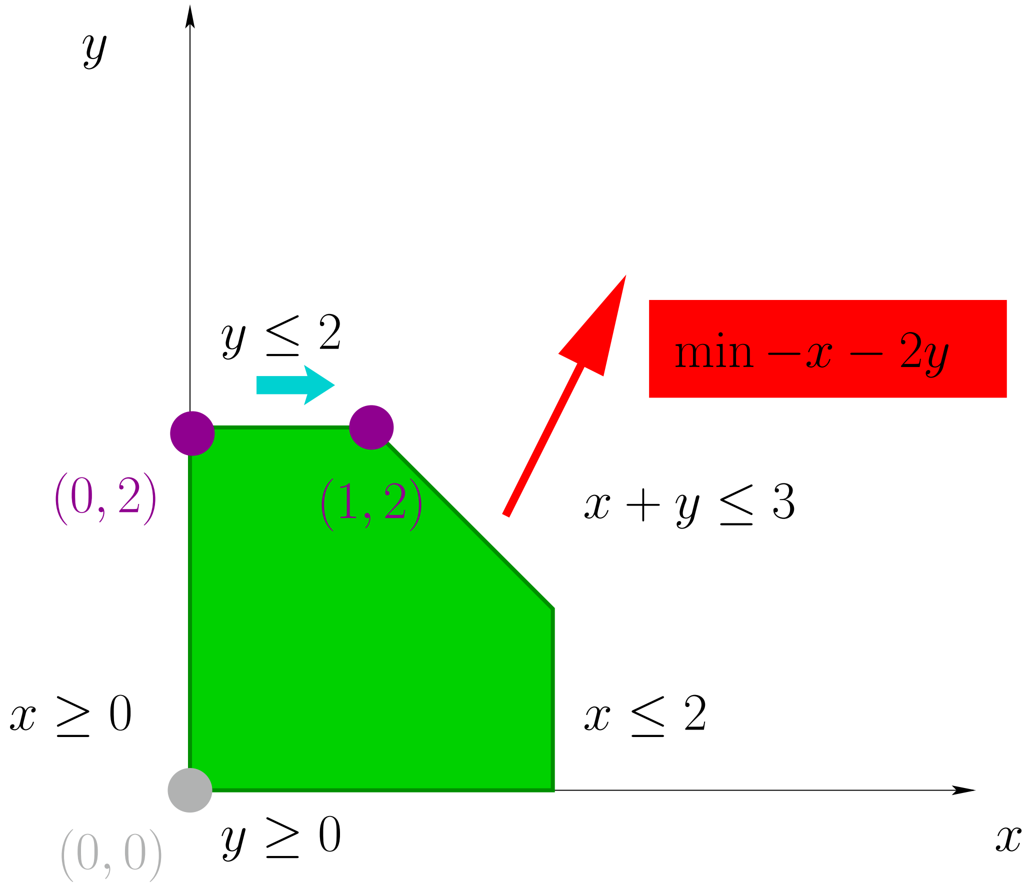
- Let us now increase value of variable x

$$\begin{cases} \min \ -4 - x + 2s_3 & \\ s_1 = 1 + s_3 - x & \text{Limits new value to} \leq 1 \\ s_2 = 2 - x & \text{Limits new value to} \leq 2 \\ y = 2 - s_3 & \text{Does not limit new value} \end{cases}$$

- Best possible new value for x is $\min(2, 1) = 1$
- Variable s_1 leaves the basis and variable x enters

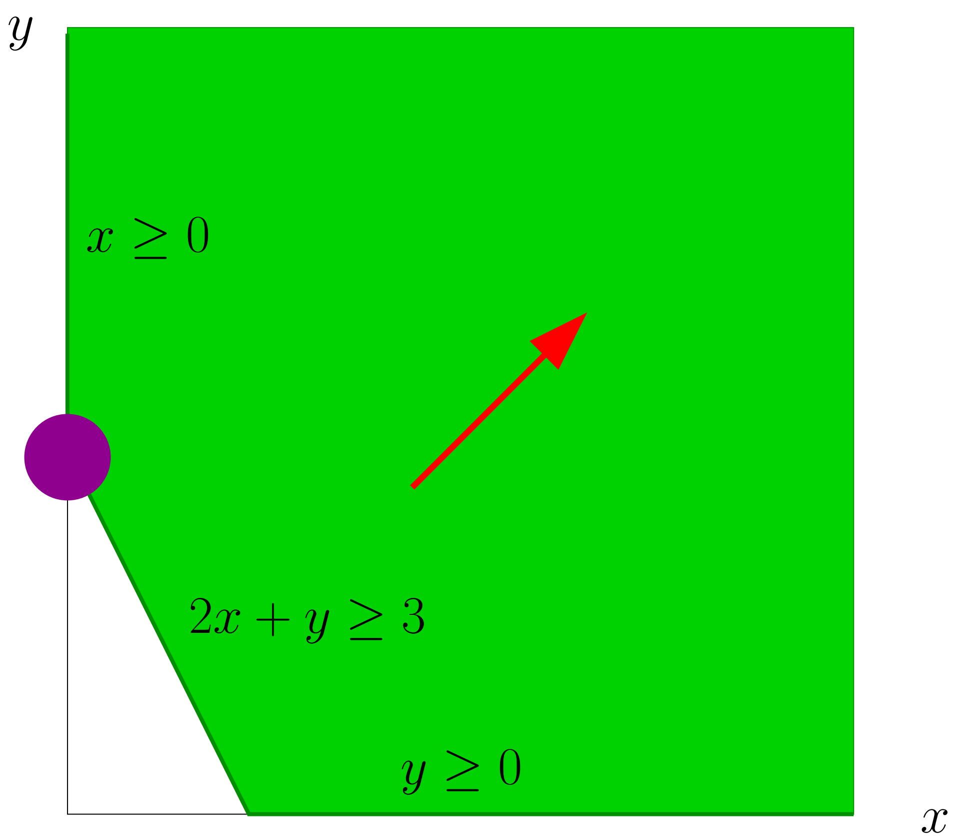
$$\begin{cases} \min \ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases} \Longrightarrow \begin{cases} \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ s_2 = 1 - s_3 + s_1 \\ y = 2 - s_3 \end{cases}$$

Improving the Basic Solution



Unboundedness

- Unboundedness is detected when
  the new value for the chosen non-basic variable is not bounded.



$$\max\ x + y$$
$$2x + y \geq 3$$
$$x, y \geq 0$$

$$\Downarrow$$

$$\begin{cases} \min -x - y \\ -2x - y + s = -3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \min -3 + x - s \\ y = 3 - 2x + s \end{cases}$$

Outline of the Simplex Algorithm

1. `Initialization:` Pick a feasible basis.

2. `Pricing:` If all reduced costs are ≥ 0,
   then return OPTIMAL.
   Else pick a non-basic variable with reduced cost < 0.

3. `Ratio test:` Compute best value for improving non-basic variable respecting non-negativity constraints of basic variables.
   If best value is not bounded,
   then return UNBOUNDED.
   Else select basic variable for exchange with improving non-basic variable.

4. `Update:` Update the tableau and go to 2.

Finding an Initial Basis

- Note that to optimize

$$\begin{aligned}&\min c^T x\\ &Ax = b\\ &x \geq 0\end{aligned}$$

  initially we need a feasible basis at step 1.

  Steps 2-4 of previous procedure are called phase II of simplex algorithm

- Phase I looks for a feasible basis

- We can get a feasible basis with the same procedure by solving another LP for which phase I is trivial

- Let us assume wlog. that $b \geq 0$

- Introduce new artificial variables z and solve

$$\begin{aligned}&\min 1^T z\\ &Ax + z = b\\ &x, z \geq 0\end{aligned}$$

Finding an Initial Basis

$$[LP] \begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad [LP'] \begin{array}{l} \min 1^T z \\ Ax + z = b \\ x, z \geq 0 \end{array} \quad \text{where } b \geq 0$$

- LP' is feasible, and a trivial feasible basis is $\mathcal{B} = (z)$
- LP' cannot be unbounded: $z \geq 0$ implies $1^T z \geq 0$

  So LP' has optimal solution with objective value ≥ 0
- If x^* is feasible solution to LP then $(x, z) = (x^*, 0)$ is optimal solution to LP' with objective value 0
- If $(x, z) = (x^*, z^*)$ is optimal solution to LP' with objective value 0 then $z^* = 0$ and so x^* is feasible solution to LP

Finding an Initial Basis

$$[LP] \begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \implies [LP'] \begin{array}{l} \min 1^T z \\ Ax + z = b \\ x, z \geq 0 \end{array} \quad \text{where } b \geq 0$$

- LP is feasible iff optimum of LP' is 0
- Still: how can we get a feasible basis for LP?
- Assume that optimum of LP' is 0. Then:

  1. If all artificial variables are non-basic,
     then an optimal basis for LP' is a feasible basis for LP
  2. Any basic artificial variable can be made non-basic
     by Gaussian elimination (since A has full rank)

Finding an Initial Basis

- Improvement: use slack variables instead of artificial variables in the initial basis whenever possible

- Alternative phase I approaches do not introduce new variables and work by minimizing the sum of infeasibilities:

$$\min \left\{ \sum_{\beta_i < 0} \beta_i \mid \mathcal{B} \text{ basis with basic solution } \beta \right\}$$

Finding an Initial Basis

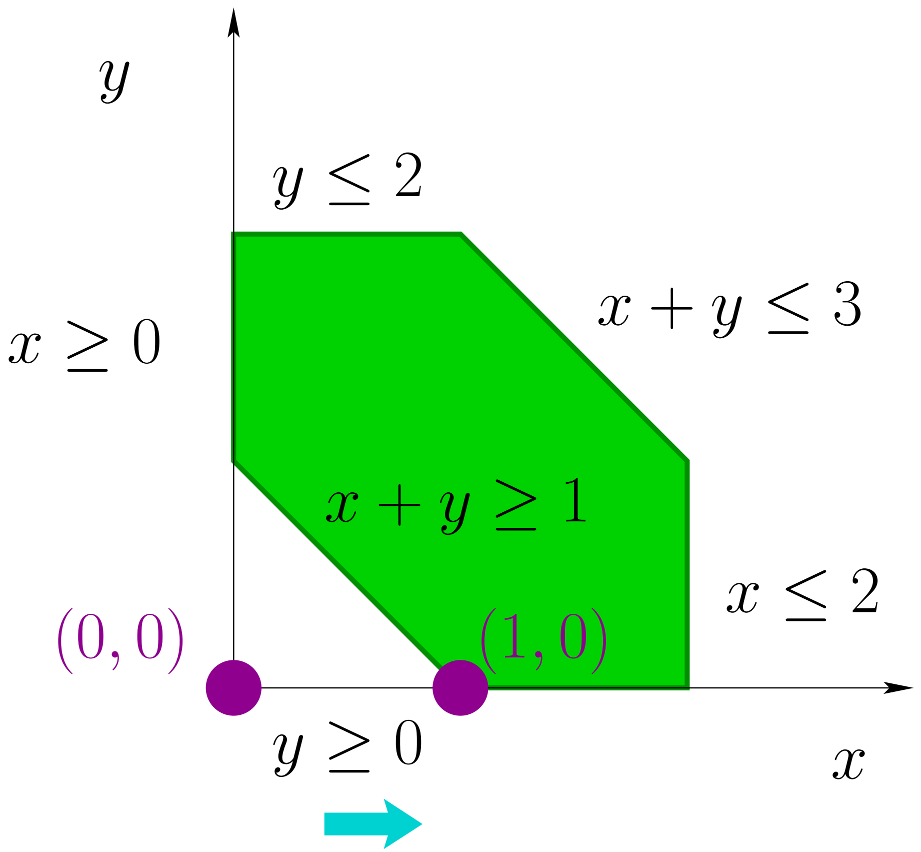
$$\begin{cases} \min\ -x - 2y \\ 1 \le x + y \le 3 \\ 0 \le x \le 2 \\ 0 \le y \le 2 \end{cases} \Rightarrow \begin{cases} \min -x - 2y \\ x + y + s_1 = 3 \\ x + y - s_2 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases} \Rightarrow \begin{cases} \min\ z_1 \\ x + y + s_1 = 3 \\ x + y - s_2 + z_1 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

Finding an Initial Basis

$$\begin{cases} \min 1 - x - y + s_2 \\ s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases} \Rightarrow \begin{cases} \min z_1 \\ s_1 = 2 + z_1 - s_2 \\ x = 1 - z_1 - y + s_2 \\ s_3 = 1 + z_1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

Feasible tableau $\begin{cases} s_1 = 2 - s_2 \\ x = 1 - y + s_2 \\ s_3 = 1 + y - s_2 \\ s_4 = 2 - y \end{cases}$

Finding an Initial Basis



Big M Method

- Alternative to phase I + phase II approach
- LP is changed as follows, where M is a "big number"

$$\begin{array}{lcll} \min c^T x & & \min c^T x + M \cdot 1^T z & \\ Ax = b & \Longrightarrow & Ax + z = b & \text{where } b \geq 0 \\ x \geq 0 & & x, z \geq 0 & \end{array}$$

- Again by taking the artificial variables we get an initial feasible basis
- The search of a feasible basis for the original problem is not blind wrt. cost
- Problems:
  - If M is a fixed big number,
    then the algorithm becomes numerically unstable
  - If M is kept symbolically,
    then handling costs becomes more expensive

Big M Method

$$\begin{cases} \min \; -x - 2y \\ 1 \le x + y \le 3 \\ 0 \le x \le 2 \\ 0 \le y \le 2 \end{cases} \Rightarrow \begin{cases} \min -x - 2y \\ x + y + s_1 = 3 \\ x + y - s_2 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases} \Rightarrow \begin{cases} \min \; -x - 2y + Mz \\ x + y + s_1 = 3 \\ x + y - s_2 + z = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

Big M Method

$$\begin{cases} \min M + (-1-M)x + (-2-M)y + Ms_2 \\ s_1 = 3 - x - y \\ z = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} \min x - 2 - 2s_2 + (M+2)z \\ s_1 = 2 + z - s_2 \\ y = 1 - x - z + s_2 \\ s_3 = 2 - x \\ s_4 = 1 + z + x - s_2 \end{cases}$$

Then we could drop the artificial variable z and continue the optimization.

Termination and Complexity

- A step of the simplex algorithm is degenerate
  if the increment of the chosen non-basic variable is 0
- At each step of the simplex algorithm:

  cost improvement $=$ reduced cost $\cdot$ increment (of chosen non-basic var)
- If the step is degenerate then there is no cost improvement
- But degenerate steps can only happen with degenerate bases
- Assume no degenerate bases occur.

  Then there is a strict improvement from a base to the next one

  So simplex terminates, as bases cannot be repeated

  No. steps is at most exponential: there are $\leq \binom{n}{m}$ bases

  Tight bound for pathological cases (Klee-Minty cube)

  In practice the cost is polynomial

Termination and Complexity

- When there is degeneracy simplex may loop forever

- Termination guaranteed with anticycling rules, e.g. Bland's rule:

  Assume there is a fixed ordering of variables.

  Pricing: among non-basic vars with reduced cost < 0, take the least one

  Ratio test: among tight basic vars, take the least one

Termination with Bland's Rule

PROOF:

States of simplex algorithm determined by bases.

To prove termination, enough to prove we can't repeat bases

Let us prove termination by contradiction.

Assume there is a cycle: $\mathcal{B}_k, ..., \mathcal{B}_t, \mathcal{B}_{t+1}$ such that $\mathcal{B}_k = \mathcal{B}_{t+1}$

Var x_j is fickle if it is in some, but not all, bases of the cycle

For all ratio tests in cycle, entering variable takes value 0

Hence pivoting steps do not change basic solution:
basic solution is the same for all bases of the cycle

So fickle variables have value 0 in basic solution

Termination with Bland's Rule

Let x_r be the largest fickle variable

Let $l \in \{k, ..., t\}$ be such that $x_r \in \mathcal{B}_l$ and $x_r \in \mathcal{R}_{l+1}$

Let $x_r = \sum_{x_j \in \mathcal{R}_l} \lambda_j x_j$ be the respective row in $\mathcal{B}_l$'s tableau

Let $x_s \in \mathcal{R}_l$ be the non-basic variable that is swapped with x_r in $\mathcal{B}_l$

Let $d_l(x_j)$ be the reduced cost of a variable x_j in $\mathcal{B}_l$

Since x_s is entering the basis, $d_l(x_s) < 0$ and $\lambda_s < 0$

Moreover, x_s is fickle too, and hence $x_s \prec x_r$

Termination with Bland's Rule

Let $\mathcal{B}_p$ be the first basis after $\mathcal{B}_{l+1}$ where x_r gets basic again: $x_r \in \mathcal{R}_p$ and $x_r \in \mathcal{B}_{p+1}$

Let $d_p(x_j)$ be the reduced cost of a variable x_j in $\mathcal{B}_p$

Since x_r is entering the basis, $d_p(x_r) < 0$

Moreover $d_p(x_s) \geq 0$:

- If $x_s \in \mathcal{R}_p$: by Bland's rule and $x_s \prec x_r$
- If $x_s \in \mathcal{B}_p$: reduced costs of basic vars are null

Termination with Bland's Rule

Let γ_l be the value of the objective function at the basic solution of $\mathcal{B}_l$
Then for any x such that $Ax = b$: $c^T x = \gamma_l + \sum d_l(x_j)x_j$

Let γ_p be the value of the objective function at the basic solution of $\mathcal{B}_p$
Then for any x such that $Ax = b$: $c^T x = \gamma_p + \sum d_p(x_j)x_j$

As basic solution is the same all the time: $\gamma_l = \gamma_p$
Hence for any x such that $Ax = b$: $\sum d_l(x_j)x_j = \sum d_p(x_j)x_j$

If $x_s = t$ and $x_j = 0$ for all $x_j \in \mathcal{R}_l, j \neq s$ then $x_{\mathcal{B}_l} = B_l^{-1}b - B_l^{-1}a_s t$. So:

$$\begin{aligned}
\sum_{x_j \in \mathcal{B}_l} d_l(x_j)x_j + \sum_{x_j \in \mathcal{R}_l} d_l(x_j)x_j &= \sum_{x_j \in \mathcal{B}_l} d_p(x_j)x_j + \sum_{x_j \in \mathcal{R}_l} d_p(x_j)x_j \\
0 + d_l(x_s)t &= \sum_{x_{k_i} \in \mathcal{B}_l} d_p(x_{k_i})(\beta_i - \alpha_s^i t) + d_p(x_s)t
\end{aligned}$$

where $\beta = B_l^{-1}b$ and $\alpha_s = B_l^{-1}a_s$

Termination with Bland's Rule

Hence $d_l(x_s) = -\sum_{x_{k_i} \in \mathcal{B}_l} d_p(x_{k_i})\alpha_s^i + d_p(x_s)$

As $d_l(x_s) < 0$ and $d_p(x_s) \geq 0$, it must be $\displaystyle\sum_{x_{k_i} \in \mathcal{B}_l} d_p(x_{k_i})\alpha_s^i > 0$

There must exist $x_{k_i} \in \mathcal{B}_l$ such that $d_p(x_{k_i})\alpha_s^i > 0$

So $d_p(x_{k_i}) \neq 0$ and $x_{k_i} \notin \mathcal{B}_p$. As $x_{k_i} \in \mathcal{B}_l$, x_{k_i} is fickle. Now:

- $x_{k_i} = x_r$: $d_p(x_r) < 0$ and $\alpha_s^i > 0$ implies $d_p(x_{k_i})\alpha_s^i < 0$!!!
- $x_{k_i} \prec x_r$: as we didn't chose x_{k_i} to enter $\mathcal{B}_p$, $d_p(x_{k_i}) \geq 0$

  Since $d_p(x_{k_i})\alpha_s^i > 0$, we have $d_p(x_{k_i}) > 0$ and $\alpha_s^i > 0$

  But x_{k_i} is fickle, so its basic value at $\mathcal{B}_l$ is 0

  By the ratio rule, x_{k_i} has ratio 0, so it could leave $\mathcal{B}_l$

  Contradiction! $x_{k_i} \prec x_r$ and x_r was chosen to leave $\mathcal{B}_l$

Pricing Strategies

1. Full pricing

   Choose the variable with the most negative reduced cost

2. Partial pricing

   Make a list with the indices of the P variables
   with the most negative reduced costs.

   In following iterations choose variables from the list
   until reduced costs are all ≥ 0

Pricing Strategies

3. Best-improvement pricing

   Let θ_k be the increment for a non-basic variable x_k with reduced cost $d_k < 0$. Choose the variable j such that

   $$|d_j| \cdot \theta_j = \max\{|d_k| \cdot \theta_k \ \text{ such that } \ d_k < 0, k \in \mathcal{R}\}$$

4. Normalized pricing.

   Let $n_k = ||\alpha_k||$ (in practice $n_k = \sqrt{1 + ||\alpha_k||^2}$)

   where α_k is the column in the tableau of variable x_k.

   Take criteria 1. or 2. but using $\frac{d_k}{n_k}$ instead of d_k

5. Other more sophisticate normalized pricing strategies:

   steepest edge, devex

Bounded Variables

- LP solvers implement a variant of the simplex algorithm that handles bounds more efficiently for LP's of the form

$$\begin{aligned}&\min c^T x\\&Ax = b\\&\ell \le x \le u\end{aligned}$$

- ℓ_i may be $-\infty$ and/or u_i may be $+\infty$
- Bounds are incorporated into pricing and ratio test
- Now non-basic variables will take values at the lower or the upper bound

Bounded Variables

$$\begin{aligned} &\min -x - 2y \\ &x + y \le 3 \\ &0 \le x \le 2 \\ &0 \le y \le 2 \end{aligned} \implies \begin{aligned} &\min -x - 2y \\ &x + y + s = 3 \\ &0 \le x \le 2 \\ &0 \le y \le 2 \\ &s \ge 0 \end{aligned} \implies \begin{aligned} &\min -x - 2y \\ &s = 3 - x - y \\ &0 \le x \le 2 \\ &0 \le y \le 2 \\ &s \ge 0 \end{aligned}$$

- Initially non-basic variables x, y are at lower bound
- We choose variable x in pricing

Bounded Variables

$$
\begin{cases}
\min -x - 2y \\
s = 3 - x - y & \text{Limits new value to } \leq 3 \text{ as } s \geq 0 \\
0 \leq x \leq 2 & \text{Limits new value to } \leq 2 \text{ as } x \leq 2 \\
0 \leq y \leq 2 \\
s \geq 0
\end{cases}
$$

- Best possible new value for x is $\min(3, 2) = 2$
- Bound flip: x is still non-basic, but is now at upper bound

$$
\begin{cases}
\min -x - 2y \\
s = 3 - x - y \\
0 \leq x \leq 2 \\
0 \leq y \leq 2 \\
s \geq 0
\end{cases}
$$

Bounded Variables

- Pricing considers the bound status of non-basic variables
- A non-basic variable x_j with reduced cost d_j can improve the cost function
  - if x_j is at lower bound and $d_j < 0$; or
  - if x_j is at upper bound and $d_j > 0$
- Choose y in pricing:

$$\left\{\begin{array}{ll} \min -x - 2y & \\ s = 3 - x - y & \text{Limits new value to } \leq 1 \text{ as } s \geq 0 \\ 0 \leq x \leq 2 & \\ 0 \leq y \leq 2 & \text{Limits new value to } \leq 2 \text{ as } y \leq 2 \\ s \geq 0 & \end{array}\right.$$

- Best possible new value for y is $\min(1, 2) = 1$

Bounded Variables

- Usual pivoting step now:

$$s = 3 - x - y \;\;\Rightarrow\;\; y = 3 - x - s$$

$$\left\{\begin{array}{l} \min -x - 2y \\ s = 3 - x - y \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{array}\right.$$

Bounded Variables

- Choose x in pricing. To respect bounds for y:

$$0 \leq y(x) \leq 2$$
$$0 \leq 3 - x \leq 2$$
$$\text{(since } x \text{ decreases its value, } 0 \leq y(x) \text{ is OK)}$$
$$3 - x \leq 2$$
$$1 \leq x$$

$$\left\{ \begin{array}{ll} \min -6 + x + 2s & \\ y = 3 - x - s & \text{Limits new value to } \geq 1 \\ 0 \leq x \leq 2 & \text{Limits new value to } \geq 0 \\ 0 \leq y \leq 2 & \\ s \geq 0 & \end{array} \right.$$

- Best possible new value for x is $\max(1, 0) = 1$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \;\;\Rightarrow\;\; x = 3 - y - s$$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s$$

$$\left\{\begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \min -3 + s - y \\ x = 3 - y - s \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{array}\right.$$

- Since upper bound of y was tight, now y is set to its upper bound
- Optimal solution: $(x, y, s) = (1, 2, 0)$ with cost -5
- Now reading the basic solution and its cost is more involved!

Bounded Variables