Solution

Universitat Politècnica de Catalunya. Departament d'Arquitectura de Computadors.
Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

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Problem 1

Let X be a random variable equal to the failure time of a system. The reliability of the system, $R(n)$, is defined as $R(n) = P(X > n)$. In other words, it is the probability that the system does not fail during, at least, n steps.

1.A (1.5 points) Assume the system of figure 1, consisting of a sensor which can fail at the end of every day with probability $\alpha = 1/5$. Compute the reliability $R_1(n)$. Use it to compute the reliability in 5 days, $R_1(5)$.

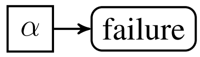

Figure 1: Single sensor.

Problem 2

Assume the system with redundancy of figure 2, consisting now of 2 sensors, as the previous one, working in parallel. The system fails only when both sensors have failed. Note that both sensors can fail simultaneously the same day.

2.A (1.5 points) Formulate a DTMC of the system and use it to compute the reliability $R_2(n)$. Use it to compute the reliability in 5 days, $R_2(5)$.

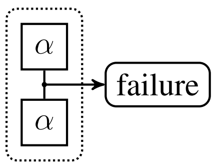

Figure 2: Two sensors in parallel.

2.B (1.5 points) Compute the expected time until failure.

2.C (1.5 points) Assume that each sensor consumes an average energy of 80 Joules per day (0 Joules if failed). Compute the expected energy consumed by the system until failure. Hint: consider the expected value of the random variables:

$$n_0 = \{\text{number of visits to a state with 2 sensors working (0 failed) before failure}\}$$
$$n_1 = \{\text{number of visits to a state with 1 sensor working (1 failed) before failure}\}$$

Note also that the expected time until failure computed in 2.B is equal to E$[n_0 + n_1]$.

Problem 3

Assume now that in the system of problem 2 sensors are replaced in one day upon failure. That is, if one sensor fails at the end of a day, it remains failed during next day, and working the day after. The same if the two sensors fail simultaneously.

3.A (1.5 points) Formulate a DTMC of the system and compute the stationary distribution.

3.B (1.5 points) Compute the expected energy consumption per day in steady state.

3.C (1 points) Compute the expected number of sensors replaced per day in steady state.

Solution

Problem 1

1.A Consider a chain with states (see figure 3):

$$\begin{aligned}&0 \text{ the sensor is working}\\&1 \text{ the sensor is failed}\end{aligned}$$

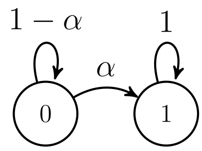

Figure 3: DTMC of 1 sensor.

Clearly:

$$\boldsymbol{P} = \begin{bmatrix} 1-\alpha & \alpha \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/5 & 1/5 \\ 0 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4/5$. Thus $\pi_0(n) = a\,\lambda_2^n$, and imposing $\pi_0(0) = 1$ we have $\pi_0(n) = (4/5)^n$. Therefore:

$$R_1(n) = P(X > n) = \pi_0(n) = (4/5)^n, \; n \geq 0 \tag{1}$$

and $R_1(5) = (4/5)^5 \approx 0.33$

Problem 2

2.A Consider a chain where the state is the number of sensors failed (see figure 4).

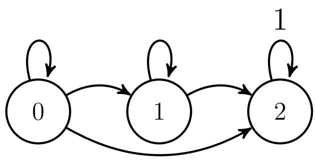

Figure 4: DTMC of 2 sensors in parallel.

Let $\beta = 1 - \alpha = 1 - 1/5 = 4/5$. We have:

$$\begin{aligned} &p_{00} = \beta^2 \qquad && p_{01} = 1 - \alpha^2 - \beta^2 = 2\,\alpha - 2\,\alpha^2 = 2\,\alpha\,(1-\alpha) = 2\,\alpha\,\beta \qquad && p_{02} = \alpha^2 \\ & && p_{11} = \beta && p_{12} = \alpha \end{aligned}$$

and thus

$$\boldsymbol{P} = \begin{bmatrix} \beta^2 & 2\,\alpha\,\beta & \alpha^2 \\ 0 & \beta & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\lambda \boldsymbol{I} - \boldsymbol{P}) = (\lambda - \beta^2)(\lambda - \beta)(\lambda - 1)$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = \beta$, $\lambda_3 = \beta^2$. Therefore $\pi_2(n) = 1 + a\,\beta^n + b\,\beta^{2n}$. Imposing $\pi_2(0) = 0$ and $\pi_2(1) = \alpha^2$ we get $b = 1$ and $a = -2$. Thus:

$$R_2(n) = 1 - \pi_2(n) = 2\,\beta^n - \beta^{2n} = 2\,(4/5)^n - (4/5)^{2n} = 2\,R_1(n) - R_1^2(n), \; n \geq 0$$

and $R_2(5) = 2\,(4/5)^5 - (4/5)^{10} \approx 0.55$ (less than double than in problem 1).

2.B We have:

$$m_{02} = \alpha^2 + \beta^2\,(1+m_{02}) + 2\,\alpha\,\beta\,(1+m_{12}) = 1 + \beta^2\,m_{02} + 2\,\alpha\,\beta\,m_{12}$$
$$m_{12} = \alpha + \beta\,(1+m_{12}) = 1 + \beta\,m_{12}$$

which yields $m_{12} = 1/\alpha$ and

$$m_{02} = \frac{1+2\,\alpha\,\beta/\alpha}{1-\beta^2} = \frac{1+2\,\beta}{1-\beta^2} = \frac{13/5}{9/25} = \frac{65}{9} = 7.\widehat{2}\text{ days.} \qquad (2)$$

2.C Clearly, $\mathrm{E}[n_0]$ will be the sojourn time in state 0: $\mathrm{E}[n_0] = 1/(1-p_{00}) = 1/(1-\beta^2) = 25/9$ days and, looking one step forward from $X(0) = 0$ we have:

$$\mathrm{E}[n_1] = \mathrm{E}[n_1|X(1)=0]\,p_{00} + \mathrm{E}[n_1|X(1)=1]\,p_{01} + \mathrm{E}[n_1|X(1)=2]\,p_{02} =$$
$$\mathrm{E}[n_1]\,\beta^2 + \frac{1}{\alpha}\,2\,\alpha\,\beta + 0$$

since $\mathrm{E}[n_1|X(1)=0] = \mathrm{E}[n_1]$, $\mathrm{E}[n_1|X(1)=1] = m_{12} = 1/\alpha$ and $\mathrm{E}[n_1|X(1)=2] = 0$. Thus

$$\mathrm{E}[n_1] = \frac{2\,\beta}{1-\beta^2} = \frac{8/5}{9/25} = \frac{40}{9}\text{ days.}$$

An easier way to compute $\mathrm{E}[n_1]$ is noting that $\mathrm{E}[n_0] + \mathrm{E}[n_1] = m_{02}$ (as suggested by the hint). Thus, using (2) we have $\mathrm{E}[n_1] = m_{02} - \mathrm{E}[n_0] = 65/9 - 25/9 = 40/9$, as before. Now we can compute the consumption of the system until failure as:

$$\text{Expected consumption} = \mathrm{E}[n_0]\,2\times 80 + \mathrm{E}[n_1]\,80 = \frac{25}{9}\,2\times 80 + \frac{40}{9}\,80 = 800\text{ J.}$$

Problem 3

3.A Now the chain is shown in figure 5.

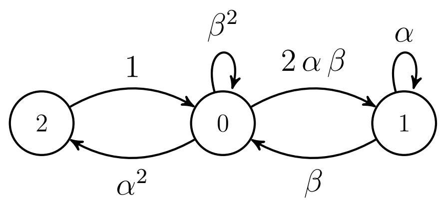

Figure 5: DTMC of 2 sensors with repair.

The chain is reversible, therefore, starting from state 0 we have:

$$\pi_0 = \frac{1}{G},\;\; \pi_1 = \frac{1}{G}\,\frac{2\,\alpha\,\beta}{\beta},\;\; \pi_2 = \frac{1}{G}\,\frac{\alpha^2}{1}$$

which yields: $G = 1 + 2\,\alpha + \alpha^2 = (1+\alpha)^2 = 36/25$ and

$$\pi_0 = 25/36$$
$$\pi_1 = 10/36$$
$$\pi_2 = 1/36$$

3.B The expected energy consumption per day in steady state is:

$$\text{Expected consumption} = \pi_0\,2\times 80 + \pi_1\,80 = 60/36\times 80 = 400/3 = 133.\widehat{3}\text{ J/day} =$$
$$133.\widehat{3}\,\frac{\text{W s}}{\text{day}}\,\frac{1\text{ day}}{24\times 60\times 60\text{ s}} \approx 1.54\text{ mW.}$$

3.C Every visit to state 0 one sensor is replaced, and every visit to state 2 two sensors are replaced. Thus, the expected number of sensors replaced per day in steady state is:

$$\text{Expected replacements} = 1\times\pi_1 + 2\times\pi_2 = 12/36 = 1/3\text{ replacements/day}$$