Algorithmic Methods for Mathematical Models

Simplex Exercises

1. Convert the following problem into standard form:

$$\begin{array}{llllll} \text{minimize} & 3x_1 & +x_2 & +4x_3 & -2x_4 & \\ \text{subject to} & x_1 & -x_2 & & +3x_4 & \leq 2 \\ & 2x_1 & & +x_3 & -x_4 & \geq 3 \\ & & 3x_2 & -x_3 & +2x_4 & = 3 \\ & x_1 \geq 0, & x_2 \leq 0, & & x_4 \geq 0 & \end{array}$$

2. (MIT, exercise 12b, chapter 2) Formulate a phase I linear program to find a feasible solution to the system and show that no feasible solution exists.

$$\begin{array}{llll} 3x_1 & +2x_2 & -x_3 & \leq -3 \\ -x_1 & -x_2 & +2x_3 & \leq -1 \\ x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0 & \end{array}$$

3. Solve the following linear program using the Simplex method. Draw the feasible region and identify, at each pivoting step, which is the point in the region that the Simplex is exploring.

   (a)

$$\begin{array}{lll} \text{maximize} & -x_1 & +x_2 \\ \text{subject to} & x_1 & -x_2 \leq 2 \\ & x_1 & +x_2 \leq 6 \\ & x_1 \geq 0, & x_2 \geq 0 \end{array}$$

   (b)

$$\begin{array}{lll} \text{maximize} & x_1 & +x_2 \\ \text{subject to} & -2x_1 & +x_2 \leq 1 \\ & x_1 & -x_2 \leq 1 \\ & x_1 \geq 0, & x_2 \geq 0 \end{array}$$

4. Solve the following linear problem using the Simplex method:

$$\begin{array}{lllllll} \text{maximize} & 6x_1 & +14x_2 & +13x_3 & & & \\ \text{subject to} & \frac{1}{2}x_1 & +2x_2 & +x_3 & +x_4 & & = 24 \\ & x_1 & +2x_2 & +4x_3 & & +x_5 & = 60 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0, & x_4 \geq 0, & x_5 \geq 0 & \end{array}$$

5. (MIT, exercise 9, chapter 2) Solve the following problem using the two phases of the simplex method:

$$\begin{array}{lllll} \text{maximize} & 2x_1 & +x_2 & +x_3 & \\ \text{subject to} & 2x_1 & +3x_2 & -x_3 & \leq 9 \\ & & 2x_2 & +x_3 & \geq 4 \\ & x_1 & & +x_3 & = 6 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0 & \end{array}$$

   Is the optimal solution unique? How can this be determined at the final simplex iteration?

6. (Luenberger, exercise 1, section 2.6) Convert the following problems to standard form:

   (a)

$$\begin{array}{ll} \text{minimize} & x + 2y + 3z \\ \text{subject to} & 2 \le x + y \le 3 \\ & 4 \le x + y \le 5 \\ & x \ge 0, \quad y \ge 0, \quad z \ge 0 \end{array}$$

   (b)

$$\begin{array}{ll} \text{minimize} & x + y + z \\ \text{subject to} & x + 2y + 3z = 10 \\ & x \ge 1, \quad y \ge 2, \quad z \ge 1 \end{array}$$

7. (MIT, exercise 14, chapter 2) Consider the linear program:

$$\begin{array}{llllll} \text{maximize} & \alpha x_1 & +2x_2 & +x_3 & -4x_4 & \\ \text{subject to} & x_1 & +x_2 & & -x_4 & = 4 + 2\Delta \quad (1) \\ & 2x1 & -x_2 & +3x_3 & -2x_4 & = 5 + 7\Delta \quad (2) \\ & x_1 \ge 0, & x_2 \ge 0, & x_3 \ge 0, & x_4 \ge 0 & \end{array}$$

   where α and Δ are viewed as parameters.

   (a) Form two new constraints as $(1') = (1) + (2)$ and $(2') = -2(1) + (2)$. Use these transformations to express the problem in canonical form with x_1 and x_2 as basic variables.
   (b) Assume $\Delta = 0$. For which values of α are x_1 and x_2 optimal basic variables in the problem?
   (c) Assume $\alpha = 3$. For which values of Δ do x_1 and x_2 form an optimal basic feasible solution.

8. (Exam January 2013) Given two real values A and B we define the Linear Program $LP_{A,B}$:

$$\begin{array}{llllll} \text{maximize} & Ax_1 & +2Bx_2 & & & \\ \text{subject to} & x_1 & +2x_2 & -3x_3 & +x_4 & = 8 \\ & & -4x_2 & +2x_3 & -2x_4 & = -12 \\ & x_1 \ge 0, & x_2 \ge 0, & x_3 \ge 0, & x_4 \ge 0 & \end{array}$$

   (a) Write an equivalent Linear Program in canonical form with basis $<x_1, x_2>$.
   (b) For which values of A and B is $<x_1, x_2>$ an optimal basis for $LP_{A,B}$?
   In this case, which is the optimal value of the objective function?
   (c) Assume now that $A \ge -3$. Among all values of A and B for which $<x_1, x_2>$ is an optimal basis, find the ones for which $LP_{A,B}$ has the largest optimal value.
   (Hint: it might help to write a Linear Program and solve it graphically.)