Algorithmic Game Theory
Homework 5 - Cooperative Games

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1 Problem 11

1.1 Simple Game Analysis

Lets analyze case by case to see if some of the option (a), (b) or (c) is a Γ_s **Simple Game**.

(a) **1.** *A Coalition X is winning in $\Gamma_s \iff X$ wins in Γ and $H[X]$ has not isolated vertices.*

Proof.

Part 1. $\implies$
This part is trivial by ***monotonicity*** *of Simple Games since if it is wining on Γ_s is winning on Γ and there should be not isolated vertices since all the players should communicate.*

Part 2. $\impliedby$
Lets assume there is no isolated vertices. But if that occurs it doesn't mean that $H[X]$ is a connected graph, because for example if $(u, v) \in E$ and $(v, z) \in E$ but $(u, z) \notin E$, the graph has not isolated vertices but it is not connected.

Therefore, (a) **is not a simple game** on Γ_s. ■

(b) **1.** *A Coalition X is winning in $\Gamma_s \iff X$ wins in Γ and $H[X]$ is connected.*

Proof.

Part 3. $\Longrightarrow$
This part is trivial by the same reason of previous analysis.

Part 4. $\Longleftarrow$
If $H[X]$ is connected means that $\forall \{u, v\} \in N, (u, v) \in E$. So every member communicate each other. At the same time X is winning in Γ, therefore X is also winning in Γ_s because both condition holds.

Therefore, (b) **is a simple game** on Γ_s. ∎

(c) **1.** *A Coalition X is winning in $\Gamma_s \iff$ there is an $Y \subseteq X$, so that Y wins in Γ and $H[Y]$ is connected.*

Proof.

Part 5. $\Longrightarrow$
Suppose that X belongs to a minimal wining coalition W^m. If that happens there cannot be any other winning coalition that is included in X, because $\forall Y \in W, Y \subsetneq X$ according to the definition of minimal winning coalition.

Therefore, (c) **is a not simple game** on Γ_s. ∎

1.2 Complexity Analysis

For the complexity analysis and taking into consideration that $X \in W$ can be decide in $Poly - time$, i could provide the following Polynomial time algorithm for deciding *empty-core* in Γ_s. In this case for (b), based on the Theorem in which we know that *A Simple Game has a non-empty core if it has a veto player*. The idea of the algorithm is trying to find that **veto**

player

Algorithm 1: Decide Γ_s has an empty core

Input : Given $H = (N, E)$
Output: If Γ_s has an empty core

$O(|N| + |E|)$ **times** $SCC \leftarrow$ Calculate **SCC** of H
n **times** **for** *each* $X \in SCC$ **do**
k **times**   **for** *each* $p \in X$ **do**
        $X \leftarrow X \setminus \{p\}$;
$Poly - time$ **if** $X \notin W$ **then** return *NON EMPTY CORE*; ;

return *EMPTY CORE*;