Stochastic Network Modeling (SNM)

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Syllabus

Parts

I. Introduction
II. Discrete Time Markov Chains (DTMC)
III. Continuous Time Markov Chains (CTMC)
IV. Queuing Theory

Evaluation

- NF = 0.1 * NP + 0.30 * max(EF, C) + 0.60 * EF
  where:
  - NF = final mark
  - EF = final theory exam
  - NP = Problems delivered by the students
  - C = average assessments mark: C = 0.5*C1 + 0.5*C2

Part I

Introduction

Outline

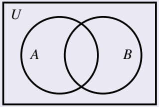
- Probability
- Stochastic Process (SP)

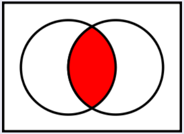
Ingredients of Probability

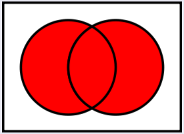
- Random experiment, e.g. toss a die.
- Outcome, ω, e.g. tossing a die can be $\omega = 2$, choosing a fruit can be $\omega =$ orange.
- Sample space or Universal set, U, set of all possible outcomes. E.g. tossing a die $U = \{1,2,3,4,5,6\}$.
- Event, A, any subset of U (e.g. tossing a die $A = \{1,2,3\}$). We say the event A occurs if the outcome of the experiment $\omega \in A$. U is the sure event, and we represent by the empty set $\emptyset$ an impossible outcome.

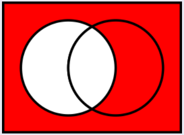
Venn Diagrams

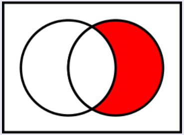
Graphical representation of events




Intersection $A \cap B$


Union $A \cup B$


Complement of A in U
$A^c = U \setminus A$


Complement of A in B
(B minus A) $A^c \cap B = B \setminus A$

source: wikipedia

Random Variable

- For simplicity it is defined a random variable (RV), X as a function that assigns a real number to each outcome in the sample space U, i.e.:

$$X : U \to \mathbb{R}$$

- We will represent the experiment by a RV, X, and the possible outcomes by its values. $X = x_i$ is the outcome $X(\omega_i) = x_i$.
- Using RVs the sample space is mapped in a subset of $\mathbb{R}$. So, in terms of X, U is a set of points of $\mathbb{R}$. The same for any event.
- Normally the definition of X comes naturally from the experiment, e.g. tossing a die: $X = \{\text{number in the toss}\}$.
- RVs can be discrete (e.g. tossing a die) or continuous (e.g. waiting time of a packet in a queue).

Probability Measure of Discrete RV

- If the sample space U of the RV X is finite (discrete RV), $U = \{x_1, \cdots x_n\}$, a probability measure is an assignment of numbers $P(x_i)$, referred to as probabilities, to each outcome x_i such that:

$$0 \le P(x_i) \le 1$$

$$P(A) = \sum_{x_i \in A} P(x_i)$$

$$P(U) = 1$$

E.g. tossing a fair die,

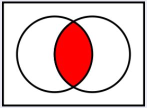
$$P(x_i) = 1/6$$

$$P(X \in \{2,4,6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Conditional Probability and Bayes Formula

- Given the the sample space U and the events $A, B \in U$ with $P(B) > 0$ the probability of A conditioned by B is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Intersection $A \cap B$

NOTE: It's common to use commas to denote set intersection, and write $P(A \cap B)$ as $P(A, B)$.

- Bayes Formula

$$P(A|B)\,P(B) = P(B|A)\,P(A) \Rightarrow P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Law of total probability

- Let B_i a partition of the sample space U ($\cup_i B_i = U$, $B_i \cap B_j = \emptyset, \forall i \neq j$), then
$$P(A) = \sum_i P(A|B_i)\, P(B_i)$$
- For conditional probabilities:
$$P(A|C) = \sum_i P(A|C \cap B_i)\, P(B_i|C)$$
- If C is independent of any of the B_i
$$P(A|C) = \sum_i P(A|C \cap B_i)\, P(B_i)$$

Probability Measure of Continuous RV

- If the sample space of the RV X is continuous (continuous RV), the events are intervals of $\mathbb{R}$. The probability measure is defined by means of the cumulative distribution function, CDF:

$$F(x) = P(X \in (-\infty, x]) = P(X \leq x)$$

- X is called absolutely continuous[a] if there exists the probability density function, PDF, such that for any interval $I = \{x \mid a \leq x \leq b\}$:

$$\int_a^b f(x)\, dx = P(X \in I) = F(b) - F(a)$$

[a] Some special distributions, called singular, do not have a PDF. One example is the Cantor distribution (see Wikipedia).

Expected value

- Given the discrete $N \in \mathbb{Z}$, respectively continuous $X \in \mathbb{R}$ RV, the expected value is:

$$\mathrm{E}[N] = \sum_{k=-\infty}^{\infty} k\, P(N = k)$$

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Variance

- The amount of dispersion of a RV X with expected value $\mu = \mathrm{E}[X]$ is measured by the Variance:

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}[(X - \mu)^2] = \mathrm{E}[X^2] - \mu^2$$

- Often it is used the standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$.

Indicator Function

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$\mathrm{E}[I(A)] = 0 \times P(I(A) = 0) + 1 \times P(I(A) = 1) = P(A)$$

Expected value of non negative RVs

- For non negative RVs, $N \geq 0$ discrete and $X \geq 0$ continous:

$$\mathrm{E}[N] = \sum_{k=0}^{\infty} k\,P(N=k) = \sum_{k=0}^{\infty} P(N>k)$$

$$\mathrm{E}[X] = \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} P(X>x)\,dx = \int_0^{\infty} (1-F(x))\,dx$$

Proof

$$N = \sum_{k=0}^{N-1} 1 = \sum_{k=0}^{\infty} I(N>k)$$

$$X = \int_0^{X} dx = \int_0^{\infty} I(X>x)\,dx$$

and take expectations.

Wald's Equation

- **Definition**: An positive integer RV $N > 0$ is a **stopping time** of a sequence $X_1, X_2, \cdots$ if the event $N = n$ is independent of $X_{n+1}, X_{n+2}, \cdots$.
  E.g. toss a die until you get 6. Let N be the number of tosses. N does not depend on the values obtained after getting 6.
- **Wald's Equation** If $X_1, X_2, \cdots$ are independent and identically distributed and N is a stopping time:

$$\boxed{\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}[X]\,\mathrm{E}[N]}$$

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- **Proof**

$$\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}\left[\sum_{n=1}^{\infty} X_n I(n \le N)\right] = \sum_{n=1}^{\infty} \mathrm{E}[X_n]\,\mathrm{E}[I(n \le N)] =$$

$$\mathrm{E}[X] \sum_{n=1}^{\infty} P(n \le N) = \mathrm{E}[X] \sum_{n=0}^{\infty} P(N > n) = \mathrm{E}[X]\,E[N]$$

$\square$