Universitat Politècnica de Catalunya. Departament d'Arquitectura de Computadors.
Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

Stochastic Network Modeling (SNM). Autumn 2017.

Second assessment, Continuous Time Markov Chains. 11/12/2017.

Problem 1

Assume a CSMA/CA system with 2 nodes. Both nodes transmit with rate $\lambda = 1/4$ packets/time unit, regardless whether they are thinking or backlogged. The transmission time is 1 time unit. Arrivals and transmissions times are exponentially distributed.

1.A (2 points) Formulate a CTMC that allows computing the throughput of each node and compute the stationary distribution. Indicate clearly the meaning of each state of the chain.

1.B (2 points) Compute the throughput of each node in packets/time unit.

1.C (2 points) Compute the probability that a node does not stay backlogged more than 4 time units.

Problem 2

Suppose that in the previous system, due to transmission delays, when a new transmission is started and both stations are not transmitting, they can start transmitting simultaneously and collide with probability p. Colliding packets are unsuccessfully transmitted, and after transmitting the packet each station becomes backlogged.

2.A (2 points) Formulate a CTMC that allows computing the throughput of each node and compute the stationary distribution in terms of p. Indicate clearly the meaning of each state of the chain.

2.B (2 points) Compute the value of p for the throughput to be reduced less than 10%.

Solution

Problem 1

1.A The nodes are indistinguishable of being in thinking or backlogged state. Thus, it is enough to remember whether some node is transmitting and, the states are (see figure 1):

0 no node is transmitting
1 any node is transmitting

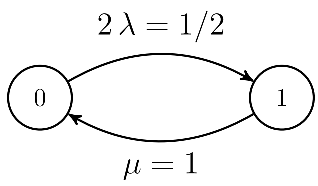

Figure 1: CTMC.

The CTMC is reversible, thus

$$\pi_0 = \frac{1}{G}$$
$$\pi_1 = \frac{1}{G}\frac{1/2}{1}$$

which yields: $G = 3/2$ and

$$\pi_0 = 2/3$$
$$\pi_1 = 1/3$$

1.B The nodes transmit only in state 1 with rate $\mu = 1$ packet/t.u. Thus, the throughput ν of each node will be

$$\nu = \frac{1}{2}\,\pi_1\,\mu = 1/6 \quad \text{packets/t.u.}$$

1.C Let's consider node n_1 as tagged, and the absorbing CTMC with states (see figure 2):

0 n_2 is transmitting
1 n_2 is not transmitting
2 n_1 is transmitting

where 0 is the initial state.

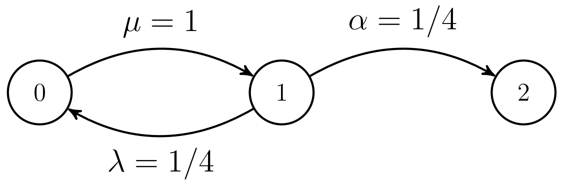

Figure 2: Absorbing CTMC.

Let T be the RV equal to the backlogged time. We have

$$P(T \le t) = \pi_2(t).$$

The rate matrix of the chain is

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 1/4 & -2/4 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the non null eigenvalues are the roots of

$$(x+1)(x+1/2) - 1/4 = x^2 + 3/2\,x + 1/4$$

which are:

$$\lambda_1 = \frac{-3-\sqrt{5}}{4}$$
$$\lambda_2 = \frac{-3+\sqrt{5}}{4}.$$

We have

$$\pi_2(t) = 1 + a\,\mathrm{e}^{\lambda_1 t} + b\,\mathrm{e}^{\lambda_2 t}$$

and

$$\pi_2(0) = 1 + a + b = 0$$
$$\pi_2'(0) = a\,\lambda_1 + b\,\lambda_2 = 0$$

which yields

$$\pi_2(t) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\,\mathrm{e}^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}\,\mathrm{e}^{\lambda_2 t}$$

and the requested probability is:

$$P(T \le 4) = \pi_2(4) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\,\mathrm{e}^{\lambda_1 4} + \frac{\lambda_1}{\lambda_2 - \lambda_1}\,\mathrm{e}^{\lambda_2 4} \approx 0{,}45.$$

Problem 2

2.A Consider the CTMC with states (see figure 3):

0 no node is transmitting
1 1 node is transmitting, no collision
2 2 nodes are transmitting, collision
3 1 node is transmitting after a collision

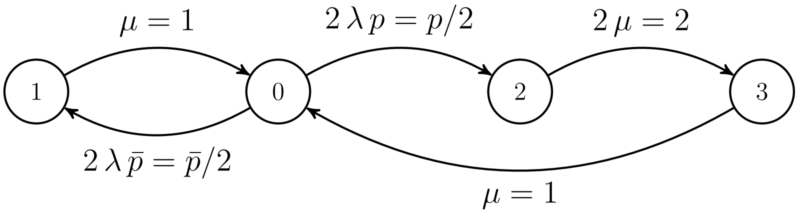

Figure 3: CTMC.

where $\bar{p} = 1 - p$. Using flux balancing we have:

$$\begin{aligned} \pi_1 &= \pi_0\,\bar{p}/2 \\ \pi_0\,p/2 &= \pi_3 \\ \pi_2\,2 &= \pi_3 \\ \sum_i \pi_i &= 1 \end{aligned}$$

which yields

$$\begin{aligned} \pi_0 &= \frac{4}{6+p} \\ \pi_1 &= \frac{2\,(1-p)}{6+p} \\ \pi_2 &= \frac{p}{6+p} \\ \pi_3 &= \frac{2\,p}{6+p}. \end{aligned}$$

2.B The throughput of one station is now:

$$\nu' = \frac{1}{2}\,\pi_1\,\mu = \frac{1-p}{6+p}$$

We want $\nu' \geq 0{,}9\,\nu = 9/60 = 3/20$, thus:

$$\frac{1-p}{6+p} \geq \frac{3}{20}$$

which yields $p \leq 2/23 \approx 0{,}09$.