Stochastic Network Modeling (SNM)

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Parts

Part III

Continuous Time Markov Chains (CTMC)

Outline

Definition of a CTMC

Properties of a continuous time MC

- The states must be a numerable set.
- Let $X(t)$ be the event {at time t the system is in state i}, then it must hold the memoryless property:

$$\boxed{\begin{aligned} & P\big(X(t_n) = i \mid X(t_1) = j, X(t_2) = k, \ldots\big) = \\ & \qquad P\big(X(t_n) = i \mid X(t_1) = j\big) \text{ for any } t_n > t_1 > t_2 > t_3 \ldots \end{aligned}}$$

Definition of a CTMC

Transition Matrix

- Transition probabilities:

$$p_{ij}(t_1, t_2) = P\Big(X(t_2) = j \mid X(t_1) = i\Big)$$

- For an homogeneous chain:

$$p_{ij}(t) = P\Big(X(t_1 + t) = j \mid X(t_1) = i\Big) =$$
$$= P\Big(X(t) = j \mid X(0) = i\Big), \ \forall t_1$$

- In matrix form (transition probability matrix):

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \ t \geq 0$$

- Notes:
  - Compare with the n-step prob. matrix of a DTMC: $\mathbf{P}(n)$.
  - $\mathbf{P}(t)$ must be a stochastic matrix (all rows add to 1).

Transition Matrix

$$p_{ij}(t) = P\big(X(t) = j \mid X(0) = i\big),\ t \geq 0$$

- We look for an equivalent 1-step prob. matrix **P** of DTMCs.
- For consistency: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$. In matrix form:

$$\lim_{t\to 0} \mathbf{P}(t) = \mathbf{I}.$$

- And assume that the following transition rates exist:

$$q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t},\ i \neq j; \quad q_{ii} = \lim_{t\to 0} \frac{p_{ii}(t) - 1}{t}$$

- In matrix form: $\mathbf{Q} = \lim_{t\to 0} \frac{\mathbf{P}(t) - \mathbf{I}}{t}$
- Note that $\sum_j p_{ij}(t) = 1 \Rightarrow p_{ii}(t) = 1 - \sum_{j\neq i} p_{ij}(t)$, thus:

$$q_{ii} = \lim_{t\to 0} \frac{p_{ii}(t) - 1}{t} = \lim_{t\to 0} \frac{-\sum_{j\neq i} p_{ij}(t)}{t} = -\sum_{j\neq i} q_{ij}$$

Definition of a CTMC

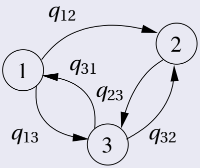
Transition Matrix

- The matrix $\mathbf{Q}$ is called the transition rate or infinitesimal generator of the chain.
- Since $q_{ii} = -\sum_{j \neq i} q_{ij}$, all the rows of $\mathbf{Q}$ add to 0.
- The rate q_{ij}, $i \neq j$ measures "how fast" the chain moves from state i to j: the higher is q_{ij}, the faster it moves from i to j.
- For $q_{ii} = -\sum_{j \neq i} q_{ij}$, the higher $-q_{ii}$ is, the faster the chain leaves state i.
- If $q_{ij} = 0, \forall j \Rightarrow q_{ii} = 0$, then i is an absorbing state: the chain "moves with rate 0 from i to other states", i.e. never leaves i.

Definition of a CTMC

State Transition Diagram

- A continuous MC is characterized by the transition rate or infinitesimal generator: the Q-matrix.
- The state transition diagram is now represented as:



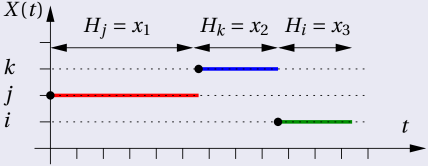
$$\Rightarrow \mathbf{Q} = \begin{bmatrix} -(q_{12}+q_{13}) & q_{12} & q_{13} \\ 0 & -q_{23} & q_{23} \\ q_{31} & q_{32} & -(q_{31}+q_{32}) \end{bmatrix}$$

- Note that now we have transition rates ($0 \leq q_{ij} < \infty$, $i \neq j$) and not probabilities.
- The rates q_{ii} are not written in the diagram, no self transitions.

Definition of a CTMC

Sojourn Time

- Sojourn or holding time: Is the RV H_k equal to the sojourn time in state k:

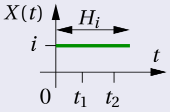


- The Markov property implies that the sojourn time is exponentially distributed with parameter q_{ii}:

$$\boxed{P(H_i \le x) = 1 - e^{q_{ii}x} \Rightarrow P(H_i > x) = e^{q_{ii}x},\ q_{ii} = -\sum_{j \neq i} q_{ij},\ x \ge 0}$$

Definition of a CTMC

The exponential distribution satisfies the Markov property



- Markov property (memoryless):

$$P\big(X(t_2) = i \mid X(t_1) = i, X(0) = i\big) =$$
$$P\big(X(t_2) = i \mid X(t_1) = i\big),\ t_2 > t_1 > 0$$

- In terms of the sojourn time:

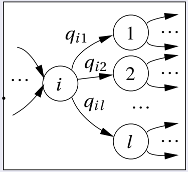
$$P\big(H_i > t_2 \mid H_i > t_1\big) = P(H_i > t_2 - t_1)$$

- But:

$$P\big(H_i > t_2 \mid H_i > t_1\big) =$$
$$\frac{P(H_i > t_2, H_i > t_1)}{P(H_i > t_1)} = \frac{P(H_i > t_2)}{P(H_i > t_1)} = \frac{e^{q_{ii} t_2}}{e^{q_{ii} t_1}} = e^{q_{ii}(t_2 - t_1)} =$$
$$P(H_i > t_2 - t_1) \quad \square$$

- The exponential distribution is the only one satisfying the memoryless property.

Exponential Jumps Description of a CTMC



Assume a process built as follows:

- Upon reaching a state i
  1. the process can jump to a state $j \in \{1, 2, \cdots l\}$.
  2. A set of independent exponential RVs, $\{H_{i1}, H_{i2}, \cdots H_{il}\}$, with parameters $\{q_{i1}, q_{i1}, \cdots q_{il}\}$ are triggered. That is, $P(H_{ij} \le t) = 1 - e^{-q_{ij} t}$, $t \ge 0$.
- If $\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij} \Rightarrow$ the process jumps to the state j. In other words, a transition occurs to state j if the RV H_{ij} is the minimum of $\{H_{i1}, H_{i2}, \cdots H_{il}\}$.

Theorem: This process is a CTMC with transition rates q_{ij}.

Definition of a CTMC

Exponential Jumps Description of a CTMC

$$P(H_{ij} \le t) = 1 - \mathrm{e}^{-q_{ij}\,t}.$$

Theorem: This process is a CTMC with transition rates q_{ij}.

Proof:

- The RV $H_i = \min\{H_{i1}, H_{i2}, \cdots H_{il}\}$ (sojourn time in state i) is exponentially distributed with parameter $q_i = \sum_j q_{ij}$: $P(H_i \le t) = 1 - \mathrm{e}^{-q_i\,t}$.
- $P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) = q_{ij} / \sum_j q_{ij}$. Thus, the transition rate to state j is:

$$\lim_{t\to 0} \frac{p_{ij}(t)}{t} = \lim_{t\to 0} \frac{P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) \times P(H_i \le t)}{t} =$$

$$\frac{q_{ij}}{\sum_j q_{ij}} \left.\frac{\partial P(H_i \le t)}{\partial t}\right|_{t=0} = \frac{q_{ij}}{\sum_j q_{ij}} \sum_j q_{ij} = q_{ij} \quad \square$$

Example: Pure Aloha System

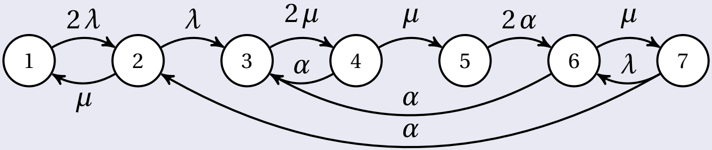
- Consider a Pure Aloha System with 2 nodes:
  - Nodes in thinking state Tx a packet in a time exponentially distributed with rate λ.
  - Transmission time is exponentially distributed with rate μ.
  - If two transmissions overlap, the packet is lost and stations become backlogged (after the packet transmission) until the packet is successfully transmitted.
  - Nodes in backlogged state Tx a packet in a time exponentially distributed with rate α.

Questions

- Build the state transition diagram.

Definition of a CTMC

Example: Pure Aloha System



State	Condition
1	T,T
2	X,T
3	C,C
4	B,C
5	B,B
6	X,B
7	T,B

Legend	
T	Thinking
X	Transmitting
C	Collided transmission
B	Backlogged