Algorithmic Game Theory
Homework 4

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1 Problem 4

1.1 (a) Valuation Function

Lets define a variable x_i in which

$$x_{ij} = \begin{cases} 1 & \text{if a copy of album } j \text{ is provided by player } i \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$v(C) = \left| \left\{ j \mid \sum_{i \in C} x_{ij} \geq k \right\} \right|$$

1.2 (b) Is the game convex?

No, it is not convex because it is not super modular.

Lets probe it by assuming that the game is super modular. A game is **super modular** if $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$.

Lets take 2 partitions C and D and lets assume that both partitions by itself has $\geq k$ copies of some album j but not each player by itself, therefore:

1. $v(C) \geq 1 \wedge v(D) \geq 1$ because at least both partition can re-recording at least the album j.

2. It can be seen by previous that $v(C \cup D) \geq 1$ because we are going to have at least 1 album recording of j because one of the 2 partition group can provide at least k copies and more if we join both.
3. Lets suppose that the combination of both partition $C \cap D$ leads to a partition in which some players don't contribute copy of album j because some of them contribute with players of partition C and some other move to form coalition with players of partition D. In this case coalition $C \cap D$ is not reaching to at least k of album j.
4. By previous statement the only thing we can assure is that $v(C \cap D) \geq 0$ by non-negative condition on Cooperative Games.
5. So we have that $v(C \cup D) + v(C \cap D) \geq 1$ and $v(C) + v(D) \geq 2$

Therefore $v(C \cup D) + v(C \cap D) \not\geq v(C) + v(D)$

Then, **it is not convex.**

1.3 (c) Shapley values

Knowing that Shapley value is additive we can think on splitting up the game as the sum of several Shapley values if they were different games, one for each album j.

Given that $S_\pi(i)$ is the list of predecessors, in this case **players** before i, that may or may not contribute with copy of album j.

We have 2 cases

- If i does not have a copy of j, then $\Phi_i(\Gamma_j) = 0$
- If i does, we need to count the number of possible permutation in which the other players have or not have $k-1$ copies of j

For that we know that if $v(S_\pi(i)) = 0$ and i is added $v(S_\pi(i) \cup \{i\}) = 1$, *if and only if* there is $\sum_{a \in S_\pi(i)} x_{aj} = k - 1$. So we have that,

$$\Phi_i(\Gamma_j) = \frac{|\{\pi | v(S_\pi(i)) = 0 \wedge v(S_\pi(i) \cup \{i\}) = 1\}|}{n!} \tag{1}$$

So, we need to count the number of other players that has copies of j.

Lets define b_j as the number of player that has copies of j.

Basically the permutation has the following form:

$$\xleftarrow[S_\pi(i)]{\qquad\qquad\qquad} i \xrightarrow[N \setminus \{S_\pi(i) \cup \{i\}\}]{\qquad\qquad\qquad}$$

In this picture, we can see that $S_\pi(i)$ has $k-1$ players with a copy of j and the rest after i does not.

Therefore we can argue that:

$$\begin{cases} |S_\pi(i)| \geq k-1 \\ n - |S_\pi(i)| \geq b_j - k \end{cases}$$

Plugin this equations together we have that

$$k-1 \leq |S_\pi(i)| \leq n - b_j + k \tag{2}$$

Lets define this amount of players as $|S_\pi(i)| = \alpha$, so

$$k-1 \leq \alpha \leq n - b_j + k \tag{3}$$

Graphically we can represent the number of players with:

$$\xleftarrow[\alpha]{\qquad\qquad\qquad} i \xrightarrow[n-\alpha-1]{\qquad\qquad\qquad}$$

This means that α_i first players have $k-1$ copies of j and the rest $n-\alpha-1$ does not.

So we need to count all possible combinations of players that has copies of j and those who don't taking into consideration that i has 1 copy.

$$\binom{b_j - 1}{k-1}\binom{n-b_j}{\alpha-k-1} \tag{4}$$

So the total number of permutations when $v(S_\pi(i)) = 0 \wedge v(S_\pi(i) \cup \{i\}) = 1$, with $|S_\pi(i)| = \alpha$ is:

$$\binom{b_j - 1}{k-1}\binom{n-b_j}{\alpha-k-1}\alpha!(n-\alpha-1)! \tag{5}$$

Therefore,

$$\Phi_i(\Gamma) = \frac{\sum_{\alpha=k-1}^{n-b_j-k} \binom{b_j-1}{k-1}\binom{n-b_j}{\alpha-k-1}\alpha!(n-\alpha-1)!}{n!} \tag{6a}$$

$$= \frac{\binom{b_j-1}{k-1}}{n!} \sum_{\alpha=k-1}^{n-b_j-k} \binom{n-b_j}{\alpha-k-1}\alpha!(n-\alpha-1)! \tag{6b}$$

Using additivity property of Shapley values if we telescope this to the sum of all games of different albums j

$$\Phi_i(\Gamma) = \Phi_i\big(\sum_{j=1}^{M}\Gamma_j\big) \tag{7a}$$

$$= \sum_{j=1}^{M}\Phi_i(\Gamma_j) \tag{7b}$$

$$= \sum_{j=1} \frac{\binom{b_j-1}{k-1}}{n!} \sum_{\alpha=k-1}^{n-b_j-k} \binom{n-b_j}{\alpha-k-1}\alpha!(n-\alpha-1)! \tag{7c}$$

Although it cannot be calculated in *Poly-time* because the size of the input can be big and the calculation could not be done in *Poly-time*, the calculation itself of the $\Phi_i(\Gamma)$ can be done in *Poly-time* assuming that **product of numbers** are constant, then **factorials** are linear. On the other hand, the combination depends on α that are a subset of the possible permutation and not the whole Set.