



# Stochastic Network Modeling (SNM)

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(DTMC)

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Research  
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## Stochastic Network Modeling (SNM)

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### Parts

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## Part II

# Discrete Time Markov Chains (DTMC)

### Outline

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- Classification of States
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- Research Example: Aloha
- Finite Absorbing Chains

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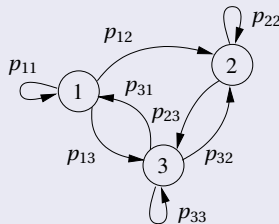
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# Definition of a DTMC

## State Transition Diagram

- We are interested in a **process that evolve in stages**.
- For the model to be tractable, it is convenient to represent the SP by giving all **possible states** (there may be  $\infty$ ), and the **possible transitions** between them:



For the model to be consistent:

$$\sum_j p_{ij} = 1$$

- Mathematically:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$



# Definition of a DTMC

## Properties of a DTMC

- The event  $X(n) = i$  (at step  $n$  the system is in state  $i$ ) must satisfy (**memoryless property**):

$$P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \dots) = P(X(n) = j \mid X(n-1) = i)$$

- If  $P(X(n) = j \mid X(n-1) = i) = P(X(1) = j \mid X(0) = i)$  for any  $n$  we have an **homogeneous** DTMC. We shall only consider homogeneous DTMC.
- We call **one-step transition probabilities** to:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.



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## Transition Matrix

- Transition probabilities:

$$p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

- In matrix form:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$



## Definition of a DTMC

## Transition Matrix

- We have

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } p_{ij} = P(X(n) = j \mid X(n-1) = i)$$

- For the model to be consistent, the probability to move from  $i$  to any state must be 1. Mathematically:

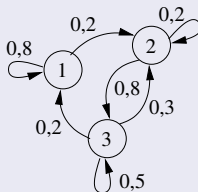
$$\sum_j p_{ij} = \sum_j P(X(n) = j \mid X(n-1) = i) = \sum_j \frac{P(X(n-1) = i \mid X(n) = j) P(X(n) = j)}{P(X(n-1) = i)} = \frac{P(X(n-1) = i)}{P(X(n-1) = i)} = 1$$

- $\mathbf{P}$  is a **stochastic matrix**, i.e. a matrix which rows sum 1.

# Definition of a DTMC

## Example

- Assume a terminal can be in **3 states**:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate  $\nu$  bps.



$$\mathbf{P} = \begin{matrix} & \begin{matrix} \text{to state} \\ 1 & 2 & 3 \end{matrix} \\ \begin{matrix} \text{from state} \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0,8 & 0,2 & 0 \\ 0 & 0,2 & 0,8 \\ 0,2 & 0,3 & 0,5 \end{bmatrix} \end{matrix}$$

- The **average transmission rate** (throughput),  $\nu_a$ , is:

$$\nu_a = P(\text{the terminal is in state 3}) \times \nu$$

# Definition of a DTMC

## Discrete Time Markov Chains (DTMC)

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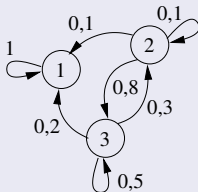
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## Absorbing Chains

- It is possible to have chains with **absorbing states**.
- A state  $i$  is absorbing if  $p_{ii} = 1$ .
- Example: State 1 is absorbing.



$$\mathbf{P} = \begin{matrix} & \begin{matrix} \text{to state} \\ 1 & 2 & 3 \end{matrix} \\ \begin{matrix} \text{from state} \\ 1 & 2 & 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0,1 & 0,1 & 0,8 \\ 0,2 & 0,3 & 0,5 \end{bmatrix} \end{matrix}$$





## Definition of a DTMC

## n-step transition probabilities

- Transition probabilities:  $p_{ij} = P(X(n) = j \mid X(n-1) = i)$
- In matrix form:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

- We define the **n-step** transition probabilities:

$$p_{ij}(n) = P(X(n) = j \mid X(0) = i)$$

$$\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

- $\mathbf{P}$  and  $\mathbf{P}(n)$  are **stochastic matrices**: Their rows sum 1.



# Definition of a DTMC

## State Probabilities

- Define the probability of being in state  $i$  at step  $n$ :

$$\pi_i(n) = P(X(n) = i)$$

- In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \dots) = (P(X(n) = 1), P(X(n) = 2), \dots).$$

- Thus, the vector  $\boldsymbol{\pi}(n)$  is the distribution of the random variable  $X(n)$ , and it is called the **state probability at step  $n$** .



## Definition of a DTMC

## State Probabilities

- State probability:

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \dots) = (P(X(n) = 1), P(X(n) = 2), \dots).$$

- Law of total prob.  $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n)$ :

$$\pi_i(n) = \sum_k P(X(n-1) = k) P(X(n) = i \mid X(n-1) = k) = \sum_k \pi_k(n-1) p_{ki}$$

$$\pi_i(n) = \sum_k P(X(0) = k) P(X(n) = i \mid X(0) = k) = \sum_k \pi_k(0) p_{ki}(n)$$

- In matrix form:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) \mathbf{P}$$

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}(n)$$

where  $\boldsymbol{\pi}(0)$  is the **initial distribution**.



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## State Probabilities

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) \mathbf{P}$$

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n$$

- Iterating

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) \mathbf{P} = \boldsymbol{\pi}(n-2) \mathbf{P} \mathbf{P} = \boldsymbol{\pi}(n-3) \mathbf{P} \mathbf{P} \mathbf{P} = \cdots = \boldsymbol{\pi}(0) \mathbf{P}^n$$

- Thus:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n = \boldsymbol{\pi}(0) \mathbf{P}^n$$



## Definition of a DTMC

## Chapman-Kolmogorov Equations

$$p_{ij}(n) = \sum_k p_{ik}(r) p_{kj}(n-r)$$

• **Proof:**

$$\begin{aligned} p_{ij}(n) &= P(X(n) = j \mid X(0) = i) = \sum_k P(X(n) = j, X(r) = k \mid X(0) = i) \\ &= \sum_k \frac{P(X(n) = j, X(r) = k, X(0) = i)}{P(X(0) = i)} \times \frac{P(X(r) = k, X(0) = i)}{P(X(r) = k, X(0) = i)} \\ &= \sum_k P(X(n) = j \mid X(r) = k, X(0) = i) P(X(r) = k \mid X(0) = i) \\ &= \sum_k P(X(n) = j \mid X(r) = k) P(X(r) = k \mid X(0) = i) \\ &= \sum_k p_{ik}(r) p_{kj}(n-r) \end{aligned}$$

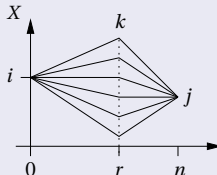


# Definition of a DTMC

## Chapman-Kolmogorov Equations

$$p_{ij}(n) = \sum_k p_{ik}(r) p_{kj}(n-r)$$

- Graphical interpretation:



- In matrix form:

$$\mathbf{P}(n) = \mathbf{P}(r) \mathbf{P}(n-r)$$



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## Chapman-Kolmogorov Equations

$$\mathbf{P}(n) = \mathbf{P}(r) \mathbf{P}(n-r)$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1) \mathbf{P}(n-1) = \mathbf{P} \mathbf{P}(n-1) = \mathbf{P}(n-1) \mathbf{P}$$

- Iterating:

$$\mathbf{P}(n) = \mathbf{P}^n$$

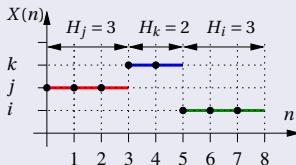
- Thus:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n$$

# Definition of a DTMC

## Sojourn or Holding Time

- Sojourn** or **holding time** in state  $k$ : Is the RV  $H_k$  equal to the number of steps that the chain remains in state  $k$  before leaving to a different state:



- The Markov property implies:

$$H_i(n) = P(H_i = n) = p_{ii}^{n-1} (1 - p_{ii}), n \geq 1$$

- Which is a **geometric** distribution with mean:

$$E[H_i] = \sum_{n=1}^{\infty} n P(H_i = n) = \frac{1}{1 - p_{ii}}.$$





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### Sojourn or Holding Time

- NOTE: We allow that:

$$p_{ii} = 0 \Rightarrow H_i(n) = I(n = 1) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise.} \end{cases}, \text{ and}$$

$$p_{ii} = 1 \Rightarrow E[H_i] = \infty \text{ (absorbing state)}.$$



# Definition of a DTMC

## Theorem

*A stochastic process is a DTMC if and only if the sojourn times are geometrically distributed.*

## Proof.

- We have seen that a DTMC has a sojourn time

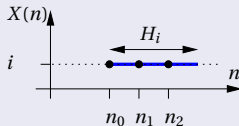
$$H_i(n) = P(H_i = n) = p_{ii}^{n-1} (1 - p_{ii}), n \geq 1$$

- Which is **geometrically** distributed.
- We need to prove that the geometric distribution satisfies the memoryless property (aka Markov property).



# Definition of a DTMC

## The geometric distribution satisfies the Markov property (1)



### Proof

- Markov property:

$$P(X(n_2) = i \mid X(n_1) = i, X(n_0) = i) = P(X(n_2) = i \mid X(n_1) = i)$$

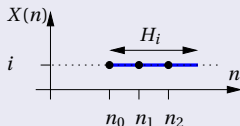
- Thus, the Markov property **in terms of the sojourn time** can be written as:

$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = P(H_i > n_2 - n_1)$$



## Definition of a DTMC

## The geometric distribution satisfies the Markov property (2)



$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = P(H_i > n_2 - n_1)$$

- Since

$$P(H_i > k) = 1 - P(H_i \leq k) = 1 - \sum_{n=1}^k p^{n-1}(1-p) = 1 - (1-p) \frac{1-p^k}{1-p} = p^k$$

- We have:

$$P(H_i > n_2 - n_0 \mid H_i > n_1 - n_0) = \frac{P(H_i > n_2 - n_0, H_i > n_1 - n_0)}{P(H_i > n_1 - n_0)} =$$

$$\frac{P(H_i > n_2 - n_0)}{P(H_i > n_1 - n_0)} = \frac{p^{n_2 - n_0}}{p^{n_1 - n_0}} = p^{n_2 - n_1} = P(H_i > n_2 - n_1) \quad \square$$



Master in Innovation and Research in Informatics (MIRI)  
Computer Networks and Distributed Systems  
**Stochastic Network Modeling (SNM)**

## Part II

# Discrete Time Markov Chains (DTMC)

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- Transient Solution
- Classification of States
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- Research Example: Aloha
- Finite Absorbing Chains

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# Transient Solution

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## Transient Solution

- If we are interested in the **transient evolution** we shall study  $\pi(n) = \pi(0) \mathbf{P}^n$ .
- If we can **diagonalize  $\mathbf{P}$** , we can obtain the transient evolution in **close form**.
- $\mathbf{P}$  can be **diagonalized** if  $\mathbf{P}$  can be decomposed as:

$$\mathbf{P} = \mathbf{L}^{-1} \mathbf{\Lambda} \mathbf{L}$$

where  $\mathbf{L}$  is some invertible matrix and  $\mathbf{\Lambda}$  is the diagonal matrix

$$\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

with  $\lambda_l$ ,  $l = 1, \dots, N$  the **eigenvalues** of  $\mathbf{P}$ .



# Transient Solution

## Eigenvalues

- The **eigenvalues**  $\lambda_l$  of a matrix  $\mathbf{A}$  are scalars that satisfy:  $\mathbf{l}\mathbf{A} = \lambda_l \mathbf{l}$  (or  $\mathbf{A}\mathbf{r} = \lambda_l \mathbf{r}$ ) for some row vectors  $\mathbf{l}$  (column vectors  $\mathbf{r}$ ), referred to as **left and right eigenvectors**, respectively.

$$\mathbf{l}\mathbf{A} = \lambda_l \mathbf{l} \Rightarrow \mathbf{l}(\mathbf{A} - \mathbf{I}\lambda_l) = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

$$\mathbf{A}\mathbf{r} = \lambda_l \mathbf{r} \Rightarrow (\mathbf{A} - \mathbf{I}\lambda_l)\mathbf{r} = 0 \Rightarrow \det(\lambda_l \mathbf{I} - \mathbf{A}) = 0$$

- Thus,  $\lambda_l$  solve the **characteristic polynomial**  $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$ .
- Note that, in general, **left and right eigenvectors** are different, but eigenvalues are the same (they solve the same **characteristic polynomial**).
- A matrix can be **diagonalized** if all eigenvalues are single (multiplicity = 1). If a matrix cannot be diagonalized it is called **defective**.



# Transient Solution

## Determinants

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{aligned} &+ a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{21} a_{32} a_{13} \\ &- a_{31} a_{22} a_{13} - a_{12} a_{21} a_{33} - a_{23} a_{32} a_{11} \end{aligned}$$

- **Cofactor Formula:** expanding along a row  $i$ :

$$\det \mathbf{A} = \sum_{j=1}^N a_{ij} (-1)^{i+j} \det M_{ij},$$

where the **minor matrices**  $M_{ij}$  are obtained removing the row  $i$  and column  $j$  from  $\mathbf{A}$ .  $(-1)^{i+j} \det M_{ij}$  is called the **cofactor** of  $a_{ij}$ .





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## Properties of the determinants

$$\det \mathbf{A} = \prod \text{eigenvalues of } \mathbf{A}$$

$$\text{trace } \mathbf{A} = \sum \text{eigenvalues of } \mathbf{A}$$

where  $\text{trace } \mathbf{A} = \sum \text{elements of the diagonal of } \mathbf{A}$ .



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## Transient Solution

- Assume a **finite DTMC** with  $N$  states. Then  $\mathbf{P} = \mathbf{P}^{N \times N}$ .
- Assume that  $\mathbf{P}$  can be **diagonalized**:  $\mathbf{P} = \mathbf{L}^{-1} \Lambda \mathbf{L}$ , where  $\Lambda$  is the diagonal matrix  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ , with  $\lambda_l, l = 1, \dots, N$  the eigenvalues of  $\mathbf{P}$ .
- Since  $\Lambda^n = \text{diag}(\lambda_1^n, \dots, \lambda_N^n)$ , we have that

$$\begin{aligned}\boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0) \mathbf{P}^n = \boldsymbol{\pi}(0) \mathbf{P}^n = \boldsymbol{\pi}(0) (\mathbf{L}^{-1} \Lambda^n \mathbf{L}) = \\ &\quad \boldsymbol{\pi}(0) (\mathbf{L}^{-1} \text{diag}(\lambda_1^n, \dots, \lambda_N^n) \mathbf{L})\end{aligned}$$



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## Transient Solution

- But  $\mathbf{L}^{-1} \text{diag}(\lambda_1^n, \dots, \lambda_N^n) \mathbf{L}$  are linear combinations of  $\lambda_1^n, \dots, \lambda_N^n$ . Thus, the probability of being in state  $i$  is given by:

$$\pi_i(n) = (\boldsymbol{\pi}(n))_i = \sum_{l=1}^N a_i^{(l)} \lambda_l^n$$

where the **unknown coefficients**  $a_i^{(l)}$  can be obtained solving the system of equations:

$$\sum_{l=1}^N a_i^{(l)} \lambda_l^n = (\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0) \mathbf{P}^n)_i, n = 0, \dots, N-1$$



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## Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

- We want the probability of being in state 2 in  $n$  steps starting from state 1:  $\pi_2(n)$  with  $\boldsymbol{\pi}(0) = [1 \ 0]$ .



## Transient Solution

## Solution

- It can be easily found that the **eigenvalues** of  $\mathbf{P}$  are  $\lambda_1 = 1$  and  $\lambda_2 = 2/5$ .

$$\pi_2(n) = \lambda_1^n a + b \lambda_2^n = a + b(2/5)^n$$

- Imposing the **boundary conditions**  $\pi_i(n) = (\boldsymbol{\pi}(0) \mathbf{P}^n)_i$ :

$$\pi_2(0) = a + b = ([1 \quad 0] \mathbf{P}^0)_2 = (\mathbf{P}^0)_{12} = 0$$

$$\pi_2(1) = a + b(2/5) = ([1 \quad 0] \mathbf{P}^1)_2 = (\mathbf{P})_{12} = 1/5$$

we have that  $a = 1/3$ ,  $b = -1/3$ , thus:

$$\pi_2(n) = 1/3 - 1/3 (2/5)^n, \quad n \geq 0$$

$$\pi_1(n) = 1 - \pi_2(n) = 2/3 + 1/3 (2/5)^n, \quad n \geq 0$$

# Transient Solution

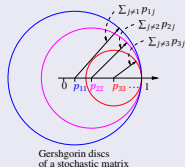
## Eigenvalues of a Stochastic Matrix

- $\mathbf{P}$  has an eigenvalue equal to 1 ( $\mathbf{P}\mathbf{x} = \lambda\mathbf{x}$ , for  $\lambda = 1$ ).

**Proof:**  $\mathbf{P}\mathbf{e} = \mathbf{e}$ , where  $\mathbf{e} = [1 \ 1 \ \dots]^T$  is a column vector of 1 (all rows of  $\mathbf{P}$  add to 1). □

- All eigenvalues of  $\mathbf{P}$  are  $|\lambda_i| \leq 1$ .

**Proof:** Using Gerschgorin's theorem *The eigenvalues of a matrix  $\mathbf{P}_{n \times n}$  lie within the union of the  $n$  circular disks with center  $p_{ii}$  and radius  $\sum_{j \neq i} |p_{ij}|$  in  $\mathbb{C}$ . Since  $\sum_j p_{ij} = 1$ , the property is proved.* □

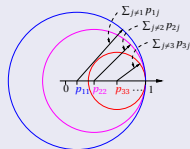


- The eigenvalue  $\lambda = 1$  is single if  $\mathbf{P}$  is irreducible (Perron-Frobenius theorem).  $\mathbf{P}$  is irreducible if all states communicate: for some  $n$ ,  $p_{ij}(n) = (\mathbf{P}^n)_{ij} > 0$ ,  $\forall i, j$ .

# Transient Solution

## Proof of Gerschgorin's theorem

**Gerschgorin's theorem:** *The eigenvalues of a matrix  $\mathbf{P}_{n \times n}$  lie within the union of the  $n$  circular disks with center  $p_{ii}$  and radius  $\sum_{j \neq i} |p_{ij}|$  in  $\mathbb{C}$ .*



Gerschgorin discs  
of a stochastic matrix

**Proof:** From  $\mathbf{P}\mathbf{x} = \lambda\mathbf{x}$  we have

$$\sum_j p_{ij} x_j = \lambda x_i \quad \forall i \in \{1, \dots, n\}.$$

We choose  $i$  such that  $|x_i| = \max_j |x_j|$ . Thus,

$\sum_{j \neq i} p_{ij} x_j = \lambda x_i - p_{ii} x_i$ , and

$$|\lambda - p_{ii}| = \left| \sum_{j \neq i} p_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} \left| p_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} |p_{ij}|$$

and the equation  $|\mathbf{x} - \mathbf{c}| \leq r$ ,  $\mathbf{x}, \mathbf{c} \in \mathbb{C}$ ,  $r \in \mathbb{R}$  is a disk of center  $\mathbf{c}$  and radius  $r$  in  $\mathbb{C}$ . □



# Transient Solution

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## Chain with a Defective Matrix

- What if  $\mathbf{P}$  cannot be diagonalized? (**defective** matrix).
- Let  $\lambda_l$ ,  $l = 1, \dots, L$  be the eigenvalues of  $\mathbf{P}^{N \times N}$ , each with multiplicity  $k_l$  ( $k_l \geq 1$ ,  $\sum_l k_l = N$ ), and a possible eigenvalue  $\lambda_1 = 0$  with multiplicity  $k_1$ . Then [1]:

$$\pi_j(n) = \sum_{m=0}^{k_1-1} a_j^{(1,m)} I(n=m) + \sum_{l=2}^L \lambda_l^n \sum_{m=0}^{k_l-1} a_j^{(l,m)} n^m, \\ 1 \leq j \leq N, n \geq 0$$

$I(n=m)$  is the indicator func.:  $I(n) = 1$  if  $n = m$ ,  $I(n) = 0$  if  $n \neq m$ .

- [1] Llorenç Cerdà-Alabern. *Transient Solution of Markov Chains Using the Uniformized Vandermonde Method*. Tech. rep. UPC-DAC-RR-XCSD-2010-2. Universitat Politècnica de Catalunya, Dec. 2010. URL: [https://www.ac.upc.edu/app/research-reports/html/research\\_center\\_index-XCSD-2010,en.html](https://www.ac.upc.edu/app/research-reports/html/research_center_index-XCSD-2010,en.html).





# Transient Solution

## Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

- We want the probability of being in state 1 in  $n$  steps starting from state 1:  $\pi_1(n)$  with  $\pi_1(0) = 1$ .
- It can be easily found that the **eigenvalues** of  $\mathbf{P}$  are  $\lambda_1 = 1$  and  $\lambda_2 = 1/4$  with multiplicity 2. We guess:

$$\pi_1(n) = a + 1/4^n(b + cn)$$

- Imposing  $\pi_1(0) = 1$ ,  $\pi_1(1) = 3/4$ ,  $\pi_1(2) = (3/4)^2$ , we have:

$$\pi_1(n) = \frac{4}{9} + \frac{1}{4^n} \left( \frac{5}{9} + \frac{2}{3} n \right)$$



## Part II

# Discrete Time Markov Chains (DTMC)

### Outline

- Definition of a DTMC
- Transient Solution
- **Classification of States**
- Steady State
- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains

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## Objective

- Identify the different **types of behavior** that the chain can have.
- Introduce the concepts of **first passage probability** and **mean recurrence time**.



# Classification of States

## Irreducibility

- A state  $j$  is said to **communicate** with  $i$ ,  $i \leftrightarrow j$ , if  $p_{ij}(m_1) > 0$ ,  $p_{ji}(m_2) > 0$  for some  $m_1, m_2 \geq 0$ .
- We define an **irreducible closed set, ICS**  $C_k$  as a set where all states communicate with each other, and have no transitions to other states out of the set:  

$$i \leftrightarrow j, \forall i, j \in C_k \text{ and } p_{ij} = 0, \forall i \in C_k, j \notin C_k$$
 (note that for  $i \in C_k, j \notin C_k$  we have:  $p_{ij}(2) = \sum_k p_{ik} p_{kj} = 0$ , since  $p_{ik} = 0$  if  $k \notin C_k$ , and  $p_{kj} = 0$  if  $k \in C_k$ . Thus,  $p_{ij}(n) = 0, \forall n$ .)
- An **absorbing state** form an ICS of only one element. This state,  $i$ , must have  $p_{ii} = 1, p_{ij} = 0 \forall j \neq i$ .
- Transient states** do not belong to any ICS.
- A MC is **irreducible** if all the states form a unique ICS.



## Classification of States

## Irreducibility

- Assume a MC has  **$M$  ICSs**: By properly numbering the states, we can write  $\mathbf{P}$  as an  $M$  block diagonal matrix with the probabilities of the transient states in the last rows.
- Example**, if  $M = 3$ :

$$\mathbf{P} = \begin{array}{|c|c|c|c|} \hline \mathbf{P}_1 & & & \\ \hline & \mathbf{P}_2 & & \\ \hline & & \mathbf{P}_3 & \\ \hline \text{at least} & & & \mathbf{T} \\ \text{one} & & & \\ > 0 & & & \\ \hline \end{array}$$

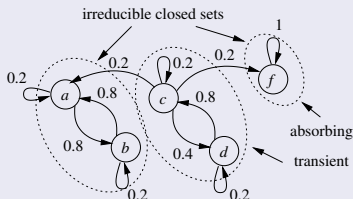
$$\Rightarrow \boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n = \boldsymbol{\pi}(0)$$

$$\begin{array}{|c|c|c|c|} \hline \mathbf{P}_1^n & & & \\ \hline & \mathbf{P}_2^n & & \\ \hline & & \mathbf{P}_3^n & \\ \hline \text{at least} & & & \mathbf{T}^n \\ \text{one} & & & \\ > 0 & & & \\ \hline \end{array}$$

- Note that **the  $M$  sub-matrices are stochastic** (their rows sum 1).

# Classification of States

## Example



$$\mathbf{P} = \begin{array}{c} \begin{matrix} a & b & f & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ f \\ c \\ d \end{matrix} \left[ \begin{array}{cc|cc|cc} 0,2 & 0,8 & 0 & 0 & 0 \\ 0,8 & 0,2 & 0 & 0 & 0 \\ \hline 0 & 0 & 1,0 & 0 & 0 \\ 0,2 & 0 & 0,2 & 0,2 & 0,4 \\ 0 & 0 & 0 & 0,8 & 0,2 \end{array} \right] \end{array}$$

$$\mathbf{P}^\infty = \begin{array}{c} \begin{matrix} a & b & f & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ f \\ c \\ d \end{matrix} \left[ \begin{array}{cc|cc|cc} 0,5 & 0,5 & 0 & 0 & 0 \\ 0,5 & 0,5 & 0 & 0 & 0 \\ \hline 0 & 0 & 1,0 & 0 & 0 \\ 0,25 & 0,25 & 0,5 & 0 & 0 \\ 0,25 & 0,25 & 0,5 & 0 & 0 \end{array} \right] \end{array}$$

- What is the meaning of the probabilities in  $\mathbf{P}^\infty$ ? (recall that  $(\mathbf{P}^n)_{ij} = p_{ij}(n) = P(X(n) = j \mid X(0) = i)$ ).



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$$\mathbf{P} = \begin{array}{|c|c|c|} \hline \mathbf{P}_1 & & \mathbf{0} \\ \hline & \mathbf{P}_2 & \\ \hline \mathbf{0} & & \mathbf{P}_3 \\ \hline \text{at least} & & \\ \text{one } > 0 & & \mathbf{T} \\ \hline \end{array}$$

**Theorem** *The multiplicity of the eigenvalue  $\lambda = 1$  is equal to the number of irreducible closed sets.*

**Proof** The characteristic polynomial of  $\mathbf{P}$  is equal to the product of the characteristic polynomials of the sub-matrices  $\mathbf{P}_i$  and  $\mathbf{T}$ . Since  $\mathbf{P}_i$  are irreducible stochastic, each will have a single eigenvalue equal to 1. For the transitorial states it must be  $\lim_{n \rightarrow \infty} \mathbf{T}^n = \mathbf{0}$ . Thus, all the eigenvalues of  $\mathbf{T}$  must be  $|\lambda| < 1$ .  $\square$

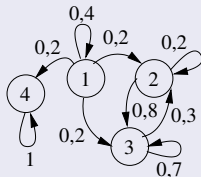
**NOTE:** in the closed form solution there is only one unknown associated with  $\lambda = 1$ , otherwise  $\sum_{m=0}^{k_l-1} a_j^{(l,m)} n^m$  will diverge as  $n \rightarrow \infty$  (i.e.  $a_j^{(l,m)} = 0, m > 0$ ), and  $a_j^{(l,0)} = \lim_{n \rightarrow \infty} \pi_j(n)$ .



# Classification of States

## Transient and Recurrent

- **Recurrent**: States that, being visited, they are visited again with probability 1. They are visited an infinite number of times when  $n \rightarrow \infty$ .
- **Transient**: States that, being visited, have a probability  $< 1$  of never being visited again. They are visited a finite number of times when  $n \rightarrow \infty$ .
- **Absorbing**: A single (recurrent) state where the chain remains with probability = 1.



State 1 is **transient**  
States 2 and 3 are **recurrent**  
State 4 is **absorbing**





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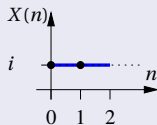
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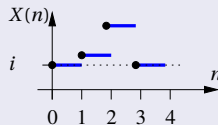
## First Passage (Transition) Probabilities

- To derive a classification criteria, we shall study the distribution of the number of steps to **go for the first time from a state  $i$  another state  $j$** . Definition:

$$f_{ij}(n) = P\left(\begin{array}{l} \text{first transition into state } i \\ \text{in } n \text{ steps starting from } i \end{array}\right)$$



first transition in 1 step



first transition in 3 steps

- Do **not confuse** with the  $n$ -step transition probability  $p_{ij}(n)$ , where the state  $i$  can be visited in the intermediate states.



## Classification of States

Relation between  $f_{ii}(n)$  and  $p_{ii}(n)$ 

- $f_{ii}(n)$  and  $p_{ii}(n)$  satisfy:

$$f_{ii}(1) = p_{ii}(1)$$

$$p_{ii}(n) = \sum_{l=1}^n f_{ii}(l) p_{ii}(n-l), n \geq 1$$

- The probability that the MC **eventually enters state  $i$  starting from  $i$**  is given by:

$$f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n)$$

- If  $f_{ii} = 1$  we say  $i$  is a **recurrent state**.
- If  $f_{ii} < 1$  we say  $i$  is a **transient state**.



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## Generalization to Any State Pair

- Analogously to  $f_{ii}(n)$ , we define the probability of the **first passage to state  $j$  starting from any state  $i$**  in  $n$  steps:  $f_{ij}(n)$ .
- $f_{ij}(n)$  and  $p_{ij}(n)$  satisfy:

$$p_{ij}(n) = \sum_{l=1}^n f_{ij}(l) p_{ij}(n-l), \quad n \geq 1$$



## Classification of States

## Recursive Equation for the First Passage Probabilities

- Recall that the The probability that the MC **eventually enters state  $j$  starting from  $i$**  is given by:  $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$
- $f_{ij}$**  can be computed as follows: Assume we are in  $i$ . With probability  $p_{ij}$  we will go to  $j$  in one step. Otherwise, we will go to  $k$ ,  $k \neq j$ , and then we will reach  $j$  with probability  $f_{kj}$ . Thus:

$$f_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}$$

- If there are more than 1 **absorbing states**, we can compute the probability to reach them using this method (if there is only 1, say  $j$ , then  $f_{ij} = 1, \forall i$ ).



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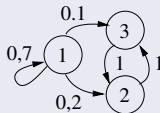
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## Example: Recurrence Times Using the Definition



$$\mathbf{P} = \begin{bmatrix} 0,7 & 0,2 & 0,1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f_{21}(n) = f_{31}(n) = 0$$

$$f_{11}(n) = 0,7 I(n=1)$$

$$f_{22}(n) = f_{33}(n) = I(n=2)$$

$$f_{23}(n) = f_{32}(n) = I(n=1)$$

$$f_{11} = 0,7$$

$$f_{12} = f_{13} = 1 \quad f_{22} = f_{23} = 1$$

$$f_{32} = f_{33} = 1 \quad f_{21} = f_{31} = 0$$

$$f_{12}(n) = \begin{cases} 0,2, & n=1 \\ 0,7^{n-1} 0,2 + 0,7^{n-2} 0,1, & n>1 \end{cases}$$

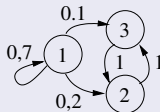
$$f_{13}(n) = \begin{cases} 0,1, & n=1 \\ 0,7^{n-1} 0,1 + 0,7^{n-2} 0,2, & n>1 \end{cases}$$

- State 1 is **transient**. States 2 and 3 are **recurrent**.



# Classification of States

## Example: First Passage Probability Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0,7 & 0,2 & 0,1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$f_{12} = p_{11}f_{12} + p_{12} + p_{13}f_{32}$$

- Clearly  $f_{32} = 1$ , thus:

$$f_{12} = 0,7f_{12} + 0,2 + 0,1 \times 1 \Rightarrow f_{12} = 1$$

as before.



# Classification of States

## Mean Recurrence Time

- If  $f_{ii} = 1$  we say  $i$  is a **recurrent state**.
- If  $f_{ii} < 1$  we say  $i$  is a **transient state**.
- When  $f_{ii} = 1$ , we define the **mean recurrence time**  $m_{ii}$  as:

$$m_{ii} = \sum_{n=1}^{\infty} n f_{ii}(n)$$

- $m_{ii}$  is the **average number of steps to eventually reach  $i$  starting from  $i$** . If  $f_{ii} < 1$  (**transient state**) then we define  $m_{ii} = \infty$ .
- Classification of **recurrent states** ( $f_{ii} = 1$ ):
  - If  $m_{ii} = \infty$  the state is **null recurrent**: it takes an  $\infty$  time to reach the state after leave it. Can only happen in chains with an infinite number of states.
  - If  $m_{ii} < \infty$  the state is **positive recurrent**: the state is reached in a finite time after leave it.



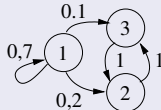
# Classification of States

## Property of States

In **finite MC**:

- 1 States can be only of type positive recurrent or transient.
- 2 At least one state must be positive recurrent.
- 3 There are not null recurrent states.

• **Example:**



- State 1 is transient. States 2 and 3 are positive recurrent.





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## Generalization to Any State Pair

- When  $f_{ij} = 1$ , the average number of steps to eventually reach  $j$  starting from  $i$ ,  $m_{ij}$  is given by:

$$m_{ij} = \sum_{n=1}^{\infty} n f_{ij}(n)$$

- If state  $j$  can not be reached starting from state  $i$  with probability one (if  $f_{ij} < 1$ ), then we define  $m_{ij} = \infty$ .



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## Recursive Equation for the Mean Recurrence Time

- Recall that the **mean recurrence time**  $m_{ij} = \sum_{n \geq 1} n f_{ij}(n)$  is the average number of steps to eventually reach  $j$  starting from  $i$ , i.e. it is the mean first passage time from state  $i$  to  $j$ .
- When  $f_{ij} = 1$ ,  $m_{ij}$  can be computed as follows: Assume we are in  $i$ . With probability  $p_{ij}$  we will go to  $j$  in one step. Otherwise, we will go to  $k$ ,  $k \neq j$ , and then it will take  $m_{kj}$  steps to reach  $j$ . Thus:

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} (1 + m_{kj}) = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

since  $\sum_j p_{ij} = 1$ .

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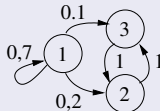
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## Example: Mean Recurrence Time Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0,7 & 0,2 & 0,1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$m_{12} = p_{12} + p_{11} (1 + m_{12}) + p_{13} (1 + m_{32}) = 1 + p_{11} m_{12} + p_{13} m_{32}$$

- Clearly  $m_{32} = 1$ , thus:

$$m_{12} = 1 + 0,7 m_{12} + 0,1 \times 1 \Rightarrow m_{12} = 11/3.$$



# Classification of States

## Periodic states

- A recurrent state  $j$  is **periodic** with period  $d > 1$  if  $j$  can only be reached after leaving it with a multiple of  $d$  steps.
- If  $d = 1$  the state is aperiodic.
- Any **periodic irreducible chain** can be partitioned in  $d$  **cyclic classes**  $C_0, \dots, C_{d-1}$  such that at each step a transition occur from class  $C_i$  to  $C_{(i+1) \bmod d}$ .
- By properly numerating the states, the transition matrix can be written as (the sub-matrices  $\mathbf{A}_i$  may not be square):

$$\mathbf{P} = \begin{matrix} & \begin{matrix} C_0 & C_1 & C_2 & \dots & C_{d-1} \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \\ \dots \\ C_{d-1} \end{matrix} & \left[ \begin{array}{ccccc} 0 & \mathbf{A}_1 & 0 & \dots & 0 \\ 0 & 0 & \mathbf{A}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{A}_{d-1} & 0 & 0 & \dots & 0 \end{array} \right] \end{matrix}$$

# Classification of States

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#### Probabilities

#### Relation between $f_{ii}(n)$ and $p_{ii}(n)$

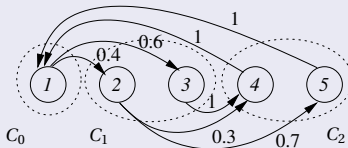
#### Generalization to Any State Pair

#### Recursive Equation for the First Passage Probabilities

#### Example: Recurrence Times Using the Definition

#### Example: First Passage Probabilities Using

## Example



$$P = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \end{bmatrix}, P^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \end{bmatrix}, P^4 = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \dots$$

- In periodic chains  $P^n$  does not converge.



# Stochastic Network Modeling (SNM)

## Part II

### Discrete Time Markov Chains (DTMC)

#### Outline

- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State
- Reversed Chain
- Reversible Chains
- Research Example: Aloha
- Finite Absorbing Chains

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## Limiting Distribution

- Probability of being in state  $i$  at step  $n$ :

$$\pi_i(n) = P(X(n) = i).$$

In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \dots).$$

- The evolution of the chain depends on the initial distribution  $\boldsymbol{\pi}(0)$ .
- If we are interested in the **transient** evolution we shall study

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0) \mathbf{P}^n.$$

- If we are interested the **steady state** we shall be interested in the **limiting distribution** (if the limit exists):

$$\boldsymbol{\pi}(\infty) = (\pi_1(\infty), \pi_2(\infty), \dots)$$



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## Limiting Distribution

Assume an **irreducible** chain with **positive recurrent** states.

- With infinite steps, we look for a probability converging to a value that depends only on the final state:

$$\pi_j(\infty) = \sum_i \pi_i(0) \lim_{n \rightarrow \infty} p_{ij}(n), \forall j \text{ and for any } \pi(0),$$

which implies:

$$\pi_j(\infty) = \lim_{n \rightarrow \infty} p_{ij}(n) \sum_i \pi_i(0) = p_{ij}(\infty), \forall j \Rightarrow$$

$$\mathbf{P}(\infty) = \lim_{n \rightarrow \infty} \mathbf{P}^n = \begin{bmatrix} \boldsymbol{\pi}(\infty) \\ \vdots \\ \boldsymbol{\pi}(\infty) \end{bmatrix}$$

- If this limit exists, we call  $\mathbf{P}(\infty)$  the **limiting matrix**, and  $\boldsymbol{\pi}(\infty)$  the **limiting distribution**.





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## Example

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0.764 & 0.168 & 0.068 \\ 0.760 & 0.170 & 0.070 \\ 0.752 & 0.174 & 0.074 \end{bmatrix}$$

$$\mathbf{P}^4 = \begin{bmatrix} 0.7628 & 0.1686 & 0.0686 \\ 0.7620 & 0.1690 & 0.0690 \\ 0.7604 & 0.1698 & 0.0698 \end{bmatrix}$$

$$\mathbf{P}^8 = \begin{bmatrix} 0.762500 & 0.168750 & 0.068750 \\ 0.762499 & 0.168750 & 0.068750 \\ 0.762497 & 0.168752 & 0.068752 \end{bmatrix}$$

...

$$\Rightarrow \boldsymbol{\pi}(\infty) = (0.76250, 0.16875, 0.06875)$$



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## Stationary distribution

- We have:

$$\begin{aligned}\pi_i(n) &= P(X(n) = i) = \sum_k P(X(n-1) = k) P(X(n) = i \mid X(n-1) = k) \\ &= \sum_k \pi_k(n-1) p_{ki}\end{aligned}$$

- In matrix form:  $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1) \mathbf{P}$
- If  $\pi_i(n) = \pi_i(n-1) = \pi_i \forall i$ , we call  $\pi_i$  the **stationary probability of state  $i$** , and  $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ , the **stationary distribution** of the chain.
- In matrix form (**Global balance equations**):

$$\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}$$

$$\boldsymbol{\pi} \mathbf{e} = 1, \mathbf{e} = [1 \quad 1 \quad \dots]^T$$

- Thus, the stationary distribution is the left-hand eigenvector corresponding to the unit eigenvalue of  $\mathbf{P}$ .
- $\boldsymbol{\pi}(n) = \boldsymbol{\pi} \Rightarrow \boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n) \mathbf{P} = \boldsymbol{\pi} \mathbf{P} = \boldsymbol{\pi} \Rightarrow \boldsymbol{\pi}(k) = \boldsymbol{\pi}, k \geq n$



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## Stationary distribution

- Do not confuse the **limiting distribution**  $\pi(\infty)$  and the **stationary distribution**  $\pi = \pi \mathbf{P}$ .
- $\pi(\infty)$  and  $\pi$  **may not be the same**, e.g. in **periodic chains**  $\pi(\infty)$  does not exist ( $\mathbf{P}$  does not converge), but we can compute the stationary distribution.
- Example:** the periodic chain

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

has the stationary distribution

$$\pi = [1/3 \quad 1/3 \quad 1/3].$$

## Steady State

## Numerical Solution

- Replace one equation method:

$$\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}$$

$$\boldsymbol{\pi} \mathbf{e} = 1, \mathbf{e} = [1 \quad 1 \quad \dots]^T$$

- We solve the equation  $\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}) = 0$  replacing the last equation by  $\boldsymbol{\pi} \mathbf{e} = 1$ :

$$\boldsymbol{\pi} \begin{bmatrix} p_{11} - 1 & p_{12} & \dots p_{1n-1} & 1 \\ p_{21} & p_{22} - 1 & \dots p_{2n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \dots p_{nn-1} & 1 \end{bmatrix} = [0 \quad 0 \quad \dots \quad 0 \quad 1]$$



## Steady State

## Numerical Solution

- **Replace one equation method:**  $\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$
- With **octave** (matlab clone):

```
octave:1> P=[0.8,0.15,0.05;0.7,0.2,0.1;0.5,0.3,0.2];
octave:2> s=size(P,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [eye(s,s-1)-P(1:s,1:s-1), ones(s,1)]
ans =
0.762500 0.168750 0.068750
```

- With **R**

```
> P <- matrix(nc=3, byr=T, c(0.8,0.15,0.05,0.7,0.2,0.1,0.5,0.3,0.2))
> s <- nrow(P)
> solve(t(cbind(P[,1:(s-1)]-diag(nr=s,nc=s-1), rep(1,s))),
+ c(rep(0,s-1),1))
[1] 0.76250 0.16875 0.06875
```

NOTE:  $\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P} \Rightarrow \boldsymbol{\pi}^T = \mathbf{P}^T \boldsymbol{\pi}^T$ . The transpose operator in R is `t()`.



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### Ergodic Chains

**Ergodic state** positive recurrent and aperiodic state.

**Ergodic chain** if all states are ergodic.

**Theorem:** All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent, and aperiodic/periodic with the same period [1, chapter XV].

**Consequences:**

- **Finite aperiodic and irreducible** chains are **ergodic** (since all states are positive recurrent).
- **Infinite aperiodic and irreducible** chains can be:
  - **Ergodic:** all the states are positive recurrent (stable chains).
  - **Non ergodic:** all states are null recurrent or transient (unstable chains).

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd Edition*. Wiley, 1968.



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## Theorems for ergodic chains

- Both stationary and limiting distribution exist and are equal,  $\pi = \pi(\infty)$ .
- In stationary regime (when  $\pi(n) \mathbf{P} = \pi(n)$ ), the **mean number of steps the system remains in state  $j$**  during  $k$  steps is given by

$$k\pi_j$$

thus,  $\pi_j$  is the average fraction of a step the chain remains in state  $j$  in stationary regime.

- In stationary regime the **mean recurrence time** (mean number of steps between two consecutive visits to state  $j$ ) is given by

$$m_{jj} = 1/\pi_j$$

The last properties are also valid for periodic chains.



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## Theorems for ergodic chains (proofs)

- Both stationary and limiting distribution exist and are equal,  $\pi = \pi(\infty)$ .

- Proof**

For an **aperiodic irreducible** chain with **positive recurrent** states:

$$\begin{cases} \pi(\infty) &= \pi(0) \mathbf{P}(\infty) \\ \mathbf{P}(\infty) &= \lim_{n \rightarrow \infty} \mathbf{P}^n = \begin{bmatrix} \pi(\infty) \\ \dots \\ \pi(\infty) \end{bmatrix} \end{cases} \Rightarrow$$

$$\pi(\infty) \mathbf{P} = (\pi(0) \lim_{n \rightarrow \infty} \mathbf{P}^n) \mathbf{P} = \pi(0) \mathbf{P}(\infty) = \pi(\infty)$$

$$\Rightarrow \begin{cases} \pi(\infty) \mathbf{P} = \pi(\infty) \\ \pi(\infty) \mathbf{e} = 1 \end{cases} \quad \pi(\infty) \text{ satisfies the GBE} \Rightarrow \pi = \pi(\infty)$$





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## Theorems for ergodic chains (proofs)

- In stationary regime (when  $\boldsymbol{\pi}(n) \mathbf{P} = \boldsymbol{\pi}(n)$ ), the **mean number of steps the system remains in state  $j$**  during  $k$  steps is given by

$$k\pi_j.$$

- Proof**

Assume the chain in stationary regime at time  $t = 0$  ( $\boldsymbol{\pi}(0) \mathbf{P} = \boldsymbol{\pi}(0)$ ), and let  $j(k)$  be the number of visits to  $j$  in  $k$  steps:  $j(k) = \sum_{i=0}^{k-1} I(X(i) = j)$  ( $I(A)$  is the indicator function:  $I(A) = 1$  if  $A$  occurs,  $I(A) = 0$  otherwise):

$$E[j(k)] = \sum_{i=0}^{k-1} E[I(X(i) = j)] = \sum_{i=0}^{k-1} P(X(i) = j) = k\pi_j \quad \square$$



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## Theorems for ergodic chains (proofs)

- In stationary regime the **mean recurrence time** (mean number of steps between two consecutive visits to state  $j$ ) is given by

$$m_{jj} = 1/\pi_j$$

- Proof**

Let  $j(k)$  be the number of visits to  $j$  in  $k$  steps:

$$\pi_j = \lim_{k \rightarrow \infty} \frac{j(k)}{k} = \lim_{k \rightarrow \infty} \frac{1}{k/j(k)} = 1/m_{jj} \quad \square$$

# Steady State

## Global balance equations

- Why are they called Global balance equations?

$$\left. \begin{aligned} \boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P} &\Rightarrow \pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij} \\ \sum_{i=0}^{\infty} p_{ji} = 1 &\Rightarrow \pi_j \sum_{i=0}^{\infty} p_{ji} = \pi_j \end{aligned} \right\} \Rightarrow \sum_{i=0}^{\infty} \pi_i p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji}$$

$\sum_{i=0}^{\infty} \pi_i p_{ij} \Rightarrow$  Frequency of **transitions entering state  $j$**

$\pi_j \sum_{i=0}^{\infty} p_{ji} \Rightarrow$  Frequency of **transitions leaving state  $j$**

- In **stationary regime**, the frequency of transitions leaving state  $j$  is equal to the frequency of transitions entering state  $j$ .



## Steady State

## Flux Balancing

- Define the **flux**  $F_{uv}$  from state  $u$  to  $v$ :

$$F_{uv} = \pi_u p_{uv}$$

- and the flux from set of states  $U$  to  $V$ :

$$F(U, V) = \sum_{u \in U} \sum_{v \in V} F_{uv}$$

- From the Global balance equations we have:

$$\sum_{i=0}^{\infty} \pi_i p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji} \Rightarrow \sum_{i \in U} F_{ij} + \sum_{i \notin U} F_{ij} = \sum_{i \in U} F_{ji} + \sum_{i \notin U} F_{ji}$$

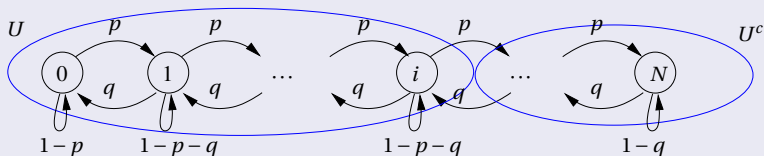
- Adding for  $j \in U$ :

$$\sum_{j \in U} \sum_{i \in U} F_{ij} + \sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \in U} F_{ji} + \sum_{j \in U} \sum_{i \notin U} F_{ji} \Rightarrow \sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \notin U} F_{ji}$$

$$\Rightarrow F(U, U^c) = F(U^c, U)$$

## Steady State

## Solution Using Flux Balancing



- Flux balancing  $\Rightarrow p\pi_i = q\pi_{i+1}$
- Iterating:  $\pi_1 = \rho \pi_0, \pi_2 = \rho \pi_1 = \rho^2 \pi_0, \dots, \Rightarrow$

$$\pi_i = \rho^i \pi_0, i = 1, \dots, N \quad \text{where: } \rho = \frac{p}{q},$$

- Normalizing:  $\sum_{i=0}^N \pi_i = 1$

$$\pi_0 = \frac{1 - \rho}{1 - \rho^{N+1}}, \quad p \neq q$$

$$\pi_0 = \frac{1}{N+1}, \quad p = q$$



## Part II

# Discrete Time Markov Chains (DTMC)

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# Reversed Chain

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## Definition

- Let  $X(n)$  be an **ergodic** MC. The chain  $X^r(n) = X(-n)$  is referred to as the **time reversal chain** of  $X(n)$ .
- Example**, consider a possible sample path of  $X(n)$ :

$$\cdots (i_0, n_0), (i_1, n_1), (i_2, n_2), \cdots$$

The same path in the time reversal chain  $X^r(n)$  would be:

$$\cdots (i_2, -n_2), (i_1, -n_1), (i_0, -n_0), \cdots$$

## Properties

- Let  $p_{ij}$ ,  $p_{ij}^r$  be the transition probabilities of  $X(n)$  respectively  $X^r(n)$ , and  $\pi_i$ ,  $\pi_i^r$  the stationary distributions of  $X(n)$  respectively  $X^r(n)$ , then:

$$\pi_i = \pi_i^r$$

- Proof:** the mean time in each state is the same for both chains.  $\square$
- However, **in general**  $p_{ij} \neq p_{ij}^r$ . For example,  $X(n)$  may be able to jump from state  $i$  to  $j$ , but not from  $j$  to  $i \Rightarrow X^r(n)$  can jump from  $j$  to  $i$ , but not from  $i$  to  $j$ .
- But it must be  $p_{ii} = p_{ii}^r$ , since self-state transitions are the same in the direct and reversed chains.



## Computation of $p_{ij}^r$

- The transition probabilities in the time reversal chain ( $p_{ji}^r$ ) satisfy:

$$\pi_i p_{ij} = \pi_j p_{ji}^r$$

- Proof** Assume the chain in **steady state**. We have:

$$\begin{aligned} P\{X(n+1) = j, X(n) = i\} &= P\{X^r(-n) = i, X^r(-n-1) = j\} = \\ &= P\{X^r(n+1) = i, X^r(n) = j\} \Rightarrow \end{aligned}$$

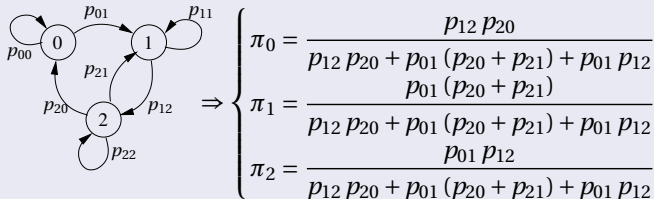
$$\begin{aligned} P\{X(n) = i\} P\{X(n+1) = j \mid X(n) = i\} &= \\ P\{X(n) = j\} P\{X^r(n+1) = i \mid X^r(n) = j\} &\Rightarrow \pi_i p_{ij} = \pi_j p_{ji}^r. \quad \square \end{aligned}$$

- We can **compute  $p_{ji}^r$**  using the **reversed balance equations**:  

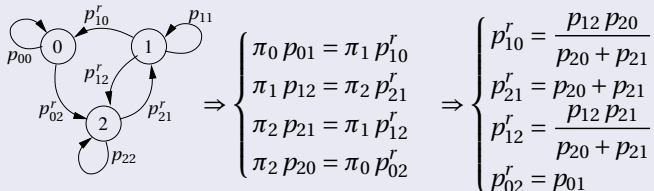
$$\pi_i p_{ij} = \pi_j p_{ji}^r \Rightarrow \sum_{i \in U} \sum_{j \in V} \pi_i p_{ij} = \sum_{i \in U} \sum_{j \in V} \pi_j p_{ji}^r \Rightarrow$$

$$F(U, V) = F^r(V, U)$$

## Example



- Time reversal chain:





## Part II

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## Definition

- A chain is reversible if:

$$p_{ij} = p_{ij}^r$$

- This equality implies the **reversibility balance equations**:

$$\pi_i p_{ij} = \pi_j^r p_{ji}^r \Rightarrow F(U, V) = F^r(U, V)$$

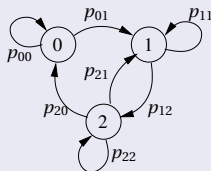
- Using both reversed ( $F^r(U, V) = F(V, U)$ ) and reversibility balance equations, the following relation holds for a reversible chain (**detailed balance equations**):

$$F(U, V) = F(V, U)$$

- NOTE: Compare with the **global balance equations**:  
 $F(U, U^C) = F(U^C, U)$ .

## Definition of path

- Define a **path** as a possible sequence of transitions of the chain. For example, in the figure it could be  $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$ .



- We denote the **sequence of states** of one path  $l$  as:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \dots \rightsquigarrow (l,m)$$

- For instance, if  $l$  is  $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$ , then  $(l,1) = 0$ ,  $(l,2) = 0$ ,  $(l,3) = 1$ ,  $(l,4) = 2$ .
- We say that a path is **closed** if it starts and ends in the same state. For instance, a path starting and ending in state  $(l,1)$ :

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \dots \rightsquigarrow (l,m) \rightsquigarrow (l,1)$$



# Reversible Chains

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## Kolmogorov Criteria

- Take a **closed path**  $l$  with  $m \geq 0$  transitions, i.e.:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m) \rightsquigarrow (l,1), m \geq 0$$

- The chain is **reversible** iff for all  $l$ :

$$p_{(l,1)(l,2)} p_{(l,2)(l,3)} \cdots p_{(l,m)(l,1)} = p_{(l,1)(l,m)} p_{(l,m)(l,m-1)} \cdots p_{(l,2)(l,1)}$$

- Proof:**
  - If the chain is reversible  $\pi_i p_{ij} = \pi_j p_{ji}$  (detailed balance equations):  $\Rightarrow \pi_{(l,k)} p_{(l,k)(l,k+1)} = \pi_{(l,k+1)} p_{(l,k+1)(l,k)}$
  - Multiplying for  $k = 1, 2, \dots, m$  and simplifying we obtain the previous relation.  $\square$

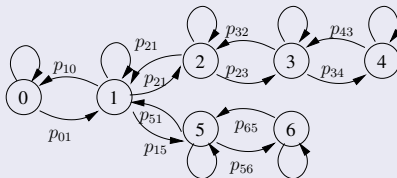
## Kolmogorov Criteria. Corollary

- A reversible chain must satisfy:

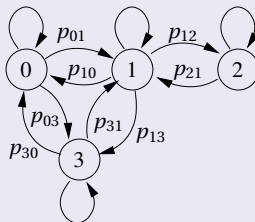
$$p_{ij} > 0 \Rightarrow p_{ji} > 0$$

$$p_{ij} = 0 \Rightarrow p_{ji} = 0$$

- An **ergodic tree chain is always reversible**. We define a tree chain as chain having no cycles, i.e. the only possible transitions are bidirectional arcs between states, and self transitions.



## Kolmogorov Criteria. Example



- The chain is **reversible** iff:

$$p_{01} p_{13} p_{30} = p_{10} p_{03} p_{31}$$



## Product Form Solution

- Let  $X(n)$  be a reversible MC with space state  $S \Rightarrow$  the **stationary probabilities** of  $X(n)$  can be computed as follows:
- Choose a state  $s \in S$ ,
- For every other state  $i \in S$ ,  $i \neq s$  look for a possible path  $l_i$  from state  $s$  to state  $i$ :

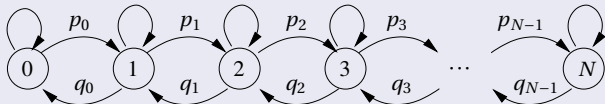
$$s = (l_i, 1) \rightsquigarrow (l_i, 2) \rightsquigarrow \dots \rightsquigarrow (l_i, m_{l_i}) = i, m_{l_i} \geq 1$$

- The stationary probabilities are given by:

$$\pi_i = \frac{\psi_i}{\sum_{j \in S} \psi_j}, i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \prod_{k=1}^{m_{l_i}-1} \frac{p_{(l_i, k)(l_i, k+1)}}{p_{(l_i, k+1)(l_i, k)}}, & i \neq s \end{cases}$$

- Proof** Use the detailed balance equations  $\pi_i p_{ij} = \pi_j p_{ji}$ . □

## Birth and Death Chains



- **Birth and death chains are reversible.**
- Applying the product form solution for reversible chains we obtain the general solution of birth death chains.

Choosing  $s = 0$ :

$$\pi_i = \frac{\psi_i}{\sum_{j=0}^N \psi_j}, i \geq 0 \quad \text{where } \psi_i = \begin{cases} 1, & i = 0 \\ \prod_{k=0}^{i-1} \frac{p_k}{q_k}, & i = 1, \dots, N \end{cases}$$



# Reversible Chains

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Chains

Truncated Reversible  
Chain

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Finite

## Truncated Reversible Chain

- Consider a reversible MC  $X$  with a stationary distribution  $\pi_i$ .
- Suppose that **we truncate the chain  $X$**  and we obtain another irreducible chain  $X'$ .
- Then,  $X'$  is also reversible with stationary distribution:

$$\pi'_i = \frac{\pi_i}{G}, \quad \sum_k \pi'_k = 1$$



## Part II

# Discrete Time Markov Chains (DTMC)

### Outline

- Definition of a DTMC
- Transient Solution
- Classification of States
- Steady State
- Reversed Chain
- Reversible Chains
- **Research Example: Aloha**
- Finite Absorbing Chains

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## Access Protocol (see the paper of Kleinrock and Lam [1])

- **Pure Aloha:**
  - Broadcast radio system.
  - **Single hop** system (all stations are in coverage).
  - Whenever a station has a frame ready, it is transmitted.
  - If two or more frames Tx overlap in time there is a **collision**, otherwise the frame is received correctly.
  - Colliding frames are reTx after a **random time (backoff)**.
- **Slotted Aloha:**
  - Time is slotted.
  - Tx can only occur at the beginning of a slot.
  - Collisions occur when 2 or more stations Tx in the same slot.

[1] Leonard Kleinrock and Simon Lam. “**Packet Switching in a Multiaccess Broadcast Channel: Performance Evaluation**”. In: *Communications, IEEE Transactions on* 23.4 (1975), pp. 410–423.



# Research Example: Aloha

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## Analysis with finite population

### Assumptions

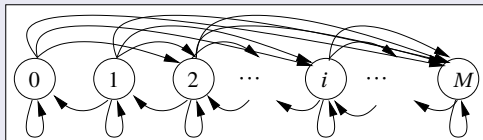
- **Slotted** Aloha.
- **Acks** are sent immediately after the reception of a frame, and are never lost.
- $M$  nodes with a **buffer** of 1 frame.
- The **nodes** can be in 2 states:
  - **Thinking**: when the buffer is empty
  - **Backlogged**: when there is a frame in the buffer.
- A thinking node generate one frame in each slot with probability  $\sigma$ . When a frame collides, the frame is stored and the node becomes backlogged.
- A backlogged node ReTx the frame in each slot with probability  $\nu$ .

## Research Example: Aloha

## Markov Chain

- The system **state**,  $X(n)$ , is the **number of backlogged nodes**:

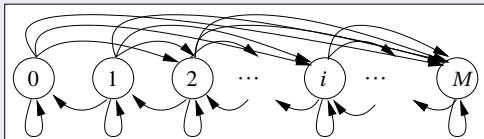
$$p_{ij} = P(X(n) = j \text{ baklogged} \mid X(n-1) = i \text{ baklogged})$$



## Research Example: Aloha

## Transition probabilities

$$p_{ij} = P(X(n) = j \text{ baklogged} \mid X(n-1) = i \text{ baklogged})$$



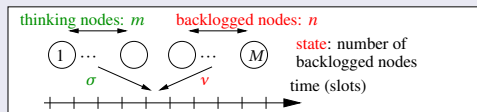
- 0 for  $j < i - 1$ .
- for  $j = i - 1$ : no thinking Tx and only 1 backlogged Tx.
- for  $j = i$ :
  - ① no thinking Tx and none or more than 1 backlogged Tx,
  - ② only 1 thinking Tx and no backlogged Tx.
- for  $j = i + 1$ : 1 thinking and 1 or more backlogged Tx.
- for  $j > i + 1$ :  $j - i$  thinking Tx, regardless of backlogged Tx.





## Research Example: Aloha

## Transition probabilities



- Define the probabilities:
  - Arrivals:**  $Q_a(m, n)$ , Probability of  $m$  thinking nodes Tx in a slot given that  $n$  nodes are backlogged:

$$Q_a(m, n) = P\left(\begin{matrix} m \text{ think.} \\ \text{nodes Tx} \end{matrix} \mid \begin{matrix} n \text{ nodes are} \\ \text{backlogged} \end{matrix}\right) = \binom{M-n}{m} \sigma^m (1-\sigma)^{M-n-m}$$

- Retransmissions:**  $Q_r(m, n)$ , Probability of  $m$  backlogged nodes Tx in a slot given that  $n$  nodes are backlogged:

$$Q_r(m, n) = P\left(\begin{matrix} m \text{ backl.} \\ \text{nodes Tx} \end{matrix} \mid \begin{matrix} n \text{ nodes are} \\ \text{backlogged} \end{matrix}\right) = \binom{n}{m} \nu^m (1-\nu)^{n-m}$$



## Research Example: Aloha

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## Transition probabilities

- 0 for  $j < i - 1$ .
- for  $j = i - 1$ : no thinking Tx and only 1 backlogged Tx.
- for  $j = i$ :
  - ① no thinking Tx and none or more than 1 backlogged Tx,
  - ② only 1 thinking Tx and no backlogged Tx.
- for  $j = i + 1$ : 1 thinking and 1 or more backlogged Tx.
- for  $j > i + 1$ :  $j - i$  thinking Tx, regardless of backlogged Tx.

$$p_{ij} = \begin{cases} 0, & j < i - 1 \\ Q_a(0, i) Q_r(1, i), & j = i - 1 \\ Q_a(0, i) (1 - Q_r(1, i)) + Q_a(1, i) Q_r(0, i), & j = i \\ Q_a(1, i) (1 - Q_r(0, i)), & j = i + 1 \\ Q_a(j - i, i), & j > i + 1 \end{cases}$$



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## Stationary distribution

- Solving the global balance equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}$$

$$\boldsymbol{\pi} \mathbf{e} = 1$$

- We obtain the probability of having  $i$  backlogged nodes:

$$\pi_i = P(i \text{ backlogged nodes})$$

NOTE: there is **no closed form solution** of the chain. The matrix  $\mathbf{P}$  must be constructed using the expression of  $p_{ij}$ , and solved numerically.



## Throughput

- Define the probabilities:

$$P_{succ}(i) = P(\text{successful Tx} \mid i \text{ backlogged})$$

- The **normalized throughput**, i.e. proportion of steps with a successful transmission) is:

$$S = \sum_{i=0}^M P(\text{successful Tx} \mid i \text{ backlogged}) P(i \text{ backlogged}) = \sum_{i=0}^M P_{succ}(i) \pi_i$$

- For a slot to be successful: (i) 1 thinking Tx and no backlogged Tx, or (ii) no thinking Tx and 1 backlogged Tx:

$$P_{succ}(i) = Q_a(1, i) Q_r(0, i) + Q_a(0, i) Q_r(1, i)$$



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## Notes on the throughput

$$S = \sum_{i=0}^M P_{succ}(i) \pi_i$$

- For the **special case**  $\sigma = \nu$  (thinking Tx with the same probability as backlogged):  $P_{succ}(i) = M\sigma(1-\sigma)^{M-1}$ , which does not depend on  $i$ , thus:  $S = M\sigma(1-\sigma)^{M-1}$ .
- The **offered load** (i.e. proportion of arrivals per slot)  $G$  is now:  $G = M\sigma$ , thus:

$$S = G \left(1 - \frac{G}{M}\right)^{M-1} \Rightarrow \lim_{M \rightarrow \infty} S = G e^{-G}$$

- We conclude that the **infinite population model** is the limit of the finite population if backlogged Tx with the same probability as thinking, and  $M \rightarrow \infty$ .

# Research Example: Aloha

## Discrete Time Markov Chains (DTMC)

### Definition of a DTMC

### Transient Solution

### Classification of States

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### Reversible Chains

### Research Example: Aloha

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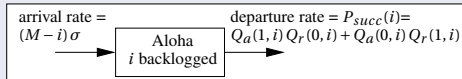
#### Markov Chain

#### Transition probabilities

#### Stationary distribution

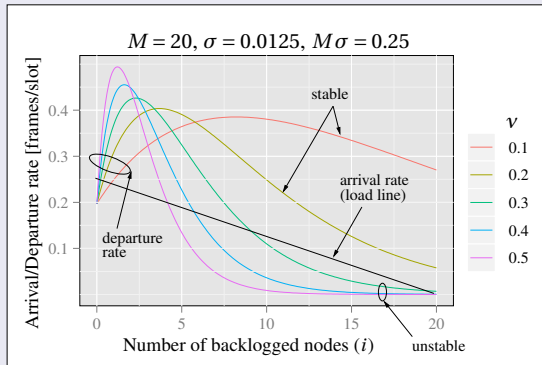
#### Throughput

## Dynamics



Note on the arrival rate (expected value of a binomial distribution):

$$\sum_{k=0}^{M-i} k \binom{M-i}{k} \sigma^k (1-\sigma)^{M-i-k} = (M-i)\sigma$$



Solving the chain:

$$S = \sum_{i=0}^M P_{succ}(i) \pi_i$$

| $v$ | $S$      |
|-----|----------|
| 0.1 | 2.38e-01 |
| 0.2 | 2.42e-01 |
| 0.3 | 1.30e-02 |
| 0.4 | 4.98e-04 |
| 0.5 | 1.90e-05 |



# Research Example: Aloha

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## Stabilizing Aloha

- The **retransmission probabilities** must adapt in accordance with the state of the system.
- Example: **binary exponential backoff** (ethernet). The retransmission rate at retransmission  $i$  is adapted as  $\nu = 2^{-i}$ . Thus, the higher are the number of retransmission trials  $i$ , the lower (exponentially) is the retransmission rate.