Simple Games

Fall 2020

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- A simple game can be described by a pair $(N, \mathcal{W})$:
  - N is a set of players,
  - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of winning coalitions, those coalitions X with $v(X) = 1$.
  - $\mathcal{L} = \mathcal{C}_N \backslash \mathcal{W}$ is the set of losing coalitions those coalitions X with $v(X) = 0$.

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- Members of $N = \{1, \ldots, n\}$ are called players or voters.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N:

- *winning coalitions* $\mathcal{W}$.
- *losing coalitions* $\mathcal{L}$.
- *minimal winning coalitions* $\mathcal{W}^m$
  $\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$
- *maximal losing coalitions* $\mathcal{L}^M$
  $\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L\}$

This provides us with many representation forms for simple games.

Weighted voting games

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  $$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$
- WVG can be represented by a tuple of integers $(q; w_1, \ldots, w_n)$.
  as any weighted game admits such an integer realization, [Carreras and Freixas, Math. Soc.Sci., 1996]

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  Both are simple games
- A simple game Γ is a vector weighted voting game if there are WVGs $\Gamma_1, \ldots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_k$.

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  The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.
  - Assume it is given by $(q; w_1, w_2, w_3, w_4)$.
  - We have $w_1 + w_2 \geq q$ and $w_3 + w_4 \geq q$.
  - Thus $\max\{w_1, w_2\} \geq q/2$ and $\max\{w_3, w_4\} \geq q/2$,
  - So, $\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q$ which cannot be.

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  - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
    Any set that is not contained in C wins!
  - The intersection of the above games describes Γ.
    A winning coalition cannot be a subset of any losing coalition.
- The dimension of a simple games is the minimum number of WVGs that allows its representation as VWVG

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A representation as WVGs

The game Γ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}\{2, 4\}\}$
- This gives four WVG, according to the previous construction

$$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0].$$

Input representations

- Simple Games
  $(N, \mathcal{W})$: extensive wining, $(N, \mathcal{W}^m)$: minimal wining
  $(N, \mathcal{L})$: extensive losing, $(N, \mathcal{L}^M)$ maximal losing
  (N, C): monotone circuit winning
  (N, F): monotone formula winning,
- Weighted voting games: $(q; w_1, \ldots, w_n)$
- Vector weighted voting games:
  $(q_1; w_1^1, \ldots, w_n^1), \ldots, (q_k; w_1^k, \ldots, w_n^k)$

All numbers are integers

The core of simple games

The core of simple games

- It is standard to assume that the grand coalition forms, even if the simple game is not superadditive.
- A player is a veto player if $v(C) = 0$, for any $C \subseteq N \setminus \{i\}$.
- Ex: Consider the unanimity game (N, v) where $v(C) = 0$, if $C \neq N$ and $v(N) = 1$.
  The game indeed is a simple game and can be described in (minimal) winning form by $(N, \{N\})$.
  In the unanimity game all players are veto players.

The core of simple games

Theorem

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- If Γ has a veto player i.
  - Consider the payoff $x_i = 1$ and $x_j = 0$, for $j \neq i$
  - For C with $i \in C$, $v(C) = 1$ and $x(C) = 1$.
  - For C with $i \notin C$, $v(C) = 0$ and $x(C) = 0$.
  - Thus, x is in the core.
- If Γ does not have a veto player and non-empty core.
  - Consider a payoff x that is in the core.
  - $x(N) = v(N) = 1$, so there exists i with $x_i > 0$.
  - So, $x(N \setminus \{i\}) < 1$. But, $v(N \setminus \{i\}) = 1$ as i is not a veto player.
  - Thus, x is not in the core.

Is the core empty?

- Determining if the core is empty or not can be done by checking for every player whether it is a veto player or not.
- For this it is enough to check whether $v(N \setminus \{i\}) = 0$.
- For reasonable v, polynomial time computable, this can be done in poly time

Shapley value and Banzhaf index

- Player i is pivotal for coalition C if $v(C) = 1$ and $v(C \setminus \{i\}) = 0$.
- The sum counts those the terms for which the player is pivotal.

$$\varphi_i(\Gamma) = \frac{1}{n!}|\{i \text{ is pivotal for } S_\pi(i)\}|$$

- $\varphi_i(\Gamma)$ is the probability that the arrival of player i turns a losing coalition into a winning one.
- The Banzhaf value gives the probability of this fact over random coalitions.
  Players in $N \setminus \{i\}$ select to be or not in the coalition tossing a fair coin.

Problems on simple games

In general we state a property $\mathtt{P}$, for simple games, and consider the associated decision problem which has the form:

Name: IsP
Input: *A simple game/WVG/VWVG* Γ
Question: *Does* Γ *satisfy property* $\mathtt{P}$*?*

Four properties

A simple game $(N, \mathcal{W})$ is

- strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a weighted voting game.
- a vector weighted voting game.

IsStrong: Simple Games

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The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

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Theorem

The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

- First observe that, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

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which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \;\wedge\; N \setminus S \subseteq L_2$$

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- which is equivalent to there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.

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- This can be checked in polynomial time, given $\mathcal{L}^M$.

IsStrong: explicit winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given $(N, \mathcal{W})$, for $i \in N$ consider the family $\mathcal{W}_i = \{X \setminus \{i\} \mid X \in \mathcal{W}\}$ and $R = \cup_{i \in N} \mathcal{W}_i$.
- All the coalitions in $R \setminus W$ are losing coalitions.
- Furthermore for a coalition $X \in \mathcal{L}^M$ and $i \notin X$, $X \cup \{i\} \in \mathcal{W}$.
- Thereofore, $\mathcal{L}^M \subseteq R \setminus W$ and $(R \setminus W)^M = \mathcal{L}^M$.
- Then, we compute $\mathcal{L}^M$ from $\mathcal{W}$ in polynomial time and then use the algoritm for the maximal losing form.

IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The ISSTRONG *problem is coNP-complete when the input game is given in explicit minimal winning form.*

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The IsStrong *problem is coNP-complete when the input game is given in explicit minimal winning form.*

- The property can be expressed as

$$\forall S \; [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and $\mathcal{W}^m$.
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

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We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m).

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- Now assume that $P \subseteq N$ satisfies

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- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m).

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- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m.
- Therefore, (N, C) has a set splitting iff (N, C^m) is not strong.

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

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- As before, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.

  Taking into account the definitions, the IsProper problem is polynomial time solvable for the explicit forms

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- equivalent to there are two minimal winning coalitions W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$.
- Which can be checked in polynomial time when $\mathcal{W}^m$ is given.

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Theorem

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- Therefore IsProper belongs to coNP.

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- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

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- Which can be obtained in polynomial time.
- Besides, Γ is strong iff Γ' is proper.

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Question: Is there $S \subseteq \{1, \ldots, n\}$ *for which*

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i.$$

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Question: Is there $S \subseteq \{1, \ldots, n\}$ *for which*

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i.$$

Observe that, for any instance of the PARTITION problem in which the sum of the n input numbers is odd, the answer must be NO.

Weighted voting games

Theorem

The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game* $(q; w)$, *are coNP-complete.*

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The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game* $(q; w)$, *are coNP-complete.*

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation $(2; 1, 1, 1)$ is both proper and strong.

Hardness

We transform an instance $x = (x_1, \ldots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

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- Function f can be computed in polynomial time provided q does.
- Independently of q, when $x_1 + \cdots + x_n$ is *odd*, x is a NO input for partition, but $f(x)$ is a YES instance of ISSTRONG or ISPROPER.

IsStrong

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$.

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- If S and $N \setminus S$ are losing coalitions in $f(x)$.
  If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning.
  Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of x.

IsProper

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- When $f(x)$ is not proper

$$\exists S \subseteq N : \sum_{i \in S} x_i \geq s \wedge \sum_{i \notin S} x_i \geq s,$$

and thus $\sum_{i \in S} x_i = s$.

Explicit forms

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit winning or losing form.*

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We can obtain $\mathcal{W}^m$ and $\mathcal{L}^M$ in polynomial time.
Once this is done we write, in polynomial time, the LP

$$
\begin{array}{lll}
\texttt{min } q & & \\
\texttt{subject to} & w(S) \geq q & \texttt{if } S \in W^m \\
 & w(S) < q & \texttt{if } S \in L^M \\
 & 0 \leq w_i & \texttt{for all } 1 \leq i \leq n \\
 & 0 \leq q &
\end{array}
$$

IsWeighted: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

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- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.

IsWeighted: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\wedge_{i \in S} x_i)$$

IsWheigthed: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

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IsWheigthed: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

IsWheigthed: Maximal loosing

- we can prove a similar result given (N, L^M).
- The dual of game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where
  $S \in \mathcal{W}^d$ iff $N \setminus S \notin \mathcal{W}$.
- Observe that Γ is weighted iff Γ^d is weighted.
- We can compute a monotone CNF formula describing the loosing coalitions of Γ. Negating this formula we get a DNF on negated variables. Replacing $\overline{x}_i$ by y_i we get a DNF describing $\mathcal{W}^d$.
- As the formula can be computed in polynomial time the result follows.

Power indices in weighted voting games

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Power indices in weighted voting games

- Before analyzing the complexity of computing the Shapley value or the Banhzaf index on weighted voting games, let us analyze the complexity two player properties.
- Let $\Gamma = (N, \mathcal{W})$ and $i \in N$
  - i is a veto player if $v(X) = 0$, for any $X \subseteq N \setminus \{i\}$.

Power indices in weighted voting games

- Before analyzing the complexity of computing the Shapley value or the Banhzaf index on weighted voting games, let us analyze the complexity two player properties.
- Let $\Gamma = (N, \mathcal{W})$ and $i \in N$
    - i is a veto player if $v(X) = 0$, for any $X \subseteq N \setminus \{i\}$.
    - i is a dummy player if $v(X) = v(X \cup \{i\}$, for any $X \subseteq N \setminus \{i\}$.

IsVeto on WVG

Name: IsVeto
Input: *A WVG $(q; w)$ and a player i*
Question: *Is player i a veto player?*

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Proof

As we have already shown, it is enought to see that $N \setminus \{i\} \in \mathcal{L}$, i.e., that $w(N \setminus \{i\}) < q$.

Empty core on WVG

Theorem

A simple game has non-empty core iff it has a veto player.

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Theorem

For WVG, checking if the core is empty can be done in polynomial time.

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Proof

If a player i is not a dummy player, there is a coalition X such that $w(X) < q$ and $w(X \cup \{i\}) \geq q$. So, the problem belongs to co-NP.

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If a player i is not a dummy player, there is a coalition X such that $w(X) < q$ and $w(X \cup \{i\}) \geq q$. So, the problem belongs to co-NP. The hardness follows from a reduction from EvenPartition, the Partition problem with the additional restriction that the sum of the weights is even.

IsDummy on WVG

Proof (cont)

- Let $I = (a_1, \ldots, a_k,)$ be an instance of $\textsc{EvenPartition}$.

IsDummy on WVG

Proof (cont)

- Let $I = (a_1, \ldots, a_k,)$ be an instance of $\textsc{EvenPartition}$.
- We define the WVG $\Gamma(I)$ having $k + 1$ players, $N = \{1, \ldots, k, k + 1\}$.
  - The weight of player $i \in N$ is $2a_i$
  - $w_{k+1} = 1$
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    - $w_{k+1} = 1$
    - $q = 2A + 1$, where $A = \sum_{i=1}^{k} a_k$
- We associate to I the pair $(\Gamma(I), k+1)$, which can, trivially, be constructed in polynomial time.

IsDummy on WVG

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Claim

If I is a "yes" instance of EVENPARTITION, then $k+1$ is not a dummy in $\Gamma(I)$.

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- In this case, there is $X \subseteq \{1, \ldots, k\}$ such that $a(X) = A$.
- Then, $w(X) = 2A$ and $w(X \cup \{k+1\}) = 2A + 1$.

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- As $w(k+1) = 1$, $w(X) \geq 2K$, but as all the values are integers, $w(X) = 2K$.

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- As $w(k+1) = 1$, $w(X) \geq 2K$, but as all the values are integers, $w(X) = 2K$.
- Finally as $w(i) = 2a_i$, we get that $a(X) = K$.

Recall: Shapley value and Banzhaf index

- Player i is pivotal for C, if $v(C) = 1$ and $v(C \setminus \{i\}) = 0$.
- The Shapley value of $i \in N$ is

$$\varphi_i(\Gamma) = \frac{1}{n!} |\{\pi \mid i \text{ is pivotal for } S_\pi(i)\}|$$

- and the Banzhaf value is

$$\beta_i(\Gamma) = \frac{1}{2^n} |\{S \mid i \text{ is pivotal for } S\}|$$

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- i is a veto player iff $\varphi_i(\Gamma) = \beta_i(\Gamma) = 0$.

Theorem

For a WVG, computing the Shapley or the Banzhaf value is NP-hard.

In fact it is known that both problems are #P-complete.

Computing Shapley and Banzhaf values

Theorem

Given an n-player WVG $\Gamma = (q; w)$ and a player i, we can compute $\beta_i(\Gamma)$ and $\varphi_i(\Gamma)$ in time $O(n^2 w_{\max})$ and $O(n^3 w_{\max})$, respectively.

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Given an n-player WVG $\Gamma = (q; w)$ and a player i, we can compute $\beta_i(\Gamma)$ and $\varphi_i(\Gamma)$ in time $O(n^2 w_{\max})$ and $O(n^3 w_{\max})$, respectively.

Proof

- W.l.o.g we can assume that $i = n$.
- If $w_n = 0$, player n is a dummy player, therefore $\beta_n(\Gamma) = \varphi_n(\Gamma) = 0$.

Computing Shapley and Banzhaf values

Proof (cont)

- Whenever n is pivotal for a coalition X, $|X| = x + 1$, it is pivotal for all permutations in which the agents in $X \setminus \{n\}$ appear in the first x positions, followed by n.
- There are exactly $x!(n - x - 1)!$ such permutations $(0! = 1)$.
- For each $x \in \{0, \dots, n - 1\}$, let N_x be the number of x-element subsets of $N \setminus \{n\}$ that have weight $W \in \{q - w_n, \dots, q - 1\}$.
- Therefore, the Shapley can be expressed as

$$\varphi_i(\Gamma) = \frac{1}{n!} \sum_{x=0}^{n-1} x!(n - x - 1)! N_x.$$

References

- Haris Aziz, *Algorithmic and complexity aspects of simple coalitional games* (Ph.D.thesis). Computer Science Dept.,U. Warwick, 2009.
- Josep Freixas, Xavier Molinero, Martin Olsen, Maria Serna: *On the complexity of problems on simple games*. RAIRO - Operations Research 45(4): 295-314 (2011)

Computing Shapley and Banzhaf values

Proof (cont)

- The N_x values can be computed using dynamic programming.
- Define $Y[j, W, x]$ to be the number of x-elements subsets of $\{1, \ldots, j\}$ that have weight W; $j \in \{1, \ldots, n-1\}$ and $x \in \{0, \ldots, n-1\}$, and $W \in \{0, \ldots, w(N)\}$.

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- Observe that

$$N_x = Y[n-1, q-w_n, x] + \cdots + Y[n-1, q-1, x]$$

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- For, $x = 0$ and $j \in \{1, \ldots, n-1\}$, we have

$$Y[j, W, 0] = \begin{cases} 1 & \text{if } W = 0 \\ 0 & \text{otherwise} \end{cases}$$

Computing Shapley and Banzhaf values

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- For, $j = 1$ and $x \in \{1, \dots, n-1\}$, we have

$$Y[1, W, x] = \begin{cases} 1 & \text{if } W = w_1 \text{ and } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

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- And the recurrence

$$Y[j, W, x] = Y[j-1, W, x] + Y[j-1, W - w_j, x-1]$$

The first term counts those that do not contain j and the second one those containing j.

Computing the Banzhaf value

Proof (cont)

- For the Banzhaf value we need only to count the number of subsets of $N \setminus \{n\}$ have weight at least $q - w_n$ and at most $q - 1$.
- This number can be computed from a simplified recurrence without the parameter x.