Basics on Linear Programming

Combinatorial Problem Solving (CPS)

Javier Larrosa   Albert Oliveras   Enric Rodríguez-Carbonell

April 3, 2020

Linear Programs (LP's)

- A linear program is an optimization problem of the form

$$\begin{aligned}&\min c^T x\\ &A_1 x \leq b_1\\ &A_2 x = b_2\\ &A_3 x \geq b_3\\ &x \in \mathbb{R}^n\end{aligned}$$

$$c \in \mathbb{R}^n, b_i \in \mathbb{R}^{m_i}, A_i \in \mathbb{R}^{m_i \times n}, i = 1, 2, 3$$

- x is the vector of variables
- $c^T x$ is the cost or objective function
- $A_1 x \leq b_1$, $A_2 x = b_2$ and $A_3 x \geq b_3$ are the constraints

Notes on the Definition of LP

- Solving minimization or maximization is equivalent:

$$\max\{\ f(x) \mid x \in S\ \} \;\; = \;\; -\min\{\ -f(x) \mid x \in S\ \}$$

- Satisfiability problems are a particular case:
  take arbitrary cost function, e.g., $c = 0$

Equivalent Forms of LP's (1)

- This form is not the most convenient for algorithms
  WLOG we can transform such a problem as follows

1. Split $=$ constraints into $\geq$ and $\leq$ constraints

$$\begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &A_2 x = b_2 \\ &A_3 x \geq b_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &A_2 x \leq b_2 \\ &A_2 x \geq b_2 \\ &A_3 x \geq b_3 \end{aligned}$$

Now all constraints are $\leq$ or $\geq$

Equivalent Forms of LP's (2)

2. Transform $\geq$ constraints into $\leq$ constraints
   by multiplying by -1

$$\begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &A_2 x \geq b_2 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &-A_2 x \leq -b_2 \end{aligned}$$

Now all constraints are $\leq$

Equivalent Forms of LP's (3)

3. Replace variables x by $y - z$, where y, z are vectors of fresh variables, and add constraints $y \geq 0$, $z \geq 0$

$$\begin{aligned} &\min c^T x \\ &Ax \leq b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\min c^T y - c^T z \\ &Ay - Az \leq b \\ &y, z \geq 0 \end{aligned}$$

Now all constraints are $\leq$ and all variables have to be ≥ 0

Equivalent Forms of LP's (4)

4. Add a slack variable to each $\leq$ constraint to convert it into $=$

$$\begin{aligned} &\min c^T x \\ &Ax \leq b \\ &x \geq 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} &\min c^T x \\ &Ax + s = b \\ &x, s \geq 0 \end{aligned}$$

Now all constraints are $=$ and all variables have to be ≥ 0

Equivalent Forms of LP's (5)

Example:

$$\begin{aligned} &\min -x - 2y \\ &x + y \leq 3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\min -x - 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Equivalent Forms of LP's (6)

- In the end we get a problem in standard form:

$$\begin{aligned}&\min c^T x\\&Ax = b\\&x \geq 0\end{aligned}$$

$$c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, n \geq m, \mathrm{rank}(A) = m$$

- These transformations are not strictly necessary
(they increase no. of constraints and variables),
but are convenient in a first formulation of the algorithms
- Often variables are identified with columns of the matrix,
and constraints are identified with rows

Methods for Solving LP's

- Simplex algorithms
- Interior-point algorithms

Methods for Solving LP's

- Simplex algorithms
- Interior-point algorithms

Basic Definitions (1)

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

- Any vector x such that $Ax = b$ is called a solution
- A solution x satisfying $x \geq 0$ is called a feasible solution
- An LP with feasible solutions is called feasible;
  otherwise it is called infeasible
- A feasible solution x^* is called optimal
  if $c^T x^* \leq c^T x$ for all feasible solution x
- A feasible LP with no optimal solution is unbounded

Basic Definitions (2)

$$
\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

- $(x, y, s_1, s_2, s_3) = (-1, -1, 5, 3, 3)$ is solution but not feasible
- $(x, y, s_1, s_2, s_3) = (1, 1, 1, 1, 1)$ is a feasible solution

Basic Definitions (3)

$$\begin{aligned}
&\max x + \beta y \\
&x + y + s_1 = \alpha \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

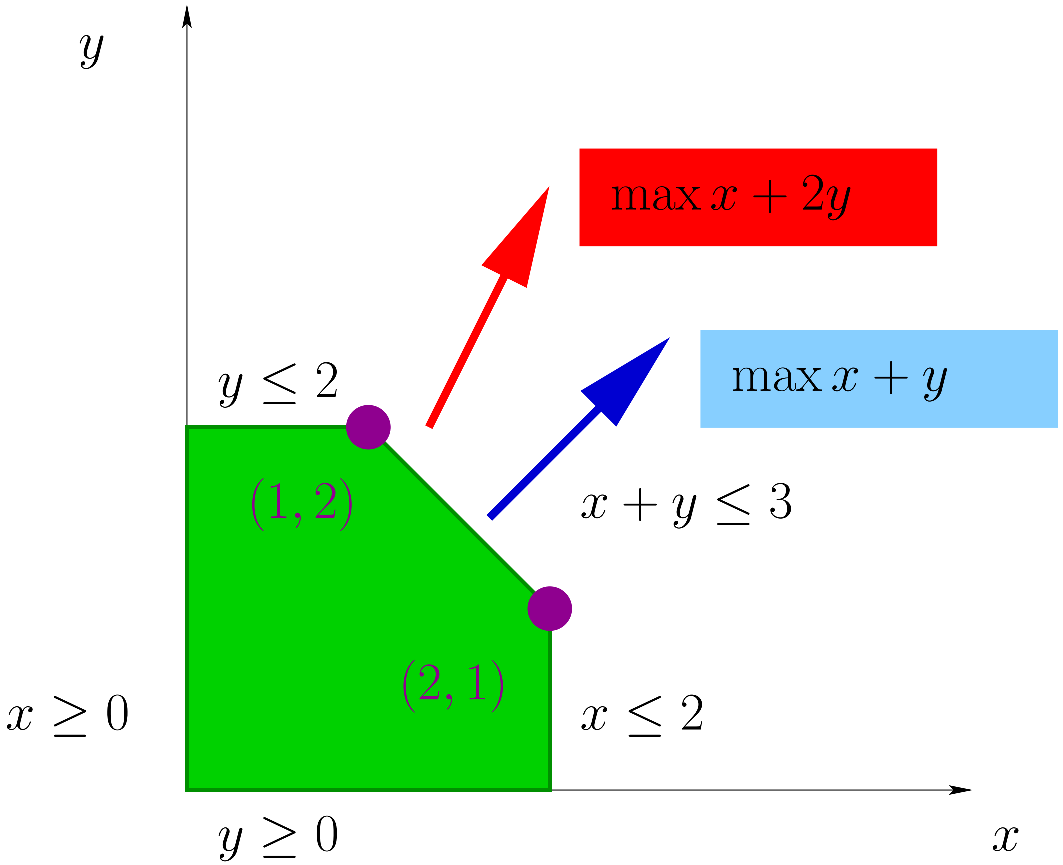
- If $\alpha = -1$ the LP is not feasible
- If $\alpha = 3, \beta = 2$ then
  $(x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)$ is the (only) optimal solution

Basic Definitions (3)

$$\begin{aligned}
&\max x + \beta y \\
&x + y + s_1 = \alpha \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}$$

- If $\alpha = -1$ the LP is not feasible
- If $\alpha = 3, \beta = 2$ then
  $(x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)$ is the (only) optimal solution
- There may be more than one optimal solution:
  If $\alpha = 3$ and $\beta = 1$ then
  $\{(1, 2, 0, 1, 0), (2, 1, 0, 0, 1), (\frac{3}{2}, \frac{3}{2}, 0, \frac{1}{2}, \frac{1}{2})\}$ are optimal

Basic Definitions (4)

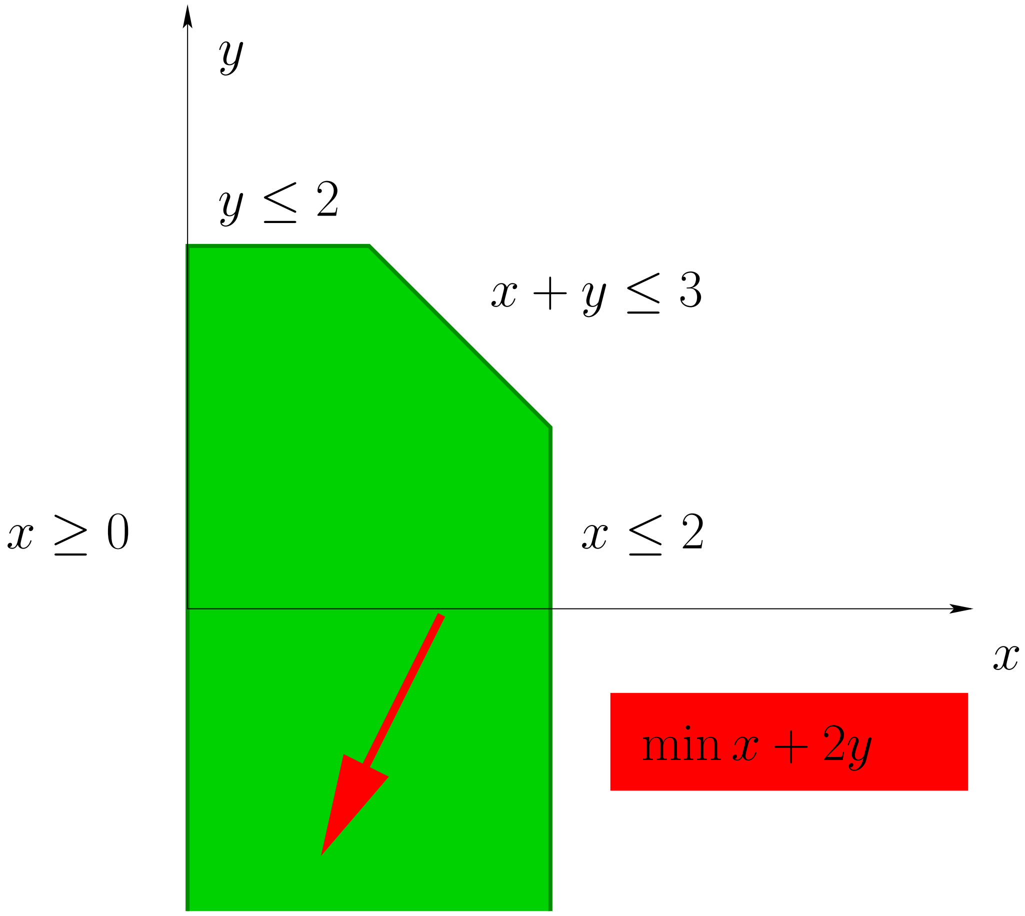


Basic Definitions (5)

$$\min x + 2y$$

$$x + y \leq 3$$
$$0 \leq x \leq 2$$
$$y \leq 2$$

Unbounded LP



Basic Definitions (6)

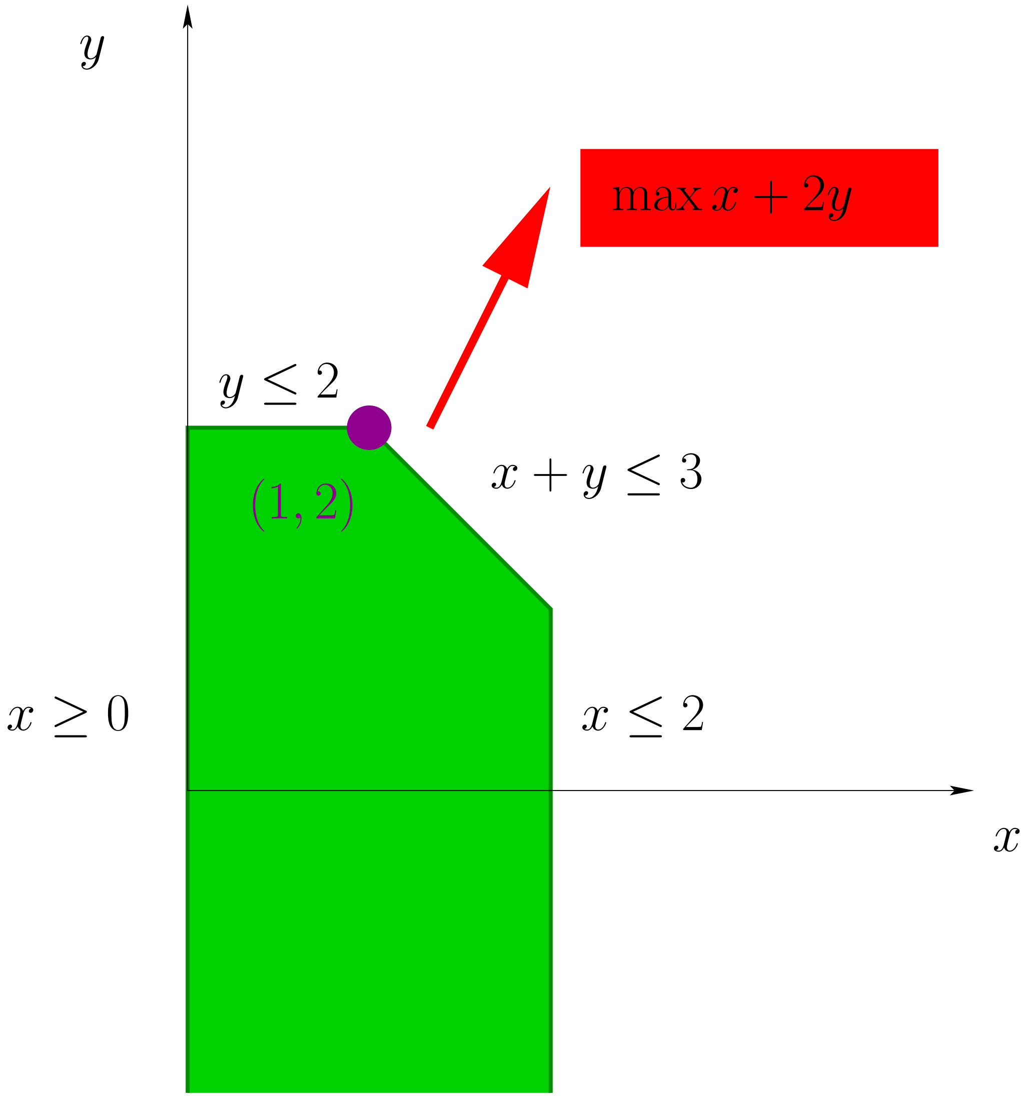
$\max x + 2y$

$x + y \leq 3$

$0 \leq x \leq 2$

$y \leq 2$

LP is bounded, but set of feasible solutions is not



Bases (1)

Let us denote by a_1, ..., a_n the columns of A

Recall that $n \geq m,\ \mathrm{rank}(A) = m$.

- A matrix of m columns $(a_{k_1}, ..., a_{k_m})$ is a basis if the columns are linearly independent
- Note that a basis is a square matrix!
- If $(a_{k_1}, ..., a_{k_m})$ is a basis, then the variables $(x_{k_1}, ..., x_{k_m})$ are called basic
- We usually denote

  by $\mathcal{B}$ the list of indices $(k_1, ..., k_m)$, and
  by $\mathcal{R}$ the list of indices $(1, 2, ..., n) - \mathcal{B}$; and

  by B the matrix $(a_i \mid i \in \mathcal{B})$, and
  by R the matrix $(a_i \mid i \in \mathcal{R})$

  $x_{\mathcal{B}}$ the basic variables, $x_{\mathcal{R}}$ the non-basic ones

Bases (2)

$$\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad
A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

- (s_1, s_2, x) do not form a basis
- (s_1, s_2, s_3) form a basis, where $x_{\mathcal{B}} = (s_1, s_2, s_3)$, $x_{\mathcal{R}} = (x, y)$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Bases (3)

- If B is a basis, then the following holds

$$Bx_{\mathcal{B}} + Rx_{\mathcal{R}} = b$$

  Hence:

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

  Non-basic variables determine values of basic ones

- If non-basic variables are set to 0, we get the solution

$$x_{\mathcal{R}} = 0, x_{\mathcal{B}} = B^{-1}b$$

  Such a solution is called a basic solution

- If a basic solution satisfies $x_{\mathcal{B}} \geq 0$ then it is called a basic feasible solution, and the basis is feasible

Bases (4)

Basis (s_1, s_2, s_3) is feasible

$$\max x + 2y$$
$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases} \qquad \sigma_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \qquad \sigma_{\mathcal{R}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Bases (5)

Basis (x, y, s_1) is not feasible

$$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} x = 2 - s_2 \\ y = 2 - s_3 \\ s_1 = -1 + s_2 + s_3 \end{cases} \qquad \sigma_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad \sigma_{\mathcal{R}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Bases (6)

A basis is called degenerate
when at least one component of its basic solution $x_{\mathcal{B}}$ is null

$$\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 4 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad
B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\qquad
R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

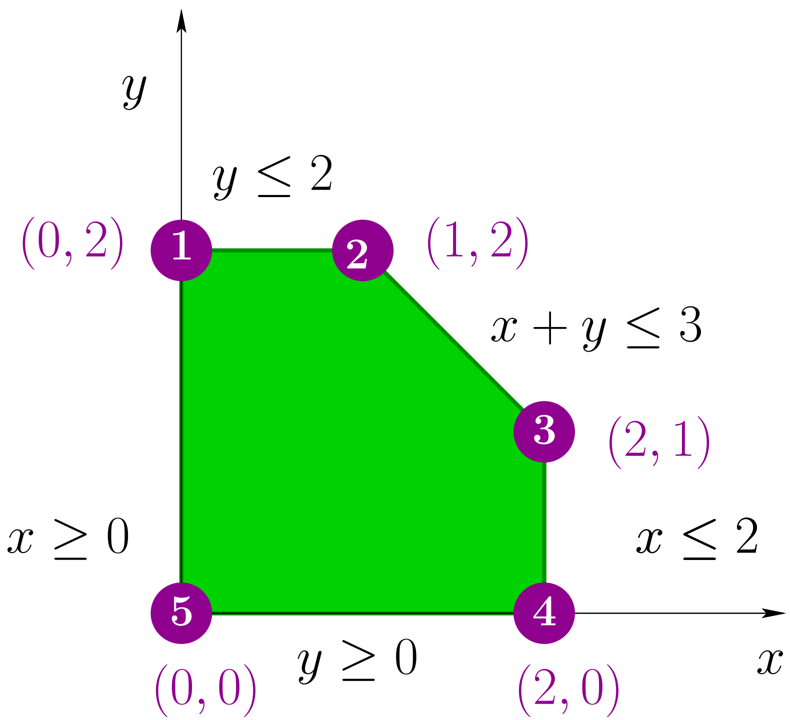
$$\begin{cases} x = 2 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = s_1 - s_3 \end{cases}
\qquad
\sigma_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Geometry of LP's (1)

- Set of feasible solutions of an LP is a convex polyhedron
- Basic feasible solutions are
  vertices of the convex polyhedron

Geometry of LP's (2)

- $$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$
- $x_{\mathcal{B}_1} = (y, s_1, s_2)$
- $x_{\mathcal{B}_2} = (x, y, s_2)$
- $x_{\mathcal{B}_3} = (x, y, s_3)$
- $x_{\mathcal{B}_4} = (x, s_1, s_3)$
- $x_{\mathcal{B}_5} = (s_1, s_2, s_3)$



Geometry of LP's (3)

- **Theorem (Minkowski-Weyl)**

  Let P be an LP.

  A point x is a feasible solution to P

  iff

  there exist basic feasible solutions $v_1, ..., v_r \in \mathbb{R}^n$ and vectors $r_1, ..., r_s \in \mathbb{R}^n$ such that

$$x = \sum_{i=1}^{r} \lambda_i v_i + \sum_{j=1}^{s} \mu_j r_j$$

  for certain λ_i, μ_j such that $\sum_{i=1}^{r} \lambda_i = 1$ and $\lambda_i, \mu_j \geq 0$.

Possible Outcomes of an LP (1)

- **Theorem (Fundamental Theorem of Linear Programming)**

  Let P be an LP.

  Then exactly one of the following holds:

  1. P is infeasible
  2. P is unbounded
  3. P has an optimal basic feasible solution

  It is sufficient to investigate basic feasible solutions!

Possible Outcomes of an LP (2)

Proof: Assume P feasible and with optimal solution x^*.

Let us see we can find a basic feasible solution as good as x^*.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

Possible Outcomes of an LP (2)

Proof: Assume P feasible and with optimal solution x^*.

Let us see we can find a basic feasible solution as good as x^*.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^r \lambda_i^* v_i + \sum_{j=1}^s \mu_j^* r_j$$

where $\sum_{i=1}^r \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^r \lambda_i^* c^T v_i + \sum_{j=1}^s \mu_j^* c^T r_j$$

- If there is j such that $c^T r_j < 0$ then objective value can be decreased by taking μ_j^* larger. Contradiction!

Possible Outcomes of an LP (2)

Proof: Assume P feasible and with optimal solution x^*.

Let us see we can find a basic feasible solution as good as x^*.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is j such that $c^T r_j < 0$ then objective value can be decreased by taking μ_j^* larger. Contradiction!

- Otherwise $c^T r_j \geq 0$ for all j. Assume $c^T x^* < c^T v_i$ for all i.

$$c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*$$

Possible Outcomes of an LP (2)

Proof: Assume P feasible and with optimal solution x^*.

Let us see we can find a basic feasible solution as good as x^*.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is j such that $c^T r_j < 0$ then objective value can be decreased by taking μ_j^* larger. Contradiction!

- Otherwise $c^T r_j \geq 0$ for all j. Assume $c^T x^* < c^T v_i$ for all i.

$$c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*$$

Contradiction! Thus there is i such that $c^T x^* \geq c^T v_i$; in fact, $c^T x^* = c^T v_i$ by the optimality of x^*.