Games with pure equilibria

Fall 2020

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Best response dynamics

Consider a strategic game $\Gamma = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, \ldots, n\}$

- PNE are defined as the fix point among mutually best responses.

Best response dynamics

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- It seems natural to consider variants of the process of local changes to try to get a PNE.

Best response dynamics

Consider a strategic game $\Gamma = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, \ldots, n\}$

- PNE are defined as the fix point among mutually best responses.
- It seems natural to consider variants of the process of local changes to try to get a PNE.
- Consider the algorithm:
  - choose $s \in A_1 \times \cdots \times A_n$
  - while s is not a NE do
    choose $i \in \{1, \ldots, n\}$ such that $s_i \notin BR(s_{-i})$
    Set s_i to be an action in $BR(s_{-i})$
- The process looks similar to local search algorithms. Is there any difference?
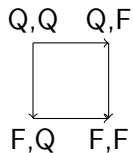
Best response graph

- The Nash dynamics or Best Response graph $G = (V, E)$ of Γ:
  - $V = A_1 \times \cdots \times A_n$
  - $(s, s') \in E$ iff $s' = (s_{-i}, s_i')$ for $i \in N$, $s_i \notin BR(s_{-i})$ and $s_i' \in BR(s_{-i})$.
- Performing local search on the best response graph
  - Does it produce a PNE?
  - If so, how much time?
  - Let's look to some examples.

Prisonner's dilemma

	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

Prisonner's dilemma

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Bach and Stravinsky

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B,B   B,S

S,B   S,S
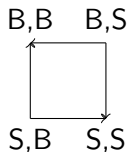
Matching Pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Matching Pennies

	Head	Tail
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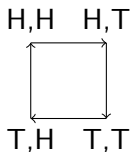
Other games

- Congestion games?

In those games we cannot get the best response graph in polynomial time.

However we can perform a local improvement step in polynomial time.

Although, even assuring convergence, it might take exponential time to reach a NE.

Best response graph: properties

- A NE is a sink (a node with out-degree 0) in the best response graph.
- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE.

Best response graph: properties

- A NE is a sink (a node with out-degree 0) in the best response graph.
- The existence of a cycle in the best response graph does not rule out the existence of a PNE.
- If the best response graph is acyclic, the game has a PNE. Furthermore, best response dynamics converges to a PNE, maybe with a lot of time.

1. Best response dynamics

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3. Congestion games

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Potential games

(Monderer and Shapley 96)

- Consider a strategic game $\Gamma = (N, A_1, \ldots, A_n, u_1, \ldots, u_n)$.
  Let $S = A_1 \times \cdots \times A_n$.

Potential games

(Monderer and Shapley 96)

- Consider a strategic game $\Gamma = (N, A_1, \ldots, A_n, u_1, \ldots, u_n)$.
  Let $S = A_1 \times \cdots \times A_n$.
- A function $\Phi : S \to \mathbb{R}$ is an exact potential function for Γ if

  $$\forall\, i \in N,\, \forall\, s \in S,\, \forall\, s_i' \in A_i :\; u_i(s) - u_i(s_{-i}, s_i') = \Phi(s) - \Phi(s_{-i}, s_i')$$

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  $$\text{or } (u_i(s) - u_i(s_{-i}, s_i'))(\Phi(s) - \Phi(s_{-i}, s_i')) > 0$$

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  or $(u_i(s) - u_i(s_{-i}, s_i'))(\Phi(s) - \Phi(s_{-i}, s_i')) > 0$

- Γ is a potential game if it admits a potential function.

Prisonner's dilemma

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Prisonner's dilemma

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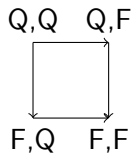
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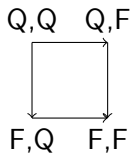
Φ	Quiet	Fink
Quiet	1	2
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Prisonner's dilemma

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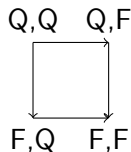
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Φ is an exact potential function

Bach and Stravinsky
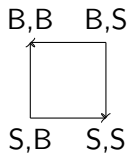
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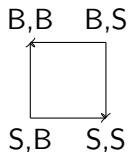
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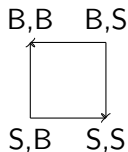
Φ is an exact potential function

Matching Pennies

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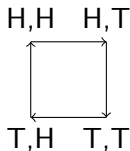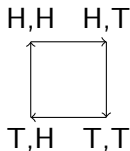
This is not a potential game

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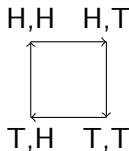
This is not a potential game
The property on Φ cannot hold along a cycle in the best response graph.

Potential games

Theorem

A strategic game is a potential game iff the best response graph is acyclic

Potential games

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Proof.

- Let G be the best response graph of Γ.

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- Let G be the best response graph of Γ.
- The existence of a potential function Φ and the fact that, for each pair of connected strategy profiles in G, at least one player improves, implies the non existence of cycles in G.

Potential games

Theorem

A strategic game is a potential game iff the best response graph is acyclic

Proof.

- Let G be the best response graph of Γ.
- The existence of a potential function Φ and the fact that, for each pair of connected strategy profiles in G, at least one player improves, implies the non existence of cycles in G.
- If G is acyclic, a topological sort of the graph provides a potential function for Γ.

$\square$

Potential games

Theorem

Any potential game has a PNE

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Proof.

As the best response graph is acyclic it must have a sink. $\qquad\square$

Potential games

Theorem

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As the best response graph is acyclic it must have a sink. □

We have a way to show that a game has a PNE by showing that it is a potential game.

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Congestion games

A congestion game

- is defined on a finite set E of resources.
- There is a delay function d mapping $E \times \mathbb{N}$ to the integers.
- Player's actions are subsets of E (all or some).
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \left(\sum_{e \in a_i} d(e, f(a_1, \ldots, a_n, e)) \right)$$

being $f(a_1, \ldots, a_n, e) = |\{i \mid e \in a_i\}|$.

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being $f(a_1, \ldots, a_n, e) = |\{i \mid e \in a_i\}|$.
- A singleton congestion game has $A_i = \{\{r\} \mid e \in E\}$.

An example of a congestion game

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An example of a congestion game

- We have a factory with two end production lines, each having a cutting and a packing unit. Orders are cut down and then packed.

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An example of a congestion game

- We have a factory with two end production lines, each having a cutting and a packing unit. Orders are cut down and then packed.
- We have 3 orders that have to be send to one of the end production lines.

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An example of a congestion game

- We have a factory with two end production lines, each having a cutting and a packing unit. Orders are cut down and then packed.
- We have 3 orders that have to be send to one of the end production lines.
- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.

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An example of a congestion game

- We have a factory with two end production lines, each having a cutting and a packing unit. Orders are cut down and then packed.
- We have 3 orders that have to be send to one of the end production lines.
- The cutting machine on the first line takes 1 hour to process a single order, 2 hours to process 2 and 4 hours to process 3. The cutting machine on the second line takes 4, 5 and 9 hours respectively.
- The packing machine on the first line takes 2 additional hours to pack a single order, 3 hours to pack 2 and 7 hours to pack 3. The packing machine on the second line takes instead 0, 2 and 9 hours respectively.

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An example of a congestion game

- We have 4 resources C_1, C_2, P_1, P_2 and 3 players $N = \{1, 2, 3\}$
- $A_i = \{\{C_1, P_1\}, \{C_2, P_2\}\}$, $i = 1, 2, 3$
- Delay functions are defined by the processing times.

	1	2	3
C_1	1	2	4
C_2	4	5	9
P_1	2	3	7
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Does this game have a PNE?

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Rosenthal's theorem

Theorem (Rosenthal 73)

Every congestion game is a potential game,

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Rosenthal's theorem

Theorem (Rosenthal 73)

Every congestion game is a potential game,

- For a strategy profile $s = (a_1, \ldots, a_n)$, define

$$\Phi(s) = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k)$$

where $r(s) = \cup_{i \in N} a_i$.

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Let us show that Φ is a potential function.

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- Let $s = (a_1, \ldots, a_n)$. Fix a player i and let $a_i' \subseteq E$ and $s' = (s_{-i}, a_i')$. We have

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- Let $s = (a_1, \ldots, a_n)$. Fix a player i and let $a_i' \subseteq E$ and $s' = (s_{-i}, a_i')$. We have

$$c_i(s) - c_i(s_{-i}, a_i') = \left(\sum_{e \in a_i} d(e, f(s, e)) \right) - \left(\sum_{e' \in a_i'} d(e, f(s', e')) \right)$$

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$$
\Phi(s) - \Phi(s') = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e' \in r(s')} \sum_{k=1}^{f(s',e')} d(e', k)
$$

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Cost difference

- Note that

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Cost difference

- Note that
  - $e \in a_i \cap a_i'$: $f(s, e) = f(s', e)$
  - $e \notin a_i$ and $e \notin a_i'$: $f(s, e) = f(s', e)$

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Cost difference

- Note that
  - $e \in a_i \cap a_i'$: $f(s, e) = f(s', e)$
  - $e \notin a_i$ and $e \notin a_i'$: $f(s, e) = f(s', e)$

$$c_i(s) - c_i(s_{-i}, s_i') = \left(\sum_{e \in a_i} d(e, f(s, e)) \right) - \left(\sum_{e' \in a_i'} d(e, f(s', e')) \right)$$

$$= \sum_{e \in a_i, e \notin a_i'} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s', e'))$$

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Potential difference

- Furthermore,

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Potential difference

- Furthermore,
  - $e \in a_i$ and $e \notin a_i'$: $f(s, e) = f(s', e) + 1$
  - $e \notin a_i$ and $e \in a_i'$: $f(s, e) + 1 = f(s', e)$

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Potential difference

- Furthermore,
  - $e \in a_i$ and $e \notin a'_i$: $f(s, e) = f(s', e) + 1$
  - $e \notin a_i$ and $e \in a'_i$: $f(s, e) + 1 = f(s', e)$

$$\Phi(s) - \Phi(s') = \sum_{e \in r(s)} \sum_{k=1}^{f(s,e)} d(e, k) - \sum_{e \in r(s')} \sum_{k=1}^{f(s',e)} d(e, k)$$

$$= \sum_{e \in a_i, e \notin a'_i} \left[\sum_{k=1}^{f(s',e)+1} d(e, k) - \sum_{k=1}^{f(s',e)} d(e, k) \right]$$

$$+ \sum_{e \notin a_i, e \in a'_i} \left[\sum_{k=1}^{f(s,e)} d(e, k) - \sum_{k=1}^{f(s,e)+1} d(e, k) \right]$$

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Potential difference

$$
= \sum_{e \in a_i, e \notin a_i'} [\sum_{k=1}^{f(s',e)+1} d(e, k) - \sum_{k=1}^{f(s',e)} d(e, k)]
$$

$$
+ \sum_{e \notin a_i, e \in a_i'} [\sum_{k=1}^{f(s,e)} d(e, k) - \sum_{k=1}^{f(s,e)+1} d(e, k)]
$$

$$
= \sum_{e \in a_i, e \notin a_i'} d(e, f(s', e) + 1) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s, e) + 1)
$$

$$
= \sum_{e \in a_i, e \notin a_i'} d(e, f(s, e)) - \sum_{e \notin a_i, e \in a_i'} d(e, f(s', e))
$$

$$
= c_i(s) - c_i(s_{-i}, s_i')
$$

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Network congestion games

- A network congestion game is a congestion game defined by a directed graph G and a collection of pairs of vertices (s_i, t_i).
  - The set of resources are the arcs in G.
  - The actions, for player i, are the $s_i \rightarrow t_i$ paths on G.
- A network congestion game is symmetric when $s_i = s$ and $t_i = t$, for $i \in N$.

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An example of a network congestion game

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An example of a network congestion game

- There are three players.
- and a network (with a delay function on arcs)

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- There are three players.
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- Player's objective: going from $s = A$ to $t = B$ as fast as possible.

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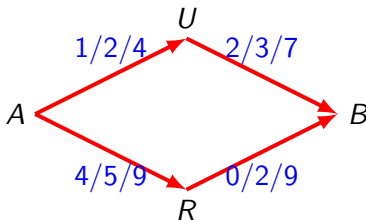Computing a PNE in congestion games

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- There are three players.
- and a network (with a delay function on arcs)

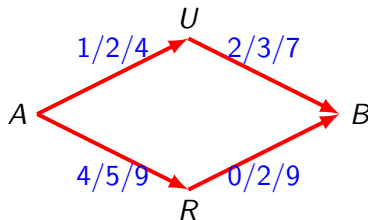- Player's objective: going from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
- A NE?

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An example of a weighted network congestion game

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An example of a weighted network congestion game

- There are three players with weights 1,1,2

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An example of a weighted network congestion game

- There are three players with weights 1,1,2
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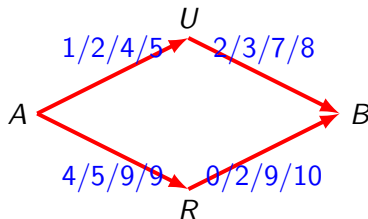# An example of a weighted network congestion game

- There are three players with weights 1,1,2
- and a network (with a delay function on arcs)

- Player's objective: send w_i units from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
- A NE?

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Results on convergence time

Theorem (Fabrikant, Papadimitriou, Talwar (STOC 04))

There exist network congestion games with an initial strategy profile from which all better response sequences have exponential length.

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Theorem (Fabrikant, Papadimitriou, Talwar (STOC 04))

There exist network congestion games with an initial strategy profile from which all better response sequences have exponential length.

Theorem (Ieong, McGrew, Nudelman, Shoham, Sun (AAAI 05))

In singleton congestion games all best response sequences have length at most $n^2 m$.

Complexity classification?

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Optimization problem

An optimization problem is a structure $\Pi = (I, \text{sol}, m, \text{goal})$, where

- I is the input set to Π;
- $\text{sol}(x)$ is the set of feasible solutions for an input x.
- m is an integer measure defined over pairs (x, y), $x \in I$ and $y \in \text{sol}(x)$.
- goal is the optimization criterium MAX or MIN.

An optimization problem is a function problem whose goal, with respect to an instance x, is to find an optimum solution, that is, a feasible solution y such that

$$y = \text{goal}\{(m(x, y') \mid y' \in \text{sol}(x)\}.$$

Example: Given a graph and two vertices, obtain a path joining them with minimum length.

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- A local search problem is an optimization problem with a neighborhood structure defined on the solution set $\mathcal{N}(\text{sol}(x))$.

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- A local optimum is a solution such that all its neighbors have equal or worse cost.

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(Johnson, Papadimitriou, Yannakakis, FOCS 85)
A local search problem belongs to PLS (Polynomial Local Search)

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- finding initial feasible solution $s \in \mathrm{sol}(x)$,
- computing the objective measure $m(x, y)$,
- checking whether a solution is a local optimum and if not finding a better solution in the neighborhood.

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PLS reductions

A PLS reduction from $(\Pi_1, \mathcal{N}_1)$ to $(\Pi_2, \mathcal{N}_2)$ is

- a polynomial time computable function $f : I_{\Pi_1} \to I_{\Pi_2}$ and
- a polynomial time computable function
  $g : \text{sol}_{\Pi_2}(f(x)) \to \text{sol}_{\Pi_1}(x)$, for $x \in I_{\Pi_1}$ such that
- if $s_2 \in \text{sol}_{\Pi_2}(f(x))$ locally optimal then $g(s_2)$ is locally optimal.

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- if $s_2 \in \mathrm{sol}_{\Pi_2}(f(x))$ locally optimal then $g(s_2)$ is locally optimal.

- If a local opt.of Π_2 is easy to find then a local opt.of Π_1 is easy to find.

- If a local opt.of Π_1 is hard to find then a local opt.of Π_2 is hard to find.

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- If a local opt.of Π_2 is easy to find then a local opt.of Π_1 is easy to find.

- If a local opt.of Π_1 is hard to find then a local opt.of Π_2 is hard to find.

A PLS problem $(\Pi, \mathcal{N})$ is PLS-complete if every problem in PLS is PLS-reducible to $(\Pi, \mathcal{N})$.

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PLS complete problems

- MAX-SAT (maximum satisfiability) problem
  - Given a Boolean formula in conjunctive normal form with a positive integer weight for each clause.
  - A solution is an assignment of the value 0 or 1 to all variables.
  - Its weight, to be maximized, is the sum of the weights of all satisfied clauses.
  - As neighborhood consider the Flip-neighborhood, where two assignments are neighbors if one can be obtained from the other by fliiping the value of a single variable.

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PLS complete problems

- MaxCut problem.
    - Given a graph $G = (V, E)$ with non-negative edge weights.
    - A feasible solution is a partition of V into two sets A and B.
    - The objective is to maximize the weight of the edges between the two sets A and B.
    - In the Flip-neighborhood two solutions are neighbors if one can be obtained from the other by moving a single vertex from one set to the other.

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Computing a PNE in congestion games is PLS-complete.

- The problem belongs to PLS taking as neighborhood the Nash dynamics because the Rosenthal's potential function can be evaluated in polynomial time.

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- We provide a reduction from MaxCut under the Flip-neigborhood.

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- Let $(G, E, (w_e)_{e \in E})$ be an instance of MaxCut, define a congestion game as follows

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- Let $(G, E, (w_e)_{e \in E})$ be an instance of MaxCut, define a congestion game as follows
    - For each edge e, we add resources e^a and e^b, with delay 0 if used by only one player and delay w_e if used by more players.

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  - The players correspond to the nodes in V, $v \in V$ has strategies $S_v^a = \{e^a \mid e \text{ incident to } v\}$ and $S_v^b = \{e^b \mid e \text{ incident to } v\}$

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    - Solutions (A, B) of MaxCut corresponds to strategy S_v^a for $v \in A$ and S_v^b for $v \in B$.

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  - Solutions (A, B) of MaxCut corresponds to strategy S_v^a for $v \in A$ and S_v^b for $v \in B$.
  - Furthermore, the local optima of the MaxCut instance coincide with the Nash equilibria of the congestion game.

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  - Solutions (A, B) of MaxCut corresponds to strategy S_v^a for $v \in A$ and S_v^b for $v \in B$.
  - Furthermore, the local optima of the MaxCut instance coincide with the Nash equilibria of the congestion game.
- We have a PLS-reduction from MaxCut.

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2. Potential games

3. Congestion games

4. References

Reference

B. Vöcking, Congestion Games: Optimization in Competition