Radomized Algorithms Solution to Hw5*.

[28.-]

1. Recall from class that $Y_i \in G(\frac{m-i+1}{m})$. Moreover the $\{Y_i\}$ are independent. Then $\mathbf{Var}[Y] = \sum_i \mathbf{Var}[Y_i] = \sum_{i=1}^{m-1} \frac{mi}{(m-i)^2} \leq n^2 \sum_{i=1}^{n} \frac{1}{i^2} \leq \frac{n^2\pi^2}{6}$. Using Chebyshev we get,
   $\mathbf{Pr}[|Y-\mu| \geq c\mu| \geq cm\ln m] \leq \frac{\mathbf{Var}[Y]}{(cm\ln m)^2} = \frac{\pi^2}{6c^2H_m}$

2. Let $n = m\ln m + cm$. Define a rv W_k counting the number of balls in bin k where we throw n bills, then $\mathbf{E}[W_j] = \ln m + c$ $\mathbf{Pr}[W_k \leq 0] \leq e^{-\mathbf{E}[W_k]/2} \leq e^{-(\ln m + c)/2}$.
   Using union-bound, $\mathbf{Pr}[\text{some bin is}\emptyset] \leq m\mathbf{Pr}[W_k = 0] = e^{\ln m - (\ln m + c)/2}$. Taking $c = \ln m - 2\ln(1/\delta)$ we get the desired bound.

3. The Chebyshev and Chernoff bounds are weaker than the ones presented in the class using a new set of r.v. and union-bound. **The main issue in this exercise is that Chernoff bounds are fine when probabilities are small and we do not want the tightest bounds, but it may be weak when probabilities are larger and we want tighter bounds.**