Universitat Politècnica de Catalunya. Departament d'Arquitectura de Computadors.
Master in Innovation and Research in Informatics (MIRI). Computer Networks and Distributed Systems.

Stochastic Network Modeling (SNM). Autumn 2018.

First assessment, Discrete Time Markov Chains. 5/11/2018.

Problem 1

Assume a slotted Aloha system with 3 nodes, n_1, n_2, n_3. All nodes transmit with probability $\sigma = 2/3$ when they are thinking and $\nu = 1/3$ when backlogged. Node n_1 has priority, and upon transmitting simultaneously with other nodes, the transmission of n_1 is successful, while the others go to backlogged. Thus, n_1 is never backlogged. Let the state be the number of backlogged nodes.

1.A (1.5 points) Draw the transition diagram and derive the one step transition probabilities.

1.B (1.5 points) Compute the stationary distribution.

1.C (1.5 point) Compute the throughput S_1, S_2, S_3, of each node (expected number of successful packets transmitted per slot).

1.D (1 point) Let $N \geq 1$ be the random variable equal to number of transmissions that one of the non priority nodes needs to successfully transmit a packet. Compute E$[N]$.

1.E (1 point) Compute probability, P_1, that when a node goes into backlogged state the chain enters into state 1 (in steady state).

1.F (1 point) Compute probability, P_2, that when a node goes into backlogged state the chain enters into state 2 (in steady state).

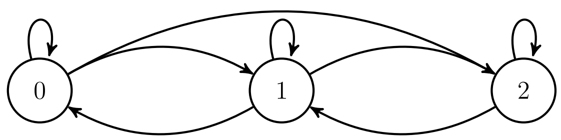
1.G (1.5 point) Let $T \geq 1$ be the random variable equal to number of slots since a node becomes backlogged until it successfully transmits the packet. Compute E$[T]$. Hint: consider a chain with one absorbing state, and use the results of the 2 previous items.

1.H (1 point) Compute the expected number of successful transmissions that the other 2 nodes will do while the node is backlogged in the previous item.

Solution

Problem 1

1.A



			state transition	
n_1	n_2	n_3	$0 \to$	$1\ (n_2 = \text{B}) \to$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	0
0	1	1	2	2
1	0	0	0	1
1	0	1	1	2
1	1	0	1	1
1	1	1	2	2

legend: 1 Tx, 0 no Tx

$p_{00} = \bar{\sigma}^3 + 3\,\sigma\,\bar{\sigma}^2$ $\quad p_{01} = 2\,\sigma^2\,\bar{\sigma}$

$p_{02} = \sigma^2$

$p_{10} = \nu\,\bar{\sigma}^2$ $\quad p_{11} = \bar{\sigma}\,\bar{\nu} + \bar{\sigma}\,\nu\,\sigma + \sigma\,\bar{\nu}\,\bar{\sigma}$

$p_{12} = \sigma^2\,\bar{\nu} + \sigma\,\nu$

$p_{21} = 2\,\nu\bar{\nu}\bar{\sigma}$ $\quad p_{22} = 1 - p_{21}$

where $\bar{\sigma} = 1 - \sigma$, $\bar{\nu} = 1 - \nu$.

$$P = \begin{bmatrix} 7/27 & 8/27 & 4/9 \\ 1/27 & 12/27 & 14/27 \\ 0 & 4/27 & 23/27 \end{bmatrix}.$$

1.B $\pi = \begin{bmatrix} 1/94 & 20/94 & 73/94 \end{bmatrix}$.

1.C

$$S_1 = \sigma = 2/3$$
$$S_2 = S_3 = \pi_0\,\sigma\,\bar\sigma^2 + \pi_1\,(\sigma\,\bar\sigma\,\bar\nu + \nu\,\bar\sigma\,\bar\sigma)/2 + \pi_2\,\nu\,\bar\nu\,\bar\sigma = 11/141.$$

1.D The transmission rate of one of the non priority nodes is:

$$G = \pi_0\,\sigma + \pi_1\,(\sigma + \nu)/2 + \pi_2\,\nu = 35/94.$$

Thus, the number of transmissions per packet is:

$$\mathrm{E}[N] = G/S_2 = 105/22 \approx 4{,}8.$$

1.E The rate entering into backlogged state is:

$$B = \pi_0\,(p_{01} + 2\,p_{02}) + \pi_1\,p_{12} = 52/423$$

and the rate entering into state 1:

$$B_1 = \pi_0\,p_{01} = 4/1269.$$

Thus, the requested probability is

$$P_1 = B_1/B = 1/39.$$

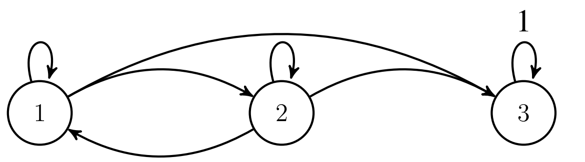
1.F The rate entering into state 2 is:

$$B_2 = \pi_0\,2\,p_{02} + \pi_1\,p_{12} = 152/1269.$$

Thus, the requested probability is

$$P_2 = B_2/B = 38/39.$$

1.G Let 3 be the absorbing state: the tagged backlogged node successfuly transmit the packet. Taking the backlogged node in state 1 as the tagged node we have:



$$\begin{array}{lll} p_{11} = \bar\sigma\,\bar\nu + \bar\sigma\,\nu\,\sigma + \sigma\,\bar\nu\,\bar\sigma & p_{12} = \sigma^2\,\bar\nu + \sigma\,\nu & p_{13} = \nu\,\bar\sigma^2 \quad \text{(like before)} \\ p_{21} = \nu\bar\nu\bar\sigma & p_{22} = 1 - 2\,\nu\bar\nu\bar\sigma & p_{23} = \nu\bar\nu\bar\sigma \end{array}$$

where $\bar\sigma = 1 - \sigma$, $\bar\nu = 1 - \nu$.

$P = \begin{bmatrix} 4/9 & 14/27 & 1/27 \\ 2/27 & 23/27 & 2/27 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 4/9 & 14/27 \\ 2/27 & 23/27 \end{bmatrix}$, and the fundamental matrix: $N = (I - Q)^{-1} = \dfrac{27}{32}\begin{bmatrix} 4 & 14 \\ 2 & 15 \end{bmatrix}$. Thus, the time to absorption stating from states 1 and 2 are: $\tau = N\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \dfrac{27}{32}\begin{bmatrix} 18 \\ 17 \end{bmatrix}$ and

$$\mathrm{E}[T] = \tau_1\,P_1 + \tau_2\,P_2 = 747/52 = 14{,}36.$$

1.H The priority node $N_1 = \sigma\,\mathrm{E}[T] = 249/26 = 9{,}58$.
The non priority node will be the number of successful transmissions from state 1 plus the number of those from state 2. That is, the number of visits to state $1 \times \sigma\,\bar\nu\,\bar\sigma$ plus the number of visits to state $2 \times p_{21}$: $N_2 = (P_1\,n_{11} + P_2\,n_{21})\,\sigma\,\bar\nu\bar\sigma + (P_1\,n_{12} + P_2\,n_{22})\,p_{21} = 31/26 \approx 1{,}19$.