INTRODUCTION TO ANOVA

Statistical Inference

What is ANOVA?

- ANOVA is short for ANalysis Of VAriance
- Used with 3 or more groups to test for MEAN DIFFS.
- E.g., wine study with 4 groups:
  - Reference (the reference soil, the common)
  - Env 1 (the first alternative)
  - Env 2 (the second alternative)
  - Env 4 (the third alternative)
- Level is value, kind or amount of IV
- Treatment Group is people who get specific treatment or level of IV
- Treatment Effect is size of difference in means

Rationale for ANOVA (1)

- We have at least 3 means to test, e.g., H_0: $\mu_1 = \mu_2 = \mu_3$.
- Could take them 2 at a time, but really want to test all 3 (or more) at once.
- Instead of using a mean difference, we can use the **variance of the group means** about the **grand mean** over all groups.
- Logic is just the same as for the t-test. Compare the observed variance among means (observed difference in means in the t-test) to what we would expect to get by chance.

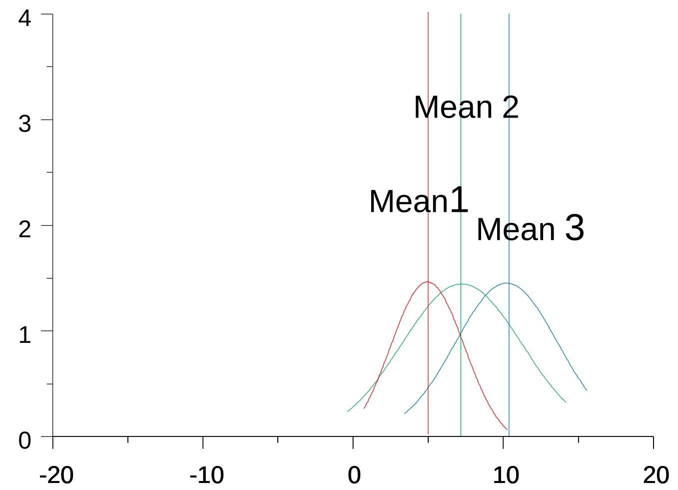
Rationale for ANOVA (2)

- HO : $\mu_1 = \mu_2 = \ldots = \mu_5$ vs. HA : not all are equal.
  - H01: $\mu_1 = \mu_2$
  - H02: $\mu_2 = \mu_3$
  - H03: $\mu_3 = \mu_4$
  - H04: $\mu_4 = \mu_5$
  - H05: $\mu_1 = \mu_3$
  - H06: $\mu_2 = \mu_4$
  - H07: $\mu_3 = \mu_5$
  - H08: $\mu_1 = \mu_4$
  - H09: $\mu_2 = \mu_5$
  - H010: $\mu_1 = \mu_5$
- Reject any null hypotheses implies reject the initial null hypothesis.
  - High computational effort.
  - Increases the type I error (reject an null hypothesis being true).

Rationale for ANOVA (3)

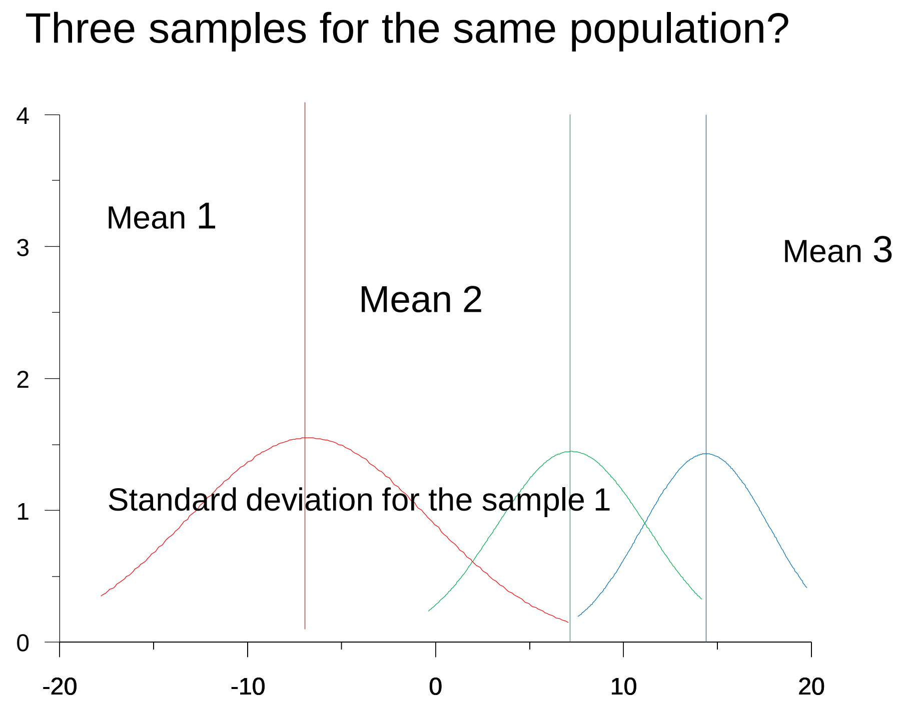
- Suppose we drew 3 samples from the same population.
- Note that the means from the 3 groups are not exactly the same, but they are close, so the variance among means will be small.

Three samples for the same population



Rationale for ANOVA (4)

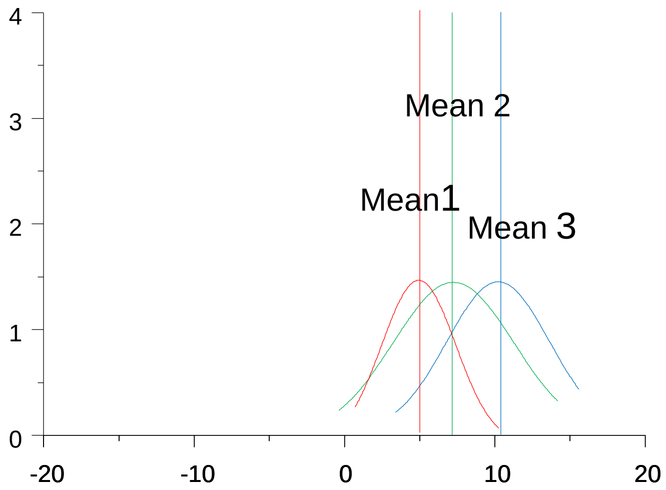
- Suppose we sample people from 3 different populations.
- Note that the sample means are far away from one another, so the variance among means will be large.



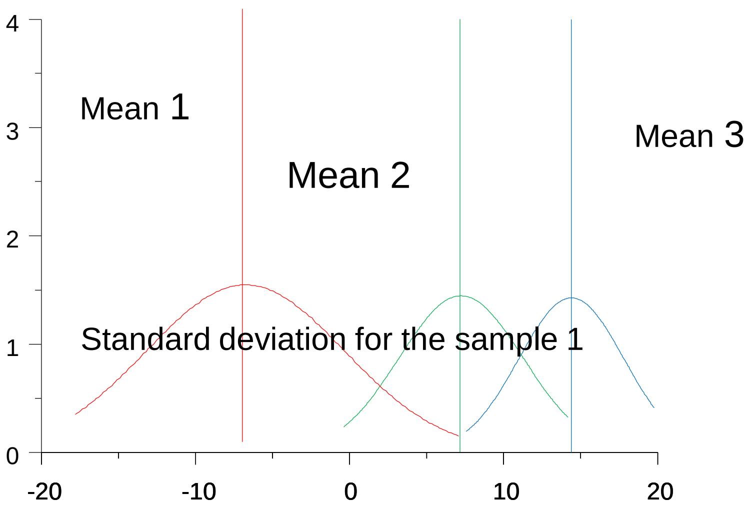
Rationale for ANOVA (5)

- Suppose we complete a study and find the following results (either graph). How would we know or decide whether there is a real effect or not?

Three samples for the same population



Three samples for different populations



- To decide, we can compare our observed variance in means to what we would expect to get on the basis of chance given no true difference in means.

Definitions

- The Grand Mean, $\bar{\bar{X}} = \bar{X}_G$, taken over all observations.
- The mean of a specific level $\bar{X}_{A_1}$(level 1 in this case).
- The observation of the i^{th} element X_i.

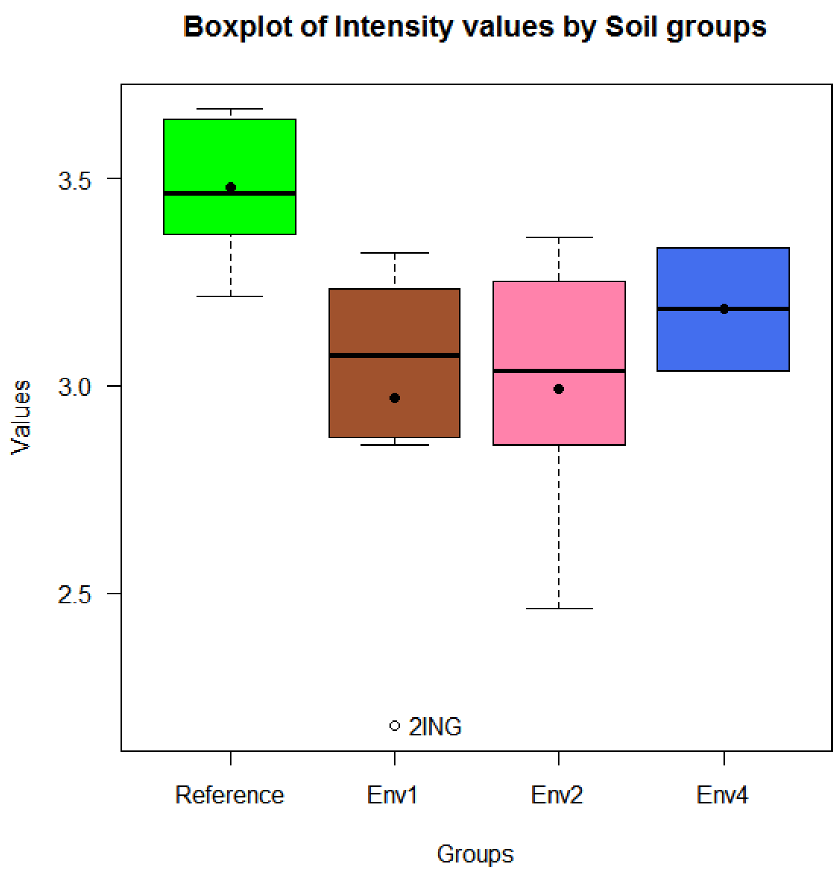
Example:
Is important the floor for the wine?

- The dataset

wine

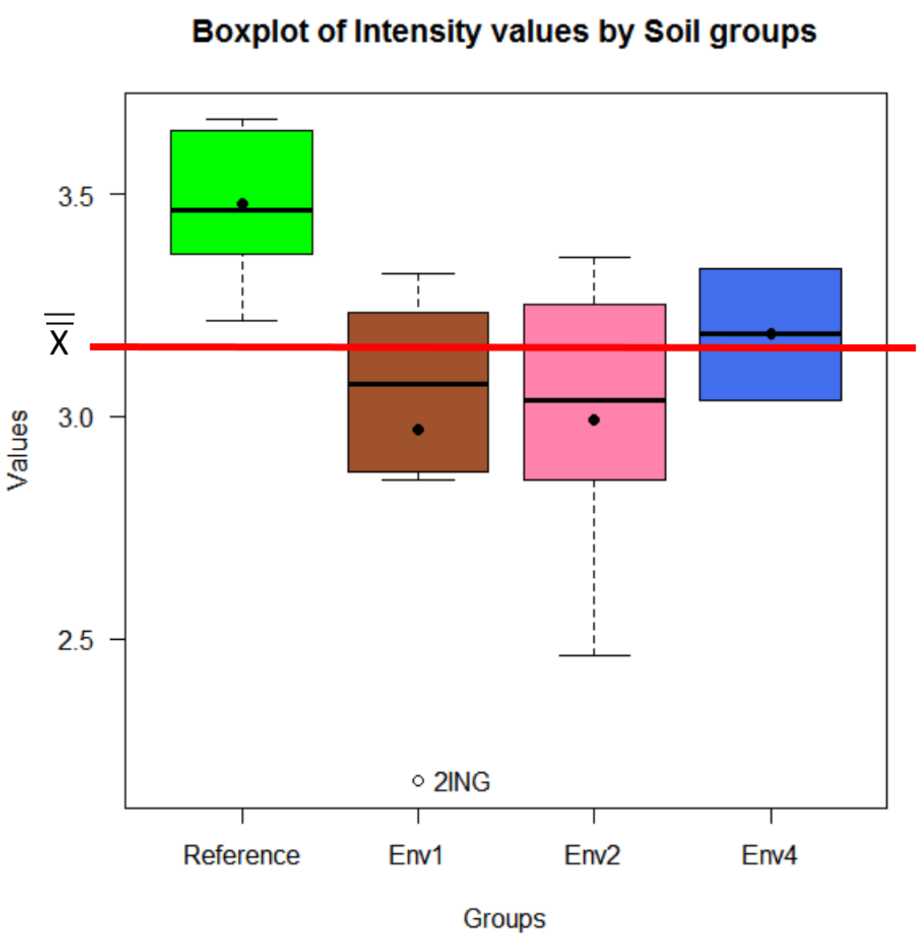
	Label	Soil	Odor.Intensity.before.shaking	Aroma.quality.before.shaking	Fruity.before.shaking	Flower.before.shaking	Spice.before.shak
2EL	Saumur	Env1	3.074	3.000	2.714	2.280	1.
1CHA	Saumur	Env1	2.964	2.821	2.375	2.280	1.
1FON	Bourgueuil	Env1	2.857	2.929	2.560	1.960	2.
1VAU	Chinon	Env2	2.808	2.593	2.417	1.913	2.
1DAM	Saumur	Reference	3.607	3.429	3.154	2.154	2.
2BOU	Bourgueuil	Reference	2.857	3.111	2.577	2.040	2.
1BOI	Bourgueuil	Reference	3.214	3.222	2.962	2.115	2.
3EL	Saumur	Env1	3.120	2.852	2.500	2.200	2.
DOM1	Chinon	Env1	2.857	2.815	2.808	1.923	2.
1TUR	Saumur	Env2	2.893	3.000	2.571	1.846	1.
4EL	Saumur	Env2	3.250	3.286	2.714	1.926	1.
PER1	Saumur	Env2	3.393	3.179	2.769	2.038	1.
2DAM	Saumur	Reference	3.179	3.286	2.778	2.231	1.
1POY	Saumur	Reference	3.071	3.107	2.731	2.120	1.
1ING	Bourgueuil	Env1	3.107	3.143	2.846	2.185	1.
1BEN	Bourgueuil	Reference	2.929	3.179	2.852	2.000	2.
2BEA	Chinon	Reference	3.036	3.179	3.037	2.231	1.
1ROC	Chinon	Env2	3.071	2.926	2.741	2.000	1.
2ING	Bourgueuil	Env1	2.643	2.786	2.536	1.889	1.
T1	Saumur	Env4	3.696	3.192	2.833	1.826	2.
T2	Saumur	Env4	3.708	2.926	2.520	2.040	2.

Example:
Is important the floor for the wine?



- We obtain information from different wines.
- There are different features (factors) that can determine the wine quality.
- We want to analyze the intensity feature.
- The factor is the floor, with 4 different possible values (levels).

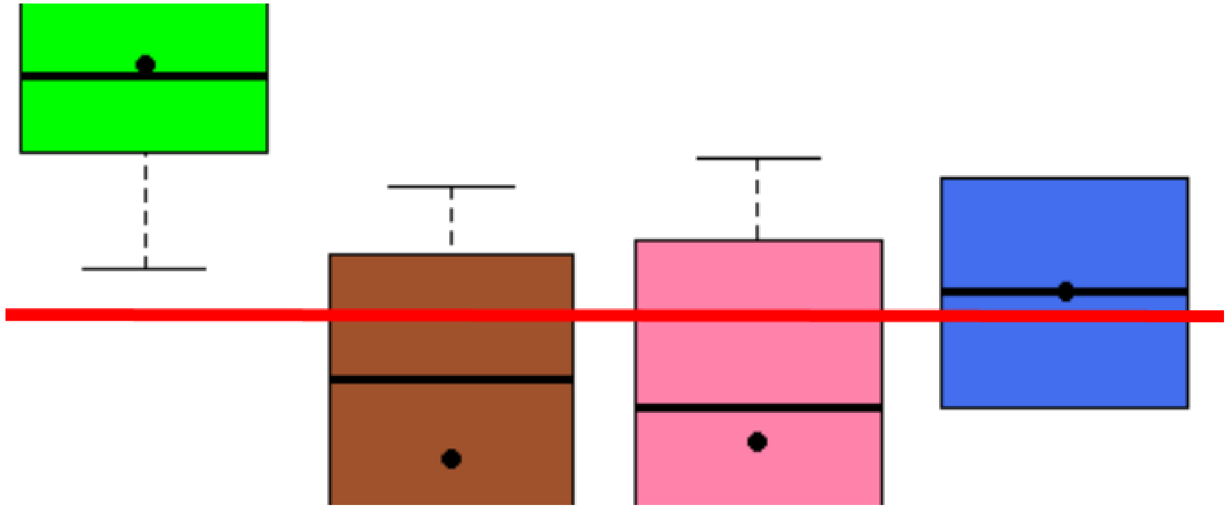
Example:
Is important the floor for the wine?



- The overall mean of the entire sample was 3.166
- This is called the "grand" mean, and is often denoted by $\bar{\bar{X}}$.
- If H0 were true then we'd expect the group means to be close to the grand mean.

Example:
Is important the floor for the wine?

- The ANOVA test is based on the combined distances from $\bar{\bar{X}}$.
- If the combined distances are large, that indicates we should reject H0.



The ANOVA statistic

SSB, MSE

The Anova Statistic

- To combine the differences from the grand mean we
  - Square the differences
  - Multiply by the numbers of observations in the groups
  - Sum over the groups
- "SSB" = Sum of Squares Between groups

$$\mathrm{SSB} = N_{\mathrm{Re}\,ference}\left(\overline{X}_{\mathrm{Re}\,fenrenc} - \overline{\overline{X}}\right)^2 + N_{Env1}\left(\overline{X}_{Env1} - \overline{\overline{X}}\right)^2 + N_{Env2}\left(\overline{X}_{Env2} - \overline{\overline{X}}\right)^2 + N_{Env4}\left(\overline{X}_{Env4} - \overline{\overline{X}}\right)^2$$

- Where the $\bar{X}_*$ are the group means.
- Note: This looks a bit like a variance.

How big is to reject?

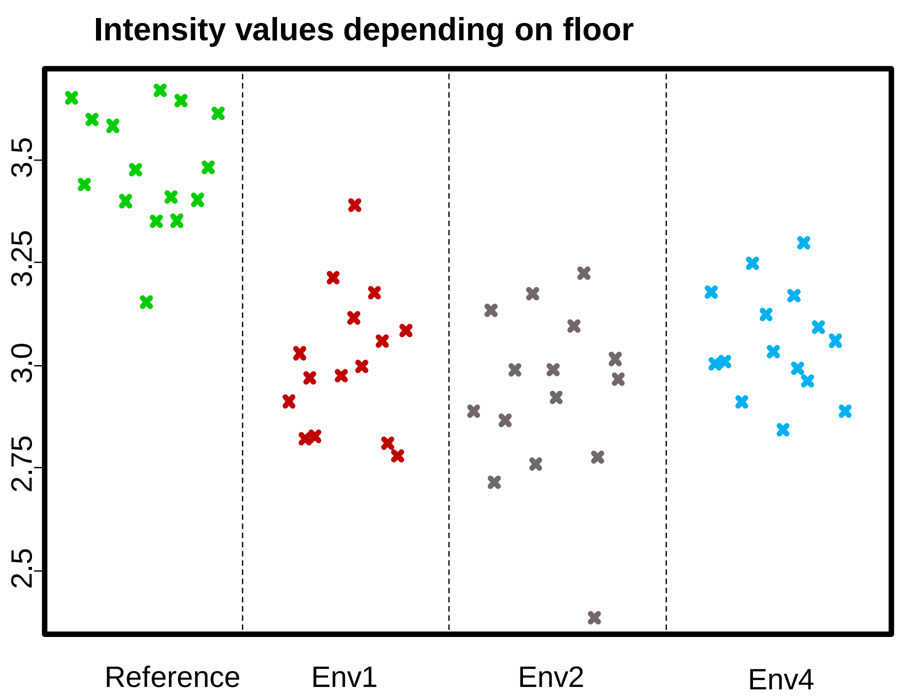
- For the wine data, SSB = 1.108

- Is that big enough to reject H0?

- As with the t test, we compare the statistic to the variability of the individual observations.

- In ANOVA the variability is estimated by the Mean Square Error, or MSE

MSE: Mean Square Error

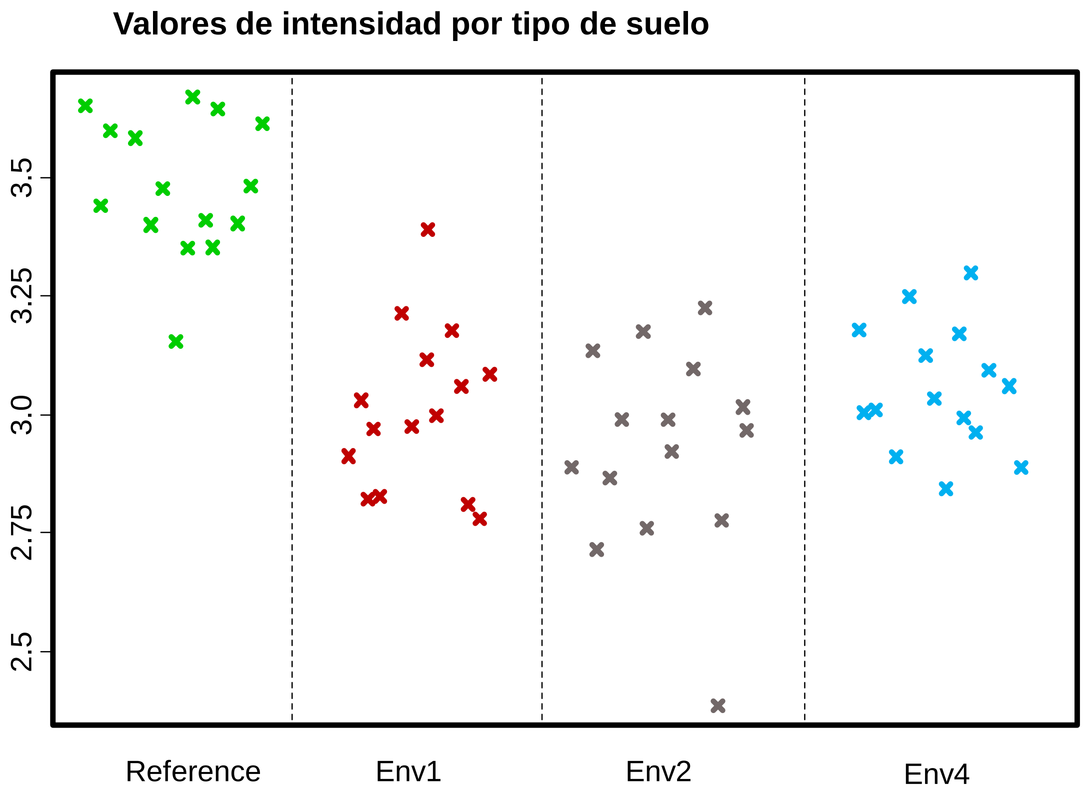


- The Mean Square Error is a measure of the variability after the group effects have been taken into account.

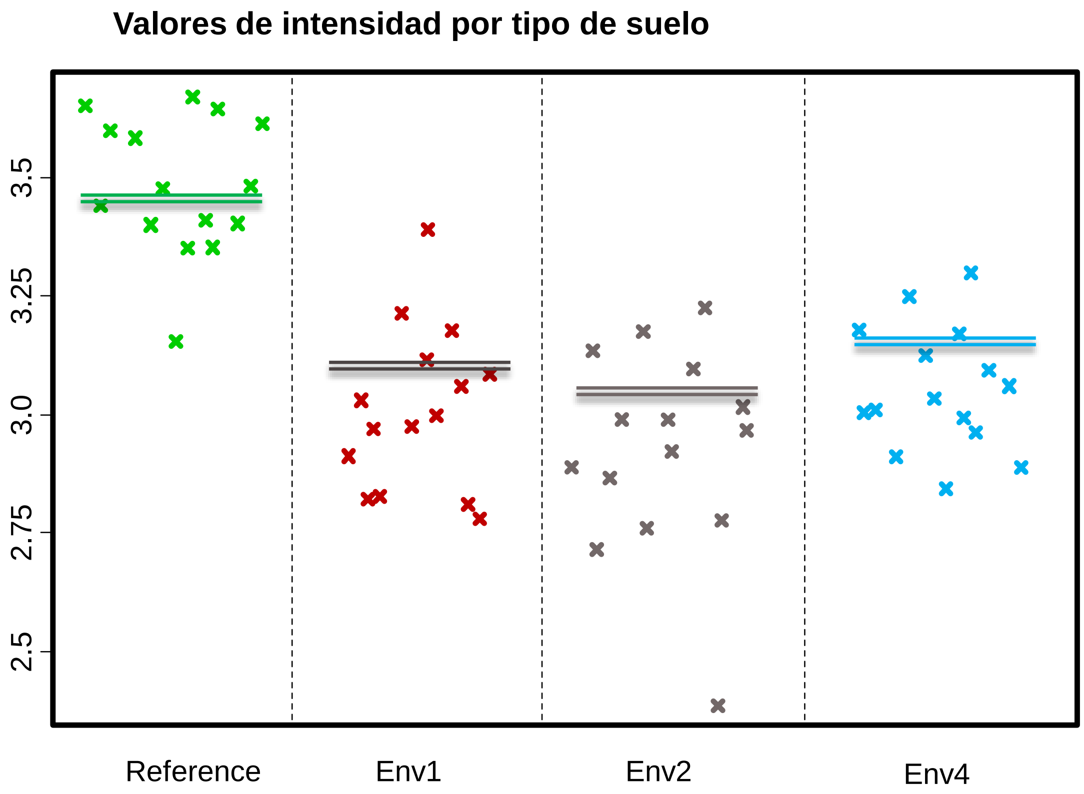
$$MSE = \frac{1}{N-K}\sum_{j}\sum_{i}\left(x_{ij} - \overline{X}_{j}\right)^2$$

- where xij is the ith observation in the jth group.

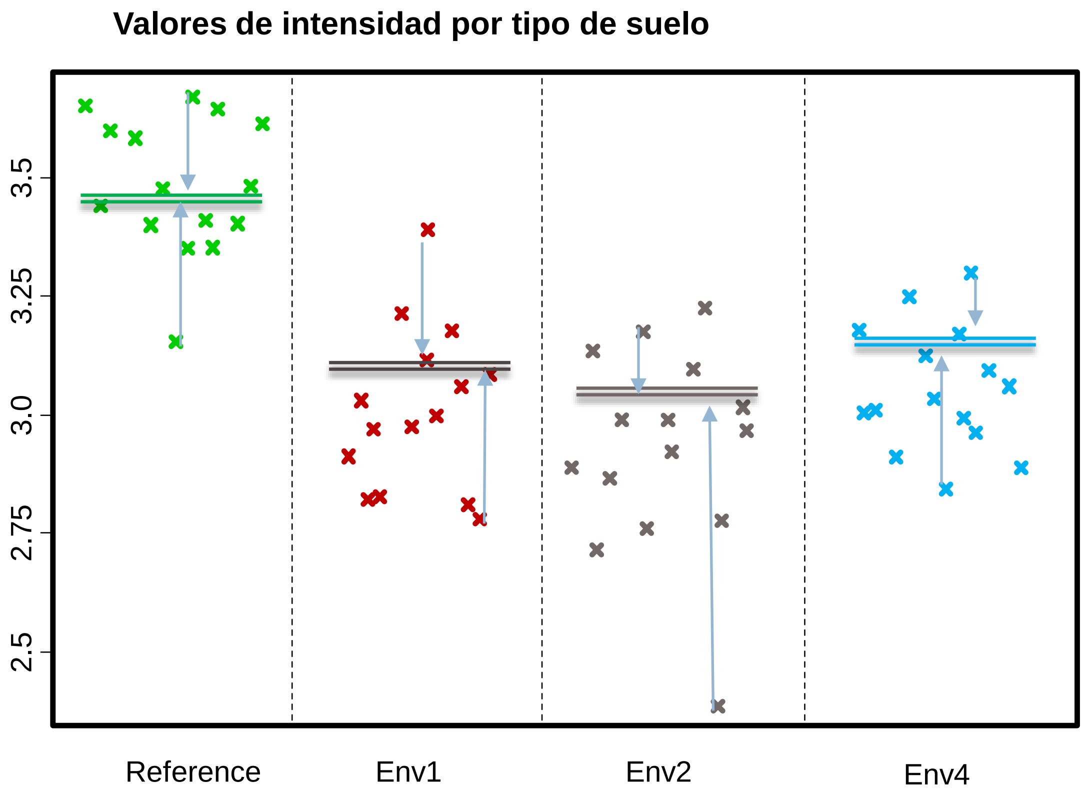
MSE: Mean Square Error



MSE: Mean Square Error



MSE: Mean Square Error



We can break the total variance in a study into meaningful pieces that correspond to treatment effects and error. That's why we call this Analysis of Variance.

Notes on MSE

- If there are only **two groups**, the MSE is equal to the pooled estimate of variance used in the equal-variance **t test.**
- ANOVA assumes that all the **group variances are equal**.
- Other options should be considered if group variances differ by a factor of 2 or more.

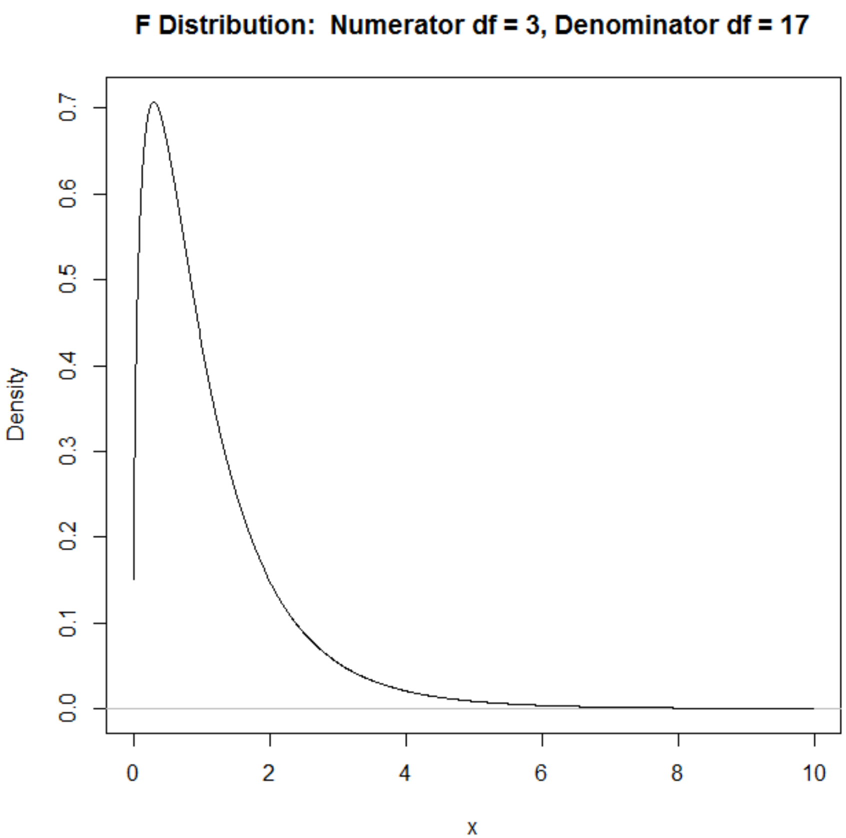
ANOVA F Test

- The ANOVA F test is based on the F statistic

$$F = \frac{SSB/(K-1)}{MSE}$$

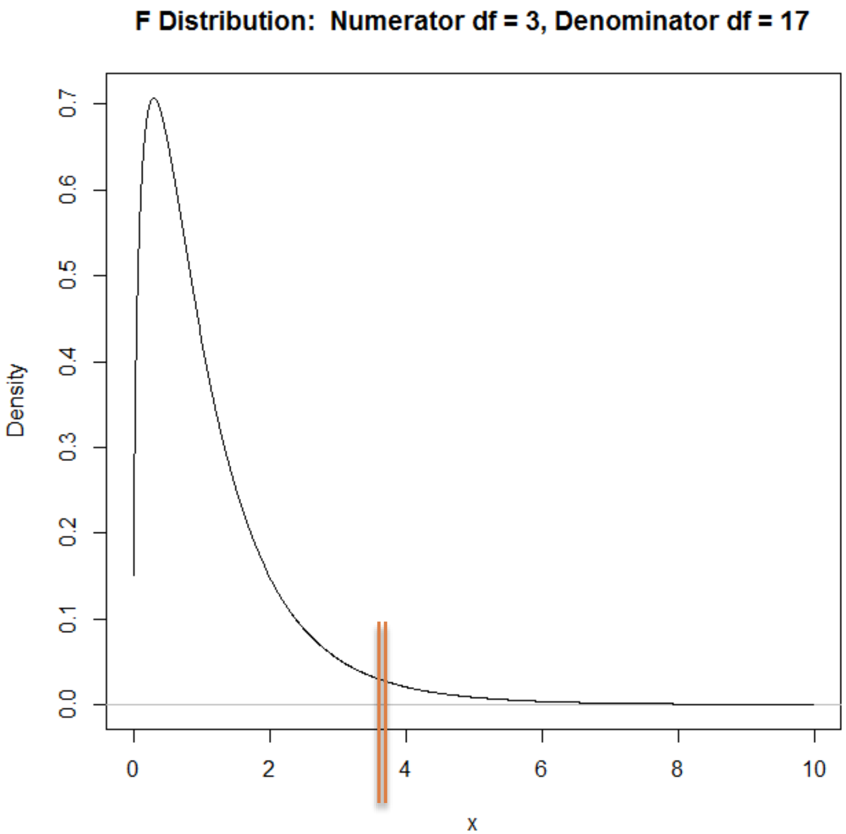
  - where K is the number of groups.
  - N is the total number of observations
- Under H_0 the F statistic has an "F" distribution, with K-1 and N-K degrees of freedom (N is the total number of observations)

Wine Data: F test p-value



- To get a p-value we compare our F statistic to an F(3, 17) distribution.

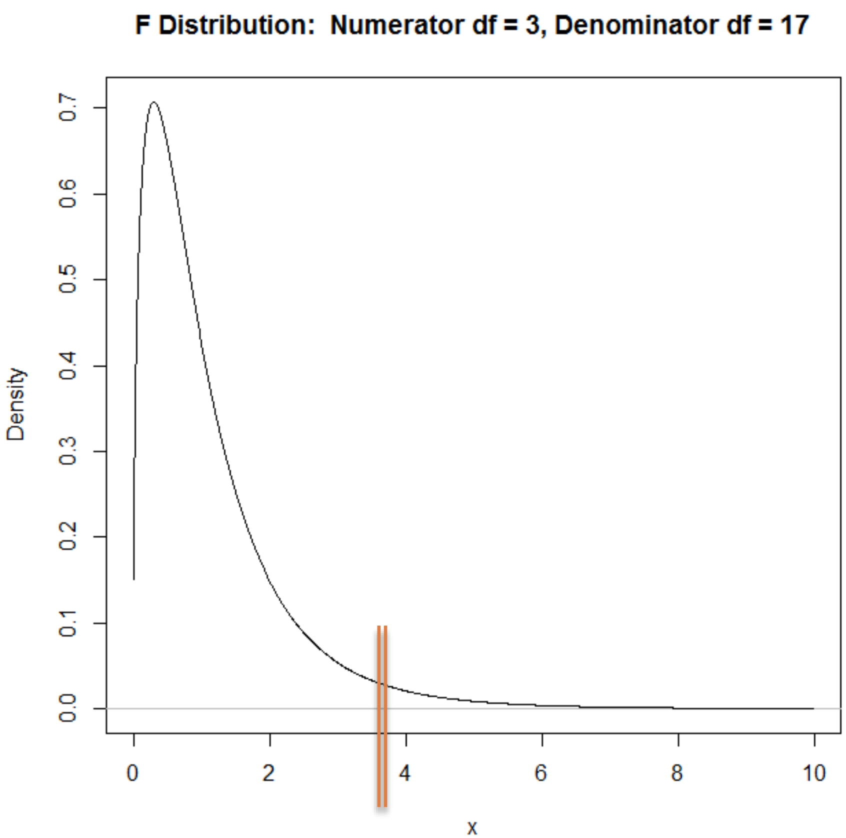
Wine Data: F test p-value



- To get a p-value we compare our F statistic to an F(3, 17) distribution.
- In our example

$$F = \frac{1.108/3}{0.0981} = 3.766$$

Wine Data: F test p-value



- To get a p-value we compare our F statistic to an F(3, 17) distribution.
- In our example

$$F = \frac{1.108/3}{0.0981} = 3.766$$

$$\mathrm{P}(\mathrm{F}(3,17) > 3.766) = 0.0306$$

- The p-value is 0.0306
- We reject H_0

Tabla de ANOVA

- Results are often displayed using an ANOVA Table

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Intra groups	3	1.108	0.3694	3.766	0.0306 *
Between groups	17	1.668	0.0981		

Sum of Squares Between (*SSB*)

Mean Square Error (*MSE*)

F Statistic

p value

R ANOVA table

- Rcmdr> AnovaModel.2 <- aov(Intensity ~ Soil, data=wine)
- Rcmdr> summary(AnovaModel.2)

```
Rcmdr>  AnovaModel.2 <- aov(Intensity ~ Soil, data=wine)

Rcmdr>  summary(AnovaModel.2)
            Df Sum Sq Mean Sq F value Pr(>F)
Soil         3  1.108  0.3694   3.766 0.0306 *
Residuals   17  1.668  0.0981
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example using R

- Continuing with the example with the Wine data, build a model that relates two independent variables such as Soil (soil where wine is grown) and label (the label of the wine). We want to analyze two dependent variables, such as intensity and aroma.

Assumptions of ANOVA

- The observations within each sample must be **independent.**
  - Durbin Watson test
  - dwtest(OurModel, alternative = "two.sided")
- The populations from which the samples are selected must be **normal.**
  - Shapiro test
  - shapiro.test(Pop1), do for all the populations.
- The populations from which the samples are selected must have **equal variances** (homogeneity of variance)
  - Breusch Pagan test
  - lmtest::bptest(OurModel)

Homoscedasticity test

- Our decision rule is as follows using the 5% level of significance:
  - H0 (Null Hypothesis): Homoscedasticity
  - HA (Alternative Hypothesis): Heteroscedasticity

Homocedasticity

lmtest::bptest(AnovaModel.3)

BP = 2.7267, df = 3, p-value = ***0.4357***

We accept H0

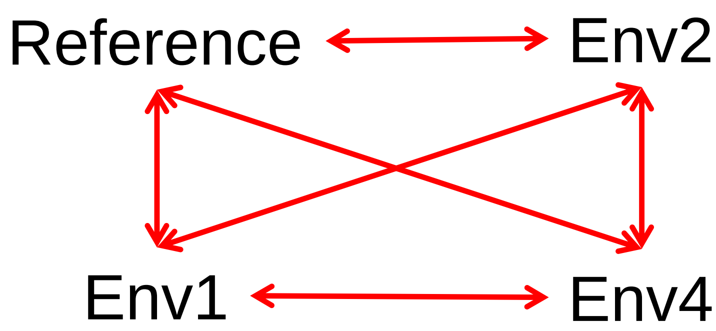
- The recommended method for correcting heteroscedasticity is **redefining the variables** (ex. Log).

Multiple comparisons

Post-hoc testing

Multiple Comparisons

- Post-hoc testing usually involves multiple comparisons.
- For example, if the data contain 4 groups, then 6 different pairwise comparisons can be made



Multiple Comparisons

- Each time a hypothesis test is performed at significance level α, there is probability α of rejecting in error.
- Performing multiple tests increases the chances of rejecting in error.

The problem of the multiple comparisons

- If one test is performed at the 5% level, there is only a 5% chance of incorrectly rejecting the null hypothesis if the null hypothesis is true.
- For 100 tests where all null hypotheses are true, the expected number of incorrect rejections is 5.

The Binomial Probability Distribution

- For a binomial experiment with:
  - **n** trials and
  - probability **p** of success on a given trial, q = 1 – p.
  - the probability of **k** successes in n trials is

$$P(x=k)=C_k^n p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k} \text{ for } k=0,1,2,\ldots n.$$

$$\text{Recall} \quad C_k^n = \frac{n!}{k!(n-k)!}$$

with $n! = n(n-1)(n-2)\ldots(2)1$ and $0! \equiv 1$.

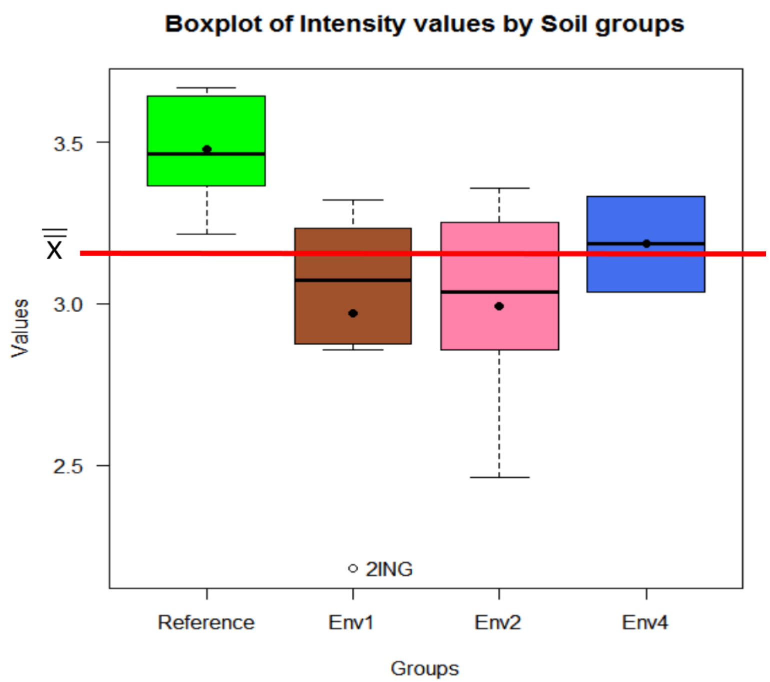
Multiple Comparisons

- If the tests are independent, the probability of at least one incorrect rejection is 99.4%. These errors are called false positives or Type I errors.

- P(at least one rejection) = 1-P(no rejections)
- 1-P(x=0) = 1-0.0059 = .994

$$P(x=0)=\frac{100!}{0!(100-0)!}0.05^{0}0.95^{100-0}$$

Post-hoc Testing



	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Intra groups	3	1.108	0.3694	3.766	0.0306 *
Between groups	17	1.668	0.0981		

- Analyzing the wine intensity regarding the soil.
- The ANOVA shows good evidence (p = 0.0306) that the means are not all the same.
- Which means are different?
- Can directly compare the subgroups using "post hoc" tests.

Post-hoc Testing

- If the t-test is significant, you have a difference in population means.
- If the F-test is significant, you have a difference in population means. But you don’t know where.
- With 3 means, could be A=B>C or A>B>C or A>B=C.

Post-hoc Testing

- ANOVA just says that the means differ, but not which ones. We have to do additional tests to determine.
- When are post hoc tests done? As the name implies after an ANOVA
  - But only after a rejection of the null hypothesis.
  - Only if there are 3 more treatments; $k > 2$. If only 2 treatments we can just do a t-test.

Post-hoc Testing

- Post hoc tests are going to let us go back through our data and compare individual treatments 2 at a time:
  - Bonferroni correction
  - Duncan's new multiple range test
  - Friedman test
  - Newman–Keuls method
  - Scheffé's method
  - Tukey's range test
  - Dunnett's test

Bonferroni Correction

- The Bonferroni correction is a simple way to adjust for the multiple comparisons. We calculate a new α value.
  - Perform each test at significance level α.
  - Divide the α by the number of tests performed.
    - $\alpha' = \alpha/m$

- Is a conservative approach. Useful when the number of groups is small.

Example Bonferroni: wine

```
data(wine, package="FactoMineR")
with(wine,
{
pairwise.t.test(Intensity,Soil,p.adj="bonf")
})
```

Pairwise comparisons using t tests with pooled SD

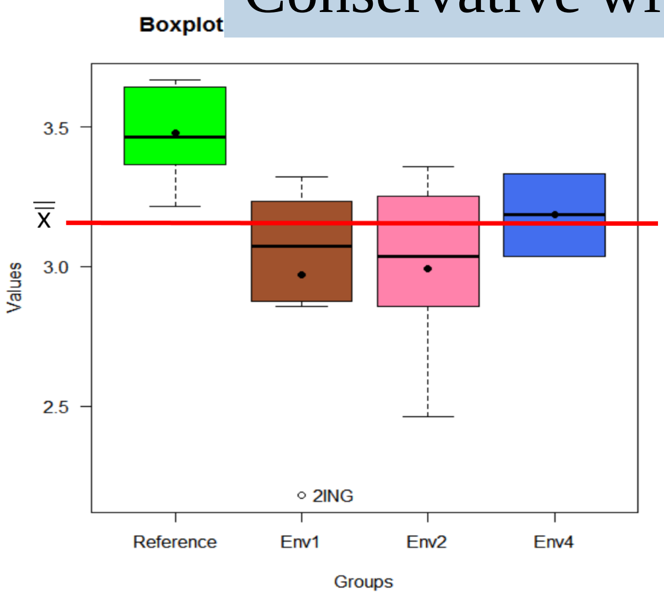
data: Intensity and Soil

Conservative with large number of tests.

2

Env1	0.044	-	-
Env2	0.100	1.000	-
Env4	1.000	1.000	1.000

P value adjustment method: bonferroni



Least Significant Difference test

- The computation is very similar to the equal-variance *t* test.
- Compute an equal-variance *t* test, but replace the pooled variance (s^2) with the *MSE*.

Least Significant Difference test

- Remember, equal variance test

$$\frac{\bar{y}_A - \bar{y}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} > t_{1-\alpha,n}$$

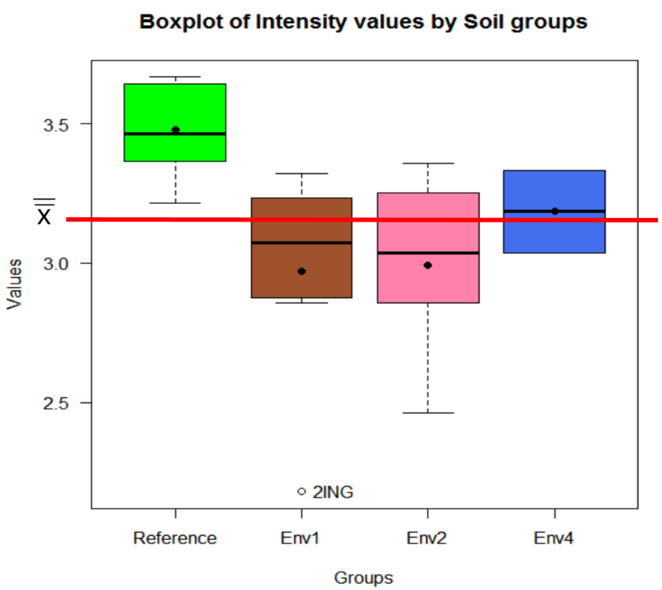
- Remember, MSE

$$MSE = \frac{1}{N-K}\sum_j \sum_i \left(x_{ij} - \bar{X}_j\right)^2$$

$$|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}| \geq t_{\alpha/2}\sqrt{s_{\mathrm{W}}^2\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Example LSD: wine

```
data(wine, package="FactoMineR")
with(wine,
{
pairwise.t.test(Intensity,Soil,p.adj="none")
})
```



Pairwise comparisons using t tests with pooled SD

data: Intensity and Soil

	Reference	Env1	Env2
Env1	0.0074	-	-
Env2	0.0166	0.9052	-
Env4	0.2565	0.4048	0.4728

P value adjustment method: none

Tukey

- $|\bar{X}_i - \bar{X}_j| \geq q(\propto, k, df)\sqrt{\frac{MSE}{n}}$

Equal number of samples

- $|\bar{X}_i - \bar{X}_j| \geq q(\propto, k, df)\sqrt{\frac{MSE}{n}(\frac{1}{n_i} + \frac{1}{n_j})}$
- $k = \#groups, df = N - k$

Studentized range distribution

$$\boldsymbol{q = \frac{(\bar{y}_{\max} - \bar{y}_{\min})}{S\sqrt{2/n}}}$$

Example Tukey: wine

```
data(wine, package="FactoMineR")
with(wine,
{
  aov_model=aov(Intensity~Soil)
  TukeyHSD(aov_model)
})
```

```
Tukey multiple comparisons of means
  95% family-wise confidence level
Fit: aov(formula = Intensity ~ Soil)
$Soil
```

When confidence intervals are needed or sample sizes are not equal.

	diff	lwr	upr	p.adj
Env1-Reference	-0,5094286	-0,9853299	-0,03353	0,033641
Env2-Reference	-0,4872571	-1,0085809	0,034067	0,071501
Env4-Reference	-0,2948571	-1,0087091	0,418995	0,650531
Env2-Env1	0,02217143	-0,4991523	0,543495	0,999342
Env4-Env1	0,21457143	-0,4992805	0,928423	0,827774

To know more

- Part III: Stochastic Processes Chapter 11.2: Regression of **Probability and Statistics for Computer Scientists** (2014 Ed.)