The DPLL algorithm

Combinatorial Problem Solving (CPS)

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Overview of the session

- Designing an efficient SAT solver
- DPLL: A Bit of History
- Abstract DPLL:
  - Rules
  - Examples
  - Theoretical Results

Designing an efficient SAT solver

INPUT: formula F in **CNF**

OUTPUT:

- If F is SAT: YES $+$ model
- If F is UNSAT: NO $+$ refutation (proof of unsatisfiability)

Two possible methods:

- resolution-based:
  - \- not direct to obtain model
  - $+$ straightforward to give refutation
- DPLL-based:
  - $+$ straightforward to obtain model
  - \- not direct to give refutation

Due to their efficiency, DPLL-based solvers are the method of choice

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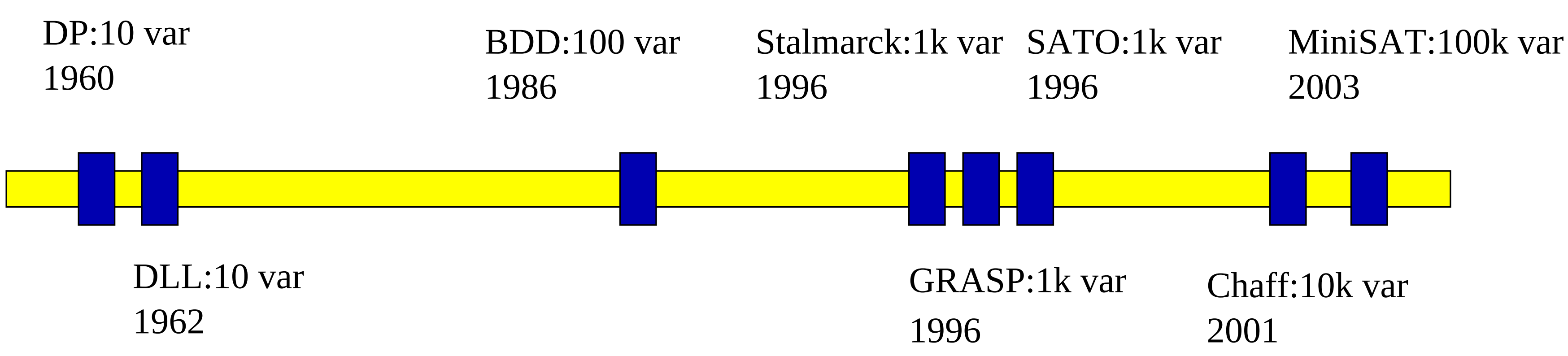
DPLL - A Bit of History

- Original DPLL was incomplete method for FOL satisfiability
- First paper (Davis and Putnam) in 1960: memory problems
- Second paper (Davis, Logemann and Loveland) in 1962:
  Depth-first-search with backtracking
- Late 90's and early 00's improvements make DPLL efficient:
  - Break-through systems: GRASP, SATO, Chaff, MiniSAT



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- Abstract DPLL:
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Our Abstraction of DPLL

- Given F in CNF, DPLL tries to build assignment M s.t. $M \models F$
- Assignments M are represented as sequences of literals (those to be true):
  EXAMPLE: sequence $p\overline{q}r$ is $M(p) = 1$, $M(q) = 0$, $M(r) = 1$
  (overlining bar $^-$ may be used to represent negation, like $\neg$)
  - Order in M matters
  - No literal appears twice in M
  - No contradictory literals in M
- Sequences may have decision literals, denoted l^{d}.
- We will introduce a transition system modelling DPLL
- States in the transition system are pairs $M \parallel F$,
  where M is a (partial) assignment and F is a CNF
- The algorithm starts with an empty assignment
- The rules in the transition system indicate which steps
  $M \parallel F \Longrightarrow M' \parallel F'$
  are allowed.

Abstract DPLL - Rules

Extending the model:

Decide

$$M \parallel F \implies M\, l^{\mathsf{d}} \parallel F \;\; \textbf{if} \begin{cases} l \text{ or } \bar{l} \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

UnitProp $$M \parallel F, C \vee l \implies M\ l \parallel F, C \vee l \quad \textbf{if} \begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$$

Abstract DPLL - Rules (2)

Repairing the model:

Fail

$$M \parallel F, C \implies \mathit{fail} \;\; \textbf{if} \begin{cases} M \models \neg C \\ M \text{ contains no decision literals} \end{cases}$$

Backtrack

$$M\, l^{\mathsf{d}}\, N \parallel F, C \implies M\, \bar{l} \parallel \;\; F, C \;\; \textbf{if} \begin{cases} M\, l^{\mathsf{d}}\, N \models \neg C \\ N \text{ contains no decision lits} \end{cases}$$

Abstract DPLL - Example 1

$$\emptyset \parallel \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} \quad \Longrightarrow$$

Abstract DPLL - Example 1

$$\emptyset \ \| \ \overline{1} \vee 2, \ \overline{3} \vee 4, \ \overline{5} \vee \overline{6}, \ 6 \vee \overline{5} \vee \overline{2} \quad \Longrightarrow \quad \text{(Decide)}$$

Abstract DPLL - Example 1

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\begin{array}{rcl}
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Abstract DPLL - Example 1

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{\color{red}1^{\mathsf{d}}}\ 2\ {\color{red}3^{\mathsf{d}}} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & \text{(UnitProp)} \\
{\color{red}1^{\mathsf{d}}}\ 2\ {\color{red}3^{\mathsf{d}}}\ 4 \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow &
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{\color{red}1^{\mathsf{d}}}\ 2\ {\color{red}3^{\mathsf{d}}}\ 4 & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & \text{(Decide)}
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1^{\mathsf{d}} & \| & \overline{1} \lor 2, & \overline{3} \lor 4, & \overline{5} \lor \overline{6}, & 6 \lor \overline{5} \lor \overline{2} & \Longrightarrow & \text{(UnitProp)} \\
1^{\mathsf{d}}\ 2 & \| & \overline{1} \lor 2, & \overline{3} \lor 4, & \overline{5} \lor \overline{6}, & 6 \lor \overline{5} \lor \overline{2} & \Longrightarrow & \text{(Decide)} \\
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1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ 5^{\mathsf{d}}\ \overline{6} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow &
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\emptyset \;\|\; \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & (\textsf{Decide}) \\
{\color{red}1^{\mathsf{d}}} \;\|\; \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & (\textsf{UnitProp}) \\
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{\color{red}1^{\mathsf{d}}}\ 2\ {\color{red}3^{\mathsf{d}}}\ 4\ {\color{red}5^{\mathsf{d}}} \;\|\; \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & (\textsf{UnitProp}) \\
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1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ 5^{\mathsf{d}}\ \overline{6} & \| & \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & \text{(Backtrack)} \\
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1^{\mathsf{d}}\; 2\; 3^{\mathsf{d}} \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & \text{(UnitProp)} \\
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1^{\mathsf{d}}\; 2\; 3^{\mathsf{d}}\; 4\; 5^{\mathsf{d}} \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow & \text{(UnitProp)} \\
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1^{\mathsf{d}}\; 2\; 3^{\mathsf{d}}\; 4\; \overline{5}\; 6^{\mathsf{d}} \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} & &
\end{array}
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Abstract DPLL - Example 1

$\emptyset$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(Decide)
1^{d}	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(UnitProp)
$1^{\mathsf{d}}\ 2$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(Decide)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(UnitProp)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(Decide)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ 5^{\mathsf{d}}$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(UnitProp)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ 5^{\mathsf{d}}\ \overline{6}$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(Backtrack)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ \overline{5}$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$	$\Longrightarrow$	(Decide)
$1^{\mathsf{d}}\ 2\ 3^{\mathsf{d}}\ 4\ \overline{5}\ 6^{\mathsf{d}}$	$\parallel$	$\overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}$		Final state found!

Abstract DPLL - Example 2

$$\emptyset \;\|\; \overline{1} \vee 2 \vee 3,\;\; 1,\;\; \overline{2} \vee 3,\;\; \overline{2} \vee \overline{3},\;\; 2 \vee 3,\;\; 2 \vee \overline{3} \qquad \Longrightarrow$$

Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} \qquad \Longrightarrow \qquad \text{(UnitProp)}$$

Abstract DPLL - Example 2

$$
\begin{array}{lllll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & (\textsf{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow &
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{lllll}
\emptyset & \| & \overline{1} \lor 2 \lor 3,\ \ 1,\ \ \overline{2} \lor 3,\ \ \overline{2} \lor \overline{3},\ \ 2 \lor 3,\ \ 2 \lor \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 & \| & \overline{1} \lor 2 \lor 3,\ \ 1,\ \ \overline{2} \lor 3,\ \ \overline{2} \lor \overline{3},\ \ 2 \lor 3,\ \ 2 \lor \overline{3} & \Longrightarrow & \text{(Decide)}
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcl}
\emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\
1 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{Decide}) \\
1\ {\color{red}2}^{\mathsf{d}} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow &
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcllll}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\;\; 1,\;\; \overline{2} \vee 3,\;\; \overline{2} \vee \overline{3},\;\; 2 \vee 3,\;\; 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 & \| & \overline{1} \vee 2 \vee 3,\;\; 1,\;\; \overline{2} \vee 3,\;\; \overline{2} \vee \overline{3},\;\; 2 \vee 3,\;\; 2 \vee \overline{3} & \Longrightarrow & \text{(Decide)} \\
1\; {\color{red}2^{\mathsf{d}}} & \| & \overline{1} \vee 2 \vee 3,\;\; 1,\;\; \overline{2} \vee 3,\;\; \overline{2} \vee \overline{3},\;\; 2 \vee 3,\;\; 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)}
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rclcl}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & (\textsf{UnitProp}) \\
1 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & (\textsf{Decide}) \\
1\ {\color{red}2^{\mathsf{d}}} & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & (\textsf{UnitProp}) \\
1\ {\color{red}2^{\mathsf{d}}}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow &
\end{array}
$$

Abstract DPLL - Example 2

$$\begin{array}{rcl} \emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\ 1 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{Decide}) \\ 1\ 2^{\mathsf{d}} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\ 1\ 2^{\mathsf{d}}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\mathsf{Backtrack}) \end{array}$$

Abstract DPLL - Example 2

$$
\begin{array}{rcl@{\qquad}c@{\qquad}l}
\emptyset & \parallel & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 & \parallel & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(Decide)} \\
1\ {\color{red}2^{\mathsf{d}}} & \parallel & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ {\color{red}2^{\mathsf{d}}}\ 3 & \parallel & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(Backtrack)} \\
1\ \overline{2} & \parallel & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow &
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcl}
\emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(Decide)} \\
1\ 2^{\mathsf{d}} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ 2^{\mathsf{d}}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(Backtrack)} \\
1\ \overline{2} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)}
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcl}
\emptyset \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\
1 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow & (\mathsf{Decide}) \\
1\ {\color{red}2^{\mathsf{d}}} \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\
1\ {\color{red}2^{\mathsf{d}}}\ 3 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow & (\mathsf{Backtrack}) \\
1\ \overline{2} \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow & (\mathsf{UnitProp}) \\
1\ \overline{2}\ 3 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} & \Longrightarrow &
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcl}
\emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(Decide)} \\
1\ {\color{red}2^{\mathsf{d}}} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ {\color{red}2^{\mathsf{d}}}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(Backtrack)} \\
1\ \overline{2} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ \overline{2}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & \text{(Fail)}
\end{array}
$$

Abstract DPLL - Example 2

$$
\begin{array}{rcllc}
\emptyset & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(Decide)} \\
1\ {\color{red}2^{\mathsf{d}}} & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ {\color{red}2^{\mathsf{d}}}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(Backtrack)} \\
1\ \overline{2} & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(UnitProp)} \\
1\ \overline{2}\ 3 & \| & \overline{1} \vee 2 \vee 3,\ \ 1,\ \ \overline{2} \vee 3,\ \ \overline{2} \vee \overline{3},\ \ 2 \vee 3,\ \ 2 \vee \overline{3} & \Longrightarrow & \text{(Fail)} \\
& \mathit{fail} & & &
\end{array}
$$

Abstract DPLL

- There are no infinite sequences of the form $\emptyset \parallel F \implies \dots$
- If $\emptyset \parallel F \implies^* M \parallel F$ with state $M \parallel F$ final, then
  - F is satisfiable
  - M is a model of F
- If $\emptyset \parallel F \implies^* \mathit{fail}$ then F is unsatisfiable

Hence the transition system gives a decision procedure for SAT

Bibliography - Some further reading

- Martin Davis, Hilary Putnam. *A Computing Procedure for Quantification Theory.* J. ACM 7(3): 201-215 (1960)

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