Notes on Stochastic Network Modeling (SNM)

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Contents

Preface

These notes were prepared for the subject *Stochastic Network Modeling (SNM)* from the *Master in Innovation and Research in Informatics (MIRI)*, Universitat Politècnica de Catalunya.

Further reading: The book of Mor [3], Nelson [7] or Trivedi [8] are excellent books on Markov Chains and cover most of the course. The book of Kemeny [4] covers absorbing DTMC.

Part I

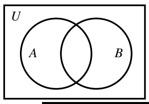
Introduction

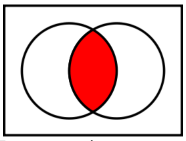
Chapter 1
Probability Review

1.1 Ingredients of Probability

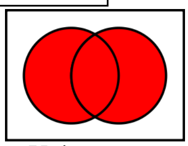
- **Random experiment**, e.g. toss a die.
- **Outcome,** ω, e.g. tossing a die can be $\omega = 2$, choosing a fruit can be $\omega =$ orange.
- **Sample space or Universal set,** U, set of all possible outcomes. E.g. tossing a die $U = \{1,2,3,4,5,6\}$.
- **Event,** A, any subset of U (e.g. tossing a die $A = \{1,2,3\}$). We say the event A occurs if the outcome of the experiment $\omega \in A$. U is the **sure event**, and we represent by the empty set $\emptyset$ an **impossible outcome**.

Venn Diagrams

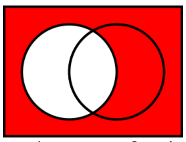
Graphical representation of events 



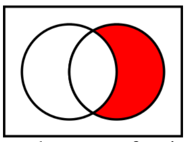
Intersection $A \cap B$



Union $A \cup B$



Complement of A in U
$A^c = U \setminus A$



Complement of A in B
(B minus A)
$A^c \cap B = B \setminus A$

source: wikipedia

Random Variable

- For simplicity it is defined a **random variable (RV),** X as a function that assigns a real number to each outcome in the sample space U, i.e.:

$$X : U \to \mathbb{R} \tag{1.1}$$

- We will represent the experiment by a RV, X, and the possible outcomes by its values. $X = x_i$ **is the outcome** $X(\omega_i) = x_i$.
- Using RVs the sample space is mapped in a subset of $\mathbb{R}$. So, in terms of X, U **is a set of points of** $\mathbb{R}$**. The same for any event.**
- Normally the definition of X **comes naturally from the experiment**, e.g. tossing a die: $X = \{\text{number in the toss}\}$.
- RVs can be **discrete** (e.g. tossing a die) or **continuous** (e.g. waiting time of a packet in a queue).

Probability Measure

- If the sample space U of the RV X is **finite** (**discrete RV**), $U = \{x_1, \cdots x_n\}$, a **probability measure** is an assignment of numbers $P(x_i)$, referred to as **probabilities**, to each **outcome** x_i such that:

$$\begin{aligned} &0 \le P(x_i) \le 1 \\ &P(A) = \sum_{x_i \in A} P(x_i) \\ &P(U) = 1 \end{aligned} \tag{1.2}$$

E.g. tossing a fair die,

$$\begin{aligned} &P(x_i) = 1/6 \\ &P(X \in \{2,4,6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned} \tag{1.3}$$

- If the sample space of the RV X is continuous (**continuous RV**), the events are intervals of $\mathbb{R}$. The probability measure is defined by means of the **cumulative distribution function, CDF**:

$$F(x) = P(X \in (-\infty, x]) = P(X \le x) \tag{1.4}$$

- X is called absolutely continuous[1] if there exists the **probability density function, PDF**, such that for any interval $I = \{x \mid a \le x \le b\}$:

$$\int_a^b f(x)\,dx = P(X \in I) = F(b) - F(a) \tag{1.5}$$

Conditional Probability and Bayes Formula

- Given the the sample space U and the **events** $A, B \in U$ with $P(B) > 0$ the **probability of** A **conditioned by** B is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1.6}$$

NOTE: It's common to use commas to denote set intersection, and write $P(A \cap B)$ **as** $P(A,B)$.

- **Bayes Formula**

$$P(A|B)\,P(B) = P(B|A)\,P(A) \Rightarrow P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} \tag{1.7}$$

Law of total probability

- Let B_i a **partition** of the sample space U ($\cup_i B_i = U$, $B_i \cap B_j = \emptyset, \forall i \ne j$), then

$$P(A) = \sum_i P(A|B_i)\,P(B_i) \tag{1.8}$$

- For **conditional probabilities**:

$$P(A|C) = \sum_i P(A|C \cap B_i)\,P(B_i|C) \tag{1.9}$$

- If C is **independent** of any of the B_i

$$P(A|C) = \sum_i P(A|C \cap B_i)\,P(B_i) \tag{1.10}$$

1.2 Expected value

- Given the discrete $N \in \mathbb{Z}$, respectively continuous $X \in \mathbb{R}$ RV, the **expected value** is:

$$\begin{aligned} \mathrm{E}[N] &= \sum_{k=-\infty}^{\infty} k\,P(N = k) \\ \mathrm{E}[X] &= \int_{-\infty}^{\infty} x\,f(x)\,dx \end{aligned} \tag{1.11}$$

[1] Some special distributions, called singular, do not have a PDF. One example is the Cantor distribution (see Wikipedia).

Example A number $X_1 \in \{1,2,\cdots 6\}$ is obtained tossing a dice. Then, a number $X_2 \in [0,\infty]$ is obtained exponentially distributed with parameter X_1. Compute $f(x_1,x_2)$, $f(x_2)$ and $\mathrm{E}[X_2]$.
Note: Exponential distribution with parameter α:

$$f(x) = \alpha\, \mathrm{e}^{-\alpha x}, x \in [0,\infty], \mathrm{E}[X] = \frac{1}{\alpha}. \tag{1.12}$$

Solution:

$$f(x_1,x_2) = f(x_2|x_1)\, P(x_1) = x_1\, \mathrm{e}^{-x_1 x_2}\, \frac{1}{6}, \begin{cases} x_1 \in \{1,2,\cdots 6\} \\ x_2 \in [0,\infty] \end{cases}$$

$$f(x_2) = \sum_{x_1} f(x_2|x_1)\, P(x_1) = \frac{1}{6} \sum_{n=1}^{6} n\, \mathrm{e}^{-n x_2}, x_2 \in [0,\infty]$$

$$\mathrm{E}[X_2] = \frac{1}{6} \sum_{n=1}^{6} \int_{x_2=0}^{\infty} x_2\, n\, \mathrm{e}^{-n x_2} = \frac{1}{6} \sum_{n=1}^{6} \frac{1}{n} = \frac{49}{120}$$

1.3 Variance

- The amount of dispersion of a RV X with expected value $\mu = \mathrm{E}[X]$ is measured by the **Variance**:

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}[(X-\mu)^2] = \mathrm{E}[X^2] - \mu^2 \tag{1.13}$$

- Often it is used the **standard deviation** $\sigma = \sqrt{\mathrm{Var}(X)}$.

1.4 Indicator Function

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.14}$$

Therefore:

$$\mathrm{E}[\mathbf{I(A)}] = 0 \times P(I(A)=0) + 1 \times P(I(A)=1) = \mathbf{P(A)} \tag{1.15}$$

1.5 Expected value of non negative RVs

- For **non negative** RVs, $N \geq 0$ discrete and $X \geq 0$ continous:

$$\begin{aligned} \mathrm{E}[N] &= \sum_{k=0}^{\infty} k\, P(N=k) = \sum_{k=0}^{\infty} P(N>k) \\ \mathrm{E}[X] &= \int_0^{\infty} x\, f(x)\, dx = \int_0^{\infty} P(X>x)\, dx = \int_0^{\infty} (1-F(x))\, dx \end{aligned} \tag{1.16}$$

$$\begin{aligned} N &= \sum_{k=0}^{N-1} 1 = \sum_{k=0}^{\infty} I(N>k) \\ X &= \int_0^{X} dx = \int_0^{\infty} I(X>x) dx \end{aligned} \tag{1.17}$$

and take expectations.

1.6 Wald's Equation

- **Definition**: An positive integer RV $N > 0$ is a **stopping time** of a sequence $X_1, X_2, \cdots$ if the event $N = n$ is independent of $X_{n+1}, X_{n+2}, \cdots$.
  E.g. toss a die until you get 6. Let N be the number of tosses. N does not depend on the values obtained after getting 6.
- **Wald's Equation** If $X_1, X_2, \cdots$ are independent and identically distributed and N is a stopping time:

$$\boxed{\mathrm{E}\left[\sum_{n=1}^{N} X_n\right] = \mathrm{E}[X]\,\mathrm{E}[N] \qquad (1.18)}$$

Proof.

$$\begin{aligned} \mathrm{E}\left[\sum_{n=1}^{N} X_n\right] &= \mathrm{E}\left[\sum_{n=1}^{\infty} X_n\, I(n \leq N)\right] = \\ &\sum_{n=1}^{\infty} \mathrm{E}[X_n]\, \mathrm{E}[I(n \leq N)] = \\ &\mathrm{E}[X] \sum_{n=1}^{\infty} P(n \leq N) = \\ &\mathrm{E}[X] \sum_{n=0}^{\infty} P(N>n) = \mathrm{E}[X]\, E[N] \end{aligned}$$ □

1.7 Probability in $\mathbb{R}^k$

If we have a set of k RV $\boldsymbol{X} = (X_1, \cdots X_k)$ taking values in $\mathbb{R}^k$ ($\boldsymbol{X} \in \mathbb{R}^k$), we define the **joint distribution**:

- Discrete RV

$$P(\boldsymbol{x}) = P(x_1, \cdots x_k) = P(X_1 = x_1, \cdots X_k = x_k) \tag{1.19}$$

- Continuos RV:
  - **cumulative distribution function, CDF**:

$$F(\boldsymbol{x}) = F(x_1, \cdots x_k) = P(X_1 \in (-\infty, x_1], \cdots X_k \in (-\infty, x_k]) \tag{1.20}$$

  - with **joint density** function $f(\boldsymbol{x}) = f(x_1, \cdots x_k)$ (if exists):

$$\begin{aligned} F(\boldsymbol{x}) &= F(x_1, \cdots x_k) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(x_1, \cdots x_k)\, dx_k \cdots dx_1 \\ f(\boldsymbol{x}) &= f(x_1, \cdots x_k) = \frac{\partial^k F(x_1, \cdots x_k)}{\partial x_1 \cdots \partial x_k} \end{aligned} \tag{1.21}$$

Marginal distributions Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X} \in \mathbb{R}^k, \boldsymbol{X}_1 \in \mathbb{R}^r, \boldsymbol{X}_2 \in \mathbb{R}^{k-r}$, $1 \leq r < k$:

- Discrete RV

$$P(\boldsymbol{x}_2) = \sum_{x_1} \cdots \sum_{x_r} P(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{1.22}$$

- Continuos RV

$$f(\boldsymbol{x}_2) = \int_{x_1} \cdots \int_{x_r} f(\boldsymbol{x}_1, \boldsymbol{x}_2)\, dx_1 \cdots dx_r \tag{1.23}$$

Independent RV

- Discrete RV

$$\begin{aligned} P(\boldsymbol{x}) &= P(x_1, \cdots x_k) = \\ &P(X_1 = x_1, \cdots X_k = x_k) = \\ &P(X_1 = x_1) \cdots P(X_k = x_k) \end{aligned} \tag{1.24}$$

- Continuos RV

$$\begin{aligned} F(\boldsymbol{x}) &= F(x_1, \cdots x_k) = F_{X_1}(x_1) \cdots F_{X_k}(x_k) \\ f(\boldsymbol{x}) &= f(x_1, \cdots x_k) = f_{X_1}(x_1) \cdots f_{X_k}(x_k) \end{aligned} \tag{1.25}$$

Conditional Distribution

- Let $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X} \in \mathbb{R}^k, \boldsymbol{X}_1 \in \mathbb{R}^r, \boldsymbol{X}_2 \in \mathbb{R}^{k-r}$, the r-dimensional distribuion of $\boldsymbol{X}_1$ **conditioned by** $\boldsymbol{X}_2 = \boldsymbol{x}_2$, $P(\{\boldsymbol{X}_2 = \boldsymbol{x}_2\}) > 0$ is:

$$F(\boldsymbol{X}_1|\boldsymbol{X}_2) = P(\boldsymbol{X}_1 \leq \boldsymbol{x}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \frac{P(\boldsymbol{X}_1 \leq \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2)}{P(\boldsymbol{X}_2 = \boldsymbol{x}_2)}.$$

If $\boldsymbol{X}$ is **discrete** with probability $P(\boldsymbol{x}_1, \boldsymbol{x}_2)$ or absolutely **continuous** with density $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$:

$$\begin{aligned} P(\boldsymbol{x}_1|\boldsymbol{x}_2) &= \frac{P(\boldsymbol{x}_1, \boldsymbol{x}_2)}{P(\boldsymbol{x}_2)} \\ f(\boldsymbol{x}_1|\boldsymbol{x}_2) &= \frac{f(\boldsymbol{x}_1, \boldsymbol{x}_2)}{f(\boldsymbol{x}_2)} \end{aligned} \tag{1.26}$$

Composition of marginals and conditionals Using the previous formulas we can compute ($\boldsymbol{X}$ can be a mixture of discrete and continuous RV):
Law of total probability

- If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are **discrete** RV: $P(\boldsymbol{x}_2) = \sum_{\boldsymbol{x}_1} P(\boldsymbol{x}_2|\boldsymbol{x}_1)\, P(\boldsymbol{x}_1)$
- If $\boldsymbol{x}_1$ is **discrete** and $\boldsymbol{x}_2$ is **cont.**: $f(\boldsymbol{x}_2) = \sum_{\boldsymbol{x}_1} f(\boldsymbol{x}_2|\boldsymbol{x}_1)\, P(\boldsymbol{x}_1)$
- If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are **cont.**: $f(\boldsymbol{x}_2) = \int_{\boldsymbol{x}_1} f(\boldsymbol{x}_2|\boldsymbol{x}_1)\, f(\boldsymbol{x}_1)\, d\boldsymbol{x}_1$
- If $\boldsymbol{x}_1$ is **cont.** and $\boldsymbol{x}_2$ is **discrete**: $P(\boldsymbol{x}_2) = \int_{\boldsymbol{x}_1} P(\boldsymbol{x}_2|\boldsymbol{x}_1)\, f(\boldsymbol{x}_1)\, d\boldsymbol{x}_1$

Conditional expected value

- Given $X \in \mathbb{R}$, $\boldsymbol{Y} \in \mathbb{R}^k$ with density $f(x, \boldsymbol{y})$:

$$\begin{aligned} \mathrm{E}[X \mid \boldsymbol{Y} = \boldsymbol{y}] &= \int_{\mathbb{R}} x\, f(x|\boldsymbol{y})\, dx \\ \mathrm{E}[X] &= \int_{\mathbb{R}^k} \mathrm{E}[X \mid \boldsymbol{Y} = \boldsymbol{y}]\, f(\boldsymbol{y})\, d\boldsymbol{y} \end{aligned} \tag{1.27}$$

where the **marginal** $f(\boldsymbol{y}) = \int_{x=-\infty}^{\infty} f(x, \boldsymbol{y}) dx$ and the **conditional** $f(x|\boldsymbol{y}) = f(x, \boldsymbol{y}) / f(\boldsymbol{y})$.
Thus, the law of total probability also applies to expected value, and it is known as **law of total expectation**.

Chapter 2

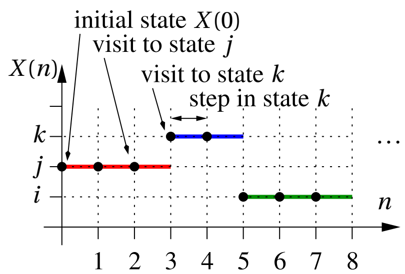
Stochastic Process (SP)

2.1 Introduction

- **Sequence of RVs** $\{X(t)\}_{t \geq 0}$.
- $X(t)$ is the **state** at time t.
- The **state** $X(t)$ can be **continuous** or **discrete**.
- The **index** can be **continuous** or **discrete**. We shall use n for the **index**, and refer to it as **steps** when it is **discrete**, and t and refer to it as **time** when it is **continuous**.
- We call a possible sequence of states of the SP the **sample function** (or sample path) of the SP.

Sample Path

- Possible evolution (**sample path**) of a **discrete state**, **discrete time** SP $\{X(n)\}_{n \geq 0}$:



- To characterize the stochastic process we would need the distribution and **joint probabilities** of the $\{X(n)\}_{n \geq 0}$ RVs:

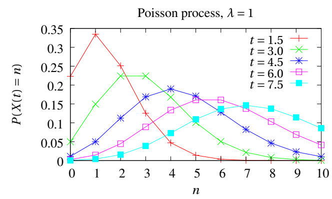
$$P\left(X(n) = i, X(n-1) = k, \cdots X(0) = j\right) \tag{2.1}$$

Example 2: Poisson Process

- It is a discrete state continuous time SP.
- It counts the number of events ocurred in a time interval.
- Often used to build models of other stochastic processes.
- Definition: The number of "events" in any interval of length t, $X(t)$, is **Poisson distributed** with mean λt, i.e.

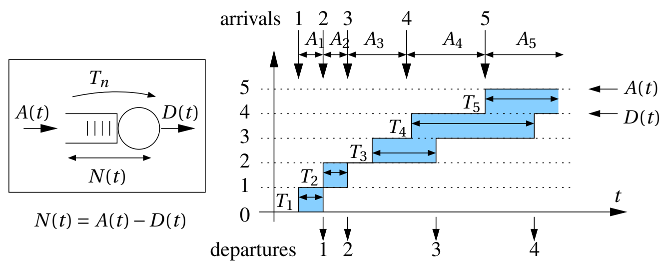
$$\boxed{\begin{aligned} &P(X(t+s) - X(s) = n) = P(X(t) - X(0) = n) = \\ &P(X(t) = n) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}} \tag{2.2}$$

where we assume $X(0) = 0$.



Example 3: Queue with Poisson Arrivals

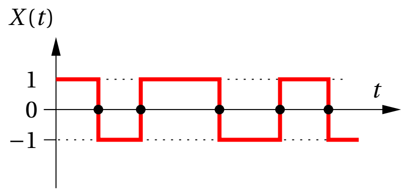
- The queue arrivals, $A(t)$, are modeled as a **Poisson process** with mean λt. Each event model an arrival.



Example 4: Telegraph signal

- The signal is modeled as a **Poisson process** with mean λt such that $X(0) = 1$ or $X(0) = -1$ with equal probability of 1/2 and:

$$X(t) = \begin{cases} 1 & \text{if the number of events in } (0, t] \text{ is even} \\ -1 & \text{if the number of events in } (0, t] \text{ is odd} \end{cases} \tag{2.3}$$



2.2 Analysis of Stochastic Processes

- **Signal Theory**: Normally interested in the **spectral analysis** of the signal. The basic tool is the **Fourier transform** of the **auto-correlation function** of the process (**energy spectral density**). We will not do this analysis.

$R(t) = \mathrm{E}[X(\tau)\,X(\tau - t)]$	autocorrelation
$F(f) = \mathscr{F}[R(t)] = \int_{-\infty}^{\infty} R(t)\,\mathrm{e}^{-j2\pi f t}\,\mathrm{d}t$	Fourier transform (**energy spectral density**)

(2.4)

- **Computer Networks**: Normally interested in probabilistic models using **Markov Chains** and **Queueing Theory**.

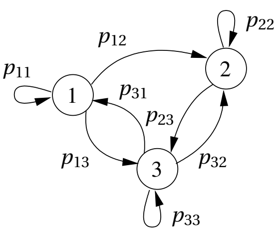
Part II

Discrete Time Markov Chains

Chapter 3
Definition of a DTMC

3.1 State Transition Diagram

- We are interested in a **process that evolve in stages**.
- For the model to be tractable, it is convenient to represent the SP by giving all **possible states** (there may be ∞), and the **possible transitions** between them:



For the model to be consistent: $\sum_{\forall j} p_{ij} = 1$

- Mathematically:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)} \tag{3.1}$$

3.2 Properties of a DTMC

- The event $X(n) = i$ (at step n the system is in state i) must satisfy (**memoryless property**):

$$\boxed{\begin{aligned} P(X(n) = j \mid X(n-1) = i, X(n-2) = k, \cdots) = \\ P(X(n) = j \mid X(n-1) = i)\end{aligned}} \tag{3.2}$$

- If $P\big(X(n) = j \mid X(n-1) = i\big) = P\big(X(1) = j \mid X(0) = i\big)$ for any n we have an **homogeneous** DTMC. We shall only consider homogeneous DTMC.
- We call **one-step transition probabilities** to:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)} \tag{3.3}$$

- The SP is called a Markov Process (MP) or Markov Chain (MC) depending on the state being continuous or discrete.

3.3 Transition Matrix

- Transition probabilities:

$$\boxed{p_{ij} = P\big(X(n) = j \mid X(n-1) = i\big)} \tag{3.4}$$

- In matrix form:

$$\boxed{\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}} \tag{3.5}$$

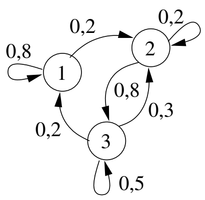
- For the model to be consistent, the probability to move from i to any state must be 1. Mathematically:

$$\begin{gathered} \sum_{\forall j} p_{ij} = \sum_{\forall j} P\big(X(n) = j \mid X(n-1) = i\big) = \\ \sum_{\forall j} \frac{P\big(X(n-1) = i \mid X(n) = j\big)\, P\big(X(n) = j\big)}{P\big(X(n-1) = i\big)} = \frac{P\big(X(n-1) = i\big)}{P\big(X(n-1) = i\big)} = 1 \end{gathered} \tag{3.6}$$

- $\mathbf{P}$ is a **stochastic matrix**, i.e. a matrix which rows sum 1.

Example

- Assume a terminal can be in **3 states**:
  - State 1: Idle.
  - State 2: Active without sending data.
  - State 3: Active and sending data at a rate v bps.



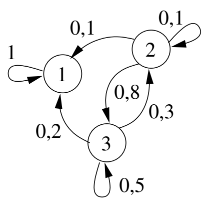
$$\mathbf{P} = \begin{array}{c} \textbf{to state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0{,}8 & 0{,}2 & 0 \\ 0 & 0{,}2 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 1 \;\; \textbf{from} \\ 2 \;\; \textbf{state} \\ 3 \end{array} \tag{3.7}$$

- The **average transmission rate** (throughput), v_a, is:

$$v_a = P\big(\text{the terminal is in state } 3\big) \times v \tag{3.8}$$

3.4 Absorbing Chains

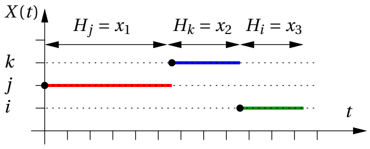
- It is possible to have chains with **absorbing states**.
- A state i is absorbing if $p_{ii} = 1$.
- Example: State 1 is absorbing.



$$\mathbf{P} = \begin{array}{c} \textbf{to state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}2 & 0{,}3 & 0{,}5 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 1 \;\; \textbf{from} \\ 2 \;\; \textbf{state} \\ 3 \end{array} \tag{3.9}$$

3.5 Sojourn or Holding Time

- **Sojourn** or **holding time** in state k: Is the RV H_k equal to the number of steps that the chain remains in state k before leaving to a different state:



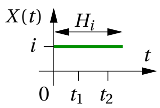
- The Markov property implies:

$$H_i(n) = P\left(H_i = n\right) = p_{ii}^{n-1}\,(1-p_{ii}),\, n \geq 1 \tag{3.10}$$

- Which is a **geometric** distribution with mean:

$$\mathrm{E}[H_i] = \sum_{n=1}^{\infty} n\,P\left(H_i = n\right) = \frac{1}{1-p_{ii}}. \tag{3.11}$$

The geometric distribution satisfies the Markov property



Proof.

- Markov property:
$P\left(X(n_2) = i \mid X(n_1) = i, X(n_0) = i\right) = P\left(X(n_2) = i \mid X(n_1) = i\right)$
- Thus, the Markov property **in terms of the sojourn time** can be written as:

$$P\left(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\right) = P\left(H_i > n_2 - n_1\right) \tag{3.12}$$

- Since

$$P\left(H_i > k\right) = 1 - P\left(H_i \leq k\right) = 1 - \sum_{n=1}^{k} p^{n-1}(1-p) = 1-(1-p)\frac{1-p^k}{1-p} = p^k \tag{3.13}$$

- We have:

$$P\left(H_i > n_2 - n_0 \mid H_i > n_1 - n_0\right) = \frac{P\left(H_i > n_2 - n_0, H_i > n_1 - n_0\right)}{P\left(H_i > n_1 - n_0\right)} = \frac{P\left(H_i > n_2 - n_0\right)}{P\left(H_i > n_1 - n_0\right)} = \frac{p^{n_2 - n_0}}{p^{n_1 - n_0}} = p^{n_2 - n_1} = P\left(H_i > n_2 - n_1\right) \tag{3.14}$$

□

3.6 n-step transition probabilities

- Transition probabilities: $p_{ij} = P\left(X(n) = j \mid X(n-1) = i\right)$
- In matrix form:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{3.15}$$

- We define the **n-step** transition probabilities:

$$p_{ij}(n) = P\left(X(n) = j \mid X(0) = i\right) \tag{3.16}$$

$$\mathbf{P}(n) = \begin{bmatrix} p_{11}(n) & p_{12}(n) & \cdots \\ p_{21}(n) & p_{22}(n) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{3.17}$$

- $\mathbf{P}$ and $\mathbf{P}(n)$ are **stochastic matrices**: Their rows sum 1.

3.7 State Probabilities

- Define the probability of being in state i at step n:

$$\pi_i(n) = P\left(X(n) = i\right) \tag{3.18}$$

- In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots) = (P\left(X(n) = 1\right), P\left(X(n) = 2\right), \cdots). \tag{3.19}$$

- Thus, the vector $\boldsymbol{\pi}(n)$ is the distribution of the random variable $X(n)$, and it is called the **state probability at step** n.
- Law of total prob. $P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n)$:

$$\pi_i(n) = \sum_k P\left(X(n-1) = k\right) P\left(X(n) = i \mid X(n-1) = k\right) = \sum_k \pi_k(n-1)\,p_{ki}$$
$$\pi_i(n) = \sum_k P\left(X(0) = k\right) P\left(X(n) = i \mid X(0) = k\right) = \sum_k \pi_k(0)\,p_{ki}(n) \tag{3.20}$$

- In matrix form:

$$\begin{aligned} \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(n-1)\,\mathbf{P} \\ \boldsymbol{\pi}(n) &= \boldsymbol{\pi}(0)\,\mathbf{P}(n) \end{aligned} \tag{3.21}$$

where $\boldsymbol{\pi}(0)$ is the **initial distribution**.

- Iterating

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P} = \boldsymbol{\pi}(n-2)\,\mathbf{P}\,\mathbf{P} = \boldsymbol{\pi}(n-3)\,\mathbf{P}\,\mathbf{P}\,\mathbf{P} = \cdots = \boldsymbol{\pi}(0)\,\mathbf{P}^n \tag{3.22}$$

- Thus:

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n \tag{3.23}$$

3.8 Chapman-Kolmogorov Equations

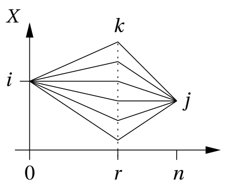
$$p_{ij}(n) = \sum_k p_{ik}(r)\;p_{kj}(n-r) \tag{3.24}$$

Proof.

$$\begin{aligned} p_{ij}(n) &= P\big(X(n) = j \mid X(0) = i\big) = \\ &\sum_k P\big(X(n) = j, X(r) = k \mid X(0) = i\big) \\ &= \sum_k \frac{P\big(X(n) = j, X(r) = k, X(0) = i\big)}{P\big(X(0) = i\big)} \times \\ &\frac{P\big(X(r) = k, X(0) = i\big)}{P\big(X(r) = k, X(0) = i\big)} = \\ &\sum_k P\big(X(n) = j \mid X(r) = k, X(0) = i\big) \times \\ &P\big(X(r) = k \mid X(0) = i\big) = \\ &\sum_k P\big(X(n) = j \mid X(r) = k\big)\,P\big(X(r) = k \mid X(0) = i\big) \\ &= \sum_k p_{ik}(r)\,p_{kj}(n-r) \end{aligned}$$

□

- Graphical interpretation:



- In matrix form:

$$\boxed{\mathbf{P}(n) = \mathbf{P}(r)\,\mathbf{P}(n-r)} \tag{3.25}$$

- Particularly:

$$\mathbf{P}(n) = \mathbf{P}(1)\,\mathbf{P}(n-1) = \mathbf{P}\,\mathbf{P}(n-1) = \mathbf{P}(n-1)\,\mathbf{P} \tag{3.26}$$

- Iterating:

$$\boxed{\mathbf{P}(n) = \mathbf{P}^n} \tag{3.27}$$

- Thus:

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n} \tag{3.28}$$

Chapter 4 Transient Solution

4.1 Close Form Solution

- If we are interested in the **transient evolution** we shall study $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n$.
- If we can **diagonalize P**, we can obtain the transient evolution in **close form**.
- **P** can be **diagonalized** if **P** can be decomposed as:

$$\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L} \tag{4.1}$$

where **L** is some invertible matrix and Λ is the diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \tag{4.2}$$

with λ_l, $l = 1, \cdots N$ the **eigenvalues** of **P**.

- Assume a **finite DTMC** with N states. Then $\mathbf{P} = \mathbf{P}^{N\times N}$.
- Assume that **P** can be **diagonalized**: $\mathbf{P} = \mathbf{L}^{-1}\,\Lambda\,\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of **P**.
- But $\mathbf{L}^{-1}\,\mathrm{diag}(\lambda_1^n, \cdots \lambda_N^n)\,\mathbf{L}$ are linear combinations of $\lambda_1^n, \cdots \lambda_N^n$. Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(n) = (\boldsymbol{\pi}(n))_i = \sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n} \tag{4.3}$$

where the **unknown coefficients** $a_i^{(l)}$ can be obtained solving the system of equations:

$$\sum_{l=1}^{N} a_i^{(l)}\,\lambda_l^n = (\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i,\ n = 0, \cdots N-1 \tag{4.4}$$

4.2 Eigenvalues

- The **eigenvalues** λ_l of a matrix **A** are scalars that satisfy: $\boldsymbol{l}\,\mathbf{A} = \lambda_l\,\boldsymbol{l}$ (or $\mathbf{A}\,\boldsymbol{r} = \lambda_l\,\boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* **and** *right* **eigenvectors**, respectively.

$$\begin{aligned} \boldsymbol{l}\,\mathbf{A} = \lambda_l\,\boldsymbol{l} &\Rightarrow \boldsymbol{l}\,(\mathbf{A} - \mathbf{I}\lambda_l) = 0 \Rightarrow \det(\lambda_l\,\mathbf{I} - \mathbf{A}) = 0 \\ \mathbf{A}\,\boldsymbol{r} = \lambda_l\,\boldsymbol{r} &\Rightarrow (\mathbf{A} - \mathbf{I}\lambda_l)\boldsymbol{r} = 0 \Rightarrow \det(\lambda_l\,\mathbf{I} - \mathbf{A}) = 0 \end{aligned} \tag{4.5}$$

- Thus, λ_l solve the **characteristic polynomial** $\det(\lambda\,\mathbf{I} - \mathbf{A}) = 0$.
- Note that, in general, *left* **and** *right* **eigenvectors** are different, but eigenvalues are the same (they solve the same **characteristic polynomial**).
- A matrix can be **diagonalized** if all eigenvalues are single (multiplicity = 1). If a matrix cannot be diagonalized it is called **defective**.

4.3 Determinant

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21} \tag{4.6}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{matrix} +a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{21}\,a_{32}\,a_{13} \\ -a_{31}\,a_{22}\,a_{13} - a_{12}\,a_{21}\,a_{33} - a_{23}\,a_{32}\,a_{11} \end{matrix} \tag{4.7}$$

- **Cofactor Formula**: expanding along a row i:

$$\det\mathbf{A} = \sum_{j=1}^{N} a_{ij}\,(-1)^{i+j}\det M_{ij}, \tag{4.8}$$

where the **minor matrices** M_{ij} are obtained removing the row i and column j from **A**. $(-1)^{i+j}\det M_{ij}$ is called the **cofactor** of a_{ij}.

$$\det\mathbf{A} = \prod \text{eigenvalues of } \mathbf{A} \tag{4.9}$$

$$\mathrm{trace}\ \mathbf{A} = \sum \text{eigenvalues of } \mathbf{A} \tag{4.10}$$

where trace $\mathbf{A} = \sum$ elements of the diagonal of **A**.

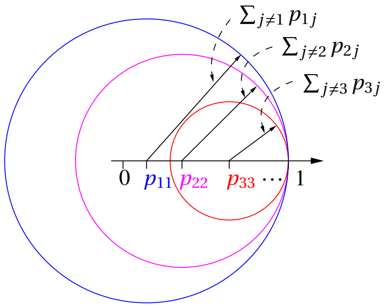
4.4 Eigenvalues of a Stochastic Matrix

- **P** has **an eigenvalue equal to 1** ($\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 1$).

  Proof: $\mathbf{P}\,\mathbf{e} = \mathbf{e}$, where $\mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}$ is a column vector of 1 (all rows of **P** add to 1).

- All eigenvalues of **P** are $|\lambda_l| \le 1$.

Proof. Using Gerschgorin's theorem *The eigenvalues of a matrix* $\mathbf{P}_{n\times n}$ *lie within the union of the n circular disks with center* p_{ii} *and radius* $\sum_{j\ne i}|p_{ij}|$ in $\mathbb{C}$. Since $\sum_j p_{ij} = 1$, the property is proved.



Gershgorin discs
of a stochastic matrix

■

Proof. of Gerschgorin's theorem
From $\mathbf{P}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ we have

$$\sum_j p_{ij}\,x_j = \lambda\,x_i \quad \forall i \in \{1,\ldots,n\}. \tag{4.11}$$

We choose i such that $|x_i| = \max_j |x_j|$. Thus, $\sum_{j\neq i} p_{ij}\,x_j = \lambda\,x_i - p_{ii}\,x_i$, and

$$|\lambda - p_{ii}| = \left|\sum_{j\neq i} p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i}\left|p_{ij}\,\frac{x_j}{x_i}\right| \le \sum_{j\neq i} |p_{ij}| \tag{4.12}$$

and the equation $|\mathbf{x}-\mathbf{c}| \le \mathbf{r}$, $x,c \in \mathbb{C}, r \in \mathbb{R}$ is a disk of center c and radius r in $\mathbb{C}$. ■

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

- We want the probability of being in state 2 in n steps starting from state 1: $\pi_2(n)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Solution

- It can be easily found that the **eigenvalues** of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 2/5$.

$$\pi_2(n) = \lambda_1^n a + b\,\lambda_2^n = a + b\,(2/5)^n \tag{4.13}$$

- Imposing the **boundary conditions** $\pi_i(n) = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i$:

$$\begin{aligned} \pi_2(0) &= a + b = (\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{P}^0)_2 = (\mathbf{P}^0)_{12} = 0 \\ \pi_2(1) &= a + b\,(2/5) = (\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{P}^1)_2 = (\mathbf{P})_{12} = 1/5 \end{aligned} \tag{4.14}$$

we have that $a = 1/3$, $b = -1/3$, thus:

$$\boxed{\begin{aligned} &\pi_2(n) = 1/3 - 1/3\,(2/5)^n, \quad n \ge 0 \\ &\pi_1(n) = 1 - \pi_2(n) = 2/3 + 1/3\,(2/5)^n, \quad n \ge 0 \end{aligned}} \tag{4.15}$$

4.5 Chain with a Defective Matrix

- What if $\mathbf{P}$ cannot be diagonalized? (**defective** matrix).
- Let λ_l, $l = 1,\cdots L$ be the eigenvalues of $\mathbf{P}^{N\times N}$, each with multiplicity k_l ($k_l \ge 1$, $\sum_l k_l = N$), and a possible eigenvalue $\lambda_1 = 0$ with multiplicity k_1. Then [1]:

$$\boxed{\begin{aligned} \pi_j(n) = \sum_{m=0}^{k_1-1} a_j^{(1,m)}\,I(n=m) + \sum_{l=2}^{L} \lambda_l^n \sum_{m=0}^{k_l-1} a_j^{(l,m)}\,n^m, \\ 1 \le j \le N, n \ge 0 \end{aligned}} \tag{4.16}$$

$I(n=m)$ is the indicator func.: $I(n) = 1$ if $n = m$, $I(n) = 0$ if $n \neq m$.

Example

- Assume a DTMC with

$$\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix} \tag{4.17}$$

- We want the probability of being in state 1 in n steps starting from state 1: $\pi_1(n)$ with $\pi_1(0) = 1$.
- It can be easily found that the **eigenvalues** of $\mathbf{P}$ are $\lambda_1 = 1$ and $\lambda_2 = 1/4$ with multiplicity 2. We guess:

$$\pi_1(n) = a + 1/4^n(b + c\,n) \tag{4.18}$$

- Imposing $\pi_1(0) = 1$, $\pi_1(1) = 3/4$, $\pi_1(2) = (3/4)^2$, we have:

$$\boxed{\pi_1(n) = \frac{4}{9} + \frac{1}{4^n}\left(\frac{5}{9} + \frac{2}{3}\,n\right)} \tag{4.19}$$

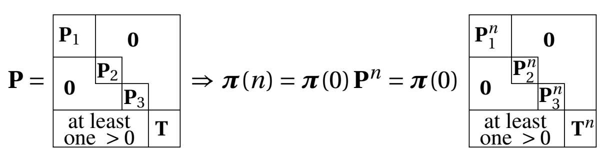
Chapter 5
Classification of States

Objective

- Identify the different **types of behavior** that the chain can have.
- Introduce the concepts of **first passage probability** and **mean recurrence time**.

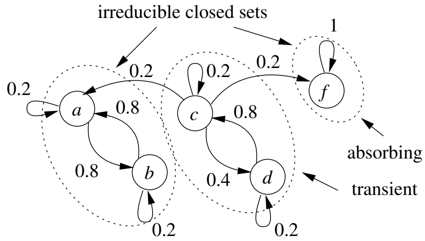
5.1 Irreducibility

- A state j is said to **communicate** with i, $i \leftrightarrow j$, if $p_{ij}(m_1) > 0$, $p_{ji}(m_2) > 0$ for some $m_1, m_2 \ge 0$.
- We define an **irreducible closed set, ICS** C_k as a set where all states communicate with each other, and have no transitions to other states out of the set:
$i \leftrightarrow j, \forall i,j \in C_k$ and $p_{ij} = 0, \forall i \in C_k, j \notin C_k$
(note that for $i \in C_k, j \notin C_k$ we have: $p_{ij}(2) = \sum_k p_{ik}\,p_{kj} = 0$, since $p_{ik} = 0$ if $k \notin C_k$, and $p_{kj} = 0$ if $k \in C_k$. Thus, $p_{ij}(n) = 0, \forall n$.)
- An **absorbing state** form an ICS of only one element. This state, i, must have $p_{ii} = 1$, $p_{ij} = 0\,\forall j \neq i$.
- **Transient states** do not belong to any ICS.
- A MC is **irreducible** if all the states form a unique ICS.
- Assume a MC has M **ICSs**: By properly numbering the states, we can write $\mathbf{P}$ as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- **Example**, if $M = 3$:



- Note that **the M sub-matrices are stochastic** (their rows sum 1).

Example



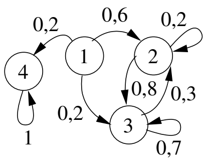
$$
\mathbf{P} = \begin{array}{c} \\ a \\ b \\ f \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & f & c & d \end{array} \\
\left[\begin{array}{cc|c|cc}
0{,}2 & 0{,}8 & 0 & 0 & 0 \\
0{,}8 & 0{,}2 & 0 & 0 & 0 \\
\hline
0 & 0 & 1{,}0 & 0 & 0 \\
\hline
0{,}2 & 0 & 0{,}2 & 0{,}2 & 0{,}4 \\
0 & 0 & 0 & 0{,}8 & 0{,}2
\end{array}\right]
\end{array}
$$

$$
\mathbf{P}^{\infty} = \begin{array}{c} \\ a \\ b \\ f \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & f & c & d \end{array} \\
\left[\begin{array}{cc|c|cc}
0{,}5 & 0{,}5 & 0 & 0 & 0 \\
0{,}5 & 0{,}5 & 0 & 0 & 0 \\
\hline
0 & 0 & 1{,}0 & 0 & 0 \\
\hline
0{,}25 & 0{,}25 & 0{,}5 & 0 & 0 \\
0{,}25 & 0{,}25 & 0{,}5 & 0 & 0
\end{array}\right]
\end{array}
$$

- What is the meaning of the probabilities in $\mathbf{P}^{\infty}$? (recall that $(\mathbf{P}^n)_{ij} = p_{ij}(n) = P\big(X(n) = j \mid X(0) = i\big)$).

5.2 Transient and Recurrent

- **Recurrent**: States that, being visited, they are visited again with probability 1. They are visited an infinite number of times when $n \to \infty$.
- **Transient**: States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $n \to \infty$.
- **Absorbing**: A single (recurrent) state where the chain remains with probability = 1.

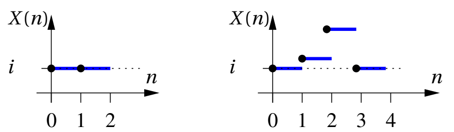


State 1 is transient
States 2 and 3 are recurrent
State 4 is absorbing

5.3 First Passage (Transition) Probabilities

- To derive a classification criteria, we shall study the distribution of the number of steps to **go for the first time from a state i another state j**. Definition:

$$
f_{ii}(n) = P\begin{pmatrix}\text{first transition into state } i \\ \text{in } n \text{ steps starting from } i\end{pmatrix} \tag{5.1}
$$



first transition in 1 step   first transition in 3 steps

- Do **not confuse** with the n-step transition probability $p_{ii}(n)$, where the state i can be visited in the intermediate states.

Relation between $f_{ii}(n)$ and $p_{ii}(n)$

- $f_{ii}(n)$ and $p_{ii}(n)$ satisfy:

$$
\begin{aligned}
f_{ii}(1) &= p_{ii}(1) \\
p_{ii}(n) &= \sum_{l=1}^{n} f_{ii}(l)\, p_{ii}(n-l),\ n >= 1
\end{aligned} \tag{5.2}
$$

- The probability that the MC **eventually enters state i starting from i** is given by:

$$
f_{ii} = \sum_{n=1}^{\infty} f_{ii}(n) \tag{5.3}
$$

- If $f_{ii} = 1$ we say i is a **recurrent state**.
- If $f_{ii} < 1$ we say i is a **transient state**.

5.4 Mean Recurrence Time

- When $f_{ii} = 1$, we define the **mean recurrence time** m_{ii} as:

$$
m_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}(n) \tag{5.4}
$$

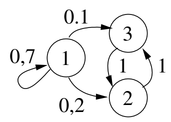
- m_{ii} is the **average number of steps to eventually reach i starting from i**. If $f_{ii} < 1$ (**transient state**) then we define $m_{ii} = \infty$.
- Classification of **recurrent states** ($f_{ii} = 1$):
  - If $m_{ii} = \infty$ the state is **null recurrent**: it takes an ∞ time to reach the state after leave it. Can only happen in chains with an infinite number of states.
  - If $m_{ii} < \infty$ the state is **positive recurrent**: the state is reached in a finite time after leave it.

5.5 Property of States

In **finite MC**:

1. States can be only of type positive recurrent or transient.
2. At least one state must be positive recurrent.
3. There are not null recurrent states.

- **Example**:



- State 1 is transient. States 2 and 3 are positive recurrent.

Generalization to Any State Pair

- Analogously to $f_{ii}(n)$, we define the probability of the **first passage to state j starting from any state i** in n steps: $f_{ij}(n)$.
- $f_{ij}(n)$ and $p_{ij}(n)$ satisfy:

$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l)\, p_{ij}(n-l),\ n \geq 1 \tag{5.5}$$

- **When** $f_{ij} = 1$, the average number of steps to eventually reach j starting from i, m_{ij} is given by:

$$\boxed{m_{ij} = \sum_{n=1}^{\infty} n\, f_{ij}(n)} \tag{5.6}$$

- If state j can not be reached starting from state i with probability one (**if** $f_{ij} < 1$), then we define $m_{ij} = \infty$.

5.6 Recursive Equation for the First Passage Probabilities

- Recall that the The probability that the MC **eventually enters state j starting from** i is given by: $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$
- f_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then we will reach j with probability f_{kj}. Thus:

$$\boxed{f_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}} \tag{5.7}$$

- If there are more than 1 **absorbing states**, we can compute the probability to reach them using this method (if there is only 1, say j, then $f_{ij} = 1, \forall i$).

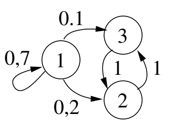
5.7 Recursive Equation for the Mean Recurrence Time

- Recall that the **mean recurrence time** $m_{ij} = \sum_{n \geq 1} n\, f_{ij}(n)$ is the average number of steps to eventually reach j starting from i, i.e. it is the mean first passage time from state i to j.
- **When** $f_{ij} = 1$, m_{ij} can be computed as follows: Assume we are in i. With probability p_{ij} we will go to j in one step. Otherwise, we will go to k, $k \neq j$, and then it will take m_{kj} steps to reach j. Thus:

$$\boxed{m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}(1 + m_{kj}) = 1 + \sum_{k \neq j} p_{ik}\, m_{kj}} \tag{5.8}$$

since $\sum_j p_{ij} = 1$.

Example: Recurrence Times Using the Definition



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f_{21}(n) = f_{31}(n) = 0$$
$$f_{11}(n) = 0{,}7\, I(n=1)$$
$$f_{22}(n) = f_{33}(n) = I(n=2)$$
$$f_{23}(n) = f_{32}(n) = I(n=1)$$

$$f_{12}(n) = \begin{cases} 0{,}2, & n = 1 \\ 0{,}7^{n-1}\,0{,}2 + 0{,}7^{n-2}\,0{,}1, & n > 1 \end{cases}$$

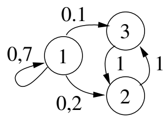
$$f_{13}(n) = \begin{cases} 0{,}1, & n = 1 \\ 0{,}7^{n-1}\,0{,}1 + 0{,}7^{n-2}\,0{,}2, & n > 1 \end{cases}$$

$$f_{11} = 0{,}7$$
$$f_{12} = f_{13} = 1 \qquad f_{22} = f_{23} = 1$$
$$f_{32} = f_{33} = 1 \qquad f_{21} = f_{31} = 0$$

$$\mathbf{M} = (m_{ij}) = \begin{bmatrix} \infty & 11/3 & 12/3 \\ \infty & 2 & 1 \\ \infty & 1 & 2 \end{bmatrix}$$

- State 1 is **transient**. States 2 and 3 are **recurrent**.

Example: First Passage Probability Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

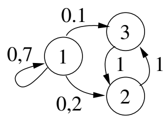
$$f_{12} = p_{11} f_{12} + p_{12} + p_{13} f_{32} \tag{5.9}$$

- Clearly $f_{32} = 1$, thus:

$$f_{12} = 0{,}7\, f_{12} + 0{,}2 + 0{,}1 \times 1 \Rightarrow \mathbf{f_{12} = 1} \tag{5.10}$$

as before.

Example: Mean Recurrence Time Using Recursion



$$\mathbf{P} = \begin{bmatrix} 0{,}7 & 0{,}2 & 0{,}1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- We have:

$$m_{12} = p_{12} + p_{11}(1 + m_{12}) + p_{13}(1 + m_{32}) = 1 + p_{11}\, m_{12} + p_{13}\, m_{32} \tag{5.11}$$

- Clearly $m_{32} = 1$, thus:

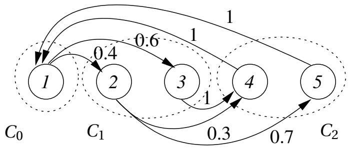
$$m_{12} = 1 + 0{,}7\, m_{12} + 0{,}1 \times 1 \Rightarrow \mathbf{m_{12} = 11/3}. \tag{5.12}$$

5.8 Periodic states

- A recurrent state j is **periodic** with period $d > 1$ if j can only be reached after leaving it with a multiple of d steps.
- If $d = 1$ the state is aperiodic.
- Any **periodic irreducible chain** can be partitioned in d **cyclic classes** $C_0, \cdots C_{d-1}$ such that at each step a transition occur from class C_i to $C_{(i+1) \bmod d}$.
- By properly numerating the states, the transition matrix can be written as (the sub-matrices $\mathbf{A}_i$ may not be square):

$$\mathbf{P} = \begin{array}{c} \\ C_0 \\ C_1 \\ \cdots \\ C_{d-1} \end{array} \begin{array}{c} \begin{array}{ccccc} C_0 & C_1 & C_2 & \cdots & C_{d-1} \end{array} \\ \begin{bmatrix} 0 & \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{A}_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{d-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array} \tag{5.13}$$

Example



$$\mathbf{P} = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \end{bmatrix},$$

$$\mathbf{P}^4 = \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 0 & 0 & 0 & 0.72 & 0.28 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \cdots$$

- In periodic chains $\mathbf{P}^n$ **does not converge**.

Chapter 6
Steady State

6.1 Limiting Distribution

- Probability of being in state i at step n:

$$\pi_i(n) = P\big(X(n) = i\big). \tag{6.1}$$

In vector form (row vector)

$$\boldsymbol{\pi}(n) = (\pi_1(n), \pi_2(n), \cdots). \tag{6.2}$$

- The evolution of the chain depends on the initial distribution $\boldsymbol{\pi}(0)$.
- If we are interested in the **transient** evolution we shall study

$$\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n. \tag{6.3}$$

- If we are interested the **steady state** we shall be interested in the **limiting distribution** (if the limit exists):

$$\boldsymbol{\pi}(\infty) = (\pi_1(\infty), \pi_2(\infty), \cdots) \tag{6.4}$$

Assume an **irreducible** chain with **positive recurrent** states.

- With infinite steps, we look for a probability converging to a value that depends only on the final state:

$$\pi_j(\infty) = \sum_i \pi_i(0) \lim_{n\to\infty} p_{ij}(n), \forall j \text{ and for any } \boldsymbol{\pi}(0), \tag{6.5}$$

which implies:

$$\pi_j(\infty) = \lim_{n\to\infty} p_{ij}(n) \sum_i \pi_i(0) = p_{ij}(\infty), \forall j \Rightarrow$$

$$\mathbf{P}(\infty) = \lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \boldsymbol{\pi}(\infty) \\ \cdots \\ \boldsymbol{\pi}(\infty) \end{bmatrix} \tag{6.6}$$

- If this limit exists, we call $\mathbf{P}(\infty)$ the **limiting matrix**, and $\boldsymbol{\pi}(\infty)$ the **limiting distribution**.

Example

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0.764 & 0.168 & 0.068 \\ 0.760 & 0.170 & 0.070 \\ 0.752 & 0.174 & 0.074 \end{bmatrix}$$

$$\mathbf{P}^4 = \begin{bmatrix} 0{,}7628 & 0{,}1686 & 0{,}0686 \\ 0{,}7620 & 0{,}1690 & 0{,}0690 \\ 0{,}7604 & 0{,}1698 & 0{,}0698 \end{bmatrix}$$

$$\mathbf{P}^8 = \begin{bmatrix} 0.762500 & 0.168750 & 0.068750 \\ 0.762499 & 0.168750 & 0.068750 \\ 0.762497 & 0.168752 & 0.068752 \end{bmatrix}$$

$$\cdots$$

$$\Rightarrow \boldsymbol{\pi}(\infty) = (0.76250, 0.16875, 0.06875)$$

6.2 Stationary distribution

- We have:

$$\pi_i(n) = P\big(X(n) = i\big) = \sum_k P\big(X(n-1) = k\big)\, P\big(X(n) = i \mid X(n-1) = k\big) = \sum_k \pi_k(n-1)\, p_{ki} \tag{6.7}$$

- In matrix form: $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(n-1)\,\mathbf{P}$
- If $\pi_i(n) = \pi_i(n-1) = \pi_i\ \forall i$, we call π_i the **stationary probability of state** i, and $\boldsymbol{\pi} = (\pi_1, \pi_2, \cdots)$, the **stationary distribution** of the chain.
- In matrix form (**Global balance equations**):

$$\boxed{\begin{aligned} &\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P} \\ &\boldsymbol{\pi}\,\mathbf{e} = 1, \mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}} \end{aligned}} \tag{6.8}$$

- Thus, the stationary distribution is the left-hand eigenvector corresponding to the unit eigenvalue of $\mathbf{P}$.
- $\boldsymbol{\pi}(n) = \boldsymbol{\pi} \Rightarrow \boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n)\,\mathbf{P} = \boldsymbol{\pi}\,\mathbf{P} = \boldsymbol{\pi} \Rightarrow \boxed{\boldsymbol{\pi}(k) = \boldsymbol{\pi}, k \geq n}$
- Do not confuse the **limiting distribution** $\boldsymbol{\pi}(\infty)$ and the **stationary distribution** $\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}$.
- $\boldsymbol{\pi}(\infty)$ and $\boldsymbol{\pi}$ **may not be the same**, e.g. in **periodic chains** $\boldsymbol{\pi}(\infty)$ does not exists ($\mathbf{P}$ does not converge), but we can compute the stationary distribution.

- **Example:** the periodic chain

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{6.9}$$

has the stationary distribution

$$\boldsymbol{\pi} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}. \tag{6.10}$$

6.3 Numerical Solution

Replace one equation method

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$
$$\boldsymbol{\pi}\mathbf{e} = 1, \mathbf{e} = \begin{bmatrix} 1 & 1 & \cdots \end{bmatrix}^{\mathrm{T}}$$

We solve the equation $\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}) = 0$ replacing the last equation by $\boldsymbol{\pi}\mathbf{e} = 1$:

$$\boldsymbol{\pi} \begin{bmatrix} p_{11} - 1 & p_{12} & \cdots p_{1n-1} & 1 \\ p_{21} & p_{22} - 1 & \cdots p_{2n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots p_{nn-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{6.11}$$

Examples

- **Replace one equation method**: $\mathbf{P} = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.7 & 0.2 & 0.1 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$
- With **octave** (matlab clone):

```
octave:1> P
    =[0.8,0.15,0.05;0.7,0.2,0.1;0.5,0.3,0.2];
octave:2> s=size(P,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [eye(s,s-1)-P(1:s,1:s-1), ones(s,1)]
ans =
0.762500   0.168750   0.068750
```

- With **R**

```
> P <- matrix(nc=3, byr=T,
+ c(0.8,0.15,0.05,0.7,0.2,0.1,0.5,0.3,0.2))
> s <- nrow(P)
> solve(t(cbind(P[,1:(s-1)]-diag(nr=s,nc=s-1),
+ rep(1,s))), c(rep(0,s-1),1))
[1] 0.76250 0.16875 0.06875
```

NOTE: $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \Rightarrow \boldsymbol{\pi}^{\mathrm{T}} = \mathbf{P}^{\mathrm{T}}\boldsymbol{\pi}^{\mathrm{T}}$. The transpose operator in R is `t()`.

6.4 Global balance equations

- Why are they called Global balance equations?

$$\left.\begin{array}{ll} \boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P} \Rightarrow & \pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij} \\ \sum_{i=0}^{\infty} p_{ji} = 1 \Rightarrow & \pi_j \sum_{i=0}^{\infty} p_{ji} = \pi_j \end{array}\right\} \Rightarrow \sum_{i=0}^{\infty} \pi_i p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji} \tag{6.12}$$

$$\begin{array}{ll} \sum_{i=0}^{\infty} \pi_i p_{ij} & \Rightarrow \text{Frequency of } \textbf{transitions entering state } j \\ \pi_j \sum_{i=0}^{\infty} p_{ji} & \Rightarrow \text{Frequency of } \textbf{transitions leaving state } j \end{array} \tag{6.13}$$

- In **stationary regime**, the frequency of transitions leaving state j is equal to the frequency of transitions entering state j.

Flux Balancing

- Define the **flux** F_{uv} from state u to v:

$$F_{uv} = \pi_u p_{uv} \tag{6.14}$$

- and the flux from set of states U to V:

$$F(U,V) = \sum_{u \in U} \sum_{v \in V} F_{uv} \tag{6.15}$$

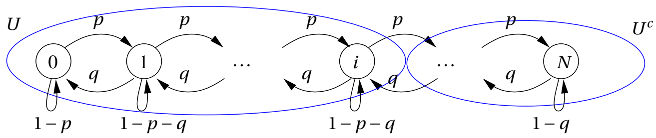
- From the Global balance equations we have:

$$\sum_{i=0}^{\infty} \pi_i p_{ij} = \pi_j \sum_{i=0}^{\infty} p_{ji} \Rightarrow \sum_{i \in U} F_{ij} + \sum_{i \notin U} F_{ij} = \sum_{i \in U} F_{ji} + \sum_{i \notin U} F_{ji} \tag{6.16}$$

- Adding for $j \in U$:

$$\sum_{j \in U} \sum_{i \in U} F_{ij} + \sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \in U} F_{ji} + \sum_{j \in U} \sum_{i \notin U} F_{ji} \Rightarrow$$
$$\sum_{j \in U} \sum_{i \notin U} F_{ij} = \sum_{j \in U} \sum_{i \notin U} F_{ji} \Rightarrow \boxed{F(U,U^c) = F(U^c,U)}$$

Solution Using Flux Balancing



- Flux balancing $\Rightarrow$ $p\pi_i = q\pi_{i+1}$
- Iterating: $\pi_1 = \rho\pi_0$, $\pi_2 = \rho\pi_1 = \rho\rho\pi_0, \cdots, \Rightarrow$

$$\boxed{\pi_i = \rho^i \pi_0, i = 1, \cdots, N} \quad \text{where: } \rho = \frac{p}{q},$$

- Normalizing: $\sum_{i=0}^{N} \pi_i = 1$ $\boxed{\begin{array}{ll} \pi_0 = \dfrac{1-\rho}{1-\rho^{N+1}}, & p \neq q \\ \pi_0 = \dfrac{1}{N+1}, & p = q \end{array}}$

6.5 Ergodic Chains

Ergodic state positive recurrent and aperiodic state.
Ergodic chain if all states are ergodic.
Theorem: All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent, and aperiodic/periodic with the same period [2, chapter XV].
Consequences:

- **Finite aperiodic and irreducible** chains are **ergodic** (since all states are positive recurrent).
- **Infinite aperiodic and irreducible** chains can be:
  - **Ergodic**: all the states are positive recurrent (stable chains).
  - **Non ergodic**: all states are null recurrent or transient (unstable chains).

6.5.1 Theorems for ergodic chains

- $\boldsymbol{\pi} = \boldsymbol{\pi}(\infty)$

  Proof. For an **aperiodic irreducible** chain with **positive recurrent** states:

$$\begin{cases} \boldsymbol{\pi}(\infty) & = \boldsymbol{\pi}(0)\,\mathbf{P}(\infty) \\ \mathbf{P}(\infty) & = \lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \boldsymbol{\pi}(\infty) \\ \cdots \\ \boldsymbol{\pi}(\infty) \end{bmatrix} \end{cases} \Rightarrow$$

$$\boldsymbol{\pi}(\infty)\,\mathbf{P} = (\boldsymbol{\pi}(0) \lim_{n\to\infty} \mathbf{P}^n)\,\mathbf{P} = \boldsymbol{\pi}(0)\,\mathbf{P}(\infty) = \boldsymbol{\pi}(\infty)$$

$$\Rightarrow \begin{cases} \boldsymbol{\pi}(\infty)\,\mathbf{P} = \boldsymbol{\pi}(\infty) \\ \boldsymbol{\pi}(\infty)\,\mathbf{e} = 1 \end{cases} \quad \boldsymbol{\pi}(\infty) \text{ satisfies the GBE} \Rightarrow \boldsymbol{\pi} = \boldsymbol{\pi}(\infty) \tag{6.17}$$

  □

- In stationary regime (when $\boldsymbol{\pi}(n)\,\mathbf{P} = \boldsymbol{\pi}(n)$), the **mean number of steps the system remains in state** j during k steps is given by

$$\boxed{k\pi_j.} \tag{6.18}$$

  Proof. Assume the chain in stationary regime at time $t = 0$ ($\pi(0)\,\mathbf{P} = \boldsymbol{\pi}(0)$), and let $j(k)$ be the number of visits to j in k steps: $j(k) = \sum_{i=0}^{k-1} I(X(i) = j)$ ($I(A)$ is the indicator function: $I(A) = 1$ if A occurs, $I(A) = 0$ otherwise):

$$\mathrm{E}[j(k)] = \sum_{i=0}^{k-1} \mathrm{E}[I(X(i) = j)] = \sum_{i=0}^{k-1} P(X(i) = j) = k\pi_j \tag{6.19}$$

  □

- In stationary regime the **mean recurrence time** (mean number of steps between two consecutive visits to state j) is given by

$$\boxed{m_{jj} = 1/\pi_j} \tag{6.20}$$

  Proof. Let $j(k)$ be the number of visits to j in k steps:

$$\pi_j = \lim_{k\to\infty} \frac{j(k)}{k} = \lim_{k\to\infty} \frac{1}{k/j(k)} = 1/m_{jj} \tag{6.21}$$

  □

Chapter 7
Reversed Chain

Definition

- Let $X(n)$ be an **ergodic** MC. The chain $X^r(n) = X(-n)$ is referred to as the **time reversal chain** of $X(n)$.
- **Example**, consider a possible sample path of $X(n)$:

$$\cdots (i_0, n_0), (i_1, n_1), (i_2, n_2), \cdots \tag{7.1}$$

  The same path in the time reversal chain $X^r(n)$ would be:

$$\cdots (i_2, -n_2), (i_1, -n_1), (i_0, -n_0), \cdots \tag{7.2}$$

Properties

- Let p_{ij}, p^r_{ij} be the transition probabilities of $X(n)$ respectively $X^r(n)$, and π_i, π^r_i the stationary distributions of $X(n)$ respectively $X^r(n)$, then:

$$\boxed{\pi_i = \pi^r_i} \tag{7.3}$$

- **Proof**: the mean time in each state is the same for both chains.
- However, **in general** $p_{ij} \neq p^r_{ij}$. For example, $X(n)$ may be able to jump from state i to j, but not from j to $i \Rightarrow X^r(n)$ can jump from j to i, but not from i to j.
- But it must be $p_{ii} = p^r_{ii}$, since self-state transitions are the same in the direct and reversed chains.

7.1 Computation of p^r_{ij}

The transition probabilities in the time reversal chain (p^r_{ji}) satisfy:

$$\pi_i\, p_{ij} = \pi_j\, p^r_{ji} \tag{7.4}$$

Proof. Assume the chain in **steady state**. We have:

$$\begin{aligned} & P\{X(n+1) = j, X(n) = i\} = \\ & \quad P\{X^r(-n) = i, X^r(-n-1) = j\} = \\ & \quad P\{X^r(n+1) = i, X^r(n) = j\} \Rightarrow \\ & \quad P\{X(n) = i\}\, P\{X(n+1) = j \mid X(n) = i\} = \\ & \quad P\{X(n) = j\}\, P\{X^r(n+1) = i \mid X^r(n) = j\} \Rightarrow \\ & \quad \pi_i\, p_{ij} = \pi_j\, p^r_{ji}. \end{aligned}$$

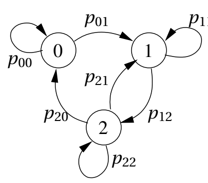
□

We can **compute** p^r_{ji} using the **reversed balance equations**:

$$\pi_i\, p_{ij} = \pi_j\, p^r_{ji} \Rightarrow \sum_{i\in U} \sum_{j\in V} \pi_i\, p_{ij} = \sum_{i\in U} \sum_{j\in V} \pi_j\, p^r_{ji} \Rightarrow$$

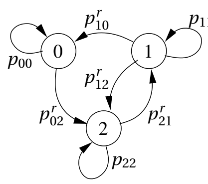
$$\boxed{F(U,V) = F^r(V,U)} \tag{7.5}$$

Example



$$\Rightarrow \begin{cases} \pi_0 = \dfrac{p_{12}\,p_{20}}{p_{12}\,p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\,p_{12}} \\ \pi_1 = \dfrac{p_{01}\,(p_{20} + p_{21})}{p_{12}\,p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\,p_{12}} \\ \pi_2 = \dfrac{p_{01}\,p_{12}}{p_{12}\,p_{20} + p_{01}\,(p_{20} + p_{21}) + p_{01}\,p_{12}} \end{cases}$$

Time reversal chain:



$$\Rightarrow \begin{cases} \pi_0\, p_{01} = \pi_1\, p^r_{10} \\ \pi_1\, p_{12} = \pi_2\, p^r_{21} \\ \pi_2\, p_{21} = \pi_1\, p^r_{12} \\ \pi_2\, p_{20} = \pi_0\, p^r_{02} \end{cases} \Rightarrow \begin{cases} p^r_{10} = \dfrac{p_{12}\, p_{20}}{p_{20}+p_{21}} \\ p^r_{21} = p_{20}+p_{21} \\ p^r_{12} = \dfrac{p_{12}\, p_{21}}{p_{20}+p_{21}} \\ p^r_{02} = p_{01} \end{cases}$$

Chapter 8
Reversible Chains

Definition

- A chain is reversible if:

$$p_{ij} = p^r_{ij} \tag{8.1}$$

- This equality implies the **reversibility balance equations**:

$$\pi_i\, p_{ij} = \pi^r_i\, p^r_{ij} \Rightarrow F(U,V) = F^r(U,V) \tag{8.2}$$

- Using both reversed ($F^r(U,V) = F(V,U)$) and reversibility balance equations, the following relation holds for a reversible chain (**detailed balance equations**):

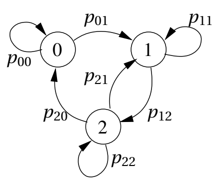
$$F(U,V) = F(V,U) \tag{8.3}$$

- NOTE: Compare with the **global balance equations**: $F(U,U^C) = F(U^C,U)$.

8.1 Kolmogorov Criteria

Definition of path

- Define a **path** as a possible sequence of transitions of the chain. For example, in the figure it could be $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$.



- We denote the **sequence of states** of one path l as:

$$(\mathbf{l},1) \rightsquigarrow (\mathbf{l},2) \rightsquigarrow \cdots \rightsquigarrow (\mathbf{l},m) \tag{8.4}$$

- For instance, if l is $0 \rightsquigarrow 0 \rightsquigarrow 1 \rightsquigarrow 2$, then $(\mathbf{l},1) = 0$, $(\mathbf{l},2) = 0$, $(\mathbf{l},3) = 1$, $(\mathbf{l},4) = 2$.
- We say that a path is **closed** if it starts and ends in the same state. For instance, a path stating and ending in state $(\mathbf{l},\mathbf{1})$:

$$(\mathbf{l},\mathbf{1}) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m) \rightsquigarrow (\mathbf{l},\mathbf{1}) \tag{8.5}$$

Kolmogorov Criteria

- Take a **closed path** l with $m \geq 0$ transitions, i.e.:

$$(l,1) \rightsquigarrow (l,2) \rightsquigarrow \cdots \rightsquigarrow (l,m) \rightsquigarrow (l,1),\ m \geq 0 \tag{8.6}$$

- The chain is **reversible iff for all** l:

$$\begin{aligned} &p_{(l,1)(l,2)}\, p_{(l,2)(l,3)} \cdots p_{(l,m)(l,1)} = \\ &p_{(l,1)(l,m)}\, p_{(l,m)(l,m-1)} \cdots p_{(l,2)(l,1)} \end{aligned} \tag{8.7}$$

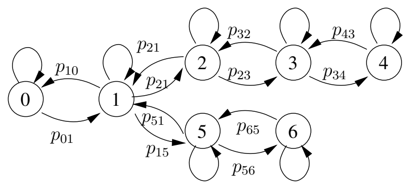
- **Proof**:
  - If the chain is reversible $\pi_i\, p_{ij} = \pi_j\, p_{ji}$ (detailed balance equations): $\Rightarrow \pi_{(l,k)}\, p_{(l,k)(l,k+1)} = \pi_{(l,k+1)}\, p_{(l,k+1)(l,k)}$
  - Multiplying for $k = 1,2\cdots m$ and simplifying we obtain the previous relation.

Corollary

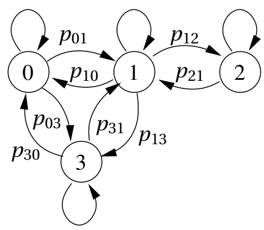
- A reversible chain must satisfy:

$$\begin{aligned} p_{ij} > 0 \Rightarrow p_{ji} > 0 \\ p_{ij} = 0 \Rightarrow p_{ji} = 0 \end{aligned} \tag{8.8}$$

- An **ergodic tree chain is always reversible**. We define a tree chain as chain having no cycles, i.e. the only possible transitions are bidirectional arcs between states, and self transitions.



Example



- The chain is **reversible iif**:

$$p_{01}\, p_{13}\, p_{30} = p_{10}\, p_{03}\, p_{31}$$

8.2 Product Form Solution

- Let $X(n)$ be a reversible MC with space state $S \Rightarrow$ the **stationary probabilities** of $X(n)$ can be computed as follows:
- Choose a state $\mathbf{s} \in S$,
- For every other state $\mathbf{i} \in S$, $i \neq s$ look for a possible path l_i from state s to state i:

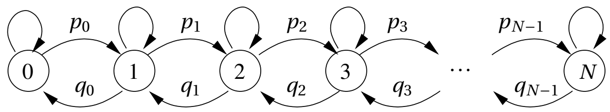
$$\mathbf{s} = (l_i,1) \rightsquigarrow (l_i,2) \rightsquigarrow \cdots \rightsquigarrow (l_i,m_{l_i}) = \mathbf{i},\ m_{l_i} \geq 1 \tag{8.9}$$

- The stationary probabilities are given by:

$$\pi_i = \frac{\psi_i}{\sum_{j\in S} \psi_j},\ i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \displaystyle\prod_{k=1}^{m_{l_i}-1} \frac{p_{(l_i,k)(l_i,k+1)}}{p_{(l_i,k+1)(l_i,k)}}, & i \neq s \end{cases} \tag{8.10}$$

- **Proof** Use the detailed balance equations $\pi_i\, p_{ij} = \pi_j\, p_{ji}$.

8.3 Birth and Death Chains



- **Birth and death chains are reversible**.
- Applying the product form solution for reversible chains we obtain the general solution of birth death chains. Choosing $s=0$:

$$\pi_i = \frac{\psi_i}{\sum_{j=0}^{N}\psi_j},\ i \geq 0 \quad \text{where } \psi_i = \begin{cases} 1, & i=0 \\ \prod_{k=0}^{i-1}\frac{p_k}{q_k}, & i=1,\cdots,N \end{cases} \tag{8.11}$$

Truncated Reversible Chain

- Consider a reversible MC X with a stationary distribution π_i.
- Suppose that **we truncate the chain** X and we obtain another irreducible chain X'.
- Then, X' is also reversible with stationary distribution:

$$\pi'_i = \frac{\pi_i}{G}, \quad \sum_k \pi'_k = 1 \tag{8.12}$$

Chapter 9
Research Example: Aloha

Access Protocol (see the paper of Kleinrock and Lam [5]).

- **Pure Aloha:**
  - Broadcast radio system.
  - **Single hop** system (all stations are in coverage).
  - Whenever a station has a frame ready, it is transmitted.
  - If two or more frames Tx overlap in time there is a **collision**, otherwise the frame is received correctly.
  - Colliding frames are reTx after a **random time** (**backoff**).
- **Slotted Aloha:**
  - Time is slotted.
  - Tx can only occur at the beginning of a slot.
  - Collisions occur when 2 or more stations Tx in the same slot.

9.1 Analysis with finite population

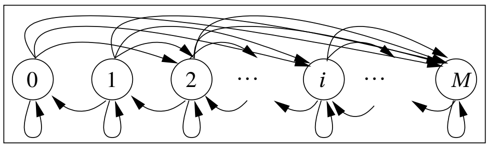
Assumptions

- **Slotted** Aloha.
- **Acks** are sent immediately after the reception of a frame, and are never lost.
- M nodes with a **buffer** of 1 frame.
- The **nodes** can be in 2 states:
  - Thinking: when the buffer is empty
  - Backlogged: when there is a frame in the buffer.
- A thinking node generate one frame in each slot with probability σ. When a frame collides, the frame is stored and the node becomes backlogged.
- A backlogged node ReTx the frame in each slot with probability ν.

Markov Chain

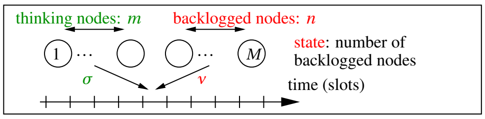
- The system **state**, $X(n)$, is the **number of backlogged nodes**:

$$p_{ij} = P\left(X(n) = j \text{ baklogged} \mid X(n-1) = i \text{ baklogged}\right) \tag{9.1}$$



Transition probabilities

- 0 for $j < i-1$.
- for $j = i-1$: no thinking Tx and only 1 backlogged Tx.
- for $j = i$:
  1. no thinking Tx and none or more than 1 backlogged Tx,
  2. only 1 thinking Tx and no backlogged Tx.
- for $j = i+1$: 1 thinking and 1 or more backlogged Tx.
- for $j > i+1$: $j-i$ thinking Tx, regardless of backlogged Tx.



In order to compute the previous events, define the probabilities:

- **Arrivals:** $Q_a(m,n)$, Probability of m thinking nodes Tx in a slot given that n nodes are backlogged:

$$Q_a(m,n) = P\left(\begin{smallmatrix} m \text{ think.} \\ \text{nodes Tx} \end{smallmatrix} \mid \begin{smallmatrix} n\text{nodes are} \\ \text{backlogged} \end{smallmatrix}\right) = \binom{M-n}{m}\sigma^m(1-\sigma)^{M-n-m} \tag{9.2}$$

- **Retransmissions:** $Q_r(m,n)$, Probability of m backlogged nodes Tx in a slot given that n nodes are backlogged:

$$Q_r(m,n) = P\left(\begin{smallmatrix} \mathbf{m} \text{ backl.} \\ \text{nodes Tx} \end{smallmatrix} \mid \begin{smallmatrix} \mathbf{n} \text{ nodes are} \\ \text{backlogged} \end{smallmatrix}\right) = \binom{n}{m}\nu^m(1-\nu)^{n-m} \tag{9.3}$$

- and we have:

$$p_{ij} = \begin{cases} 0, & j < i-1 \\ Q_a(0,i)\,Q_r(1,i), & j = i-1 \\ Q_a(0,i)\,(1-Q_r(1,i)) + Q_a(1,i)\,Q_r(0,i), & j = i \\ Q_a(1,i)\,(1-Q_r(0,i)), & j = i+1 \\ Q_a(j-i,i), & j > i+1 \end{cases} \tag{9.4}$$

9.1.1 Stationary distribution

- Solving the global balance equations:

$$\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{\pi}\,\mathbf{P} \\ \boldsymbol{\pi}\,\mathbf{e} &= 1 \end{aligned} \tag{9.5}$$

- We obtain the probability of having i backlogged nodes:

$$\pi_i = P\left(i \text{ backlogged nodes}\right) \tag{9.6}$$

NOTE: there is **no closed form solution** of the chain. The matrix $\mathbf{P}$ must be constructed using the expression of p_{ij}, and solved numerically.

9.2 Throughput

- Define the probabilities:

$$P_{succ}(i) = P\left(\text{successful Tx} \mid i \text{ backlogged}\right) \tag{9.7}$$

- The **normalized throughput**, i.e. proportion of steps with a successful transmission) is:

$$S = \sum_{i=0}^{M} P\left(\text{successful Tx} \mid i \text{ backlogged}\right) \times$$
$$P\left(i \text{ backlogged}\right) = \sum_{i=0}^{M} P_{succ}(i)\,\pi_i \tag{9.8}$$

- For a slot to be successful: (i) 1 thinking Tx and no backlogged Tx, or (ii) no thinking Tx and 1 backlogged Tx:

$$P_{succ}(i) = Q_a(1,i)\,Q_r(0,i) + Q_a(0,i)\,Q_r(1,i) \tag{9.9}$$

Notes on the throughput

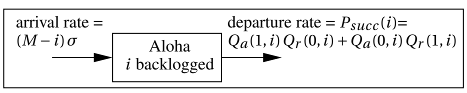
$$S = \sum_{i=0}^{M} P_{succ}(i)\,\pi_i \tag{9.10}$$

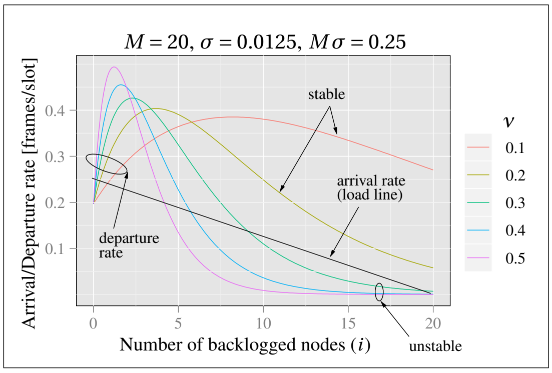
- For the **special case** $\sigma = \nu$ (thinking Tx with the same probability as backlogged): $P_{succ}(i) = M\sigma(1-\sigma)^{M-1}$, which does not depend on i, thus: $S = M\sigma(1-\sigma)^{M-1}$.
- The **offered load** (i.e. proportion of arrivals per slot) G is now: $G = M\sigma$, thus:

$$S = G\left(1 - \frac{G}{M}\right)^{M-1} \Rightarrow \lim_{M\to\infty} \mathbf{S} = \mathbf{G}\mathrm{e}^{-\mathbf{G}} \tag{9.11}$$

- We conclude that the **infinite population model** is the limit of the finite population if backlogged Tx with the same probability as thinking, and $M \to \infty$.

9.2.1 Dynamics





Note on the arrival rate (expected value of a binomial distribution):

$$\sum_{k=0}^{M-i} k \binom{M-i}{k} \sigma^k (1-\sigma)^{M-i-k}$$
$$= (\mathbf{M}-\mathbf{i})\,\sigma$$

Solving the chain: $S = \sum_{i=0}^{M} P_{succ}(i)\,\pi_i$

ν	S
0.1	2.38e-01
0.2	2.42e-01
0.3	**1.30e-02**
0.4	**4.98e-04**
0.5	**1.90e-05**

9.2.2 Stabilizing Aloha

- The **retransmission probabilities** must adapt in accordance with the state of the system.
- Example: **binary exponential backoff** (ethernet). The retransmission rate at retransmission i is adapted as $\nu = 2^{-i}$. Thus, the higher are the number of retransmission trials i, the lower (exponentially) is the retransmission rate.

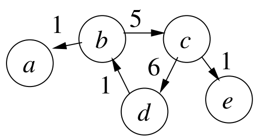
Chapter 10
Finite Absorbing Chains

Canonical Form

- Let $\mathbf{P}^{rxr}$ be the transition probability matrix of a chain with r states: s **transient** states and $r-s$ **absorbing** states. We can write $\mathbf{P}^{rxr}$ in the **canonical** form:

$$\boxed{\mathbf{P}^{rxr} = \begin{bmatrix} \mathbf{Q}^{s\times s} & \mathbf{R}^{s\times r-s} \\ \mathbf{0}^{r-s\times s} & \mathbf{I}^{r-s\times r-s} \end{bmatrix} \tag{10.1}}$$

Example



$$\mathbf{P} = \begin{array}{c} \\ b \\ c \\ d \\ a \\ e \end{array} \begin{array}{ccc|cc} b & c & d & a & e \end{array} \atop \left[\begin{array}{ccc|cc} 0 & 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

10.1 Results

- Define:

$$\mathbf{n_{ij}} = \begin{Bmatrix} \text{number of steps in state } j \text{ before} \\ \text{absorption, starting from state } i \end{Bmatrix},$$
$$\mathbf{t_i} = \begin{Bmatrix} \text{number of steps in transient states before} \\ \text{absorption, starting from state } i \end{Bmatrix}, \tag{10.2}$$
$$\mathbf{b_{ij}} = P\big(\text{probability to be absorbed } j \text{ starting } i\big)$$

- Then:

$$\begin{aligned}\{\mathrm{E}[n_{ij}]\} &= \mathbf{N} = (\mathbf{I}-\mathbf{Q})^{-1} \\ \{\mathrm{Var}[n_{ij}]\} &= \mathbf{N}\,(2\mathbf{N}_{\text{diag}}-\mathbf{I}) - \mathbf{N}\text{sqr} \\ \{\mathrm{E}[t_i]\} &= \boldsymbol{\tau} = \mathbf{N}\mathbf{e} \\ \{\mathrm{Var}[t_i]\} &= (2\mathbf{N}-\mathbf{I})\,\boldsymbol{\tau} - \boldsymbol{\tau}_{\text{sqr}} \\ \{b_{ij}\} &= \mathbf{B} = \mathbf{N}\mathbf{R}.\end{aligned} \tag{10.3}$$

where $\{a_{ij}\}$ is a matrix with a_{ij} as element ij and $\mathbf{e}$ is a column vector of 1s. $\mathbf{N}$ is called the **fundamental matrix**.

Proofs

- $\{\mathrm{E}[n_{ij}]\} = \mathbf{N} = (\mathbf{I}-\mathbf{Q})^{-1}$

Proof.
$$\mathrm{E}[n_{ij}] = \sum_{k\in A} p_{ik}\delta_{ij} + \sum_{k\in T} p_{ik}\mathrm{E}[n_{kj}+\delta_{ij}] = \delta_{ij} + \sum_{k\in T} p_{ik}\mathrm{E}[n_{kj}]$$
$$\Rightarrow \{\mathrm{E}[n_{ij}]\} = \mathbf{N} = \mathbf{I}+\mathbf{Q}\mathbf{N} \Rightarrow \mathbf{N} = (\mathbf{I}-\mathbf{Q})^{-1}$$

where A is the set of absorbing states and T is the set of transient states.

Notation: $\delta_{ij} = I(i=j) = \begin{cases} 1, & i=j, \\ 0, & \text{otherwise.}\end{cases}$ □

- $\{\mathrm{Var}[n_{ij}]\} = \mathbf{N}\,(2\mathbf{N}_{\text{diag}}-\mathbf{I}) - \mathbf{N}_{\text{sqr}}$

Proof.
$$\mathrm{Var}[n_{ij}] = \mathrm{E}[n_{ij}^2] - \mathrm{E}[n_{ij}]^2 \Rightarrow \{\mathrm{Var}[\mathbf{n_{ij}}]\} = \{\mathrm{E}[\mathbf{n_{ij}^2}]\} - \mathbf{N}_{\text{sqr}}$$
$$\mathrm{E}[\mathbf{n_{ij}^2}] = \sum_{k\in A} p_{ik}\delta_{ij}^2 + \sum_{k\in T} p_{ik}\mathrm{E}[(n_{kj}+\delta_{ij})^2] = \sum_{k\in A} p_{ik}\delta_{ij} + \sum_{k\in T} p_{ik}(\mathrm{E}[n_{kj}^2] + 2\mathrm{E}[n_{kj}]\delta_{ij} + \delta_{ij}) = \delta_{ij} + \sum_{k\in T}(p_{ik}\mathrm{E}[n_{kj}^2] + 2\,p_{ik}\mathrm{E}[n_{kj}]\delta_{ij}) \Rightarrow$$
$$\{\mathrm{E}[\mathbf{n_{ij}^2}]\} = \mathbf{I}+\mathbf{Q}\{\mathrm{E}[n_{ij}^2]\} + 2\,(\mathbf{Q}\mathbf{N})_{\text{diag}} = (\mathbf{I}-\mathbf{Q})^{-1}(\mathbf{I}+2(\mathbf{Q}\mathbf{N})_{\text{diag}}) = \mathbf{N}(\mathbf{I}+2(\mathbf{N}-\mathbf{I})_{\text{diag}}) = \mathbf{N}(2\mathbf{N}_{\text{diag}}-\mathbf{I})$$ □

- $\{\mathrm{E}[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\mathbf{e}$

Proof.
$$\mathrm{E}[t_i] = \sum_{k\in T}\mathrm{E}[n_{ij}] \Rightarrow \{E[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\mathbf{e}$$ □

- $\{\mathrm{Var}[t_i]\} = (2\mathbf{N}-\mathbf{I})\,\boldsymbol{\tau} - \boldsymbol{\tau}_{\text{sqr}}$

Proof.
$$\mathrm{Var}[t_i] = \mathrm{E}[t_i^2] - \mathrm{E}[t_i]^2 \Rightarrow \{\mathrm{Var}[\mathbf{t_i}]\} = \{\mathrm{E}[\mathbf{t_i^2}]\} - \tau_{\text{sqr}}$$
$$\mathrm{E}[\mathbf{t_i^2}] = \sum_{k\in A} p_{ik} + \sum_{k\in T} p_{ik}\mathrm{E}[(t_k+1)^2] = \sum_{k\in A} p_{ik} + \sum_{k\in T} p_{ik}(\mathrm{E}[t_k^2] + 2\mathrm{E}[t_k]+1) = 1 + \sum_{k\in T}(p_{ik}\mathrm{E}[t_k^2] + 2\,p_{ik}\mathrm{E}[t_k]) \Rightarrow$$
$$\{\mathrm{E}[\mathbf{t_i^2}]\} = \mathbf{e}+\mathbf{Q}\{\mathrm{E}[t_i^2]\} + 2\,\mathbf{Q}\,\tau = (\mathbf{I}-\mathbf{Q})^{-1}(\mathbf{e}+2\,\mathbf{Q}\,\tau) = \mathbf{N}\,(\mathbf{e}+2\,\mathbf{Q}\,\tau) = \tau + 2\mathbf{N}\mathbf{Q}\,\tau = \tau + 2\,(\mathbf{N}-\mathbf{I})\tau = (2\mathbf{N}-\mathbf{I})\tau$$ □

- $\{b_{ij}\} = \mathbf{B} = \mathbf{N}\mathbf{R}$

Proof.
$$b_{ij} = p_{ij} + \sum_{k\in T} p_{ik}b_{kj},\ j\in A \Rightarrow \{b_{ij}\} = \mathbf{B} = \mathbf{R}+\mathbf{Q}\mathbf{B} \Rightarrow \mathbf{B} = (\mathbf{I}-\mathbf{Q})^{-1}\mathbf{R} = \mathbf{N}\mathbf{R}.$$ □

10.2 Extension of the Results

- The previous results can be generalized to any group of states of $\mathbf{P}$:
- A set S is referred to as **open** if the chain can reach some state of S^c starting from any state of S. Let

$$\mathbf{Q} = \{p_{ij}, i\in S, j\in S\} \tag{10.4}$$
$$\mathbf{R} = \{p_{ij}, i\in S, j\in S^c\} \tag{10.5}$$

Let assume that the process starts from $i\in S$. Define:

$$\mathbf{n_{ij}} = \begin{Bmatrix} \text{number of steps in state } j \text{ before} \\ \text{leaving } S\text{, starting from state } i \end{Bmatrix},$$
$$\Rightarrow \{\mathrm{E}[n_{ij}]\} = \mathbf{N} = (\mathbf{I}-\mathbf{Q})^{-1}.$$

- Similarly for the other results, e.g. $\boldsymbol{\tau} = \{\mathrm{E}[t_i]\} = \mathbf{N}\mathbf{e}$ and $\mathbf{B} = \{b_{ij}\} = \mathbf{N}\mathbf{R}$.

10.3 Inverse of a matrix

Cofactors

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^{\mathrm{T}} \tag{10.6}$$

where $\mathbf{C}^{\mathrm{T}}$ is the transposed cofactor matrix: $c_{ij} = (-1)^{i+j}\det\mathbf{M}_{ij}$, and $\mathbf{M}_{ij}$ are the minor matrices obtained removing the row i and column j from $\mathbf{A}$.

Gaussian Elimination

Do the transformation:

$$[\mathbf{A}\,|\,\mathbf{I}] \to \left[\mathbf{I}\,|\,\mathbf{A}^{-1}\right] \tag{10.7}$$

using the elementary row operations:

- Swapping two rows.
- Multiplying a row by a nonzero number.
- Adding a multiple of one row to another row.

Part III

Continous Time Markov Chains

Chapter 11
Definition of a CTMC

11.1 Properties of a continuous time MC

- The states must be a numerable set.
- Let $X(t)$ be the event $\{$at time t the system is in state $i\}$, then it must hold the **memoryless property**:

$$\boxed{\begin{aligned} &P\big(X(t_n)=i \mid X(t_1)=j, X(t_2)=k,\ldots\big) = \\ &\qquad P\big(X(t_n)=i \mid X(t_1)=j\big) \text{ for any } t_n > t_1 > t_2 > t_3 \ldots \end{aligned}} \tag{11.1}$$

11.2 Transition Matrix

- **Transition probabilities**:

$$p_{ij}(t_1,t_2) = P\big(X(t_2)=j \mid X(t_1)=i\big) \tag{11.2}$$

- For an **homogeneous chain**:

$$\begin{aligned} p_{ij}(t) &= P\big(X(t_1+t)=j \mid X(t_1)=i\big) = \\ &= P\big(X(t)=j \mid X(0)=i\big),\ \forall t_1 \end{aligned} \tag{11.3}$$

- In matrix form (**transition probability matrix**):

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots \\ p_{21}(t) & p_{22}(t) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix},\ t \geq 0 \tag{11.4}$$

- Notes:
  - Compare with the n-step prob. matrix of a DTMC: $\mathbf{P}(n)$.
  - $\mathbf{P}(t)$ must be a **stochastic matrix** (all rows add to 1).
- We look for an equivalent 1-step prob. matrix $\mathbf{P}$ of DTMCs.
- For consistency: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}$. In matrix form:

$$\lim_{\mathbf{t}\to\mathbf{0}} \mathbf{P}(\mathbf{t}) = \mathbf{I}. \tag{11.5}$$

- And assume that the following **transition rates** exist:

$$q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t},\ i \neq j; \quad q_{ii} = \lim_{t\to 0} \frac{p_{ii}(t)-1}{t} \tag{11.6}$$

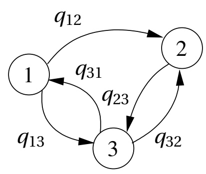
- In matrix form: $\mathbf{Q} = \lim_{t\to 0} \dfrac{\mathbf{P}(t)-\mathbf{I}}{t}$
- Note that $\sum_j p_{ij}(t) = 1 \Rightarrow p_{ii}(t) = 1 - \sum_{j\neq i} p_{ij}(t)$, thus:

$$\mathbf{q_{ii}} = \lim_{t\to 0} \frac{p_{ii}(t)-1}{t} = \lim_{t\to 0} \frac{-\sum_{j\neq i} p_{ij}(t)}{t} = -\sum_{\mathbf{j}\neq\mathbf{i}} \mathbf{q_{ij}} \tag{11.7}$$

- The matrix $\mathbf{Q}$ is called the **transition rate or infinitesimal generator** of the chain.
- Since $q_{ii} = -\sum_{j\neq i} q_{ij}$, **all the rows of Q add to 0**.
- The rate q_{ij}, $i \neq j$ measures "how fast" the chain moves from state i to j: the higher is q_{ij}, the faster it moves from i to j.
- For $q_{ii} = -\sum_{j\neq i} q_{ij}$, the higher $-q_{ii}$ is, the faster the chain leaves state i.
- If $\mathbf{q_{ij}} = \mathbf{0}, \forall \mathbf{j} \Rightarrow \mathbf{q_{ii}} = \mathbf{0}$, then i is an **absorbing state**: the chain "moves with rate 0 from i to other states", i.e. never leaves i.

11.3 State Transition Diagram

- A continuous MC is characterized by the transition rate or infinitesimal generator: the Q-matrix.
- The state transition diagram is now represented as:

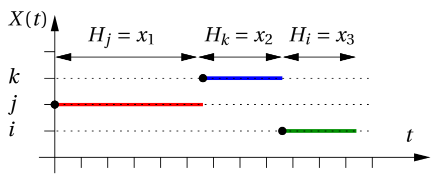


$$\Rightarrow \mathbf{Q} = \begin{bmatrix} -(q_{12}+q_{13}) & q_{12} & q_{13} \\ 0 & -q_{23} & q_{23} \\ q_{31} & q_{32} & -(q_{31}+q_{32}) \end{bmatrix}$$

- Note that now we have **transition rates** ($0 \leq q_{ij} < \infty$, $i \neq j$) **and not probabilities**.
- The **rates q_{ii} are not written** in the diagram, **no self transitions**.

11.4 Sojourn Time

- Sojourn or holding time: Is the RV H_k equal to the sojourn time in state k:



- The Markov property implies that **the sojourn time is exponentially distributed with parameter q_{ii}**:

$$\boxed{P\big(H_i \leq x\big) = 1 - e^{q_{ii}x} \Rightarrow P\big(H_i > x\big) = e^{q_{ii}x},\ q_{ii} = -\sum_{j\neq i} q_{ij},\ x \geq 0} \tag{11.8}$$

The exponential distribution satisfies the Markov property

- Markov property (**memoryless**):

$$\begin{aligned} &P\big(X(t_2)=i \mid X(t_1)=i, X(0)=i\big) = \\ &P\big(X(t_2)=i \mid X(t_1)=i\big),\ t_2 > t_1 > 0 \end{aligned} \tag{11.9}$$

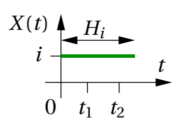


Figure 11.1: Sojourn time.

- In terms of the sojourn time:
  $P\big(H_i > t_2 \mid H_i > t_1\big) = P\big(H_i > t_2 - t_1\big)$

- But:

$$P(H_i > t_2 \mid H_i > t_1) = \frac{P(H_i > t_2, H_i > t_1)}{P(H_i > t_1)} = \frac{P(H_i > t_2)}{P(H_i > t_1)} = \frac{e^{q_{ii}t_2}}{e^{q_{ii}t_1}} = e^{q_{ii}(t_2-t_1)} = P(H_i > t_2 - t_1)$$

- The **exponential distribution is the only one satisfying the memoryless property**.

11.5 Exponential Jumps Description of a CTMC

Assume a process built as follows:

- Upon reaching a state i
  1. the process can jump to a state $\mathbf{j} \in \{\mathbf{1,2,\cdots l}\}$.
  2. A set of **independent exponential RVs**, $\{H_{i1}, H_{i2}, \cdots H_{il}\}$, with parameters $\{q_{i1}, q_{i1}, \cdots q_{il}\}$ are triggered. That is, $P(H_{ij} \le t) = 1 - \mathbf{e}^{-q_{ij}t}$, $t \ge 0$.
- If $\min\{\mathbf{H_{i1}}, \mathbf{H_{i2}}, \cdots \mathbf{H_{il}}\} = \mathbf{H_{ij}} \Rightarrow$ the process jumps to the state $\mathbf{j}$. In other words, a transition occurs to state j if the RV H_{ij} is the minimum of $\{H_{i1}, H_{i2}, \cdots H_{il}\}$(see figure 11.2).

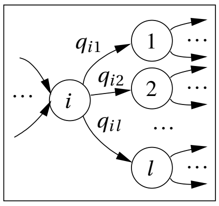


Figure 11.2: Jumps in a CTMC.

Theorem: This process is a CTMC with transition rates q_{ij}.

$$\mathbf{P(H_{ij} \le t) = 1 - e^{-q_{ij}t}}. \tag{11.10}$$

Proof. • The RV $H_i = \min\{H_{i1}, H_{i2}, \cdots H_{il}\}$ (sojourn time in state i) is **exponentially distributed** with parameter $q_i = \sum_j q_{ij}$: $\mathbf{P(H_i \le t) = 1 - e^{-q_i t}}$.

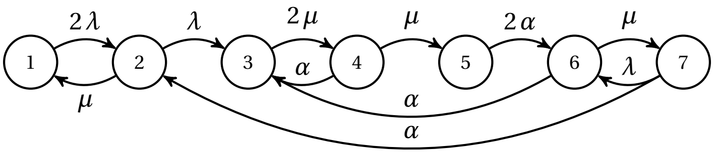
- $\mathbf{P}(\min\{\mathbf{H_{i1}}, \mathbf{H_{i2}}, \cdots \mathbf{H_{il}}\} = \mathbf{H_{ij}}) = \mathbf{q_{ij}} / \sum_\mathbf{j} \mathbf{q_{ij}}$. Thus, the **transition rate to state** j is:

$$\lim_{t\to 0} \frac{p_{ij}(t)}{t} = \lim_{t\to 0} \frac{P(\min\{H_{i1}, H_{i2}, \cdots H_{il}\} = H_{ij}) \times P(H_i \le t)}{t} = \frac{q_{ij}}{\sum_j q_{ij}} \left. \frac{\partial P(H_i \le t)}{\partial t} \right|_{t=0} = \frac{q_{ij}}{\sum_j q_{ij}} \sum_j q_{ij} = \mathbf{q_{ij}} \tag{11.11}$$

□

Example: Pure Aloha System

- Consider a **Pure Aloha System** with **2 nodes**:
  - Nodes in **thinking state** Tx a packet in a time exponentially distributed with rate λ.
  - **Transmission time** is exponentially distributed with rate μ.
  - If two transmissions overlap, the packet is lost and stations become backlogged (after the packet transmission) until the packet is successfully transmitted.
  - Nodes in **backlogged state** Tx a packet in a time exponentially distributed with rate α.



State	Condition	Legend	
1	T,T	T	Thinking
2	X,T	X	Transmitting
3	C,C	C	Collided transmission
4	B,C	B	Backlogged
5	B,B		
6	X,B		
7	T,B		

Chapter 12
Transient Solution

12.1 Chapman-Kolmogorov Equations

- **Chapman-Kolmogorov**: $p_{ij}(t) = \sum_k p_{ik}(t-\alpha) p_{kj}(\alpha)$, $0 \le \alpha \le t$
- Thus:

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_k \left\{ \frac{p_{ik}(t+\Delta t-\alpha) - p_{ik}(t-\alpha)}{\Delta t} p_{kj}(\alpha) \right\}$$

- Taking the limit

$$\alpha \to t, \Delta t \to 0 \Rightarrow \begin{cases} p_{ik}(t-\alpha) \to 0, & i \ne k \\ p_{ik}(t-\alpha) \to 1, & i = k \end{cases}$$

and using:

$$\begin{cases} q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)}{t}, & i \ne j \\ q_{ij} = \lim_{t\to 0} \frac{p_{ij}(t)-1}{t}, & i = j \end{cases}$$

we have:

$$\frac{\partial p_{ij}(t)}{\partial t} = \sum_k q_{ik} p_{kj}(t),\ t \ge 0, \forall i,j \tag{12.1}$$

- In matrix form:

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{Q}\,\mathbf{P}(t),\ t \ge 0 \tag{12.2}$$

known as the master equations of a CTMC.

- The solution of the previous matrix differential equation is the **exponential matrix**:

$$\mathbf{P(t)} = e^{\mathbf{Q}t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \mathbf{I} + \mathbf{Q}\,t + \frac{\mathbf{Q}^2 t^2}{2!} + \frac{\mathbf{Q}^3 t^3}{3!} + \cdots,\ t \ge 0 \tag{12.3}$$

- Due to rounding errors, the previous series is difficult to compute numerically (the powers of $\mathbf{Q}$ have positive and negative entries).

12.2 State Probabilities

- Define the probability of being in state i at time t:

$$\boxed{\pi_i(t) = P\big(X(t) = i\big) \qquad (12.4)}$$

- In vector form (row vector)

$$\boldsymbol{\pi}(t) = (\pi_1(t), \pi_2(t), \cdots). \qquad (12.5)$$

- Clearly:

$$\pi_i(t) = \sum_k P\big(X(0) = k\big)\, P\big(X(t) = i \mid X(0) = k\big) = \sum_k \pi_k(0)\, p_{ki}(t) \qquad (12.6)$$

- In matrix form:

$$\boxed{\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t},\ t \ge 0 \qquad (12.7)}$$

  where $\boldsymbol{\pi}(0)$ is the **initial distribution**.

- **NOTE**: Compare with **DTMC**

$$\boxed{\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\,\mathbf{P}^n,\ n \ge 0 \qquad (12.8)}$$

12.3 Transient Solution

- If we are interested in the **transient evolution** we shall study $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t}$, $t \ge 0$.
- Assume a **finite CTMC** with N states (infinitesimal generator $\mathbf{Q}^{N\times N}$).
- Assume that $\mathbf{Q}$ can be **diagonalized**: $\mathbf{Q} = \mathbf{L}^{-1}\Lambda\mathbf{L}$, where Λ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_N)$, with λ_l, $l = 1, \cdots N$ the eigenvalues of $\mathbf{Q}$.
- **NOTE**: the **eigenvalues** λ_l of a matrix $\mathbf{A}$ are scalars that satisfy: $\boldsymbol{l}\mathbf{A} = \lambda_l\,\boldsymbol{l}$ (or $\mathbf{A}\,\boldsymbol{r} = \lambda_l\,\boldsymbol{r}$) for some row vectors $\boldsymbol{l}$ (column vectors $\boldsymbol{r}$), referred to as *left* **and** *right* **eigenvectors**, respectively. Thus, solve the **characteristic polynomial** $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.
- Since

$$\mathbf{P}(t) = \mathrm{e}^{\mathbf{Q}\,t} = \sum_{i=0}^{\infty} \frac{(\mathbf{Q}\,t)^i}{i!} = \sum_{i=0}^{\infty} \frac{(\mathbf{L}^{-1}\Lambda\,\mathbf{L}\,t)^i}{i!} = \mathbf{L}^{-1}\,\mathrm{diag}\left(\sum_{i=0}^{\infty}\frac{(\lambda_1\,t)^i}{i!}, \cdots\right)\mathbf{L} = \mathbf{L}^{-1}\,\mathrm{diag}(\mathrm{e}^{\lambda_1\,t}, \cdots)\,\mathbf{L} \qquad (12.9)$$

  we have that

$$\boxed{\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t} = \boldsymbol{\pi}(0)\,\mathbf{L}^{-1}\,\mathrm{diag}(\mathrm{e}^{\lambda_1\,t}, \mathrm{e}^{\lambda_2\,t}, \cdots \mathrm{e}^{\lambda_L\,t})\,\mathbf{L} \qquad (12.10)}$$

- Thus, the probability of being in state i is given by:

$$\boxed{\pi_i(t) = (\boldsymbol{\pi}(t))_i = \sum_{l=1}^{N} a_i^{(l)}\,\mathrm{e}^{\lambda_l\,t},\ t \ge 0 \qquad (12.11)}$$

where the **unknown coefficients** $a_i^{(l)}$ can be obtained solving the system of equations:

$$\boxed{\left.\frac{\partial^n \pi_i(t)}{\partial t^n}\right|_{t=0} = (\boldsymbol{\pi}(0)\,\mathbf{Q}^n)_i,\ n = 0, \cdots N-1 \qquad (12.12)}$$

NOTE: Compare with $(\boldsymbol{\pi}(n))_i = (\boldsymbol{\pi}(0)\,\mathbf{P}^n)_i$, $n = 0, \cdots N-1$

12.3.1 Eigenvalues of an Infinitesimal Generator

- $\mathbf{Q}$ has **an eigenvalue equal to 0** ($\mathbf{Q}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$, for $\lambda = 0$, $\boldsymbol{x} \ne \mathbf{0}$).
  Proof: $\mathbf{Q}\,\mathbf{e} = \mathbf{0}$, where $\mathbf{e} = (1, 1, \cdots)^{\mathrm{T}}$ is a column vector of 1 (all rows of $\mathbf{Q}$ add to 0).
- The eigenvalue $\lambda = 0$ **is single if Q is irreducible** (Perron-Frobenius theorem). $\mathbf{Q}$ is irreducible if all states communicate: for $t > 0$, $p_{ij}(t) > 0$, $\forall i, j$.
- All eigenvalues of $\mathbf{Q}$ are $\lambda_l \le 0$.
  Proof: Using Gerschgorin's theorem (see figure 12.1) and the fact that the rows of $\mathbf{Q}$ add to 0.

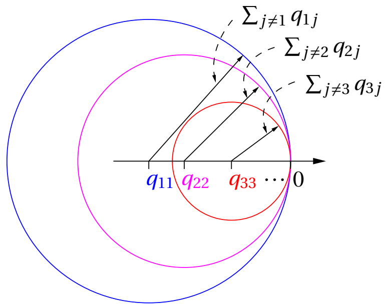


Figure 12.1: Gerschgorin discs of a CTMC

Example

- Assume a CTMC with

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 1/2 & -1/2 \end{bmatrix}$$

- We want the probability of being in state 2 at time t starting from state 1: $\pi_2(t)$ with $\boldsymbol{\pi}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Solution

- It can be easily found that the **eigenvalues** of $\mathbf{Q}$ are $\lambda_1 = 0$ and $\lambda_2 = -3/2$.

$$\pi_2(t) = a\mathrm{e}^{\lambda_1\,t} + b\mathrm{e}^{\lambda_2\,t} = a + b\mathrm{e}^{-(3/2)\,t} \qquad (12.13)$$

- Imposing the **boundary conditions**:

$$\pi_2(0) = a + b = (\boldsymbol{\pi}(0)\,\mathbf{Q}^0)_2 = (\boldsymbol{\pi}(0)\,\mathbf{I})_2 = (\boldsymbol{\pi}(0))_2 = 0$$
$$\left.\frac{\partial \pi_2(t)}{\partial t}\right|_{t=0} = b(-3/2) = (\boldsymbol{\pi}(0)\,\mathbf{Q})_2 = \mathbf{Q}_{12} = 1 \qquad (12.14)$$

  we have that $a = 2/3$, $b = -2/3$, thus:

$$\pi_2(t) = 2/3\,(1 - \mathrm{e}^{-(3/2)\,t}), \quad t \ge 0 \qquad (12.15)$$

12.3.2 Chain with a Defective Matrix

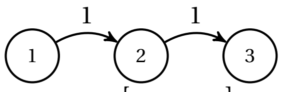
- What if $\mathbf{Q}$ cannot be diagonalized? (**defective** matrix).
- Let λ_l, $l = 1, \cdots L$ be the eigenvalues of $\mathbf{Q}^{N\times N}$, each with multiplicity k_l ($k_l \geq 1$, $\sum_l k_l = N$). Then [1]:

$$\boxed{\pi_j(t) = \sum_{l=1}^{L} \mathrm{e}^{\lambda_l t} \sum_{m=0}^{k_l - 1} a_j^{(l,m)}\, t^m} \qquad (12.16)$$

where $a_j^{(l,m)}$ are constants. So, exponentials associated with eigenvalues λ_l of multiplicity $k_l > 1$ are multiplied by polynomials in t of degree $k_l - 1$.

Example

- Assume the CTMC:



$$\pi(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (12.17)$$

- We have $\lambda_1 = 0$ and $\lambda_2 = -1$ with multiplicity 2. Thus:

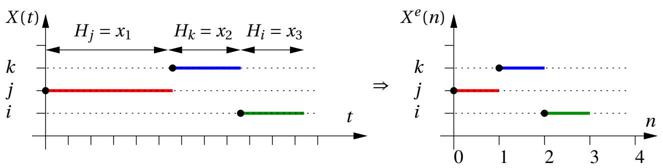
$$\pi_3(t) = a + \mathrm{e}^{-t}(b + c\,t) \qquad (12.18)$$

- We have that $a = 1$, because state 3 is absorbing. Imposing $\pi_3(0) = 0$ and $\pi_3'(0) = 0$, we have $b = c = -1$, and

$$\boxed{\pi_3(t) = 1 - \mathrm{e}^{-t}(1 + t),\ t \geq 0} \qquad (12.19)$$

Chapter 13
Embedded MC of a CTMC

13.1 Definition



- We form a discrete time process $X^e(n)$**, called the *Embedded MC (EMC)***, by looking a CTMC at the transition time instants.

Theorem: This process is a **DTMC** with transition probabilities:

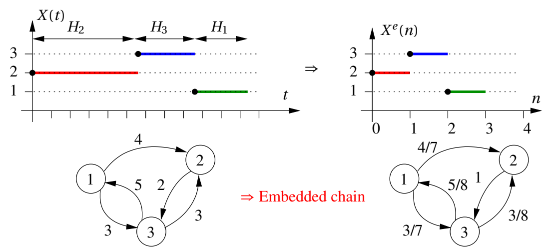
$$p^e_{ij} = \begin{cases} 0, & i = j \\ \dfrac{q_{ij}}{\sum_{j \neq i} q_{ij}}, & i \neq j \end{cases} \qquad (13.1)$$

- **NOTE**: If i is **absorbing** ($q_{ii} = 0$), we define $p^e_{ii} = 1$.

Proof.

- The EMC satisfies the **memoryless** property.
- Since we look the system only upon transition to a different state: $p^e_{ii} = 0$. NOTE: it might be $p^e_{ii} \neq 0$ if we look at transitions that end up in the same state.
- The probability that there is a transition from state i to j in the CTMC is the probability that the exponentially distributed RV with parameter q_{ij} is the **minimum from the independent exponentially distributed RVs** with parameters $\{q_{ik}\}_{k\neq i}$. This probability is $q_{ij} / \sum_{k\neq i} q_{ik}$.

Example



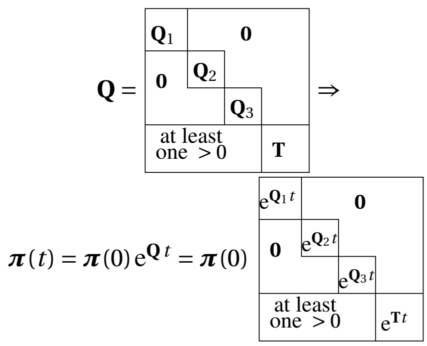
$$\mathbf{Q} = \begin{bmatrix} -7 & 4 & 3 \\ 0 & -2 & 2 \\ 5 & 3 & -8 \end{bmatrix} \Rightarrow \quad \mathbf{P}_e = \begin{bmatrix} 0 & 4/7 & 3/7 \\ 0 & 0 & 1 \\ 5/8 & 3/8 & 0 \end{bmatrix} \qquad (13.2)$$

- Each **transition** in the CTMC is a transition in the EMC.
- One step in i in the EMC is a **sojourn time** H_i in the CTMC.

Chapter 14
Classification of States

14.1 Irreducibly

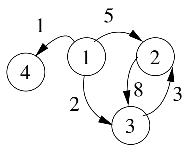
- A state j is said to **communicate** with i, $i \leftrightarrow j$, if $p_{ij}(t_1) > 0$, $p_{ji}(t_2) > 0$ for some $t_1 \geq 0$, $t_2 \geq 0$.
- We define an **irreducible closed set, ICS** C_k as a set where all states communicate with each other, and have no transitions to other states out of the set:
$i \leftrightarrow j, \forall i, j \in C_k$ and $q_{ij} = 0, \forall i \in C_k, j \notin C_k$
- An **absorbing state** form an ICS of only one element. This state, i, must have $q_{ij} = 0\ \forall i, j$.
- **Transient states** do not belong to any ICS.
- A MC is **irreducible** if all the states form a unique ICS.
- Assume a MC has M **ICSs**: By properly numbering the states, we can write $\mathbf{P}$ as an M block diagonal matrix with the probabilities of the transient states in the last rows.
- **Example**, if $M = 3$:



- Note that **the** M **sub-matrices are infinitesimal generators** (their rows add to 0).

14.2 Transient and Recurrent

- **Recurrent**: States that, being visited, have a probability > 0 of being visited again. They are visited an infinite number of times when $t \to \infty$.
- **Transient**: States that, being visited, have a probability > 0 of never being visited again. They are visited a finite number of times when $t \to \infty$.
- **Absorbing**: A single (recurrent) state where the chain remains with probability = 1.



State 1 is transient
States 2 and 3 are recurrent
State 4 is absorbing

- To derive a classification criteria, we shall study the **embedded MC (EMC)**, and proceed as in DTMC: Let $f^e_{ij}(n)$ the first passage prob. of the EMC, and $f^e_{ij} = \sum_{n=1}^{\infty} f^e_{ij}(n)$.
- If $f^e_{ii} = 1$ we say i is a **recurrent state**.
- If $f^e_{ii} < 1$ we say i is a **transient state**.
- When $f^e_{ij} = 1$, we define the **mean recurrence time of the EMC** $m^e_{ij} = \sum_{n=1}^{\infty} n\, f^e_{ij}(n)$. **NOTE**: in **steps**, not time units.
- If $m^e_{ii} = \infty$ the state is **null recurrent**.
- If $m^e_{ii} < \infty$ the state is **positive recurrent**.
- **NOTEs**: (i) Even if the EMC is periodic, **there are not periodic CTMC** (it has no sense). (ii) f^e_{ij} **and** m^e_{ij} can be computed using the **recursive equations** for DTMC.

14.3 Mean recurrence time of the CTMC

- If the chain is in i at a time t, it takes an **expected time to leave** i equal to $1/(-q_{ii}) = 1/\sum_{j\neq i} q_{ij}$ (**sojourn time exponentially distributed** with rate $q_i = -q_{ii} = \sum_{j\neq i} q_{ij}$).
- Thus, if the chain is **in state** i, it takes a **mean time to enter state** j (**mean first passage time**):

$$\boxed{m_{ij} = \frac{1}{q_i} + \sum_{k\neq i} p^e_{ik}\, m_{kj}} \tag{14.1}$$

- Since: $\mathbf{p^e_{ij}} = \begin{cases} 0, & i = j \\ \dfrac{q_{ij}}{\sum_{j\neq i} q_{ij}} = \dfrac{q_{ij}}{q_i}, & i \neq j \end{cases}$ we have:

$$\boxed{m_{ij} = \frac{1}{q_i} + \sum_{k\neq i} p^e_{ik}\, m_{kj} = \frac{1}{q_i} + \sum_{k\neq i} \frac{q_{ik}}{q_i} m_{kj} \text{ [time units]}} \tag{14.2}$$

Chapter 15
Steady State

15.1 Limiting Distribution

- The probability to be in state i at time t is:

$$\pi_i(t) = P\big(X(t) = i\big) = \sum_k \pi_k(0)\, p_{ki}(t),\ t \geq 0 \tag{15.1}$$

- In matrix form:

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t},\ t \geq 0 \tag{15.2}$$

- Assume that the following limit exists:

$$\boldsymbol{\pi}(\infty) = \lim_{t\to\infty} \boldsymbol{\pi}(t) = \lim_{t\to\infty} \boldsymbol{\pi}(0)\,\mathbf{P}(t) = \boldsymbol{\pi}(0) \lim_{t\to\infty} \mathrm{e}^{\mathbf{Q}\,t} \tag{15.3}$$

- for any $\boldsymbol{\pi}(0)$, which implies

$$\lim_{t\to\infty} \mathrm{e}^{\mathbf{Q}\,t} = \mathbf{P}(\infty) = \begin{bmatrix} \boldsymbol{\pi}(\infty) & \cdots & \boldsymbol{\pi}(\infty) \end{bmatrix}^{\mathrm{T}} \tag{15.4}$$

- If this limit exists, we call $\mathbf{P}(\infty)$ the **limiting matrix**, and $\boldsymbol{\pi}(\infty)$ the **limiting distribution**.
- $\mathbf{P}(\infty) = \begin{bmatrix} \boldsymbol{\pi}(\infty) & \cdots & \boldsymbol{\pi}(\infty) \end{bmatrix}^{\mathrm{T}}$ does not exist if the CTMC has more than one irreducible closed set (each ICS will converge to a diagonal block, and $\boldsymbol{\pi}(\infty)$ will depend on $\boldsymbol{\pi}(0)$).

15.2 Stationary Distribution

- We have: $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\,\mathrm{e}^{\mathbf{Q}\,t}$, $t \geq 0$.
- In steady state the probabilities do not change. We look for a probability vector $\boldsymbol{\pi} = \boldsymbol{\pi}(t_1)$ satisfying: $\boldsymbol{\pi}(t_1)\,\mathrm{e}^{\mathbf{Q}\,t} = \boldsymbol{\pi}(t_1)$. In other words, for $t \geq t_1$ the probability vector reach the steady state $\boldsymbol{\pi}$, and do not change anymore. Thus:

$$\boxed{\boldsymbol{\pi}\,\frac{\partial \mathrm{e}^{\mathbf{Q}\,t}}{\partial t} = \boldsymbol{\pi}\,\mathbf{Q}\,\mathrm{e}^{\mathbf{Q}\,t} = \mathbf{0}} \tag{15.5}$$

- and we obtain that the **stationary distribution $\boldsymbol{\pi}$ can be computed with the Global balance equations**:

$$\boxed{\begin{aligned} &\boldsymbol{\pi}\,\mathbf{Q} = \mathbf{0} \\ &\boldsymbol{\pi}\,\mathbf{e} = 1,\ \mathbf{e}^{\mathrm{T}} = (1, 1, \cdots) \end{aligned}} \tag{15.6}$$

- **NOTE**: Compare with DTMC $\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}$, $\boldsymbol{\pi}\,\mathbf{e} = 1$.

15.3 Numerical Solution

- **Replace one equation method**:

$$\boxed{\begin{aligned} &\boldsymbol{\pi}\,\mathbf{Q} = \mathbf{0} \\ &\boldsymbol{\pi}\,\mathbf{e} = 1,\ \mathbf{e}^{\mathrm{T}} = (1, 1, \cdots) \end{aligned}} \tag{15.7}$$

- We solve the equation $\boldsymbol{\pi}\,\mathbf{Q} = 0$ replacing the last equation by $\boldsymbol{\pi}\,\mathbf{e} = 1$:

$$\boxed{\boldsymbol{\pi} \begin{bmatrix} q_{11} & q_{12} & \cdots q_{1n-1} & 1 \\ q_{21} & q_{22} & \cdots q_{2n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ q_{n1} & q_{n2} & \cdots q_{nn-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}} \tag{15.8}$$

- **Replace one equation method**:

$$\mathbf{Q} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad \begin{aligned} &\boldsymbol{\pi}\,\mathbf{Q} = 0 \\ &\boldsymbol{\pi}\,\mathbf{e} = 1 \end{aligned}$$

- Solving with **octave** (matlab clone):

```
octave:1> Q=[-2,1,1;1,-2,1;1,1,-2];
octave:2> s=size(Q,1); # number of rows.
octave:3> [zeros(1,s-1),1] / ...
> [Q(1:s,1:s-1), ones(s,1)]
ans =
   0.33333   0.33333   0.33333
```

- With **R**

```
> Q <- matrix(nc=3, byr=T, c
    (-2,1,1,1,-2,1,1,1,-2))
> s <- nrow(Q)
> solve(t(cbind(Q[,1:(s-1)], rep(1,s))), c(rep
    (0,s-1),1))
[1] 0.3333333 0.3333333 0.3333333
```

15.4 Global balance equations

- Why are they called Global balance equations?

$$\left.\begin{array}{ll} \boldsymbol{\pi}\,\mathbf{Q}=\mathbf{0}\Rightarrow & \sum_{i=0}^{\infty}\pi_i\,q_{ij}=0 \\ \sum_{i=0}^{\infty}q_{ji}=0\Rightarrow & \pi_j\sum_{i=0}^{\infty}q_{ji}=0 \end{array}\right\}\Rightarrow \pi_j\sum_{i=0}^{\infty}q_{ji}=\sum_{i=0}^{\infty}\pi_i q_{ij} \quad (15.9)$$

$$\begin{array}{ll} \sum_{i=0}^{\infty}\pi_i\,q_{ij} & \Rightarrow\text{Frequency of } \textbf{transitions entering state } j \\ \pi_j\sum_{i=0}^{\infty}q_{ji} & \Rightarrow\text{Frequency of } \textbf{transitions leaving state } j \end{array} \quad (15.10)$$

- In **stationary regime**, the frequency of transitions leaving state j is equal to the frequency of transitions entering state j.

15.4.1 Solving using flux balancing

- Define the **flux** F_{uv} from state u to v:

$$\mathbf{F_{uv}}=\pi_{\mathbf{u}}\,\mathbf{q_{uv}} \quad (15.11)$$

- and the flux from set of states U to V:

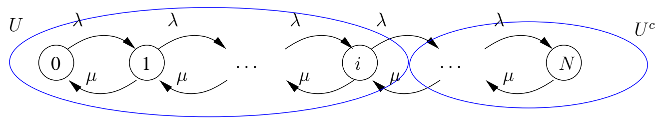
$$\mathbf{F(U,V)}=\sum_{\mathbf{u}\in\mathbf{U}}\sum_{\mathbf{v}\in\mathbf{V}}\mathbf{F_{uv}} \quad (15.12)$$

- From the Global balance equations, and reasoning exactly as in DTMC:

$$F(U,U^c)=F(U^c,U) \quad (15.13)$$

- **NOTE**: Same equation as in DTMC, changing p_{ij} by q_{ij}.

Example: Birth-dead Process



- Flux balancing $\Rightarrow \lambda\pi_i=\mu\pi_{i+1}$
- Iterating:

$$\pi_i=\pi_0\,\rho^i,\ i=0,1,\cdots N-1,\ \rho=\frac{\lambda}{\mu} \quad (15.14)$$

- Normalizing:

$$\pi_0=\frac{1-\rho}{1-\rho^N} \quad (15.15)$$

15.5 Ergodic Chains

- **Ergodic state**: positive recurrent ($f_{ii}^e=1$, $m_{ii}^e<\infty$).
- **Ergodic chain** if all states are ergodic.
- **Theorem:** All states of an irreducible Markov chain are of the same type: Transient or positive/null recurrent (see [2, chapter XV]).
- **Consequences**:
  - **Finite irreducible** chains are **ergodic** (since all states are positive recurrent).
  - **Infinite irreducible** chains can be:
    * **Ergodic**: all the states are positive recurrent (stable chains).
    * **Non ergodic**: all states are null recurrent or transient (unstable chains).

Theorems for ergodic chains

- $\boldsymbol{\pi}=\boldsymbol{\pi}(\infty)$. Proof: $\boldsymbol{\pi}(\infty)$ satisfies the GBE.
- In stationary regime (when $\boldsymbol{\pi}=\boldsymbol{\pi}\,e^{\mathbf{Q}\,t}$), the **mean number of time the system remains in state** j during T time units is given by

$$T\,\pi_j \quad (15.16)$$

  thus, π_j is the fraction of time the chain remains in state j. The proof is analogous to DTMC.
- **NOTE**: The relation of DTMC between **mean recurrence time** and stationary probabilities does not hold for CTMC. I.e., the mean number of time units between two consecutive visits to state j, m_{jj}**, cannot be computed as** $1/\pi_j$. It must be computed with the **recursive equations** (slide 21).

15.6 Reversible Chains

- Let $X(t)$ be an **ergodic** MC. The chain $X^r(t)=X(-t)$ is referred to as the **time reversal chain** of $X(t)$.
- The same results obtained for DTMC reversed chains apply to CTMC, changing p_{ij} by q_{ij}:
  - The reversed chain transition rates q_{ij}^r, given by:

$$\pi_i\,q_{ij}=\pi_j\,q_{ji}^r \quad (15.17)$$

  satisfy the **reversed balance equations**: $F(U,V)=F^r(V,U)$
- A chain is **reversible** if:

$$q_{ij}=q_{ij}^r \quad (15.18)$$

- Reversible chains satisfy the **detailed balance equations**:

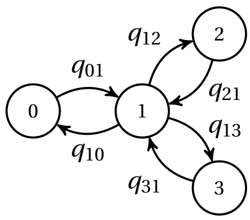
$$\boxed{F(U,V) = F(V,U), \forall (V,U), V \cap U = \emptyset \qquad (15.19)}$$

- The same results obtained for DTMC Reversible Chains apply to CTMC: **Kolmogorov Criteria** and **Product Form Solution** for the stationary distribution (changing p_{ij} by q_{ij}).
- E.g. the **stationary probabilities** are given by:
  - Choose a state $\mathbf{s} \in S$,
  - For every other state $\mathbf{i} \in S$, $i \neq s$ look for a possible path l_i from state s to state i:

$$\mathbf{s} = (l_i,1) \rightsquigarrow (l_i,2) \rightsquigarrow \cdots \rightsquigarrow (l_i,m_{l_i}) = \mathbf{i},\ m_{l_i} \geq 1$$

$$\pi_i = \frac{\psi_i}{\sum_{j\in S} \psi_j}, i \in S \quad \text{where } \psi_i = \begin{cases} 1, & i = s \\ \prod_{k=1}^{m_{l_i}-1} \frac{q_{(l_i,k)(l_i,k+1)}}{q_{(l_i,k+1)(l_i,k)}}, & i \neq s \end{cases} \qquad (15.20)$$

15.6.1 Example



- An ergodic tree is always reversible, thus

$$\pi_0 = \frac{1}{G}, \pi_1 = \frac{1}{G}\frac{q_{01}}{q_{10}}, \pi_2 = \frac{1}{G}\frac{q_{01}}{q_{10}}\frac{q_{12}}{q_{21}}, \pi_3 = \frac{1}{G}\frac{q_{01}}{q_{10}}\frac{q_{13}}{q_{31}}. \quad (15.21)$$

- Normalizing:

$$G = 1 + \frac{q_{01}}{q_{10}} + \frac{q_{01}}{q_{10}}\frac{q_{12}}{q_{21}} + \frac{q_{01}}{q_{10}}\frac{q_{13}}{q_{31}} \quad (15.22)$$

Chapter 16
Semi-Markov Process

- Define the continuous RV H_i equal to the **sojourn time** in state i.
- In a **semi-Markov process** we leave the H_i distribution to be **arbitrary**. If H_i is exponentially distributed, we have a CTMC.
- **NOTE**: **If H_i is not exponentially distributed**, considering only the current state does not satisfy the Markov property (memoryless) since **the evolution of the process depends on the current state and the sojourn time in the state:** (i, t_i).
- If we consider (i, t_i) as the state, the state would satisfy the Markov property, but we would have a **Markov process** (since t_i is not a discrete RV).

16.1 Embedded MC (EMC) of a semi-Markov process

- **Embedded MC** (EMC) of the process: **We only look at the state transition instants**.
  - **The EMC is a DTMC** with transition probabilities p^e_{ij}.
  - The **time step is variable**.
  - There are not self transitions ($p^e_{ii} = 0$), unless we look at some memoryless event that produce a self transition.
- **Theorem**: let π^e_i and π_i be the stationary distribution of the EMC and the semi-Markov process respectively. Let $\mathbf{E}[H_i]$ be the mean sojourn time in state i, then:

$$\boxed{\pi_i = \frac{\pi^e_i \mathrm{E}[H_i]}{\sum_j \pi^e_j \mathrm{E}[H_j]} \qquad (16.1)}$$

NOTE: By *stationary distribution* for the semi-Markov process we mean to the long-run proportion of time that the process is in each state.

Proof.

- **For n steps of the EMC**, define:
  - $f_i(n)$: **proportion of time** the process is in state i.
  - $N_i(n)$: **number of visits** to state i.
  - $H_i(l)$: **sojourn time** in state i in the **visit** number l.

$$f_i(n) = \frac{\sum_{l=1}^{N_i(n)} H_i(l)}{\sum_j \sum_{l=1}^{N_j(n)} H_j(l)} =$$

$$\frac{\frac{N_i(n)}{n} \sum_{l=1}^{N_i(n)} \frac{H_i(l)}{N_i(n)}}{\sum_j \frac{N_j(n)}{n} \sum_{l=1}^{N_j(n)} \frac{H_j(l)}{N_j(n)}} \Rightarrow$$

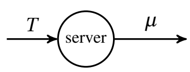
$$\pi_i = \frac{\pi^e_i \mathrm{E}[H_i]}{\sum_j \pi^e_j \mathrm{E}[H_j]}$$

- since:

$$\lim_{n\to\infty} f_i(n) = \pi_i$$
$$\lim_{n\to\infty} \sum_{l=1}^{N_i(n)} \frac{H_j(l)}{N_i(n)} = \mathrm{E}[H_i]$$
$$\lim_{n\to\infty} \frac{N_i(n)}{n} = \pi^e_i. \qquad \square$$

16.1.1 Example

Suppose the system:



- Packets arrive deterministically every T time units.
- Upon a packet arrival it goes immediately into service if the server is empty, and it is lost if the server is busy.
- Services are exponentially distributed with rate μ.

$$\text{Define a semi-Markov process with states} \begin{cases} (0) & \text{server empty,} \\ (1) & \text{server busy.} \end{cases} \qquad (16.2)$$

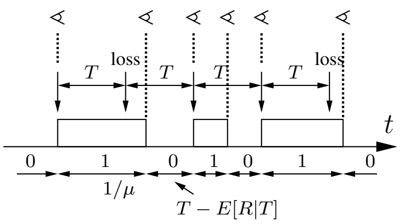
1. Derive the EMC and stationary distribution of the EMC and continuous time process.

2. Compute the throughput and loss probability.

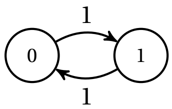
Hint: The distribution of an event R exponentially distributted with rate μ, given that occurs in an interval $t \in [0,T]$, is

$$F_R(t|T) = \frac{P(R \le t, R \le T)}{P(R \le T)} = \frac{P(R \le t)_{t\in[0,T]}}{P(R \le T)} = \frac{1-e^{-\mu t}}{1-e^{-\mu T}},\ t \in [0,T], \text{ and}$$
$$E[R|T] = \int_0^T (1-F_R(t|T))\,dt = \frac{1-\alpha-\alpha\mu T}{\mu(1-\alpha)}, \text{where } \alpha = e^{-\mu T}.$$

Solution



EMC:



- $\pi_0^e = \pi_1^e = 1/2$,
- $E[H_0] = T - E[R|T] = \dfrac{T\mu-(1-\alpha)}{\mu(1-\alpha)}, E[H_1] = \dfrac{1}{\mu}$.

And the continuous time process:

$$G = \pi_0^e E[H_0] + \pi_1^e E[H_1] = \frac{T}{2(1-\alpha)}\ [\frac{\text{time units}}{\text{step}}]$$
$$\pi_0 = \frac{\pi_0^e E[H_0]}{G} = \frac{\mu T-(1-\alpha)}{\mu T}, \pi_1 = \frac{\pi_1^e E[H_1]}{G} = \frac{1-\alpha}{\mu T} \tag{16.3}$$

$$\text{Throughput: } S = \mu\pi_1 = \frac{1-\alpha}{T}\ (\text{check } S = \frac{1}{E[H_0]+E[H_1]})$$
$$\text{Loss probability: } S = \frac{1}{T}(1-p_L), p_L = 1-ST = \alpha = e^{-\mu T}. \tag{16.4}$$

16.2 Embedded MC of a CTMC

- Assume that the semi-Markov process is a **CTMC** (sojourn times are exponentially distributed).
- The transition probabilities p_{ij}^e of the EMC, and the stationary distribution π_i of the CTMC are given by:

$$\begin{cases} p_{ij}^e = q_{ij}/q_i, & i \ne j \\ p_{ij}^e = 0, & i = j \end{cases} \qquad \pi_i = \frac{\pi_i^e/q_i}{\sum\limits_k \pi_k^e/q_k} \tag{16.5}$$

where: $q_i = \sum\limits_{k\ne i} q_{ik} = -q_{ii}$.

Proof.

- The equations for p_{ij}^e was proven in a previous section.
- The distribution of the **sojourn time in state** i **in the CTMC** is the distribution of the **minimum of independent exponentially distributed RV** with parameters $\{q_{ik}\}_{k\ne i}$. This distribution is **exponentially distributed with parameter** $\sum_{k\ne i} q_{ik}$. Thus $E[H_i] = 1/\sum_{k\ne i} q_{ik} = 1/q_i$, $q_i = \sum_{k\ne i} q_{ik}$.
- Substituting:

$$\pi_i = \frac{\pi_i^e E[H_i]}{\sum_j \pi_j^e E[H_j]} = \frac{\pi_i^e/q_i}{\sum_j \pi_j^e/q_j} \tag{16.6}$$

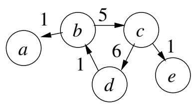
where: $q_i = \sum_{k\ne i} q_{ik} = -q_{ii}$. □

Chapter 17
Finite Absorbing Chains

17.1 Canonical Form

- Let $\mathbf{Q}^{rxr}$ be the transition probability matrix of a chain with a set S of s **transient** states and a set A of $r-s$ **absorbing** states. We can write $\mathbf{Q}^{rxr}$ in the **canonical** form:

$$\mathbf{Q}^{rxr} = \begin{bmatrix} \mathbf{T}^{s\times s} & \mathbf{R}^{s\times r-s} \\ \mathbf{0}^{r-s\times s} & \mathbf{0}^{r-s\times r-s} \end{bmatrix} \tag{17.1}$$



$$\mathbf{P} = \begin{array}{c} \\ b \\ c \\ d \\ a \\ e \end{array} \begin{array}{c} \begin{array}{ccc|cc} b & c & d & a & e \end{array} \\ \left[\begin{array}{ccc|cc} -6 & 5 & 0 & 1 & 0 \\ 0 & -7 & 6 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{17.2}$$

17.2 Results

- Define:

$$\mathbf{n_{ij}} = \left\{\begin{matrix}\text{time in state } j \text{ before absorption,} \\ \text{starting from state } i\end{matrix}\right\},$$
$$\mathbf{t_i} = \left\{\begin{matrix}\text{time in transient states before} \\ \text{absorption, starting from state } i\end{matrix}\right\}, \tag{17.3}$$
$$\mathbf{b_{ij}} = P\big(\text{probability to be absorbed } j \text{ starting } i\big).$$

- Then:

$$\{E[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}$$
$$\{E[t_i]\} = \boldsymbol{\tau} = \mathbf{Ne} \tag{17.4}$$
$$\{b_{ij}\} = \mathbf{B} = \mathbf{NR}$$

where $\{a_{ij}\}$ is a matrix with a_{ij} as element ij and $\mathbf{e}$ is a column vector of 1s. $\mathbf{N}$ is called the **fundamental matrix**.

Proof.
Let S be the set of **transient** states and A the set of **absorbing** states.

- $\{\mathbf{E}[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}$

$$\mathrm{E}[n_{ij}] = \frac{\delta_{ij}}{-q_{ii}} + \sum_{\substack{k\neq i\\ k\in S}} p^e_{ik}\,\mathrm{E}[n_{kj}] =$$

$$\frac{\delta_{ij}}{-q_{ii}} + \sum_{\substack{k\neq i\\ k\in S}} \frac{q_{ik}}{-q_{ii}}\,\mathrm{E}[n_{kj}] \Rightarrow$$

$$-\sum_{k\in S} q_{ik}\,\mathrm{E}[n_{kj}] = \delta_{ij} \Rightarrow$$

$$-\mathbf{T}\{\mathrm{E}[n_{ij}]\} = \mathbf{I} \Rightarrow \{\mathrm{E}[\mathbf{n_{ij}}]\} = -\mathbf{T^{-1}} = \mathbf{N}$$

- $\{\mathbf{E}[t_i]\} = \boldsymbol{\tau} = \mathbf{N}\mathbf{e}$

$$\mathrm{E}[t_i] = \sum_{k\in S} \mathrm{E}[n_{ij}] \Rightarrow \{\mathbf{E}[\mathbf{t_i}]\} = \boldsymbol{\tau} = \mathbf{N}\mathbf{e} \tag{17.5}$$

- $\{b_{ij}\} = \mathbf{B} = \mathbf{N}\mathbf{R}$

$$b_{ij} = p^e_{ij} + \sum_{\substack{k\neq i\\ k\in S}} p^e_{ik}\,b_{kj} =$$

$$\frac{q_{ij}}{-q_{ii}} + \sum_{\substack{k\neq i\\ k\in S}} \frac{q_{ij}}{-q_{ii}}\,b_{kj},\ i\in S,\ j\in\mathscr{A} \Rightarrow$$

$$-\sum_{k\in S} q_{ik}\,b_{kj} = q_{ij} \Rightarrow -\mathbf{T}\{b_{ij}\} = \mathbf{R} \Rightarrow$$

$$\{\mathbf{b_{ij}}\} = \mathbf{B} = -\mathbf{T^{-1}}\,\mathbf{R} = \mathbf{N}\mathbf{R}. \qquad \square$$

17.3 Extension of the Results

- The previous results can be generalized to any group of states of $\mathbf{Q}$:
- A set S is referred to as **open** if the chain can reach some state of S^c starting from any state of S. Let

$$\begin{aligned} \mathbf{T} &= \{q_{ij},\ i\in S,\ j\in S\} \\ \mathbf{R} &= \{q_{ij},\ i\in S,\ j\in S^c\} \end{aligned} \tag{17.6}$$

Let assume that the process starts from $i\in S$. Define:

$$\begin{aligned} \mathbf{n_{ij}} &= \left\{\begin{matrix}\text{number of steps in state } j \text{ before} \\ \text{leaving } S\text{, starting from state } i\end{matrix}\right\}, \\ &\Rightarrow \boxed{\{\mathrm{E}[n_{ij}]\} = \mathbf{N} = -\mathbf{T}^{-1}.} \end{aligned} \tag{17.7}$$

- Similarly for the other results, e.g. $\boldsymbol{\tau} = \{\mathrm{E}[t_i]\} = \mathbf{N}\mathbf{e}$ and $\mathbf{B} = \{b_{ij}\} = \mathbf{N}\mathbf{R}$.

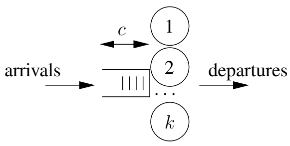
Part IV
Queuing Theory

Chapter 18
Introduction

Kendal Notation: A/S/k[/c/p]

$$\boxed{\text{A/S/k[/c/p]}} \tag{18.1}$$

- **A**: arrival process,
- **S**: service process,
- **k**: number of servers,
- **c**: maximum number in the system (number of servers + queue size). Note: some authors use the queue size.
- **p**: population.
  If “c” or “p” are missing, they are assumed to be **infinite**.



Common arrivals/departures processes

- **G**: general (non specific process is assumed),
- **M**: Markovian (exponentially or geometrically distributed),
- **D**: deterministic,
- **P**: Poisson (discrete RV, N, equal to the number of arrivals exponentially dist. in a time t):

$$P_p(N = n, t) = \frac{(\lambda\, t)^n\, e^{-\lambda\, t}}{n!},\ n \geq 0, t \geq 0. \tag{18.2}$$

- **Er**: Erlang (continuous RV equal to the time t that last n arrivals exponentially dist.):

$$f_e(t) = \lambda\, P_p(N = n-1, t) = \frac{\lambda^n\, t^{n-1}\, e^{-\lambda\, t}}{(n-1)!},\ t \geq 0, n \geq 1 \tag{18.3}$$

Examples

- **M/M/1**: M. arr. / M. serv. / 1 server, ∞ queue and population.
- **M/G/1**: M. arr. / Gen. serv. / 1 server, ∞ queue and population.

Chapter 19
Fundamental Theorems

19.1 Little Theorem

- Define the stochastic processes:
  - $A(t)$: number of arrivals $[0, t]$.
  - T_n: time in the system (response time) for customer n.
  - $N(t)$: number in the system at time t.
- And the mean values:
  - Mean number of customers in the system:

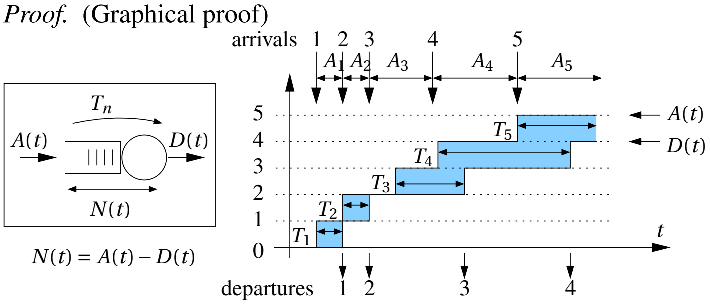
$$\mathbf{N} = \lim_{\mathbf{t}\to\infty} \frac{\mathbf{1}}{\mathbf{t}} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{N(s)\, ds} \tag{19.1}$$

  - Arrival rate: $\lambda = \lim_{t\to\infty} A(t)\,/\,t$
  - Mean time in the system: $T = \lim_{t\to\infty} \left(\sum_n T_n\right) / A(t)$
- The following relation follows:

$$\boxed{N = \lambda\, T} \tag{19.2}$$

Mnemonic: NAT (Number = Arrivals x Time).

Proof. (Graphical proof)

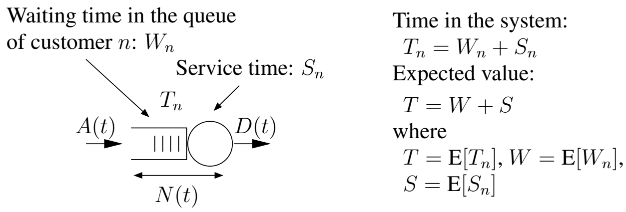


- From the graph we have:

$$\frac{1}{t}\int_0^t N(s)\, ds = \frac{1}{t}\sum_{i=1}^{A(t)} T_i = \frac{A(t)}{t}\,\frac{\sum_{i=1}^{A(t)} T_i}{A(t)} \tag{19.3}$$

- Taking the limit $t \to \infty$: $\boxed{N = \lambda\, T}$ □

Application to the waiting line and the server



- We can apply the Little theorem to the **waiting line** and the **server**:
- **Mean number** of customers **in the queue**: $N_Q = \lambda\, W$.
- **Mean number** of customers **in the server**: $N_S = \rho = \lambda\, S$.

Mean number in the Server

- In a **single server queue** (even if not Markovian):

$$\boxed{\begin{gathered}\rho = N_S = \mathrm{E}[N_S(t)] = \lambda\, \mathrm{E}[S] \\ \mathrm{E}[N_S(t)] = 0 \times \pi_0 + 1 \times (1 - \pi_0) = 1 - \pi_0 \Rightarrow \pi_0 = 1 - \rho\end{gathered}} \tag{19.4}$$

- $\rho = N_S = \lambda\, \mathbf{E}[S] = 1 - \pi_0$ is the proportion of time the system is busy, in other words, is the **server utilization or load**.

19.2 PASTA Theorem

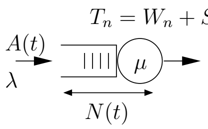
PASTA: Poisson Arrivals See Time Averages

- The mean time the chain is in state i is $\pi_i \Rightarrow$ using **PASTA**, the **probability that a Markovian arrival see the system in state i is π_i** (proof: see [9]).
- The equivalent theorem in **discrete time** is the **arrival theorem, RASTA**: Random Arrivals See Time Averages: the **probability that a random arrival see the system in state i is π_i**.

Example of PASTA

- Assume that a system can have, at most, N **customers** (e.g $N-1$ in the queue and 1 in service).
- Assume that an arrival is **lost** when the system is full.
- By **PASTA** the proportion of Poisson arrivals that see the system full, and are lost, is equal to the proportion of time the system has N in the system, π_N.
- Thus, **the loss probability is π_N**.

Chapter 20
The M/M/1 Queue



- Markovian **arrivals** with rate $\lambda \Rightarrow$ the **interarrival time** is exponentially distributed with mean $1/\lambda$:

$$P\{A_n \le x\} = 1 - e^{-\lambda x},\ x \ge 0 \tag{20.1}$$

$\Rightarrow$ $A(t)$ **is a Poisson process**: $P(A(t) = i) = \dfrac{(\lambda\, t)^i}{i!} e^{-\lambda t}$, $i \ge 0$, $t \ge 0$

- **Markovian Services** with rate $\mu \Rightarrow$ **service time** exponentially distributed with mean $1/\mu$:

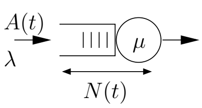
$$P\{S_n \le x\} = 1 - e^{-\mu x},\ x \ge 0 \tag{20.2}$$

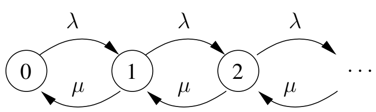
y

Q-matrix

- The process $N(t) = \{$number in the system at time $t \ge 0\}$ **is a CTMC**.

  OBSERVATION: for a non Markovian service, the process $N(t)$ would not be a MC! State transition diagram:





- Q-matrix:

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & \mu & -(\mu+\lambda) & \lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{20.3}$$

Stationary Distribution

- Solving the M/M/1 queue using flux balancing (or the general solution of a reversible chain):

$$\boxed{\pi_i = (1-\rho)\,\rho^i,\ i = 0, \cdots, \infty \qquad (20.4)}$$

where $\rho = \frac{\lambda}{\mu}$

Properties

- Mean **customers in the system**:

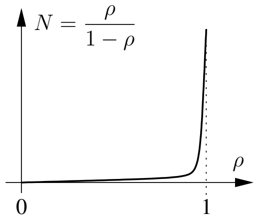
$$N = \lim_{t\to\infty} \frac{1}{t}\int_0^t N(s)\,ds = \sum_{i=0}^{\infty} i\,\pi_i = \sum_{i=0}^{\infty} i\,(1-\rho)\,\rho^i = \frac{\rho}{1-\rho} \tag{20.5}$$

- Mean **time in the system** (response time):
  Little: $N = \lambda\, T \Rightarrow T = \dfrac{N}{\lambda} = \dfrac{\rho}{\lambda\,(1-\rho)} = \dfrac{1}{\mu-\lambda}$
- Mean **time in the queue**: $W = T - \dfrac{1}{\mu} = \dfrac{\rho}{\mu-\lambda}$
- Mean **Number in the queue**: $N_Q = \lambda\, W = \dfrac{\rho^2}{1-\rho}$
- Mean **number in the server**: $N_s = N - N_Q = \rho$

  NOTE: $\pi_0 = 1-\rho$

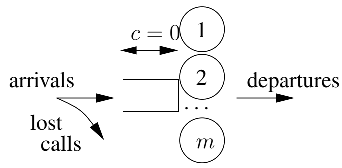
Stability

- N and T are proportional to $1/(1-\rho) \Rightarrow$
  when $\rho \to 1 \Rightarrow N, T \to \infty$.
- The process $N(t)$ is **positive recurrent**, **null recurrent** or **transient** according to whether $\rho = \lambda/\mu$ is below, equal or greater than 1, respectively.

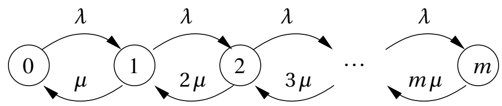


Example: Loss probability in a telephone switching center

- Hypothesis: Switching center with m **circuits** and "lost call", infinite population, Markovian arrivals with rate λ and exponentially distributed call duration with mean $1/\mu \Rightarrow$ **M/M/m/m** queue.



- Since the minimum of i independent and identically exponentially distributed RV with parameter **service time** is exponentially distributed with parameter $i\,\mu$:



- Stationary Distribution of the queue M/M/m/m:

- Solving using the **general solution of a reversible chain**:

$$\text{Define } \rho_k = \frac{\lambda}{(k+1)\,\mu},\ k = 0,\cdots,m-1$$
$$\pi_0 = \frac{1}{G},\ \pi_i = \frac{1}{G}\prod_{k=0}^{i-1}\rho_k = \frac{1}{G}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!},\ 0 < i \le m \Rightarrow \tag{20.6}$$

$$\boxed{\pi_i = \frac{1}{G}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!},\ 0 \le i \le m.\ G = \sum_{k=0}^{m}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}.} \tag{20.7}$$

- Using **PASTA** Theorem (Poisson Arrivals See Time Average): the **loss call probability** is the probability that the queue is in state m: π_m, **"Erlang B Formula"**.

Chapter 21
M/G/1 Queue

- The process $N(t) = \{\text{number in the system at time } t \ge 0\}$ in general it is not a MC (it is so only if G is Markovian).
- We can build a **semi-Markov process** observing the system at **departure times** t_n (note that t_n are also the service completion times). Define the discrete time process:

$$\mathbf{X(n)} = \{\text{number in the system at time } \mathbf{t_n \ge 0, n = 0, 1, \cdots}\} \tag{21.1}$$

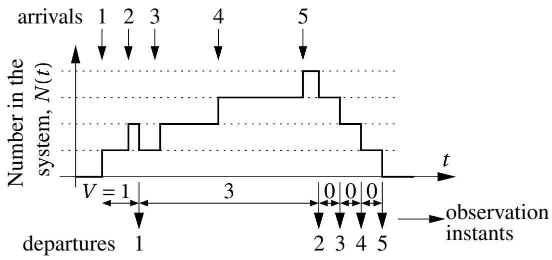
- **Theorem:** The process $X(n)$ **is a DTMC**.
- **Proof**: $X(n)$ only depends on the number of **arrivals in non overlapping intervals**. Since arrivals are Markovian, this is a **memoryless** process.
- **NOTE**: Looking at **departure times** the chain may have **self transitions** (in contrast to observing at transition times): we can have the same number in the system after a departure.

Transition Probability Matrix

- Let $f_S(x), x \ge 0$ be the **service time** density function.
- Define the RV $V =$ **{number of arrivals during a service time}**, and the probabilities: $v_i = P\{V = i\}$.
- Conditioning on the service duration:

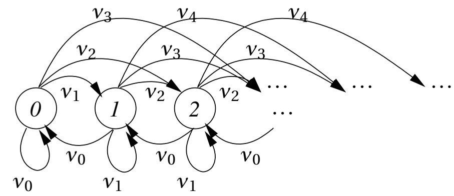
$$v_i = \int_{x=0}^{\infty} P\{i \text{ arrivals in time } x \mid S = x\}\ f_S(x)\,dx \Rightarrow \tag{21.2}$$

$$\boxed{v_i = \int_{x=0}^{\infty} \frac{(\lambda x)^i}{i!} e^{-\lambda x} f_S(x)\,\mathrm{d}x} \tag{21.3}$$



- $v_i = P\{\text{number of arrivals during a service time} = i\} \Rightarrow$

$$\boxed{p_{ij} = \begin{cases} 0, & j < i-1 \quad (N(t) \text{ can only be decreased by 1}) \\ v_j, & i = 0, j \ge 0 \quad (i = 0 \to \text{the queue was empty}) \\ v_{j-i+1}, & i > 0, j \ge i-1 \quad (i > 0 \to \text{the queue was busy}) \end{cases}} \tag{21.4}$$



$$\boxed{p_{ij} = \begin{cases} 0, & j < i-1 \\ v_j, & i = 0, j \ge 0 \\ v_{j-i+1}, & i > 0, j \ge i-1 \end{cases} \Rightarrow \mathbf{P} = \begin{bmatrix} v_0 & v_1 & v_2 & v_3 & \cdots \\ v_0 & v_1 & v_2 & v_3 & \cdots \\ 0 & v_0 & v_1 & v_2 & \cdots \\ 0 & 0 & v_0 & v_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}} \tag{21.5}$$

- Stationary distribution: $\pi = \pi\mathbf{P}, \pi\mathbf{e} = 1$.

Properties of the stationary distribution ($\pi = \pi\mathbf{P}, \pi\mathbf{e} = 1$)

- Using the **"Level Crossing Law"** theorem: a queue with **unitary arrivals and departures** satisfies:

$$\boxed{\begin{aligned} &P\{\textbf{an arriving customer finds } i \text{ in the system}\} = \\ &\quad P\{\textbf{a departing customer leaves } i \text{ in the system}\} \Rightarrow \end{aligned}} \tag{21.6}$$

$$\boxed{\pi_i = P\{\text{an arriving customer find } i \text{ in the system}\}} \tag{21.7}$$

- Using **PASTA**:

$$\boxed{\begin{aligned} \pi_i = P\{&\text{there are } i \text{ customers in the} \\ &\text{system at an arbitrary time}\} \end{aligned}} \tag{21.8}$$

So, in an M/G/1 the stationary distribution of the EMC obtained observing the departures, is the stationary distribution of the continuous time process.

Proof. Level Crossing Law Theorem

- Define:
  - $\mathbf{A_i(t)} = \{\text{number of arrivals finding } i \text{ in the system at } t \ge 0\}$
  - $\mathbf{D_i(t)} = \{\text{number of departures leaving } i \text{ in the system at } t \ge 0\}$
  - $\mathbf{P}\{\text{a customer finds } i \text{ in the system}\} = \lim_{t\to\infty} A_i(t)/A(t)$
  - $\mathbf{P}\{\text{a customer leave } i \text{ in the system}\} = \lim_{t\to\infty} D_i(t)/D(t)$
- An arriving customer that finds i in the system produce a transition $i \to i+1$. A customer leaving i in the system produce a transition $i+1 \to i$.
- Since arrivals and departures are unitary, the number of transitions $i \to i+1$ and $i+1 \to i$ can only differ in 1: $|A_i(t) - D_i(t)| \le 1$. Note that $N(t) = A(t) - D(t)$.
- For a **stable queue**: $A(t) - D(t) < \infty$
- We have:
  - $A_i(t) = \{\text{number of arrivals finding } i \text{ customer in the system}\}$

- $D_i(t)$ = {number of departures leaving i customers in the system}
- P{a customer finds i in the system} $= \lim_{t\to\infty} A_i(t)/A(t)$
- P{a customer leave i in the system} $= \lim_{t\to\infty} D_i(t)/D(t)$
- $A_i(t) - D_i(t) \in \{0,1\}$, $N(t) = A(t) - D(t) < \infty$.
- $\lim_{t\to\infty} A(t) = \infty$, $\lim_{t\to\infty} D(t) = \infty$.

• Thus:

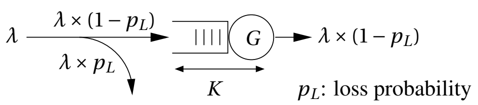
$$\lim_{t\to\infty}\left\{\frac{A_i(t)}{A(t)} - \frac{D_i(t)}{D(t)}\right\} =$$

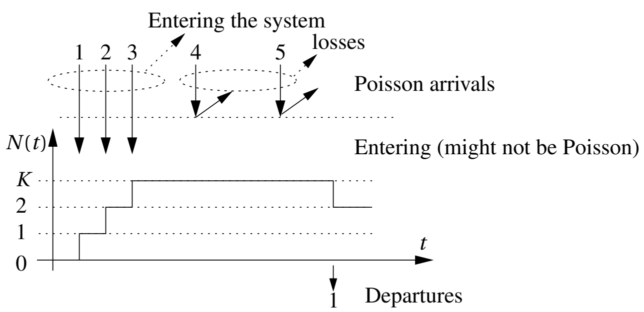
$$\lim_{t\to\infty}\left\{\frac{A_i(t)}{A(t)} - \frac{D_i(t)}{A(t)} - \left(\frac{D_i(t)}{D(t)} - \frac{D_i(t)}{A(t)}\right)\right\} =$$

$$\lim_{t\to\infty}\left\{\frac{A_i(t)-D_i(t)}{A(t)} - \frac{D_i(t)}{D(t)}\frac{A(t)-D(t)}{A(t)}\right\} = 0 \qquad \square$$

21.1 M/G/1/K Queue

Problem Formulation





Stationary Distribution

- Using the **general solution of an M/G/1/K** we obtain the stationary distribution of the number in the system left by a **departing** customer: π_i^d.
- By the **Level Crossing Law** this is the stationary distribution of the number in the system found by the **successful arrivals**:

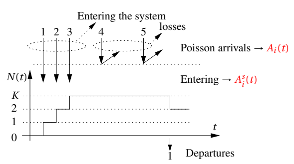
$$\pi_i^s = \pi_i^d,\ i = 0,1,\cdots \mathbf{K-1}. \tag{21.9}$$

and

$$\pi_i^s = P(\text{a customer entering the system finds } i) \tag{21.10}$$

- **NOTE**: a departing customer cannot leave the system full (nor an arrival can enter the system when it is full).

Loss Probability



Define:

- $A_i^a(t)$: Number of **arrivals** (lost or not) finding i in the system.
- $A_i^s(t)$: Number of **successful arrivals** finding i in the system.
- π_i^a, π_i^s the stationary distribution of the embedded Markov chains $A_i^a(t)$, $A_i^s(t)$. By **PASTA** π_i^a is also the stationary distribution of the continuous time process. Thus,

$\pi_i^s = P(\text{a customer entering the system finds } i),\ i = 0,1,\cdots \mathbf{K-1} \Rightarrow$

$$\pi_i^s = \lim_{t\to\infty}\frac{A_i^s(t)}{\sum_{k=0}^{K-1}A_k^s(t)}\frac{\sum_{k=0}^{K}A_k^a(t)}{\sum_{k=0}^{K}A_k^a(t)} = \frac{\pi_i^a}{\sum_{k=0}^{K-1}\pi_i^a} = \frac{\pi_i^a}{1-\pi_K^a} = \frac{\pi_i^a}{1-p_L}, \Rightarrow$$

$$\pi_i^a = \pi_i^s(1-p_L) = \pi_i^d(1-p_L),\ i = 0,1,\cdots \mathbf{K-1} \tag{21.11}$$

- Applying **Little**: $\rho_s = \mathbf{E}[N_s] = 1-\pi_0 = \lambda(1-p_L)\mathbf{E}[S] = \rho(1-p_L)$. Where $\rho = \lambda\mathbf{E}[S]$ and π_0 is the proportion of time the server is empty.
- Using **PASTA**: $\pi_0 = \pi_0^a$ (Poisson arrivals). Using $\pi_i^a = \pi_i^d(1-p_L)$:

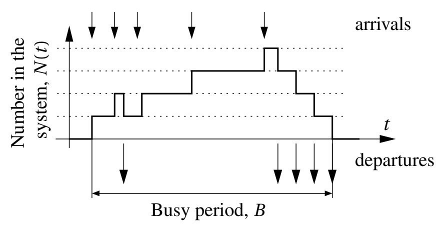
$$\left.\begin{array}{l} 1-\pi_0 = 1-\pi_0^a = 1-\pi_0^d(1-p_L) \\ 1-\pi_0 = \rho(1-p_L) \end{array}\right\} \Rightarrow$$

$$p_L = \frac{\rho+\pi_0^d-1}{\rho+\pi_0^d},\ \rho = \lambda\mathrm{E}[S] \tag{21.12}$$

- Where π_0^d is computed using the general solution of an M/G/1/K.

21.2 M/G/1 Busy Period

Expected Length of a Busy Period

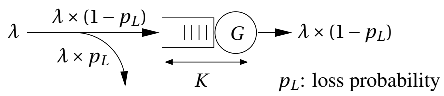


- Define the RV:
  - **Busy period,** B.
  - **Idle period,** I. Poisson arrivals with rate $\lambda \Rightarrow \mathrm{E}[\mathbf{I}] = \mathbf{1}/\lambda$
- Clearly:

$$\text{System load } \rho = \lambda\mathrm{E}[S] = \frac{\mathrm{E}[B]}{\mathrm{E}[I]+\mathrm{E}[B]} \Rightarrow \tag{21.13}$$

$$\mathrm{E}[B] = \frac{1}{\lambda}\frac{\rho}{1-\rho} \tag{21.14}$$

21.3 M/G/1/K Busy Period



- **Busy period,** B.
- **Idle period,** I. Poisson arrivals with rate $\lambda \Rightarrow \mathrm{E}[\mathbf{I}] = \mathbf{1}/\lambda$
- Clearly:

$$\text{System load } \rho_s = \lambda\,(1-p_L)\,\mathrm{E}[S] = \frac{\mathrm{E}[B]}{\mathrm{E}[I]+\mathrm{E}[B]} \Rightarrow$$

$$\boxed{\mathrm{E}[B] = \frac{1}{\lambda}\,\frac{\rho\,(1-p_L)}{1-\rho\,(1-p_L)},\ \rho = \lambda\,\mathrm{E}[S]} \tag{21.15}$$

- Or, in terms of $\pi_0 = \pi_0^d\,(1-p_L)$:

$$\text{System load } \rho_s = 1-\pi_0 = \frac{\mathrm{E}[B]}{\mathrm{E}[I]+\mathrm{E}[B]} \Rightarrow$$

$$\boxed{\mathrm{E}[B] = \frac{1}{\lambda}\,\frac{1-\pi_0}{\pi_0}} \tag{21.16}$$

21.4 M/G/1 Delays

M/G/1 Mean Time in the Queue

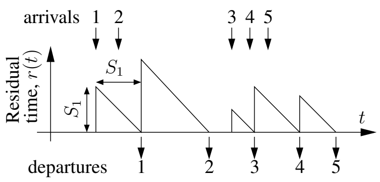
- **Method of the moments**: Using **PASTA**, the **mean time in the queue** (W) for an arriving customer, is the mean time to finish the current service (**mean residual time,** R) plus the **mean time to service the customers in the queue** ($\mathbf{E}[S]\,N_Q$):

$$\boxed{W = R + \mathrm{E}[S]\,N_Q} \tag{21.17}$$

- Using **Little for the queue length**:

$$\boxed{N_Q = \lambda\,W \Rightarrow W = R + E[S]\,\lambda\,W \Rightarrow W = \frac{R}{1-\rho},\ \rho = \lambda\,E[S].} \tag{21.18}$$

M/G/1 Residual Time



- From the figure (note the **right triangles with two equal cathetus**), we have:

$$\boxed{\mathbf{R} = \frac{1}{t}\int_0^t r(\tau)\,d\tau = \frac{1}{t}\sum_{i=1}^{A(t)}\frac{S_i^2}{2} = \frac{1}{2}\,\frac{A(t)}{t}\sum_{i=1}^{A(t)}\frac{S_i^2}{A(t)} \underset{t\to\infty}{=} \frac{1}{2}\lambda\,\mathrm{E}[S^2]} \tag{21.19}$$

- For instance, for an M/M/1

$$\mathrm{E}[S^2] = \mathrm{Var}(S) + \mathrm{E}[S]^2 = \frac{1}{\mu^2} + \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}, \tag{21.20}$$

thus, the residual time is:

$$R = \frac{1}{2}\lambda\,\mathrm{E}[S^2] = \frac{\lambda}{\mu^2} = \frac{\rho}{\mu},\ \rho = \frac{\lambda}{\mu}. \tag{21.21}$$

- We can check that $\mathrm{E}[R|\mathrm{S\ idle}] = 0$ and $\mathrm{E}[R|\mathrm{S\ busy}] = 1/\mu$, thus

$$R = \mathrm{E}[R|\mathrm{S\ idle}]\,\pi_0 + \mathrm{E}[R|\mathrm{S\ busy}]\,(1-\pi_0) = \frac{\rho}{\mu},\ \rho = 1-\pi_0, \tag{21.22}$$

as expected.

Pollaczek-Khinchin, P-K formula

- We have:

$$\begin{aligned} W &= \frac{R}{1-\rho},\ \rho = \lambda\,E[S] \\ R &= \frac{1}{2}\lambda\,\mathrm{E}[S^2] \end{aligned} \tag{21.23}$$

- Substituting we get the **Pollaczek-Khinchin, P-K formula**:

$$\boxed{\mathbf{W} = \frac{\lambda\,\mathrm{E}[S^2]}{2(1-\rho)},\ \rho = \lambda\,\mathrm{E}[S]} \tag{21.24}$$

- Mean **time in the system** (response time):

$$\boxed{T = \mathrm{E}[S] + W = \mathrm{E}[S] + \frac{\lambda\,\mathrm{E}[S^2]}{2\,(1-\rho)}} \tag{21.25}$$

- For an **M/M/1** queue: $\mathrm{E}[S^2] = \dfrac{2}{\mu^2} \Rightarrow W = \dfrac{\rho}{\mu\,(1-\rho)}$
- For an **M/D/1** queue: $E[S^2] = \dfrac{1}{\mu^2} \Rightarrow W = \dfrac{\rho}{2\,\mu\,(1-\rho)}$
- **Observation**: The M/D/1 has the minimum value of $E[S^2] \Rightarrow$ is a lower bound of W, T, N_Q and N for an M/G/1.

P-K Formula Does Not Apply to an M/G/1/K Queue

- **P-K formula is not applicable** to an **M/G/1/K** queue because the **customers entering the system** might not be Poisson. Thus, they **does not observe the mean residual time**.
- **Example**: Customers entering an **M/G/1/1** queue (0 queue size) observe the system always empty. Thus, in an M/G/1/1 queue the expected time in the queue is $W = 0$ (P-K formula does not apply), and the expected time in the system is $T = \mathbf{E}[S]$ (mean service time).
- **With an M/G/1/K** we can compute $N = \sum_{n=1}^{K} n\,\pi_n^a$, and use Little: $N = \lambda\,(1-p_L)\,T$. For instance, for an M/G/1/1 we have $\pi_0^d = 1$, and $N = 0\,\pi_0^a + 1\,\pi_1^a = \pi_1^a = p_L$. Thus, $p_L = \frac{\rho+\pi_0^d-1}{\rho+\pi_0^d} = \frac{\rho}{\rho+1}$, and $T = \frac{N}{\lambda\,(1-p_L)} = \frac{p_L}{\lambda\,(1-p_L)} = \frac{\rho}{\lambda} = \mathrm{E}[S]$, as expected.

Chapter 22
Queues in Tandem

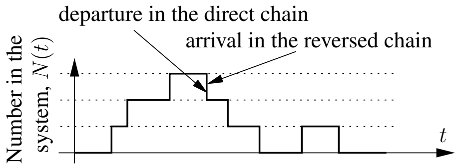
22.1 Burke theorem

- The **departure process in an M/M/m** queue, $1 \le m \le \infty$, is a **Poisson** process with the same parameter than the arrival process.
- At each time t, the **number of customers in the system** is independent of the sequence of departures previous to t.

- Relation between the arrival and departure process: $\boxed{\text{The departure process in a reversible queue has the same joint distribution than the arrival process.}}$

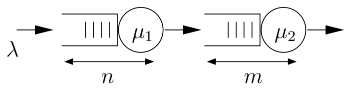
Proof.

- If the queue is reversible: $q_{ij} = q^r_{ij} \Rightarrow$ the arrival process in the reversed chain has the same distribution than the arrival process in the direct chain,
- but: □



- The queue **M/M/m is reversible** ⇒ The **departures are Poisson** with the same parameter than the arrivals.
- The arrivals in the reversed chain previous to t are Markovian, thus, independent of the number of customers in the system after t. This implies that the **departures** in the direct chain are **independent of the number in the system** before t.

22.2 Tandem M/M/m Queues



- Define the chain:

$$\mathbf{X}(\mathbf{n},\mathbf{m}) = \{n \text{ in the system } 1,\ m \text{ in the system } 2\} \tag{22.1}$$

- The **stationary distribution** is the **product of the stationary distributions of the isolated queues**:

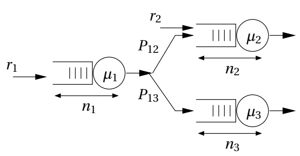
$$\boxed{\pi_{nm} = (1-\rho_1)\,\rho_1^n\,(1-\rho_2)\rho_2^m,\ \rho_1 = \lambda/\mu_1,\ \rho_2 = \lambda/\mu_2 \quad (22.2)}$$

- *Proof.* Using Burke, the departures of system 1 are Poisson and the number in the system 1 is independent of the previous departures (arrivals to system 2), thus, independent from the number of customers in system 2. □

Chapter 23

Networks of Queues

23.1 Feed Forward Queues



- Suppose **M/M/1 queues** with outside arrivals with rate r_i randomly forwarded with probabilities P_{ij} (see figure).
- The network has the following **product form solution**:

$$\boxed{\begin{aligned}&\pi(n_1, n_2, \cdots n_K) = (1-\rho_1)\,\rho_1^{n_1}\,(1-\rho_2)\,\rho_2^{n_2}\cdots(1-\rho_k)\rho_k^{n_k},\\ &\rho_i = \lambda_i/\mu_i.\end{aligned} \quad (23.1)}$$

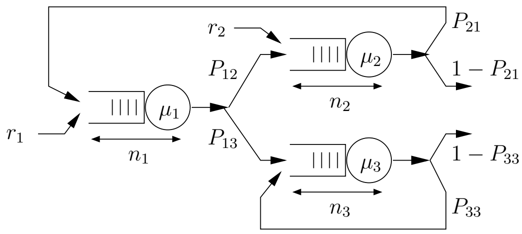
- The rates λ_i are computed solving: $\boxed{\lambda_i = r_i + \sum_j \lambda_j P_{ji}}$.
- Stability condition: $\rho_i < 1$.

Proof. (draft)

- Burke theorem.
- **Superposition of Poisson** processes with rates λ_i is Poisson with rate $\sum_i \lambda_i$.
- A **Poisson** process with rate λ **randomly split** with probabilities p_i, $\sum_i p_i = 1$, produce Poisson processes with rates $p_i\,\lambda$. □

23.2 Jackson Theorem

- Suppose **M/M/m queues**. In queue i the customers **arrive** from outside with rate r_i and **depart** to queue j with probability P_{ij}, or leave the system with probability $1-\sum_j P_{ij}$:



- The network has the following **product form solution**:

$$\boxed{\pi(n_1, n_2, \cdots n_K) = \pi_{n_1}\,\pi_{n_2}\cdots\pi_{n_k} \quad (23.2)}$$

where π_{n_i} is the solution of the queue i with arrival rates λ_i obtained solving:

$$\boxed{\lambda_i = r_i + \sum_j \lambda_j P_{ji} \quad (23.3)}$$

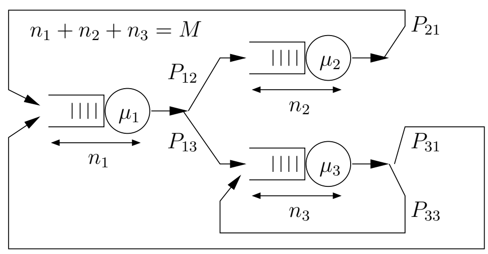
- Stability condition: $\rho_i = \lambda_i/\mu_i < 1$.
- For example, for M/M/1 queues:

$$\boxed{\pi(n_1, n_2, \cdots n_K) = (1-\rho_1)\,\rho_1^{n_1}\,(1-\rho_2)\,\rho_2^{n_2}\cdots(1-\rho_k)\rho_k^{n_k} \quad (23.4)}$$

- **Proof**: The solution satisfies the global balance equations.
- **NOTE**: The proof is different from feed forward queues, since routing loops make arrivals not necessarily Poisson.

23.3 Closed Networks of Queues

- **M/M/m networks** without arrivals and departures to outside of the system:



Jackson Theorem for Closed Networks of Queues

- The network has the following **product form solution**:

$$\boxed{\pi(n_1, n_2, \cdots n_K) = \frac{1}{G}\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k},\ \rho_i = \lambda_i/\mu_i. \qquad (23.5)}$$

- Where the rates λ_i are any solution to the equations:

$$\boxed{\lambda_i = \sum_j \lambda_j P_{ji} \quad \text{(in matrix form: } \boldsymbol{\lambda} = \boldsymbol{\lambda}\mathbf{P}\text{)} \qquad (23.6)}$$

- And the normalization factor is given by:

$$\boxed{G = \sum_{n_1+n_2+\cdots+n_k=M} \rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k} \qquad (23.7)}$$

- **Proof**: The solution satisfies the global balance equations.
- **NOTE**: the equation $n_1+n_2+\cdots+n_k = M$ has $\binom{M+k-1}{M} = \binom{M+k-1}{k-1}$ solutions (ways to allocate M items in k boxes).

Chapter 24
Matrix Geometric Method

24.1 Squared coefficient of variation

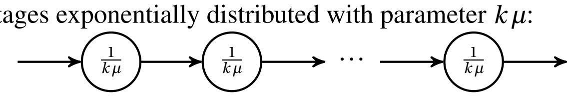
- **Idea**: almost all distributions can be approximated by a mixture of exponentials.
- **Squared coefficient of variation**, characterization of a distribution variability (for distributions with $E[X] > 0$):

$$\boxed{C_X^2 = \frac{\mathrm{Var}(X)}{E[X]^2} = \frac{E[X^2] - E[X]^2}{E[X]^2} = \frac{E[X^2]}{E[X]^2} - 1 \qquad (24.1)}$$

- **Deterministic** distribution: $C_D^2 = 0$.
- **Exponential** distribution: $E[X] = 1/\mu$, $\mathrm{Var}(X) = 1/\mu^2$. Thus $C_{\text{exp}}^2 = 1$.
- What if we want a distribution more *deterministic* than an exponential, $C_X^2 < 1$? or with larger variability, $C_X^2 > 1$?

24.2 $C_X^2 < 1$: Erlang-k

- k stages exponentially distributed with parameter $k\mu$:



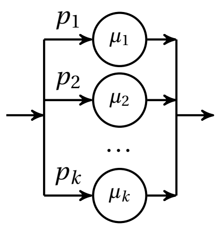
$$f_E(t) = \frac{(k\mu)^k t^{k-1} e^{-k\mu t}}{(k-1)!},\ t \geq 0, k \geq 1 \qquad (24.2)$$

$$E[\mathbf{X}] = k\frac{1}{k\mu} = \frac{1}{\mu}$$

$$\mathrm{Var}(\mathbf{X}) = k \times \mathrm{Var}(\exp(k\mu)) = k\frac{1}{(k\mu)^2} = \frac{1}{k\mu^2} \qquad (24.3)$$

$$\mathbf{C_X^2} = \frac{\mathrm{Var}[X]}{E[X]^2} = \frac{1}{k} < 1$$

24.3 $C_X^2 > 1$: Hyper-exponential



$$f_H(t) = \sum_{i=1}^{k} p_i \mu_i e^{-\mu_i t},\ \sum_{i=1}^{k} p_i = 1,\ t \geq 0$$

$$E[\mathbf{X}] = \sum_{i=1}^{k} p_i \frac{1}{\mu_i}, E[X^2] = \sum_{i=1}^{k} p_i \frac{2}{\mu_i^2}$$

$$\mathrm{Var}(\mathbf{X}) = E[X^2] - E[X]^2 =$$

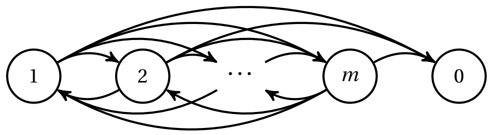
$$\sum_{i=1}^{k} p_i \frac{2}{\mu_i^2} - \left(\sum_{i=1}^{k} p_i \frac{1}{\mu_i}\right)^2 =$$

$$\left(\sum_{i=1}^{k} p_i \frac{1}{\mu_i}\right)^2 + \sum_{i=1}^{k}\sum_{j\neq i} p_i p_j \left(\frac{1}{\mu_i} - \frac{1}{\mu_j}\right)^2$$

$$\mathbf{C_X^2} = \frac{\mathrm{Var}[X]}{E[X]^2} > 1$$

24.4 Phase type distribution

- General mixture of exponentials.
- The service ends upon reaching the absorbing state.
- Can approximate arbitrary distributions.
- Representation: $PH(\mathbf{a}, \mathbf{T})$.



$$\mathbf{Q}^{m+1\times m+1} = \begin{bmatrix} \mathbf{T}^{m\times m} & \mathbf{c}^{m\times 1} \\ \mathbf{0}^{1\times m} & 0 \end{bmatrix} \qquad (24.4)$$

$$\text{Initial prob.} \begin{bmatrix} \mathbf{a}^{1\times m} & a_0 \end{bmatrix}. \qquad (24.5)$$

$$\begin{aligned} \mathbf{f_{PH}(t)} &= \mathbf{a}e^{\mathbf{T}t}\mathbf{c},\ t \geq 0 \\ E[\mathbf{X^k}] &= k!\,\mathbf{a}(-\mathbf{T}^{-1})^k\mathbf{e} \end{aligned} \qquad (24.6)$$

where $\mathbf{e}$ is a column vector of 1s.

24.5 Quasi Birth Death Processes

- Assume a two dimensional MC with **states** (n,i) (e.g. an **M/PH/1** queue). We call n the **level** and i the **phase**. We group the states of the **stationary distribution**:

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \cdots \end{bmatrix}$$

$$\begin{cases} \boldsymbol{\pi}_0 = \begin{bmatrix} (0,0) & (0,1) & \cdots (0,m') \end{bmatrix} & \textbf{initial part (level 0)} \\ \boldsymbol{\pi}_i = \begin{bmatrix} (i,1) & \cdots & (i,m) \end{bmatrix}, i \geq 1 & \textbf{repetitive part (level } i \geq 1\textbf{)} \end{cases} \qquad (24.7)$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{L}_0 & \mathbf{F}_0 & & & \\ \mathbf{B}_0 & \mathbf{L} & \mathbf{F} & & \\ & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ & & \mathbf{B} & \mathbf{L} & \cdots \\ & & & \cdots & \cdots \end{bmatrix} \tag{24.8}$$

$$\begin{cases} \mathbf{B} & \text{governs the transitions to previous level} \\ \mathbf{L} & \text{governs the change of phase inside a level} \\ \mathbf{F} & \text{governs the transitions to next level} \end{cases} \tag{24.9}$$

24.6 Matrix Geometric Solution

- Due to similarity with an M/M/1 ($\pi_i = \pi_0 \rho^i$) we guess for the repetitive part:

$$\boxed{\boldsymbol{\pi}_{i+1} = \boldsymbol{\pi}_1 \mathbf{R}^i,\ i \geq 0} \tag{24.10}$$

which gives:

$$\begin{aligned} &\boldsymbol{\pi}_1 \mathbf{F} + \boldsymbol{\pi}_2 \mathbf{L} + \boldsymbol{\pi}_3 \mathbf{B} = \mathbf{0} \Rightarrow \\ &\boldsymbol{\pi}_1 \mathbf{F} + \boldsymbol{\pi}_1 \mathbf{R}\mathbf{L} + \boldsymbol{\pi}_1 \mathbf{R}^2 \mathbf{B} = \mathbf{0} \Rightarrow \\ &\boxed{\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2 \mathbf{B} = \mathbf{0}} \end{aligned} \tag{24.11}$$

- Isolating $\mathbf{R}$ we have that it can be found iterating

$$\boxed{\mathbf{R}_{n+1} = -(\mathbf{F} + \mathbf{R}_n^2 \mathbf{B})\mathbf{L}^{-1},} \tag{24.12}$$

starting e.g. with $\mathbf{R}_0 = \mathbf{I}$.

- Better iterative algorithms can be found in [6].

```
##
## Basic iterative algorithm to compute the
   matrix R
##
## B, L, F: repetitive part matrices
##
invL <- -solve(L) # -1/L
C1 <- F %*% invL # -F/L
C2 <- B %*% invL # -B/L
R <- diag(nrow(B))
epsilon <- 1e-15
MaxIter <- 500
IterB <- 1
while(IterB < MaxIter) {
    prev <- R
    R <- C1 + R %*% R %*% C2 # -(F + R^2 B)/L
    if(max(abs(prev-R)) < epsilon) { break }
    IterB = IterB + 1
}
```

- Solving $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$:

$$\begin{aligned} &\mathbf{Q} = \begin{bmatrix} \mathbf{L}_0 & \mathbf{F}_0 & & & \\ \mathbf{B}_0 & \mathbf{L} & \mathbf{F} & & \\ & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ & & \mathbf{B} & \mathbf{L} & \cdots \\ & & & \cdots & \cdots \end{bmatrix} \\ &\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \boldsymbol{\pi}_3 & \cdots \end{bmatrix} \mathbf{Q} = 0 \\ &\boldsymbol{\pi}_{i+1} = \boldsymbol{\pi}_1 \mathbf{R}^i,\ i \geq 0. \end{aligned} \tag{24.13}$$

- Thus:

$$\left.\begin{aligned} &\boldsymbol{\pi}_0 \mathbf{L}_0 + \boldsymbol{\pi}_1 \mathbf{B}_0 = \mathbf{0} \\ &\boldsymbol{\pi}_0 \mathbf{F}_0 + \boldsymbol{\pi}_1 \mathbf{L} + \boldsymbol{\pi}_1 \mathbf{R}\mathbf{B} = \mathbf{0} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 \end{bmatrix} \begin{bmatrix} \mathbf{L}_0 & \mathbf{F}_0 \\ \mathbf{B}_0 & \mathbf{L} + \mathbf{R}\mathbf{B} \end{bmatrix} = \mathbf{0}} \tag{24.14}$$

- and the normalization condition:

$$\boldsymbol{\pi}_0 \mathbf{e}_0 + \sum_{i=0}^{\infty} \boldsymbol{\pi}_1 \mathbf{R}^i \mathbf{e}_1 = 1 \Rightarrow \boldsymbol{\pi}_0 \mathbf{e}_0 + \boldsymbol{\pi}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbf{e}_1 = 1 \Rightarrow$$

$$\boxed{\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_0 \\ (\mathbf{I} - \mathbf{R})^{-1} \mathbf{e}_1 \end{bmatrix} = 1} \tag{24.15}$$

where $\mathbf{e}_i$ are column vectors of 1s of appropriate size.

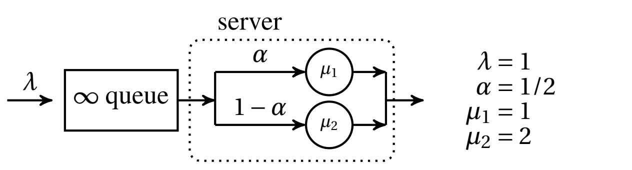
```
##
## Basic algorithm to compute the initial
   probabilities
##
## B0, L0, F0: initial part matrices
## B, L, F: repetitive part matrices
##
IMRinv <- solve(diag(nrow(B))-R) # 1/(I-R)
M0 <- rbind(cbind(L0, F0), cbind(B0, L + R %*% B)
   )

## Normalization column
NE <- c(rep(1, nrow(L0)), IMRinv %*% rep(1,nrow(
   IMRinv)))

## solve using the replace 1 equation method
M0 <- cbind(NE, M0[,2:ncol(M0)]) # replace first
   column of M0 by NE
stat <- solve(t(M0), c(1, rep(0, nrow(M0)-1)))
```

24.6.1 Example

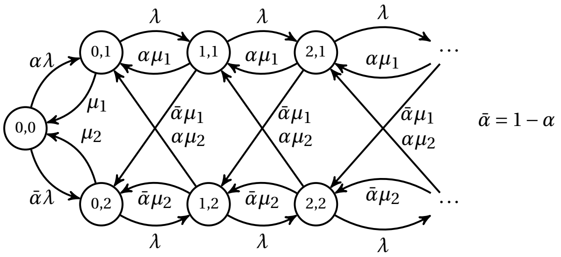
- Consider an M/G/1 queue where service time is hyper-exponentially distributed:



- Derive the rate matrix, Q, ordering the states lexicographically. Identify the states that form the initial and repetitive part. Identify the submatrices that would be used for a matrix geometric solution: $\mathbf{B}_0$, $\mathbf{L}_0$, $\mathbf{F}_0$, $\mathbf{B}$, $\mathbf{L}$, $\mathbf{F}$.

- Solve the Chain using the matrix geometric method. Compute the number in the system. Check it with the PK formula.

Solution



$$
\mathbf{P} = \begin{array}{c}
\begin{matrix} 0,0 & 0,1 & 0,2 & 1,1 & 1,2 & 2,1 & 2,2 & \cdots \end{matrix} \\
\left[\begin{array}{c|cc|cc|cc|c}
-\lambda & \alpha\lambda & \bar{\alpha}\lambda & 0 & 0 & 0 & 0 & \cdots \\ \hline
\mu_1 & -(\lambda+\mu_1) & 0 & \lambda & 0 & 0 & 0 & \cdots \\
\mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & \cdots \\ \hline
0 & \alpha\mu_1 & \bar{\alpha}\mu_1 & -(\lambda+\mu_1) & 0 & \lambda & 0 & \cdots \\
0 & \alpha\mu_2 & \bar{\alpha}\mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & \cdots \\ \hline
0 & 0 & 0 & \alpha\mu_1 & \bar{\alpha}\mu_1 & -(\lambda+\mu_1) & 0 & \cdots \\
0 & 0 & 0 & \alpha\mu_2 & \bar{\alpha}\mu_2 & 0 & -(\lambda+\mu_2) & \cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}
$$

$$\Rightarrow \mathbf{B}_0 = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathbf{L}_0 = \begin{bmatrix} -\lambda \end{bmatrix}, \mathbf{F}_0 = \begin{bmatrix} \alpha\lambda & \bar{\alpha}\lambda \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \alpha\mu_1 & \bar{\alpha}\mu_1 \\ \alpha\mu_2 & \bar{\alpha}\mu_2 \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} -(\lambda+\mu_1) & 0 \\ 0 & -(\lambda+\mu_2) \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

- Iterating $\mathbf{R}_{n+1} = -(\mathbf{F} + \mathbf{R}_n^2\,\mathbf{B})\,\mathbf{L}^{-1}$ we get:

$$\mathbf{R} = \begin{bmatrix} 5/7 & 1/7 \\ 1/7 & 3/7 \end{bmatrix}$$

- Using $\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 \end{bmatrix} \begin{bmatrix} \mathbf{L}_0 & \mathbf{F}_0 \\ \mathbf{B}_0 & \mathbf{L} + \mathbf{R}\mathbf{B} \end{bmatrix} = \mathbf{0}$, $\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_0 \\ (\mathbf{I}-\mathbf{R})^{-1}\,\mathbf{e}_1 \end{bmatrix} = 1$ we get:

$$\begin{bmatrix} \boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/28 & 1/14 \end{bmatrix}$$

- Number in the system:

$$N = \sum_{n=1}^{\infty} n\,\boldsymbol{\pi}_1\,\mathbf{R}^n\,\mathbf{e}_1 = \boldsymbol{\pi}_1\,(\mathbf{I}-\mathbf{R})^{-2}\,\mathbf{e}_1 = \frac{13}{4}$$

- Using the PK Formula:

$$\mathrm{E}[S] = \frac{\alpha}{\mu_1} + \frac{1-\alpha}{\mu_2} = \frac{1}{4}, \rho = \lambda\,\mathrm{E}[S] = \frac{3}{4}, \mathrm{E}[S^2] = \frac{2\,\alpha}{\mu_1^2} + \frac{2\,(1-\alpha)}{\mu_2^2} = \frac{5}{4}$$

thus,

$$T = \mathrm{E}[S] + \frac{\lambda\,\mathrm{E}[S^2]}{2\,(1-\rho)} = \frac{13}{4}, N = \lambda\,T = \frac{13}{4}, \text{ as expected.}$$

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Table with some distributions

Distribution	Parametres	Density	Mean	Variance	Characteristic Function
Bernoulli	$0 \le p \le 1$ $q = 1-p$	$p^k\,(1-p)^{1-k}$ $k = 0,1$	p	$p\,(1-p)$	$q + p\,\mathrm{e}^{i\,t}$
Binomial	$0 \le p \le 1$ $q = 1-p$	$\binom{n}{k} p^k\,(1-p)^{n-k}$ $k = 0,1,\dots n$	$n\,p$	$n\,p\,(1-p)$	$(q + p\,\mathrm{e}^{i\,t})^n$
Geometric	$0 \le p \le 1$ $q = 1-p$	$p\,(1-p)^k$ $k \ge 0$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-q\,\mathrm{e}^{i\,t}}$
Negative binomial	$r > 0$ $0 \le p \le 1$ $q = 1-p$	$\binom{k+r-1}{k} p^r\,q^k$ $k \ge 0$	$r\,\frac{1-p}{p}$	$r\,\frac{1-p}{p^2}$	$\left(\frac{p}{1-q\,\mathrm{e}^{i\,t}}\right)^r$
Poisson	$\lambda > 0$	$\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}, \quad k \ge 0$	λ	λ	$\exp\left\{\lambda\,(\mathrm{e}^{i\,t}-1)\right\}$
Normal $N(\mu,\sigma)$	$\mu \in \mathbb{R}$ $\sigma > 0$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$ $x \in \mathbb{R}$	μ	σ^2	$\exp\left\{\mu\,i\,t - \frac{t^2\,\sigma^2}{2}\right\}$
Uniform	$a < b$	$\frac{1}{b-a}, \quad a \ge x \ge b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{\mathrm{e}^{i\,t\,b} - \mathrm{e}^{i\,t\,a}}{i\,t\,(b-a)}$
Exponential	α	$\alpha\,\mathrm{e}^{-\alpha\,x}, \quad x \ge 0$	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2}$	$\left(1 - \frac{i\,t}{\alpha}\right)^{-1}$
Gamma $\gamma(n,\alpha)$	$\alpha > 0$, $n > 0$	$\frac{\alpha^n\,x^{n-1}\,\mathrm{e}^{-\alpha\,x}}{\Gamma(n)}, \quad x \ge 0$	$\frac{n}{\alpha}$	$\frac{n}{\alpha^2}$	$\left(1 - \frac{i\,t}{\alpha}\right)^{-n}$
Beta $\beta(p,q)$	$p > 0$, $q > 0$	$\frac{x^{p-1}\,(1-x)^{q-1}}{\mathrm{B}(p,q)}$, $0 \ge x \ge 1$	$\frac{p}{p+q}$	$\frac{p\,q}{(p+q)^2(p+q+1)}$	

$$\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt, \Gamma(n) = (n-1)! \quad \mathrm{B}(p,q) = \int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt, \mathrm{B}(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$