Radomized Algorithms Problems (29-34) Fall 2019. (Due Thurs. Dec. 10)

29.- (MU 7.1) Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix:

$$P = \begin{bmatrix} 0 & 3/10 & 1/10 & 3/5 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/10 & 7/10 & 1/10 & 1/10 \\ 9/10 & 1/10 & 0 & 0 \end{bmatrix}$$

(a) Find the stationary distribution of the Markov Chain.

(b) Find the probability of being in state 3 after 32 steps if the chain begins at state 0.

(c) Find the probability of being in state 3 after 128 steps if the chain begins at a state chosen uniformly at random from the four states.

(d) Suppose that the chain begins in state 0. What is the smallest value of t for which $\max_i |P^t_{0,i} - \pi^*_i| \leq 0.01$? , where π^*_i is the stationary distribution.

30.- A fair coin is tossed repeatedly and independently. Use a Markov chain to find the expected number of tosses until the pattern HTH appears.

31.- Discuss the irreducibility and the periodicity of the following Markov chains:

$$(a)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix};\ (b)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix};\ (c)\ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$(d)\ P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 0 \end{pmatrix};\ (e)\ P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix};$$

32.- * I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

(a) If the probability of rain is p, what is the probability that I get wet?

(b) If the current forecast shows a $p = 0.6$, how many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

(*Hint: Use a MC*)

33.- * (MU 7.22) A cat and a mouse take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $(m^2 n)$ on the expected time before the cat eats the mouse. (Hint: Consider a Markov chain whose states are the ordered pair (a, b), where a is the position of the cat and b is a position of the mouse.)

34.- * Consider the 3-SAT algorithm, Given a set a Boolean variables $X = \{x_1, \ldots x_n\}$, and conjunctive normal form (CNF) formula ϕ with m clauses, where each clause has exactly 3 literals, find a truth assignment $A : \phi \to \{0, 1\}$ such that $A(\phi) = 1$. For example, in $\phi = (x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_4 \vee x_3)$ a possible truth assignment is $A(x_1) = 1, A(x_2) = 1, A(x_3) = 0, A(x_4) = 0$. A difference with the 2-SAT, the 3-SAT is NP-complete. Consider the following extension of the 2-SAT algorithm to 3-SAT (using a Markov Chain $Y = \{Y_0, Y_1, Y_2, \ldots\}$) to find an assignment A^*, if there is one.

> Given 3-SAT input ϕ on $X = \{x_i\}_{i=1}^n$ and $\{C_j\}_{j=1}^m$
> Make $\forall 1 \le i \le n$ $A(x_i) = 1$
> **while** ϕ contains unsatisfied clause and $\le c$ steps **do**
>   pick and unsatisfied clause C_j
>   choose u.a.r. one of the 2 literals and flip their value
>   **if** ϕ is satisfied now **OUTPUT** the new truth assignment
> **end while**
> **OUTPUT** ϕ is unsatisfiable

As in the 2-SAT case we should start by bounding the expected number of steps to find a satisfying assignment if ϕ is satisfiable, unfortunately we get that in expectation, it takes $c = \Theta(2^n)$ steps. Because the Markov Chain Y has more probability to move towards 0 than towards n, and the larger c is, the more quickly it will move towards 0.

Prove that if h_j is the expected number of steps to reach absorbing state n when starting from any state j, then $h_j = \Theta(2^n)$.

34.- * (3-SAT continuation) Let us consider a different strategy for the design of a Markov Chain to randomly get an optimal assignment A^* to solve the 3-SAT problem: If the does not find the assignment in a small number of steps, reset the process again. The following Monte-Carlo algorithm with probability p finds a satisfying assignment A^* from an initial random assignment A_1, providing the input has such a satisfying assignment A^*, with $p \ge \frac{a}{\sqrt{n}}(\frac{3}{4})^n$, and constant $a = \frac{\sqrt{3}}{8\sqrt{\pi j}}$. Notice the number of generated A_1 the algorithm tries, before finding a A^*, follows a geometric distribution $G(p)$.

Given 3-SAT input ϕ on $X = \{x_i\}_{i=1}^n$ and $\{C_j\}_{j=1}^m$
for repeat c steps **do**
  Chose u.a.r. an A_1
  for $\leq 3n$ steps **do**
    while ϕ contains unsatisfied clause **do**
      pick and unsatisfied clause C_j
      choose u.a.r. one of the 3 literals and flip their value
      if ϕ is satisfied now **then**
        OUTPUT A^* and stop
      end if
    end while
  end for
end for
OUTPUT ϕ is unsatisfiable

Let $p =$ the probability that the algorithm starting from a random A_1, gets optimal assignment A^* in $\leq 3n$ steps. Let p_j be the probability that the algorithm obtains A^* in $\leq 3n$ steps, starting from a random A_1 that has j variables assigned differently of A^*, that is we start with a *distance between assignments* A_1 and the optimal A^* of j different positions, we should denote that as $A^* \ominus A_1 = j$.

(a) Prove that each p_i we get the bound $p_i \geq \frac{\sqrt{3}}{8\sqrt{\pi j}} \frac{1}{\sqrt{j}2^j}$. (Hint:define the events F_i that we move $j + 2i$ steps, of which exactly i steps do not increase the distance between assignments)

(b) Give a lower bound for the probability p. (Hint: Define the eents H_j that $A^* \ominus A_1 = j$ for $0 \leq j \leq n$.)

34.- * The lollipop graph on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices, as shown in the figure below. The node u is a part of both the clique and the path. Let v denote the other end of the path.

(a) Show that the expected covering time of a random walk starting at v is $O(n^2)$.

(b) Show that the expected covering time for a random walk starting at u is $O(n^3)$.

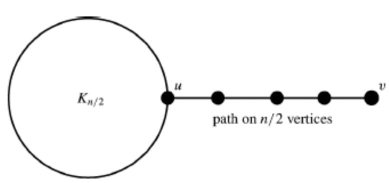

Figure 1: Lollipop