Network Creation Games

Fall 2020

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- A strategy s_u of player u is a subset $s_u \in V - \{u\}$ that represents the set of nodes for which u pays for a link.
- As a result of a strategy profile $s = (s_1, ..., s_n)$ a graph $G(s) = (V, E)$ is created so that $E = \{(u, v) | u \in s_v \vee v \in s_u\}$

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- n players, think of them as vertices $V = \{1, ..., n\}$ in an undirected graph
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- The goal of the player u is to minimize a cost function:

$$c_u(s) = \text{creation cost} + \text{usage cost}$$

Individual cost

Let $s = (s_1, \ldots, s_n)$ and $G = G(s)$. The cost for player u:

$$c_u(s) = \text{creation cost } + \text{usage cost}$$

- Creation cost: $\alpha|s_u|$
- Usage cost:
  - SumGame (Fabrikant et al. PODC 2003)
    Sum over all distances: $\sum_{v \in V} d_G(u, v)$
    This is an average-case approach to the usage cost
  - MaxGame (Demaine et al. PODC 2007)
    Maximum over all distances: $\max_{v \in V} d_G(u, v)$
    A worst-case approach to the usage cost

$$\text{SumGame} : c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u, v)$$

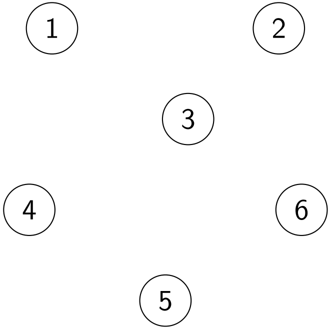
Social cost

- Creation cost: $\alpha|E(G)|$
- Usage cost:
  - SumGame: $\sum_{u,v\in V} d_G(u,v)$
  - MaxGame: $\max_{u,v\in V} d_G(u,v)$

$$\text{SumGame: } C(s) = \alpha|E| + \sum_{u,v\in V} d_G(u,v)$$

$$\text{MaxGame: } C(s) = \alpha|E| + \max_{u,v\in V} d_G(u,v)$$

An example



An example

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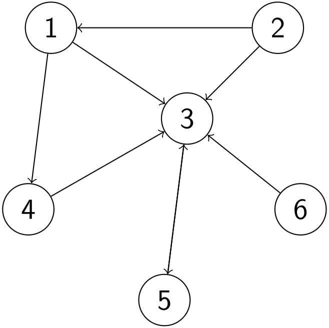
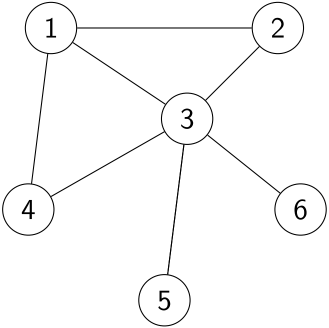


An arrow indicates who bought the edge

An example

$s = (\{3,4\},\{1,3\},\{5\},\{3\},\{3\},\{3\})$ and $G(s)$



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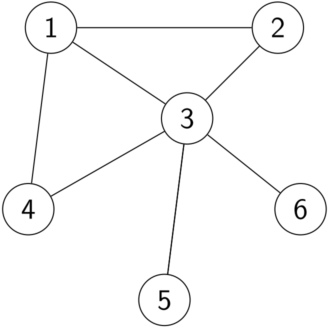

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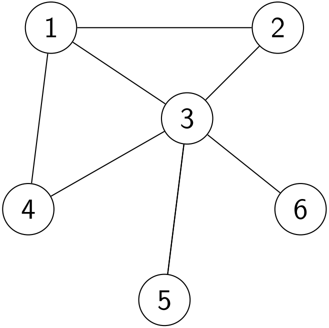

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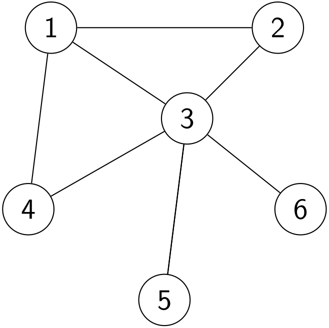
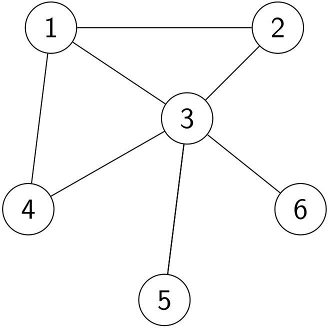


$c_1(s) = 2\alpha + 1 + 1 + 1 + 2 + 2 = 2\alpha + 7 \ldots$
$c(s) = 7\alpha + (7 + 8 + 5 + 8 + 9 + 9) = 7\alpha + 56$

An example: MaxGame

$s = (\{3,4\},\{1,3\},\{5\},\{3\},\{3\},\{3\})$ and $G(s)$



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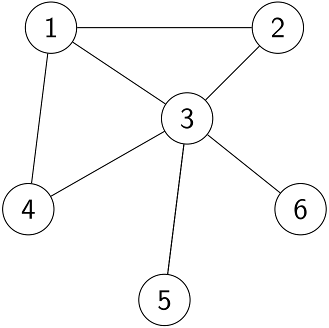

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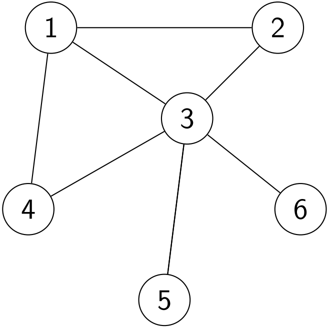

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$c(s) = 7\alpha + 2$

What to study?

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- What are the social optima?
- What network topologies are formed? What families of equilibrium graphs can one construct for a given α?
- How efficient are they? Price of Anarchy/Stability?

We will cover some results on SumGames

1. General model
2. Sum Game

Optimal/Equilibrium topologies

$$c_u(s) = \alpha|s_u| + \sum_{v \in V} d_G(u, v)$$

$$C(s) = \alpha|E| + \sum_{u,v \in V} d_G(u, v)$$

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- When is it better to add/remove an edge?

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- Can an edge be created by more than two players? No
- We study them as a function of α
- When is it better to add/remove an edge?
- Can the graph be disconnected? No,
  $c_u(s) = \infty$ if $G(s)$ is not connected

Add an edge?

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- When is it better to add an edge?
- Set $d = d_G(u, v) > 1$ and let ${s'}_u = s_u \cup \{v\}$

$$c_u(s_{-u}, {s'}_u) - c_u(s) = \alpha + 1 - d + \sum_{w \in V, w \neq u} (d_{G'}(u, w)) - d_G(u, w))$$

$$\leq \alpha + 1 - d \leq 0$$

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- $d > \alpha$

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- $d > \alpha$ which implies Nash topologies have diameter $\leq \alpha$.

Computing a Best Response

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  - The game has $n + 1$ players $\{0, 1, \ldots, n\}$, choose α so that $1 < \alpha < 2$, the player will be player 0. The strategy is defined as follows:
  - Compute an orientation of E and define s_{-0} accordingly.

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  - Compute an orientation of E and define s_{-0} accordingly.
- As $1 < \alpha < 2$, player 0 will like to buy edges to any player at distance > 2.
- So, in the BR graphs the radius of vertex 0 must be ≤ 2.
- Hence, $c_0(s_{-0}, s_0') = (\alpha + 1)|s_0'| + 2(n - |s_0'|)$

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- To get radius ≤ 2, $s'_0 \subseteq V$ must be a dominating set of $G = (V, E)$.
- The BR strategies for vertex 0 are the dominating sets of G having minimum size.
- Computing a minimum size dominating set is NP-hard, so
- Computing a BR in the sum game is NP-hard

Optimal topologies

$$c(s) = \alpha|E| + \sum_{u,v \in V} d_G(u, v)$$

- When two vertices u, v are not adjacent $d_G(u, v) \geq 2$.
- When two vertices u, v are adjacent $d_G(u, v) = 1$.
- Therefore for any strategy profile s,

$$C(s) = \alpha|E| + \sum_{u,v \in V} d_G(u, v) \geq \alpha|E| + 2(\sum_{u,v \in V} 1 - |E|)$$

$$C(s) \geq \alpha|E| - 2|E| + 2n(n-1) = 2n(n-1) + (\alpha - 2)|E|$$

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- Holds with equality on graphs with diameter ≤ 2.

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- This function has different minima depending on whether $(\alpha - 2)$ is positive or negative.
- When $\alpha = 2$, the optimal cost is independent of the number of edges in the graph. So,
- For $\alpha = 2$, any graph with diameter ≤ 2 has optimal cost.

Optimal topologies

- For all s,
  $C(s) \geq 2n(n-1) + (\alpha - 2)|E| \geq 2n(n-1) + (\alpha - 2)(n-1)$
- When $\alpha > 2$, to make the cost minimum we have to take the minimum number of edges in G. Of course the graph must be connected. So,

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- For $\alpha > 2$, the star S_n is the unique optimal topology.

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- When $\alpha < 2$, to make the cost minimum we have to take the maximum number of edges in G. Then,
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Nash topologies

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- K_n (Clique) is the unique Nash topology for $\alpha < 1$
- S_n (Star) is a Nash topology for $\alpha \geq 1$
  although they might be other PNE

PoA: $\alpha < 1$

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- $PoA = PoS = 1$

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- Any Nash equilibrium must have diameter ≤ 2, so S_n is a Nash topology with the worst social cost.

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$$PoA = \frac{c(S_n)}{c(K_n)} = \frac{(n-1)(\alpha - 2 + 2n)}{n(n-1)\frac{\alpha-2}{2} + 2}$$
$$= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} < \frac{4}{2+\alpha} \leq \frac{4}{3}$$

PoA: $\alpha > n^2$

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$$PoA = \frac{c(T_n)}{c(S_n)} \leq \frac{\alpha(n-1) + (n-1)(n-1)}{\alpha(n-1) + 1 + 2n(n-1)} = O(1)$$

General Upper bound for PoA: $\alpha < n^2$

- For a worst NE topology G,
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- Furthermore, $|E| = O(\frac{n^2}{\sqrt{\alpha}})$ (see [Fabrikant et al. 2003])
- $C(G) \leq \alpha O(\frac{n^2}{\sqrt{\alpha}}) + n(n-1)2\sqrt{\alpha} = O(\sqrt{\alpha}n^2)$

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- $d_G(u,v) < 2\sqrt{\alpha}$, otherwise u will be willing to connect to the node in the center of the shortest path from u to v to be closer by $-\sqrt{\alpha}$ to $\sqrt{\alpha}$ nodes.
- Furthermore, $|E| = O(\frac{n^2}{\sqrt{\alpha}})$ (see [Fabrikant et al. 2003])
- $C(G) \leq \alpha O(\frac{n^2}{\sqrt{\alpha}}) + n(n-1)2\sqrt{\alpha} = O(\sqrt{\alpha}n^2)$
- $C(S_n) = \Omega(n^2)$

General Upper bound for PoA: $\alpha < n^2$

- For a worst NE topology G,
  $C(G) = \alpha|E| + \sum_{u,v \in V} d_G(u,v)$
- $d_G(u,v) < 2\sqrt{\alpha}$, otherwise u will be willing to connect to the node in the center of the shortest path from u to v to be closer by $-\sqrt{\alpha}$ to $\sqrt{\alpha}$ nodes.
- Furthermore, $|E| = O(\frac{n^2}{\sqrt{\alpha}})$ (see [Fabrikant et al. 2003])
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- $C(S_n) = \Omega(n^2)$
- Thus $PoA = O(\sqrt{\alpha})$

PoA: Conjectures

PoA on trees ≤ 5 [Fabrikant et al. 2003]

Constant PoA conjecture: For all α, $PoA = O(1)$.

Tree conjecture: for all $\alpha > n$, all NE are trees.

$O(1)$ PoA conjecture: large α

$PoA = O(1)$	
$\alpha > n^{\frac{3}{2}}$	[Lin 2003]
$\alpha > 12n \log n$	[Albers et al. 2014]
$\alpha > 273n$	[Mihalak, Schlegel, 2013]
$\alpha > 65n$	[Mamageishivii et al. 2015]
$\alpha > 17n$	[Àlvarez, Messegue 2017]
$\alpha > 4n - 13$	[Bilo, Lezner 2018]
$\alpha > (1 + \epsilon)n$	[Àlvarez, Messegue 2019]

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[Àlvarez, Messegue 2019] On the price of Anarchy for High-Price links, *15th Conference on Web and Internet Economics*, WINE 2019, 316– 329

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$O(1)$ PoA conjecture: small α

$PoA = O(1)$	
$\alpha = O(1)$	[Fabrikant et al. 2003]
$\alpha = O(\sqrt{n})$	[Lin 2003]
$\alpha = O(n^{1-\delta})$, $\delta \geq 1/\log n$	[Demaine et al. 2007]