Homework 4 - Exercises Cooperative Game Theory

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Exercise 2

- Consider a cooperative game which is defined on an undirected graph $G = (V, E)$
- The players are the vertices in the graph
- $S \subseteq V, v(S) = |\{u \in V | N(u) \cap S \neq \emptyset\}|$
- As usual $N(u) = \{v | (u, v) \in E\}$.

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a. Is the valuation function monotone?
   - Let $C \subseteq D$, then

$$\{u \in V | N(u) \cap C \neq \emptyset\} \subseteq \{u \in V | N(u) \cap D \neq \emptyset\}$$

   - It follows that $v(C) \subseteq v(D)$ so $v(\cdot)$ is monotone!

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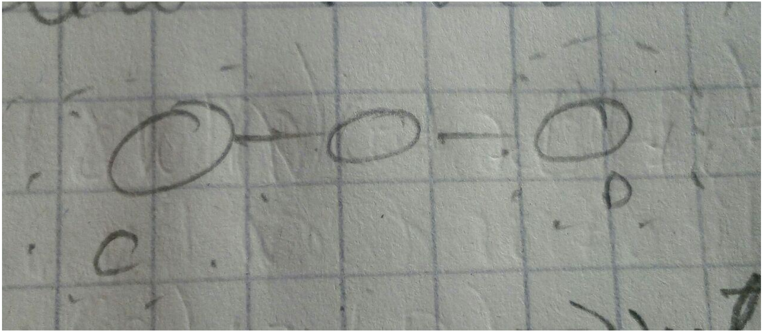
a. Is the valuation function superadditive?



- Clearly no, as

$$v(C \cup D) = 1 \ngeq 2 = 1 + 1 = v(C) + v(D)$$

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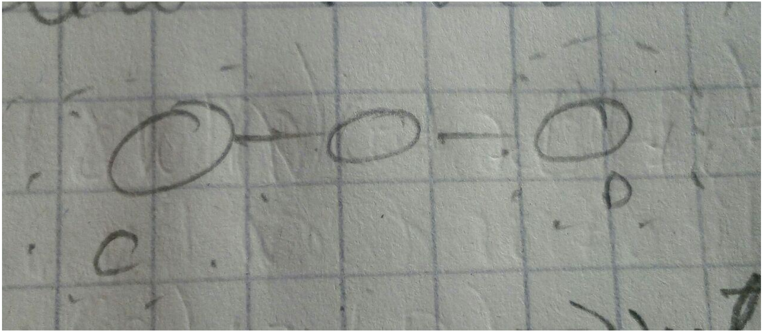
a. Is the valuation function supermodular?



- Clearly no, as

$$v(C \cup D) + v(C \cap D) = 1 + 0 \not\geq 2 = 1 + 1 = v(D) + v(C)$$

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b. Is the core empty? Can this property be decided in polynomial time?

  - Let x_i be the payoff for every node v_i, s.t. $x \in$ Core:

$$\sum_{i=1}^{n} x_i = n, \;\; x_i \geq \deg(v_i), \;\; \sum_{i=1}^{n} \deg(v_i) = 2|E|$$

  - We obtain then following condition on G if the core is non-empty:

$$n = \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} \deg(v_i) = 2|E| \Rightarrow n/2 \geq |E|$$

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b. Is the core empty? Can this property be decided in polynomial time?
   - So the core is not empty if $n/2 \geq |E|$
   - This is, if there is a node with degree ≥ 2, the core is empty
   - This is decidable in polynomial time by iterating on all nodes and edges