Global Constraints

Combinatorial Problem Solving (CPS)

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  at most one of the Boolean variables $x_1, \ldots, x_n$ is set to true.

- The dual graph translation does not work well in practice.

AC for Non-binary Problems

- Can be naturally extended from the binary case
- Value $a \in d_i$ is AC wrt. (non-binary) constraint $c \in C$ iff there exists an assignment τ (the support of a) such that:
  - τ assigns a value to exactly the variables in $\mathrm{scope}(c)$
  - $\tau[x_i] = a$
  - $c(\tau)$ holds
- Constraint $c \in C$ is AC iff every $a \in d_i$ of every $x_i \in \mathrm{scope}(c)$ has a support in c
- A CSP is AC if all its constraints are AC
- For non-binary constraints, arc consistency is also called hyperarc consistency, generalized arc consistency or domain consistency

Example

- Consider the constraint $3x + 2y + z > 3$ over $x, y, z \in \{0, 1\}$
- Value 1 for x is AC: $\tau = (x \mapsto 1, y \mapsto 1, z \mapsto 1)$ is a support
- Value 0 for x is not AC: it does not have any support.
- Hence, the constraint is not AC

Example

- Note that AC depends on the syntax
- Consider $x_1 \in \{1, 2\}$, $x_2 \in \{1, 2\}$, $x_3 \in \{1, 3, 4\}$
- Case 1: constraints are $x_i \neq x_j$ for all $i < j$
  - All constraints are arc-consistent
- Case 2: there is only one constraint $\texttt{alldiff}(x_1, x_2, x_3)$
  - Value 1 for x_1 is AC
    because $\tau = (x_1 \mapsto 1, x_2 \mapsto 2, x_3 \mapsto 3)$ is a support for it.
  - Value 1 for x_3 is not AC: does not have any support
  - Hence, the constraint is not AC

Enforcing AC: $\texttt{Revise}(i, c)$

- Natural extension of binary case
- Removes values from the domain of x_i without a support in c

// Let $(x_1, \ldots, x_{i-1}, x_i, x_{i+1} \ldots, x_k)$ be the scope of c
function $\texttt{Revise}(i, c)$
  change $:= \text{false}$
  for each $a \in d_i$ **do**
  if $\forall_{a_1 \in d_1, \ldots, a_{i-1} \in d_{i-1}, a_{i+1} \in d_{i+1}, \ldots, a_k \in d_k} \neg c(x_1 \leftarrow a_1, \ldots, x_i \leftarrow a, \ldots, x_k \leftarrow a_k)$
    remove a **from** d_i
    change $:= \text{true}$
  return change

- The time complexity of $\texttt{Revise}(i, c)$ is $O(k \cdot |d_1| \cdots |d_k|)$
(assuming that evaluating a constraint takes linear time in the arity)

AC-3

- The natural extension of binary AC-3
- $(i, c) \in Q$ means that
  "we cannot guarantee that all domain values of x_i have a support in c"

procedure $\texttt{AC3}(X, D, C)$
$\quad Q := \{(i, c) \mid c \in C, x_i \in \text{scope}(C)\}$
$\quad$ **while** $Q \neq \emptyset$ **do**
$\quad\quad (i, c) := \texttt{Fetch}(Q)$ // selects and removes
$\quad\quad$ **if** $\texttt{Revise}(i, c)$ **then**
$\quad\quad\quad Q := Q \cup \{(j, c')|\;\; c' \in C, c' \neq c, j \neq i, \{x_i, x_j\} \subseteq \text{scope}(c')\}$

- Let $m = \max_i\{|d_i|\}$, $e = |C|$ and $k = \max_c\{|\text{scope}(c)|\}$
- Time complexity: $O(e \cdot k^3 \cdot m^{k+1})$
- Space complexity: $O(e \cdot k)$

AC for non-binary constraints

- Enforcing AC with generic algorithms is exponentially expensive in the maximum arity of the CSP
- Only practical with constraints of very small arity
- Is it possible to develop constraint-specific algorithms?

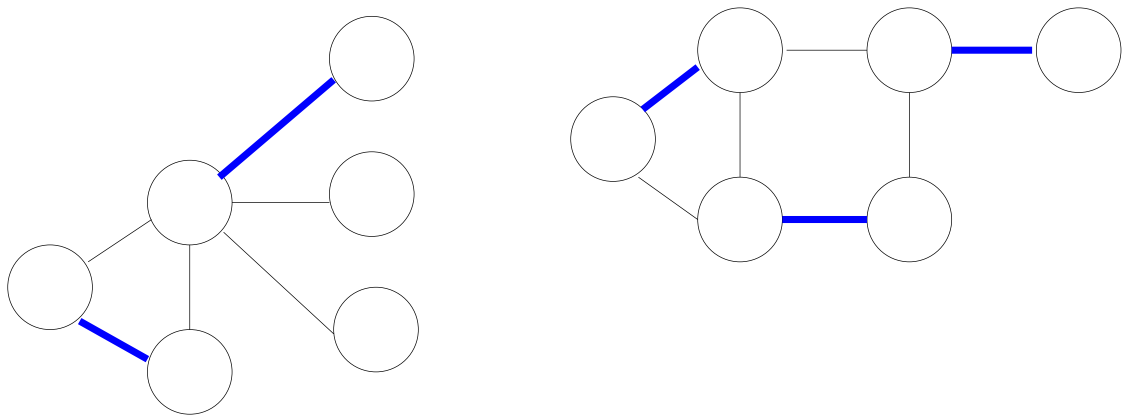
procedure `Revise`(c)
// removes every arc-inconsistent value $a \in d_i$ for all $x_i \in X(c)$
endprocedure

- Next: `alldiff` constraint
- ... but first a diversion to matching theory

Begin Matching Theory

Definitions

- Given a graph $G = (V, E)$,
  a matching M is a set of pairwise non-incident edges
- A vertex is matched or covered if it is an endpoint of some $e \in M$,
  and it is free otherwise
- A maximum matching is a matching
  that contains the largest possible number of edges

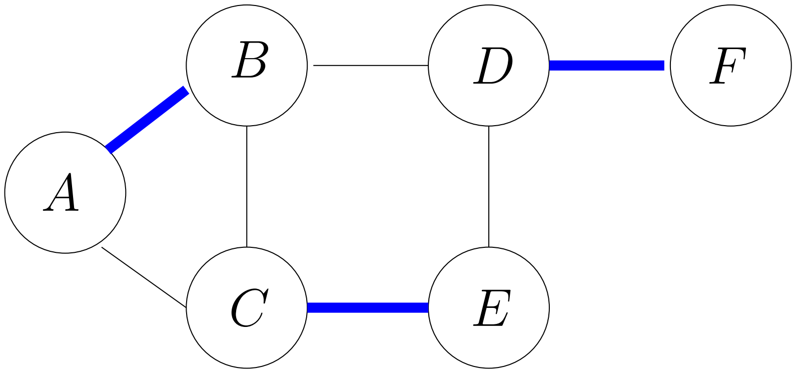


(edges in the matching, in blue)

- In particular, a perfect matching matches all vertices of the graph

Example

- We have to organize one round of a football league.
  Compatibility relation between teams is given by a graph



Perfect matchings $\leftrightarrow$ feasible arrangements of matches

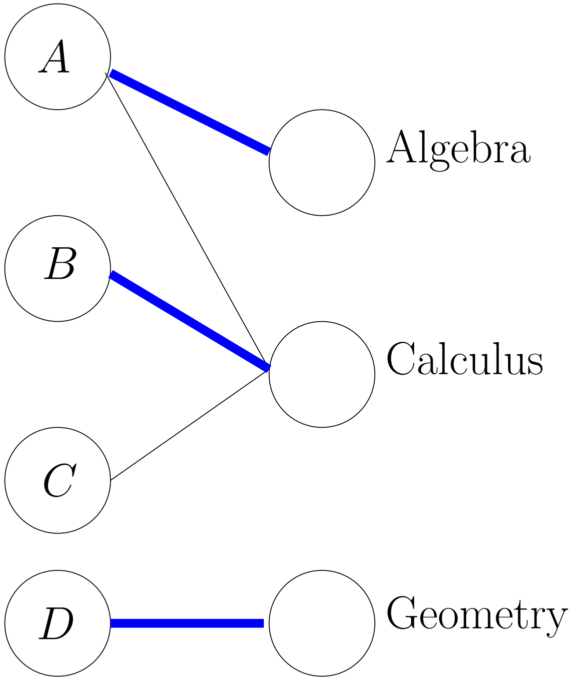
Bipartite Matching

- Graph $G = (V, E)$ is bipartite
  if there is a partition (L, R) of V (i.e., $L \cup R = V, L \cap R = \emptyset$)
  such that each $e \in E$ connects a vertex in L to one in R
- Now focus on maximum bipartite matching problem:
  given a bipartite graph, find a matching of maximum size
- From now on, assume $|V| \leq 2|E|$
  (isolated vertices can be removed)

Example (I)

- Assignment problem:
  - n workers, m tasks
  - list of pairs (w, t) meaning: "worker w can do task t"

  Maximum matchings tell how to assign tasks to workers so that the maximum number of tasks are carried out



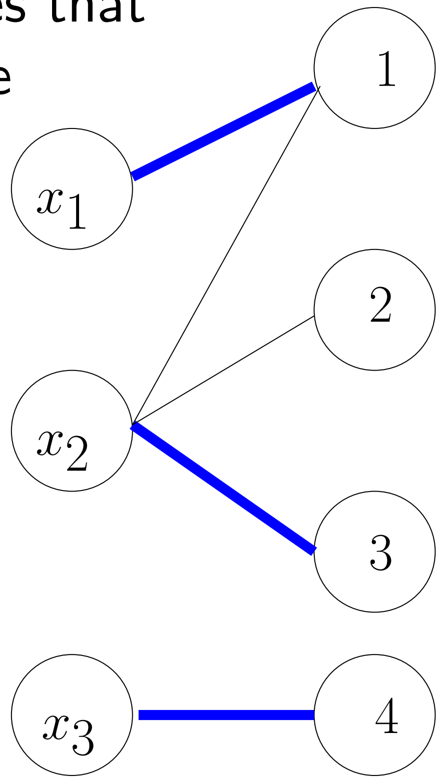
Example (II)

- We have n variables $x_1, ..., x_n$

  Variable x_i can take values in $D_i \subseteq \mathbb{Z}$ finite $(1 \leq i \leq n)$

  Constraint $\texttt{alldifferent}(x_1, ..., x_n)$ imposes that variables should take different values pairwise

$$
\begin{array}{rcl}
D_1 & = & \{1\} \\
D_2 & = & \{1, 2, 3\} \\
D_3 & = & \{4\}
\end{array}
$$



Matchings covering $x_1, \ldots, x_n$ correspond to solutions to $\texttt{alldifferent}(x_1, ..., x_n)$

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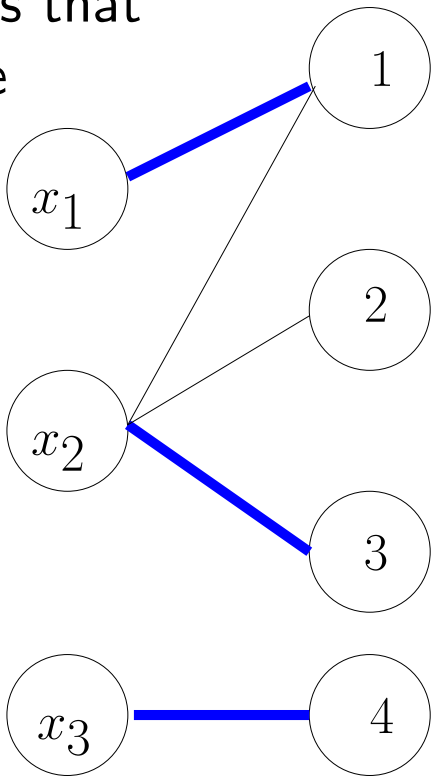


  Matchings covering $x_1, \ldots, x_n$ correspond to solutions to $\texttt{alldifferent}(x_1, ..., x_n)$

- Note that matchings covering $x_1, \ldots, x_n$ are maximum. However, a maximum matching may not cover $x_1, \ldots, x_n$

Augmenting Paths

Let M be a matching of $G = (V, E)$ **(not necessarily bipartite)**.

We view paths as sequences of edges rather than sequences of vertices.

- An alternating path is
  a simple path in which the edges belong alternatively to M and not to M.

- An alternating cycle is
  a cycle in which the edges belong alternatively to M and not to M.

- An augmenting path is
  an alternating path that starts and ends at different free vertices.

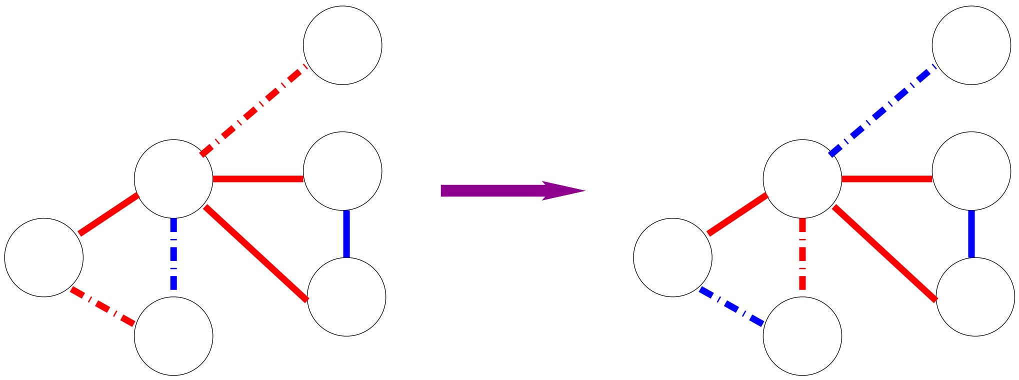
- **Berge's Lemma**. A matching is maximum if and only if
  it does not have any augmenting path.

Properties (I)

- An alternating cycle has as many edges in M as not in M
- An augmenting path has 1 more edge not in M than in M
- Given two sets $A, B \subseteq X$:
  - their difference is $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
  - their symmetric difference is $A \oplus B = (A - B) \cup (B - A)$

If P is an augmenting path wrt. M,
then $M \oplus P$ is a matching and $|M \oplus P| = |M| + 1$

I.e., if we paint edges $\in M$ in blue and edges $\notin M$ in red, then flipping the colors of P results in a valid matching



Proof of Berge's Lemma (I)

Let us prove the contrapositive:

G has a matching larger than M if and only if
G has an augmenting path wrt. M

$(\Leftarrow)$ Just proved in the last slide.

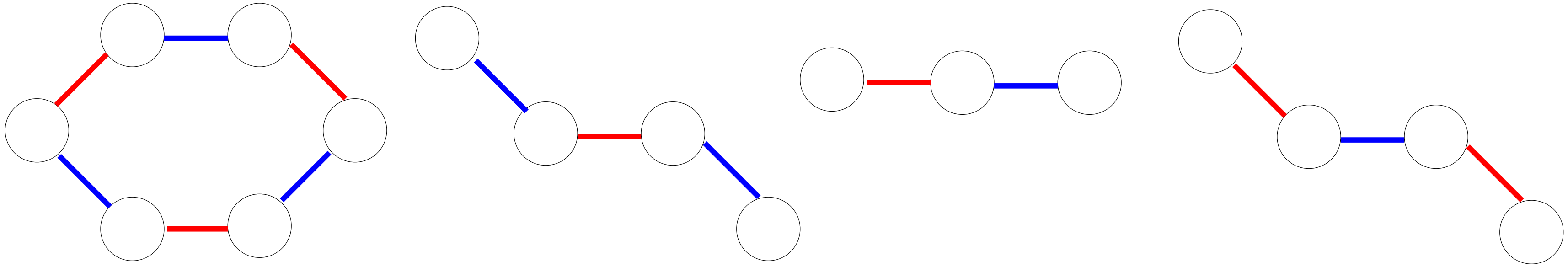
Proof of Berge's Lemma (II)

($\Rightarrow$) Let M' be a matching in G larger than M.

Each vertex of $M \oplus M'$ has degree at most two:
incident with ≤ 1 edge from M and ≤ 1 edge from M'

So $M \oplus M'$ is a vertex-disjoint union of simple paths and cycles.

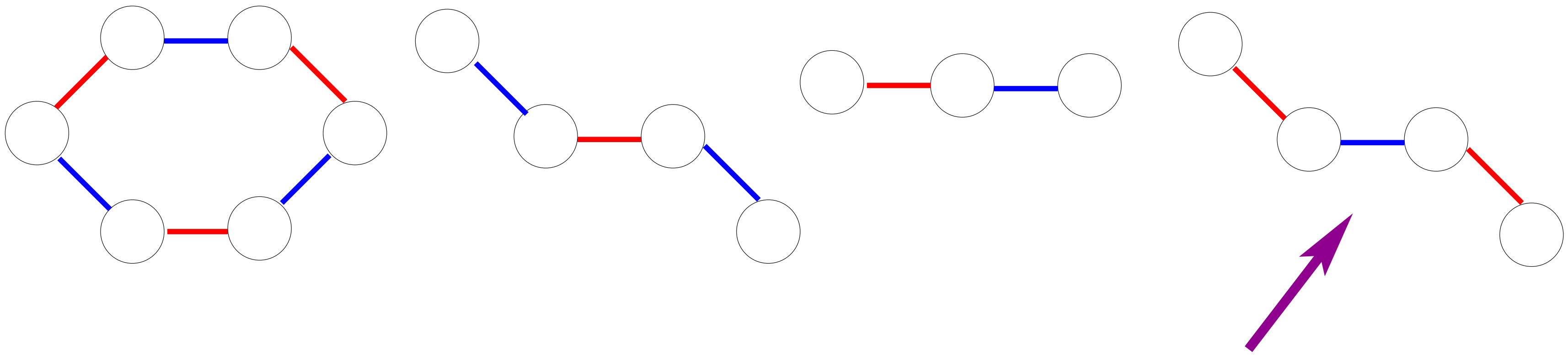
Furthermore, paths and cycles in $M \oplus M'$ are alternating
(wrt. M, and wrt. M')



Edges $\in M$, $\notin M'$
Edges $\in M'$, $\notin M$

Proof of Berge's Lemma (III)

($\Rightarrow$) (cont.) Since $|M'| > |M|$, $M \oplus M'$ must contain at least one connected component that has more edges from M' than from M.



Such a component is a simple path in G
that starts and ends at different vertices with edges $\notin M$.

The extreme vertices are free.

So the path is augmenting.

Aug. Paths in Bipartite Graphs

- **Idea:** Starting from the empty matching, increase the size of the current matching by finding augmenting paths
- Now assume the graph is bipartite.
- For finding augmenting paths, do the following:

  1. Mark vertices as matched or free.
  2. Start DFS (Depth First Search) or BFS (Breadth First Search) from each of the free vertices in L.
  3. Traverse edges $\notin M$ from L to R.
  4. Traverse edges $\in M$ from R to L.
  5. Stop successfully if a free vertex from R is reached.
  6. Stop with failure if search terminates without finding a free vertex from R.

- Cost: $O(|E|)$

Algorithm

```
int MAX_BIPARTITE_MATCHING(bipartite_graph G) {
  M = ∅;
  P = AUG_PATH(G, M);
  while (P != NULL) {
    M = M ⊕ P;
    P = AUG_PATH(G, M);
  }
  return M.size();
}
```

- Cost: $O(|V||E|)$
  - Each iteration costs $O(|E|)$
  - At each iteration 2 new vertices are matched (one from L and one from R)

    So at most $\min(|L|, |R|) = O(|V|)$ iterations suffice

Example

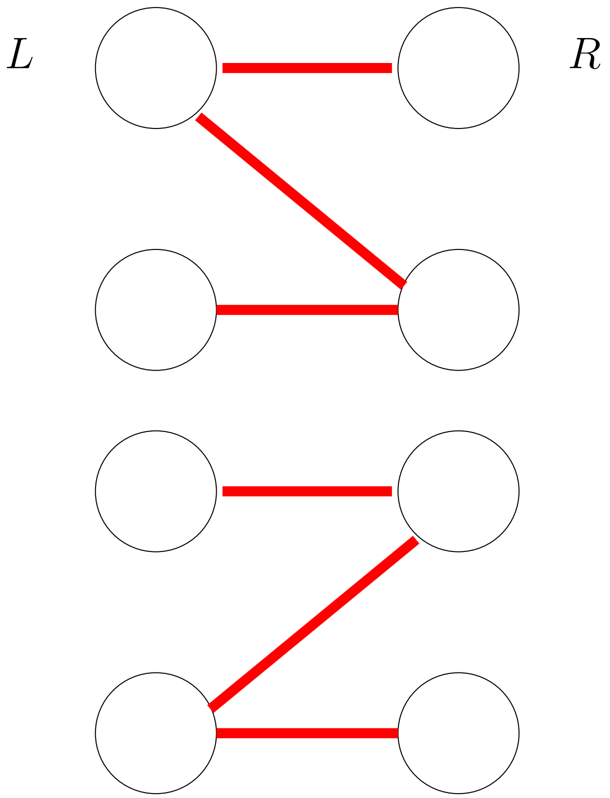
Bipartite graph $G = (L \cup R, E)$

Initially matching M is empty.

Blue edges: $e \in M$

Red edges: $e \notin M$

Let us look for an augmenting path using DFS.



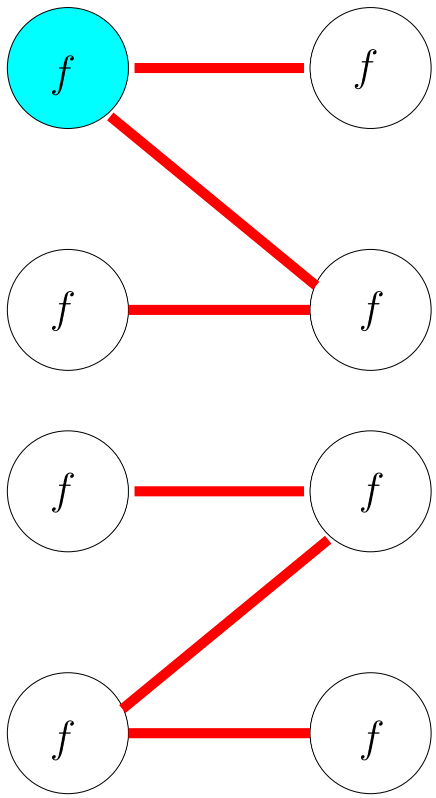
Example

Mark vertices as matched (m) or free (f).

Start at a free vertex in L.

Left $\rightarrow$ right: red edges

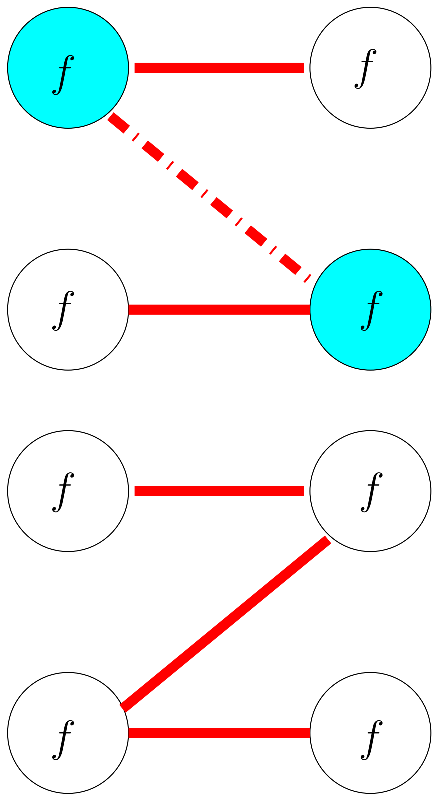
Right $\rightarrow$ left: blue edges



Example

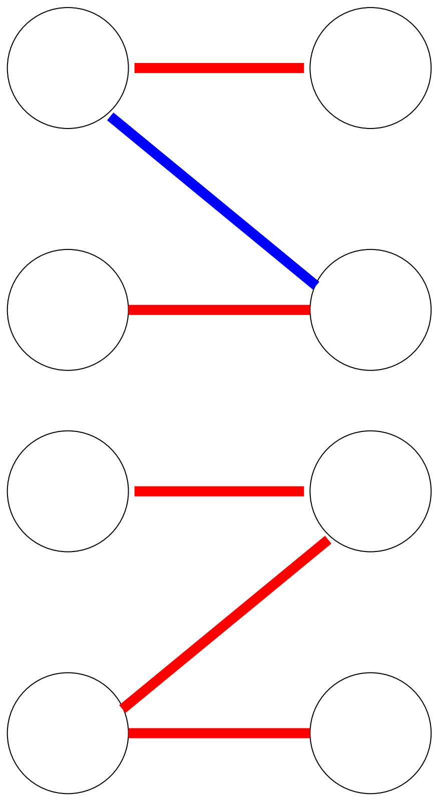
Found a free vertex in R.

Found an augmenting path.



Example

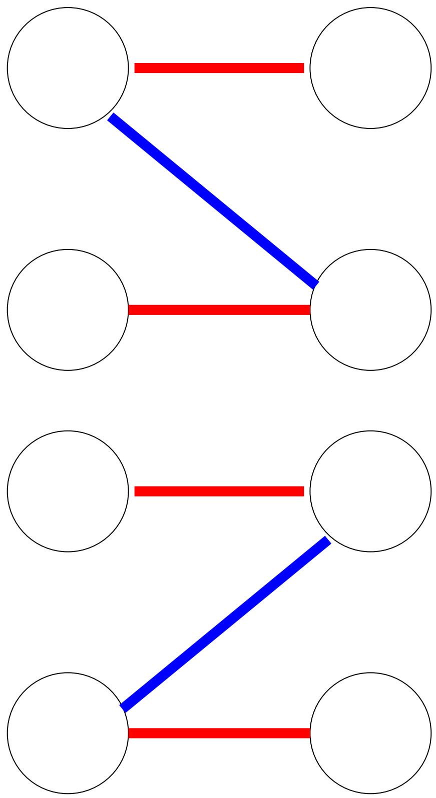
Flip colors of augmenting path and a new M is obtained



Example

Let us look for another augmenting path.

By symmetry.



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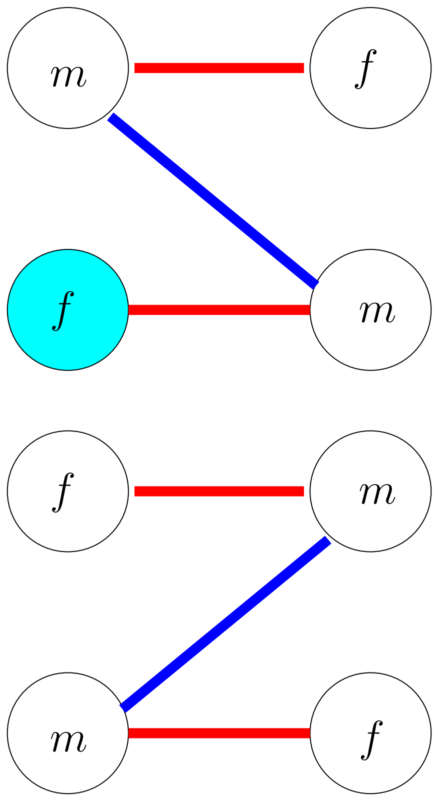
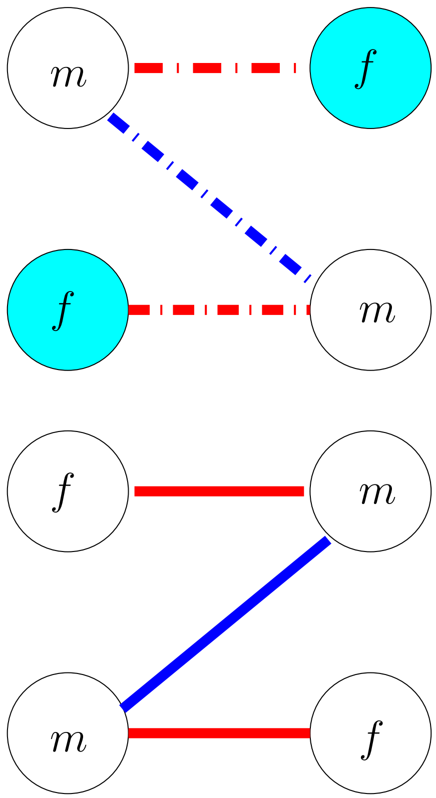


Example

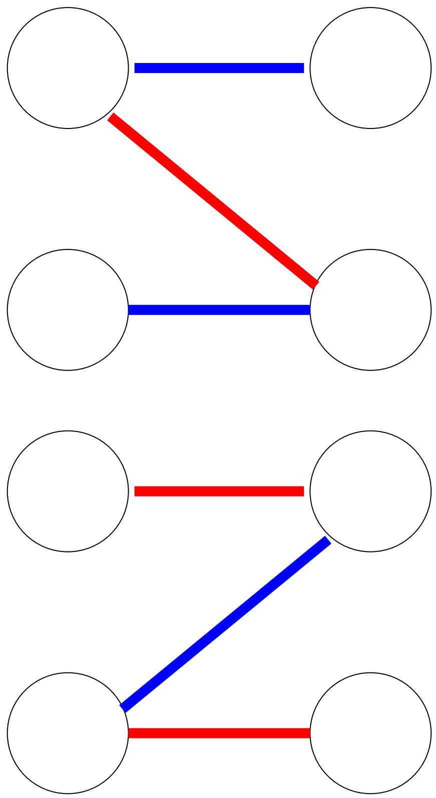


Found a free vertex in R.

Found an augmenting path.

Example

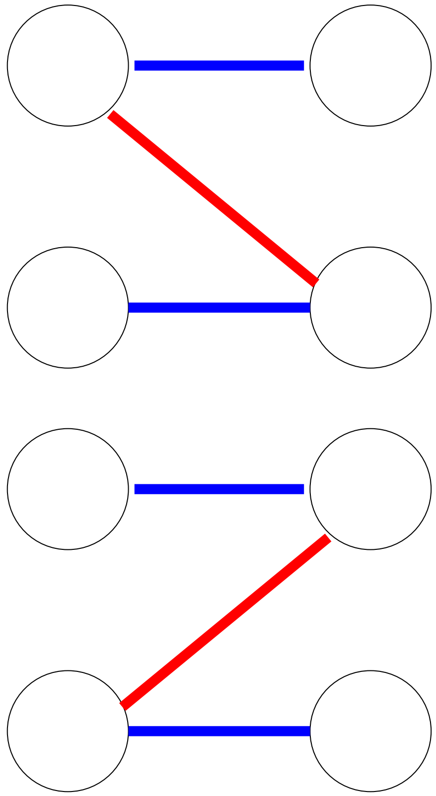
Flip colors of augmenting path and a new M is obtained



Example

By symmetry.

No more augmenting paths,
M is a maximum matching



Hopcroft-Karp Algorithm

- If $P_1, \ldots, P_k$ are vertex-disjoint augmenting paths wrt. M, then $M \oplus (P_1 \cup \cdots \cup P_k)$ is a matching of $|M| + k$ edges
- **Idea:** instead of finding 1 augmenting path per iteration, let us find a maximal set of vertex-disjoint shortest augmenting paths

  This reduces the number of iterations from $O(|V|)$ to $O(\sqrt{|V|})$

```
int HOPCROFT_KARP(bipartite_graf G) {
  M = ∅;
  S = MAXIMAL_SET_VD_SHORTEST_AUG_PATHS(G, M);
  while (S != ∅) {
    M = M ⊕ ⋃{ P | P ∈ S };
    S = MAXIMAL_SET_VD_SHORTEST_AUG_PATHS(G, M);
  }
  return M.size();
}
```

Max. Vertex-Disjoint Shortest AP

- Let us find a maximal set of vertex-disjoint shortest augmenting paths
- Let l be the length of the shortest augmenting paths wrt. M

  Goal: compute a maximal (not necessarily maximum) set of vertex-disjoint augmenting paths of length l

- Phase 1: compute length l and augmenting paths of length l

  1. BFS but start simultaneously at all free vertices in L
  2. Traverse edges $\notin M$ from L to R
  3. Traverse edges $\in M$ from R to L
  4. If a free vertex is found in R:
     current distance is l, the length of the shortest augmenting paths
  5. Complete BFS after finding all free vertices in R at distance l

Max. Vertex-Disjoint Shortest AP

- We need augmenting paths to be vertex-disjoint
- Phase 2: ensure vertex-disjointness and maximality

  Let X be the set of all free vertices in R at distance l

  1. Compute DFS from $u \in X$ to the free vertices in L, using the BFS distances to guide the search:
     - the DFS is only allowed to follow edges that lead to an *unused* vertex in the previous distance layer
     - the DFS must alternate between matched and unmatched edges.
  2. Once an augmenting path is found, mark its vertices as *used* and continue the DFS from the next $u \in X$.

- Cost of Phase 1: $O(|E|)$ (1 single BFS!)
- Cost of Phase 2: $O(|E|)$ (1 single DFS!)

Progress in Hopcroft-Karp

- **Theorem.** Let:
  - $l =$ length of a shortest augmenting path wrt. M
  - $P_1, \ldots, P_k =$ a maximal set of vertex-disjoint shortest augmenting paths wrt. M
  - $M' = M \oplus (P_1 \cup \ldots \cup P_k)$
  - $P =$ a shortest augmenting path with respect to M'

  Then $|P| > l$.
- I.e., from one iteration to the next one,
  the length of the shortest augmenting path increases

Progress in Hopcroft-Karp

Proof. Let us consider two cases:

1. P is vertex-disjoint from $P_1, \ldots, P_k$. By contradiction.

   Since P is an augmenting path wrt. M' and is vertex-disjoint from $P_1, \ldots, P_k$, P is an augmenting path wrt. M.

   Then $|P| \geq l$.

   If $|P| = l$, then P is a shortest augmenting path wrt. M.

   But this contradicts the maximality of $P_1, \ldots, P_k$.

   So $|P| > l$.

Progress in Hopcroft-Karp

2. P is not vertex-disjoint from $P_1, \ldots, P_k$.

   By def., $M' = M \oplus (P_1 \cup \ldots \cup P_k)$.

   So $M \oplus M' = (M \oplus M) \oplus (P_1 \cup \ldots \cup P_k) = P_1 \cup \ldots \cup P_k$.

   So $H := M \oplus M' \oplus P = (P_1 \cup \ldots \cup P_k) \oplus P$.

   But H is a set of vertex-disjoint cycles and simple paths.

   And $|M' \oplus P| - |M| = |M' \oplus P| - |M'| + |M'| - |M| = k + 1$

   So there are at least $k + 1$ simple paths in H that use more edges from $M' \oplus P$ than from M.

   Each of these is an augmenting path wrt. M.

   So H contains $\geq k + 1$ vertex-disjoint augmenting paths with respect to M, each of which of length $\geq l$.

Progress in Hopcroft-Karp

(cont.) So $|H| = |(P_1 \cup \ldots \cup P_k) \oplus P| \geq (k+1)l$.

Hence $|(P_1 \cup \ldots \cup P_k) - P| \ + \ |P - (P_1 \cup \ldots \cup P_k)| \ \geq \ (k+1)l$

As $P_1, \ldots, P_k$ are vertex-disjoint and have length l,
they contribute to $|(P_1 \cup \ldots \cup P_k) - P|$ with at most kl distinct edges.

So $P - (P_1 \cup \ldots \cup P_k)$ contributes with at least l edges to the inequality.

I.e., $|P - (P_1 \cup \ldots \cup P_k)| \geq l$. So

$|P| = |P \cap (P_1 \cup \ldots \cup P_k)| + |P - (P_1 \cup \ldots \cup P_k)| \geq l + |P \cap (P_1 \cup \ldots \cup P_k)|$

Now let us see that $|P \cap (P_1 \cup \ldots \cup P_k)| \geq 1$

Let v be a vertex shared by P and some P_i.

As P_i is an augmenting path wrt. M, there is an edge $e \in P_i - M$ with endpoint v.
So $e \in M'$ and e is the only edge of M' with endpoint v.

As v is matched in M', $v \in P$ and P is an augmenting path wrt. M',
there is a unique edge in $P \cap M'$ with endpoint v, which must be e.

So we have that $e \in P \cap P_i$, that $|P \cap (P_1 \cup \ldots \cup P_k)| \geq 1$, and $|P| > l$

Complexity of Hopcroft-Karp

- We already know that each iteration takes $O(|E|)$ time.
- **Theorem.** Hopcroft-Karp runs in $O(\sqrt{|V|}|E|)$ time.
  (actually, in $O(\sqrt{\min(|L|,|R|)}|E|)$ time)
- Best known algorithm for bipartite matching.
- **Lema.** Hopcroft-Karp takes at most $2\sqrt{\min(|L|,|R|)}$ iterations.
- **Proof.** Wlog. let us assume that $|L| \leq |R|$.
  After $\sqrt{|L|}$ iterations:

  1. either the algorithm terminated because
     a maximum matching was found, or
  2. a matching M was obtained for which the shortest augmenting path
     (wrt. M) has length $\geq 2\sqrt{|L|}+1$

Complexity of Hopcroft-Karp

- **Proof.** (contd.)

  Assume 2. Let M' be a maximum matching of G.

  $M' \oplus M$ contains at least $|M'| - |M|$ vertex-disjoint augmenting paths with respect to M.

  Each of those paths has length $\geq 2\sqrt{|L|} + 1$.

  Since each vertex of $M' \oplus M$ has degree ≤ 2, $M' \oplus M$ is a vertex-disjoint union of simple paths and cycles. As the graph G is bipartite:

  1. In a simple path $P \subseteq M' \oplus M$ of odd length, the number of vertices from L is $(|P| + 1)/2$, which is $\geq |P|/2$.
  2. In a simple path $P \subseteq M' \oplus M$ of even length, the number of vertices from L is $|P|/2$ or $1+|P|/2$, which is $\geq |P|/2$.
  3. In a cycle $C \subseteq M' \oplus M$ (thus, of even length, as it is alternating), the number of vertices from L is $|C|/2$.

  By vertex-disjointness, there are $\geq \frac{|M' \oplus M|}{2}$ vertices from L in $M' \oplus M$.
  So in L there are at least $\frac{|M' \oplus M|}{2}$ different vertices.

Complexity of Hopcroft-Karp

- **Proof.** (contd.)

  Thus

  $2|L| \geq |M' \oplus M| \geq (|M'| - |M|)(2\sqrt{|L|} + 1)$

  So $|M'| - |M| \leq \frac{2|L|}{2\sqrt{|L|}+1} = \frac{2\sqrt{|L|}\sqrt{|L|}}{2\sqrt{|L|}+1} \leq \frac{(2\sqrt{|L|}+1)\sqrt{|L|}}{2\sqrt{|L|}+1} = \sqrt{|L|}$.

  Hence, after another at most $\sqrt{|L|}$ iterations,
  the algorithm is guaranteed to find a maximum matching.

Example

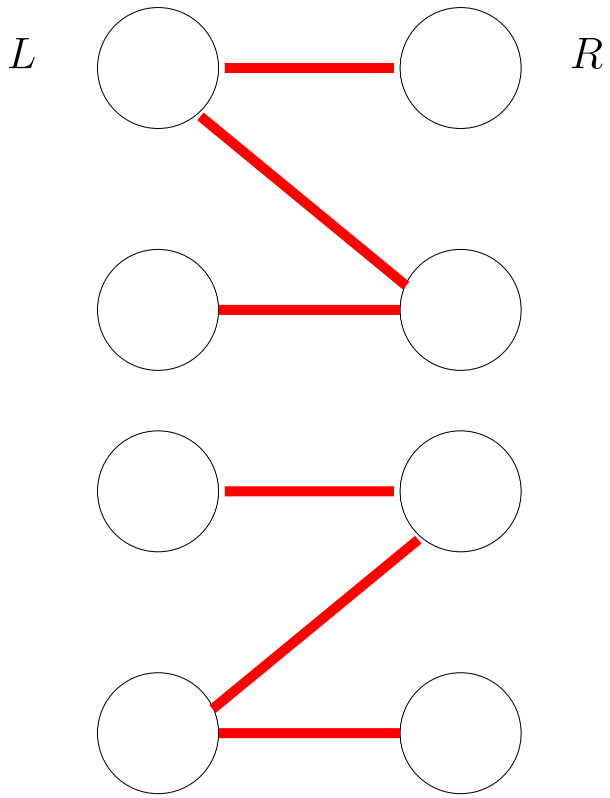
Bipartite graph $G = (L \cup R, E)$

Initially matching M is empty.

Blue edges: $e \in M$

Red edges: $e \notin M$

Let us look for a maximal set of shortest augmenting paths using BFS.



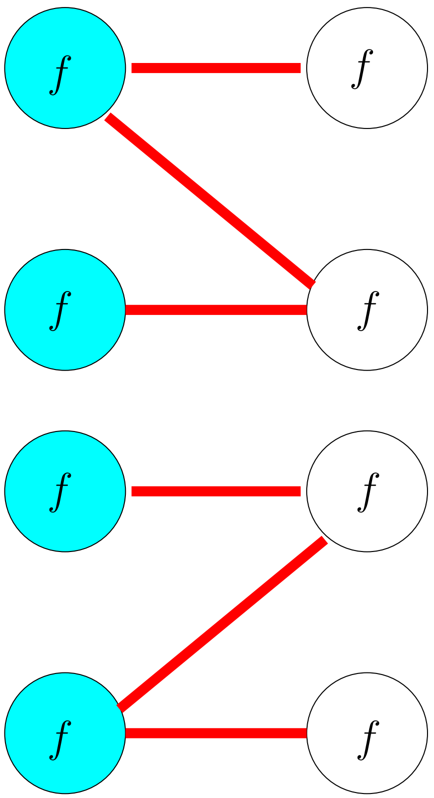
Example

Mark vertices as matched (m) or free (f).

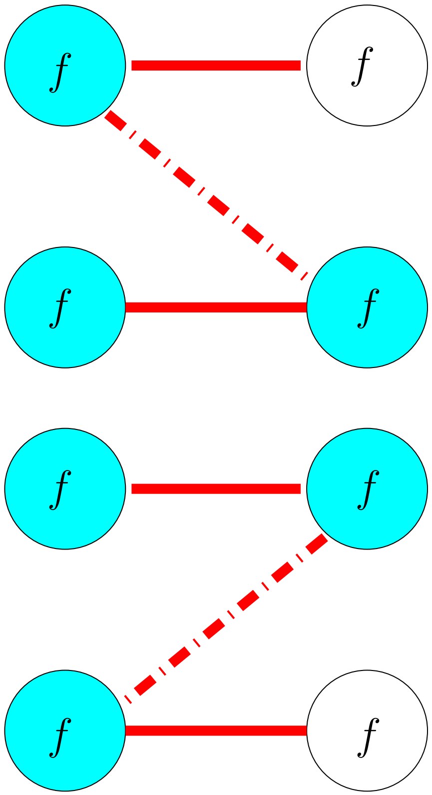
Start at **all** free vertices in L.

Left $\rightarrow$ right: red edges

Right $\rightarrow$ left: blue edges



Example



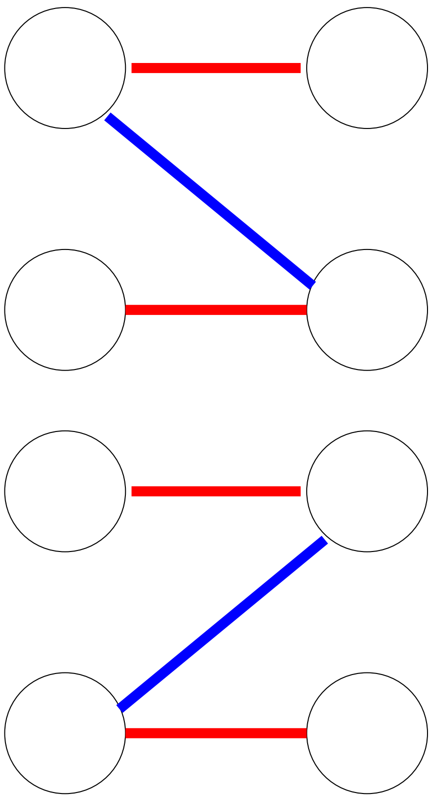
Shortest augmenting path has length 1.

Found all free vertices in R at distance 1.

Found maximal set of shortest aug. paths.
(note that it is not maximum)

Example

Flip colors of augmenting paths and new M is obtained



Example

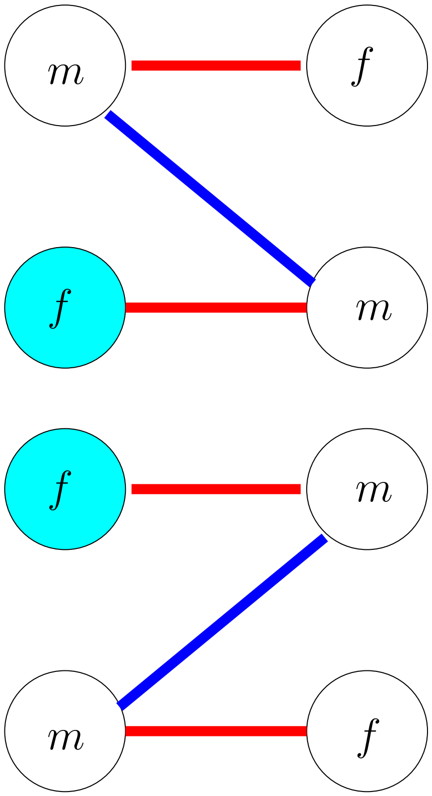
Another maximal set of shortest augmenting paths?

Mark vertices as matched (m) or free (f).

Start at all free vertices in L.

Left $\rightarrow$ right: red edges

Right $\rightarrow$ left: blue edges

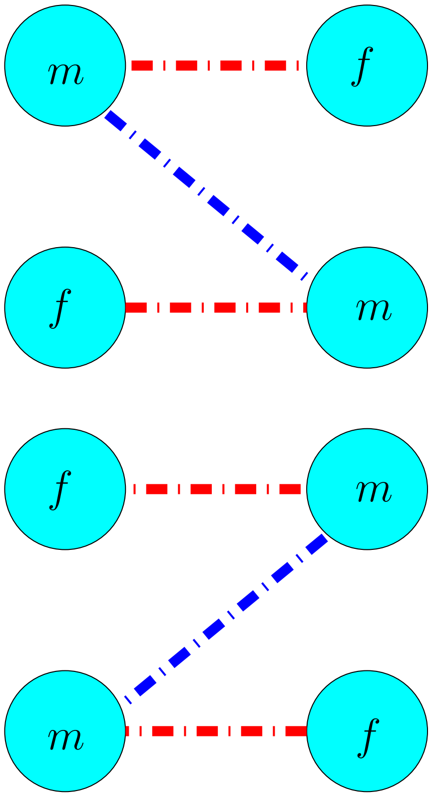


Example

Shortest augmenting path has length 3.

Found all free vertices in R at distance 3.

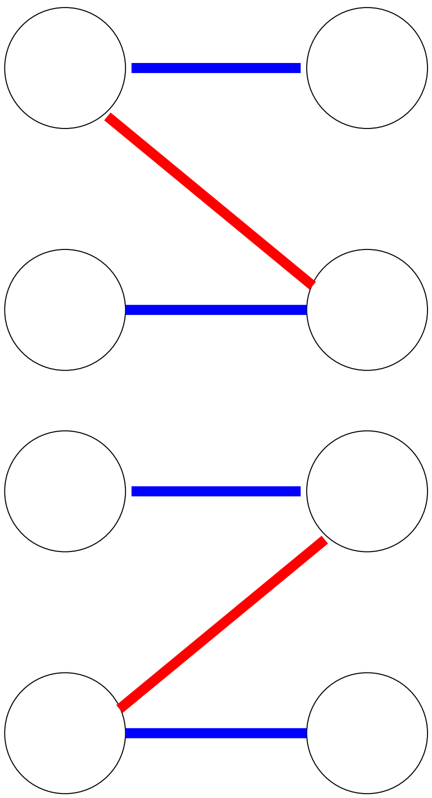
Found maximal set of shortest aug. paths



Example

Flip colors of augmenting path and a new M is obtained

No more augmenting paths, M is a maximum matching



End Matching Theory

Arc Consistency for `alldiff`

[reminder]

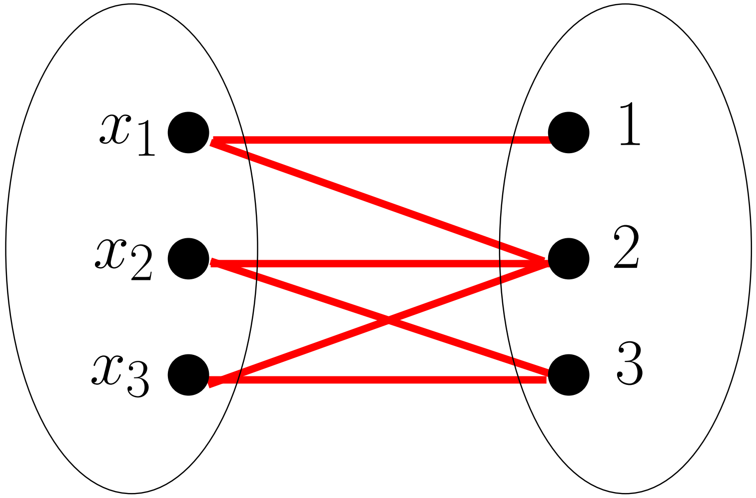
- Consider $x_1 \in \{1, 2\}$, $x_2 \in \{2, 3\}$, $x_3 \in \{2, 3\}$ and the constraint $\texttt{alldiff}(x_1, x_2, x_3)$
  - Value 1 for x_1 is AC since $\tau = (x_1 \mapsto 1, x_2 \mapsto 2, x_3 \mapsto 3)$ is a support for it.
  - Value 2 for x_1 is not AC: it does not have any support (no room left for x_2, x_3)
  - After enforcing AC: $x_1 \in \{1\}$, $x_2 \in \{2, 3\}$, $x_3 \in \{2, 3\}$

Value Graph of `alldiff`

- Given variables $X = \{x_1, \ldots, x_n\}$ with domains $D_1, \ldots, D_n$, the value graph of $\texttt{alldiff}(x_1, \ldots, x_n)$ is the bipartite graph $G = (X \cup \bigcup_{i=1}^{n} D_i, E)$ where $(x_i, v) \in E$ iff $v \in D_i$

$\texttt{alldiff}(x_1, x_2, x_3)$

$$\begin{array}{rcl} D_1 & = & \{1, 2\} \\ D_2 & = & \{2, 3\} \\ D_3 & = & \{2, 3\} \end{array}$$

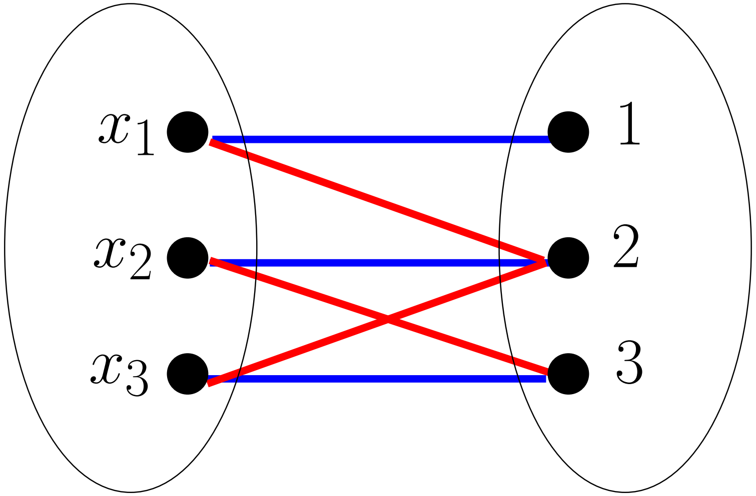


Solutions and Matchings

- We say a matching M covers a set S iff every vertex in S is covered (i.e, is an endpoint of an edge in M)
- Solutions to $\texttt{alldiff}(X) =$ matchings covering X

$$\texttt{alldiff}(x_1, x_2, x_3)$$

$$\begin{array}{lcl}
D_1 & = & \{1,2\} \\
D_2 & = & \{2,3\} \\
D_3 & = & \{2,3\}
\end{array}
\qquad
\begin{array}{l}
x_1 = 1 \\
x_2 = 2 \\
x_3 = 3
\end{array}$$



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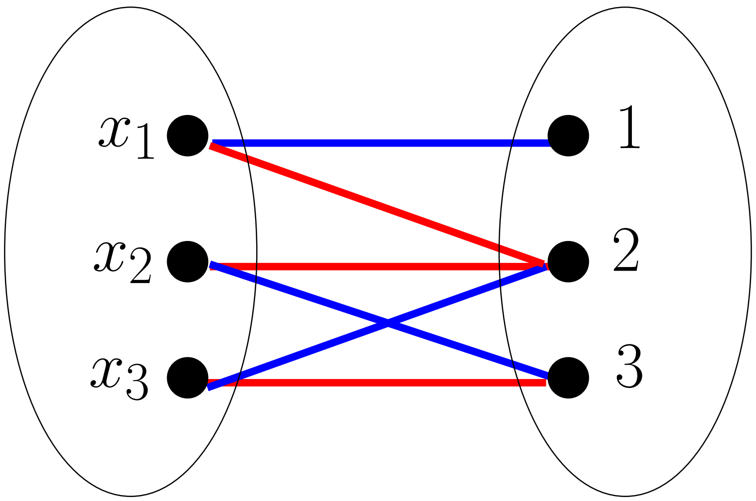

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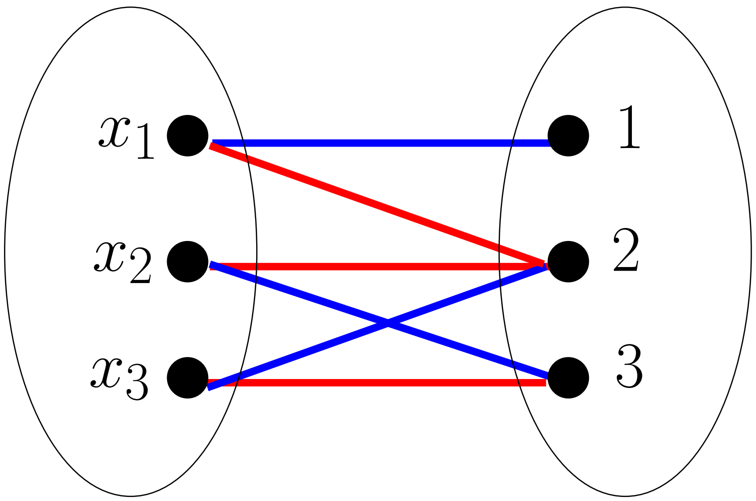

- A matching covering X is a maximum matching
- There are solutions to $\texttt{alldiff}(X)$ iff size of maximum matchings is $|X|$

Solutions and Matchings

- Algorithm for checking feasibility of $\texttt{alldiff}(X)$:
  (with Hopcroft-Karp, in time $O(dn\sqrt{n})$, where $n = |X|$, $d = \max_i\{|D_i|\}$)

```
// Returns true iff there is a solution to alldiff(X)
// G is the value graph of alldiff(X)
M = COMPUTE_MAXIMUM_MATCHING(G)
if ( |M| < |X| ) return false

return true
```

Solutions and Matchings

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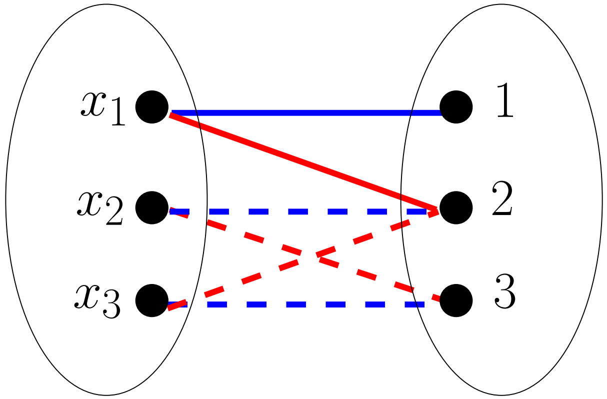
if ( |M| < |X| ) return false
else REMOVE_EDGES_FROM_GRAPH(G, M)
return true
```

- But in addition to check feasibility we want to find arc-inconsistent values
- Assume alldiff(X) has a solution. Then:
  value v from the domain of variable x is arc-inconsistent iff
  there is no solution to alldiff(X) that assigns value v to x iff
  there is no matching covering X that contains edge (x, v) iff
  there is no maximum matching that contains edge (x, v)
- So we have to remove the edges not contained in any maximum matching
- Next: we'll extend the algorithm to do so using the maximum matching M

Filtering

- We want to remove the edges not contained in any maximum matching
- We will identify the complementary set:
  the edges contained in some maximum matching
- We say an edge is vital if it belongs to all maximum matchings
- **Theorem.** Let M be an arbitrary maximum matching.
  An edge belongs to some maximum matching iff
  - it is vital; or
  - it belongs to an alternating cycle wrt. M; or
  - it belongs to an even-length simple alternating path starting at a free vertex wrt. M



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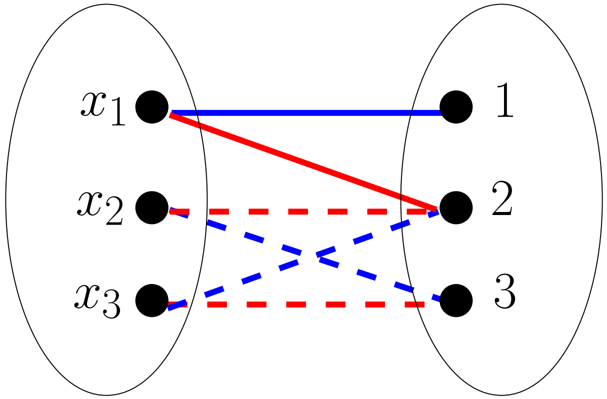


Filtering

Proof: $\Leftarrow$) Let us consider all cases:

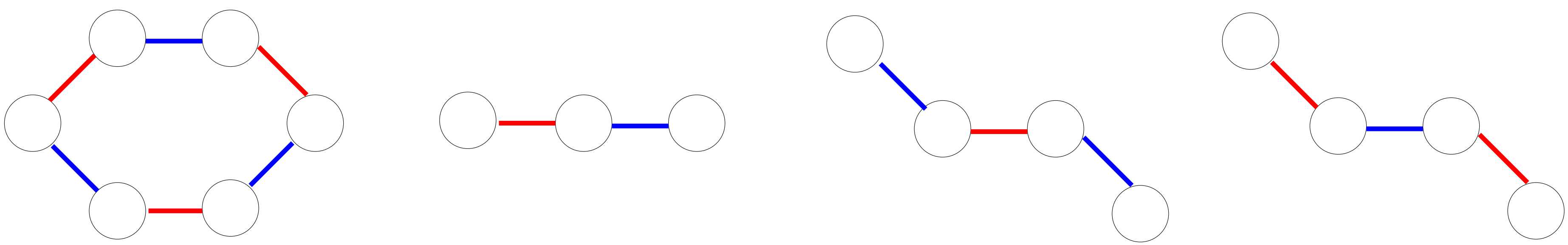
- If edge e is vital,
  then by definition it belongs to a maximum matching
- If e belongs to an alternating cycle P wrt. maximum matching M,
  then M and $M \oplus P$ are maximum matchings,
  one contains e and the other does not
- Similary if e belongs to an even-length path starting at a free vertex
  that is alternating wrt. maximum matching M

Filtering

Proof: $\Rightarrow$) Let e be an edge that belongs to a maximum matching.
Let us assume that e is not vital.

Two cases:

- Suppose $e \in M$. Since e is not vital, there exists a maximum matching M' such that $e \notin M'$. Then $e \in M \oplus M'$.
  But $M \oplus M'$ is a vertex-disjoint union of:



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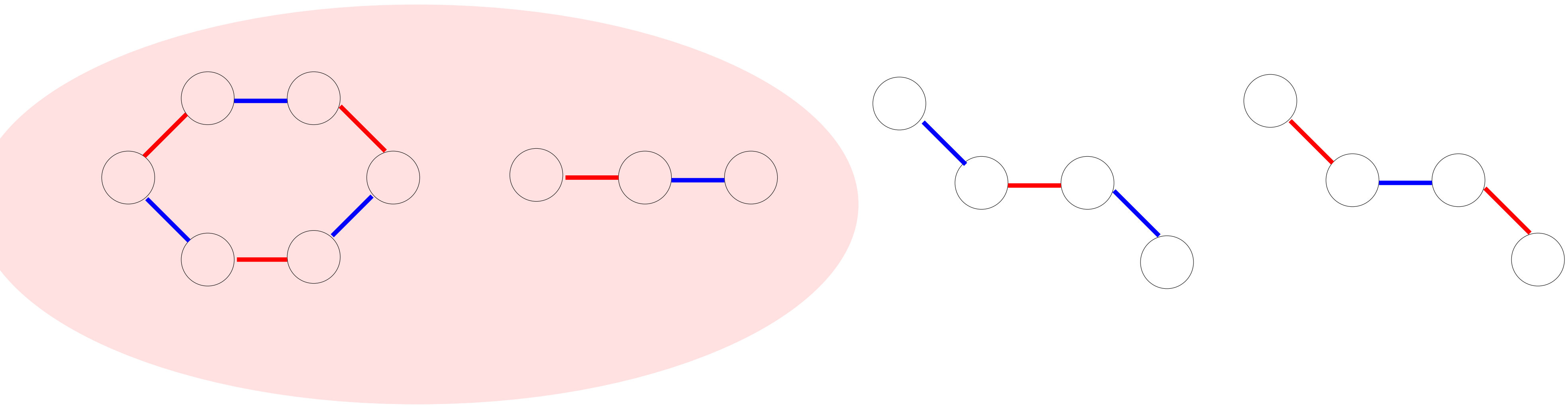


Recall that M, M' are maximum matchings

Filtering

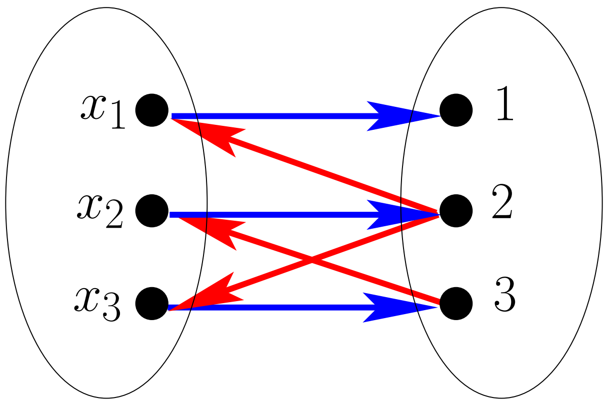
Proof: $\Rightarrow)$ Let e be an edge that belongs to a maximum matching.
Let us assume that e is not vital.

Two cases:

- Suppose $e \notin M$. Let M' be a maximum matching such that $e \in M'$ (which exists by hypothesis). Then the same argument as before applies.

Orienting Edges

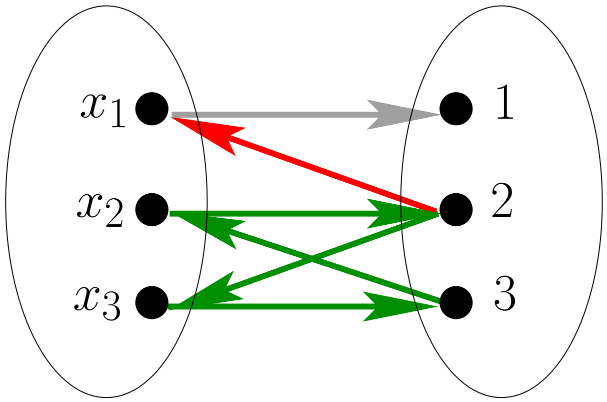
- It simplifies things to orient edges:
  - Edges $e \in M$ are oriented from left to right
  - Edges $e \notin M$ are oriented from right to left



Orienting Edges

- **Corollary.** Let M be an arbitrary maximum matching.
  An edge belongs to some maximum matching iff
  - it belongs to a cycle, or
  - it belongs to a simple path starting at a free vertex wrt. M, or
  - it is vital

  in the oriented graph.



Removing Arc-Inconsistent Edges

- We will actually identify AC edges, and the remaining ones will be non-AC
- An edge (u, v) belongs to a cycle in a digraph G iff u, v belong to the same strongly connected component (SCC) of G

```
REMOVE_EDGES_FROM_GRAPH(G, M)

  0)  Mark all edges in G as UNUSED

  1)  Compute SCC's, and
      mark as USED edges with vertices in same SCC

  2)  Do a depth-first search from free vertices, and
      mark as USED edges in simple paths starting at free vertices

  3)  Mark UNUSED edges of M as VITAL

  4)  Remove remaining UNUSED edges
```

- Time complexity: linear in the size of the value graph

Computing SCC's

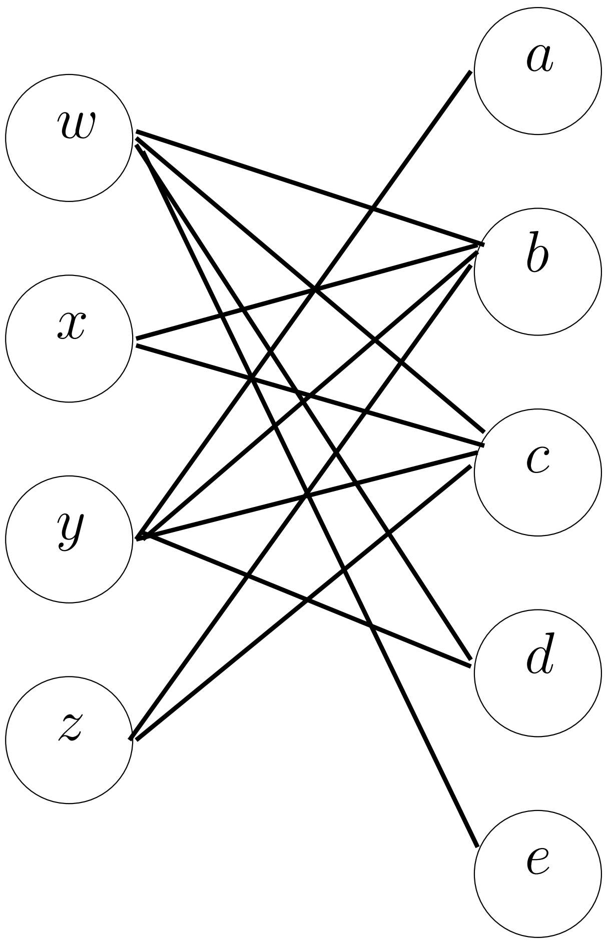
- Given a directed graph $G = (V, E)$,
  SCC's can be computed in time $O(|V| + |E|)$,
  e.g. with Kosaraju's algorithm:

  1. Do DFS
  2. Reverse the direction of the edges
  3. Do DFS in reverse chronological order of finish times wrt. step 1.
  4. Each tree in the previous DFS forest is a SCC

Example

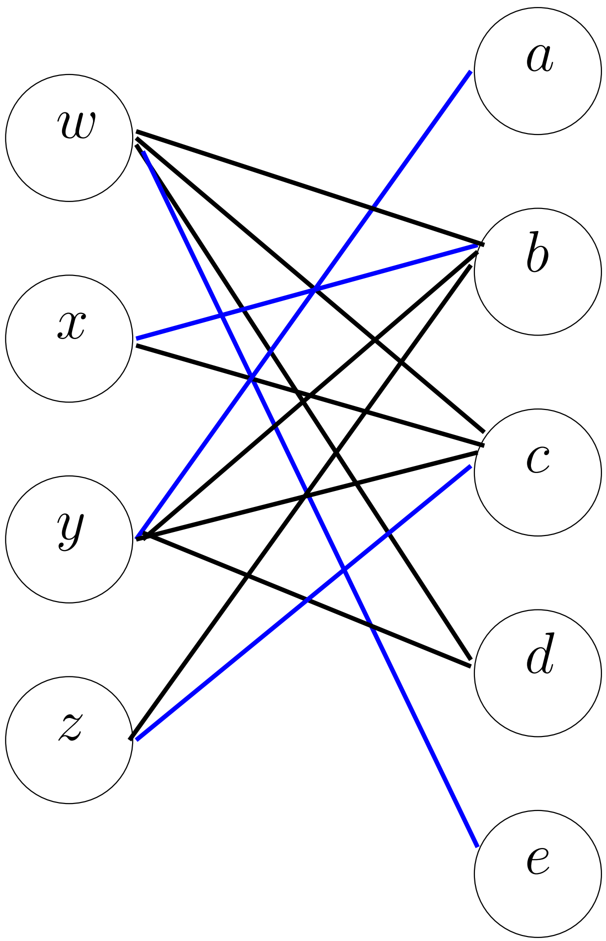
- Variables $\{w, x, y, z\}$
- Domains

  $d(w) = \{b, c, d, e\}$,
  $d(x) = \{b, c\}$,
  $d(y) = \{a, b, c, d\}$,
  $d(z) = \{b, c\}$



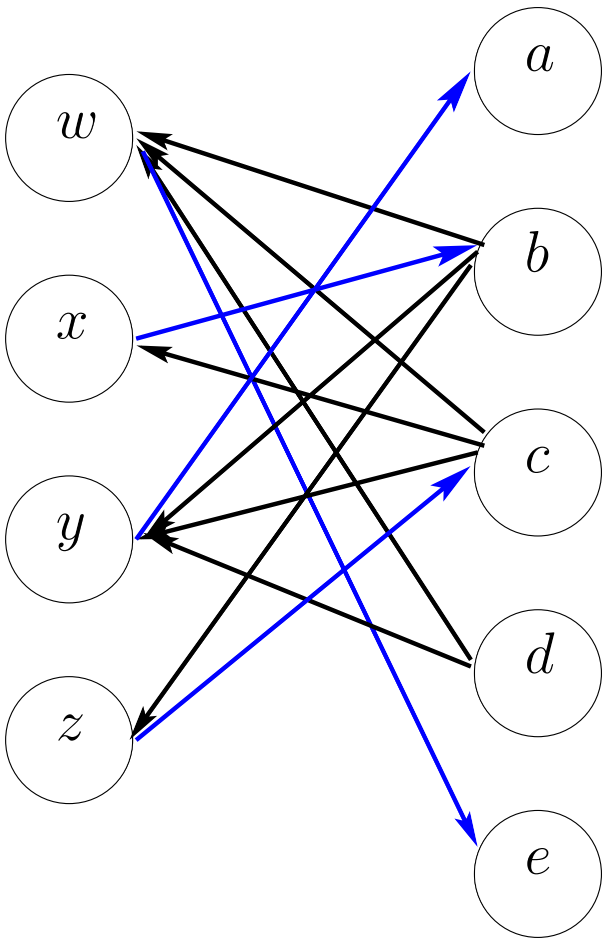
Example

- We assume we already have a maximum matching
- All variables are covered



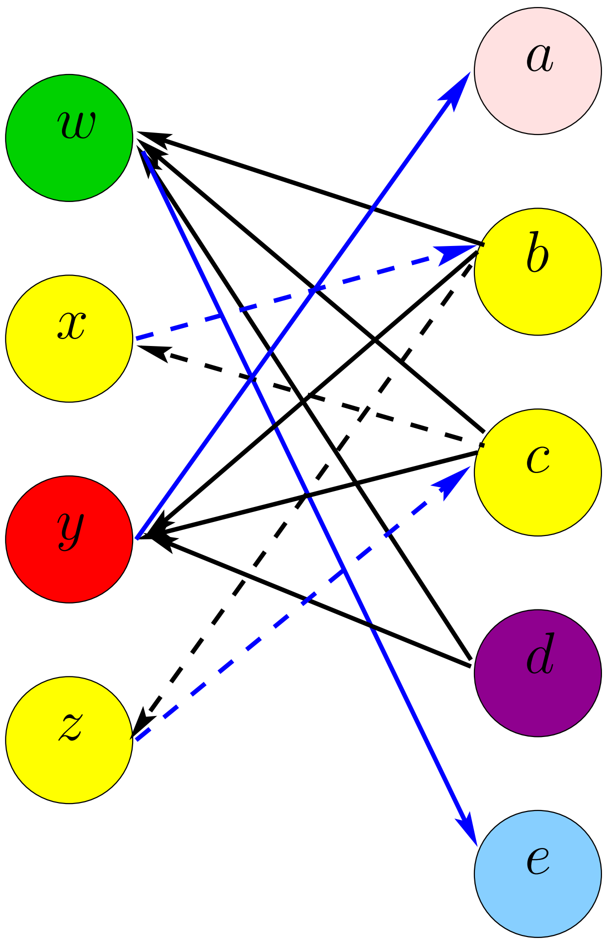
Example

- Direct the edges



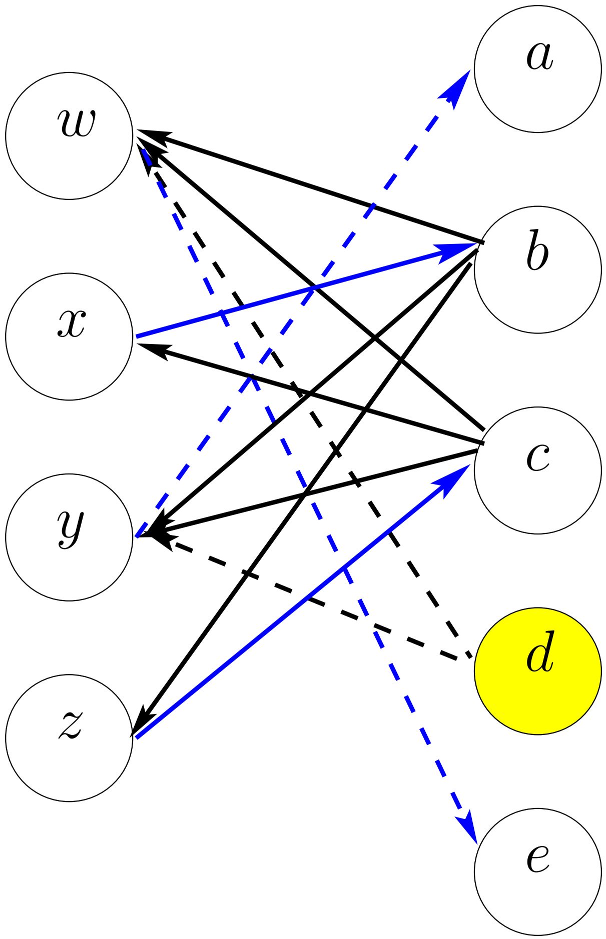
Example

- Compute SCC's



Example

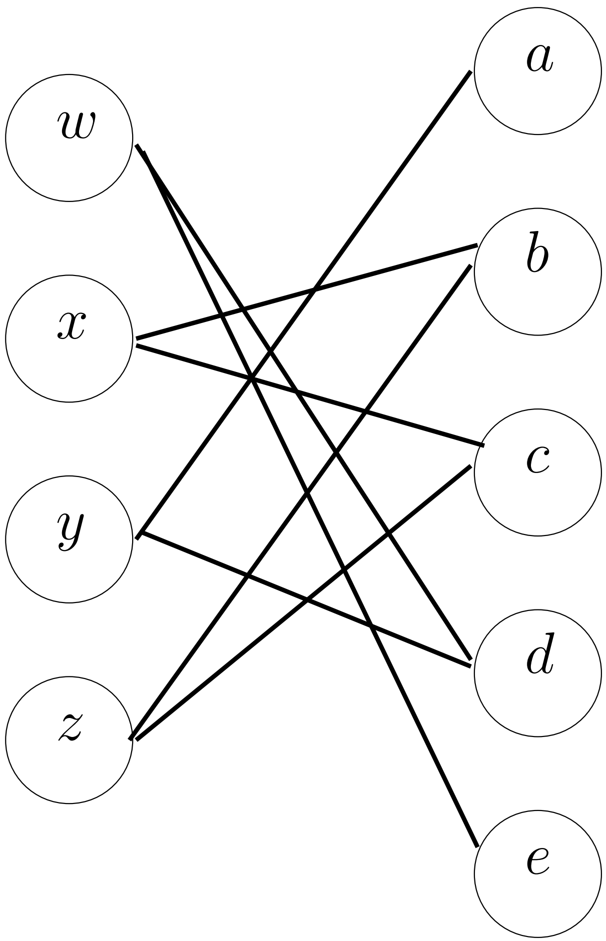
- Compute all simple paths starting at a free vertex



Example

- Remove unused edges that are not vital
- After enforcing arc consistency:

  $d(w) = \{d, e\}$,
  $d(x) = \{b, c\}$,
  $d(y) = \{a, d\}$,
  $d(z) = \{b, c\}$



Complexity

- Consider CSP with a single constraint `alldiff`$(x_1, \dots, x_k)$ where $m = \max_i\{|D_i|\}$
- Cost of enforcing AC with AC-3: $O(k^3 m^{k+1})$
- Cost of enforcing AC with bipartite matching: $O(km\sqrt{k})$
  - Cost of constructing maximum matching: $O(km\sqrt{k})$
  - Cost of removing edges: $O(km)$