Cooperative Game Theory: Solution concepts

Fall 2020

References

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- Notation: N, set of players, $C, S, X \subseteq N$ are coalitions.

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- For a set of players N, a coalition is any subset of N.
  N is the grand coalition.
- A partition of N is a splitting of all the players into disjoint coalitions.

Characteristic Function Games

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  - normalized: $v(\emptyset) = 0$,
  - non-negative: $v(C) \geq 0$, for any $C \subseteq N$, and
  - monotone: $v(C) \leq v(D)$, for any C, D such that $C \subseteq D$
- Example: $N = \{A, B, C\}$ and

$\mathcal{C}_N$	$\emptyset$	A	B	C	AB	AC	BC	ABC
v	0	12	0	0	18	18	18	24

Partition Function Games

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$\mathcal{P}_N$	$\emptyset$, ABC		AB, C		AC, B		BC, A		A,B,C		
C	$\emptyset$	ABC	AB	C	AC	B	BC	A	A	B	C
v	0	24	18	0	18	0	18	0	12	6	0

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  in such games, each coalition can be identified with the profit it obtains by choosing its best action
- We restrict in this course to focus on characteristic function games, and use the term coalition game to refer to such a game.

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- The payoff of each group is the maximum quantity of ice-cream the members of the group can buy by pooling all their money.
- The ice-cream can be shared arbitrarily within the group.

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- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 750, v(\{C, P\}) = 750, v(\{M, P\}) = 500$
- $v(\{C, M, P\}) = 1000$

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- $x = (x_1, ..., x_n)$ is a payoff vector, which distributes the value of each coalition in P:
  - $x_i \geq 0$, for all $i \in N$
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Outcome:example

Suppose $v(\{1,2,3\}) = 9$ and $v(\{4,5\}) = 4$

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- $((\{1,2,3\},\{4,5\}),(3,3,3,3,1))$ is an outcome
- $((\{1,2,3\},\{4,5\}),(2,3,2,3,3))$ is NOT an outcome as transfers between coalitions are not allowed

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An outcome (P, x) is called an imputation if it satisfies individual rationality:

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Notation: we denote $\sum_{i \in A} x_i$ by $x(A)$

1. Definitions
2. Stability notions

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- Let us present some stability notions related to outcomes or imputations.
- To simplify the presentation we consider superadditive games.

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$$v(C \cup D) \geq v(C) + v(D),$$

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- Example: $v(C) = |C|^2$

$$v(C \cup D) = (|C| + |D|)^2 \geq |C|^2 + |D|^2 = v(C) + v(D)$$

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  i.e., an outcome is a vector $x = (x_1, \ldots, x_n)$ with $x(N) = v(N)$

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Ice-cream pots: $w = (500, 750, 100)$ and $p = (\$7, \$9, \$11)$

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(200, 200, 350) is not stable!

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  $((\{1,2,3\},\{4,5\}),(3,3,3,3,1))$ is NOT in the core
  as $x(\{2,4\}) = 6$ and $v(\{2,4\}) = 7$
- no subgroup of players can deviate so that each member of the subgroup gets more.

Ice-cream game: Core

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- $(250, 250, 250)$ is in the core: alone or in pairs do not get more.
- (750, 0, 0) is also in the core:
  Marcie and Pattie cannot get more on their own!

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- Let $\Gamma = (N, v)$, where $N = \{1, 2, 3\}$ and $v(C) = 1$ if $|C| > 1$ and $v(C) = 0$ otherwise.

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- Consider an outcome (P, x).

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- Thus the core of Γ is empty.

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  - But $((\{1, 2\}, \{3, 4\}), (1/2, 1/2, 1/2, 1/2))$ is in the core

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  - $1/3$-core is non-empty: $(1/3, 1/3, 1/3) \in 1/3$-core
  - ϵ-core is empty for any $\epsilon < 1/3$:
    $x_i \geq 1/3$, for some $i = 1, 2, 3$, so $x(N \setminus \{i\}) \leq 2/3$,
    $v(N \setminus \{i\}) = 1$

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- How do we divide payoffs in a fair way?

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  the resulting outcome is fair!
- Can we generalize this idea?

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- In the previous slide we have $\Phi_1 = \Phi_2 = 10$

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- Φ_i is i's average marginal contribution to the coalition of its predecessors, over all permutations
- Suppose that we choose a permutation of players uniformly at random, then Φ_i is the expected marginal contribution of player i to the coalition of his predecessors

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- Two players i and j are said to be symmetric in Γ if

$$v(C \cup \{i\}) = v(C \cup \{j\}), \text{ for any } C \subseteq N \setminus \{i, j\}$$

Shapley value: Axiomatic Characterization

Properties of the Shapley value:

- Efficiency: $\Phi_1 + ... + \Phi_n = v(N)$
- Dummy: if i is a dummy, $\Phi_i = 0$
- Symmetry: if i and j are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i((\Gamma_1) + \Phi_i(\Gamma_2)$

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Theorem

The Shapley value is the only payoff distribution scheme that has properties (1) - (4)

$\Gamma = \Gamma_1 + \Gamma_2$ is the game (N, v) with $v(C) = v_1(C) + v_2(C)$

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- We are usually interested in algorithms whose running time is polynomial in n

- So what can we do? subclasses?

Checking Non-emptiness of the Core: Superadditive Games

- An outcome in the core of a superadditive game satisfies the following constraints:

$$x_i \geq 0 \text{ for all } i \in N$$

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- A linear feasibility program, with one constraint for each coalition: $2^n + n + 1$ constraints

Superadditive Games: Computing the Least Core

- Starting from the linear feasibility problem for the core

$$
\begin{aligned}
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- A minimization program, rather than a feasibility program

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  Convergence guaranteed by Law of Large Numbers