Radomized Algorithms Solution to Problemes 1 (*) Fall 2019.

3.- (*)

(a) The probability of any subset is the probability of the product of the n heads and tails which are flipped to generate it. Due to the independence and that the uar, the product is $1/2^n$, each subset is equally likely.

(b) Use the chain rule: $\mathbf{Pr}[X \subseteq Y] = \sum_x \underbrace{\mathbf{Pr}[X \subseteq Y | X = x]}_{(*)} \mathbf{Pr}[X = x]$.

The $(*) = \prod_{i \in x} 1/2 = 1/2^{|x|}$. Therefore,
$\mathbf{Pr}[X \subseteq Y] = \sum_x \frac{1}{2^{|x|}} \mathbf{Pr}[X = x] = \sum_{j=1}^n \frac{1}{2^j} \mathbf{Pr}[|X| = j] = \sum_{j=1}^n \binom{n}{j} \frac{1}{2^{j+n}}$
(Alternative solution: If H=1 and T=0:
$\mathbf{Pr}[X \leq Y] = \mathbf{Pr}[X_i \leq 1] \mathbf{Pr}[Y_i = 1] + \mathbf{Pr}[x_i \leq 0] \mathbf{Pr}[Y_i = 0] = 3/4$, so
$\mathbf{Pr}[X \subseteq Y] = \prod_{i=1}^n \mathbf{Pr}[X_i \leq Y_i] = (3/4)^n$).
$\mathbf{Pr}[X \cup Y = S] = \prod_{i=1}^n \mathbf{Pr}[X_i + Y_i > 0] = (3/4)^n$.

4.- Let N be a uniform number in the given range and let E_k be the event that N is divisible by k. Notice in the range of 10^6. the number of integer divisible by k is $\lfloor 10^6/k \rfloor$. Then, if D is a set of divisors, for any $k \in D$

$$\mathbf{Pr}[E_k] = \frac{\lfloor 10^6/k \rfloor}{10^6}.$$

If we consider $D = \{4, 6, 9\}$. Then $\mathbf{Pr}[E_4 \cup E_6 \cup E_9] = \mathbf{Pr}[E_4] + \mathbf{Pr}[E_6] + \mathbf{Pr}[E_9] - \mathbf{Pr}[E_4 \cap E_6] - \mathbf{Pr}[E_4 \cap E_9] - \mathbf{Pr}[E_6 \cap E_9] + \mathbf{Pr}[E_4 \cap E_6 \cap E_9]$
Notice $\mathbf{Pr}[E_4 \cap E_6] = \mathbf{Pr}[E_{12}]$, $\mathbf{Pr}[E_4 \cap E_9] = \mathbf{Pr}[E_{36}]$, $\mathbf{Pr}[E_9 \cap E_6] = \mathbf{Pr}[E_{18}]$,
$\mathbf{Pr}[E_4 \cap E_6 \cap E_9] = \mathbf{Pr}[E_{36}]$
$\mathbf{Pr}[E_4 \cup E_6 \cup E_9] = \frac{1}{10^6}(250000 + 166666 + 111111 - 83333 - 27777 - 55555 + 27777) = 0.38$

5.- (*) Fix a r. If $r = i$ then $m_n, \ldots, m_{r+1}$ yield tails and m_r is heads, which happens with probability

$$\frac{1}{r} \prod_{j=n}^{r+1} (1 - \frac{1}{j}) = \frac{1}{r} \frac{n-1}{n} \frac{n-2}{n-1} \cdots \frac{r}{r+1} = \frac{1}{n}.$$

Therefore the algorithm generates r, uar.

7.- (*)

(a) Probabilities

- Four of a kind (ex. 7,7,7,7, $*$): The number of ways to choose the rank is $\binom{13}{1} = 13$, then the number of ways to draw the other three cards of that rank is $\binom{4}{4} = 1$, finally the number of ways to choose the odd card is the number of ways to choose a card from the remaining 48 cards, $\binom{48}{41} = 48$. Therefore the probability of selecting one four of a kind is $13 \times 4/\binom{52}{5} = 0.000240096$
- Having The number of possible flush that we can have is choosing 5 cards from the same suit $\binom{13}{5}$ and 4 possible suits, so favorable cases $= 4\binom{13}{5}$, so probability is $4\binom{13}{5}/\binom{52}{5} = 0.00198079$
- A straight flush never can start by the values 4,3,2 (the last possible straight flush is 5,4,3,2,A) Therefore the possible number of straight flush are 10 initial positions and 4 possible suits, 40. The probability of having one of those is $40/\binom{52}{5} = 0.0000153908$

(b) The probability one of the other players does not have a straight flush $(1 - 40/\binom{52}{5})$ the probability none of them has a straight flush is $(1 - 40/\binom{52}{5})^3$. the probability at least one has a straight flush is $1 - (1 - 40/\binom{52}{5})^3 = 0.0000461717$