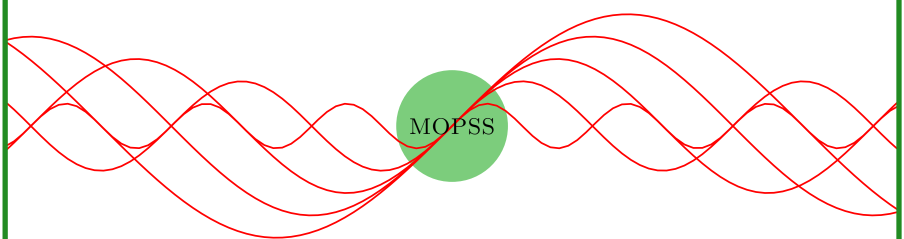
Warm up

MOPSS

15 February 2025

Mathematics Olympiad

Problem Solving Sessions



Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

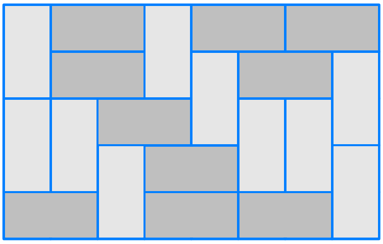


Figure 1: India BMath 2006 (a tiling of a 5×8 rectangle with non-overlapping dominoes), Example 1.1

List of problems and examples

§1 Warm up

Example 1.1 (India BMath 2006)**.** A domino is a 2 by 1 rectangle. For what integers m and n, can one cover an m by n rectangle with non-overlapping dominoes?

Walkthrough —

(a) If an $m \times n$ rectangle admits a covering by non-overlapping dominos, then show that at least one of the integers m, n has to be even.

(b) If at least one of m, n is even, then prove that an $m \times n$ rectangle admits a covering by non-overlapping dominos.

Solution 1. In the following, an $m \times n$ rectangle is to be thought as an $m \times n$ rectangular grid.

To be able to cover an $m \times n$ rectangle by non-overlapping dominoes, it is necessary for the product mn to be even, and hence, at least one of m, n is even. Indeed, if an $m \times n$ rectangle admits a covering using k non-overlapping dominoes, then those dominoes together cover $2k$ unit squares, and this yields that $2k = mn$.

Moreover, when at least one of m, n is even, an $m \times n$ rectangle can be covered by non-overlapping dominoes by covering each row by $m/2$ (resp. each column by $n/2$) non-overlapping dominos if m (resp. n) is even.

This shows that an $m \times n$ rectangle can be covered by non-overlapping dominoes if and only if at least one of m, n is even. ■

Remark. The above conclusion shows that an $m \times n$ rectangle admits a covering by non-overlapping dominoes if and only if it admits a covering by non-overlapping dominoes in the *most obvious manner*, i.e. a covering by non-overlapping dominoes such that all of them are either horizontal or vertical (cf. [**Bru10**, p. 6]).

The following problem is a more general version of Example 1.1.

Exercise 1.2. [**Eng98**, Problem 8, Chapter 2, p. 26] Show that an $m \times n$ rectangle admits a covering by non-overlapping $k \times 1$ rectangles if and only if k divides m or k divides n.

Exercise 1.3. [**Eng98**] [**Bru10**, p. 4] Consider an $n \times n$ chessboard and remove the squares at the end of one diagonal. Determine whether the mutilated chessboard admits a covering by non-overlapping dominos.

Example 1.4 (India RMO 2003 P7)**.** Consider the set $X = \{1, 2, 3, \dots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C, $\text{prod}(C)$ denotes the product of all numbers in C,

(c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.

Summary — It is equivalent to finding a subset B of $\{1, \dots, 10\}$, other than $\emptyset, \{1, \dots, 10\}$, such that $\text{prod}(B)^2$ divides 10! and the quotient $10!/\text{prod}(B)^2$ is minimized. To do so,

(a) write down the prime power factorization of 10!,

(b) throw in enough elements in B so that $\text{prod}(B)$ is maximized, and $\text{prod}(B)^2$ divides 10!.

Walkthrough —

(a) Observe that it is enough to find a nonempty proper subset B of $\{1, 2, \dots, 10\}$ such that $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum.

(b) Writing down the prime power factorization of 10!, deduce that B does not contain 7, it contains a multiple of 5, and also a multiple of 2 and a multiple of 3.

(c) Prove that B contains exactly one multiple of 5, and not more that two multiples of 3.

(d) Show that B is equal to one of the subsets $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$.

(e) Show that any of these three subsets also have the stated property.

First, let's work on it. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\mathrm{prod}(A)}{\mathrm{prod}(B)} = \frac{10!}{(\mathrm{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\mathrm{prod}(B)^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\mathrm{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, and hence, B does not contain 7. Moreover, B contains a multiple of 5, otherwise $(\mathrm{prod}(B \cup \{5\}))^2$ would divide $10!$ and $\mathrm{prod}(B \cup \{5\})$ would be strictly larger than $\mathrm{prod}(B)$, which contradicts the choice of B. Similarly, B also contains a multiple of 2 and a multiple of 3. Note that B contains exactly one multiple of 5 (since $5^3 \nmid 10!$). Since $(\mathrm{prod}(B))^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum, B is equal to one of the following sets

- $\{5, 3, 2, 2^3\}$, $\{5, 3, 2, 2^3, 1\}$, $\{5, 6, 2, 2^3\}$, $\{5, 6, 2, 2^3, 1\}$, $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$ if B contains 5,
- $\{10, 3, 2^3\}$, $\{10, 3, 2^3, 1\}$, $\{10, 3, 2, 2^2\}$, $\{10, 3, 2, 2^2, 1\}$, $\{10, 6, 2^2\}$, $\{10, 6, 2^2, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2, 2^2\}$, $\{10, 9, 2, 2^2, 1\}$ if B contains 10.

For any of the above sets, the product of its elements is equal to 240, 480, or 720. So B is equal to one of the sets $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$.

Also note that if B denotes one of these subsets of $\{1, \ldots, 10\}$, then $\mathrm{prod}(B)^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum.

This proves that $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$ are precisely all the subsets of $\{1, \ldots, 10\}$ having the required property. Thus we could take $A = \{1, 2, 3, 4, 5, 6, 7\}, B = \{8, 9, 10\}$ for instance. ♣

Remark. Note that the above discussion provides more than what has been required. After observing that $\mathrm{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, one may show that there is a subset B with $\mathrm{prod}(B)$ equal to $2^4 \cdot 3^2 \cdot 5$ (for instance, $B = \{8, 9, 10\}$),

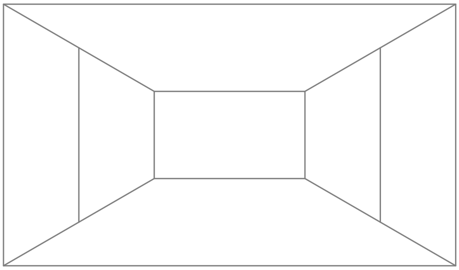


Figure 2: India RMO 2014, Example 1.5

and then conclude.

Solution 2. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\mathrm{prod}(A)}{\mathrm{prod}(B)} = \frac{10!}{(\mathrm{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\mathrm{prod}(B)^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\mathrm{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$. If $B = \{8, 9, 10\}$, then $\mathrm{prod}(B)$ is equal to $2^4 \cdot 3^2 \cdot 5$. Hence, $A = \{1, \ldots, 7\}, B = \{8, 9, 10\}$ are two disjoint nonempty subsets of $X = \{1, \ldots, 10\}$ satisfying the required conditions. ■

Remark. Don't be surprised that it took a bit long to arrive at the above solution. It is often the case. Further, it is a standard practice to write down a complete solution as the final one, without any reference to the prior attempts (possibly several). Those attempts have their important role in providing insights, which may lead to a solution. Here, the details of those attempts have not been hidden from you, in order to take you along the journey. However, I would like to highlight that a *solution* to a problem has to be complete, and at the same time, has to be free from the prior thoughts that have no direct role to play in that solution, though they might have played a significant role in gaining insight.

Example 1.5 (India RMO 2014d P6)**.** In Fig. 2, can the numbers $1, 2, 3, 4, \ldots, 18$ be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?

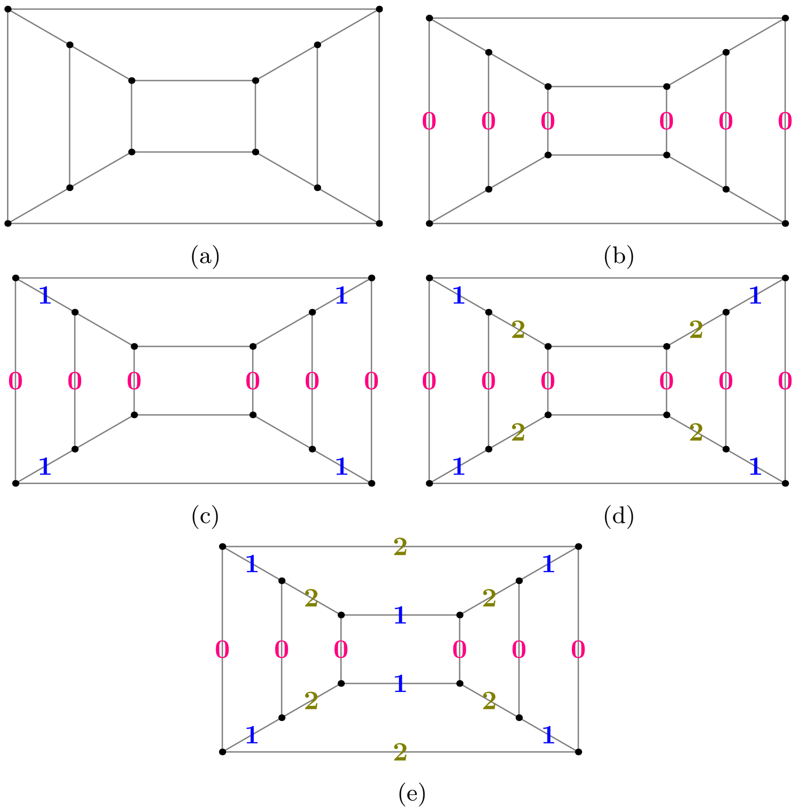


Figure 3: India RMO 2014, Example 1.5

Summary — Since there are 18 line segments, it follows that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \ldots, 18$ satisfying the required condition.

Solution 3. Note that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \ldots, 18$ satisfying the required condition (by replacing the 0's (resp. 1's, 2's) by the six integers among $1, 2, \ldots, 18$ which are congruent to 0 (resp. $1, 2$) modulo 3, and such a replacement can be carried out since there are six elements among $1, 2, \ldots, 18$ congruent to $i \bmod 3$ for any $i \in \{0, 1, 2\}$). We now show that such an arrangement of $0, 1, 2$ exists. First, put 0's on all the vertical segments as in Fig. 3b, and then put 1's on the 'diagonal' segments as shown in Fig. 3c. This forces to put 2's on the 'diagonal' segments as shown in Figure 3d, which in turn, forces to write 1's and 2's on the horizontal segments as in Figure 3e. Note that the sum of the numbers (as in Figure 3e) on the three line segments meeting at each point is divisible by 3. So this gives an arrangement of $0, 1, 2$ satisfying the desired properties, then $1, 2, \ldots, 18$ can be arranged satisfying the given conditions (as described above). ■

Example 1.5 leads to the following question.

Question 1.6. Under which conditions, does a k-regular graph admit an edge coloring by the k-th roots of unity such that the sum of the colors incident at any vertex equals to zero?

Example 1.7 (India RMO 2017b P1)**.** Consider a chessboard of size 8 units $\times$ 8 units (i.e. each small square on the board has a side length of 1 unit). Let S be the set of all the 81 vertices of all the squares on the board. What is the number of line segments whose vertices are in S, and whose length is a positive integer? (The segments need not be parallel to the sides of the board.)

Summary — A segment having vertices in S and length a positive integer, is horizontal or vertical, or the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board. To count such right-angled triangles, note that they cannot have a too large hypotenuse.

Walkthrough —

(a) Determine the number of the horizontal segments with vertices in S and whose lengths are positive integers.

(b) By symmetry, the number of such vertical segments is equal to the above.

(c) To determine the slanted ones, note that such a slanted segment is the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board, and have integer lengths. Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$.

(d) Does a symmetry argument help? For instance, flipping around a diagonal, and then flipping around an axis (i.e. a line parallel to one of the sides of the board and dividing the board in two equal halves).

Solution 4. Note that within each horizontal line, there are $8-\ell+1$ horizontal segments of length ℓ for any $1 \leq \ell \leq 8$. This shows that the number of the horizontal segments with vertices in S and whose lengths are positive integers is equal to

$$8 \times (1 + 2 + \cdots + 8)) = \frac{1}{2}8^2 \cdot 9.$$

By symmetry, the number of such vertical segments is also equal to $\frac{1}{2}8^2 \cdot 9$. Hence, there are $8^2 \cdot 9$ segments parallel to the sides of the board, which have vertices in S and whose lengths are positive integers.

Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have the side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$. The number of such right-angled triangles, having the side lengths equal to $(3, 4, 5)$, is equal to

$$4 \times (8 - 3 + 1) \times (8 - 4 + 1).$$

The number of such right-angled triangles, having the side lengths equal to the remaining triples, can be expressed in a similar way. This shows that the total count of such triangles is

$$2 \times 4 \times ((8 - 3 + 1) \times (8 - 4 + 1) + (8 - 6 + 1) \times (8 - 8 + 1)) = 360.$$

Hence, the number of the line segments with the stated property is

$$8^2 \cdot 9 + \frac{1}{2} \cdot 360 = 756.$$

■

Remark. Note that the above argument considers the line segments whose endpoints are distinct. There are 81 line segments having equal end-points and end-points lying in S.

Example 1.8 (India RMO 2016g P2)**.** On a stormy night ten guests came to dinner party and left their shoes outside the room in order to keep the carpet

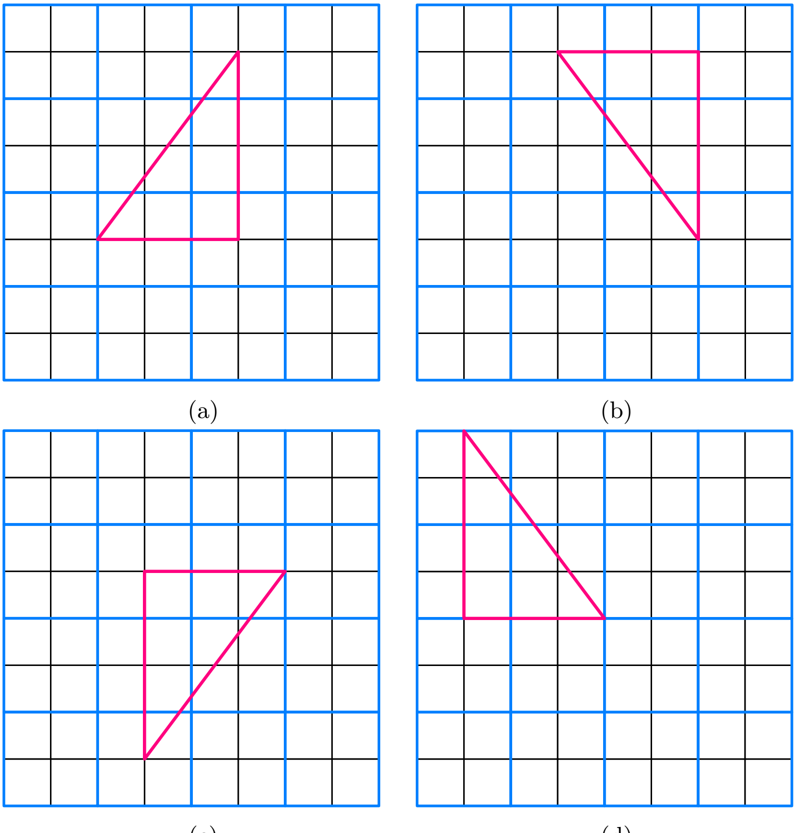


Figure 4: India RMO 2017 — Several configurations of triangles with side lengths $(3, 4, 5)$, Example 1.7

clean. After the dinner there was a blackout, and the guests leaving one by one, put on at random, any pair of shoes big enough for their feet. (Each pair of shoes stays together). Any guest who could not find a pair big enough spent the night there. What is the largest number of guests who might have had to spend the night there?

> **Walkthrough** — What happens if a person having shoe of the smallest size wears a shoe of the largest size, and next, if a person having shoe of the second smallest size wears a shoe of the second largest size, and it continues?

Solution 5. If a person having shoe of the k-th smallest size wears a shoe of the k-th largest size for $k = 1, 2, 3, 4, 5$, and if the sizes of the shoes of the ten persons are distinct, then none of the remaining five persons will not find a big enough shoe.

We claim that if k persons have left for some $1 \leq k \leq 4$, then one of the remaining $10 - k$ persons will find a big enough shoe. Let us denote the sizes of the shoes of the guests by $s_1, s_2, \ldots, s_{10}$. If the claim is false, then for some $1 \leq k \leq 4$, there are $10 - k$ guests whose shoes have sizes larger than some $10 - k$ numbers among $s_1, \ldots, s_{10}$. So, some $10 - k$ numbers among $s_1, \ldots, s_{10}$ are larger than some $10 - k$ numbers among $s_1, \ldots, s_{10}$. It follows that

$$10 - k + 10 - k \leq 10,$$

which shows that $k \geq 5$. This contradicts the bound $k \leq 4$. This proves the claim. Consequently, at most five guests may need to spend the night there.

We conclude that the largest number of the guests who might need to spend the night there is five. ■

Example 1.9 (Moscow MO 2015 Grade 11 Day 1 P5). Prove that it is impossible to put the integers from 1 to 64 (using each integer once) into an 8×8 table so that for any 2×2 square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$, the difference $ad - bc$ is equal to 1 or -1.

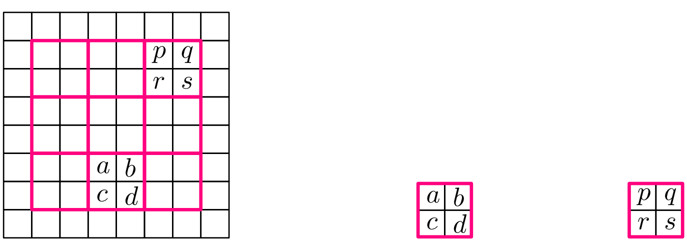


Figure 5: $ad - bc = \pm 1, ps - qr = \pm 1$, Example 1.9

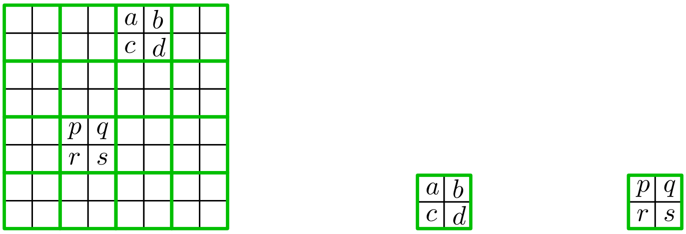


Figure 6: $ad - bc = \pm 1, ps - qr = \pm 1$, Example 1.9

Remark.

- Given a 2×2 square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$, the difference $ad - bc$ is equal to

  the product of the diagonal terms
  − the product of the anti-diagonal terms.

  Let us call this difference *the determinant* of the 2×2 square.
- For instance,
  - $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has determinant equal to -2,
  - $\begin{pmatrix} 8 & 9 \\ 7 & 12 \end{pmatrix}$ has determinant equal to $96 - 63 = 33$,
  - $\begin{pmatrix} 13 & 14 \\ 5 & 7 \end{pmatrix}$ has determinant equal to $91 - 70 = 21$.
  - Did you notice that if the determinant is odd, then the diagonal entries are odd or the anti-diagonal entries are odd?
- We need to show that there is no filling of an 8×8 table using the integers from 1 to 64, using each integer once, such that any 2×2 square (such squares have been marked in Fig. 5, Fig. 6, note that there $9 + 16 = 25$ such 2×2 squares.) has a determinant equal to 1 or -1. Equivalently[a], no matter how one may fill an 8×8 table using using the integers from 1 to 64, using each integer once, some 2×2 square has to have a determinant other than $1, -1$.

[a]Is the equivalence clear? Try to think about it!

Summary — If such a filling exists, then divide the 8×8 table into 16 pairwise disjoint 2×2 squares (as in Fig. 7). Due to parity constraints, each square contains precisely two evens along its diagonal or anti-diagonal, and their product is at most one more than the product of the odd entries. Consequently, for any of these 16 squares, the product of its even entries is less than the product of the successors of its odd entries. Multiplying across the squares gives a contradiction.

Walkthrough —

(a) Assume that such a filling exists.

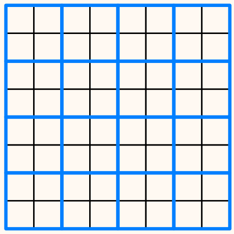
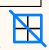


Figure 7: Moscow MO 2015 Grade 11 Day 1 P5, Example 1.9

(b) Recall that the determinant of a 2×2 square $\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$ is

$$\begin{aligned} &\text{the product of the diagonal terms} \\ &- \text{the product of the anti-diagonal terms.} \end{aligned}$$

(c) Note that $\boxed{\text{even} - \text{even} \neq \pm 1, \text{odd} - \text{odd} \neq \pm 1}$, and hence any square contains two odd numbers along the diagonal  or on the anti-diagonal .

(d) Divide the 8×8 table into 16 pairwise disjoint 2×2 squares.

(e) Each of these 16 squares contains at least two odd integers, and hence, they together contain at least 32 odd integers.

(f) Conclude that each of these 16 squares contains precisely two odd integers, and precisely two even integers.

(g) Consider a square among them. It is of the form

$$\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \text{ with } a, d \text{ both odd, and } b, c \text{ both even,}$$

or of the form

$$\begin{array}{|c|c|} \hline b & a \\ \hline d & c \\ \hline \end{array} \text{ with } a, d \text{ both odd, and } b, c \text{ both even.}$$

(h) The product of its even entries is at most one more than the product of its odd entries.

(i) Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b + 1)(c + 1)$ holds.

(j) This shows that

the product of two evens between 1 and 64
< the product of
two (possibly different) evens between 1 and 64.

(k) Multiply all the even entries of the 16 squares to obtain

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1+1) \cdot (3+1) \cdot \ldots \cdot (63+1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

Solution 6. Let us assume that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1.

Claim — Any 2×2 square contains at least two odd integers.

Proof of the claim. Since the difference of two integers can be odd only when they are of different parity (i.e. one of them is odd, and the other is even), it follows that for any 2×2 square, the product of its diagonal entries and the product of its anti-diagonal entries are of different parity, and hence of these two products is odd, and consequently, the diagonal entries are odd or the anti-diagonal entries are odd. In particular, any 2×2 square contains at least two odd integers. □

Let us divide the 8×8 table into 16 pairwise disjoint 2×2 squares (as in Fig. 7).

Claim — Each of these 16 squares contains exactly two even integers, lying along its diagonal or anti-diagonal.

Proof of the claim. By the previous Claim, each of these 16 squares contains at least two odd integers, and they contain at least $16 \times 2 = 32$ odd integers. Since there are precisely 32 odd integers between 1 and 64, it follows that each of these 16 squares contains exactly two odd integers along its diagonal or anti-diagonal, and hence exactly two even integers along its anti-diagonal or diagonal. □

Since the determinant of any 2×2 square is 1 or -1, it follows that for any of the 16 squares as in Fig. 7, the product of its even entries is at most one more than the product of its odd entries. Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b + 1)(c + 1)$ holds. Consequently, for any of the 16 squares as in Fig. 7, the product of its even entries is less than the product of the successors of its odd entries. This implies that the product of the even entries of all the 16 squares is less than the product of the successors of the odd entries of these boxes. Note that the even entries of these squares are the even integers lying between 1 and 64, so are the successors of the odd entries of these squares. It follows that

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1+1) \cdot (3+1) \cdot \ldots \cdot (63+1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

This contradicts the assumption that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1. Hence, no such filling is possible. ■

Example 1.10 (India RMO 2018b P4). Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

Summary — It relies on the fact that one can find enough points on the plane such that no three of them are collinear and no four of them are concyclic.

Walkthrough —

(a) There is an upper bound on the number of the red points in terms of the number of the white points. This gives an upper bound on the total number of points, which is 100, in terms of the number of the white points.

(b) Use this bound to guess the least possible number of the white points, which would turn out to be 10.

(c) Begin with 10 white points on the plane in *general position*, and then, introduce enough red points to construct a configuration of 100 points with the stated properties.

Solution 7. Let n denote the number of white points. Since each red point is the centre of a circle passing through at least three white points, it follows that the number of red points is at most $\binom{n}{3}$. This shows that

$$n + \binom{n}{3} \geq 100.$$

Note that $n \mapsto n + \binom{n}{3}$ defines an increasing function on the nonnegative integers. Observe that

$$9 + \binom{9}{3} = 93, \quad 10 + \binom{10}{3} = 130.$$

This implies that $n \geq 10$.

We claim that there is a configuration of 100 points on the plane such that it admits a coloring using two colors, red and white, such that precisely 10 points are colored white, and that each red point is the centre of a circle passing through at least three white points. Indeed, consider 10 points on the plane such that no three of them are collinear and no four of them are concyclic [1]. Color these 10 points white. These white points have $\binom{10}{3} = 120$ subsets of size 3. Consider only 90 such subsets of the white points, and for any such subset of size 3, color the center of the circle passing through them red. Since no three white points are collinear and no four white points are concyclic, it follows that there are precisely 90 pairwise distinct red points. So, the red and the white points together form a set of 100 points such that each red point is the centre of a circle passing through at least three white points. ■

References

[**Bru10**] Richard A. Brualdi. *Introductory combinatorics*. Fifth. Pearson Prentice Hall, Upper Saddle River, NJ, 2010, pp. xii+605. isbn: 978-0-13-602040-0; 0-13-602040-2 (cited p. 3)

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[1] Why does such a collection exist? This could be intuitively clear, but can you write down a precise proof? Does induction help?