Auxiliary configuration

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Auxiliary configuration

Example 1.1 (India RMO 1998 P6)**.** Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

With some trial-and-error, one can find the following example

$$T = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, d, g\}, \{c, e, f\}\},$$

and then check that it has the desired properties.

Walkthrough. Suppose we were not able to make a guess for such a set T [1]. In that situation, an approach would be to assume that such a collection T exists, and then try to find out some additional properties of T **using the hypothesis** that T has the stated properties.

- This approach has the **disadvantage** that the argument would rely on the assumption that such a set T exists, which may not be the case [2].
- However, the **advantage** is that we may be able to make further conclusions about such a putative set T, which may in turn allow us to correctly determine such a set T (or even all such sets T), or to even conclude that no such T exists (of course, depending on the problem).

(a) Show that the size of T is equal to 7.

(b) Relabel the elements of A as $1, 2, \dots, 7$.

(c) Assume that T contains $\{1, 2, 3\}$.

(d) Show that some of the 3-subsets lying in T, other than $\{1, 2, 3\}$, intersects with $\{1, 2, 3\}$ at exactly one element.

(e) Reordering the elements of A if necessary, prove that T contains $\{1, 4, 5\}$.

(f) Next, prove that T contains $\{1, 6, 7\}$, $\{2, 4, 6\}$, $\{3, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 5, 6\}$, and conclude that

$$T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

[1] Yes, do assume that we were not that clever!
[2] The problem, as stated, indicates the existence of such a T, but this does not suffice.

(g) **Wait!** We have **only** proved that a putative T is equal to the above up to a reordering of the elements of A. This **does not guarantee** the existence of any collection T having the desired properties.

(h) Thus, it still remains to find a collection T with the prescribed properties. Does the above collection of 3-subsets of A work?

Solution 1. Suppose T is a set consisting of some size 3-subsets of A such that any size two subset of A is contained in exactly one element of T. Consider the set

$$X = \{(P, Q) \mid P \subseteq Q, |P| = 2, Q \in T\}.$$

Note that X contains precisely $3 \cdot |T|$ many elements. For each subset P of A of size two, there is precisely one element (P, Q) in X. In other words, the map $(P, Q) \mapsto P$ from $X \to \binom{A}{2}$ is a bijection. It follows that $|T| = \frac{1}{3}\binom{7}{2} = 7$, provided there is a set T with the stated properties.

Note that if such a set T exists and it contains $\{a, b, c\}$, then we claim that some of the remaining 3-subsets lying in T has nonempty intersection with $\{a, b, c\}$. Otherwise, the remaining subsets would be subsets of $A \setminus \{a, b, c\}$, and hence T would contain at most $1 + \binom{7-3}{3} = 1 + 4 < 7 = |T|$ elements. This proves the claim. Hence, some of the 3-subsets lying in T, other than $\{a, b, c\}$, intersects with $\{a, b, c\}$ at exactly one element. [3]

Let's relabel the elements of A as $1, 2, \dots, 7$ for simplicity. The set T contains $\{1, 2, 3\}$ and some of the remaining elements of T intersects with $\{1, 2, 3\}$ at exactly one element. By reordering the elements of A if necessary, we assume that an element of T intersects $\{1, 2, 3\}$ at $\{1\}$. Note that this element contains none of $2, 3$. By reordering the elements of A once again if required, we assume that T contains $\{1, 4, 5\}$. Since T contains an element containing $\{1, 6\}$, it follows that $\{1, 6, 7\}$ lies in T. Since T contains an element containing $\{2, 4\}$, it follows that $\{2, 4, 6\}$ or $\{2, 4, 7\}$ lies in T. Reordering $6, 7$ if necessary, we assume that T contains $\{2, 4, 6\}$. Since T contains an element containing $\{3, 4\}$, it follows that $\{3, 4, i\}$ lies in T for some $1 \leq i \leq 7$ with $i \notin \{1, 2, 5, 6\}$, i.e., T contains $\{3, 4, 7\}$. Since T contains an element containing $\{2, 5\}$, it follows that $\{2, 5, 7\}$ lies in T. Since T contains an element containing $\{3, 5\}$, it follows that $\{3, 5, 6\}$ lies in T. Using that $|T| = 7$, it follows that

$$T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

Note that the above collection does have the desired properties. ■

[3] Having proved this statement, one may realize that it is much easier to establish! Indeed, such a collection T is nonempty, and hence contains the set $\{a, b, c\}$ up to a reordering of the elements of A. By hypothesis, some element of T contains $\{a, d\}$, and that element intersects with $\{a, b, c\}$ at precisely one element. This argument can be used to replace the longer argument above. It should be noted that the above argument introduces a **crucial idea**, namely, to determine the number of certain objects (here the number of elements of T), it is often helpful to take a detour by counting the number of objects of another type (here the number of pairs of the form (P, Q) satisfying suitable conditions). A similar idea is discussed in **??**.

> **Remark.** The above argument also shows that up to a reordering of the elements of A, such a set T is equal to
>
> $$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{3,4,7\},\{2,5,7\},\{3,5,6\}\}.$$

Example 1.2 (India RMO 2023a P4)**.** The set X of N four-digit numbers formed from the digits $1,2,3,4,5,6,7,8$ satisfies the following condition: *for any two different digits from* $1,2,3,4,5,6,7,8$*, there exists a number in* X *which contains both of them.* Determine the smallest possible value of N.

First, let's work on it. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X, we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1,\dots,8\}$ which does not occur as a digit in x. Note that after this modification of an element of X, the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1,\dots,8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X, it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1,2,\dots,8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1,\dots,8\}$ such that any size two subsets of $\{1,\dots,8\}$ is contained in some element of X.

Consider the set

$$A := \{(P,Q) \mid |P| = 2, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $\binom{4}{2}|X| = 6|X|$. By hypothesis, it follows that A contains at least $\binom{8}{2} = 28$ elements. Hence, X contains at least 5 elements.

With some effort, one can find the following set

$$\{\{1,2,3,4\},\{5,6,7,8\},\{1,5,2,6\},\{3,7,4,8\},\{1,7,8,2\},\{3,5,6,4\}\}.$$

So far, we have proved that X contains at least 5 elements. Based on this information and the above example only (possibly together with the fact that we are not able to come up with a set consisting of five size 4 subsets of $\{1,\dots,8\}$ having the stated property), we are not in a position to conclude that X has cardinality 6.

In fact, a modification of the above argument does prove that $|X| \geq 6$. Indeed, consider the set

$$A' := \{(P,Q) \mid |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1,\dots,8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1, \ldots, 8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements. ♣

Solution 2. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X, we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \ldots, 8\}$ which does not occur as a digit in x. Note that after this modification of an element of X, the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \ldots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X, it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \ldots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subsets of $\{1, \ldots, 8\}$ is contained in some element of X.

Consider the set

$$A := \{(P, Q) \mid |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $4|X|$. Also note that for any element $i \in \{1, \ldots, 8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1, \ldots, 8\}$ containing i, and the union of no two size 4 subsets of $\{1, \ldots, 8\}$ contains all the size two subsets of $\{1, \ldots, 8\}$ containing i. By hypothesis, it follows that A contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the set

$$\{\{1,2,3,4\}, \{5,6,7,8\}, \{1,5,2,6\}, \{3,7,4,8\}, \{1,7,8,2\}, \{3,5,6,4\}\}$$

has size 6 and it consists of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subset of $\{1, \ldots, 8\}$ is contained in one such size 4 subset.

This proves that the smallest possible value of N is 6. ■