Coloring proofs

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

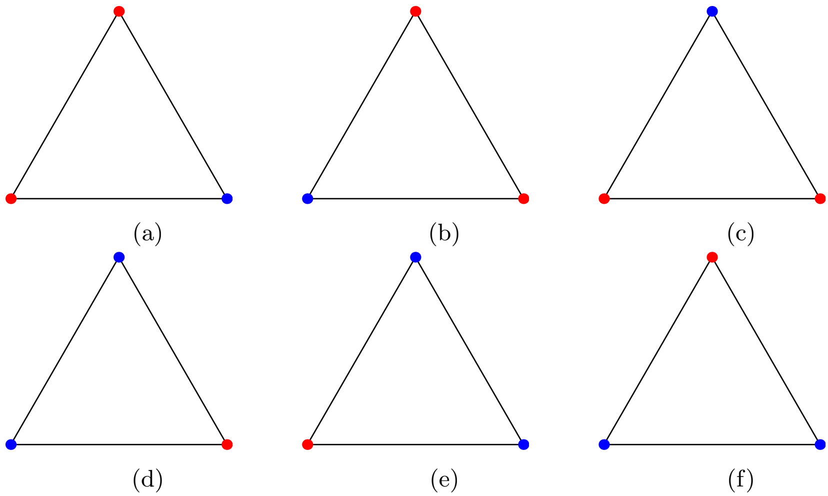

Figure 1: Two monochromatic points are one unit apart, Example 1.1

List of problems and examples

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§1 Coloring proofs

See [**Sob13**, §3.2], [**Eng98**, Chapter 2].

§1.1 Two coloring

Example 1.1. If each point of the plane is colored red or blue, then there are two points of the same color at distance 1 from each other.

Summary — Consider a "bigger" or an "auxiliary" structure such that its size puts some constraint on it (for instance, by the pigeonhole principle), which would lead to the result. See Fig. 1.

Solution 1. Consider an equilateral triangle with side-length 1. Then at least two of its vertices are of the same color, which proves the result. ■

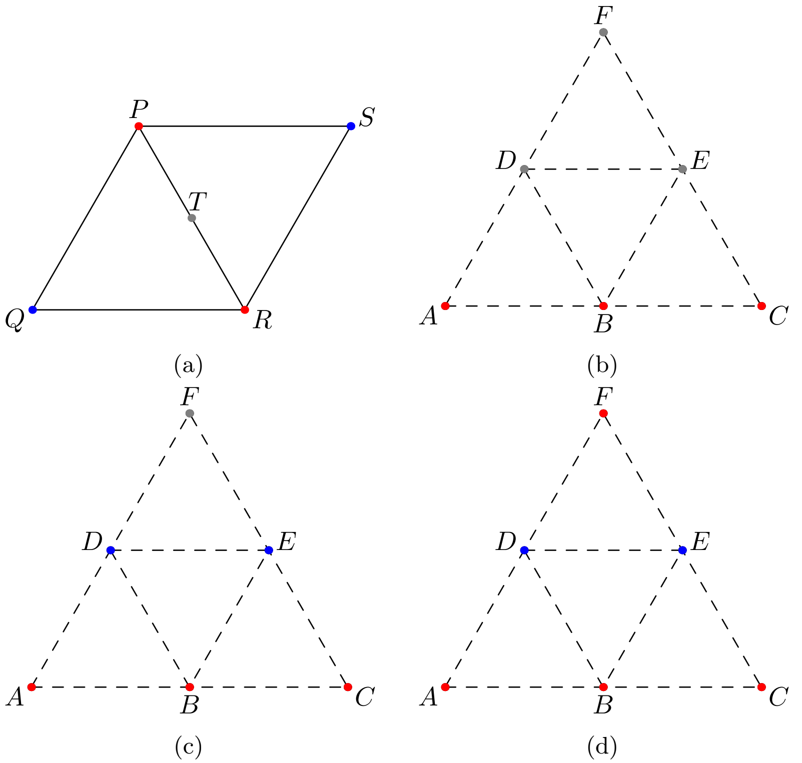

Figure 2: Example 1.2

Example 1.2. Suppose that to every point of the plane a colour, either red or blue, is associated.

1. Show that if there is no equilateral triangle with all vertices of the same colour then there must exist three points A, B and C of the same colour such that B is the mid-point of AC.

2. Use the above to conclude that there must be an equilateral triangle with all vertices of the same colour.

> **Summary** — Consider a "bigger" or an "auxiliary" structure such that its size puts some constraint on it (for instance, by the pigeonhole principle), which would lead to the result. See Fig. 2.

Solution 2. Let us prove the first part. Suppose there is no equilateral triangle with all vertices of the same color. So any equilateral triangle has two vertices of the same color. Let PQR be an equilateral triangle such that P, R are of the same color. Without loss of generality[1], assume that P, R are red. Note that Q is blue. Consider the equilateral triangle PRS as in Fig. 2a. Then S is blue. Let T denote the mid-point of PR. If T is red (resp. blue), then we could take (A, B, C) equal to (P, T, R) (resp. (Q, T, S)).

Suppose there is no equilateral triangle with all vertices of the same color. By the first part, there exist three points A, B, C such that B is the mid-point of AC (as in Fig. 2b). Let F be a point on the plane such that ACF is an equilateral triangle. Let D (resp. E) denote the mid-point of AF (resp. CF). Since ABD is equilateral, D is blue (as in Fig. 2c). Similarly, E is also blue (as in Fig. 2c). Hence F is red (as in Fig. 2d). So the triangle ACF has the required properties. Therefore, assuming that there is no equilateral triangle with all vertices of the same color, we proved that there is an equilateral triangle with all vertices of the same color. This shows that there is an equilateral triangle with all vertices of the same color. ■

Example 1.3 (Bay Area MO 2007)**.** The points of the plane are colored in red and blue so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

Solution 3. If not, then choose two points A, B from the plane which are of different color as in Fig. 3. Let C denote the mid-point of AB. Without loss of generality, we assume that B, C are of the same color, say red. Then A is blue. Draw a parallelogram $BCDE$. Note that if E were red, then considering the parallelogram $BCDE$, it would follow that D is red, and then considering the parallelogram $ACED$, we would obtain that A is red. Hence, E is blue. Since BC, DE are parallel and have the same length, so are AC and DE, and hence $ACDE$ is a parallelogram. Considering the parallelogram $ACDE$ once again, it follows that D is blue. Since A, D, E are blue, C cannot be red. Hence all points of the plane are of the same color. ■

§1.2 Three coloring

Example 1.4. Suppose that each point in the plane is colored red, green or blue. Prove that either there are two points of the same color a distance 1 unit apart, or there is an equilateral triangle of side length $\sqrt{3}$ all of whose vertices are of the same color.

> **Summary** — Consider a "bigger" or an "auxiliary" structure such that its size puts some constraint on it (for instance, by the pigeonhole principle), which would lead to the result. See Figs. 4 and 5.

Solution 4. Note that it suffices to prove the claim below.

[1] Is it clear that there is *no loss of generality*?

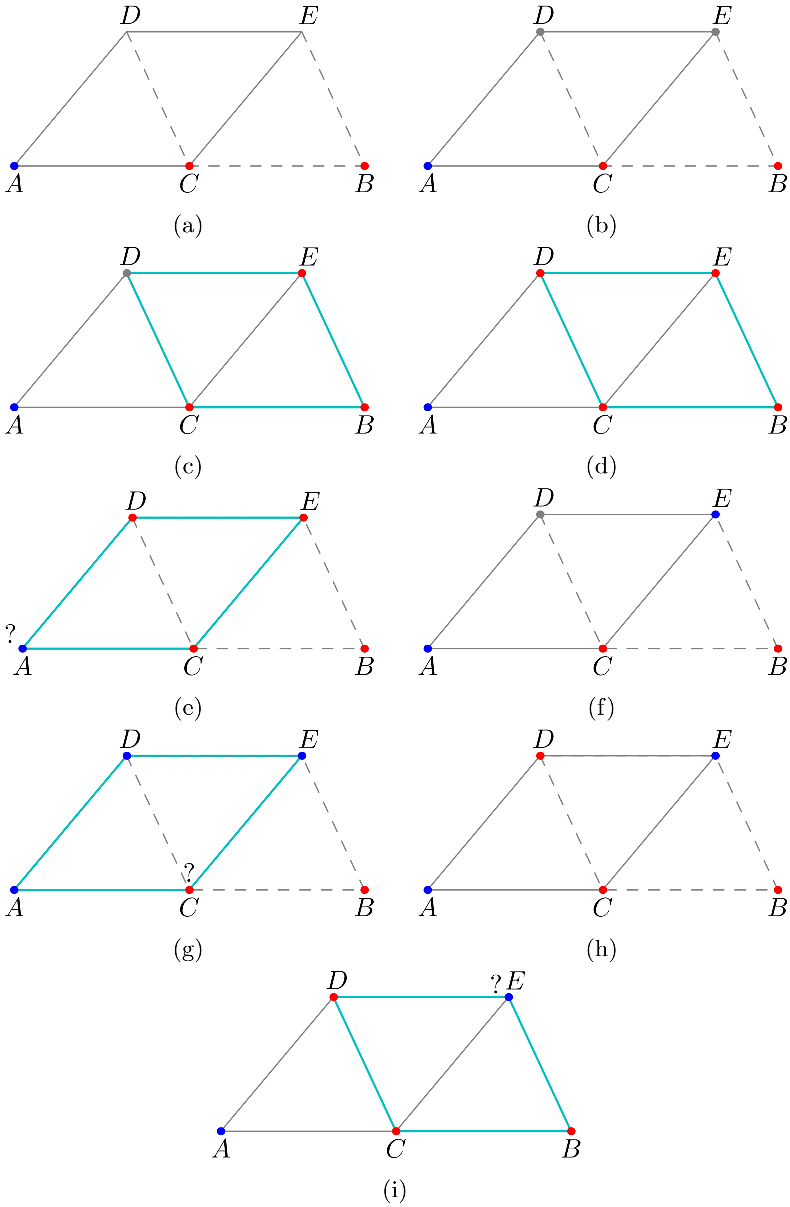

Figure 3: Bay Area MO 2007 P2, Example 1.3

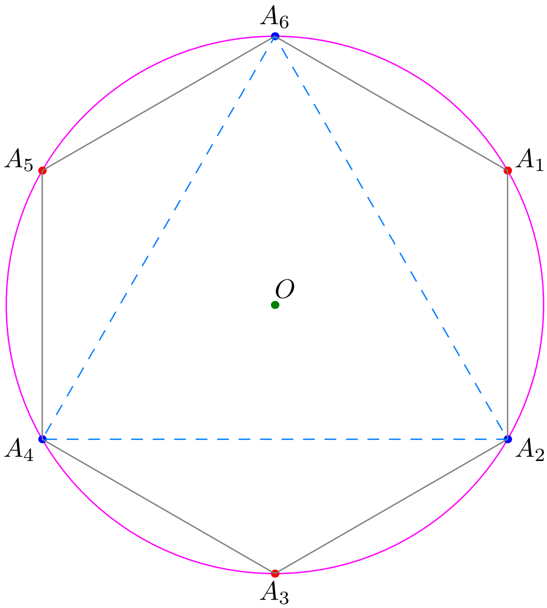

Figure 4: Example 1.4

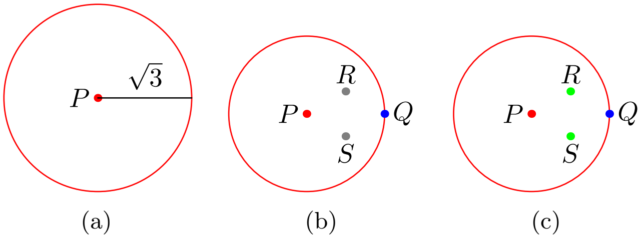

Figure 5: Remark 1

Claim — Suppose that each point in the plane is colored red, green or blue. If no two points, which are one unit apart, are of the same color, then there is an equilateral triangle of side length $\sqrt{3}$ all of whose vertices are of the same color.

Proof of the Claim. Assume that no two points which are one unit apart are of the same color. Choose a point O from the plane and without loss of generality assume that O is green. Then draw a unit circle with centre at O, and a regular hexagon circumscribed in this unitcircle as shown in Fig. 4. By the assumption, none of its vertices are green, and hence a vertex is red or blue. Since the edges of the hexagon are of length one, it follows that the colors of its vertices are red and green alternatively. Then the triangle $A_2A_4A_6$ is an equilateral triangle of side length $\sqrt{3}$ and all of its vertices are of the same color. This proves the claim. □

■

Remark 1. In fact, one can show that when the points of the plane are three colored, then there are two monochromatic points that are one unit apart.

Claim — Suppose that each point in the plane is colored red, green or blue. Then there are two monochromatic points that are one unit apart.

Proof of the Claim. On the contrary, let us assume that no two points, which are one unit apart, are monochromatic. Let us consider a point P on the plane. Suppose it is red, and consider the circle of radius $\sqrt{3}$ with centre at P.

In fact, assuming each point in the plane is colored red, green or blue, one can show that there are two monochromatic points that are one unit apart. Otherwise, we choose a point P, suppose it is red, and consider the circle $\mathcal{C}$ of radius $\sqrt{3}$ with centre at P (as in Fig. 5a).

SubClaim — All points on the circumference of $\mathcal{C}$ are red.

Proof of the SubClaim. On the contrary, let us suppose that there is a point on the circumference of $\mathcal{C}$ which is red. Consider one such point Q. Suppose it is blue. Consider the points R, S (as in Fig. 5b), which lie on the opposite sides of PQ such that RS is perpendicular to PQ and PRS is an equilateral triangle. Note that the point R has to be green. Indeed, if R is red (resp. green), then the points P, R (resp. Q, R) are monochromatic and one unit apart, which contradicts our assumption. Similarly, the point S also has to be green (as in Fig. 5c). Note that R, S are one unit apart. Since they are monochromatic, we obtain a contradiction to our assumption. This proves the subclaim. □

Consider any chord of $\mathcal{C}$ of length one unit. The end-points of the chord are red by the SubClaim. The claim follows. □

§1.3 Further coloring problems

Example 1.5 (India RMO 2017a P4)**.** Consider n^2 unit squares in the xy-plane centred at the point (i, j) with integer coordinates for $1 \leq i \leq n, 1 \leq j \leq n$. It is required to colour each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, the three squares with centres at (i, k), (j, k), (j, ℓ) have distinct colours. What is the least possible number of colours needed?

Remark. For such problems, it is often useful to first work out a special case.

Walkthrough —

(a) First, work out a simple case in order to gain insight for the general case.

(b) One may consider the squares below a suitable diagonal.

(c) Extend to the general case!

First, let's work on it. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 6a have pairwise distinct colors. Let's apply the colors $1, \ldots, 8$ to these squares as in Fig. 6b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let's apply the colors $3, \ldots, 9$ to these squares as in Fig. 6c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let's apply another set of $8 - 3 + 1$ new colors (for example, $5, \ldots, 10$) to these squares as in Fig. 6d. One may continue this process to yield the coloring as in Fig. 6e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 2, as in Fig. 7a. In the third last row, the colors $3, 4$ may be used as in Fig. 7b. In the fourth last row, the colors smaller than 7 may be used as in Fig. 7c. One may continue this process to yield the coloring as in Fig. 7d. Note that the coloring as in Fig. 7d **does satisfy** the required condition. ♣

Remark. The **preceding argument alone** does not guarantee that the least possible number of colors have been used in Fig. 7d.

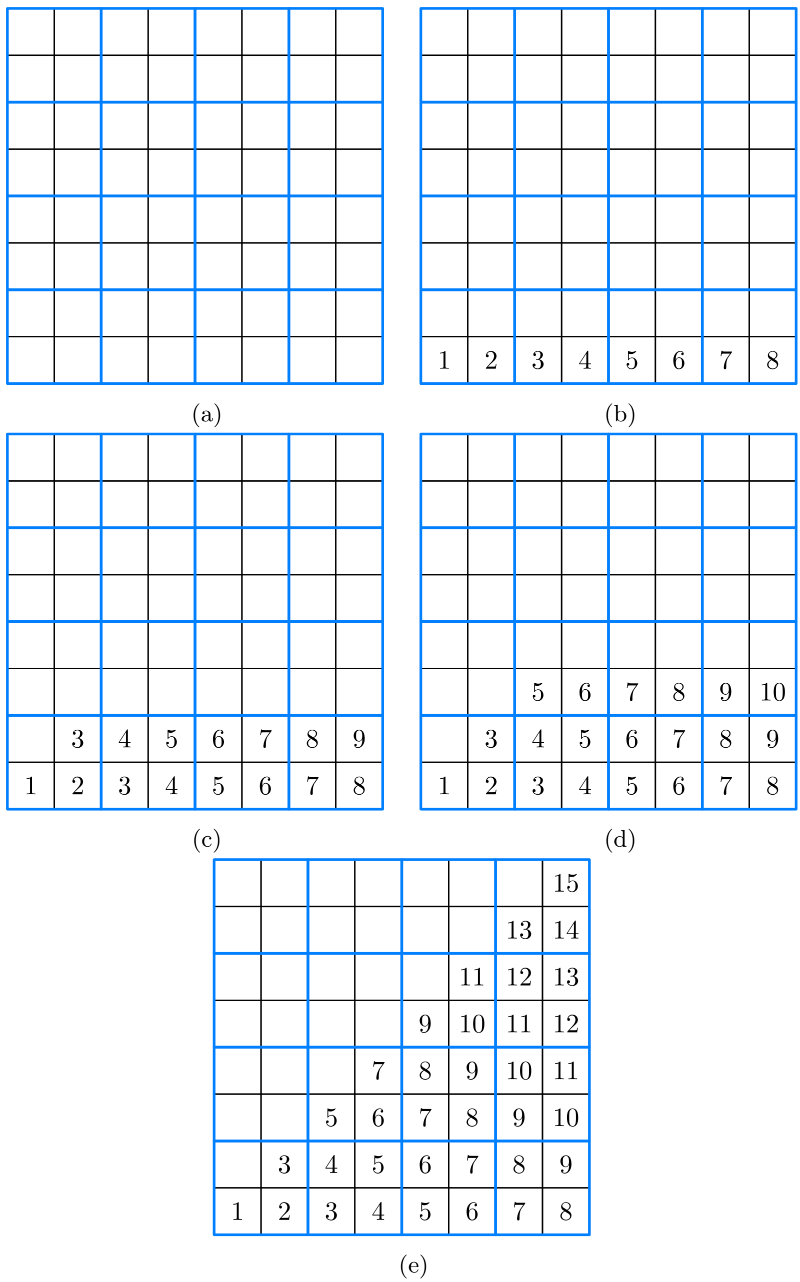

Figure 6: India RMO 2017, Example 1.5

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
		5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(a)

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(b)

							15
						13	14
					11	12	13
				9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(c)

8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(d)

Figure 7: India RMO 2017, Example 1.5

Skip this part for the first reading —

Remark. The **preceding argument alone** does not guarantee that the least possible number of colors have been used in Fig. 7d. It could have been the case that in our very first attempt of working out a special case, we might have proceeded in way which would have used more colors than what is required. For instance, consider the following approach.

In fact, that is what I (the author) did in a prior attempt, and then I could not prove that the number of colors that one may require to color the squares satisfying the required condition is at least as large as the number of colors used in the approach below. This led to think again, which yield the argument above, in which case,

¶ A prior attempt that used more colors than the minimum number of colors required. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 8a have pairwise distinct colors. Let us apply the colors $1, \dots, 8$ to these squares as in Fig. 8b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let us apply the colors $9, \dots, 15$ to these squares as in Fig. 8c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let us apply another set of $8 - 3 + 1$ new colors (for example, $16, \dots, 21$) to these squares as in Fig. 8d. One may continue this process to yield the coloring as in Fig. 8e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 8, as in Fig. 9a. In the third last row, the colors $14, 15$ may be used as in Fig. 9b. In the fourth last row, the colors smaller than 22 may be used as in Fig. 9c. One may continue this process to yield the coloring as in Fig. 9d. Note that the coloring as in Fig. 9d does satisfy the required condition. ♣

Now that we have gained some experience, we may proceed to the general case as follows.

Bogus Solution. Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying to the right of the above-mentioned diagonal and contained in a row except the bottom one, have colors different from the unit squares contained in the rows below and lying to the right of that diagonal. Hence, we require at least

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{1}{2}n(n+1)$$

colors.

Moreover, similar to the coloring as in Fig. 9, we may color the n^2 unit squares using $\frac{1}{2}n(n+1)$ colors so that required condition is satisfied.

This proves that the least number of colors required is $\frac{1}{2}n(n+1)$.

Question. Where does the above go wrong?

Now that we have gained some idea, we may proceed to the general case as follows.

Solution 5. If $n = 1$, then using one color works. Let us assume[2] that $n \geq 2$.

Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying on the rightmost column except the bottom square, have colors different from those of the squares lying on the bottom row. Hence, we require at least

$$n + (n-1) = 2n - 1$$

colors.

Next, we show that there is a coloring of the n^2 unit squares using $2n-1$ colors satisfying the required condition. Indeed, if the colors $i, i+1, \ldots, i+n-1$ are applied to the squares lying in the i-th last row in an increasing order from the left to the right for all $1 \leq i \leq n$, then the squares in any row have distinct colors and so are the squares in any column, and hence, it is a coloring satisfying the required condition.

This shows that the least number of colors required is $2n-1$. ■

References

[Eng98] Arthur Engel. *Problem-solving strategies.* Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 2)

[2] The reason for considering the case $n = 1$ separately will become clear from the rest of the argument. Is it clear what would go wrong with the rest of the argument if $n = 1$?

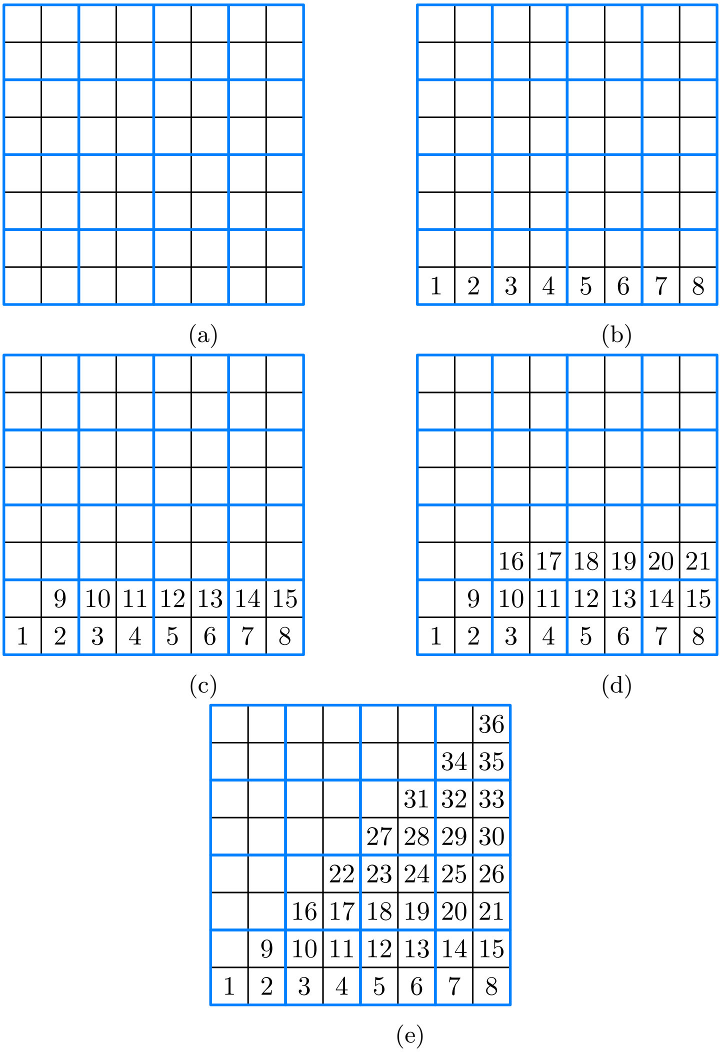

Figure 8: India RMO 2017, Example 1.5

							36
						34	35
					31	32	33
				27	28	29	30
			22	23	24	25	26
		16	17	18	19	20	21
8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8

(a)

							36
						34	35
					31	32	33
				27	28	29	30
			22	23	24	25	26
14	15	16	17	18	19	20	21
8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8

(b)

							36
						34	35
					31	32	33
				27	28	29	30
19	20	21	22	23	24	25	26
14	15	16	17	18	19	20	21
8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8

(c)

29	30	31	32	33	34	35	36
28	29	30	31	32	33	34	35
26	27	28	29	30	31	32	33
23	24	25	26	27	28	29	30
19	20	21	22	23	24	25	26
14	15	16	17	18	19	20	21
8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8

(d)

Figure 9: India RMO 2017, Example 1.5

[**Sob13**] Pablo Soberón. *Problem-solving methods in combinatorics*. An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. isbn: 978-3-0348-0596-4; 978-3-0348-0597-1. doi: 10.1007/978-3-0348-0597-1. url: http://dx.doi.org/10.1007/978-3-0348-0597-1 (cited p. 2)