Counting in two different ways

MOPSS

24 February 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

1.1 Example (India BStat-BMath 2014) 2

§1 Counting in two different ways

See [**AF13**, §7], [**AE11**, §3.3].

Example 1.1 (India BStat-BMath 2014)**.** A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \cdots + a_{100} = c_1 + c_2 + \cdots + c_{99}.$$

Solution 1. For $1 \leq i \leq 100$, denote the i-th student by s_i. For $1 \leq j \leq 99$, let C_j denote the set of students having at least j friends. Note that for any $1 \leq i \leq 100$,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for $1 \leq j \leq 99$, 1_{C_j} denotes the map, defined on $\{s_1, s_2, \ldots, s_{100}\}$, given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over $1 \leq i \leq 100$, and interchanging the order of summation, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &= \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i) \\ &= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}| \\ &= \sum_{j=1}^{99} c_j. \end{aligned}$$

This completes the proof. ∎

> **Remark 1.** The following is **somewhat naive**, and **does require** additional explanation to be included (at which step(s)?). The following explains (with some effort from readers' end, of course!) why the stated result should hold, and it may also help to arrive at the above solution. However, the following

lacks some details.

Walkthrough —

$$
\begin{array}{cccccccc}
 & c_1 \text{ bullets} & c_2 \text{ bullets} & c_3 \text{ bullets} & \cdots & c_j \text{ bullets} & \cdots & \\
 & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
s_1 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_1 \text{ bullets} \\
s_2 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_2 \text{ bullets} \\
s_3 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_3 \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_i & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_i \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_{100} & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_{100} \text{ bullets}
\end{array}
$$

Denote the students by $s_1, s_2, \ldots, s_{100}$ and write $s_1, s_2, \ldots, s_{100}$ in a vertical column. Then for each $1 \leq i \leq 100$, put a_i bullets next to s_i (as shown above). Then the sum $a_1 + \cdots + a_{100}$ is equal to the total number of bullets. It turns out that the number of bullets in the j-th column is equal to c_j for any $1 \leq j \leq 99$, proving that the total number of bullets is also equal to $c_1 + c_2 + c_3 + \cdots + c_{99}$.

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)

[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. ISBN: 9780817681548. URL: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)