Inclusion-exclusion principle

MOPSS

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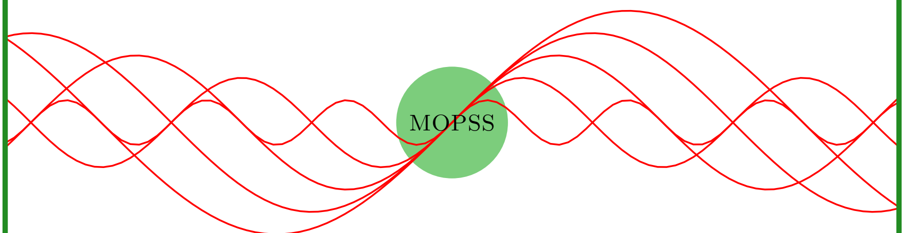
Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Inclusion-exclusion principle

Example 1.1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution 1. Let A (resp. B) denote the set of integers not exceeding 1000 that are divisible by 7 (resp. 11). Then the size of $A \cup B$ is equal to

$$\#A + \#B - \#A \cap B = \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor = 142 + 90 - 12 = 220.$$

■

Example 1.2 (India RMO 2015f P6)**.** From the list of natural numbers $1, 2, 3, \ldots,$ suppose we remove all multiples of 7, 11 and 13.

- At which position in the resulting list does the number 1002 appear?
- What number occurs in the position 3600?

Solution 2. Let S denote the set of all positive integers none of which is divisible by 7, 11 or 13. Note that the sum of $\#\{x \in S \,|\, x \leq 1002\}$ and the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to 1002. Moreover, by the inclusion-exclusion principle, the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to

$$\left\lfloor \frac{1002}{7} \right\rfloor + \left\lfloor \frac{1002}{11} \right\rfloor + \left\lfloor \frac{1002}{13} \right\rfloor - \left\lfloor \frac{1002}{7 \cdot 11} \right\rfloor - \left\lfloor \frac{1002}{11 \cdot 13} \right\rfloor - \left\lfloor \frac{1002}{13 \cdot 7} \right\rfloor + \left\lfloor \frac{1002}{7 \cdot 11 \cdot 13} \right\rfloor$$
$$= 143 + 91 + 77 - 13 - 7 - 11 + 1 = 281.$$

Note that 1002 belongs to S and hence it appears at the $1002 - 281 = 721$st position.

Suppose n occurs at the 3600th position. So the number of positive integers $\leq n$ divisible by 7, 11 or 13 is equal to $n - 3600$. Using the inclusion-exclusion principle, we obtain

$$n - 3600 = \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor - \left\lfloor \frac{n}{7 \cdot 11} \right\rfloor - \left\lfloor \frac{n}{11 \cdot 13} \right\rfloor - \left\lfloor \frac{n}{13 \cdot 7} \right\rfloor + \left\lfloor \frac{n}{7 \cdot 11 \cdot 13} \right\rfloor. \tag{1}$$

This gives

$$\begin{aligned}
&\frac{n}{7} - 1 + \frac{n}{11} - 1 + \frac{n}{13} - 1 - \frac{n}{7\cdot 11} - \frac{n}{11\cdot 13} - \frac{n}{13\cdot 7} + \frac{n}{7\cdot 11\cdot 13} - 1\\
&\le n - 3600\\
&\le \frac{n}{7} + \frac{n}{11} + \frac{n}{13} - \frac{n}{7\cdot 11} + 1 - \frac{n}{11\cdot 13} + 1 - \frac{n}{13\cdot 7} + 1 + \frac{n}{7\cdot 11\cdot 13},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&-4\\
&\le n\left(1 - \frac{1}{7} - \frac{1}{11} - \frac{1}{13} + \frac{1}{7\cdot 11} + \frac{1}{11\cdot 13} + \frac{1}{13\cdot 7} - \frac{1}{7\cdot 11\cdot 13}\right) - 3600\\
&\le 3.
\end{aligned}$$

Noting that

$$\left(1 - \frac{1}{7} - \frac{1}{11} - \frac{1}{13} + \frac{1}{7\cdot 11} + \frac{1}{11\cdot 13} + \frac{1}{13\cdot 7} - \frac{1}{7\cdot 11\cdot 13}\right) = \frac{6\cdot 10\cdot 12}{7\cdot 11\cdot 13},$$

we obtain

$$-4\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12} \le n - 5\cdot 7\cdot 11\cdot 13 \le 3\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12},$$

which yields

$$-\frac{7\cdot 11\cdot 13}{180} \le n - 5\cdot 7\cdot 11\cdot 13 \le \frac{7\cdot 11\cdot 13}{240},$$

and consequently, any solution of the Eq. (1) satisfies

$$-5 \le n - 5\cdot 7\cdot 11\cdot 13 \le 4.$$

Note that $n = 5\cdot 7\cdot 11\cdot 13$ is the unique solution to

$$n\left(1 - \frac{1}{7} - \frac{1}{11} - \frac{1}{13} + \frac{1}{7\cdot 11} + \frac{1}{11\cdot 13} + \frac{1}{13\cdot 7} - \frac{1}{7\cdot 11\cdot 13}\right) - 3600 = 0,$$

and it is a multiple of the pairwise coprime integers $7, 11, 13$. Hence, it is a solution to Eq. (1). Also note that $5\cdot 7\cdot 11\cdot 13 - 1$ is also a solution to Eq. (1). Moreover, no integer lying in $[-6 + 5\cdot 7\cdot 11\cdot 13, 4 + 5\cdot 7\cdot 11\cdot 13]$, other than $5\cdot 7\cdot 11\cdot 13 - 1$ and $5\cdot 7\cdot 11\cdot 13$ is a solution to Eq. (1). Observe that $5\cdot 7\cdot 11\cdot 13 - 1$ lies in S.

We conclude that the number that occurs in the 3600th position is $5\cdot 7\cdot 11\cdot 13 - 1 = 5004$. ■

Example 1.3 (India RMO 2019b P5)**.** There is a pack of 27 distinct cards, and each card has three values on it. The first value is a shape from $\{\Delta, \square, \odot\}$; the second value is a letter from $\{A, B, C\}$; and the third value is a number from $\{1, 2, 3\}$. In how many ways can we choose an unordered set of 3 cards from the pack, so that no two of the chosen cards have two matching values. For example we can chose $\{\Delta A1, \Delta B2, \odot C3\}$. But we cannot choose $\{\Delta A1, \square B2, \Delta C1\}$.

Solution 3. Let $\mathcal{A}$ denote the set of ordered tuples (u, v, w) with $u, v, w \in (\mathbb{Z}/3\mathbb{Z})^3$ such that no two among u, v, w have two equal coordinates. For $u \in (\mathbb{Z}/3\mathbb{Z})^3$, let $\mathcal{A}_u$ denote the set of ordered tuples lying in $\mathcal{A}$ with the first coordinate equal to u. Note that $(u, v, w) \mapsto (u, v, w) - (u, u, u)$ defines a bijection between $\mathcal{A}_u$ and $\mathcal{A}_0$. Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot |\mathcal{A}_0|$. Note that

$$\begin{aligned}\mathcal{A}_0 = \{(0, v, w) \,|\, v \notin \langle e_1\rangle \cup \langle e_2\rangle \cup \langle e_3\rangle, w \notin \langle e_1\rangle \cup \langle e_2\rangle \cup \langle e_3\rangle,\\ w \notin (v + \langle e_1\rangle) \cup (v + \langle e_2\rangle) \cup (v + \langle e_3\rangle)\},\end{aligned}$$

and hence

$$\begin{aligned}&|\mathcal{A}_0|\\ &= \sum_{v\notin\langle e_1\rangle\cup\langle e_2\rangle\cup\langle e_3\rangle} \left|(\langle e_1\rangle \cup \langle e_2\rangle \cup \langle e_3\rangle)^c \bigcap ((v + \langle e_1\rangle) \cup (v + \langle e_2\rangle) \cup (v + \langle e_3\rangle))^c\right|.\end{aligned}$$

For any subset S of $(\mathbb{Z}/3\mathbb{Z})^3$ and an element $v \in (\mathbb{Z}/3\mathbb{Z})^3$, we have

$$\begin{aligned}|S^c \cap (v + S)^c| &= 27 - |(S^c \cap (v + S)^c)^c|\\ &= 27 - |S \cup (v + S)|\\ &= 27 - |S| - |v + S| + |S \cap (v + S)|\\ &= 27 - 7 - 7 + |S \cap (v + S)|\\ &= 13 + |S \cap (v + S)|.\end{aligned}$$

Henceforth, S denotes the subset $\langle e_1\rangle \cup \langle e_2\rangle \cup \langle e_3\rangle$ of $(\mathbb{Z}/3\mathbb{Z})^3$, and v denotes an element of $(\mathbb{Z}/3\mathbb{Z})^3$ lying outside S.

First, let us consider the case when v has exactly two nonzero coordinates. It follows that v lies in exactly one of $\langle e_1\rangle - \langle e_2\rangle$, $\langle e_2\rangle - \langle e_3\rangle$, $\langle e_3\rangle - \langle e_1\rangle$. Observe that the set $S \cap (v + S)$ is equal to the union of $\langle e_i\rangle \cap (v + \langle e_j\rangle)$ for $1 \le i, j \le n$. It follows that $S \cap (v + S)$ has size two.

Now, let us consider the case when all coordinates of v are nonzero. Note that $\langle e_i\rangle \cap (v + \langle e_j\rangle)$ is empty for any $1 \le i, j \le n$. Hence, so is the set $S \cap (v + S)$.

Note that the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having exactly two nonzero coordinates is $3 \cdot 2 \cdot 2 = 12$, and the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having all coordinates nonzero is $2^3 = 8$. It follows that

$$|\mathcal{A}_0| = 12 \cdot (13 + 2) + 8 \cdot (13 + 0) = 284.$$

Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot 284 = 1278$. ■