INMOTC 2025 (MP region)

Algebra

11 January 2025

List of problems and examples

§1 Problems

Example 1.1 (Moscow MO 1946 Grades 7–8 P5)**.** Prove that after completing the multiplication and collecting the terms

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$$

has no monomials of odd degree.

> **Summary** — What happens if x is replaced by $-x$?

Example 1.2. Let n be an even positive integer, and let $p(x)$ be a polynomial of degree n such that $p(k) = p(-k)$ for $k = 1, 2, \ldots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

> **Walkthrough** — Note that the polynomial $p(x) - p(-x)$ has degree $< n$ because n is even. Observe that it has at least n roots.

> **Remark.** What would happen if n is not assumed to be even?

Example 1.3. Determine the remainder when $x + x^9 + x^{25} + x^{49} + x^{81} + x^{121}$ is divided by $x^3 - x$.

Example 1.4 (Moscow MO 2015 Grade 9 P6)**.** Do there exist two polynomials with integer coefficients such that each of them has a coefficient with absolute value exceeding 2015, but no coefficient of their product has absolute value exceeding 1?

> **Summary** — Try to come up with enough polynomials $g_1(x), g_2(x), g_3(x), \ldots$ and $h_1(x), h_2(x), h_3(x), \ldots$ such that each of the products $g_1g_2g_3\ldots$ and $h_1h_2h_3\ldots$ have at least one coefficient which is **large in absolute value**, and all the coefficients of the product $(g_1g_2g_3\ldots)(h_1h_2h_3\ldots)$ are at most 1 in absolute value.

§2 Factorization and roots

Example 2.1. Let $g(x)$ and $h(x)$ be polynomials with real coefficients such that

$$g(x)(x^2 - 3x + 2) = h(x)(x^2 + 3x + 2)$$

and $f(x) = g(x)h(x) + (x^4 - 5x^2 + 4)$. Prove that $f(x)$ has at least four real roots.

Example 2.2 (USAMO 1975 P3). [**PutnamBeyond**] A polynomial $P(x)$ of degree n satisfies

$$P(k) = \frac{k}{k+1} \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Find $P(n+1)$.

Example 2.3. Let $P(x)$ be a polynomial with real coefficients. Assume that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist polynomials g, h with real coefficients such that

$$P = g^2 + h^2.$$

Walkthrough —

(a) Show that the real roots of P have even multiplicity.

(b) Conclude that P can be expressed as a product of monic quadratic polynomials with real coefficients having nonreal roots, and even powers of degree one polynomials with real coefficients.

(c) Show that a monic quadratic polynomial with real coefficients having nonreal roots is the sum of the squares of two polynomials with real coefficients.

§3 Roots of unity

Example 3.1 (USAMO 2014 P1). Let a, b, c, d be real numbers such that $b-d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4+ax^3+bx^2+cx+d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

Example 3.2. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeroes lie on the unit circle. Show that all the zeroes of $P(x)$ are roots of unity, that is, $P(x)$ divides $(x^n - 1)^k$ for some positive integers n, k.

Example 3.3 (USAMO 1976 P5). If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

Example 3.4 (Leningrad Math Olympiad 1991). A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form

$$a_k + a_{k+p} + a_{k+2p} + \ldots$$

is the same for any $k = 1, 2, 3, \ldots$. For instance the sequence

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2$$

is a 3-balanced. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

Summary — Consider the polynomial $\sum_{i=1}^n a_i x^n$.

§4 Growth of polynomials

Example 4.1 (India RMO 2015b P3)**.** Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

§5 Size of the roots

Example 5.1. Find all polynomials P (with complex coefficients) satisfying

$$P(x)P(x+2) = P(x^2).$$

Summary — Note that if α is a root of P, then so are α^2 and $(\alpha-2)^2$. Also note that if $\alpha \neq 1$, then $|(\alpha-2)^2| > |\alpha|$. Conclude that $P(x) = c(x-1)^n$.

Example 5.2 (INMO 2018 P4)**.** Find all polynomials $P(x)$ with real coefficients such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Walkthrough —

(a) Show that if α is a root of $P(x)$, then $P(x)$ vanishes at $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$, where β_1, β_2 are the roots of $x^2 + x + 1 = \alpha$.

(b) If α is nonzero, then show that one of $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$ is larger than α in absolute value.

Example 5.3. Does there exist a polynomial $f(x)$ satisfying

$$xf(x-1) = (x+1)f(x)?$$

§6 Differentiation and double roots

Lemma 1

Let $P(x)$ be a polynomial with complex coefficients, and α be a complex number. Then α is a double root of $P(x)$ (i.e., $(x-\alpha)^2$ divides $P(x)$) if and only if it is a root of $P(x)$ and $P'(x)$.

Example 6.1. Let $P(x), Q(x)$ be polynomials with complex coefficients such that they have the same set of roots with possibly different multiplicities. Suppose that $P+1, Q+1$ also have the same set of roots with possibly different multiplicities. Show that $P = Q$.

Walkthrough —

(a) Assume that $\deg P \geq \deg Q$.

(b) Denote these two set of roots by S_1, S_2. Considering multiplicities, show that

$$2\deg P - |S_1| - |S_2| \leq \deg P' = \deg P - 1,$$

which yields

$$|S_1| + |S_2| > \deg P.$$

(c) Note that $P - Q$ vanishes at the elements of $S_1 \cup S_2$, which has size larger than the degree of $P - Q$.

§7 Crossing the x-axis/Intermediate value theorem

Example 7.1 (China TST 1995). Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1.$$

They fill in real numbers to empty boxes in turn. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?

§8 Lagrange interpolation

Example 8.1. If a polynomial of degree n takes rationals to rationals on $n+1$ points, then show that it is a rational polynomial.

Example 8.2 (USAMO 2002 P3). Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Example 8.3. For each positive integer $n \geq 1$, determine all monic polynomials of degree n whose roots are all real, for which every coefficient is either 1 or -1.

Walkthrough —

(a) Consider the average of the squares of the roots, and show that it is small (and consequently, smaller than their geometric mean) if the polynomial has degree ≥ 4.

(b) Repeat the argument for degree three polynomials.

(c) Finding the degree one and degree two Polynomials is easy.

§9 Integer divisibility

Lemma 2

If P is a polynomial with integer coefficients and a, b are integers, then $P(a) - P(b)$ is a multiple of $a - b$.

Example 9.1 (USAMO 1974 P1)**.** Let a, b, and c denote three distinct integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

Example 9.2. Let $P(x)$ be a polynomial with integer coefficients, and let n be an odd positive integer. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that $x_2 = P(x_1), x_3 = P(x_2), \ldots, x_n = P(x_{n-1})$, and $x_1 = P(x_n)$. Prove that all the x_i's are equal.

Walkthrough — Show that

$$a_1 - a_2 \mid a_2 - a_3 \mid a_3 - a_4 \mid \cdots \mid a_n - a_1 \mid a_1 - a_2.$$

Note that sum of these differences is an odd multiple of their absolute value.

Lemma 3

Let P be a polynomial with integer coefficients. Suppose a is an integer and k is a positive integer such that $P(P(\ldots P(P(a))\ldots)) = a$, where P occurs k times. Show that $P(P(a)) = a$.

Example 9.3 (IMO 2006 P5)**.** (Dan Schwarz, Romania) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients, and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.