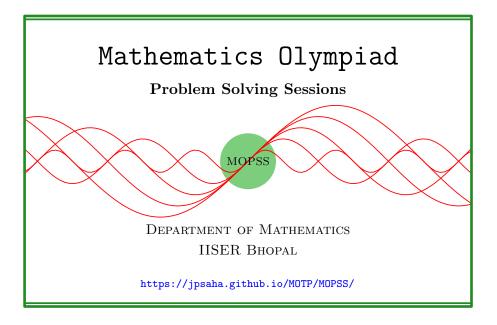
# Counting in two different ways

#### MOPSS

24 February 2025



## **Suggested readings**

- Evan Chen's
  - advice On reading solutions, available at https://blog.evanchen. cc/2017/03/06/on-reading-solutions/.
  - Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

### List of problems and examples

## §1 Counting in two different ways

See  $[AF13, \S7]$ ,  $[AE11, \S3.3]$ .

**Example 1.1** (India BStat-BMath 2014). A class has 100 students. Let  $a_i$ ,  $1 \le i \le 100$ , denote the number of friends the *i*-th student has in the class. For each  $0 \le j \le 99$ , let  $c_j$  denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \dots + a_{100} = c_1 + c_2 + \dots + c_{99}.$$

**Solution 1.** For  $1 \le i \le 100$ , denote the *i*-th student by  $s_i$ . For  $1 \le j \le 99$ , let  $C_j$  denote the set of students having at least j friends. Note that for any  $1 \le i \le 100$ ,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for  $1 \leq j \leq 99$ ,  $1_{C_j}$  denotes the map, defined on  $\{s_1, s_2, \ldots, s_{100}\}$ , given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

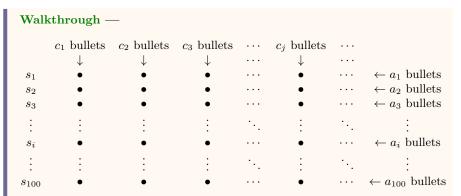
Summing over  $1 \leq i \leq 100$ , and interchanging the order of summation, we obtain

$$a_1 + a_2 + \dots + a_{100} = \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i)$$
$$= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}|$$
$$= \sum_{j=1}^{99} c_j.$$

This completes the proof.

Remark 1. The following is somewhat naive, and does require additional explanation to be included (at which step(s)?). The following explains (with some effort from readers' end, of course!) why the stated result should hold, and it may also help to arrive at the above solution. However, the following

lacks some details.



Denote the students by  $s_1, s_2, \ldots, s_{100}$  and write  $s_1, s_2, \ldots, s_{100}$  in a vertical column. Then for each  $1 \le i \le 100$ , put  $a_i$  bullets next to  $s_i$  (as shown above). Then the sum  $a_1 + \cdots + a_{100}$  is equal to the total number of bullets. It turns out that the number of bullets in the j-th column is equal to  $c_j$  for any  $1 \le j \le 99$ , proving that the total number of bullets is also equal to  $c_1 + c_2 + c_3 + \cdots + c_{99}$ .

#### References

- [AE11] TITU ANDREESCU and BOGDAN ENESCU. Mathematical Olympiad treasures. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)
- [AF13] T. Andreescu and Z. Feng. A Path to Combinatorics for Undergraduates: Counting Strategies. Birkhäuser Boston, 2013. ISBN: 9780817681548. URL: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)