Functional equations

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Functional equations

Example 1.1 (cf. Canadian Mathematical Olympiad 1969 P8, Australian Mathematics Competition 1984, India Pre-RMO 2012 P16)**.** [**TaoSolvingMathProb**]
Suppose

$$x_1, x_2, x_3, x_4, \ldots$$

is a sequence of integers satisfying the following properties:

(1) $x_2 = 2$,

(2) $x_{mn} = x_m x_n$ for all positive integers m, n,

(3) $x_m < x_n$ for any positive integers m, n with $m < n$.

Find x_{2024}. What is the value of the sum

$$x_1 + x_2 + \cdots + x_{2024}?$$

Summary — Observe that $x_{2^n} = 2^n$ for any $n \geq 1$. Combining this with the hypothesis that $\{x_n\}_{n\geq 1}$ is an increasing sequence of **positive integers**, conclude that $x_n = n$ for any $n \geq 1$.

Walkthrough —

(a) What can be said about x_4, x_8, x_{16}, x_{32}?

(b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.

(c) Can one show that $x_{2^n} = 2^n$ for any $n \geq 1$?

(d) Show that $x_m = m$ for any $m \geq 1$ (does property (3) help?).

Solution 1. From the second condition, we obtain

$$x_{2^n} = x_2^n$$

for any integer $n \geq 1$. Using the first condition, it gives

$$x_{2^n} = 2^n$$

for any integer $n \geq 1$. Since $\{x_n\}_{n\geq 1}$ is an increasing sequence of positive integers, it follows that $x_n = n$ for any positive integer n. This gives

$$x_{2024} = 2024.$$

It follows that

$$x_1 + x_2 + \cdots + x_{2024} = 1012 \cdot 2025 = 2049300.$$

■

Example 1.2 (India RMO 2006 P7)**.** Let X be the set of all positive integers greater than or equal to 8, and let $f\colon X \to X$ be a function such that $f(x+y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

> **Walkthrough** — Using the given condition, try to express $f(9)$ in terms of $f(8)$.

Solution 2. Note that

$$f(9) = f(4+5) = f(20) = f(4+16) = f(64),$$

and

$$f(8) = f(4+4) = f(16) = f(8+8) = f(64).$$

Using the given condition $f(8) = 9$, it follows that $f(9) = 9$. ■