List of problems and examples

§1 Invariance principle

Example 1.1 (Moscow MO 1959 Grade 7 Day 2 P5)**.** Consider n numbers $x_1, \ldots, x_n$, each equal to 1 or -1. Prove that if

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0, \tag{1}$$

then n is divisible by 4.

Solution 1. Since each of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ is equal to 1 or -1, using Eq. (1), it follows that the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ equal to 1 coincides with the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ equal to -1. These two numbers add up to n, and hence n is even, i.e. $n = 2m$ for some positive integer m.

Note that m is equal to the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$, which are equal to -1. In other words, m is equal to the number of *alterations*, i.e. pairs of consecutive terms of the sequence

$$x_1, x_2, \ldots, x_n, x_1,$$

whose signs differ. Since the initial and the final term of the above sequence are equal, their signs remain unchanged, and hence the number of the alterations of the above sequence has to be an even number. Since m is even, it follows that n is divisible by 4. ■

Remark. An elementary congruence argument may also be used to observe that n is even. Note that

$$\begin{aligned} n &= x_1^2x_2^2 + x_2^2x_3^2 + \cdots + x_{n-1}^2x_n^2 + x_n^2x_1^2 \\ &\equiv x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \pmod 2, \end{aligned}$$

and hence n is even.

One may also proceed as follows. For $1 \le i \le n$, write

$$\varepsilon_i := \begin{cases} 1 & \text{if } x_i \text{ and } x_{i+1} \text{ are of the same sign,} \\ -1 & \text{otherwise,} \end{cases}$$

where $x_{n+1} := x_1$. In other words,

$$\varepsilon_i = x_ix_{i+1}.$$

Note that

$$\begin{aligned}
n &= x_1^2x_2^2 + x_2^2x_3^2 + \cdots + x_{n-1}^2x_n^2 + x_n^2x_1^2 \\
&= \varepsilon_1 x_1x_2 + \varepsilon_2 x_2x_3 + \cdots + \varepsilon_{n-1}x_{n-1}x_n + \varepsilon_n x_n x_1 \\
&= (\varepsilon_1 - 1)x_1x_2 + (\varepsilon_2 - 1)x_2x_3 + \cdots + (\varepsilon_{n-1} - 1)x_{n-1}x_n + (\varepsilon_n - 1)x_nx_1 \\
&= 2 \sum_{1\leq i\leq n, \varepsilon_i = -1} (-x_ix_{i+1}) \\
&= 2 \times \text{the number of alternations of the sequence } x_1, \ldots, x_n, x_1.
\end{aligned}$$

Note that to arrive at the above, we did not use the fact that n is even. Moreover, the prior argument tells us that any (finite) sequence of ±1's, with equal initial and final terms, have an even number of alterations. Hence, n is a multiple of 4.

Upshot — At times, algebraic techniques might be quite useful.

Example 1.2. [**Eng98**, p. 5] Let $a_1, a_2, \ldots, a_n$ be n numbers such that each a_i is either 1 or -1. If

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0,$$

then prove that 4 divides n.

Solution 2. The solution relies on the following claim.

Claim — Let $y_1, y_2, \ldots, y_n$ be a sequence of integers such that each y_i is 1 or -1. Let $1 \leq m \leq n$ be an integer. Then the sum

$$y_1y_2y_3y_4 + y_2y_3y_4y_5 + \cdots + y_ny_1y_2y_3,$$

and the new sum obtained by replacing y_m by $-y_m$ in the above sum, leave the same remainder upon division[a] by 4.

[a] If b is a positive integer, and a is an integer, then there are integers q, r such that

$$a = qb + r, \quad 0 \leq r \leq b$$

hold. The integer q is called the *quotient* and r is called the *remainder*.

Note that one may divide any integer (even the negative ones) by a positive integer and obtain the quotient and the remainder. For example, if 9 is divided by 4, it leaves a remainder of 1. If -11 is divided by 4, then (after writing $-11 = (-3)\cdot 4+1$, one finds that) it leaves a remainder of 1. If -53 is divided by 4, then (after writing $-53 = (-14)\cdot 4 + 3$, one finds that) it leaves a remainder of 3.

Proof of the Claim. In the above cyclic sum, y_m appears in precisely four summands. Since the remaining summands are unchanged, it suffices to show that the sum of those four summands (denoted by $\mathcal{A}$), and the new sum obtained by replacing y_m by $-y_m$ in $\mathcal{A}$ (to be denoted by $\mathcal{B}$), leave the same remainder upon division by 4.

In fact, it reduces to showing that if $\mathscr{A}$ is a sum of four terms where each term is equal to 1 or -1, then $\mathscr{A}$ and $-\mathscr{A}$ leave the same remainder upon division by 4, or equivalently, their difference, which is $2\mathscr{A}$ is a multiple of 4, which is equivalent to saying $\mathscr{A}$ is even.

If all the terms of $\mathscr{A}$ are equal, then $\mathscr{A}$ is even. If three of the terms of $\mathscr{A}$ are 1 and the remaining one is -1, then $\mathscr{A}$ is even. Moreover, if two of the terms of $\mathscr{A}$ are 1 and the other twos are -1, then $\mathscr{A}$ is even. Further, if only one term of $\mathscr{A}$ is 1 and the others are equal to -1, then $\mathscr{A}$ is also even. The Claim follows. $\square$

In the given sum

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3,$$

we continue to replace all the a_i's which are equal to -1 by 1's. In the first step, S is divisible by 4. By the above Claim, at the end of each step, the new sum obtained is also divisible by 4. At the very last step, the sum will change to n, and hence, n is divisible by 4. $\blacksquare$

For more exercises on the invariance principle, we refer to [**Eng98**, Chapter 1].

References

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited pp. 2, 3)