

**Problem 0.1.** Show that the positive integers of the form  $4n + 3$ , that is, the integers

$$3, 7, 11, 15, 19, \dots$$

cannot be written as the sum of two perfect squares.

(a) Consider the integers

$$\begin{aligned} &0^2 + 1^2, 0^2 + 2^2, 0^2 + 3^2, 0^2 + 4^2, \dots, \\ &1^2 + 1^2, 1^2 + 2^2, 1^2 + 3^2, 1^2 + 4^2, \dots, \\ &2^2 + 1^2, 2^2 + 2^2, 2^2 + 3^2, 2^2 + 4^2, \dots, \\ &3^2 + 1^2, 3^2 + 2^2, 3^2 + 3^2, 3^2 + 4^2, \dots, \\ &4^2 + 1^2, 4^2 + 2^2, 4^2 + 3^2, 4^2 + 4^2, \dots, \end{aligned}$$

(b) Observe that upon division by 4, they leave the integers 0, 1, 2 as remainders.

$$\begin{aligned} &0^2 + 1^2 \rightsquigarrow \mathbf{1}, 0^2 + 2^2 \rightsquigarrow \mathbf{0}, 0^2 + 3^2 \rightsquigarrow \mathbf{1}, 0^2 + 4^2 \rightsquigarrow \mathbf{0}, \dots, \\ &1^2 + 1^2 \rightsquigarrow \mathbf{2}, 1^2 + 2^2 \rightsquigarrow \mathbf{1}, 1^2 + 3^2 \rightsquigarrow \mathbf{2}, 1^2 + 4^2 \rightsquigarrow \mathbf{1}, \dots, \\ &2^2 + 1^2 \rightsquigarrow \mathbf{1}, 2^2 + 2^2 \rightsquigarrow \mathbf{0}, 2^2 + 3^2 \rightsquigarrow \mathbf{1}, 2^2 + 4^2 \rightsquigarrow \mathbf{0}, \dots, \\ &3^2 + 1^2 \rightsquigarrow \mathbf{2}, 3^2 + 2^2 \rightsquigarrow \mathbf{1}, 3^2 + 3^2 \rightsquigarrow \mathbf{2}, 3^2 + 4^2 \rightsquigarrow \mathbf{1}, \dots, \\ &4^2 + 1^2 \rightsquigarrow \mathbf{1}, 4^2 + 2^2 \rightsquigarrow \mathbf{0}, 4^2 + 3^2 \rightsquigarrow \mathbf{1}, 4^2 + 4^2 \rightsquigarrow \mathbf{0}, \dots, \end{aligned}$$

(c) Show that it is **always** the case, namely, upon division by 4, the sum of two perfect squares leaves one of 0, 1, 2 as the remainder.

(d) Conclude that no integer, which leaves the remainder of 3 **upon division by 4**, can be written as the sum of two squares.