Example 1 (Putnam 2002 A2)**.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

> **Remark.** Let $\mathcal{S}$ be a sphere and $\mathcal{C}$ be a great circle on it. Then $\mathcal{C}$ divides $\mathcal{S}$ into two parts, which are called the *hemispheres defined by* $\mathcal{C}$. Any such hemisphere together with the great circle $\mathcal{C}$ is called a *closed hemisphere*. In Fig. 1, there are a few examples of closed hemispheres. Those are the grey ones together with the great circles marked in red, and the blue ones together with the great circles marked in red.

> **Summary** — Apply the pigeonhole principle.

> **Walkthrough** —
>
> **(a)** Draw a great circle passing through at least two of the five points.
>
> **(b)** At least one closed hemisphere contains at least two of the remaining three points.
>
> **(c)** Conclude!

Solution 1. Draw a great circle passing through at least two of the five points. Then at least one closed hemisphere contains at least two of the remaining three points. This proves the result. See [**AN10**, Example 3.2]. ■

References

[**AN10**] Claudi Alsina and Roger B. Nelsen. *Charming proofs*. Vol. 42. The Dolciani Mathematical Expositions. A journey into elegant mathematics. Mathematical Association of America, Washington, DC, 2010, pp. xxiv+295. ISBN: 978-0-88385-348-1 (cited p. 1)

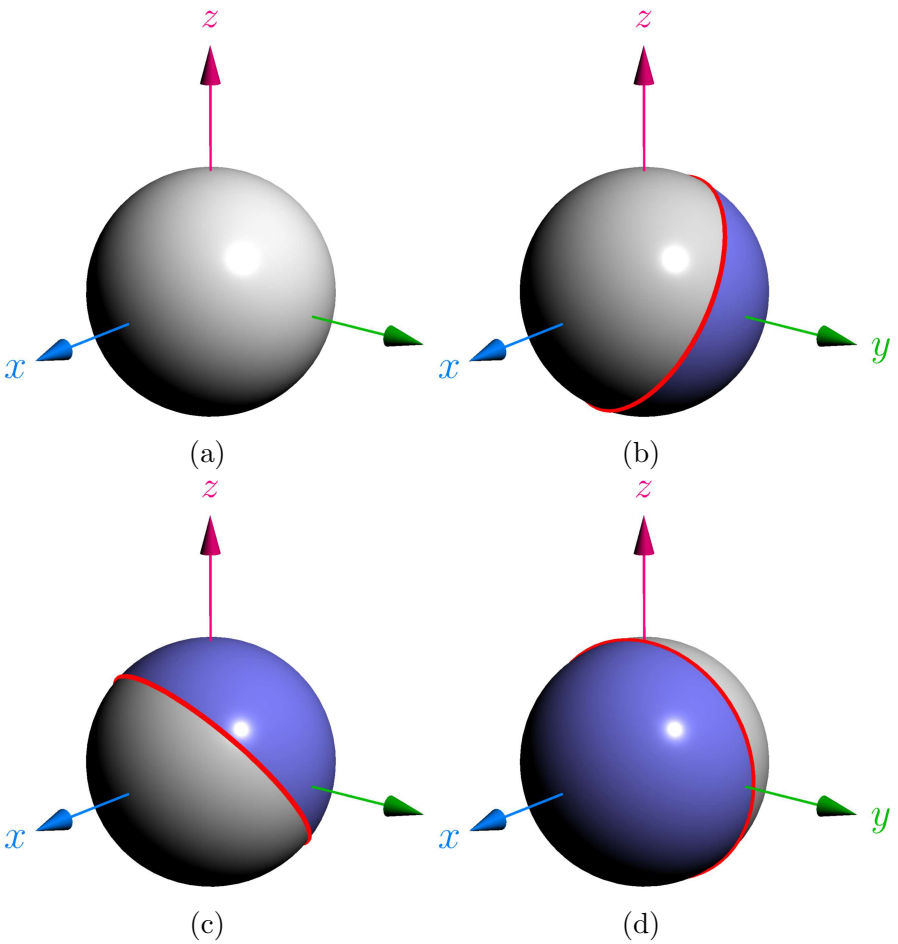

Figure 1: USA Putnam 2002 A2