1 Coloring proofs

See [**Eng98**, Chapter 2], [**Sob13**, §3.2].

Example 1.1 (India RMO 2017)**.** Consider n^2 unit squares in the xy-plane centred at the point (i, j) with integer coordinates for $1 \leq i \leq n, 1 \leq j \leq n$. It is required to colour each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, the three squares with centres at (i, k), (j, k), (j, ℓ) have distinct colours. What is the least possible number of colours needed?

Remark. For such problems, it is often useful to first work out a special case.

Walkthrough.

(a) First, work out a simple case in order to gain insight for the general case.

(b) One may consider the squares below a suitable diagonal.

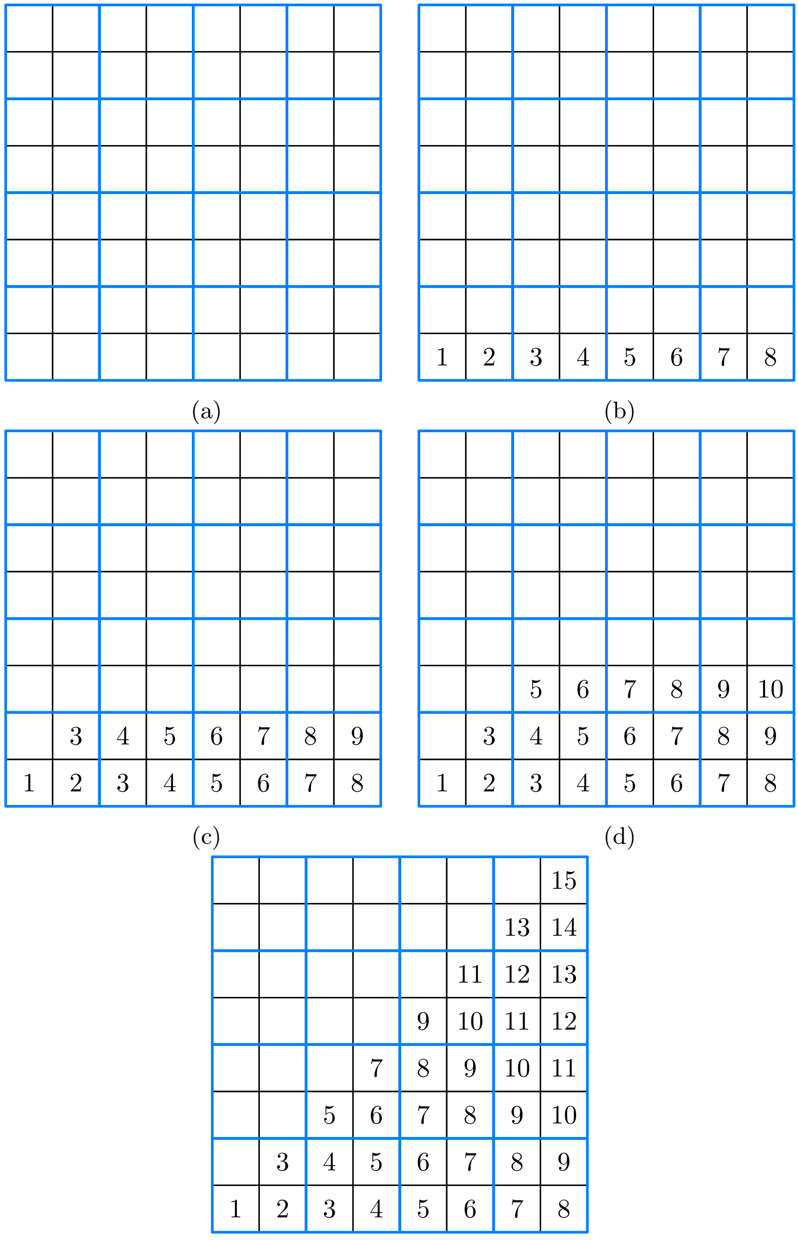
(c) Extend to the general case!

First, let's work on it. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 1.1a have pairwise distinct colors. Let's apply the colors $1, \dots, 8$ to these squares as in Fig. 1.1b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let's apply the colors $3, \dots, 9$ to these squares as in Fig. 1.1c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let's apply another set of $8 - 3 + 1$ new colors (for example, $5, \dots, 10$) to these squares as in Fig. 1.1d. One may continue this process to yield the coloring as in Fig. 1.1e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 2, as in Fig. 1.2a. In the third last row, the colors $3, 4$ may be used as in Fig. 1.2b. In the fourth last row, the colors smaller than 7 may be used as in Fig. 1.2c. One may continue this process to yield the coloring as in Fig. 1.2d. Note that the coloring as in Fig. 1.2d **does satisfy** the required condition. ♣

Remark. Now that we have gained some experience, we may proceed to the general case as follows.



(a) (b) (c) (d) (e)

Figure 1.1: India RMO 2017

2 *The material posted here and at this blog by Evan Chen are quite useful.*

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
		5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(a)

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(b)

							15
						13	14
					11	12	13
				9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(c)

8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(d)

Figure 1.2: India RMO 2017

Solution 1. If $n = 1$, then using one color works. Let us assume[1] that $n \geq 2$.

Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying on the rightmost column except the bottom square, have colors different from those of the squares lying on the bottom row. Hence, we require at least

$$n + (n - 1) = 2n - 1$$

colors.

Next, we show that there is a coloring of the n^2 unit squares using $2n - 1$ colors satisfying the required condition. Indeed, applying the colors $i, i + 1, \ldots, i + n - 1$ in the i-th last row in an increasing order from the left to the right for all $1 \leq i \leq n$, yields such a coloring.

This shows that the least number of colors required is $2n - 1$. ■

[1] The reason for considering the case $n = 1$ separately will become clear from the rest of the argument. Is it clear what would go wrong with the rest of the argument if $n = 1$?

Bibliography

[**Eng98**] Arthur Engel. *Problem-solving strategies.* Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 1)

[**Sob13**] Pablo Soberón. *Problem-solving methods in combinatorics.* An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. isbn: 978-3-0348-0596-4; 978-3-0348-0597-1. doi: `10.1007/978-3-0348-0597-1`. url: `http://dx.doi.org/10.1007/978-3-0348-0597-1` (cited p. 1)