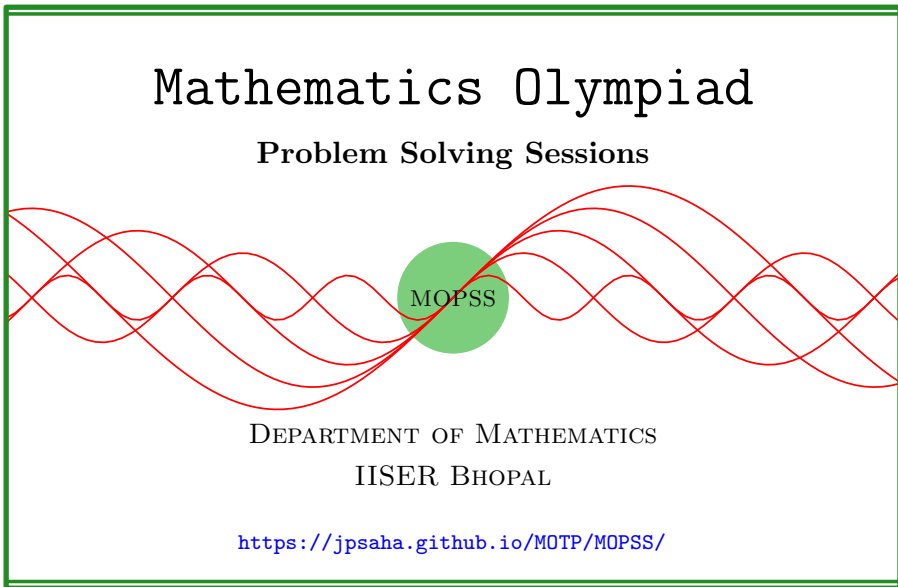


# IOQM 2023

MOPSS

16 June 2024



## Suggested readings

- **Evan Chen's**
  - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
  - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- **Evan Chen** discusses why *math olympiads* are a valuable experience for high schoolers in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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Question 1

Let  $n$  be a positive integer such that  $1 \leq n \leq 1000$ . Let  $M_n$  be the number of integers in the set

$$X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \dots, \sqrt{4n+1000}\}.$$

Let

$$a = \max\{M_n \mid 1 \leq n \leq 1000\}, \text{ and } b = \min\{M_n \mid 1 \leq n \leq 1000\}.$$

Find  $a - b$ .

Answer: 22.

**Summary** — Note that the gap between two consecutive squares increases as one **moves to the right** of the number line (because the difference between  $(n+1)^2$  and  $n^2$  is a linear polynomial in  $n$  with positive coefficients).

Walkthrough —

- (a) Given a sequence of consecutive 1000 positive integers, we need to understand how many of them are perfect squares.
- (b) Note that the gap between two consecutive squares increases as one **moves to the right**.
- (c) Draw the number line to convince yourself that

$$M_1 \geq M_2 \geq M_2 \geq \dots \geq M_{1000}.$$

(d) Note that  $31^2 < 1000 < 32^2$ , and hence

$$a = M_1 = 31 - 2 = 29.$$

(e) Also note that  $63^2 < 4001 < 64^2$ , and  $70^2 < 5000 < 71^2$ , and hence

$$b = M_{1000} = 70 - 63 = 7.$$

(f) Conclude that  $a - b = 22$ .

### Question 2

Find the number of elements in the set

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, \log_a(b) + 6 \log_b(a) = 5\}.$$

Answer: 54.

**Summary** — Quadratic equation!

### Walkthrough —

(a) Note that the condition

$$\log_a(b) + 6 \log_b(a) = 5$$

is equivalent to

$$\log_a(b) = 2, \text{ or } 3,$$

which is equivalent to  $b = a^2$  or  $b = a^3$ .

(b) This shows that the given set is equal to the union of the sets

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, b = a^2\}$$

and

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, b = a^3\}.$$

(c) Observe that

$$44^2 < 2023 < 45^2, 12^3 = 144 \times 12 < 2023, 13^3 = 169 \times 13 > 2023.$$

(d) Note that the above sets have no common elements. It follows that the number of elements of the given set is  $44 - 1 + 12 - 1 = 54$ .

**Question 3**

Let  $\alpha, \beta$  be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

Find the smallest possible value of  $\beta$ .

**Question 4**

Let  $x, y$  be positive integer such that

$$x^4 = (x - 1)(y^3 - 23) - 1.$$

Find the maximum possible value of  $x + y$ .

Answer: 7.

**Summary** — Look for some divisibility conditions.

**Walkthrough** —

- (a) Note that  $x \neq 1$  and  $x - 1$  divides 2. It follows that  $x = 2$  or  $x = 3$ .
- (b) If  $x = 2$ , then  $y^3 = 23 + 2^4 + 1 = 40$ , which is impossible since  $y$  is an integer.
- (c) If  $x = 3$ , then  $y^3 = 23 + \frac{1}{2}(3^4 + 1) = 23 + 41 = 64$ , which gives  $y = 4$ .
- (d) It follows that  $x + y = 7$ .

**Question 5**

In a triangle  $ABC$ , let  $E$  be the midpoint of  $AC$  and  $F$  be the midpoint of  $AB$ . The medians  $BE$  and  $CF$  intersect at  $G$ . Let  $Y$  and  $Z$  be the midpoints of  $BE$  and  $CF$  respectively. If the area of the triangle  $ABC$  is 480, find the area of the triangle  $GYZ$ .

Answer: 10.

**Summary** — The ratio of the areas of two triangles with equal heights equal to the ratio of their bases. One often says that the area of a triangle with constant height is proportional to its base.

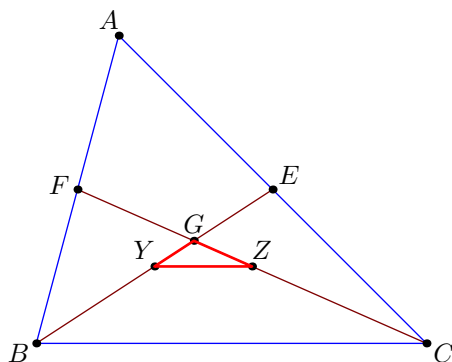


Figure 1: IOQM23 Q5, Question 5

**Walkthrough** —

- (a) The area of the triangle  $BCF$  is equal to 240.
- (b) The area of the triangle  $BCG$  is equal to  $\frac{2}{3} \times 240 = 160$ .
- (c) Note that  $BY : YG : GE$  is equal to  $3 : 1 : 2$ . This shows that  $BY : YG = 3 : 1$ . Similarly,  $CZ : ZG = 3 : 1$ .
- (d) It follows that  $YZ$  is parallel to  $BC$ , and  $YZ : BC = 1 : 4$ .
- (e) The ratio of the height of  $GYZ$  to the height of  $BCG$  is equal to  $1 : 4$ .
- (f) Consequently, the area of  $GYZ$  is equal to  $\frac{1}{4^2} \times 160 = 10$ .

**Question 6**

Let  $X$  be the set of all even positive integers  $n$  such that the measure of the angle subtended by a side at the center of some regular polygon is  $n$  degrees. Find the number of elements in  $X$ .

Answer: 16.

**Walkthrough** —

- (a) Let  $r \geq 3$  be an integer and consider a regular  $r$ -gon. Note that  $360/r$  is even if and only if  $r$  is equal to one of

$$3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 24, 36, 45, 60, 120, 360.$$

In other words, there are precisely 16 positive integers  $r \geq 3$  such that dividing 360 by  $r$  yields an even number.

- (b) It follows that the number of elements of  $X$  is equal to 16.

**Question 7**

Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

Answer: 48.

**Walkthrough** —

- (a) Fix a pair of opposite faces (say, top and the bottom one, for ease of reference), which are to be marked using 1 and 2.
- (b) Note that after marking the top and bottom face using 1 and 2 (in some order), the remaining faces are to be marked using 3, 4, 5, 6, and thus, the marking of the remaining faces correspond to the circular permutations of 3, 4, 5, 6.
- (c) This shows that the marking can be done in 6 ways.
- (d) Note that there are precisely 6 dice of the prescribed type having the same marking.
- (e) Hence, the number of distinct dice is equal to  $6 \times 2^3 = 48$ .

**Question 8**

Given a  $2 \times 2$  tile and seven dominoes ( $2 \times 1$  tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a  $2 \times 7$  rectangle using some of these tiles.

**Question 9**

Find the number of triples  $(a, b, c)$  of positive integers such that

- (a)  $ab$  is a prime,
- (b)  $bc$  is a product of two primes,
- (c)  $abc$  is not divisible by square of any prime and
- (d)  $abc \leq 30$ .

Answer: 17.

**Walkthrough** —

- (a) Suppose  $(a, b, c)$  is a pair satisfying the above conditions.
- (b) Since  $ab$  is a prime, it follows that one of  $a, b$  is equal to 1, and the remaining one is a prime.
- (c) If  $a = 1$ , then  $b, c$  are distinct primes.
- (d) If  $b = 1$ , then  $c$  is the product of two distinct primes which are not equal to  $a$ .
- (e) This shows that  $(a, b, c)$  is equal to one of

$$\begin{aligned}
 &(1, 2, 3), (1, 2, 5), (1, 2, 7), (1, 2, 11), (1, 2, 13), \\
 &(1, 3, 2), (1, 3, 5), (1, 3, 7), \\
 &(1, 5, 2), (1, 5, 3), \\
 &(1, 7, 2), (1, 7, 3), \\
 &(1, 11, 2), \\
 &(1, 13, 2), \\
 &(2, 1, 3 \cdot 5), (3, 1, 2 \cdot 5), (5, 1, 2 \cdot 3).
 \end{aligned}$$

- (f) Hence, the number of triples satisfying the above conditions is at most 16.
- (g) Since any of the above triples does satisfy the given conditions, it follows that the number of triples satisfying the given conditions is equal to 17.

**Question 10**

The sequence  $\langle a_n \rangle_{n \geq 0}$  is defined by  $a_0 = 1, a_1 = -4$  and

$$a_{n+2} = -4a_{n+1} - 7a_n$$

for  $n \geq 0$ . Find the number of positive integer divisors of  $a_{50}^2 - a_{49}a_{51}$ .

Answer: 51.

**Walkthrough** —

- (a) Compute the first few terms of the sequence to note that

$$\begin{aligned}
 a_0 &= 1, \\
 a_1 &= -4, \\
 a_2 &= 9, \\
 a_3 &= -8, \\
 a_4 &= -31,
 \end{aligned}$$

which gives

$$\begin{aligned}
 a_1^2 - a_0 a_2 &= 16 - 9 \\
 &= 7, \\
 a_2^2 - a_1 a_3 &= 81 - 32 \\
 &= 49, \\
 a_3^2 - a_2 a_4 &= 64 + 279 \\
 &= 343.
 \end{aligned}$$

This may suggest that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds for any integer  $n \geq 1$ .

(b) Indeed, it is clear for  $n = 1$ . Assuming that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds for some integer  $n \geq 1$ , note that

$$\begin{aligned}
 a_{n+1}^2 - a_n a_{n+2} &= a_{n+1}^2 - a_n(-4a_{n+1} - 7a_n) \\
 &= a_{n+1}^2 + 4a_n a_{n+1} + 7a_n^2 \\
 &= a_{n+1}^2 + 4a_n a_{n+1} + 7a_{n-1} a_{n+1} + 7^{n+1} \\
 &= a_{n+1}(a_{n+1} + 4a_n + 7a_{n-1}) + 7^{n+1} \\
 &= 7^{n+1}.
 \end{aligned}$$

This proves that

$$a_n^2 - a_{n-1} a_{n+1} = 7^n$$

holds for any integer  $n \geq 1$ .

(c) Consequently,

$$a_{50}^2 - a_{49} a_{51} = 7^{50},$$

whose number of positive integer divisors is equal to 51.

### Question 11

A positive integer  $m$  has the property that  $m^2$  is expressible in the form  $4n^2 - 5n + 16$  where  $n$  is an integer (of any sign). Find the maximum possible value of  $|m - n|$ .

Answer: 14.

**Walkthrough** —



- (a) Let  $m, n$  be integers satisfying  $m^2 = 4n^2 - 5n + 16$ , or equivalently,

$$4n^2 - 5n + (16 - m^2) = 0.$$

- (b) Since  $n$  is an integer, the discriminant of the quadratic polynomial  $4n^2 - 5n + (16 - m^2)$  in  $n$  is a perfect square, that is, there exists an integer  $\ell$  such that

$$25 - 16(16 - m^2) = \ell^2,$$

or equivalently,  $16m^2 - \ell^2 = 231$ , which gives

$$(4m - \ell)(4m + \ell) = 3 \cdot 7 \cdot 11.$$

- (c) Note that  $4m - \ell, 4m + \ell$  are odd and hence congruent to  $1, -1$  modulo 4 (in some order). Also note that  $4m - \ell < 4m + \ell$  holds.
- (d) This shows that  $(4m - \ell, 4m + \ell)$  is equal to one of

$$(1, 3 \cdot 7 \cdot 11), (3, 77), (7, 33), (11, 21),$$

and consequently,  $8m = 232, 80, 40, 32$ , which yields  $m = 29, 10, 5, 4$ .

- (e) If  $m = 29$ , then  $4n^2 - 5n - 825 = 0$ , which gives  $n = 15$ .
- (f) If  $m = 10$ , then  $4n^2 - 5n - 84 = 0$  holds, which gives  $n = -4$ .
- (g) If  $m = 5$ , then  $4n^2 - 5n - 9 = 0$  holds, implying  $n = -1$ .
- (h) If  $m = 4$ , then  $n = 0$ .
- (i) Conclude that the maximum possible value of  $|m - n|$  is equal to 14.

### Question 12

Let  $P(x) = x^3 + ax^2 + bx + c$  be a polynomial where  $a, b, c$  are integers and  $c$  is odd. Let  $p_i$  be the value of  $P(x)$  at  $x = i$ . Given that  $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$ , find the value of  $p_2 + 2p_1 - 3p_0$ .

Answer: 18.

### Walkthrough —

- (a) Note that if  $p_1 + p_2 + p_3 \neq 0$ , then using the condition that  $p_1, p_2, p_3$  are real, it follows that  $p_1 = p_2 = p_3$ .
- (b) Note that

$$p_1 = 1 + a + b + c,$$

$$p_2 = 8 + 4a + 2b + c,$$

$$p_3 = 27 + 9a + 3b + c,$$

which gives that

$$p_1 + p_2 + p_3 = 36 + 14a + 6b + 3c.$$

Since  $c$  is an odd integer, it follows that  $p_1 + p_2 + p_3 \neq 0$ . Consequently,  $p_1 = p_2 = p_3$ .

(c) This gives

$$3a + b = -7, 5a + b = -19,$$

and hence,  $a = -6, b = 11$ .

(d) This gives

$$\begin{aligned} p_2 + 2p_1 - 3p_0 &= 10 + 6a + 4b + 3c - 3c \\ &= 10 - 36 + 44 \\ &= 18. \end{aligned}$$

### Question 13

The ex-radii of a triangle are  $10\frac{1}{2}$ , 12 and 14. If the sides of the triangle are the roots of the cubic  $x^3 - px^2 + qx - r = 0$ , where  $p, q, r$  are integers, find the integer nearest to  $\sqrt{p+q+r}$ .

Answer: 58.

### Walkthrough —

(a) Denote the sides of the triangle by  $a, b, c$ . Let  $s$  denote its perimeter, and  $\Delta$  denote its area.

(b) Using

$$\Delta = (s-a)r_1 = (s-b)r_2 = (s-c)r_3,$$

we obtain

$$\Delta = \frac{21}{2}(s-a) = 12(s-b) = 14(s-c).$$

This gives

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s\Delta^3 \times \frac{2}{21} \times \frac{1}{12} \times \frac{1}{14}} = \frac{1}{42}\Delta\sqrt{s\Delta},$$

implying that

$$s\Delta = 42^2.$$

(c) Also note that

$$s-a = \frac{2}{21}\Delta, s-b = \frac{1}{12}\Delta, s-c = \frac{1}{14}\Delta,$$

yields

$$s = \left( \frac{2}{21} + \frac{1}{12} + \frac{1}{14} \right) \Delta = \frac{8+7+6}{84}\Delta = \frac{1}{4}\Delta.$$

(d) It follows that

$$s = 21, \Delta = 84.$$

(e) Moreover, note that

$$a = s - \frac{2}{21}\Delta$$

$$= 21 - 8$$

$$= 13,$$

$$b = s - \frac{1}{12}\Delta$$

$$= 21 - 7$$

$$= 14,$$

$$c = s - \frac{1}{14}\Delta$$

$$= 21 - 6$$

$$= 15.$$

(f) Since  $a, b, c$  are the roots of the cubic  $x^3 - px^2 + qx - r = 0$ , it follows that

$$-p - q - r - 1 = (-1 - a)(-1 - b)(-1 - c) = -14 \cdot 15 \cdot 16,$$

which gives

$$p + q + r = 3360 - 1 = 3359.$$

(g) Check that the integer nearest to  $\sqrt{3359}$  is 58.

### Question 14

Let  $ABC$  be a triangle in the  $xy$  plane, where  $B$  is at the origin  $(0, 0)$ . Let  $BC$  be produced to  $D$  such that  $BC : CD = 1 : 1$ ,  $CA$  be produced to  $E$  such that  $CA : AE = 1 : 2$  and  $AB$  be produced to  $F$  such that  $AB : BF = 1 : 3$ . Let  $G(32, 24)$  be the centroid of the triangle  $ABC$  and  $K$  be the centroid of the triangle  $DEF$ . Find the length  $GK$ .

Answer: 40.

### Walkthrough —

(a) Note that

$$D = 2C - B,$$

$$E = 3A - 2C,$$

$$F = 4B - 3A,$$

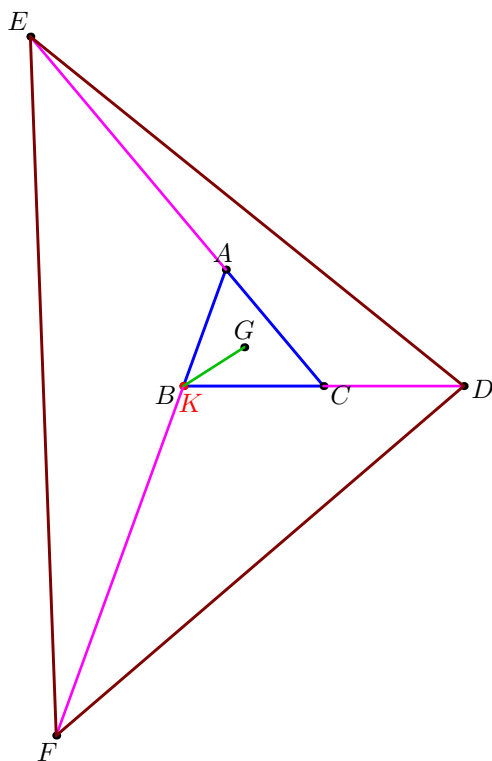


Figure 2: IOQM23 Q14, Question [14](#)

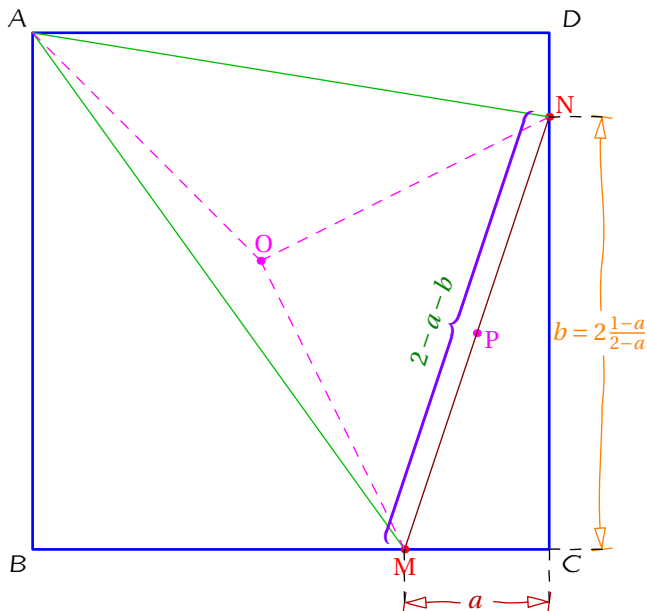


Figure 3: IOQM23 Q15, Question 15

which gives

$$K = \frac{D + E + F}{3} = B.$$

Since  $B$  is at the origin  $(0, 0)$  and  $G$  has coordinates  $(32, 24)$ , it follows that

$$GK = \sqrt{32^2 + 24^2} = 8 \cdot \sqrt{4^2 + 3^2} = 40.$$

### Question 15

Let  $ABCD$  be a unit square. Suppose  $M$  and  $N$  are points on  $BC$  and  $CD$  respectively such that the perimeter of triangle  $MCN$  is 2. Let  $O$  be the circumcentre of triangle  $MAN$ , and  $P$  be the circumcentre of triangle  $MON$ . If  $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$  for some relatively prime positive integers  $m$  and  $n$ , find the value of  $m + n$ .

Answer: 3.

## Walkthrough —

(a) Choose  $a, b$  as in Fig. 3 such that

$$a^2 + b^2 = (2 - a - b)^2,$$

which gives

$$4 - 4a = (4 - 2a)b,$$

and hence

$$b = 2 \frac{1-a}{2-a}$$

Note that

$$\begin{aligned} AM^2 &= 1 + (1-a)^2 \\ &= 2 - 2a + a^2, \\ AN^2 &= 1 + (1-b)^2 \\ &= 2 - 2b + b^2 \\ &= 2 \left( 1 - 2 \frac{1-a}{2-a} + 2 \left( \frac{1-a}{2-a} \right)^2 \right) \\ &= \frac{2}{(2-a)^2} (4 - 4a + a^2 - 4 + 6a - 2a^2 + 2 - 4a + 2a^2) \\ &= \frac{2}{(2-a)^2} (2 - 2a + a^2), \\ AM^2 + AN^2 - MN^2 &= 1 + (1-a)^2 + 1 + (1-b)^2 - (2-a-b)^2 \\ &= 2 + (1-a)^2 + (1-b)^2 - a^2 - b^2 \\ &= 4 - 2a - 2b \\ &= 4 - 2a - 4 \frac{1-a}{2-a} \\ &= 2 \left( 2 - a - \frac{2-2a}{2-a} \right) \\ &= 2 \frac{2-2a+a^2}{2-a} \\ &= \sqrt{2} AM \cdot AN. \end{aligned}$$

This gives that  $\angle MAN = 45^\circ$ . Since  $O$  is the circumcenter of  $MAN$ , it follows that  $\angle MON = 90^\circ$ . Moreover,  $OA = OM = ON$ . This implies that  $MON$  is a right-angled isosceles triangle. Since  $P$  is the circumcenter of  $MON$ , it follows that  $OPN$  is a right-angled isosceles triangle, which gives  $OP = \frac{1}{\sqrt{2}} ON$ . This implies that  $\left( \frac{OP}{OA} \right)^2 = \frac{1}{2}$ , and hence  $m + n = 3$ .