List of problems and examples

§1 Warm up

It would be good to go through [**Che24**, Chapter 1].

Example 1.1 (G. Galperin, Tournament of Towns, Autumn 1989, Junior, O Level, P4)**.** Find the solutions of the equation

$$x + \frac{1}{y + \frac{1}{z}} = \frac{10}{7} \tag{1}$$

in positive integers.

Solution 1. Let x, y, z be positive integers satisfying Eq. (1). Since $y \geq 1$ and $\frac{1}{z} > 0$, it follows that $y + \frac{1}{z} > 1$, which gives $0 < \frac{1}{y+\frac{1}{z}} < 1$. Using Eq. (1), it follows that $x = 1$, and hence $y + \frac{1}{z} = \frac{7}{3}$. By a similar argument as above[1], it follows that $y = 2$ and consequently, $z = 3$.

Moreover, for $x = 1, y = 2, z = 3$, Eq. (1) holds.

This proves that $x = 1, y = 2, z = 3$ is the only solution[2] of Eq. (1). ■

> **Remark.** Is the part *for* $x = 1, y = 2, z = 3$*, Eq.* (1) *holds* in the above argument redundant? Or, is it not so? Think about it. Further, it would be worth going through [**Che24**, Chapter 1].

Example 1.2 (IMO 1959 P1, proposed by Poland)**.** Prove that the fraction

$$\frac{21n + 4}{14n + 3}$$

is irreducible for every natural number n.

We need to show that $21n + 4, 14n + 3$ have no factor in common other than 1 for every natural number n.

[1] Write the argument instead of resorting to using "by a similar argument" unless it is clear to you. Even then, consider it as an exercise and write it down!

[2] It means $x = 1, y = 2, z = 3$ is **a solution** to Eq. (1), and that it is the **only solution**, i.e. if we are given a solution, it cannot be different from $x = 1, y = 2, z = 3$. Does the above argument prove both?

Summary — It follows from considering the greatest common divisor of the numerator and the denominator.

Walkthrough —

- The summand $21n$ from the numerator and the summand $14n$ from the denominator do not "balance well".
- One way "enforce balancing" would be to consider

$$2 \cdot 21n - 3 \cdot 14n,$$

which vanishes.
- Does the above "ad hoc thoughts" help to conclude?

Solution 2. Let n be a natural number. It is enough to show that the greatest common divisor of the integers $21n + 4, 14n + 3$ is equal to 1. Note that any common divisor of $21n + 4, 14n + 3$ divides

$$2(21n + 4) - 3(14n + 3) = -1.$$

This shows that the greatest common divisor of the integers $21n + 4, 14n + 3$ is equal to 1, completing the proof. ∎

Example 1.3 (India RMO 2015 P3). Find all fractions which can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell - 1}{4\ell - 3}$$

for some integers k, ℓ.

Solution 3. The solution relies on the following claim.

Claim — Suppose k, ℓ are integers. Then the equality

$$\frac{7k-5}{5k-3} = \frac{6\ell - 1}{4\ell - 3}$$

is equivalent to the pair (k, ℓ) being equal to one of

$$(0, 6), (1, -1), (6, -6), (13, -7), (-2, -22), (-3, -15), (-8, -10), (-15, -9). \tag{2}$$

Proof of the Claim. Suppose k, ℓ are integers. Observing that $5k - 3$ and $4\ell - 3$ are nonzero, it follows that

$$\frac{7k-5}{5k-3} = \frac{6\ell - 1}{4\ell - 3}$$

$$
\begin{aligned}
\iff \quad & (7k-5)(4\ell-3) = (5k-3)(6\ell-1) \\
\iff \quad & 28k\ell - 20\ell - 21k + 15 = 30k\ell - 18\ell - 5k + 3 \\
\iff \quad & 2k\ell + 2\ell + 16k - 12 = 0 \\
\iff \quad & k\ell + \ell + 8k - 6 = 0 \\
\iff \quad & (k+1)(\ell+8) = 14.
\end{aligned}
$$

This implies that $k+1$ is equal to

$$\pm 1, \pm 2, \pm 7, \pm 14,$$

i.e. k is equal to

$$0, 1, 6, 13, -2, -3, -8, -15. \tag{3}$$

It follows that

$$(k+1)(\ell+8) = 14$$

is equivalent to (k, ℓ) being equal to one of the pairs as in Eq. (2). This proves the Claim. $\square$

Note that if a fraction can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell-1}{4\ell-3}$$

for two integers k, ℓ, then the Claim implies that (k, ℓ) is equal to the pairs as in Eq. (2), and then k is equal to the integers as in Eq. (3), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim again), is also equal to

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}. \tag{4}$$

Further[3], observe that the preceeding argument also proves that these fractions can be written simultaneously in the forms as stated above. Indeed, if (k, ℓ) is one of the pairs as in Eq. (2), and then k is equal to the integers as in Eq. (3), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim), is also equal to the fractions as in Eq. (4).

We conclude that the fractions as in Eq. (4) are precisely all the fractions with the required property. $\blacksquare$

[3]Note that the argument needs to go on since what we have proved so far does not complete the solution. The previous step only says that if a fraction can be written simultaneously in the forms as stated above (and **a priori, it is not clear if there is even a single fraction that can be expressed simultaneously in the stated forms**), then the fraction cannot be anything other than

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}.$$

This does not guarantee if any of these fractions enjoy the stated property.

If this causes any confusion, then it would be a good idea to go through [**Che24**, Chapter 1].

References

[**Che24**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2024, pp. vi+289 (cited pp. 1, 3)