Example 1 (Moscow MO 2015 Grade 11)**.** Prove that it is impossible to put the integers from 1 to 64 (using each integer once) into an 8×8 table so that any 2×2 square, considered as a matrix, has a determinant that is equal to 1 or -1.

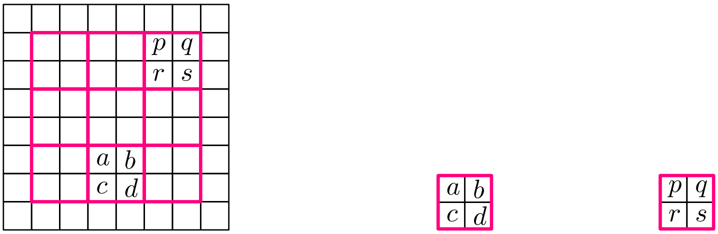

Figure 1: $ad - bc = \pm 1, ps - qr = \pm 1$

Remark.

- Given a 2×2 matrix (or array of numbers) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is
$$ad - bc,$$
i.e.
$$\begin{aligned} &\text{the product of the diagonal terms} \\ &- \text{the product of the anti-diagonal terms.} \end{aligned}$$
- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has determinant -2.

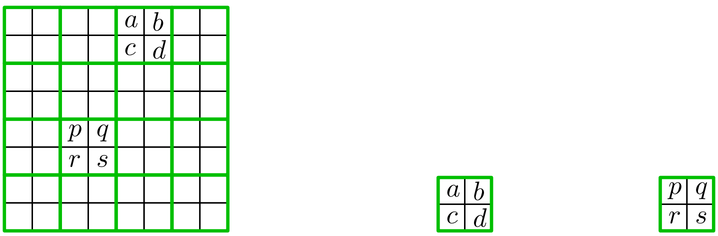

Figure 2: $ad - bc = \pm 1, ps - qr = \pm 1$

- $\begin{pmatrix} 8 & 9 \\ 7 & 12 \end{pmatrix}$ has determinant $96 - 63 = 33$.
- $\begin{pmatrix} 13 & 14 \\ 5 & 7 \end{pmatrix}$ has determinant $91 - 70 = 21$.
- We need to show that there is no filling of an 8×8 table using the integers from 1 to 64, using each integer once, such that any 2×2 square (such squares have been marked in Fig. 1, Fig. 2, note that there $9 + 16 = 25$ such 2×2 squares.) has a determinant equal to 1 or -1.

Walkthrough —

(a) Assume that such a filling exists.

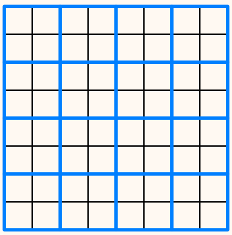

Figure 3: Moscow MO 2015 Grade 11 Day 1 P5

(b) Recall that the determinant of a 2×2 square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ is

$$\begin{aligned} &\text{the product of the diagonal terms} \\ &- \text{the product of the anti-diagonal terms.} \end{aligned}$$

(c) Note that $\boxed{\text{even} - \text{even} \neq \pm 1, \text{odd} - \text{odd} \neq \pm 1}$, and hence any square contains two odd numbers along the diagonal or on the anti-diagonal.

(d) Divide the 8×8 table into 16 pairwise disjoint 2×2 squares.

(e) Each of these 16 squares contains at least two odd integers, and hence, they together contain at least 32 odd integers.

(f) Conclude that each of these 16 squares contains precisely two odd integers, and precisely two even integers.

(g) Consider a square among them. It is of the form

$$\boxed{\begin{array}{c|c} a & b \\ \hline c & d \end{array}} \text{ with } a, d \text{ both odd, and } b, c \text{ both even,}$$

or of the form

$$\boxed{\begin{array}{c|c} b & a \\ \hline d & c \end{array}} \text{ with } a, d \text{ both odd, and } b, c \text{ both even.}$$

(h) The product of its even entries is at most one more than the product of its odd entries.

(i) Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b + 1)(c + 1)$ holds.

(j) This shows that

$$\begin{aligned} &\text{the product of two evens between 1 and 64} \\ &< \text{the product of two (possibly different) evens between 1 and 64.} \end{aligned}$$

(k) Multiply all the even entries of the 16 squares to obtain

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1 + 1) \cdot (3 + 1) \cdot \ldots \cdot (63 + 1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

Solution 1. Let us assume that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1.

Claim — Any 2×2 square contains at least two odd integers.

Proof of the claim. Since the difference of two integers can be odd only when they are of different parity (i.e. one of them is odd, and the other is even), it follows that for any 2×2 square, the product of its diagonal entries and the product of its anti-diagonal entries are of different parity, and hence of these two products is odd, and consequently, the diagonal entries are odd or the anti-diagonal entries are odd. In particular, any 2×2 square contains at least two odd integers. $\square$

Let us divide the 8×8 table into 16 pairwise disjoint 2×2 squares (as in Fig. 3).

Claim — Each of these 16 squares contains exactly two even integers, lying along its diagonal or anti-diagonal.

Proof of the claim. By the previous Claim, each of these 16 squares contains at least two odd integers, and they contain at least $16 \times 2 = 32$ odd integers. Since there are precisely 32 odd integers between 1 and 64, it follows that each of these 16 squares contains exactly two odd integers along its diagonal or anti-diagonal, and hence exactly two even integers along its anti-diagonal or diagonal. $\square$

Since the determinant of any 2×2 square is 1 or -1, it follows that for any of the 16 squares as in Fig. 3, the product of its even entries is at most one more than the product of its odd entries. Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b+1)(c+1)$ holds.

Denote the 16 squares as in Fig. 3 by $\square_1, \ldots, \square_{16}$. Denote the even entries of a square $\square_i$ by $e_{i,1}, e_{i,2}$, and denote the odd entries of a square $\square_i$ by $o_{i,1}, o_{i,2}$. We have proved that

$$e_{i,1}e_{i,2} \leq o_{i,1}o_{i,2} + 1 < (o_{i,1} + 1)(o_{i,2} + 1)$$

holds for any $1 \leq i \leq 16$. Note that the even (resp. odd) entries of all the 16 squares together gives all the even (resp. odd) integers lying between 1 and 64. Moreover, adding 1 to all the odd integers lying between 1 and 64, produces all the even integers lying between 1 and 64. Multiplying the bound $e_{i,1}e_{i,2} < (o_{i,1} + 1)(o_{i,2} + 1)$ across all the squares, we obtain

$$\prod_{1 \leq i \leq 16} e_{i,1}e_{i,2} < \prod_{1 \leq i \leq 16} (o_{i,1} + 1)(o_{i,2} + 1),$$

and consequently,

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1+1) \cdot (3+1) \cdot \ldots \cdot (63+1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

This contradicts the assumption that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1. Hence, no such filling is possible. $\blacksquare$