1 Warm up

Example 1.1 (India RMO 2003)**.** Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C, $\text{prod}(C)$ denotes the product of all numbers in C,

(c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.

Summary. It is equivalent to finding a subset B of $\{1, \ldots, 10\}$, other than $\emptyset, \{1, \ldots, 10\}$, such that $\text{prod}(B)^2$ divides $10!$ and the quotient $10!/\text{prod}(B)^2$ is minimized. To do so,

(a) write down the prime power factorization of $10!$,

(b) throw in enough elements in B so that $\text{prod}(B)$ is maximized, and $\text{prod}(B)^2$ divides $10!$.

Walkthrough.

(a) Observe that it is enough to find a nonempty proper subset B of $\{1, 2, \ldots, 10\}$ such that $\text{prod}(B)^2$ divides $10!$ and $\text{prod}(B)$ is the maximum.

(b) Writing down the prime power factorization of $10!$, deduce that B does not contain 7, it contains a multiple of 5, and also a multiple of 2 and a multiple of 3.

(c) Prove that B contains exactly one multiple of 5, and not more that two multiples of 3.

(d) Show that B is equal to one of the subsets $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$.

(e) Show that any of these three subsets also have the stated property.

First, let's work on it. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{10!}{(\text{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\operatorname{prod}(B)^2$ divides $10!$ and $\operatorname{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\operatorname{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, and hence, B does not contain 7. Moreover, B contains a multiple of 5, otherwise $(\operatorname{prod}(B \cup \{5\}))^2$ would divide $10!$ and $\operatorname{prod}(B \cup \{5\})$ would be strictly larger than $\operatorname{prod}(B)$, which contradicts the choice of B. Similarly, B also contains a multiple of 2 and a multiple of 3. Note that B contains exactly one multiple of 5 (since $5^3 \nmid 10!$). Since $(\operatorname{prod}(B))^2$ divides $10!$ and $\operatorname{prod}(B)$ is the maximum, B is equal to one of the following sets

- $\{5,3,2,2^3\}$, $\{5,3,2,2^3,1\}$, $\{5,6,2,2^3\}$, $\{5,6,2,2^3,1\}$, $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$ if B contains 5,
- $\{10,3,2^3\}$, $\{10,3,2^3,1\}$, $\{10,3,2,2^2\}$, $\{10,3,2,2^2,1\}$, $\{10,6,2^2\}$, $\{10,6,2^2,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2,2^2\}$, $\{10,9,2,2^2,1\}$ if B contains 10.

For any of the above sets, the product of its elements is equal to 240, 480, or 720. So B is equal to one of the sets $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$.

Also note that if B denotes one of these subsets of $\{1,\dots,10\}$, then $\operatorname{prod}(B)^2$ divides $10!$ and $\operatorname{prod}(B)$ is the maximum.

This proves that $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$ are precisely all the subsets of $\{1,\dots,10\}$ having the required property. Thus we could take $A = \{1,2,3,4,5,6,7\}, B = \{8,9,10\}$ for instance. ♣

Remark. Note that the above solution provides more than what has been required. After observing that $\operatorname{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, one may show that there is a subset B with $\operatorname{prod}(B)$ equal to $2^4 \cdot 3^2 \cdot 5$ (for instance, $B = \{8,9,10\}$), and then conclude.

Solution 1. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\operatorname{prod}(A)}{\operatorname{prod}(B)} = \frac{10!}{(\operatorname{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\operatorname{prod}(B)^2$ divides $10!$ and $\operatorname{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\operatorname{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$. If $B = \{8,9,10\}$, then $\operatorname{prod}(B)$ is equal to $2^4 \cdot 3^2 \cdot 5$. Hence, $A = \{1,\dots,7\}, B = \{8,9,10\}$ are two disjoint nonempty subsets of $X = \{1,\dots,10\}$ satisfying the required conditions. ■

Remark. Don't be surprised that it took a bit long to arrive at the above solution. It is often the case. Further, it is a standard practice to write down a complete solution as the final one, without any reference to the prior attempts (possibly several). Those attempts have their important role in providing insights, which may lead to a solution. Here, the details of those attempts have not been hidden from you, in order to take you along the journey. However, I would like to highlight that a *solution* to a problem has to be complete, and at the same time, has to be free from the prior thoughts that have no direct role to play in that solution, though they might have played a significant role in gaining insight.