Example 1. Show that the positive integers of the form $4n + 3$, that is, the integers

$$3, 7, 11, 15, 19, \ldots$$

cannot be written as the sum of two perfect squares.

Summary — Show that the squares leave a remainder of 0 or 1 upon division by 4. Conclude that a sum of two squares leaves a remainder of $0, 1, 2$ upon division by 4.

Walkthrough —

(a) Consider the integers

$$\begin{gathered}
0^2 + 1^2, 0^2 + 2^2, 0^2 + 3^2, 0^2 + 4^2, \ldots, \\
1^2 + 1^2, 1^2 + 2^2, 1^2 + 3^2, 1^2 + 4^2, \ldots, \\
2^2 + 1^2, 2^2 + 2^2, 2^2 + 3^2, 2^2 + 4^2, \ldots, \\
3^2 + 1^2, 3^2 + 2^2, 3^2 + 3^2, 3^2 + 4^2, \ldots, \\
4^2 + 1^2, 4^2 + 2^2, 4^2 + 3^2, 4^2 + 4^2, \ldots.
\end{gathered}$$

(b) Observe that upon division by 4, they leave the integers $0, 1, 2$ as remainders.

$$\begin{gathered}
0^2 + 1^2 \rightsquigarrow \mathbf{1}, 0^2 + 2^2 \rightsquigarrow \mathbf{0}, 0^2 + 3^2 \rightsquigarrow \mathbf{1}, 0^2 + 4^2 \rightsquigarrow \mathbf{0}, \ldots, \\
1^2 + 1^2 \rightsquigarrow \mathbf{2}, 1^2 + 2^2 \rightsquigarrow \mathbf{1}, 1^2 + 3^2 \rightsquigarrow \mathbf{2}, 1^2 + 4^2 \rightsquigarrow \mathbf{1}, \ldots, \\
2^2 + 1^2 \rightsquigarrow \mathbf{1}, 2^2 + 2^2 \rightsquigarrow \mathbf{0}, 2^2 + 3^2 \rightsquigarrow \mathbf{1}, 2^2 + 4^2 \rightsquigarrow \mathbf{0}, \ldots, \\
3^2 + 1^2 \rightsquigarrow \mathbf{2}, 3^2 + 2^2 \rightsquigarrow \mathbf{1}, 3^2 + 3^2 \rightsquigarrow \mathbf{2}, 3^2 + 4^2 \rightsquigarrow \mathbf{1}, \ldots, \\
4^2 + 1^2 \rightsquigarrow \mathbf{1}, 4^2 + 2^2 \rightsquigarrow \mathbf{0}, 4^2 + 3^2 \rightsquigarrow \mathbf{1}, 4^2 + 4^2 \rightsquigarrow \mathbf{0}, \ldots.
\end{gathered}$$

(c) Show that it is **always** the case, namely, upon division by 4, the sum of two perfect squares leaves one of $0, 1, 2$ as the remainder.

(d) Conclude that no integer, which leaves the remainder of 3 **upon division by 4**, can be written as the sum of two squares.

Solution 1. The solution relies on the following claim.

Claim — For any integer x, the integer x^2 leaves a remainder of 0 or 1 upon division by 4.

Proof of the claim. Let x be an integer. Let us consider the following cases.

1. Upon division by 4, x leaves a remainder of 0.
2. Upon division by 4, x leaves a remainder of 1.
3. Upon division by 4, x leaves a remainder of 2.
4. Upon division by 4, x leaves a remainder of 3.

In the first case, x is a multiple of 4, and hence x^2 leaves a remainder of 0 upon division by 4. Similarly, in the third case, x is a multiple[1] of 2, i.e. x is equal to $2k$, and hence x^2 is a multiple of 4.

In the second case, x is equal to $4k+1$ for some integer k. Note that

$$\begin{aligned} x^2 &= (4k+1)^2 \\ &= (4k)^2 + 2 \cdot 4k + 1 \\ &= 4(4k^2 + 2k) + 1, \end{aligned}$$

and hence x^2 leaves a remainder of 1 upon division by 4.

In the fourth case, x is equal to $4k+3$ for some integer k. Note that

$$\begin{aligned} x^2 &= (4k+3)^2 \\ &= (4k)^2 + 2 \cdot 4k \cdot 3 + 9 \\ &= 4(4k^2 + 6k + 2) + 1, \end{aligned}$$

and hence x^2 leaves a remainder of 1 upon division by 4.

This proves the claim. □

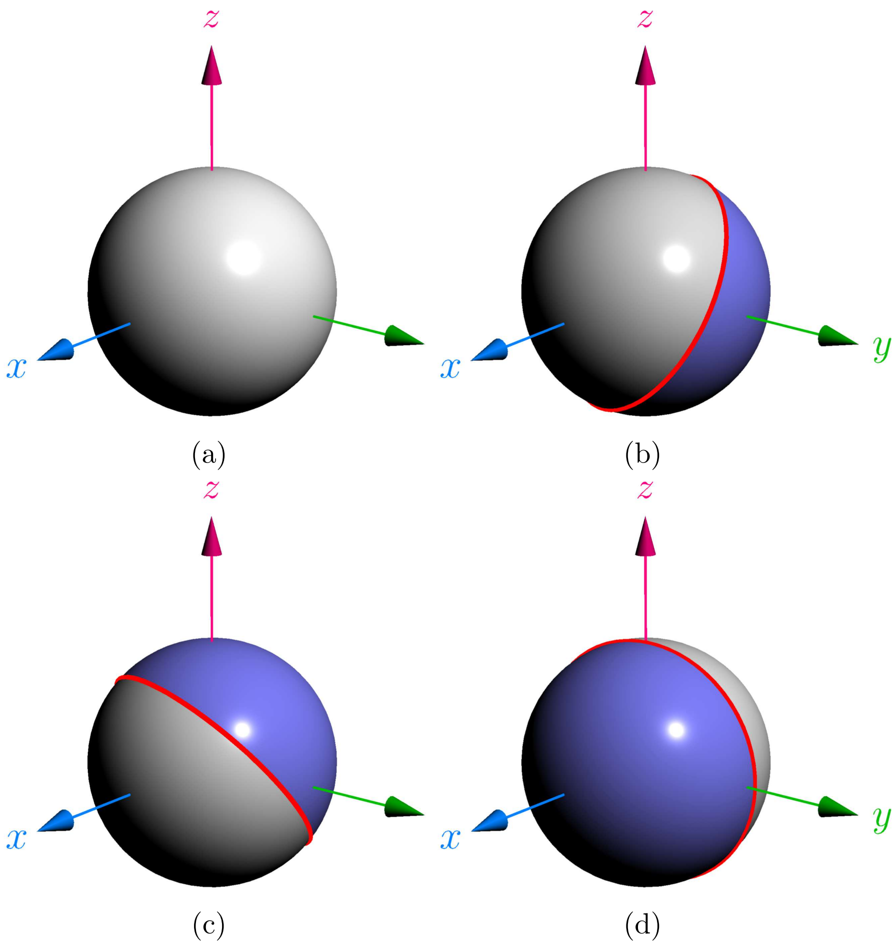
Using the claim, it follows that a sum of two squares leaves one of $0, 1, 2$ as a remainder upon division by 4. Hence, no integer of the form $4n+3$ can be expressed as a sum of two perfect squares. ■

Example 2 (Putnam 2002 A2)**.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

[1] Is it clear?

Remark. Let $\mathcal{S}$ be a sphere and $\mathcal{C}$ be a great circle on it. Then $\mathcal{C}$ divides $\mathcal{S}$ into two parts, which are called the *hemispheres defined by* $\mathcal{C}$. Any such hemisphere together with the great circle $\mathcal{C}$ is called a *closed hemisphere*. In Fig. 1, there are a few examples of closed hemispheres. Those are the grey ones together with the great circles marked in red, and the blue ones together with the great circles marked in red.

Summary — Apply the pigeonhole principle.



(a) (b) (c) (d)

Figure 1: USA Putnam 2002 A2

> **Walkthrough** —
>
> **(a)** Draw a great circle passing through at least two of the five points.
>
> **(b)** At least one closed hemisphere contains at least two of the remaining three points.
>
> **(c)** Conclude!

Solution 2. Draw a great circle passing through at least two of the five points. Then at least one closed hemisphere contains at least two of the remaining three points. This proves the result. See [**AN10**, Example 3.2]. ■

Example 3 (Moscow MO 2015 Grade 11)**.** Prove that it is impossible to put the integers from 1 to 64 (using each integer once) into an 8×8 table so that any 2×2 square, considered as a matrix, has a determinant that is equal to 1 or -1.

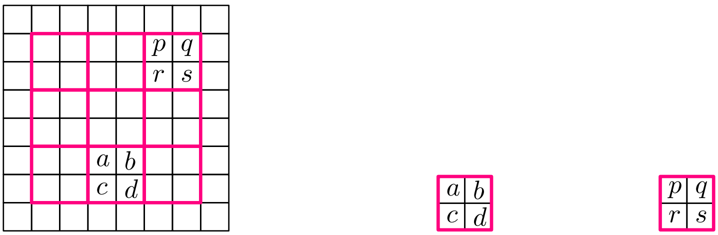


Figure 2: $ad - bc = \pm 1, ps - qr = \pm 1$

> **Remark.**
>
> - Given a 2×2 matrix (or array of numbers) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is
> $$ad - bc,$$
> i.e.
> $$\begin{aligned} &\text{the product of the diagonal terms} \\ &- \text{the product of the anti-diagonal terms.} \end{aligned}$$

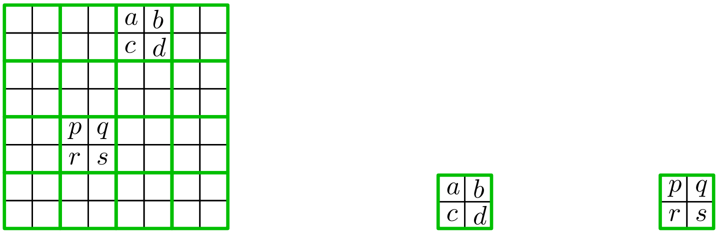


Figure 3: $ad - bc = \pm 1, ps - qr = \pm 1$

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has determinant -2.
- $\begin{pmatrix} 8 & 9 \\ 7 & 12 \end{pmatrix}$ has determinant $96 - 63 = 33$.
- $\begin{pmatrix} 13 & 14 \\ 5 & 7 \end{pmatrix}$ has determinant $91 - 70 = 21$.
- We need to show that there is no filling of an 8×8 table using the integers from 1 to 64, using each integer once, such that any 2×2 square (such squares have been marked in Fig. 2, Fig. 3, note that there $9 + 16 = 25$ such 2×2 squares.) has a determinant equal to 1 or -1.

Summary — If such a filling exists, then divide the 8×8 table into 16 pairwise disjoint 2×2 squares (as in Fig. 4). Due to parity constraints, each square contains precisely two evens along its diagonal or anti-diagonal, and their product is at most one more than the product of the odd entries. Consequently, for any of these 16 squares, the product of its even entries is less than the product of the successors of its odd entries. Multiplying across the squares gives a contradiction.

Walkthrough —

(a) Assume that such a filling exists.

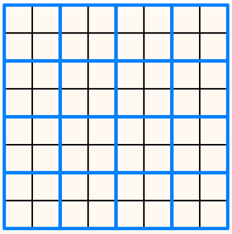


Figure 4: Moscow MO 2015 Grade 11 Day 1 P5

(b) Recall that the determinant of a 2×2 square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ is

$$\begin{aligned} &\text{the product of the diagonal terms} \\ &- \text{the product of the anti-diagonal terms.} \end{aligned}$$

(c) Note that $\boxed{\text{even} - \text{even} \neq \pm 1, \text{odd} - \text{odd} \neq \pm 1}$, and hence any square contains two odd numbers along the diagonal or on the anti-diagonal.

(d) Divide the 8×8 table into 16 pairwise disjoint 2×2 squares.

(e) Each of these 16 squares contains at least two odd integers, and hence, they together contain at least 32 odd integers.

(f) Conclude that each of these 16 squares contains precisely two odd integers, and precisely two even integers.

(g) Consider a square among them. It is of the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \text{ with } a, d \text{ both odd, and } b, c \text{ both even,}$$

or of the form

$$\begin{array}{|c|c|}\hline b & a \\ \hline d & c \\ \hline\end{array} \text{ with } a, d \text{ both odd, and } b, c \text{ both even.}$$

(h) The product of its even entries is at most one more than the product of its odd entries.

(i) Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b+1)(c+1)$ holds.

(j) This shows that

$$\begin{aligned}&\text{the product of two evens between 1 and 64}\\ <\ &\text{the product of}\\ &\qquad\qquad\text{two (possibly different) evens between 1 and 64.}\end{aligned}$$

(k) Multiply all the even entries of the 16 squares to obtain

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1+1) \cdot (3+1) \cdot \ldots \cdot (63+1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

Solution 3. Let us assume that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1.

Claim — Any 2×2 square contains at least two odd integers.

Proof of the claim. Since the difference of two integers can be odd only when they are of different parity (i.e. one of them is odd, and the other is even), it follows that for any 2×2 square, the product of its diagonal entries and the product of its anti-diagonal entries are of different parity, and hence of these two products is odd, and consequently, the diagonal entries are odd or the anti-diagonal entries are odd. In particular, any 2×2 square contains at least two odd integers. □

Let us divide the 8×8 table into 16 pairwise disjoint 2×2 squares (as in Fig. 4).

Claim — Each of these 16 squares contains exactly two even integers, lying along its diagonal or anti-diagonal.

Proof of the claim. By the previous Claim, each of these 16 squares contains at least two odd integers, and they contain at least $16 \times 2 = 32$ odd integers. Since there are precisely 32 odd integers between 1 and 64, it follows that each of these 16 squares contains exactly two odd integers along its diagonal or anti-diagonal, and hence exactly two even integers along its anti-diagonal or diagonal. □

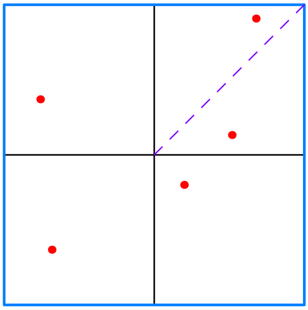


Figure 5

Since the determinant of any 2×2 square is 1 or -1, it follows that for any of the 16 squares as in Fig. 4, the product of its even entries is at most one more than the product of its odd entries. Note that for any two odd positive integers b, c, the inequality $bc + 1 < (b+1)(c+1)$ holds. Consequently, for any of the 16 squares as in Fig. 4, the product of its even entries is less than the product of the successors of its odd entries. This implies that the product of the even entries of all the 16 squares is less than the product of the successors of the odd entries of these boxes. Note that the even entries of these squares are the even integers lying between 1 and 64, so are the successors of the odd entries of these squares. It follows that

$$2 \cdot 4 \cdot \ldots \cdot 64 < (1+1) \cdot (3+1) \cdot \ldots \cdot (63+1) = 2 \cdot 4 \cdot \ldots \cdot 64.$$

This contradicts the assumption that an 8×8 table admits a filling by the integer from 1 to 64, using each integer once, such that each 2×2 square, considered as a matrix, has determinant equal to 1 or -1. Hence, no such filling is possible. ■

Example 4. Among any 5 points in a 2×2 square, show that there are two points which are at most $\sqrt{2}$ apart.

Summary — Divide the 2×2 square into suitable "boxes/pockets", so that the pigeonhole principle can be applied.

Walkthrough —

(a) Divide the 2×2 square into four unit squares.

(b) Two points among any choice of 5 points from the 2×2 square lie in one of these unit squares.

(c) Conclude!

Solution 4. Suppose we are given a set of five points in a 2×2 square. Divide the 2×2 square into four unit squares. By the pigeonhole principle, two points among those five points lie in one of these unit squares. Note that the distance between any two points lying in a unit square is at most the length of any of its diagonals. By Pythagoras' theorem, any diagonal of a unit square has length equal to $\sqrt{2}$. Consequently, two of those five points are at most $\sqrt{2}$ apart. ■

Example 5 (Austrian Junior Regional Competition 2022)**.** Determine all prime numbers p, q and r with $p + q^2 = r^4$.

Summary — Write down p in terms of q, r and factorize p, which is a prime!

Walkthrough —

(a) Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

(b) This gives $r^2 - q = 1$, and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

> **(c)** This implies that $r - 1 = 1$.
>
> **(d)** Conclude that $r = 2, q = 3, p = 7$.

Solution 5. Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

Since p is a prime and $r^2 - q < r^2 + q$ holds, it follows that $r^2 - q = 1$, and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

Since p is a prime and $r - 1 < r + 1$, this implies that $r - 1 = 1$. This gives $r = 2, q = 3, p = 7$. Since $2, 3, 7$ are primes, it follows that the only solution of the given equation in primes is

$$p = 7, q = 3, r = 2.$$

■

Example 6 (cf. Australian Mathematics Competition 1984)**.** Suppose

$$x_1, x_2, x_3, x_4, \ldots$$

is a sequence of integers satisfying the following properties:

(1) $x_2 = 2$,

(2) $x_{mn} = x_m x_n$ for all positive integers m, n,

(3) $x_m < x_n$ for any positive integers m, n with $m < n$.

Find x_{2024}.

> **Summary** — Observe that $x_{2^n} = 2^n$ for any $n \geq 1$. Combining this with the hypothesis that $\{x_n\}_{n \geq 1}$ is an increasing sequence of **positive integers**, conclude that $x_n = n$ for any $n \geq 1$.

Walkthrough —

(a) What can be said about x_4, x_8, x_{16}, x_{32}?

(b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.

(c) Can one show that $x_{2^n} = 2^n$ for any $n \geq 1$?

(d) Show that $x_m = m$ for any $m \geq 1$ (does property (3) help?).

Solution 6. From the second condition, we obtain

$$x_{2^n} = x_2^n$$

for any integer $n \geq 1$. Using the first condition, it gives

$$x_{2^n} = 2^n$$

for any integer $n \geq 1$. Since $\{x_n\}_{n\geq 1}$ is an increasing sequence of **positive integers**, it follows that $x_n = n$ for any positive integer n. This gives

$$x_{2024} = 2024.$$

■

References

[**AN10**] Claudi Alsina and Roger B. Nelsen. *Charming proofs*. Vol. 42. The Dolciani Mathematical Expositions. A journey into elegant mathematics. Mathematical Association of America, Washington, DC, 2010, pp. xxiv+295. ISBN: 978-0-88385-348-1 (cited p. 4)