List of problems and examples

§1 $a^3+b^3+c^3-3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

> **Remark.** An immediate approach would be to begin from the expression at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3+b^3+c^3-3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Note that

$$\begin{aligned}
&a^3+b^3+c^3-3abc\\
&=(a+b)^3-3ab(a+b)+c^3-3abc\\
&=(a+b)^3+c^3-3ab(a+b)-3abc\\
&=(a+b)^3+c^3-3ab(a+b+c)\\
&=(a+b+c)^3-3(a+b)c(a+b+c)-3ab(a+b+c)\\
&=(a+b+c)((a+b+c)^2-3(a+b)c-3ab)\\
&=(a+b+c)(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)\\
&=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).
\end{aligned}$$

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> **Remark.** There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X)=X^3-(a+b+c)X^2+(ab+bc+ca)X-abc.$$

Since a, b, c are the roots[1] of the equation $P(X)=0$, we obtain

$$a^3-(a+b+c)a^2+(ab+bc+ca)a-abc=0,$$

[1] If it is not clear, then the following equalities may directly be verified.

$$b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0,$$
$$c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0.$$

Adding them yields

$$a^3 + b^3 + c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) - 3abc = 0.$$

This proves that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

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The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a+b+c=0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1)**.** Factor $(x-y)^3 + (y-z)^3 + (z-x)^3$.

Solution 3. Note that if $a+b+c=0$, then $a^3+b^3+c^3 = 3abc$. This gives

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

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Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}
&(x-y)^3 + (y-z)^3 + (z-x)^3 \\
&= x^3 - 3x^2y + 3xy^2 - y^3 \\
&+ y^3 - 3y^2z + 3yz^2 - z^3 \\
&+ z^3 - 3z^2x + 3zx^2 - x^3 \\
&= -3x^2y + 3xy^2 - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
&= -3xy(x-y) - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
&= -3xy(x-y) - 3y^2z + 3zx^2 + 3yz^2 - 3z^2x \\
&= -3xy(x-y) + 3z(x^2-y^2) - 3z^2(x-y) \\
&= -3xy(x-y) + 3z(x-y)(x+y) - 3z^2(x-y) \\
&= 3(x-y)\big(-xy + z(x+y) - z^2\big) \\
&= 3(x-y)\big(-xy + zx + zy - z^2\big) \\
&= 3(x-y)\big(-x(y-z) + z(y-z)\big) \\
&= 3(x-y)(y-z)(z-x).
\end{aligned}$$

> However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2)**.** Solve the following equation for real x:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3=0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$\begin{aligned}
&(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3\\
&=3(x^2+x-2)(2x^2-x-1)(-3x^2+3)\\
&=-9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).
\end{aligned}$$

Thus the required solutions for x are

$$-2,-1,\frac{1}{2},1.$$

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Example 1.4 (India RMO 2012b P6)**.** Show that for all real numbers x, y, z such that $x+y+z=0$ and $xy+yz+zx=-3$, the expression $x^3y+y^3z+z^3x$ is a constant.

Solution 5. Consider the polynomial

$$P(t)=t^3-(x+y+z)t^2+(xy+yz+zx)t-xyz.$$

Since x, y, z are the roots[2] of the equation $P(t)=0$, we obtain

$$\begin{aligned}
x^3-(x+y+z)x^2+(xy+yz+zx)x-xyz&=0,\\
y^3-(x+y+z)y^2+(xy+yz+zx)y-xyz&=0,\\
z^3-(x+y+z)z^2+(xy+yz+zx)z-xyz&=0.
\end{aligned}$$

Using them, we obtain

$$\begin{aligned}
x^3y+y^3z+z^3x&=\big((x+y+z)x^2-(xy+yz+zx)x+xyz\big)y\\
&\quad+\big((x+y+z)y^2-(xy+yz+zx)y+xyz\big)z\\
&\quad+\big((x+y+z)z^2-(xy+yz+zx)z+xyz\big)x\\
&=(x+y+z)(x^2y+y^2z+z^2x)
\end{aligned}$$

[2]If it is not clear, then the following equalities may directly be verified.

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\begin{aligned}
&- (xy + yz + zx)(xy + yz + zx) \\
&+ xyz(x + y + z) \\
= &-(xy + yz + zx)^2 \qquad \text{(using } x + y + z = 0\text{)} \\
= &-9 \qquad \text{(using } xy + yz + zx = -3\text{).}
\end{aligned}
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This completes the proof. ■