Example 1 (cf. Australian Mathematics Competition 1984)**.** Suppose

$$x_1, x_2, x_3, x_4, \ldots$$

is a sequence of integers satisfying the following properties:

(1) $x_2 = 2$,

(2) $x_{mn} = x_m x_n$ for all positive integers m, n,

(3) $x_m < x_n$ for any positive integers m, n with $m < n$.

Find x_{2024}.

Summary — Observe that $x_{2^n} = 2^n$ for any $n \geq 1$. Combining this with the hypothesis that $\{x_n\}_{n\geq 1}$ is an increasing sequence of **positive integers**, conclude that $x_n = n$ for any $n \geq 1$.

Walkthrough.

(a) What can be said about x_4, x_8, x_{16}, x_{32}?

(b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.

(c) Can one show that $x_{2^n} = 2^n$ for any $n \geq 1$?

(d) Show that $x_m = m$ for any $m \geq 1$ (does property (3) help?).

Solution 1. From the second condition, we obtain

$$x_{2^n} = x_2^n$$

for any integer $n \geq 1$. Using the first condition, it gives

$$x_{2^n} = 2^n$$

for any integer $n \geq 1$. Since $\{x_n\}_{n\geq 1}$ is an increasing sequence of **positive integers**, it follows that $x_n = n$ for any positive integer n. This gives

$$x_{2024} = 2024.$$

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