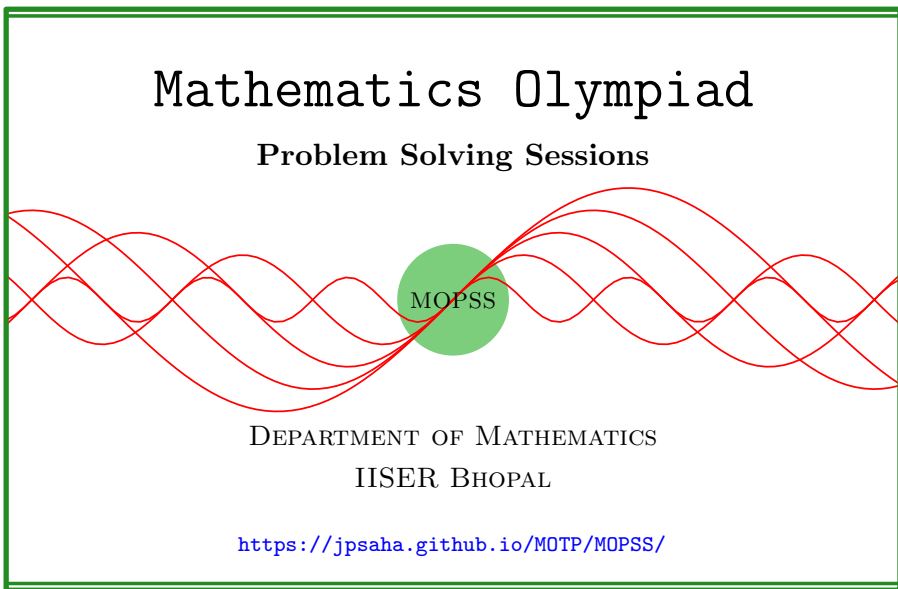


Cubic polynomials

MOPSS

3 June 2024



Suggested readings

- Evan Chen's
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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§1 Cubic polynomials

Example 1.1 (India RMO 2012b P6). Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution 1. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots¹ of the equation $P(t) = 0$, we obtain

$$x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz = 0,$$

$$y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz = 0,$$

$$z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz = 0.$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= ((x + y + z)x^2 - (xy + yz + zx)x + xyz)y \\ &\quad + ((x + y + z)y^2 - (xy + yz + zx)y + xyz)z \\ &\quad + ((x + y + z)z^2 - (xy + yz + zx)z + xyz)x \\ &= (x + y + z)(x^2y + y^2z + z^2x) \\ &\quad - (xy + yz + zx)(xy + yz + zx) \\ &\quad + xyz(x + y + z) \\ &= -(xy + yz + zx)^2 \quad (\text{using } x + y + z = 0) \\ &= -9 \quad (\text{using } xy + yz + zx = -3). \end{aligned}$$

This completes the proof. ■

¹If it is not clear, then the following equalities may directly be verified.