Polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Polynomials

Example 1.1. Factorize the polynomial $x^8 + x^4 + 1$ into factors of at most the second degree.

> **Summary** — Expressing an expression as a difference of two squares yields a factorization.

Solution 1. Note that

$$\begin{aligned}
&x^8 + x^4 + 1 \\
&= x^8 + 2x^4 + 1 - x^4 \\
&= (x^4 + 1)^4 - (x^2)^2 \\
&= (x^4 - x^2 + 1)(x^4 + 1 + x^2) \\
&= (x^4 + 2x^2 + 1 - 3x^2)(x^4 + 2x^2 + 1 - x^2) \\
&= ((x^2 + 1)^2 - (\sqrt{3}x)^2)((x^2 + 1)^2 - x^2) \\
&= (x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)(x^2 - x + 1)(x^2 + x + 1).
\end{aligned}$$

■

Example 1.2. Show that

$$2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

 The content posted here and at this blog by Evan Chen are quite useful.

Solution 2. Write[1]

$$a = y + z,$$
$$b = z + x,$$
$$c = x + y.$$

Note that

$$\begin{aligned}
&2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\\
&= 2(y+z)^2(z+x)^2 + 2(z+x)^2(x+y)^2 + 2(x+y)^2(y+z)^2\\
&- (y+z)^4 - (z+x)^4 - (x+y)^4\\
&= 2(z^2 + 2yz + y^2)(z^2 + 2zx + x^2)\\
&+ 2(x^2 + 2zx + z^2)(x^2 + 2xy + y^2)\\
&+ 2(y^2 + 2xy + x^2)(y^2 + 2yz + z^2)\\
&- (y+z)^4 - (z+x)^4 - (x+y)^4\\
&= 2\sum_{\text{cyc}} \left(x^4 + x^2(y^2 + z^2 + 2x(y+z)) + yz(2x+y)(2x+z)\right) - \sum_{\text{cyc}}(x+y)^4\\
&= 2\sum_{\text{cyc}} \left(x^4 + x^2y^2 + z^2x^2 + 2x^3(y+z) + 4x^2yz + 2xyz(y+z) + y^2z^2\right)\\
&- \sum_{\text{cyc}}(x+y)^4\\
&= 2(x^4 + y^4 + z^4) + 6(x^2y^2 + y^2z^2 + z^2x^2) + 16xyz(x+y+z) + 4\sum_{\text{cyc}} x^3(y+z)\\
&- \sum_{\text{cyc}}(x+y)^4\\
&= 2(x^4 + y^4 + z^4) + 6(x^2y^2 + y^2z^2 + z^2x^2) + 16xyz(x+y+z) + 4\sum_{\text{cyc}} x^3(y+z)\\
&- \sum_{\text{cyc}}(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4)\\
&= 16xyz(x+y+z)\\
&= (a+b+c)(a+b-c)(b+c-a)(c+a-b).
\end{aligned}$$

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[1] It is good ask the following simple and innocent question: how can one **write** a, b, c as stated above? Does it mean that given any three real numbers a, b, c, one can find real numbers x, y, z such that $a = y + z, b = z + x, c = x + y$? A crucial point to note is that one very often deals with indeterminates (aka variables) instead of real numbers. In the above, a, b, c could be indeterminates instead of being real numbers! What do we do in that case? Is it so that there are **indeterminates** x, y, z such that the six indeterminates a, b, c, x, y, z satisfy $a = y + z, b = z + x, c = x + y$? As of now, **let's not worry about any of these!**. Just keep in mind the message that at times, **we need to be quite careful about what we do!**

Remark. Please go through the footnote next to the word **Write** from the above solution. This footnote would explain that just writing **Write** $a = y + z, b = z + x, c = x + y$ requires more care! To address this issue, replace **Write** $a = y + z, b = z + x, c = x + y$ in the above solution by the following.

Consider the real numbers x, y, z defined by

$$\begin{aligned} x &= \frac{1}{2}(b + c - a), \\ y &= \frac{1}{2}(c + a - b), \\ z &= \frac{1}{2}(a + b - c). \end{aligned}$$

Note that

$$a = y + z, b = z + x, c = x + y$$

holds.

Example 1.3. Find numbers a, b, c, d for which the equation

$$\frac{2x - 7}{4x^2 + 16x + 15} = \frac{a}{x + c} + \frac{b}{x + d}$$

would be an identity.

Walkthrough — Factorize the denominator into linear factors. Then expressing the numerator as a linear combination of those factors would provide such an identity.

Solution 3. Note that

$$\frac{2x - 7}{4x^2 + 16x + 15} = \frac{2x - 7}{(2x + 3)(2x + 5)}.$$

Hence, if $2x - 7$ can be expressed as

$$p(2x + 3) + q(2x + 5),$$

then $\frac{2x-7}{4x^2+16x+15}$ can be expressed as a sum of two fractions, each having a constant in the numerator and a linear polynomial in the denominator.

One way to find if there are any such p, q, is to assume first that there are such real numbers p and q such that

$$2x - 7 = p(2x + 3) + q(2x + 5)$$

holds[2]. Substituting $x = -\frac{5}{2}$, we obtain $-2p = -12$, which gives $p = 6$. Next, substituting $x = -\frac{3}{2}$, we obtain $2q = -10$, which implies $q = -5$.

[2] and try to see what conditions get imposed on p, q. It may happen that the conditions that get imposed, may suggest that there are no such p, q. However, it may also happen that we would be able to find out which p, q would work!

Note that

$$6(2x+3)+(-5)(2x+5) = 12x+18-10x-25 = 2x-7$$

holds[3]. Using it, we obtain

$$\begin{aligned}\frac{2x-7}{4x^2+16x+15} &= \frac{2x-7}{(2x+3)(2x+5)}\\ &= \frac{6(2x+3)+(-5)(2x+5)}{(2x+3)(2x+5)}\\ &= \frac{6}{2x+5} - \frac{5}{2x+3}\\ &= \frac{3}{x+\frac{5}{2}} - \frac{\frac{5}{2}}{x+\frac{3}{2}}.\end{aligned}$$

Hence, we may take

$$a = 3, b = -\frac{5}{2}, c = \frac{5}{2}, d = \frac{3}{2}.$$

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Exercise 1.4. Are there other choices for a, b, c, d for which the identify would hold?

Example 1.5. Determine the remainder obtained upon dividing x^{100} by $x^2 - 3x + 2$.

Solution 4. Let $q(x)$ (resp. $r(x)$) denote the quotient (resp. the remainder) obtained upon dividing x^{100} by x^2-3x+2. Note that $r(x)$ is a linear polynomial, i.e. $r(x) = ax + b$ for some real numbers a, b. Then we have

$$x^{100} = q(x)(x^2 - 3x + 2) + r(x).$$

Substituting $x = 1$, it yields

$$1 = r(1) = a + b.$$

Similarly, substituting $x = 2$, it gives

$$2^{100} = r(2) = 2a + b.$$

[3]It should be noted that p, q were assumed to exist such that $p(2x+3)+q(2x+5) = 2x-7$ holds. Under this hypothesis, we obtained $p = 6, q = -5$. At this point, we cannot immediately conclude that $6(2x+3)+(-5)(2x+5) = 2x-7$ holds (unless we verify it), because if we do so, then we would do it under the same hypothesis.

- Even then, what would go wrong with that?
- Can a hypothesis (possibly combined with some of its consequences) be a justification for itself to hold? Think about this point.

This shows that
$$a = 2^{100} - 1, \quad b = 1 - a = 2 - 2^{100}.$$
Hence, the remainder obtained upon dividing x^{100} by $x^2 - 3x + 2$ is equal to
$$(2^{100} - 1)x + 2 - 2^{100}.$$

■

Example 1.6 (USAMO 1975 P3)**.** [**GA17**, Problem 151] A polynomial $P(x)$ of degree n satisfies
$$P(k) = \frac{k}{k+1} \quad \text{for } k = 0, 1, 2, \ldots, n.$$
Find $P(n+1)$.

Solution 5. Note that $xP(x+1) - x$ is a polynomial of degree $n+1$, and it vanishes at the $n+1$ integers $0, 1, 2, \ldots, n$. It follows that
$$(x+1)P(x) - x = cx(x-1)(x-2)\ldots(x-n)$$
for some nonzero real number c. Substituting $x = -1$ yields
$$1 = (-1)^{n+1}c(n+1)!,$$
which gives $c = \frac{(-1)^{n+1}}{(n+1)!}$. This implies that
$$(n+2)P(n+1) = n + 1 + (-1)^{n+1},$$
and consequently,
$$P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}.$$

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Example 1.7. Let $g(x)$ and $h(x)$ be polynomials with real coefficients such that
$$g(x)(x^2 - 3x + 2) = h(x)(x^2 + 3x + 2)$$
and $f(x) = g(x)h(x) + (x^4 - 5x^2 + 4)$. Prove that $f(x)$ has at least four real roots.

Solution 6. Note that $g(x)$ and $h(x)$ satisfy
$$g(x)(x-1)(x-2) = h(x)(x+1)(x+2),$$
which shows that
$$g(-1), g(-2), h(1), h(2)$$

are equal to 0. Also note that

$$\begin{aligned} x^4 - 5x^2 + 4 &= (x^2 - 1)(x^2 - 4) \\ &= (x-1)(x+1)(x+2)(x-2). \end{aligned}$$

Hence, the polynomials $g(x)h(x)$ and $x^4 - 5x^2 + 4$ vanish at $1, -1, 2, -2$. Consequently, f also vanishes at these four points. ■

Example 1.8. Let n be a positive integer. Show that

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Solution 7. Note that

$$\begin{aligned} &(x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) \\ &= x(x^{n-1} + x^{n-2} + \cdots + x + 1) - (x^{n-1} + x^{n-2} + \cdots + x + 1) \\ &= x^n + x^{n-1} + \cdots + x^2 + x - (x^{n-1} + x^{n-2} + \cdots + x + 1) \\ &= x^n - 1. \end{aligned}$$

■

Example 1.9. Prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

Solution 8. Note that

$$\begin{aligned} &x^{44} + x^{33} + x^{22} + x^{11} + 1 \\ &= x^{40} \cdot x^4 + x^{30} \cdot x^3 + x^{20} \cdot x^2 + x^{10} \cdot x + 1 \\ &= (x^{40} - 1)x^4 + (x^{30} - 1)x^3 + (x^{20} - 1)x^2 + (x^{10} - 1)x \\ &\quad + x^4 + x^3 + x^2 + x + 1. \end{aligned}$$

Hence, to prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by $x^4 + x^3 + x^2 + x + 1$, it suffices to show that $x^4 + x^3 + x^2 + x + 1$ divides the polynomials

$$x^{40} - 1, x^{30} - 1, x^{20} - 1, x^{10} - 1.$$

Since

$$\begin{aligned} x^5 - 1 &= (x-1)(x^4 + x^3 + x^2 + x + 1), \\ x^{10} - 1 &= (x^5 - 1)(x^5 + 1), \end{aligned}$$

it follows that $x^4 + x^3 + x^2 + x + 1$ divides $x^{10} - 1$. Moreover, the polynomial x^{10-1} divides all of

$$x^{40} - 1, x^{30} - 1, x^{20} - 1, x^{10} - 1.$$

Hence, $x^4+x^3+x^2+x+1$ divides the polynomials

$$x^{40}-1, x^{30}-1, x^{20}-1, x^{10}-1.$$

■

Exercise 1.10. Show that the polynomial $x^{580}+x^{390}+x^{326}+x^{262}+x^{198}+x^{134}+1$ is divisible by $x^6+x^5+x^4+x^3+x^2+x+1$.

Example 1.11 (Moscow MO 1946 Grades 7–8 P5)**.** Prove that after completing the multiplication and collecting the terms

$$(1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+\cdots+x^{99}+x^{100})$$

has no monomials of odd degree.

Summary — What happens if x is replaced by $-x$?

Solution 9. Let $P(x)$ denote the polynomial

$$(1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+\cdots+x^{99}+x^{100}).$$

Note that $P(x)=P(-x)$. By the Claim below, it follows that $P(x)$ has no monomials of odd degree.

Claim — Let $Q(x)$ be a polynomial satisfying $Q(x)=Q(-x)$. Then $Q(x)$ has no monomials of odd degree.

Proof of the Claim. Note that

$$Q(x)=\frac{Q(x)+Q(-x)}{2}+\frac{Q(x)-Q(-x)}{2}$$

holds. Using $Q(x)=Q(-x)$, it follows that $Q(x)=\frac{Q(x)+Q(-x)}{2}$. Consequently, $Q(x)$ has no monomials of odd degree. □

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Remark. The above decomposition of $Q(x)$ is a special case of general phenomena[a].

[a] Can you think of a few? Which **general phenomena** is referred to?!

Remark. The above solution is more elegant, and less cumbersome. Moreover, it also highlights the underlying reason, whereas the solution below obscures the conceptual viewpoint.

Solution 10. One can multiply the polynomials to note that

$$\begin{aligned}
&1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100} \\
&= 1 - x + x^2(1-x) + x^4(1-x) + x^6(1-x) + \cdots + x^{98}(1-x) + x^{100} \\
&= (1-x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) + x^{100}.
\end{aligned}$$

Using this, we obtain

$$\begin{aligned}
&(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= \big((1-x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) + x^{100}\big)(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= (1-x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&\quad + x^{100}(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{100}(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{100}(x + x^3 + x^5 + \cdots + x^{99}) \\
&\quad + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}) \\
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{101}(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) \\
&\quad + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}) \\
&= 1 + x^2 + x^4 + x^6 + \cdots + x^{98} + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}),
\end{aligned}$$

which has no monomial of odd degree. ■

The following exercise is quite similar to the Claim proved in the solution to Example 1.11.

Exercise 1.12. Let $Q(x)$ be a polynomial satisfying $Q(x) = -Q(-x)$. Then $Q(x)$ has no monomials of even degree.

The exercise below relies on Example 1.8.

Example 1.13 (Moscow MO 2015 Grade 9 P6)**.** Do there exist two polynomials with integer coefficients such that each of them has a coefficient with absolute value exceeding 2015, but no coefficient of their product has absolute value exceeding 1?

Summary — Try to come up with enough polynomials $g_1(x), g_2(x), g_3(x), \dots$ and $h_1(x), h_2(x), h_3(x), \dots$ such that each of the products $g_1 g_2 g_3 \dots$ and $h_1 h_2 h_3 \dots$ have at least one coefficient which is **large in absolute value**, and all the coefficients of the product $(g_1 g_2 g_3 \dots)(h_1 h_2 h_3 \dots)$ are at most 1 in absolute value.

Walkthrough —

(a) Try to come up with a polynomial $P(x)$ whose coefficients are at most 1 in absolute value, and it can be written as a product of enough factors (say $f_1(x), f_2(x), \dots$) such that each of such factor $f_i(x)$ admits a decomposition into the product of two polynomials $g_i(x)$ and $h_i(x)$.

(b) Can you make sure that the product of the g_i's, and the product of the h_i's have to have at least one large coefficient?

(c) For instance, would taking $g_1(x) = g_2(x) = g_3(x) = \cdots = 1 - x$ work for some suitable choice of $h_1(x), h_2(x), \dots$?

(d) Does taking

$$\begin{aligned} h_1(x) &= 1 + x, \\ h_2(x) &= 1 + x + x^2, \\ h_3(x) &= 1 + x + x^2 + x^3, \end{aligned}$$

etc. work?

(e) Note that the product of enough g_i's would have a large coefficient (namely, the coefficient of the second largest power of x). On the other hand, the product of enough h_i's would have a large coefficient (namely, the coefficient of the power of x).

(f) What can be said about the absolute value of the coefficients of the product of these two products?

The above seems to work except that having a control on the coefficients of the product $(g_1 g_2 g_3 \dots)(h_1 h_2 h_3 \dots)$ seems hard[4].

Solution 11. Consider the polynomial

$$P(x) = (1 - x)(1 - x^2)(1 - x^4)(1 - x^8) \cdots (1 - x^{2^{2016}}).$$

Since

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} < 2^n,$$

it follows that the coefficients of $P(x)$ are at most 1 in absolute value. Note that

$$P(x) = Q(x)R(x)$$

[4] Is it because it fails?

holds where

$$Q(x) = (1-x)^{2017},$$
$$R(x) = (1+x)(1+x+x^2+x^3)\cdots(1+x+x^2+\cdots+x^{2^{2016}-1}).$$

The coefficient of x^{2016} in $Q(x)$ is equal to 2017, and the coefficient of x in $R(x)$ is equal to 2016. This completes the proof. ■

References

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. ISBN: 978-3-319-58986-2; 978-3-319-58988-6. DOI: 10.1007/978-3-319-58988-6. URL: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 6)