Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 $a^3 + b^3 + c^3 - 3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

> **Remark.** An immediate approach would be to begin from the expression at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3 + b^3 + c^3 - 3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Note that

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc \\
&= (a+b)^3 - 3ab(a+b) + c^3 - 3abc \\
&= (a+b)^3 + c^3 - 3ab(a+b) - 3abc \\
&= (a+b)^3 + c^3 - 3ab(a+b+c) \\
&= (a+b+c)^3 - 3(a+b)c(a+b+c) - 3ab(a+b+c) \\
&= (a+b+c)((a+b+c)^2 - 3(a+b)c - 3ab) \\
&= (a+b+c)(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca) \\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).
\end{aligned}$$

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> **Remark.** There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a+b+c)X^2 + (ab+bc+ca)X - abc.$$

Since a, b, c are the roots[1] of the equation $P(X) = 0$, we obtain

$$\begin{aligned} a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc &= 0, \\ b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc &= 0, \\ c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc &= 0. \end{aligned}$$

Adding them yields

$$a^3 + b^3 + c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) - 3abc = 0.$$

This proves that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

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The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a + b + c = 0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1)**.** Factor $(x-y)^3 + (y-z)^3 + (z-x)^3$.

Solution 3. Note that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. This gives

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

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> **Remark.** The following proof is direct, and of course, it works.
>
> $$\begin{aligned} &(x-y)^3 + (y-z)^3 + (z-x)^3 \\ &= x^3 - 3x^2y + 3xy^2 - y^3 \\ &+ y^3 - 3y^2z + 3yz^2 - z^3 \\ &+ z^3 - 3z^2x + 3zx^2 - x^3 \\ &= -3x^2y + 3xy^2 - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \end{aligned}$$

[1] If it is not clear, then the following equalities may directly be verified.

$$
\begin{aligned}
&= -3xy(x-y) - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
&= -3xy(x-y) - 3y^2z + 3zx^2 + 3yz^2 - 3z^2x \\
&= -3xy(x-y) + 3z(x^2-y^2) - 3z^2(x-y) \\
&= -3xy(x-y) + 3z(x-y)(x+y) - 3z^2(x-y) \\
&= 3(x-y)\big(-xy + z(x+y) - z^2\big) \\
&= 3(x-y)\big(-xy + zx + zy - z^2\big) \\
&= 3(x-y)\big(-x(y-z) + z(y-z)\big) \\
&= 3(x-y)(y-z)(z-x).
\end{aligned}
$$

However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2)**.** Solve the following equation for real x:

$$(x^2+x-2)^3 + (2x^2-x-1)^3 = 27(x^2-1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2+x-2)^3 + (2x^2-x-1)^3 + (-3x^2+3)^3 = 0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$
\begin{aligned}
&(x^2+x-2)^3 + (2x^2-x-1)^3 + (-3x^2+3)^3 \\
&= 3(x^2+x-2)(2x^2-x-1)(-3x^2+3) \\
&= -9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).
\end{aligned}
$$

Thus the required solutions for x are

$$-2, -1, \frac{1}{2}, 1.$$

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Example 1.4 (India INMO 2002 P2)**.** Find the smallest positive value taken by $a^3+b^3+c^3-3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

Walkthrough —

(a) Note that $a=b=c=1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.

(b) By taking $a=1, b=2, c=1$, one can find that $a^3+b^3+c^3-3abc=4$. Next, we need determine whether $a^3+b^3+c^3-3abc$ can be equal to $1, 2, 3$ or 4 for positive integers a, b, c.

(c) Use

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= \frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2)$$

to get a lower bound on $a^3 + b^3 + c^3 - 3abc$.

Solution 5. Let a, b, c be positive integers such that $a^3 + b^3 + c^3 - 3abc$ is positive. Note that they cannot be equal, and hence at least two of them are distinct. Since $a^3 + b^3 + c^3 - 3abc$ is symmetric[2] in a, b, c, we may assume[3] that $a \neq b$.

Apart from a, b, there is another pair of two integers among a, b, c which are not equal, i.e. $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b = c$ and $c = a$ hold, and then we would have $a = b$, which is a contradiction. Note that

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc \\
&= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
&= \frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2) \\
&\geq \frac{1}{2}(a+b+c)(1^2 + 1^2) \\
&\qquad\qquad \text{(since at least two of } a-b, b-c, c-a \text{ are nonzero, and } a+b+c > 0) \\
&\geq a+b+c \\
&\geq 1+2+1 \quad (\text{since } a < b \text{ and } a, b, c \geq 1) \\
&= 4.
\end{aligned}$$

Also note that if $c > 1$, then

$$a^3 + b^3 + c^3 - 3abc > 4.$$

For $a = 1, b = 2, c = 1$, we obtain

$$a^3 + b^3 + c^3 - 3abc = 4.$$

Hence, the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$, for positive integers a, b, c, is equal to 4.

Moreover, if a, b, c are positive integers such that $a^3 + b^3 + c^3 - 3abc$ takes the value 4, then at least two of a, b, c are unequal, and the above argument shows that

$$a + b + c \leq a^3 + b^3 + c^3 - 3abc \leq 4,$$

and consequently, two of a, b, c are equal to 1 and the remaining one is equal to 2. Hence, $a^3 + b^3 + c^3 - 3abc$ takes the value 4 precisely when

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

[2] A reader unfamiliar with this term may require to look online.
[3] How we may do so? It does require a thought.

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For more exercises around this theme, we refer to [**AE11**, §1.1].

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures.* Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 101)