Cubic polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Cubic polynomials

Example 1.1 (India RMO 2012b P6)**.** Show that for all real numbers x, y, z such that $x+y+z=0$ and $xy+yz+zx=-3$, the expression $x^3y+y^3z+z^3x$ is a constant.

Solution 1. Consider the polynomial

$$P(t) = t^3 - (x+y+z)t^2 + (xy+yz+zx)t - xyz.$$

Since x, y, z are the roots[1] of the equation $P(t) = 0$, we obtain

$$\begin{aligned} x^3 - (x+y+z)x^2 + (xy+yz+zx)x - xyz &= 0, \\ y^3 - (x+y+z)y^2 + (xy+yz+zx)y - xyz &= 0, \\ z^3 - (x+y+z)z^2 + (xy+yz+zx)z - xyz &= 0. \end{aligned}$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= \big((x+y+z)x^2 - (xy+yz+zx)x + xyz\big)y \\ &\quad + \big((x+y+z)y^2 - (xy+yz+zx)y + xyz\big)z \\ &\quad + \big((x+y+z)z^2 - (xy+yz+zx)z + xyz\big)x \\ &= (x+y+z)(x^2y+y^2z+z^2x) \\ &\quad - (xy+yz+zx)(xy+yz+zx) \\ &\quad + xyz(x+y+z) \\ &= -(xy+yz+zx)^2 \qquad (\text{using } x+y+z=0) \\ &= -9 \qquad (\text{using } xy+yz+zx=-3). \end{aligned}$$

This completes the proof. ■

[1] If it is not clear, then the following equalities may directly be verified.