List of problems and examples

§1 $a^3 + b^3 + c^3 - 3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Remark. An immediate approach would be to begin from the expression at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3 + b^3 + c^3 - 3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Note that

$$\begin{aligned}
&a^3 + b^3 + c^3 - 3abc \\
&= (a + b)^3 - 3ab(a + b) + c^3 - 3abc \\
&= (a + b)^3 + c^3 - 3ab(a + b) - 3abc \\
&= (a + b)^3 + c^3 - 3ab(a + b + c) \\
&= (a + b + c)^3 - 3(a + b)c(a + b + c) - 3ab(a + b + c) \\
&= (a + b + c)((a + b + c)^2 - 3(a + b)c - 3ab) \\
&= (a + b + c)(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca) \\
&= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).
\end{aligned}$$

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Remark. There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a + b + c)X^2 + (ab + bc + ca)X - abc.$$

Since a, b, c are the roots[1] of the equation $P(X) = 0$, we obtain

$$a^3 - (a + b + c)a^2 + (ab + bc + ca)a - abc = 0,$$

[1] If it is not clear, then the following equalities may directly be verified.

$$b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0,$$
$$c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0.$$

Adding them yields

$$a^3 + b^3 + c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) - 3abc = 0.$$

This proves that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ca).$$

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The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a+b+c=0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1)**.** Factor $(x-y)^3 + (y-z)^3 + (z-x)^3$.

Solution 3. Note that if $a+b+c=0$, then $a^3+b^3+c^3 = 3abc$. This gives

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

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Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}
&(x-y)^3 + (y-z)^3 + (z-x)^3 \\
&= x^3 - 3x^2y + 3xy^2 - y^3 \\
&+ y^3 - 3y^2z + 3yz^2 - z^3 \\
&+ z^3 - 3z^2x + 3zx^2 - x^3 \\
&= -3x^2y + 3xy^2 - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
&= -3xy(x-y) - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
&= -3xy(x-y) - 3y^2z + 3zx^2 + 3yz^2 - 3z^2x \\
&= -3xy(x-y) + 3z(x^2-y^2) - 3z^2(x-y) \\
&= -3xy(x-y) + 3z(x-y)(x+y) - 3z^2(x-y) \\
&= 3(x-y)\big(-xy + z(x+y) - z^2\big) \\
&= 3(x-y)\big(-xy + zx + zy - z^2\big) \\
&= 3(x-y)\big(-x(y-z) + z(y-z)\big) \\
&= 3(x-y)(y-z)(z-x).
\end{aligned}$$

> However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2)**.** Solve the following equation for real x:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3=0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$\begin{aligned}&(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3\\&=3(x^2+x-2)(2x^2-x-1)(-3x^2+3)\\&=-9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).\end{aligned}$$

Thus the required solutions for x are

$$-2,-1,\frac{1}{2},1.$$

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Example 1.4 (India INMO 2002 P2)**.** Find the smallest positive value taken by $a^3+b^3+c^3-3abc$ for positive integers a,b,c. Find all a,b,c which give the smallest value.

Walkthrough —

(a) Note that $a=b=c=1$ won't work, not even taking all of a,b,c to be equal would be of any use. In other words, at least two of a,b,c have to be unequal.

(b) By taking $a=1,b=2,c=1$, one can find that $a^3+b^3+c^3-3abc=4$. Next, we need determine whether $a^3+b^3+c^3-3abc$ can be equal to $1,2,3$ or 4 for positive integers a,b,c.

(c) Use

$$\begin{aligned}a^3+b^3+c^3-3abc&=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\\&=\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)\end{aligned}$$

to get a lower bound on $a^3+b^3+c^3-3abc$.

Solution 5. Let a,b,c be positive integers such that $a^3+b^3+c^3-3abc$ is positive. Note that they cannot be equal, and hence at least two of them are distinct. Since $a^3+b^3+c^3-3abc$ is symmetric[2] in a,b,c, we may assume[3] that $a \neq b$.

Apart from a, b, there is another pair of two integers among a,b,c which are not equal, i.e. $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b = c$ and $c = a$ hold, and then we would have $a = b$, which is a contradiction. Note that

$$
\begin{aligned}
&a^3+b^3+c^3-3abc \\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\
&= \frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2) \\
&\geq \frac{1}{2}(a+b+c)(1^2+1^2) \\
&\qquad\qquad \text{(since at least two of } a-b, b-c, c-a \text{ are nonzero, and } a+b+c>0) \\
&\geq a+b+c \\
&\geq 1+2+1 \quad (\text{since } a<b \text{ and } a,b,c\geq 1) \\
&= 4.
\end{aligned}
$$

Also note that if $c > 1$, then

$$a^3+b^3+c^3-3abc > 4.$$

For $a = 1, b = 2, c = 1$, we obtain

$$a^3+b^3+c^3-3abc = 4.$$

Hence, the smallest positive value taken by $a^3+b^3+c^3-3abc$, for positive integers a,b,c, is equal to 4.

Moreover, if a,b,c are positive integers such that $a^3+b^3+c^3-3abc$ takes the value 4, then at least two of a,b,c are unequal, and the above argument shows that

$$a+b+c \leq a^3+b^3+c^3-3abc \leq 4,$$

and consequently, two of a,b,c are equal to 1 and the remaining one is equal to 2. Hence, $a^3+b^3+c^3-3abc$ takes the value 4 precisely when

$$(a,b,c) = (1,1,2),(1,2,1),(2,1,1).$$

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Example 1.5 (India RMO 2012b P6). Show that for all real numbers x,y,z such that $x+y+z=0$ and $xy+yz+zx=-3$, the expression $x^3y+y^3z+z^3x$ is a constant.

[2] A reader unfamiliar with this term may require to look online.
[3] How we may do so? It does require a thought.

Solution 6. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots[4] of the equation $P(t) = 0$, we obtain

$$\begin{aligned} x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz = 0, \\ y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz = 0, \\ z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz = 0. \end{aligned}$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= \big((x + y + z)x^2 - (xy + yz + zx)x + xyz\big)y \\ &\quad + \big((x + y + z)y^2 - (xy + yz + zx)y + xyz\big)z \\ &\quad + \big((x + y + z)z^2 - (xy + yz + zx)z + xyz\big)x \\ &= (x + y + z)(x^2y + y^2z + z^2x) \\ &\quad - (xy + yz + zx)(xy + yz + zx) \\ &\quad + xyz(x + y + z) \\ &= -(xy + yz + zx)^2 \qquad \text{(using } x + y + z = 0\text{)} \\ &= -9 \qquad \text{(using } xy + yz + zx = -3\text{)}. \end{aligned}$$

This completes the proof. ■

For more exercises around this theme, we refer to [**AE11**, §1.1].

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 5)

[4] If it is not clear, then the following equalities may directly be verified.