$a^3 + b^3 + c^3 - 3abc$

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 $a^3 + b^3 + c^3 - 3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= (a+b+c)\left((a+b+c)^2 - 3(ab+bc+ca)\right) \\ &= \frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right). \end{aligned}$$

> **Remark.** An immediate approach would be to begin from the expression $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3+b^3+c^3-3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Observe that

$$\begin{aligned} & a^2+b^2+c^2-ab-bc-ca \\ &= a^2+b^2+c^2+2ab+2bc+2ca-3(ab+bc+ca) \\ &= (a+b+c)^2 - 3(ab+bc+ca), \\ & 2(a^2+b^2+c^2-ab-bc-ca) \\ &= a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2 \\ &= (a-b)^2+(b-c)^2+(c-a)^2. \end{aligned}$$

Note that

$$\begin{aligned} & a^3+b^3+c^3-3abc \\ &= (a+b)^3 - 3ab(a+b) + c^3 - 3abc \\ &= (a+b)^3 + c^3 - 3ab(a+b) - 3abc \\ &= (a+b)^3 + c^3 - 3ab(a+b+c) \end{aligned}$$

$$\begin{aligned}
&= (a+b+c)^3 - 3(a+b)c(a+b+c) - 3ab(a+b+c)\\
&= (a+b+c)((a+b+c)^2 - 3(a+b)c - 3ab)\\
&= (a+b+c)(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca)\\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).
\end{aligned}$$

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> **Remark.** There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a+b+c)X^2 + (ab+bc+ca)X - abc.$$

Since a, b, c are the roots[1] of the equation $P(X) = 0$, we obtain

$$\begin{aligned}
a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc = 0,\\
b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0,\\
c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0.
\end{aligned}$$

Adding them yields

$$a^3+b^3+c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) - 3abc = 0.$$

This proves that

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

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The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a+b+c=0$, then

$$a^3+b^3+c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1)**.** Factor $(x-y)^3+(y-z)^3+(z-x)^3$.

Solution 3. Note that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. This gives

$$(x-y)^3+(y-z)^3+(z-x)^3 = 3(x-y)(y-z)(z-x).$$

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[1] If it is not clear, then the following equalities may directly be verified.

Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}
&(x-y)^3+(y-z)^3+(z-x)^3\\
&=x^3-3x^2y+3xy^2-y^3\\
&+y^3-3y^2z+3yz^2-z^3\\
&+z^3-3z^2x+3zx^2-x^3\\
&=-3x^2y+3xy^2-3y^2z+3yz^2-3z^2x+3zx^2\\
&=-3xy(x-y)-3y^2z+3yz^2-3z^2x+3zx^2\\
&=-3xy(x-y)-3y^2z+3zx^2+3yz^2-3z^2x\\
&=-3xy(x-y)+3z(x^2-y^2)-3z^2(x-y)\\
&=-3xy(x-y)+3z(x-y)(x+y)-3z^2(x-y)\\
&=3(x-y)\big(-xy+z(x+y)-z^2\big)\\
&=3(x-y)\big(-xy+zx+zy-z^2\big)\\
&=3(x-y)\big(-x(y-z)+z(y-z)\big)\\
&=3(x-y)(y-z)(z-x).
\end{aligned}$$

However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2)**.** Solve the following equation for real x:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3=0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$\begin{aligned}
&(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3\\
&=3(x^2+x-2)(2x^2-x-1)(-3x^2+3)\\
&=-9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).
\end{aligned}$$

Thus the required solutions for x are

$$-2,-1,\frac{1}{2},1.$$

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Example 1.4 (Formula of Unity/The Third Millennium 2022/2023 Qualifying Round Grade R11 P5)**.** Find all real a, b, c such that

$$27^{a^2+b+c+1}+27^{b^2+c+a+1}+27^{c^2+a+b+1}=3.$$

Solution 5. For any three real numbers a, b and c, note that

$$\begin{aligned}&27^{a^2+b+c+1}+27^{b^2+c+a+1}+27^{c^2+a+b+1}\\&\geq 3\cdot 3^{a^2+b+c+1}\cdot 3^{b^2+c+a+1}\cdot 3^{c^2+a+b+1}\\&\qquad\text{(using Example 1.1 and noting that } 3^x\geq 0 \text{ for any real number } x)\\&=3\cdot 3^{a^2+b^2+c^2+2a+2b+2c+3}\\&=3\cdot 3^{(a+1)^2+(b+1)^2+(c+1)^2}\end{aligned}$$

hold. This shows that if a,b,c are real numbers satisfying the given condition, then

$$a=b=c=-1.$$

Moreover, note that for $a=b=c=-1$, the equality

$$27^{a^2+b+c+1}+27^{b^2+c+a+1}+27^{c^2+a+b+1}=3$$

holds. Hence, the solution of the given equation is

$$a=b=c=-1.$$

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Example 1.5 (Formula of Unity/The Third Millennium 2023/2024 Qualifying Round Grade R11 P3, S. Pavlov)**.** Let a,b,c be nonzero real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=6,\quad \frac{b}{a}+\frac{c}{b}+\frac{a}{c}=2.$$

What could be the value of the expression

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}?$$

Solution 6. Write $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$. Note that

$$x+y+z=6,\quad xy+yz+zx=2.$$

This yields

$$\begin{aligned}x^3+y^3+z^3&=3+(x+y+z)(x^2+y^2+z^2-xy-yz-zx)\\&=3+(x+y+z)\left((x+y+z)^2-3(xy+yz+zx)\right)\\&=3+6\times(6^2-3\cdot 2)\\&=183.\end{aligned}$$

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Example 1.6 (India INMO 2002 P2)**.** Find the smallest positive value taken by $a^3+b^3+c^3-3abc$ for positive integers a,b,c. Find all a,b,c which give the smallest value.

Walkthrough —

(a) Note that $a = b = c = 1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.

(b) By taking $a = 1, b = 2, c = 1$, one can find that $a^3 + b^3 + c^3 - 3abc = 4$. Next, we need determine whether $a^3 + b^3 + c^3 - 3abc$ can be equal to $1, 2, 3$ or 4 for positive integers a, b, c.

(c) Use

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2) \end{aligned}$$

to get a lower bound on $a^3 + b^3 + c^3 - 3abc$.

Solution 7. Let a, b, c be positive integers such that $a^3 + b^3 + c^3 - 3abc$ is positive. Note that they cannot be equal, and hence at least two of them are distinct. Since $a^3 + b^3 + c^3 - 3abc$ is symmetric[2] in a, b, c, we may assume[3] that $a \neq b$.

Apart from the integers a and b, there is another pair of two integers among a, b, c which are not equal, i.e. $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b = c$ and $c = a$ hold, and then we would have $a = b$, which is a contradiction. Note that

$$\begin{aligned} &a^3 + b^3 + c^3 - 3abc \\ &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) \\ &\geq \frac{1}{2}(a + b + c)(1^2 + 1^2) \\ &\qquad \text{(since at least two of } a - b, b - c, c - a \text{ are nonzero, and } a + b + c > 0) \\ &\geq a + b + c \\ &\geq 1 + 2 + 1 \quad \text{(since at least two of } a - b, b - c, c - a \text{ are nonzero, and } a, b, c \geq 1) \\ &= 4. \end{aligned}$$

Also note that if $c > 1$, then

$$a^3 + b^3 + c^3 - 3abc > 4.$$

For $a = 1, b = 2, c = 1$, we obtain

$$a^3 + b^3 + c^3 - 3abc = 4.$$

[2] A reader unfamiliar with this term may require to look online.
[3] How we may do so? It does require a thought.

Hence, the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$, for positive integers a, b, c, is equal to 4.

Moreover, if a, b, c are positive integers such that $a^3 + b^3 + c^3 - 3abc$ takes the value 4, then at least two of a, b, c are unequal, and the above argument shows that

$$a + b + c \leq a^3 + b^3 + c^3 - 3abc \leq 4,$$

and consequently, two of a, b, c are equal to 1 and the remaining one is equal to 2. Hence, $a^3 + b^3 + c^3 - 3abc$ takes the value 4 precisely when

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

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For more exercises around this theme, we refer to [**AE11**, §1.1].

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8