1 Coloring proofs

See [**Eng98**, Chapter 2], [**Sob13**, §3.2].

Example 1.1. If each point of the plane is colored red or blue, then there are two points of the same color at distance 1 from each other.

Solution 1. Consider an equilateral triangle with side-length 1. Then at least two of its vertices are of the same color, which proves the result. ■

Example 1.2. Suppose that each point in the plane is colored red, green or blue. Prove that either there are two points of the same color a distance 1 unit apart, or there is an equilateral triangle of side length $\sqrt{3}$ all of whose vertices are of the same color.

Solution 2. Suppose no two points which are one unit apart are of the same color. Choose a point O from the plane and without loss of generality assume that O is green. Then draw a unit circle with centre at O, and a regular hexagon circumscribed in this unitcircle as shown in Figure 1.1. Then none of its vertices are green, hence their colors are red and green alternatively. Then the triangle $A_2A_4A_6$ is an equilateral triangle of side length $\sqrt{3}$ and all of its vertices are of the same color. ■

Remark 1.1. In fact, assuming each point in the plane is colored red, green or blue, one can show that there are two monochromatic points that are one unit apart. Otherwise, we choose a point P, suppose it is red and draw a circle of radius $\sqrt{3}$ with centre at P. Then we claim that all points on its circumference are red. If not, pick a point Q which is not red, say it is blue. So the points R, S (as in Figure 1.2) both have to be green, which is not possible by our assumption. Hence all points on its circumference are red. Then pick a chord of length one unit, whose endpoints are of the same color.

Example 1.3. Suppose that to every point of the plane a colour, either red or blue, is associated.

1. Show that if there is no equilateral triangle with all vertices of the same colour then there must exist three points A, B and C of the same colour such that B is the mid-point of AC.

2. Use the above to conclude that there must be an equilateral triangle with all vertices of the same colour.

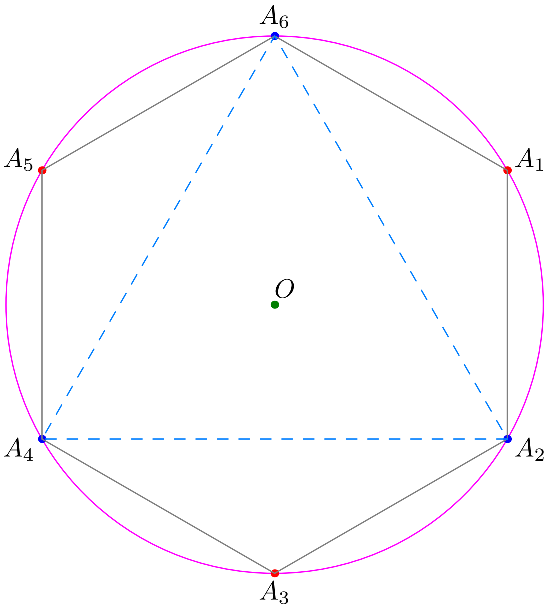


Figure 1.1: Example 1.2

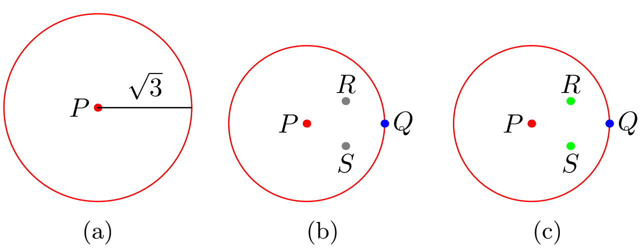


Figure 1.2: Remark 1.1

 The material posted here and at this blog by Evan Chen are quite useful.

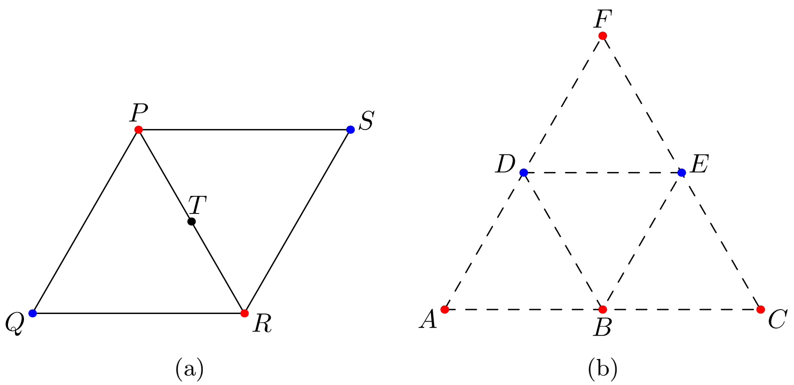


Figure 1.3: Example 1.3

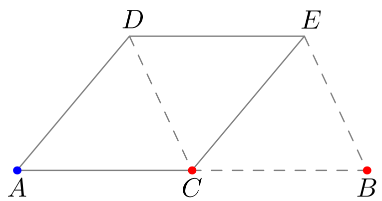


Figure 1.4: Bay Area MO 2007

Solution 3. Suppose there is no equilateral triangle with all vertices of the same color. So any equilateral triangle has two vertices of the same color. Let PQR be an equilateral triangle such that P, R are of the same color. Without loss of generality, assume that P, R are red. Note that Q is blue. Let T denote the mid-point of PR. Draw triangle PRS as in Figure 1.3a. Then Q is blue. Since T is red (resp. blue), then we could talk (A, B, C) equal to (P, T, R) (resp. (Q, T, S)).

Suppose there is no equilateral triangle with all vertices of the same color. Let A, B, C be three red points such that B is the mid-point of AC. Let F be a point on the plane such that ACF is an equilateral triangle. Let D (resp. E) denote the mid-point of AF (resp. CF). Since ABD is equilateral, D is blue. Similarly, E is also blue. Hence F is red. So the triangle ACF has the required properties. ■

Example 1.4 (Bay Area MO 2007)**.** The points of the plane are colored in red and blue so that whenever three vertices of a parallelogram are the same color, the fourth vertex is that color, too. Prove that all the points of the plane are the same color.

Solution 4. If not, then choose two points A, B from the plane which are of different color as in Figure 1.4. Let C denote the mid-point of AB. Without loss of generality, we assume that B, C are of the same color, say red. Then A is blue. Draw a parallelogram $BCDE$. Note that if E were red, then considering the parallelogram $BCDE$, it would follow that D is red, and then considering the parallelogram $ACED$, we would obtain that A is red. Hence, E is blue. Considering the parallelogram $ACDE$ once again, it follows that D is blue. Since A, D, E are blue, C cannot be red. Hence all points of the plane are of the same color. ■

Example 1.5 (India RMO 2017)**.** Consider n^2 unit squares in the xy-plane centred at the point (i, j) with integer coordinates for $1 \leq i \leq n, 1 \leq j \leq n$. It is required to colour each unit square in such a way that whenever $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, the three squares with centres at (i, k), (j, k), (j, ℓ) have distinct colours. What is the least possible number of colours needed?

Remark. For such problems, it is often useful to first work out a special case.

Walkthrough.

(a) First, work out a simple case in order to gain insight for the general case.

(b) One may consider the squares below a suitable diagonal.

(c) Extend to the general case!

First, let's work on it. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 1.5a have pairwise distinct colors. Let's apply the colors $1, \ldots, 8$ to these squares as in Fig. 1.5b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let's apply the colors $3, \ldots, 9$ to these squares as in Fig. 1.5c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let's apply another set of $8 - 3 + 1$ new colors (for example, $5, \ldots, 10$) to these squares as in Fig. 1.5d. One may continue this process to yield the coloring as in Fig. 1.5e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 2, as in Fig. 1.6a. In the third last row, the colors $3, 4$ may be used as in Fig. 1.6b. In the fourth last row, the colors smaller than 7 may be used as in Fig. 1.6c. One may continue this process to yield the coloring as in Fig. 1.6d. Note that the coloring as in Fig. 1.6d **does satisfy** the required condition. ♣

Remark. The **preceding argument *alone*** does not guarantee that the least possible number of colors have been used in Fig. 1.6d.

Now that we have gained some idea, we may proceed to the general case as follows.

Solution 5. If $n = 1$, then using one color works. Let us assume[1] that $n \geq 2$.

Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying on the rightmost column except the bottom square, have colors different from those of the squares lying on the bottom row. Hence, we require at least

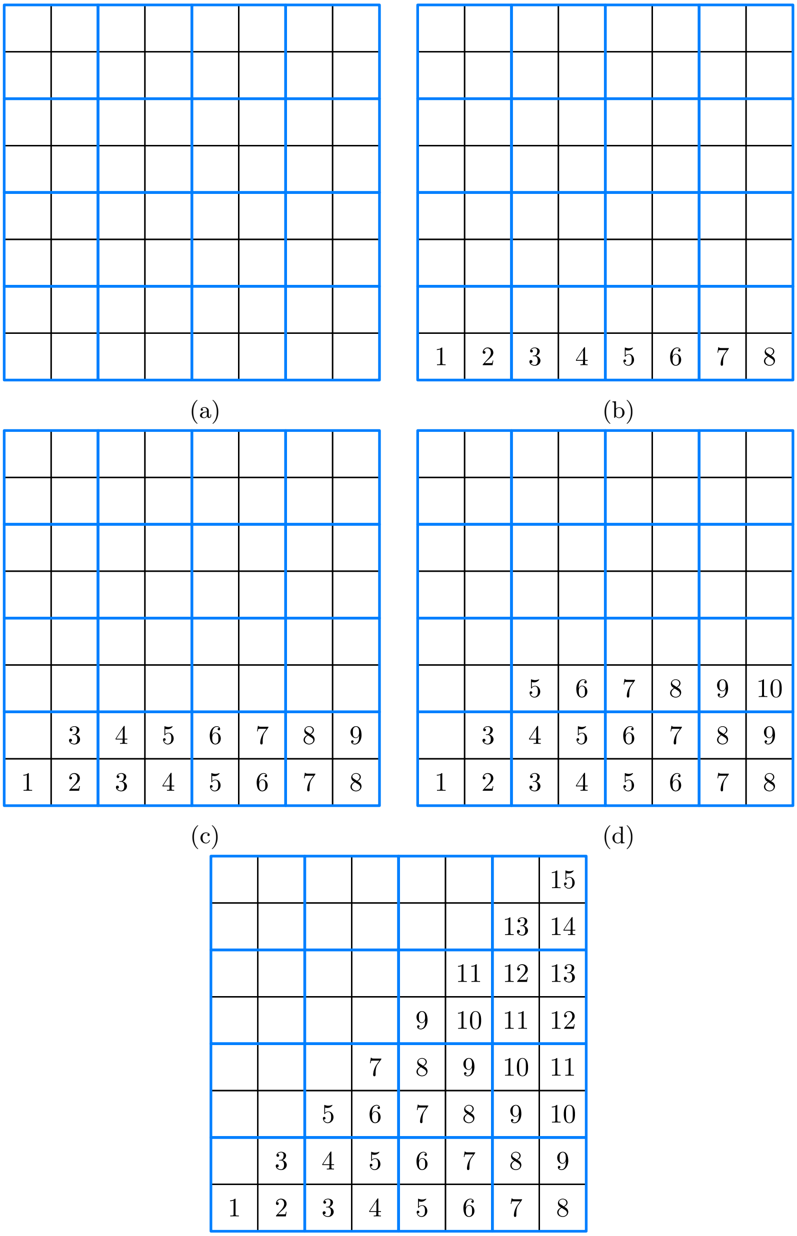
$$n + (n - 1) = 2n - 1$$

colors.

Next, we show that there is a coloring of the n^2 unit squares using $2n - 1$ colors satisfying the required condition. Indeed, if the colors $i, i+1, \ldots, i+n-1$ are applied to the squares lying in the i-th last row in an increasing order from the left to the right for all $1 \leq i \leq n$, then the squares in any row have distinct colors and so are the squares in any column, and hence, it is a coloring satisfying the required condition.

This shows that the least number of colors required is $2n - 1$. ■

[1] The reason for considering the case $n = 1$ separately will become clear from the rest of the argument. Is it clear what would go wrong with the rest of the argument if $n = 1$?



(a) (b) (c) (d) (e)

Figure 1.5: India RMO 2017

 The material posted here and at this blog by Evan Chen are quite useful.

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
		5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(a)

							15
						13	14
					11	12	13
				9	10	11	12
			7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(b)

							15
						13	14
					11	12	13
				9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(c)

8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8

(d)

Figure 1.6: India RMO 2017

Bibliography

[**Eng98**] Arthur Engel. *Problem-solving strategies.* Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 1)

[**Sob13**] Pablo Soberón. *Problem-solving methods in combinatorics.* An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. isbn: 978-3-0348-0596-4; 978-3-0348-0597-1. doi: `10.1007/978-3-0348-0597-1`. url: `http://dx.doi.org/10.1007/978-3-0348-0597-1` (cited p. 1)