Growth of polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 On the growth of polynomials

Example 1.1 (India BStat-BMath 2012)**.** Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root.

> **Summary** — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at large enough arguments.

Solution 1. Let α be a real number. Let us consider the following cases.

1. $\alpha \geq 1$,
2. $\alpha \leq 0$,
3. $0 \leq \alpha \leq 1$.

If $\alpha \geq 1$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^7(\alpha - 1) + \alpha(\alpha - 1) + 15 \\ &\geq 15. \end{aligned}$$

If $\alpha \leq 0$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^8 + (-\alpha^7) + \alpha^2 + (-\alpha) + 15 \\ &\geq 15. \end{aligned}$$

If $0 \leq \alpha \leq 1$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^8 + (1 - \alpha^7) + \alpha^2 + (1 - \alpha) + 13 \\ &\geq 13. \end{aligned}$$

It follows that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root. ■

Example 1.2 (India RMO 2015b P3)**.** Let $P(x)$ be a polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at large enough arguments.

Solution 2. Note that $P(x) = 1$ serves as a counterexample. Henceforth, let us assume that $P(x)$ is a nonconstant polynomial.

Let $Q(x), R(x)$ be polynomials with rational coefficients such that

$$P(P(x) - 2015) = P(x)Q(x) + R(x)$$

and $R(x) = 0$ or $\deg R(x) \leq \deg P(x)$. Note that $P(n)$ is positive for all integer $n \geq 1$ since the coefficients of $P(x)$ are positive integers. By the given condition, it follows that $P(n)$ divides $R(n)$ for any integer $n \geq 1$.

Claim — Let $f(x), g(x)$ be two nonzero polynomials with real coefficients. Suppose $f(x)$ is a nonconstant polynomial, and $\deg g(x) < \deg f(x)$. Then there exists an integer $n_0 \geq 1$ such that

$$f(n) > g(n)$$

for any $n \geq n_0$.

Proof of the Claim. Note that it suffices to the Claim if $f(x)$ is a monomial, that is, a power of x. Indeed, write $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ with $a_0, \ldots, a_d \in \mathbb{R}$ and d denoting the degree of f. Also write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. Note that for a positive integer n, the inequality

$$a_d n^d + a_{d-1}n^{d-1} + \cdots + a_0 > b_e n^e + b_{e-1}n^{e-1} + \cdots + b_0$$

would follow if

$$a_d n^d > \frac{a_{d-1}}{a_d} n^{d-1} + \cdots + \frac{a_0}{a_d} + \frac{b_e}{a_d} n^e + \frac{b_{e-1}}{a_d} n^{e-1} + \cdots + \frac{b_0}{a_d}$$

holds, which can be concluded provided the Claim is known in the case when f is a monomial.

Let us assume that f is a monomial. Write $f(x) = x^d$ where d denotes the degree of f, and write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. For any integer n, note that

$$\begin{aligned} f(n) - g(n) = &\left(\frac{1}{e+1} n^d - b_e n^e\right) \\ &+ \left(\frac{1}{e+1} n^d - b_{e-1} n^{e-1}\right) \\ &+ \cdots + \left(\frac{1}{e+1} n^d - b_0\right) \end{aligned}$$

$$\begin{aligned}&\geq \left(\frac{1}{e+1}n^d - |b_e|n^e\right)\\&+\left(\frac{1}{e+1}n^d - |b_{e-1}|n^{e-1}\right)\\&+\cdots+\left(\frac{1}{e+1}n^d - |b_0|\right).\end{aligned}$$

Since $d \geq e$, it follows that there exists an integer $n_0 \geq 1$ such that

$$\frac{1}{e+1}n^d - |b_e|n^e \frac{1}{e+1}n^d - |b_{e-1}|n^{e-1}, \dots, \frac{1}{e+1}n^d - |b_0|$$

are positive for any $n \geq n_0$. This proves the Claim. □

By the above Claim, it follows that $R(x)$ is the zero polynomial. This implies that

$$P(P(x) - 2015) = P(x)Q(x).$$

Since $P(x)$ is a nonconstant polynomial, it has a root z in $\mathbb{C}$. Substituting $x = z$ yields

$$P(-2015) = 0.$$

■