Example 1 (Austrian Junior Regional Competition 2022)**.** Determine all prime numbers p, q and r with $p + q^2 = r^4$.

Summary — Write down p in terms of q, r and factorize p, which is a prime!

Walkthrough —

(a) Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

(b) This gives $r^2 - q = 1$, and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

(c) This implies that $r - 1 = 1$.

(d) Conclude that $r = 2, q = 3, p = 7$.

Solution 1. Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

Since p is a prime and $r^2 - q < r^2 + q$ holds, it follows that $r^2 - q = 1$, and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

Since p is a prime and $r - 1 < r + 1$, this implies that $r - 1 = 1$. This gives $r = 2, q = 3, p = 7$. Since $2, 3, 7$ are primes, it follows that the only solution of the given equation in primes is

$$p = 7, q = 3, r = 2.$$

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