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§1 $a^3 + b^3 + c^3 - 3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Remark. An immediate approach would be to begin from the expression at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3 + b^3 + c^3 - 3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Note that

$$\begin{aligned}
 & a^3 + b^3 + c^3 - 3abc \\
 &= (a + b)^3 - 3ab(a + b) + c^3 - 3abc \\
 &= (a + b)^3 + c^3 - 3ab(a + b) - 3abc \\
 &= (a + b)^3 + c^3 - 3ab(a + b + c) \\
 &= (a + b + c)^3 - 3(a + b)c(a + b + c) - 3ab(a + b + c) \\
 &= (a + b + c)((a + b + c)^2 - 3(a + b)c - 3ab) \\
 &= (a + b + c)(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca) \\
 &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).
 \end{aligned}$$

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Remark. There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a + b + c)X^2 + (ab + bc + ca)X - abc.$$

Since a, b, c are the roots¹ of the equation $P(X) = 0$, we obtain

$$a^3 - (a + b + c)a^2 + (ab + bc + ca)a - abc = 0,$$

¹If it is not clear, then the following equalities may directly be verified.

$$\begin{aligned}b^3 - (a + b + c)b^2 + (ab + bc + ca)b - abc &= 0, \\c^3 - (a + b + c)c^2 + (ab + bc + ca)c - abc &= 0.\end{aligned}$$

Adding them yields

$$a^3 + b^3 + c^3 - (a + b + c)(a^2 + b^2 + c^2) + (ab + bc + ca)(a + b + c) - 3abc = 0.$$

This proves that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

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The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a + b + c = 0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1). Factor $(x - y)^3 + (y - z)^3 + (z - x)^3$.

Solution 3. Note that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. This gives

$$(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x).$$

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Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}&(x - y)^3 + (y - z)^3 + (z - x)^3 \\&= x^3 - 3x^2y + 3xy^2 - y^3 \\&+ y^3 - 3y^2z + 3yz^2 - z^3 \\&+ z^3 - 3z^2x + 3zx^2 - x^3 \\&= -3x^2y + 3xy^2 - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\&= -3xy(x - y) - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\&= -3xy(x - y) - 3y^2z + 3zx^2 + 3yz^2 - 3z^2x \\&= -3xy(x - y) + 3z(x^2 - y^2) - 3z^2(x - y) \\&= -3xy(x - y) + 3z(x - y)(x + y) - 3z^2(x - y) \\&= 3(x - y)(-xy + z(x + y) - z^2) \\&= 3(x - y)(-xy + zx + zy - z^2) \\&= 3(x - y)(-x(y - z) + z(y - z)) \\&= 3(x - y)(y - z)(z - x).\end{aligned}$$

However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2). Solve the following equation for real x :

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 + (-3x^2 + 3)^3 = 0.$$

Note that $x^2 + x - 2$, $2x^2 - x - 1$, $-3x^2 + 3$ add up to zero. This implies

$$\begin{aligned} & (x^2 + x - 2)^3 + (2x^2 - x - 1)^3 + (-3x^2 + 3)^3 \\ &= 3(x^2 + x - 2)(2x^2 - x - 1)(-3x^2 + 3) \\ &= -9(x + 2)(x - 1)(x - 1)(2x - 1)(x - 1)(x + 1). \end{aligned}$$

Thus the required solutions for x are

$$-2, -1, \frac{1}{2}, 1.$$

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Example 1.4 (India INMO 2002 P2). Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c . Find all a, b, c which give the smallest value.

Walkthrough —

- (a) Note that $a = b = c = 1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.
- (b) By taking $a = 1, b = 2, c = 1$, one can find that $a^3 + b^3 + c^3 - 3abc = 4$. Next, we need determine whether $a^3 + b^3 + c^3 - 3abc$ can be equal to 1, 2, 3 or 4 for positive integers a, b, c .
- (c) Use

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2) \end{aligned}$$

to get a lower bound on $a^3 + b^3 + c^3 - 3abc$.

Solution 5. Let a, b, c be positive integers such that $a^3 + b^3 + c^3 - 3abc$ is positive. Note that they cannot be equal, and hence at least two of them are

distinct. Since $a^3 + b^3 + c^3 - 3abc$ is symmetric² in a, b, c , we may assume³ that $a \neq b$.

Apart from a, b , there is another pair of two integers among a, b, c which are not equal, i.e. $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b = c$ and $c = a$ hold, and then we would have $a = b$, which is a contradiction. Note that

$$\begin{aligned}
 & a^3 + b^3 + c^3 - 3abc \\
 &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
 &= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2) \\
 &\geq \frac{1}{2}(a + b + c)(1^2 + 1^2) \\
 &\quad \text{(since at least two of } a - b, b - c, c - a \text{ are nonzero, and } a + b + c > 0) \\
 &\geq a + b + c \\
 &\geq 1 + 2 + 1 \quad \text{(since } a < b \text{ and } a, b, c \geq 1) \\
 &= 4.
 \end{aligned}$$

Also note that if $c > 1$, then

$$a^3 + b^3 + c^3 - 3abc > 4.$$

For $a = 1, b = 2, c = 1$, we obtain

$$a^3 + b^3 + c^3 - 3abc = 4.$$

Hence, the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$, for positive integers a, b, c , is equal to 4.

Moreover, if a, b, c are positive integers such that $a^3 + b^3 + c^3 - 3abc$ takes the value 4, then at least two of a, b, c are unequal, and the above argument shows that

$$a + b + c \leq a^3 + b^3 + c^3 - 3abc \leq 4,$$

and consequently, two of a, b, c are equal to 1 and the remaining one is equal to 2. Hence, $a^3 + b^3 + c^3 - 3abc$ takes the value 4 precisely when

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

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Example 1.5 (India RMO 2012b P6). Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

²A reader unfamiliar with this term may require to look online.

³How we may do so? It does require a thought.

Solution 6. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots⁴ of the equation $P(t) = 0$, we obtain

$$\begin{aligned} x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz &= 0, \\ y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz &= 0, \\ z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz &= 0. \end{aligned}$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= ((x + y + z)x^2 - (xy + yz + zx)x + xyz)y \\ &\quad + ((x + y + z)y^2 - (xy + yz + zx)y + xyz)z \\ &\quad + ((x + y + z)z^2 - (xy + yz + zx)z + xyz)x \\ &= (x + y + z)(x^2y + y^2z + z^2x) \\ &\quad - (xy + yz + zx)(xy + yz + zx) \\ &\quad + xyz(x + y + z) \\ &= -(xy + yz + zx)^2 \quad (\text{using } x + y + z = 0) \\ &= -9 \quad (\text{using } xy + yz + zx = -3). \end{aligned}$$

This completes the proof. ■

⁴If it is not clear, then the following equalities may directly be verified.