1 Warm up

Example 1.1 (India RMO 2003)**.** Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C, $\text{prod}(C)$ denotes the product of all numbers in C,

(c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.

Summary. It is equivalent to finding a subset B of $\{1, \ldots, 10\}$, other than $\emptyset, \{1, \ldots, 10\}$, such that $\text{prod}(B)^2$ divides $10!$ and the quotient $10!/\text{prod}(B)^2$ is minimized. To do so,

(a) write down the prime power factorization of $10!$,

(b) throw in enough elements in B so that $\text{prod}(B)$ is maximized, and $\text{prod}(B)^2$ divides $10!$.

Walkthrough.

(a) Observe that it is enough to find a nonempty proper subset B of $\{1, 2, \ldots, 10\}$ such that $\text{prod}(B)^2$ divides $10!$ and $\text{prod}(B)$ is the maximum.

(b) Writing down the prime power factorization of $10!$, deduce that B does not contain 7, it contains a multiple of 5, and also a multiple of 2 and a multiple of 3.

(c) Prove that B contains exactly one multiple of 5, and not more that two multiples of 3.

(d) Show that B is equal to one of the subsets $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$.

(e) Show that any of these three subsets also have the stated property.

First, let's work on it. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{10!}{(\text{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum. Note that 10! is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\text{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, and hence, B does not contain 7. Moreover, B contains a multiple of 5, otherwise $(\text{prod}(B \cup \{5\}))^2$ would divide 10! and $\text{prod}(B \cup \{5\})$ would be strictly larger than $\text{prod}(B)$, which contradicts the choice of B. Similarly, B also contains a multiple of 2 and a multiple of 3. Note that B contains exactly one multiple of 5 (since $5^3 \nmid 10!$). Since $(\text{prod}(B))^2$ divides 10! and $\text{prod}(B)$ is the maximum, B is equal to one of the following sets

- $\{5,3,2,2^3\}$, $\{5,3,2,2^3,1\}$, $\{5,6,2,2^3\}$, $\{5,6,2,2^3,1\}$, $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$ if B contains 5,
- $\{10,3,2^3\}$, $\{10,3,2^3,1\}$, $\{10,3,2,2^2\}$, $\{10,3,2,2^2,1\}$, $\{10,6,2^2\}$, $\{10,6,2^2,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2,2^2\}$, $\{10,9,2,2^2,1\}$ if B contains 10.

For any of the above sets, the product of its elements is equal to 240, 480, or 720. So B is equal to one of the sets $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$.

Also note that if B denotes one of these subsets of $\{1,\dots,10\}$, then $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum.

This proves that $\{5,3,6,2^3\}$, $\{5,3,6,2^3,1\}$, $\{5,3,6,2,2^2\}$, $\{5,3,6,2,2^2,1\}$, $\{5,9,2,2^3\}$, $\{5,9,2,2^3,1\}$, $\{10,3,6,2^2\}$, $\{10,3,6,2^2,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$, $\{10,9,2^3\}$, $\{10,9,2^3,1\}$ are precisely all the subsets of $\{1,\dots,10\}$ having the required property. Thus we could take $A = \{1,2,3,4,5,6,7\}, B = \{8,9,10\}$ for instance. ♣

Remark. Note that the above solution provides more than what has been required. After observing that $\text{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, one may show that there is a subset B with $\text{prod}(B)$ equal to $2^4 \cdot 3^2 \cdot 5$ (for instance, $B = \{8,9,10\}$), and then conclude.

Solution 1. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{10!}{(\text{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum. Note that 10! is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\text{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$. If $B = \{8,9,10\}$, then $\text{prod}(B)$ is equal to $2^4 \cdot 3^2 \cdot 5$. Hence, $A = \{1,\dots,7\}, B = \{8,9,10\}$ are two disjoint nonempty subsets of $X = \{1,\dots,10\}$ satisfying the required conditions. ■

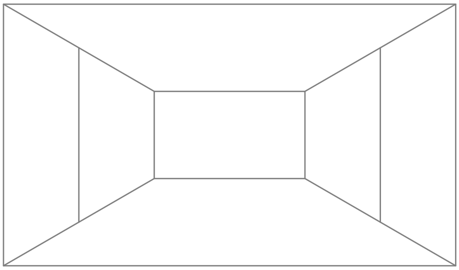

Figure 1.1: India RMO 2014

Remark. Don't be surprised that it took a bit long to arrive at the above solution. It is often the case. Further, it is a standard practice to write down a complete solution as the final one, without any reference to the prior attempts (possibly several). Those attempts have their important role in providing insights, which may lead to a solution. Here, the details of those attempts have not been hidden from you, in order to take you along the journey. However, I would like to highlight that a *solution* to a problem has to be complete, and at the same time, has to be free from the prior thoughts that have no direct role to play in that solution, though they might have played a significant role in gaining insight.

Example 1.2 (India RMO 2014)**.** In Fig. 1.1, can the numbers $1, 2, 3, 4, \dots, 18$ be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?

Summary. Since there are 18 line segments, it follows that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \dots, 18$ satisfying the required condition.

Solution 2. Note that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \dots, 18$ satisfying the required condition (by replacing the 0's (resp. 1's, 2's) by the six integers among $1, 2, \dots, 18$ which are congruent to 0 (resp. $1, 2$) modulo 3, and such a replacement can be carried out since there are six elements among $1, 2, \dots, 18$ congruent to $i \bmod 3$ for any $i \in \{0, 1, 2\}$). We now show that such an arrangement of $0, 1, 2$ exists. First, put 0's on all the vertical segments as in Fig. 1.2b, and then put 1's on the 'diagonal' segments as shown in Fig. 1.2c. This forces to put 2's on the 'diagonal' segments as shown in Figure 1.2d, which in turn, forces to write 1's and 2's on the horizontal segments as in Figure 1.2e. Note that the sum of the numbers (as in Figure 1.2e) on the three line segments meeting at each point is divisible by 3. So this gives an

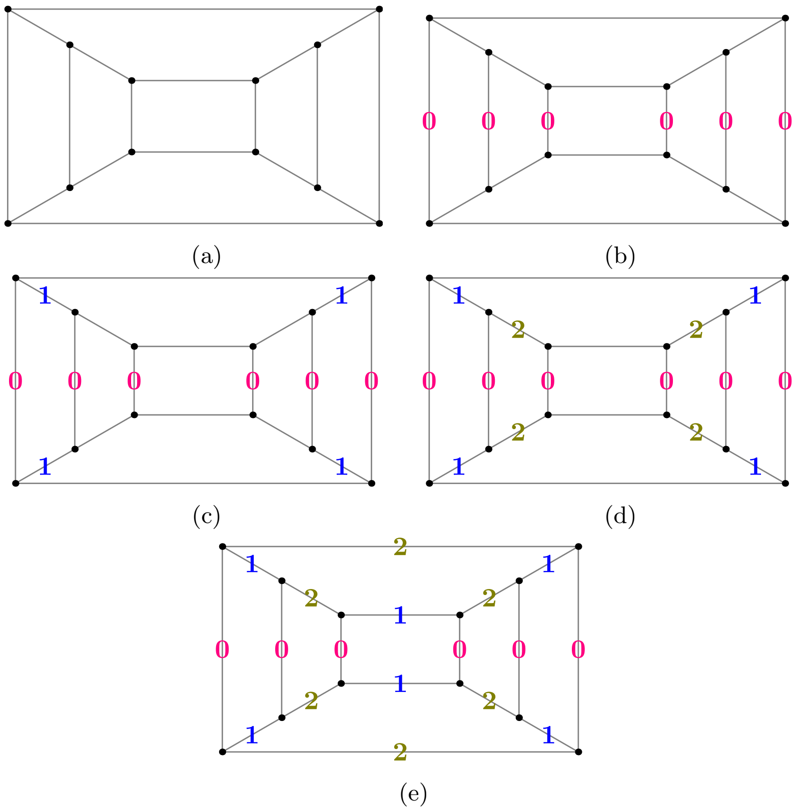

Figure 1.2: India RMO 2014

arrangement of $0, 1, 2$ satisfying the desired properties, then $1, 2, \ldots, 18$ can be arranged satisfying the given conditions (as described above). ■

The above problem leads to the following question.

Question 1.3. Under which conditions, does a k-regular graph admit an edge coloring by the k-th roots of unity such that the sum of the colors incident at any vertex equals to zero?

Example 1.4 (India RMO 2017)**.** Consider a chessboard of size 8 units $\times$ 8 units (i.e. each small square on the board has a side length of 1 unit). Let S be the set of all the 81 vertices of all the squares on the board. What is the number of line segments whose vertices are in S, and whose length is a positive integer? (The segments need not be parallel to the sides of the board.)

Summary. A segment having vertices in S and length a positive integer, is horizontal or vertical, or the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board. To count such right-angled triangles, note that they cannot have a too large hypotenuse.

Walkthrough.

(a) Determine the number of the horizontal segments with vertices in S and whose lengths are positive integers.

(b) By symmetry, the number of such vertical segments is equal to the above.

(c) To determine the slanted ones, note that such a slanted segment is the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board, and have integer lengths. Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$.

(d) Does a symmetry argument help? For instance, flipping around a diagonal, and then flipping around an axis (i.e. a line parallel to one of the sides of the board and dividing the board in two equal halves).

Solution 3. Note that within each horizontal line, there are $8-\ell+1$ horizontal segments of length ℓ for any $1 \leq \ell \leq 8$. This shows that the number of the horizontal segments with vertices in S and whose lengths are positive integers is equal to

$$8 \times (1 + 2 + \cdots + 8)) = \frac{1}{2} 8^2 \cdot 9.$$

By symmetry, the number of such vertical segments is also equal to $\frac{1}{2}8^2 \cdot 9$. Hence, there are $8^2 \cdot 9$ segments parallel to the sides of the board, which have vertices in S and whose lengths are positive integers.

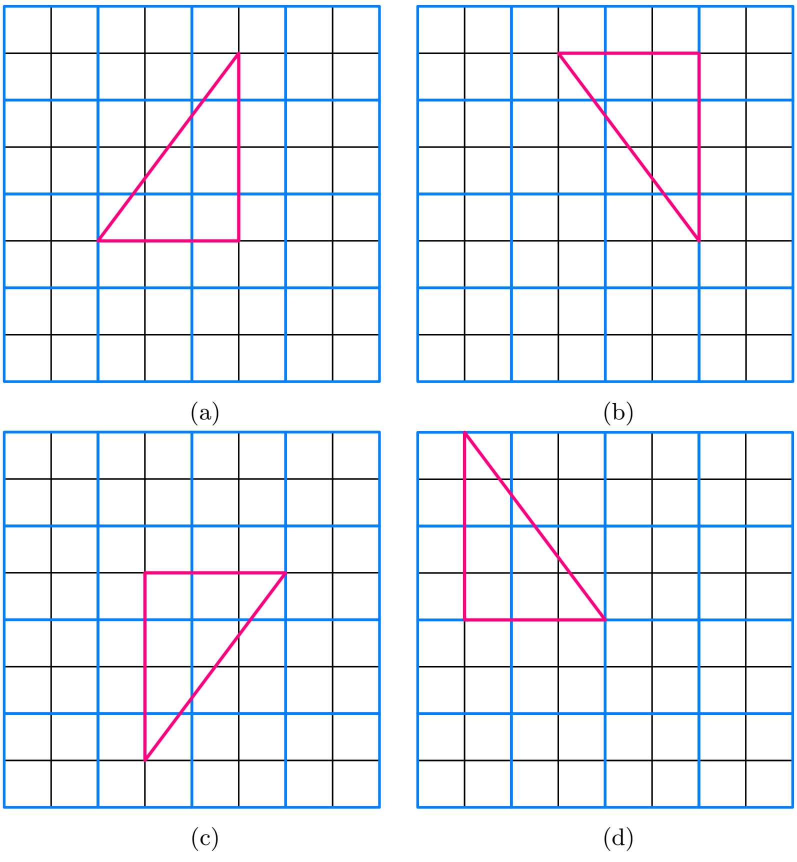

Figure 1.3: Several configurations of triangles with side lengths $(3, 4, 5)$

Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have the side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$. The number of such right-angled triangles, having the side lengths equal to $(3, 4, 5)$, is equal to

$$4 \times (8 - 3 + 1) \times (8 - 4 + 1).$$

The number of such right-angled triangles, having the side lengths equal to the remaining triples, can be expressed in a similar way. This shows that the total count of such triangles is

$$2 \times 4 \times ((8 - 3 + 1) \times (8 - 4 + 1) + (8 - 6 + 1) \times (8 - 8 + 1)) = 360.$$

Hence, the number of the line segments with the stated property is

$$8^2 \cdot 9 + 360 = 936.$$

■

Remark. Note that the above argument considers the line segments whose endpoints are distinct. There are 81 line segments having equal end-points and end-points lying in S.

Example 1.5 (India RMO 2018)**.** Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?

Summary. It relies on the fact that one can find enough points on the plane such that no three of them are collinear and no four of them are concyclic.

Walkthrough.

(a) There is an upper bound on the number of the red points in terms of the number of the white points. This gives an upper bound on the total number of points, which is 100, in terms of the number of the white points.

(b) Use this bound to guess the least possible number of the white points, which would turn out to be 10.

(c) Begin with 10 white points on the plane in *general position*, and then, introduce enough red points to construct a configuration of 100 points with the stated properties.

Solution 4. Let n denote the number of white points. Since each red point is the centre of a circle passing through at least three white points, it follows that the number of red points is at most $\binom{n}{3}$. This shows that

$$n + \binom{n}{3} \geq 100.$$

Note that $n \mapsto n + \binom{n}{3}$ defines an increasing function on the nonnegative integers. Observe that

$$9 + \binom{9}{3} = 93, \quad 10 + \binom{10}{3} = 130.$$

This implies that $n \geq 10$.

We claim that there is a configuration of 100 points on the plane such that it admits a coloring using two colors, red and white, such that precisely 10 points are colored white, and that each red point is the centre of a circle passing through at least three white points. Indeed, consider 10 points on the plane such that no three of them are collinear and no four of them are concyclic [1]. Color these 10 points white. These white points have $\binom{10}{3} = 120$ subsets of size 3. Consider only 90 such subsets of the white points, and for any such size 3 subset, color the center of the circle passing through them red. Since no three white points are collinear and no four white points are concyclic, it follows that there are precisely 90 pairwise distinct red points. So, the red and the white points together form a set of 100 points such that each red point is the centre of a circle passing through at least three white points. ■

[1] Why does such a collection exist? This could be intuitively clear, but can you write down a precise proof? Does induction help?