1 Warm up

Example 1.1 (India RMO 2003)**.** Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) $\mathrm{prod}(A)$ is divisible by $\mathrm{prod}(B)$, where for any finite set of numbers C, $\mathrm{prod}(C)$ denotes the product of all numbers in C,

(c) the quotient $\mathrm{prod}(A)/\mathrm{prod}(B)$ is as small as possible.

Summary. It is equivalent to finding a subset B of $\{1, \ldots, 10\}$, other than $\emptyset, \{1, \ldots, 10\}$, such that $\mathrm{prod}(B)^2$ divides $10!$ and the quotient $10!/\mathrm{prod}(B)^2$ is minimized. To do so,

- write down the prime power factorization of $10!$,
- throw in enough elements in B so that $\mathrm{prod}(B)$ is maximized, and $\mathrm{prod}(B)^2$ divides $10!$.

Walkthrough.

(a) Observe that it is enough to find a nonempty proper subset B of $\{1, 2, \ldots, 10\}$ such that $\mathrm{prod}(B)^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum.

(b) Writing down the prime power factorization of $10!$, deduce that B does not contain 7, it contains a multiple of 5, and also a multiple of 2 and a multiple of 3.

(c) Prove that B contains exactly one multiple of 5, and not more that two multiples of 3.

(d) Show that B is equal to one of the subsets $\{5, 3, 6, 2^3\}$, $\{10, 3, 6, 2^2\}$, $\{10, 9, 2^3\}$ of $\{1, \ldots, 10\}$.

(e) Show that any of these three subsets also have the stated property.

Solution 1. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\mathrm{prod}(A)}{\mathrm{prod}(B)} = \frac{10!}{(\mathrm{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\mathrm{prod}(B)^2$ divides $10!$ and $\mathrm{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$.

So B does not contain 7. Moreover, B contains a multiple of 5, otherwise $(\mathrm{prod}(B \cup \{5\}))^2$ would divide 10! and $\mathrm{prod}(B \cup \{5\})$ would be strictly larger than $\mathrm{prod}(B)$, which contradicts the choice of B. Similarly, B also contains a multiple of 2 and a multiple of 3. Note that B contains exactly one multiple of 5 (since $5^3 \nmid 10!$). Since $(\mathrm{prod}(B))^2$ divides 10! and $\mathrm{prod}(B)$ is the maximum, B is equal to one of the following sets

- $\{5, 3, 2^3\}, \{5, 6, 2^3\}, \{5, 3, 6, 2^3\}, \{5, 9, 2^3\}$ if B contains 5,
- $\{10, 3, 2^3\}, \{10, 6, 2^2\}, \{10, 3, 6, 2^2\}, \{10, 9, 2^3\}$ if B contains 10.

The products of the elements of these sets are equal to 120, 240, 720, 360, 240, 480, 720, 720 respectively. So B is equal to one of the sets $\{5, 3, 6, 2^3\}$, $\{10, 3, 6, 2^2\}$, $\{10, 9, 2^3\}$.

Also note that if B denotes one of the subsets $\{5, 3, 6, 2^3\}$, $\{10, 3, 6, 2^2\}$, $\{10, 9, 2^3\}$ of $\{1, \dots, 10\}$, then $\mathrm{prod}(B)^2$ divides 10! and $\mathrm{prod}(B)$ is the maximum.

This proves that $\{5, 3, 6, 2^3\}$, $\{10, 3, 6, 2^2\}$, $\{10, 9, 2^3\}$ are precisely all the subsets of $\{1, \dots, 10\}$ having the required property. Thus we could take $A = \{1, 2, 3, 4, 5, 6, 7\}, B = \{8, 9, 10\}$ for instance. ■