Problem 0.1. Show that the positive integers of the form $4n+3$, that is, the integers

$$3, 7, 11, 15, 19, \dots$$

cannot be written as the sum of two perfect squares.

(a) Consider the integers

$$\begin{aligned}
&0^2+1^2, 0^2+2^2, 0^2+3^2, 0^2+4^2, \dots,\\
&1^2+1^2, 1^2+2^2, 1^2+3^2, 1^2+4^2, \dots,\\
&2^2+1^2, 2^2+2^2, 2^2+3^2, 2^2+4^2, \dots,\\
&3^2+1^2, 3^2+2^2, 3^2+3^2, 3^2+4^2, \dots,\\
&4^2+1^2, 4^2+2^2, 4^2+3^2, 4^2+4^2, \dots.
\end{aligned}$$

(b) Observe that upon division by 4, they leave the integers $0, 1, 2$ as remainders.

$$\begin{aligned}
&0^2+1^2 \rightsquigarrow \mathbf{1}, 0^2+2^2 \rightsquigarrow \mathbf{0}, 0^2+3^2 \rightsquigarrow \mathbf{1}, 0^2+4^2 \rightsquigarrow \mathbf{0}, \dots,\\
&1^2+1^2 \rightsquigarrow \mathbf{2}, 1^2+2^2 \rightsquigarrow \mathbf{1}, 1^2+3^2 \rightsquigarrow \mathbf{2}, 1^2+4^2 \rightsquigarrow \mathbf{1}, \dots,\\
&2^2+1^2 \rightsquigarrow \mathbf{1}, 2^2+2^2 \rightsquigarrow \mathbf{0}, 2^2+3^2 \rightsquigarrow \mathbf{1}, 2^2+4^2 \rightsquigarrow \mathbf{0}, \dots,\\
&3^2+1^2 \rightsquigarrow \mathbf{2}, 3^2+2^2 \rightsquigarrow \mathbf{1}, 3^2+3^2 \rightsquigarrow \mathbf{2}, 3^2+4^2 \rightsquigarrow \mathbf{1}, \dots,\\
&4^2+1^2 \rightsquigarrow \mathbf{1}, 4^2+2^2 \rightsquigarrow \mathbf{0}, 4^2+3^2 \rightsquigarrow \mathbf{1}, 4^2+4^2 \rightsquigarrow \mathbf{0}, \dots,
\end{aligned}$$

(c) Show that it is **always** the case, namely, upon division by 4, the sum of two perfect squares leaves one of $0, 1, 2$ as the remainder.

(d) Conclude that no integer, which leaves the remainder of 3 **upon division by** 4, can be written as the sum of two squares.