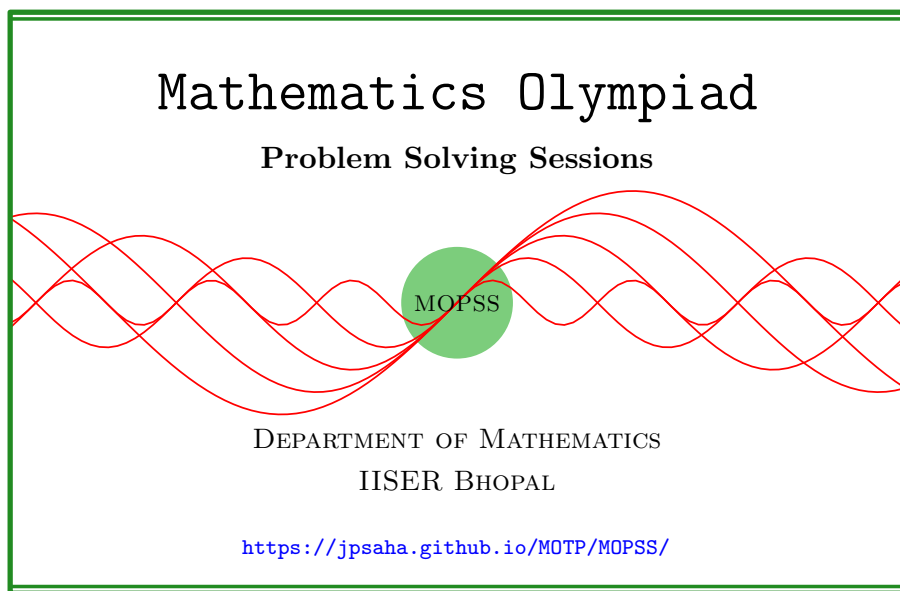


$$a^3 + b^3 + c^3 - 3abc$$

MOPSS

15 June 2024



Suggested readings

- Evan Chen's
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- Evan Chen discusses why *math olympiads* are a valuable experience for high schoolers in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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§1 $a^3 + b^3 + c^3 - 3abc$

Example 1.1. Let a, b, c be real numbers. Show that

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a + b + c) \left((a + b + c)^2 - 3(ab + bc + ca) \right) \\ &= \frac{1}{2}(a + b + c) \left((a - b)^2 + (b - c)^2 + (c - a)^2 \right). \end{aligned}$$

Remark. An immediate approach would be to begin from the expression $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3 + b^3 + c^3 - 3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Observe that

$$\begin{aligned} &a^2 + b^2 + c^2 - ab - bc - ca \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3(ab + bc + ca) \\ &= (a + b + c)^2 - 3(ab + bc + ca), \\ &2(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2 \\ &= (a - b)^2 + (b - c)^2 + (c - a)^2. \end{aligned}$$

Note that

$$\begin{aligned} &a^3 + b^3 + c^3 - 3abc \\ &= (a + b)^3 - 3ab(a + b) + c^3 - 3abc \\ &= (a + b)^3 + c^3 - 3ab(a + b) - 3abc \\ &= (a + b)^3 + c^3 - 3ab(a + b + c) \end{aligned}$$

$$\begin{aligned}
&= (a + b + c)^3 - 3(a + b)c(a + b + c) - 3ab(a + b + c) \\
&= (a + b + c)((a + b + c)^2 - 3(a + b)c - 3ab) \\
&= (a + b + c)(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca) \\
&= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).
\end{aligned}$$

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Remark. There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a + b + c)X^2 + (ab + bc + ca)X - abc.$$

Since a, b, c are the roots¹ of the equation $P(X) = 0$, we obtain

$$\begin{aligned}
a^3 - (a + b + c)a^2 + (ab + bc + ca)a - abc &= 0, \\
b^3 - (a + b + c)b^2 + (ab + bc + ca)b - abc &= 0, \\
c^3 - (a + b + c)c^2 + (ab + bc + ca)c - abc &= 0.
\end{aligned}$$

Adding them yields

$$a^3 + b^3 + c^3 - (a + b + c)(a^2 + b^2 + c^2) + (ab + bc + ca)(a + b + c) - 3abc = 0.$$

This proves that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

■

The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a + b + c = 0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Example 1.2 (Moscow MO 1940 Grades 7–8 P1). Factor $(x - y)^3 + (y - z)^3 + (z - x)^3$.

Solution 3. Note that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. This gives

$$(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x).$$

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¹If it is not clear, then the following equalities may directly be verified.

Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}
 & (x-y)^3 + (y-z)^3 + (z-x)^3 \\
 &= x^3 - 3x^2y + 3xy^2 - y^3 \\
 &+ y^3 - 3y^2z + 3yz^2 - z^3 \\
 &+ z^3 - 3z^2x + 3zx^2 - x^3 \\
 &= -3x^2y + 3xy^2 - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
 &= -3xy(x-y) - 3y^2z + 3yz^2 - 3z^2x + 3zx^2 \\
 &= -3xy(x-y) - 3y^2z + 3zx^2 + 3yz^2 - 3z^2x \\
 &= -3xy(x-y) + 3z(x^2 - y^2) - 3z^2(x-y) \\
 &= -3xy(x-y) + 3z(x-y)(x+y) - 3z^2(x-y) \\
 &= 3(x-y)(-xy + z(x+y) - z^2) \\
 &= 3(x-y)(-xy + zx + zy - z^2) \\
 &= 3(x-y)(-x(y-z) + z(y-z)) \\
 &= 3(x-y)(y-z)(z-x).
 \end{aligned}$$

However, the former solution is less cumbersome, and more elegant.

Example 1.3 (India RMO 2002 P2). Solve the following equation for real x :

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 + (-3x^2 + 3)^3 = 0.$$

Note that $x^2 + x - 2$, $2x^2 - x - 1$, $-3x^2 + 3$ add up to zero. This implies

$$\begin{aligned}
 & (x^2 + x - 2)^3 + (2x^2 - x - 1)^3 + (-3x^2 + 3)^3 \\
 &= 3(x^2 + x - 2)(2x^2 - x - 1)(-3x^2 + 3) \\
 &= -9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).
 \end{aligned}$$

Thus the required solutions for x are

$$-2, -1, \frac{1}{2}, 1.$$

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Example 1.4 (Formula of Unity/The Third Millennium 2022/2023 Qualifying Round Grade R11 P5). Find all real a, b, c such that

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3.$$

Solution 5. For any three real numbers a, b and c , note that

$$\begin{aligned}
 & 27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} \\
 & \geq 3 \cdot 3^{a^2+b+c+1} \cdot 3^{b^2+c+a+1} \cdot 3^{c^2+a+b+1} \quad (\text{using Example 1.1}) \\
 & = 3 \cdot 3^{a^2+b^2+c^2+2a+2b+2c+3} \\
 & = 3 \cdot 3^{(a+1)^2+(b+1)^2+(c+1)^2}
 \end{aligned}$$

hold. This shows that if a, b, c are real numbers satisfying the given condition, then

$$a = b = c = -1.$$

Moreover, note that for $a = b = c = -1$, the equality

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3$$

holds. Hence, the solution of the given equation is

$$a = b = c = -1.$$

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Example 1.5 (Formula of Unity/The Third Millennium 2023/2024 Qualifying Round Grade R11 P3, S. Pavlov). Let a, b, c be nonzero real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 6, \quad \frac{b}{a} + \frac{c}{b} + \frac{a}{c} = 2.$$

What could be the value of the expression

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}?$$

Solution 6. Write $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$. Note that

$$x + y + z = 6, \quad xy + yz + zx = 2.$$

This yields

$$\begin{aligned}
 x^3 + y^3 + z^3 &= 3 + (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
 &= 3 + (x + y + z)((x + y + z)^2 - 3(xy + yz + zx)) \\
 &= 3 + 6 \times (6^2 - 2 \cdot 2) \\
 &= 195.
 \end{aligned}$$

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Example 1.6 (India INMO 2002 P2). Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c . Find all a, b, c which give the smallest value.

Walkthrough —

- (a) Note that $a = b = c = 1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.
- (b) By taking $a = 1, b = 2, c = 1$, one can find that $a^3 + b^3 + c^3 - 3abc = 4$. Next, we need determine whether $a^3 + b^3 + c^3 - 3abc$ can be equal to 1, 2, 3 or 4 for positive integers a, b, c .
- (c) Use

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2) \end{aligned}$$

to get a lower bound on $a^3 + b^3 + c^3 - 3abc$.

Solution 7. Let a, b, c be positive integers such that $a^3 + b^3 + c^3 - 3abc$ is positive. Note that they cannot be equal, and hence at least two of them are distinct. Since $a^3 + b^3 + c^3 - 3abc$ is symmetric² in a, b, c , we may assume³ that $a \neq b$.

Apart from the integers a and b , there is another pair of two integers among a, b, c which are not equal, i.e. $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b = c$ and $c = a$ hold, and then we would have $a = b$, which is a contradiction. Note that

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2) \\ &\geq \frac{1}{2}(a + b + c)(1^2 + 1^2) \\ &\quad (\text{since at least two of } a - b, b - c, c - a \text{ are nonzero, and } a + b + c > 0) \\ &\geq a + b + c \\ &\geq 1 + 2 + 1 \quad (\text{since at least two of } a - b, b - c, c - a \text{ are nonzero, and } a, b, c \geq 1) \\ &= 4. \end{aligned}$$

Also note that if $c > 1$, then

$$a^3 + b^3 + c^3 - 3abc > 4.$$

For $a = 1, b = 2, c = 1$, we obtain

$$a^3 + b^3 + c^3 - 3abc = 4.$$

²A reader unfamiliar with this term may require to look online.

³How we may do so? It does require a thought.

Hence, the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$, for positive integers a, b, c , is equal to 4.

Moreover, if a, b, c are positive integers such that $a^3 + b^3 + c^3 - 3abc$ takes the value 4, then at least two of a, b, c are unequal, and the above argument shows that

$$a + b + c \leq a^3 + b^3 + c^3 - 3abc \leq 4,$$

and consequently, two of a, b, c are equal to 1 and the remaining one is equal to 2. Hence, $a^3 + b^3 + c^3 - 3abc$ takes the value 4 precisely when

$$(a, b, c) = (1, 1, 2), (1, 2, 1), (2, 1, 1).$$

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For more exercises around this theme, we refer to [**AndreescuMOTreasures**].