

**Example 1** (Austrian Junior Regional Competition 2022). Determine all prime numbers  $p, q$  and  $r$  with  $p + q^2 = r^4$ .

**Summary** — Write down  $p$  in terms of  $q, r$  and factorize  $p$ , which is a prime!

### Walkthrough.

(a) Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

(b) This gives  $r^2 - q = 1$ , and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

(c) This implies that  $r - 1 = 1$ .

(d) Conclude that  $r = 2, q = 3, p = 7$ .

**Solution 1.** Note that

$$\begin{aligned} p &= r^4 - q^2 \\ &= (r^2 - q)(r^2 + q). \end{aligned}$$

Since  $p$  is a prime and  $r^2 - q < r^2 + q$  holds, it follows that  $r^2 - q = 1$ , and hence

$$\begin{aligned} q &= r^2 - 1 \\ &= (r - 1)(r + 1). \end{aligned}$$

Since  $p$  is a prime and  $r - 1 < r + 1$ , this implies that  $r - 1 = 1$ . This gives  $r = 2, q = 3, p = 7$ . Since  $2, 3, 7$  are primes, it follows that the only solution of the given equation in primes is

$$p = 7, q = 3, r = 2.$$

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