1 Warm up

Example 1.1 (India BMath 2006)**.** A domino is a 2 by 1 rectangle. For what integers m and n, can one cover an m by n rectangle with non-overlapping dominoes?

Walkthrough.

(a) If an $m \times n$ rectangle admits a covering by non-overlapping dominos, then show that at least one of the integers m, n has to be even.

(b) If at least one of m, n is even, then prove that an $m \times n$ rectangle admits a covering by non-overlapping dominos.

Solution 1. In the following, an $m \times n$ rectangle is to be thought as an $m \times n$ rectangular grid.

To be able to cover an $m \times n$ rectangle by non-overlapping dominoes, it is necessary for the product mn to be even, and hence, at least one of m, n is even. Indeed, if an $m \times n$ rectangle admits a covering using k non-overlapping dominoes, then those dominoes together cover $2k$ unit squares, and this yields that $2k = mn$.

Moreover, when at least one of m, n is even, an $m \times n$ rectangle can be covered by non-overlapping dominoes by covering each row by $m/2$ (resp. each column by $n/2$) non-overlapping dominos if m (resp. n) is even.

This shows that an $m \times n$ rectangle can be covered by non-overlapping dominoes if and only if at least one of m, n is even. ∎

Remark 1.1. The above conclusion shows that an $m \times n$ rectangle admits a covering by non-overlapping dominoes if and only if it admits a covering by non-overlapping dominoes in the *most obvious manner*, i.e. a covering by non-overlapping dominoes such that all of them are either horizontal or vertical (cf. [**Bru10**, p. 6]).

The following problem is a more general version of Example 1.1.

Example 1.2. [**Eng98**, Problem 8, Chapter 2, p. 26] Show that an $m \times n$ rectangle admits a covering by non-overlapping $k \times 1$ rectangles if and only if k divides m or k divides n.

Figure 1.1: India BMath 2006 (a tiling of a 5×8 rectangle with non-overlapping dominoes)

Bibliography

[**Bru10**] Richard A. Brualdi. *Introductory combinatorics*. Fifth. Pearson Prentice Hall, Upper Saddle River, NJ, 2010, pp. xii+605. isbn: 978-0-13-602040-0; 0-13-602040-2 (cited p. 1)

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 1)