Linear Transformation

Why we use linear transformation

Computer Graphics: Linear transformation is used to rotate, scale, shear, and reflect 2D and 3D objects, which is essential for creating animations, video games, and 3D models in CAD software.

Cryptography: In cryptography, linear transformations can scramble messages by mapping letters to numbers, breaking them into vectors, and applying a transformation matrix for encryption and decryption purposes.

Data Science and Machine learning
Principal Component Analysis (PCA) uses linear transformations to reduce the dimensionality of large datasets, projecting the data onto a lower-dimensional space while retain important information by identifying directions of maximum variance.

Solving Systems of Equations.
Linear transformations are fundamental to solving systems of linear equations, which is core problem in many scientific and engineering disciplines.

Image Processing and Computer Vision

Linear transformations can be applied to images to perform transformations like rotation, reflection, and scaling, often by using a matrix to represent these changes.

Animation

They help in creating animated sequences by rotating and translating objects to form a sequence of images.

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LINEAR TRANSFORMATION OF A VECTOR SPACE

Definition Let V and W be vector spaces over same field $\mathbb{F}$. A function $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ is called a linear transformation from $V(\mathbb{F})$ into $W(\mathbb{F})$ if for all $x, y \in V(\mathbb{F})$ and $c \in \mathbb{F}$, we have

(a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = c\,T(x)$

Zero Transformation

A linear transformation $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ is said to be zero transformation if

$$T(x) = 0 \quad \text{for all } x \in V(\mathbb{F}).$$

Identity transformation

A linear transformation $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ is said to be Identity transformation if

$$T(x) = x \quad \text{for all } x \in V(\mathbb{F}).$$

Properties of Linear transformation

1. If T is linear transformation, then

$$T(0) = 0$$

Proof. let $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ is linear transformation

then $T(0) = T(0+0) = T(0) + T(0)$

$\Rightarrow T(0) = 0$

2. T is linear transformation if and only if

$$T(ax+y) = aT(x)+T(y) \text{ for all } x,y \in V(\mathbb{F}) \text{ \& } a \in \mathbb{F}$$

Proof suppose that T is a linear transformation then to prove that

$$T(ax+y) = aT(x)+T(y) \quad \text{——— (1)}$$

Taking the left hand side of eqn(1).

$$\begin{aligned} L.H.S &= T(ax+y) \\ &= T(ax)+T(y) \quad (\because T \text{ is linear}) \\ &= aT(x)+T(y) \quad (\because T \text{ is linear}) \end{aligned}$$

$\Rightarrow$ L.H.S = R.H.S

Conversely, suppose that the given condition i.e.

$T(ax+y) = aT(x)+T(y)$ holds then to show that T is linear transformation.

Since $a \in \mathbb{F}$, in particular $a=1$ then

$$T(x+y) = T(x)+T(y), \forall x,y \in V(\mathbb{F}) \quad \text{——— (2)}$$

Also, $y \in V(\mathbb{F})$ & $V(\mathbb{F})$ is a vector space then take $y=0$, we have

$$\begin{aligned} T(ax) &= aT(x)+T(0) \\ &= aT(x) \quad (\because \text{by property (1)}) \end{aligned}$$

$\Rightarrow T(ax) = aT(x)$ for all $x \in V(\mathbb{F})$. ——— (3)

From eqns (2) and (3),
T is a linear transformation from $V(\mathbb{F})$ to $W(\mathbb{F})$.

3 T is linear transformation if and only if for $x_1, x_2, x_3, ---, x_n \in V(\mathbb{F})$ and $a_1, a_2, ---, a_n \in \mathbb{F}$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

Proof Suppose that T is linear transformation then to show that

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i) \quad \text{——— (1)}$$

Taking the left hand side of eqn (1).

$$L.H.S = T\left(\sum_{i=1}^{n} a_i x_i\right)$$

$$= T(a_1x_1 + a_2x_2 + a_3x_3 + ---- + a_nx_n)$$

$$= a_1T(x_1) + a_2T(x_2) + a_3T(x_3) + ---- + a_nT(x_n)$$

$(\because T \text{ is linear})$

$$= \sum_{i=1}^{n} a_i T(x_i)$$

$\Rightarrow$ L.H.S = R.H.S.

Conversly, suppose that the given condition holds i.e.

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$ then to show that

T is linear transformation

Now, by property two, we can easily proof.

Example-1 Define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. Show that T is a linear transformation on $\mathbb{R}^2$.

Solution To show $T : \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}^2(\mathbb{R})$ is a linear transformation, we use definition

Let $c \in \mathbb{R}$ and $x = (a_1, a_2) \in \mathbb{R}^2$, $y = (b_1, b_2) \in \mathbb{R}^2$

Then

$$T(x+y) = T\{(a_1, a_2) + (b_1, b_2)\}$$

$$= T(a_1 + b_1, a_2 + b_2)$$

$$= (2a_1 + 2b_1 + a_2 + b_2, a_1 + b_1)$$

$$= (2a_1 + a_2, a_1) + (2b_1 + b_2, b_1)$$

$$= T(a_1, a_2) + T(b_1, b_2)$$

$$= T(x) + T(y)$$

$$\Rightarrow T(x+y) = T(x) + T(y) \text{ for all } x, y \in \mathbb{R}^2$$

Now

$$T(cx) = T(c(a_1, a_2))$$

$$= T(ca_1, ca_2)$$

$$= (2ca_1 + ca_2, ca_1)$$

$$= c(2a_1 + a_2, a_1)$$

$$= cT(a_1, a_2)$$

$$= cT(x) \quad \forall \, x \in \mathbb{R}^2 \text{ and } c \in \mathbb{R}$$

Hence T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

Example-2 :- Define a mapping $T: M_{m\times n}(\mathbb{F}) \longrightarrow M_{m\times n}(\mathbb{F})$ by $T(A) = A^t$, where A^t is the transpose of the matrix A. Then prove that T is a linear transformation.

Solution : Given that, a mapping

$$T: M_{m\times n}(\mathbb{F}) \longrightarrow M_{m\times n}(\mathbb{F}) \text{ defined by}$$

$$T(A) = A^t$$

By the definition of linear transformation, we prove T is linear transformation.

Now, let $A, B \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$

$$T(A+B) = (A+B)^t$$

$$= A^t + B^t$$

$$T(A+B) = T(A) + T(B) \quad \forall\ A, B \in M_{m\times n}(\mathbb{F})$$

Also

$$T(cA) = (cA)^t = cA^t$$

$$T(cA) = c\,T(A) \quad \text{for all } c \in \mathbb{F} \ \& \ A \in M_{m\times n}(\mathbb{F})$$

Hence T is a linear transformation

Example-3 :- Let $V = P_n(\mathbb{R})$ and $W = P_{n-1}(\mathbb{R})$.

Define $T: V \longrightarrow W$ by $T(f) = f'$, where f' denotes the derivative of f. Prove that T is a linear transformation

Solution To show $T: V \to W$ is a linear transformation we use property 2.

Let $f(x), g(x) \in V = P_n(\mathbb{R})$ and $a \in \mathbb{R}$, then

$$T(af(x)+g(x)) = (af(x)+g(x))'$$

$$= (af(x))' + g'(x) \quad \text{(by Algebra of derivative).}$$

$$= af'(x) + g'(x)$$

$$= aT(f(x)) + T(g(x))$$

$$\Rightarrow T(af(x)+g(x)) = aT(f(x)) + T(g(x))$$

Hence, by property 2, T is linear transformation

Example-4 :- For $0 \leq \theta < 2\pi$, we define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$.

Show that T_θ is a linear transformation & T_θ is called the rotation by θ.

Solution:- To show that $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, we use property 2.

Let $x = (a_1, a_2) \in \mathbb{R}^2$ & $y = (b_1, b_2) \in \mathbb{R}^2$, $a \in \mathbb{R}$, then

$$T_\theta(ax+y) = T_\theta\{a(a_1, a_2) + (b_1, b_2)\}$$

$$= T_\theta\{aa_1 + b_1, aa_2 + b_2\}$$

$$= (aa_1+b_1)\cos\theta - (aa_2+b_2)\sin\theta, (aa_1+b_1)\sin\theta + (aa_2+b_2)\cos\theta$$

$$= \{aa_1\cos\theta - aa_2\sin\theta, aa_1\sin\theta + aa_2\cos\theta\} + \{b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta\}$$

$$= a(a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta) + (b_1\cos\theta - b_2\sin\theta,\ b_1\sin\theta + b_2\cos\theta)$$

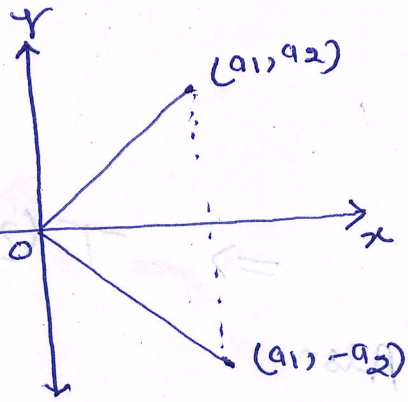
$$= a\,T_\theta(a_1, a_2) + T_\theta(b_1, b_2)$$

$T_\theta(ax+y) = a\,T_\theta(x) + T_\theta(y)$ for all $x, y \in \mathbb{R}^2$ & $a \in \mathbb{R}$.

Hence, by property 2. T_θ is a linear transformation.

Example Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$.
The mapping T is called reflection about the x-axis.
show that T is linear transformation.

Solution By the definition of linear transformation



$T(x+y) = T\{(a_1, a_2) + (b_1, b_2)\}$

where $x = (a_1, a_2)$ &
$y = (b_1, b_2)$

$\Rightarrow T(x+y) = T(a_1+b_1,\ a_2+b_2)$

$= (a_1+b_1,\ -a_2-b_2)$

$= (a_1, -a_2) + (b_1, -b_2)$

$= T(x) + T(y)$

$\Rightarrow T(x+y) = T(x) + T(y)\ \forall\ x, y \in \mathbb{R}^2.$

& $T(cx) = T(c(a_1, a_2)) = T(ca_1, ca_2)$
$= (ca_1, -ca_2) = c(a_1, -a_2) = cT(x)$

Therefore T is linear transformation

Example Define $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$. The mapping T is called the projection on the x-axis. Show that T is linear transformation on $\mathbb{R}^2$.

Solution

Let $x = (a_1, a_2) \in \mathbb{R}^2$, $y = (b_1, b_2) \in \mathbb{R}^2$ & $c \in \mathbb{R}$

then

$$T(x+y) = T\{(a_1, a_2) + (b_1, b_2)\}$$

$$= T(a_1+b_1, a_2+b_2)$$

$$= (a_1+b_1, 0)$$

$$= (a_1, 0) + (b_1, 0)$$

$$= T(a_1, a_2) + T(b_1, b_2)$$

$$= T(x) + T(y)$$

$\Rightarrow \quad T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^2$

Also

$$T(cx) = T(c(a_1, a_2))$$

$$= T(ca_1, ca_2)$$

$$= (ca_1, 0)$$

$$= c(a_1, 0)$$

$$= cT(a_1, a_2)$$

$$= cT(x)$$

$\Rightarrow \quad T(cx) = cT(x)$ for all $x \in \mathbb{R}^2$ & $c \in \mathbb{R}$.

Hence projection mapping is also linear transformation on $\mathbb{R}^2$.

By the definition of Null space

$$N(T) = \{x = (a_1, a_2, a_3) \in \mathbb{R}^3 : T(x) = 0\}$$

$$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : T(a_1, a_2, a_3) = 0\}$$

$$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : (a_1 - a_2, 2a_3) = (0, 0)\}$$

$$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_2,\ a_3 = 0\}$$

$$= \{(a_1, a_1, 0) \in \mathbb{R}^3 : a_1 \in \mathbb{R}\}$$

Now, find range of T

$$R(T) = \{T(x) : x \in \mathbb{R}^3\}$$

$$= \left\{T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3\right\}$$

$$= \{(a_1 - a_2, 2a_3) : a_1, a_2, a_3 \in \mathbb{R}\}$$

$R(T) = \mathbb{R}^2$ ($\because$ if we take $a_2 = 0$ then $(a_1, 2a_3)$ generate $\mathbb{R}^2$)

Null space (or Kernel of T)

Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be the vector spaces and let $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be the linear transformation. Then the null space (or kernel) of T is denoted by $N(T)$ and is defined by the set of all vector $x \in V(\mathbb{F})$ such that $T(x) = 0$ i.e. $N(T) = \{x \in V(\mathbb{F}) : T(x) = 0\}$

Range (or Image) of T

Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be the vector spaces and let $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be the linear transformation. Then the range (or image) of T is denoted by $R(T)$ and is defined by the subset of $W(\mathbb{F})$ consisting of all images (under T) of elements of $V(\mathbb{F})$.

i.e. $R(T) = \{T(x) : x \in V(\mathbb{F})\}$.

Remark: Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces, and let $I: V(\mathbb{F}) \longrightarrow V(\mathbb{F})$ and $T_0: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be the identity and zero transformations, respectively. Then $N(I) = 0$, $R(I) = V(\mathbb{F})$, $N(T_0) = V(\mathbb{F})$ and $R(T_0) = \{0\}$.

Example Define a linear transformation

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \text{ by } T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

find the null space and range space of T.

Solution:- Given that, the linear transformation

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \text{ by } T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

Example Define a linear transformation

$$T : M_{n\times n}(\mathbb{F}) \to \mathbb{F} \quad \text{by} \quad T(A) = tr(A),$$

where $tr(A) = \sum_{i=1}^{n} a_{ii}$ is a trace of matrix A.

Find null space and range space of T.

Solution :- Given that, linear transformation $T : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

by $\quad T(A) = tr(A)$

By the definition of null space

$$N(T) = \{ A_{n\times n} \in M_{n\times n}(\mathbb{F}) : T(A) = 0 \}$$

$$N(T) = \{ A_{n\times n} \in M_{n\times n}(\mathbb{F}) : tr(A) = 0 \}$$

And range of T given by

$$R(T) = \{ T(A) : A \in M_{n\times n}(\mathbb{F}) \}$$

$$= \{ tr(A) : A \in M_{n\times n}(\mathbb{F}) \}$$

$$= \mathbb{F}$$

Example Define a linear transformation

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R}) \quad \text{by} \quad T(f(x)) = x f(x) + f'(x)$$

, find null space and range space of T.

Solution Given that linear transformation

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R}) \text{ by}$$

$$T(f(x)) = x f(x) + f'(x)$$

By the definition of null space

$$N(T) = \{ f(x) \in P_2(\mathbb{R}) : T(f(x)) = 0 \}$$

$$= \{ f(x) \in P_2(\mathbb{R}) : x f(x) + f'(x) = 0 \}$$

$$= \{ f(x) = a_2 x^2 + a_1 x + a_0 \in P_2(\mathbb{R}) : x(a_2 x^2 + a_1 x + a_0) + (2a_2 x + a_1) = 0 \}$$

$$= \{ a_2 x^2 + a_1 x + a_0 \in P_2(\mathbb{R}) : a_2 x^3 + a_1 x^2 + (a_0 + 2a_2)x + a_1 = 0 \}$$

$$= \{ a_2 x^2 + a_1 x + a_0 \in P_2(\mathbb{R}) : a_2 = 0,\ a_1 = 0,\ a_0 = 0 \}$$

$$N(T) = \{0\}$$

Now, find the range space

$$R(T) = \{ T(f(x)) : f(x) = a_2 x^2 + a_1 x + a_0 \in P_2(\mathbb{R}) \}$$

$$= \{ x f(x) + f'(x) : f(x) = a_2 x^2 + a_1 x + a_0 \in P_2(\mathbb{R}) \}$$

$$= \{ x(a_2 x^2 + a_1 x + a_0) + 2a_2 x + a_1 : a_0, a_1, a_2 \in \mathbb{R} \}$$

$$= \{ a_2 x^3 + a_1 x^2 + (a_0 + 2a_2)x + a_1 : a_0, a_1, a_2 \in \mathbb{R} \}$$

$$R(T) = P_3(\mathbb{R})$$

Rank-Nullity Theorem (Dimension Theorem) :-

Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces and let $T: V(\mathbb{F}) \to W(\mathbb{F})$ be linear transformation. If $V(\mathbb{F})$ is finite-dimensional then

$$\boxed{\text{nullity}(T) + \text{rank}(T) = \dim(V(\mathbb{F}))}$$

$$\boxed{\text{Rank}(T) = \dim(\text{Img}\,T)} \quad \boxed{\text{Nullity}(T) = \dim(\ker T)}$$

Theorem :- Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces, and let $T: V(\mathbb{F}) \to W(\mathbb{F})$ be linear transformation. Then T is one-to-one if and only if $N(T) = \{0\}$.

Theorem :- Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces, ~~and~~ of finite and equal dimension, and let $T: V(\mathbb{F}) \to W(\mathbb{F})$ be linear transformation. Then T is one-to-one if and only if T is onto.

Example :- Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt,$$

find the rank and nullity of linear transformation T.

Solution First of all, calculate null space of T, so

$$N(T) = \{f(x) = a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R}) : T(f(x)) = 0\}$$

$$= \left\{f(x) = a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R}) : 2(2a_2x + a_1) + 3\int_0^x (a_2t^2 + a_1t + a_0)\,dt = 0\right\}$$

$$= \left\{a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R}) : 4a_2x + 2a_1 + a_2x^3 + \tfrac{3}{2}a_1x^2 + 3a_0x = 0\right\}$$

Theorem :- Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces and $T : V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be linear transformation. Then $N(T)$ and $R(T)$ are subspaces of $V(\mathbb{F})$ and $W(\mathbb{F})$ respectively.

Theorem :- Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces and let $T : V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be linear transformation. If $V(\mathbb{F})$ has a basis $\beta = \{v_1, v_2, ----, v_n\}$ then

$$R(T) = \text{span}\{T(v_1), T(v_2), -----, T(v_n)\}.$$

Nullity of linear transformation

Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces and let $T : V(\mathbb{F}) \rightarrow W(\mathbb{F})$ be linear transformation. The nullity of linear transformation T denoted by nullity (T) and is defined by the dimension of null space. i.e. number of element present in the basis of null space.

Rank of linear transformation :-

Let $V(\mathbb{F})$ and $W(\mathbb{F})$ be vector spaces and let $T : V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ be the linear transformation. The rank of linear transformation T denoted by $\text{Rank}(T)$ and is defined by the dimension of range space. i.e. number of element present in the basis of range space.

$N(T) = \{ a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R}) : a_2x^3 + \frac{3}{2}a_1x^2 + (4a_2 + 3a_0)x + 2a_1 = 0 \}$

$= \{ a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R}) : a_2 = 0, a_1 = 0, a_0 = 0 \}$

$N(T) = \{0\}$

$\Rightarrow \dim(N(T)) = 0$

Therefore $\boxed{\text{Nullity}(T) = 0} \Rightarrow T$ is one-to-one mapping

Now, find the rank of T by using rank-nullity theorem then

$$\text{nullity}(T) + \text{rank}(T) = \dim(P_2(\mathbb{R}))$$

$$\Rightarrow \quad 0 + \text{rank}(T) = 3$$

$$\Rightarrow \quad \boxed{\text{rank}(T) = 3}$$

<u>Example</u> Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2) = (a_1 + a_2, a_1),$$ find the rank and nullity of T.

<u>Solution</u> By definition of Null space,

$N(T) = \{ x = (a_1, a_2) \in \mathbb{R}^2 : T(a_1, a_2) = (0,0) \}$

$= \{ (a_1, a_2) \in \mathbb{R}^2 : (a_1 + a_2, a_1) = (0,0) \}$

$= \{ (a_1, a_2) \in \mathbb{R}^2 : (a_1 = 0, a_2 = 0) \}$

$N(T) = \{ (0,0) \}$

Dimension of Null space $= 0$

$\Rightarrow$ Nullity$(T) = 0$ $\Rightarrow$ T is one-to-one mapping.

Now, find rank of T, then

$$R(T) = \{T(a_1, a_2) : (a_1, a_2) \in \mathbb{R}^2\}$$

$$R(T) = \{(a_1 + a_2, a_1) \in \mathbb{R}^2 : a_1, a_2 \in \mathbb{R}\}$$

$\Rightarrow R(T) = \mathbb{R}^2$ $\Rightarrow$ $\dim(R(T)) = 2$

Therefore, Rank$(T) = 2$. Hence T is onto mapping.

Example Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2),$$ find the rank & nullity of T.

Solution By the definition of null space

$$N(T) = \{a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R}) : T(a_0 + a_1x + a_2x^2) = (0,0,0)\}$$

$$N(T) = \{a_0 + a_1x + a_2x^2 \in P_2(\mathbb{R}) : (a_0, a_1, a_2) = (0,0,0)\}$$

$\Rightarrow N(T) = \{0\}$ $\Rightarrow$ Nullity$(T) = 0 \Rightarrow T$ is one-to-one

Since dimension of $P_2(\mathbb{R}) = 3 = \dim(\mathbb{R}^3)$

Therefore rank$(T) = 3$. hence T is onto.

Application of Rank and Nullity theorem to systems of linear equations

Let $AX=B$ denote the matrix form of a system of m linear equations in n unknowns. Now the matrix A may be viewed as a linear mapping $A: V \longrightarrow W$ where $\dim(V)=n$ and $\dim(W)=m$. The solution of the equation $AX=B$ may be viewed as the preimage of the vector $B \in V$ under the linear mapping A. The solution of associated homogeneous system $AX=0$ may be viewed as the kernel of the linear mapping A. So by Rank Nullity Theorem

$$\dim(\ker A) = \dim V - \dim(\operatorname{Im} A)$$

$$\dim(\ker A) = n - r$$

n is exactly number of unknowns in the homogeneous system $AX=0$

Example: Let $T: R^3 \longrightarrow R^3$ be the linear mapping defined by

$$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$$

Find a basis and the dimension of

(a) the image of T

(b) the kernel of T

Solution: (a) First find the images of the usual basis of R^3. Usual basis of R^3 are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$$

$$T(1,0,0) = (1+2\times 0 - 0,\ 0+0,\ 1+0-2\times 0) = (1,0,1)$$

$$T(0,1,0) = (2,1,1)$$

$$T(0,0,1) = (-1,1,-2)$$

Image vectors span Image T

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1,0,1)$ and $(0,1,-1)$ form a basis for Image T; hence $\dim(\operatorname{Im} T) = $ Rank of $T = 2$

(b) Set $T(v) = 0$ where $v = (x,y,z)$

$$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z) = (0,0,0)$$

$$x+2y-z=0$$
$$y+z=0$$
$$x+y-2z=0$$

Solve this following Homogeneous system (By Rank method)

$$x+2y-z=0$$
$$y+z=0$$

The only free variable is z hence $\dim(\ker T) = 1$

or

$$\dim(\ker T) = n - r = \text{unknowns} - \dim(\operatorname{Im} T)$$
$$= 3 - 2 = 1$$

set $z=1$, then $y=-1$, $x=3$

Thus $(3,-1,1)$ form a basis of ker T.

Example: Consider the matrix mapping $T: R^4 \longrightarrow R^3$, where $T = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$

Find a basis and the dimension of

(a) the Image of T

(b) the Kernel of T

Solution: (a) The column space of T' is equal to Image T. Now reduce T' to echelon form:

$$T' = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\{(1,1,3),(0,1,2)\}$ is a basis of Image and $\dim(\text{Image } T) = \text{Rank of } T' = 2$

(b) Here ker T is the solution space of the homogeneous system $TX=0$. Thus reduce the matrix T of coefficient to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$x + 2y + 3z + t = 0$$
$$y + 2z - 3t = 0$$

so $\dim(\ker T) = n - r = 4 - 2 = 2$

The free variables are z and t

(i) set $z = 1$, $t = 0$ to get the solution $(1, -2, 1, 0)$

(ii) set $z = 0$, $t = 1$ to get the solution $(-7, 3, 0, 1)$

Thus $(1, -2, 1, 0)$ and $(-7, 3, 0, 1)$ form a basis for $\operatorname{Ker} T$.

Change of Basis

Matrix Representation of a linear operator (Transformation)

Let T be a linear operator (transformation) from a vector space V into itself, and suppose $S = \{u_1, u_2, u_3 \cdots u_n\}$ is a basis of V. Now $T(u_1), T(u_2) \cdots T(u_n)$ are vectors in V, and so each is a linear combination of the vectors in the basis S; say

$$T(u_1) = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$
$$T(u_2) = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T(u_n) = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

The following definition applies.

Definition

The transpose of the above matrix of coefficients, denoted by $[T]_S$, is called the matrix representation of T relative to the basis S, or simply the matrix of T in the basis S.

$$[T]_S = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Example:

Consider the linear mapping $T: R^2 \longrightarrow R^2$ defined by $T(x,y) = (3x+4y,\ 2x-5y)$ and the following bases of R^2:

$E = \{e_1, e_2\} = \{(1,0), (0,1)\}$ and $S = \{u_1, u_2\} = \{(1,2), (2,3)\}$

(a) Find the matrix A representing T relative to the basis E.

(b) Find the matrix B representing T relative to the basis S.

Ans (a) Since E is the usual basis, the rows of A are simply the coefficients in the components of $T(x,y)$, that is, using

$(a,b) = ae_1 + be_2$

$(a,b) = a(1,0) + b(0,1) = (a,0) + (0,b) = (a,b)$

$T(e_1) = T(1,0) = (3,2) = 3(1,0) + 2(0,1)$

$T(e_2) = T(0,1) = (4,-5) = 4(1,0) + 5(0,1)$

so $[A]_E = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

Note that the coefficient of the basis vectors are written as columns in the matrix representation.

(b) Let us consider an arbitrary vector $(a,b) \in R^2$ relative to the basis S. We have.

$(a,b) = xu_1 + yu_2 = x(1,2) + y(2,3)$ —— ①

$(a,b) = (x, 2x) + (2y, 3y)$

$(a,b) = (x+2y,\ 2x+3y)$

so $x + 2y = a$
$2x + 3y = b$

solving these equations, we have

$x = -3a + 2b, \quad y = 2a - b$

so from ①

$(a,b) = (-3a+2b)u_1 + (2a-b)u_2$

Then use the formula for (a,b) to find the coordinates of $T(u_1)$ and $T(u_2)$ relative to S.

$T(u_1) = T(1,2) = (11,-8) = (-3\times 11 + 2\times -8)u_1 + (2\times 11 + 8)u_2$

$(11,-8) = -49u_1 + 30u_2$

$T(u_2) = T(2,3) = (18,-11) = (-3\times 18 + 2\times -11)u_1 + (2\times 18 + 11)u_2$

$(18,-11) = -76u_1 + 47u_2$

so $[B]_S = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$

Alternativily, First find $T(u_1)$ and write it is a linear combination of the basis vectors u_1 and u_2, we have

$T(u_1) = T(1,2) = (11,-8) = x\overset{u_1}{(1,2)} + y\overset{u_2}{(2,3)}$

$\Rightarrow (11,-8) = (x,2x) + (2y,3y)$

so $x + 2y = 11$ Solve this system
$2x + 3y = -8$ $x = -49, \ y = 30$

$T(u_1) = -49u_1 + 30u_2$

Next $T(u_2) = T(2,3) = (18,-11) = x(1,2) + y(2,3)$

$(18,-11) = (x,2x) + (2y,3y)$

so $x + 2y = 18$ solving these eqn's
$2x + 3y = -11$ $x = -76, \ y = 47$

so $T(u_2) = -76 u_1 + 47 u_2$

write the coefficients of u_1 and u_2 as columns to obtain $[B]_S = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$

Example:

Let $T : R^2 \to R^2$ be defined by $T(x,y) = (2x+3y, 4x-5y)$

Find the matrix representation $[T]_S$ of T relative to the basis $S = \{u_1, u_2\} = \{(1,-2), (2,-5)\}$

Ans Let us consider an arbitrary vector $(a,b) \in R^2$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x u_1 + y u_2 = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} + \begin{bmatrix} 2y \\ -5y \end{bmatrix}$$

so
$$x + 2y = a$$
$$-2x - 5y = b$$

solving these equations, we have $x = 5a + 2b$
$y = -2a - b$

so $(a,b) = (5a+2b) u_1 + (-2a-b) u_2$

Now Find

$T(u_1) = T(1,-2) = (-4, 14) = (5 \times -4 + 2 \times 14) u_1 + (-2 \times -4 - 14) u_2$

$(-4, -14) = 8 u_1 - 6 u_2$

$T(u_2) = T(2,-5) = (-11, 33) = (5 \times -11 + 2 \times 33) u_1 + (-2 \times -11 - 33) u_2$

$(-11, 33) = 11 u_1 - 11 u_2$

so $[T]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$

Example

Let V be the vector space of functions with basis $S = \{\sin t, \cos t, e^{3t}\}$, and let $D: V \to V$ be the differential operator defined by $D(f(t)) = \frac{d}{dt} f(t)$. Represent matrix in the basis S.

Ans

$$D(\sin t) = \cos t = 0(\sin t) + 1.(\cos t) + 0(e^{3t})$$

$$D(\cos t) = -\sin t = -1(\sin t) + 0.(\cos t) + 0.(e^{3t})$$

$$D(e^{3t}) = 3e^{3t} = 0(\sin t) + 0.(\cos t) + 3.(e^{3t})$$

so
$$[D]_S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Theorem: Let $T: V \to V$ be a linear operator, and let S be a finite (basis of V. Then, for any vector v in V, $[T]_S [v]_S = [T(v)]_S$

example

Let $T: R^2 \to R^2$ be defined by $T(x,y) = (2x+3y, 4x-5y)$

Find the matrix representation $[T]_S$ of T relative to the basis $S = \{u_1, u_2\} = \{(1,-2), (2,-5)\}$.

Verify $[T]_S [v]_S = [T(v)]_S$.

Ans Let us consider an arbitrary vector $(a,b) \in R^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} = x u_1 + y u_2 = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} + \begin{bmatrix} 2y \\ -5y \end{bmatrix}$$

so
$$x + 2y = a$$
$$-2x - 5y = b$$

Solving these equations

$$x = 5a+2b \quad , \quad y = -2a-b$$

so $(a,b) = (5a+2b)u_1 + (-2a-b)u_2$ —— ①

$T(u_1) = T(1,-2) = (-4,14) = (5\times -4 + 2\times 14)u_1 + (-2\times -4 - 14)u_2$

$$(-4,14) = 8u_1 - 6u_2$$

$T(u_2) = T(2,-5) = (-11,33) = (5\times -11 + 2\times 33)u_1 + (-2\times -11 - 33)u_2$

$$(-11,33) = 11u_1 - 11u_2$$

so $[T]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}$

Let $v = (5,-7)$ and $T(5,-7) = (-11,55)$

From ①

$(5,-7) = (5\times 5 + 2\times -7)u_1 + (-2\times 5 + 7)u_2$

$= 11u_1 - 3u_2$

so $[v]_S = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$

$T(5,-7) = (-11,55) = (5\times -11 + 2\times 55)u_1 + (-2\times -11 - 55)u_2$

$= 55u_1 - 33u_2$

so $[T(v)]_S = \begin{bmatrix} 55 \\ -33 \end{bmatrix}$

$[T]_S[v]_S = \begin{bmatrix} 8 & 11 \\ -6 & -11 \end{bmatrix}\begin{bmatrix} 11 \\ -3 \end{bmatrix} = \begin{bmatrix} 55 \\ -33 \end{bmatrix} = [T(v)]_S$

Hence Theorem verified.

Change of Basis

Definition Let $S = \{u_1, u_2, \dots u_n\}$ be a basis of a vector space V, and let $S' = \{v_1, v_2 \dots v_n\}$ be another basis. (For reference, we will call S the "old" basis and S' the "new" basis.) Since S is a basis, each vector in the "new" basis S' can be written uniquely as a linear combination of the vectors in S; say,

$$v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$
$$v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

Let P be the transpose of the above matrix of coefficient; that is, let $P = [p_{ij}]$, where $p_{ij} = a_{ij}$. Then P is called the change-of-basis (or transition matrix) from the "old" basis S to the "new" basis S'.

Remark: If P is the change of basis matrix from the "old" basis to "new" basis and Q is the change of basis matrix from the "new" basis to the "old" basis S. Then $Q = P^{-1}$

Example:

Consider the following two bases of R^2:

$S = \{u_1, u_2\} = \{(1,2), (3,5)\}$ and

$S' = \{v_1, v_2\} = \{(1,-1), (1,-2)\}$

Find (i) Find the change of basis matrix P from S to the "new" basis S'.

(ii) Find the change of basis matrix Q from the "new" basis S' back to the "old" basis S.

Ans: (a) Write each of the new basis vectors of S' as a linear combination of the original basis vectors u_1 and u_2 of S.

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{u_1} + y \underbrace{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}_{u_2} \quad \text{or} \quad \begin{aligned} x + 3y &= 1 \\ 2x + 5y &= -1 \end{aligned}$$

so $x = -8$, $y = 3$

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{u_1} + y \underbrace{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}_{u_2} \quad \text{or} \quad \begin{aligned} x + 3y &= 1 \\ 2x + 5y &= -2 \end{aligned}$$

so $x = -11$, $y = 4$

Thus

$$v_1 = -8u_1 + 3u_2$$

$$v_2 = -11u_1 + 4u_2$$

and hence $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$

(b) Here we write each of the "old" basis vectors u_1 and u_2 of S as a linear combination of the "new" basis vectors v_1 and v_2 of S'.

$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\left.\begin{aligned} x + y &= 1 \\ -x - 2y &= 2 \end{aligned}\right\} \Rightarrow x = 4, y = -3$

$u_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\left.\begin{aligned} x + y &= 3 \\ -x - 2y &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 11 \\ y &= -8 \end{aligned}$

Thus

$$\begin{aligned} u_1 &= 4v_1 - 3v_2 \\ u_2 &= 11v_1 - 8v_2 \end{aligned}$$

and hence $Q = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$

Here $P^{-1} = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$, $Q = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$

so $P^{-1} = Q$

Theorem: Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then, for any vector $v \in V$, we have:

$$P[v]_{S'} = [v]_S \text{ and hence } P^{-1}[v]_S = [v]_{S'}$$

Theorem: Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then, for any linear operator T on V,

$$[T]_{S'} = P^{-1}[T]_S P$$

That is, if A and B are the matrix representation of T relative, respectivily, to S and S', then

$$B = P^{-1}AP$$

Now we write each of the basis vectors in E as a linear combination of the basis elements of S.

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}2\\1\\2\end{pmatrix} + z\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}x\\0\\x\end{pmatrix} + \begin{pmatrix}2y\\y\\2y\end{pmatrix} + \begin{pmatrix}z\\2z\\2z\end{pmatrix}$$

$$\begin{matrix}1\\0\\0\end{matrix} = \begin{matrix}x+2y+z\\y+2z\\x+2y+2z\end{matrix}$$

Solving this system of equations, Augmented matrix

$[A : B]$

$$\begin{bmatrix}1 & 2 & 1 & : & 1\\0 & 1 & 2 & : & 0\\1 & 2 & 2 & : & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1 & : & 1\\0 & 1 & 2 & : & 0\\0 & 0 & 1 & : & -1\end{bmatrix}$$

By back-substitution, we have

$$x+2y+z=1$$
$$y+2z=0$$
$$z=-1$$

$\Rightarrow x=-2,\ y=2,\ z=-1$

so $\boxed{e_1 = -2u_1 + 2u_2 - u_3}$

Similarly

$$e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}2\\1\\2\end{pmatrix} + z\begin{pmatrix}1\\2\\2\end{pmatrix}$$

solving this $x=-2,\ y=1,\ z=0$

so $\boxed{e_2 = -2u_1 + u_2}$

$$e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} = x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}2\\1\\2\end{pmatrix} + z\begin{pmatrix}1\\2\\2\end{pmatrix}$$

solving this $x=3,\ y=-2,\ z=1$

so $\boxed{e_3 = 3u_1 - 2u_2 + u_3}$

Hence
$$Q = \begin{bmatrix}-2 & -2 & 3\\2 & 1 & -2\\-1 & 0 & 1\end{bmatrix}$$

Matrix and General linear mapping

we consider the general case of linear mappings from one vector space into another. Suppose V and W are vector spaces over the same field F and, say, $\dim V = m$ and $\dim W = n$. Furthermore, suppose

$$S = \{v_1, v_2 \cdots v_m\} \text{ and } S' = \{u_1, u_2 \cdots u_n\}$$

are arbitrary but fixed bases, respectively, of V and W.

Suppose $T : V \longrightarrow W$ is a linear mapping. Then the vectors $T(v_1), T(v_2), T(v_3) \cdots T(v_m)$ belong to W, and so each is a linear combination of the basis vectors in S'; say

$$T(v_1) = a_{11}u_1 + a_{12}u_2 + \cdots a_{1n}u_n$$

$$T(v_2) = a_{21}u_1 + a_{22}u_2 + \cdots a_{2n}u_n$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$T(v_m) = a_{m1}u_1 + a_{m2}u_2 + \cdots a_{mn}u_n$$

The transpose of the above matrix of coefficient, denoted by $[T]_{S,S'}$ is called the matrix representation of F relative to the bases S and S'.

Theorem : For any vector $v \in V$, $[T]_{S,S'}[v]_S = [T(v)]_{S'}$

Theorem: Let P be the change-of-basis matrix from a basis E to a basis E' in V, and let Q be the change-of-basis matrix from a basis S to a basis S' in W. Then, for any linear map $T: V \longrightarrow W$,

$$[T]_{E',S'} = Q^{-1}[T]_{E,S}\,P$$

In other words, if A is the matrix representation of a linear mapping T relative to the bases E and S, and B is the matrix representation of T relative to the bases E' and S', then

$$B = Q^{-1}AP$$

Example: Let $T: R^3 \longrightarrow R^2$ be the linear map defined by $T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$

(a) Find the matrix of T in the following bases of R^3 and R^2.

$S = \{w_1, w_2, w_3\} = \{(1,1,1), (1,1,0), (1,0,0)\}$ and

$S' = \{u_1, u_2\} = \{(1,3), (2,5)\}$

(b) Verify $[T]_{S,S'}[v]_S = [T(v)]_{S'}$

Ans (a) First we find the coordinate of any arbitrary vector $v = (a,b) \in R^2$

$$v=\begin{bmatrix}a\\b\end{bmatrix}=x\overset{u_1}{\begin{bmatrix}1\\3\end{bmatrix}}+y\overset{u_2}{\begin{bmatrix}2\\5\end{bmatrix}} \quad \text{and so} \quad \begin{matrix}x+2y=a\\3x+5y=b\end{matrix}$$

solve for x and y in terms of a and b to get
$x=-5a+2b$, $y=3a-b$.

$$\text{so } v=\begin{bmatrix}a\\b\end{bmatrix}=(-5a+2b)\overset{u_1}{\begin{bmatrix}1\\3\end{bmatrix}}+(3a-b)\overset{u_2}{\begin{bmatrix}2\\5\end{bmatrix}} \quad \text{—— (1)}$$

$$\text{so } [v]_S=\begin{bmatrix}-5a+2b\\3a-b\end{bmatrix}$$

Again $\begin{bmatrix}a\\b\end{bmatrix}=(-5a+2b)u_1+(3a-b)u_2$ (From (1))

$$T(\omega_1)=T(1,1,1)=(3\times1+2\times1-4\times1,\ 1-5\times1+3\times1)$$
$$=(1,-1)$$
$$T(\omega_1)=(1,-1)=(-5\times1+2\times-1)\,u_1+(3\times1-(-1))\,u_2$$
$$T(1,1,1)=(1,-1)=-7u_1+4u_2$$

Similarly $T(1,1,0)=(5,-4)=-33u_1+19u_2$

$T(1,0,0)=(3,1)=-13u_1+8u_2$

$$\text{so } [T]_{S,S'}=\begin{bmatrix}-7&-33&-13\\4&19&8\end{bmatrix}$$

This is the matrix T in the bases of R^3 and R^2.

(b) First we consider an arbitrary vector $(x,y,z)\in R^3$, we write (x,y,z) as a linear combination of $\omega_1,\omega_2,\omega_3$ using unknown scalars a,b,c.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=a\begin{pmatrix}1\\1\\1\end{pmatrix}+b\begin{pmatrix}1\\1\\0\end{pmatrix}+c\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{and so} \quad \begin{matrix}a+b+c=x\\a+b=y\\a=z\end{matrix}$$

solving these equations $a=z$, $b=y-z$, $c=x-y$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (y-z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (x-y)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = v = z\omega_1 + (y-z)\omega_2 + (x-y)\omega_3 \quad \text{--- (2)}$$

$$T(v) = (3x+2y-4z, x-5y+3z) = (-5(3x+2y-4z) + 2(x-5y+3z))u_1 + (3\times(3x+2y-4z) - (x-5y+3z))u_2$$

$$T(v) = (3x+2y-4z, x-5y+3z) = (-13x-20y+26z)u_1 + (8x+11y-15z)u_2$$

$$[T(v)]_{S'} = \begin{bmatrix} -13x-20y+26z \\ 8x+11y-15z \end{bmatrix}$$

Thus $[T]_{S,S'}[v]_S = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$

$$= \begin{bmatrix} -13x-20y+26z \\ 8x+11y-15z \end{bmatrix} = [T(v)]_{S'}$$

Example: Let $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$. A mapping $T: R^3 \to R^2$ defined by $T(v) = Av$, where vectors are written as columns. Find the matrix $[T]$ that represents the mapping relative to the following bases R^3 and R^2:

(a) The usual bases of R^3 and of R^2

(b) $S = \{\omega_1, \omega_2, \omega_3\} = \{(1,1,1), (1,1,0), (1,0,0)\}$ and $S' = \{u_1, u_2\} = \{(1,3), (2,5)\}$

Usual basis of $R^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

usual basis of $R^2 = \{(1,0), (0,1)\}$

Ans

Let us take an arbitrary vector $v = (a, b) \in R^2$

write vector v as a linear combination of

Basis of $R^2 = \{u_1, u_2\} = \{(1,0), (0,1)\}$

$$v = \begin{bmatrix} a \\ b \end{bmatrix} = x u_1 + y u_2 = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $x = a$, $y = b$

$$(a, b) = a u_1 + b u_2 \quad \text{———} \quad (1)$$

Basis of $R^3 = \{w_1, w_2, w_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

Given mapping $T(v) = Av$

so put $v = w_1$

$$T(w_1) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now using (1) $(a, b) = a u_1 + b u_2$

$$(2, 1) = 2 u_1 + 1 \cdot u_2$$

Similarly $F(w_2) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+5+0 \\ 0-4+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$T(w_3) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0-3 \\ 0+0+7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

so writing the coefficients of $T(w_1), F(w_2), T(w_3)$ as columns yields $[T] = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 4 & 7 \end{bmatrix}$

Note: Relative to the usual bases, $[T]$ is the matrix A itself.

(b) Now we find the matrix representing T relative to the bases of S and S'

Let us consider an arbitrary vector $v = (a,b) \in R^2$.

write $v = (a,b)$ as a linear combination of of

$S' = \{u_1, u_2\} = \{(1,3), (2,5)\}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} u_1 \end{bmatrix} + y\begin{bmatrix} u_2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{matrix} x + 2y = a \\ 3x + 5y = b \end{matrix} \Rightarrow \begin{matrix} x = -5a + 2b \\ y = 3a - b \end{matrix}$$

so

$$(a,b) = (-5a+2b)u_1 + (3a-b)u_2 \quad \text{———} \quad (1)$$

Thus

$$T(w_1) = Aw_1 = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Now using (1)

$$(4,4) = (-5\times4 + 2\times4)u_1 + (3\times4 - 4)u_2 = -12u_1 + 8u_2$$

similarly

$$T(w_2) = Aw_2 = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$$(7,-3) = (-5\times7 + 2\times-3)u_1 + (3\times7 + 3)u_2 = -41u_1 + 24u_2$$

$$T(w_3) = Aw_3 = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(2,1) = (-5\times2 + 2\times1)u_1 + (3\times2 - 1)u_2 = -8u_1 + 5u_2$$

Now writing the coefficients of $T(w_1), T(w_2), T(w_3)$ as colums yields

$$[T] = \begin{bmatrix} -12 & -41 & -8 \\ 8 & 24 & 5 \end{bmatrix}$$

Singular and Non singular linear mapping

Let $T: V \to W$ be a linear mapping. T is said to be singular if the image of some nonzero vector v is 0, that is, if there exists $v \neq 0$ such that $T(v) = 0$. Thus $T: V \to W$ is non singular if the zero vector 0 is the only vector whose image under T is 0 i.e. $T(0)=0$ or in other words, if $\ker T = \{0\}$

Example: Determine whether or not each of the following linear maps is singular. If not, find a nonzero vector v whose image is 0.

(a) $T: R^2 \to R^2$ defined by $T(x,y) = (x-y, x-2y)$

(b) $T: R^2 \to R^2$ defined by $T(x,y) = (2x-4y, 3x-6y)$

Solution (a) Find $\ker T$ by setting $T(v)=0$, where $v=(x,y)$

$$(x-y, x-2y) = (0,0) \text{ or } \begin{array}{l} x-y=0 \\ x-2y=0 \end{array} \Rightarrow x=0, y=0$$

so $v=(x,y)=(0,0)$. Hence $v=(0,0)$ is the only vector whose image is 0. Hence T is non singular

(b) Find $\ker T$, by setting $T(v)=0$, where $v=(x,y)$

$T(x,y)=0$

$$(2x-4y, 3x-6y) = 0 \quad \text{or} \quad \begin{array}{l} 2x-4y=0 \\ 3x-6y=0 \end{array} \quad \begin{array}{l} \text{or } x-2y=0 \\ \text{or } x-2y=0 \end{array}$$

or

$x-2y=0$

Let $y=1$, then $x=2$, so $v=(2,1)$ which is non zero vector. so there exists a non zero vector whose image is 0. so T is singular

Inverse of a linear Transformation

Let $T: V \longrightarrow V$ be a linear operator (transformation). T is said to be invertible if T is both one-to-one and onto.

Theorem: Let T be a linear operator on a finite dimensional vector V. Then the following four conditions are equivalent.

(i) T is nonsingular.
(ii) T is one-to-one.
(iii) T is an onto mapping.
(iv) T is invertible.

Remark: Suppose A is a square matrix over F. Then A may be viewed as a linear operator on F^n. Since F^n has finite dimension, above theorem holds for the square matrix A. This is why the terms "non singular" and "invertible" are used interchangeably when applied to square matrices.

Example: let T be the linear operator on R^2 defined by $T(x,y) = (2x+y, 3x+2y)$

(a) Show that T is invertible

(b) Find T^{-1}.

Solution:

(a) To show that T is invertible, we need only show that T is nonsingular. Set T is non singular.

Set $T(x,y) = (0,0)$ to obtain the homogeneous system

$$2x+y=0 \quad \text{and} \quad 3x+2y=0$$

Solve for x and y to get $x=0, y=0$. Hence T is non singular and so invertible.

(b) To find a formula for T^{-1}, we set

$$T(x,y) = (s,t) \text{ and so } T^{-1}(s,t) = (x,y).$$

$$(2x+y, 3x+2y) = (s,t)$$

so
$$2x+y = s$$
$$3x+2y = t$$

Solve for x and y in terms of s and t to obtain $x = 2s-t$, $y = -3s+2t$. Thus

$$T^{-1}(s,t) = (2s-t, -3s+2t)$$

so $T^{-1}(x,y) = (2x-y, -3x+2y)$

Example: Let $T: R^2 \longrightarrow R^3$ be defined by

$$T(x,y) = (x+y,\ x-2y,\ 3x+y)$$

(a) Show that T is nonsingular.

(b) Find a formula for T^{-1}.

Solution

(a) To show nonsingular set $T(x,y) = (0,0,0)$ to find Ker T, we have

$$(x+y,\ x-2y,\ 3x+y) = (0,0,0)$$

$$x+y=0,\quad x-2y=0,\quad 3x+y=0$$

The only solution is $x=0$, $y=0$; hence T is non singular

(b) Although T is nonsingular, it is not invertible, since R^2 and R^3 have different dimension (for invertibility dimension should be same). so T^{-1} does not exist.

Application in Recurrence Relation:

Several features of the system are each measured at discrete time intervals, producing a sequence of vectors $x_0, x_1, x_2, \ldots$. The entries in x_k provides the information about the state of system at the time of the k^{th} measurement.

If there exist a ~~transformation~~ T such that $x_1 = Tx_0 = Ax_0$, where A is some matrix corresponds the transformation T, and, in general

$$x_{k+1} = T x_k \quad , \quad k = 0, 1, 2, \ldots \qquad \text{--- (1)}$$

then equation (1) is called recurrence relation or linear difference equation.

A subject of interest to demographs is the movement of populations or groups of people from one region to another. The simple model here considers the change in population of a certain city and its surrounding suburbs over a period of years.

Fix the initial year (say 2014), and denote the population of the city and suburbs that year by r_0 and s_0, respectively.

let x_0 be the population vector

$$x_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

For 2015 and the subsequent years, denote the population of the city

$$x_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \ \text{---}$$

Now suppose demographic study shows that each years about 5% of the city's population moves to suburbs (and 95% remains in city), while 3% of the suburb's population moves to city

After 1 year, the original r_0 persons in the city are now distributed between city and suburbs as

$$\begin{bmatrix} .95 r_0 \\ .05 r_0 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} \quad \text{---} \quad (2)$$

and s_0 person in suburbs in 2014 are after 1 year as $s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix}$ — (3)

The vector (2) and (3) give the population in 2015.

Thus

$$x_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

That is $x_1 = M x_0$ ~~or~~ where M is migration matrix — (4)

Eq^n (4) represents a linear transformation.

i.e. $x_1 = T x_0 = M x_0$.

Similarly in year 2016, we have

$$x_2 = T x_1 = T(T x_0) = (T \circ T) x_0 = M x_1.$$

In general $x_{k+1} = T x_k = M x_k$

OR $x_{k+1} = (\underbrace{T \circ T \circ T \circ \cdots \circ T}_{k\text{-times}}) x_0$

The seq^n $\{x_0, x_1, \ldots, x_k, \ldots\}$ describes the population in city or suburban region over a period of years.

Example: Compute the population of the region just described for the years 2015 and 2016, given the population in 2014 was 600,000 in the city and 400,000 in the suburbs.

Sol^n:- The initial population in 2014
(let) $x_0 = \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix}$, for 2015,

define $x_1 = T(x_0) = Ax_0$

where $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$

implies $x_1 = Ax_0 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix}$

$= \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix}$

Now for year 2016

$x_2 = Ix_1 = T(Tx_0) \simeq (T \circ T)(x_0)$

$= A[T(x_0)]$

$= \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix}$

$= \begin{bmatrix} 565,440 \\ 434,560 \end{bmatrix}$

Hence the population in 2016 i.e. 565,440 in city
434,560 in suburbs

Applications

There are number of mathematical models which are linear, in Business, Science, Engineering, etc. Linear models are important as natural phenomena are often liner or nearly linear when the variable involved are held within reasonable bounds. Also, linear models are more easily adapted for computer application (or calculation) than are complex non-linear model.

Linear Equations and Electrical Networks:-

Current flow in a simple electrical network can be described by a system of linear eqⁿs. A voltage source such as a battery forces a current of elections to flow through the network. When the current passes through a resistor, some of the voltage is 'used up'; by 'Ohm's law, this 'voltage drop' across a resistor is given by

$$V = RI$$

where the voltage V is measured in volts, the resistance R in Ohms and the current flow I in 'amperes'.

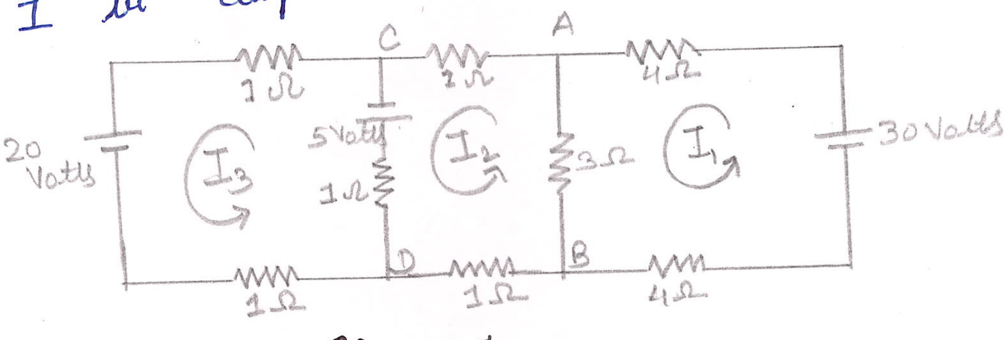


Figure 1

The network in figure 1 contains three closed loops. The current flowing in loop 1, 2, 3 are denoted by I_1, I_2 and I_3, respectively. The designated directions of such loop current are arbitrary. If a current turns out to be -ve, then the actual direction of current flow is opposite to that chosen in the figure. If the current direction shown is away from the +ve side of a battery (⊣⊢) around to the -ve side, the voltage is positive; otherwise, the voltage is -ve.

The current flow in a loop is governed by the following rule:

Kirchoff's Voltage Law:

The algebric sum of the RI voltage drops in one direction around a loop equals the algebric sum of the voltage sources in the same direction around the loop.

Example 1 Determine the loop currents in the network in figure 1.

Solution: For loop 1, the current I_1 flows through three resistors, and the sum of the RI voltage drops is

$$4I_1 + 4I_1 + 3I_1 = (4+4+3)I = 11I_1$$

Current from loop 2 also flows in part of loop 1, through the short branch between A and B. The associated RI drop there is $3I_2$ volts.

However, the current direction for the branch
AB in loop 1 is opposite to that chosen for
the flow in loop 2, so the algebric sum of
all RI drops for loop 1 is $11I_1 - 3I_2$.

Since the voltage in loop 1 is +30 volts,
Kirchhoff's Law implies that

$$11I_1 - 3I_2 = 30$$

The equation for loop 2 is

$$-3I_1 + 6I_2 - I_3 = 5$$

The term $-3I_1$ comes from the flow of the loop-1
current through the bronch AB (with a -ve voltage)
drop because the current flow there is opposite to
the flow in loop 2). The term $6I_2$ is the sum
of all resistances in loop 2, multiplied by the
loop current. The term $-I_3 = -1I_3$ comes from
the loop-3 current flowing through the 1-ohm
resistor in branch CD, in the direction opposite
to the flow in loop 2. The loop-3 equation is

$$-I_2 + 3I_3 = -25$$

Note that the 5-volt battery in branch CD is
Counted as part of both loop 2 and loop 3,
but it is -5 volts for loop 3 because of the direction
chosen for the current in loop 3. The 20-volt
battery is -ve for the same reason.

The loop currents are found by solving
the system

$$\left.\begin{aligned} 11I_1 - 3I_2 \quad &= 30 \\ -3I_1 + 6I_2 - I_3 &= 5 \\ -I_2 + 3I_3 &= -25 \end{aligned}\right\} \text{——} (1)$$

Here $A = \begin{bmatrix} 11 & -3 & 0 \\ -3 & 6 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $|A| = 160$

$A^{-1} = \frac{1}{160} \begin{bmatrix} 17 & 9 & 3 \\ 9 & 33 & 11 \\ 3 & 11 & 57 \end{bmatrix}$ $X = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$, $B = \begin{bmatrix} 30 \\ 5 \\ -25 \end{bmatrix}$

$X = A^{-1} B = \frac{1}{160} \begin{bmatrix} 17 & 9 & 3 \\ 9 & 33 & 11 \\ 3 & 11 & 57 \end{bmatrix} \begin{bmatrix} 30 \\ 5 \\ -25 \end{bmatrix}$

$X = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix}$

so $I_1 = 3$ amps, $I_2 = 1$ amps, $I_3 = -8$ amps.

Q1 Show that the mapping $T: R^2 \longrightarrow R^3$ defined as $T(x,y) = (x+y,\ x-y,\ y)$ is a linear transformation.

Q2 Let V be the vector space of real n-square matrices, and let M be a fixed non zero matrix in V. Show that $T(A) = AM + MA$ is linear transformation.

Q3 Consider the mapping $T: R^3 \longrightarrow R^2$ $T(x,y,z) = (x+1,\ y+z)$ is not linear Transformation

Q4 Let $A: R^2 \longrightarrow R^2$ be defined by the matrix

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$$A = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$$

Find the matrix B representing A relative to the basis $S = \{u_1, u_2\} = \{(1,3), (2,8)\}$.

Q5 Find the matrix representing each linear Transformation T on R^3 relative to the usual basis of R^3

(a) $T(x,y,z) = (z,\ y+z,\ x+y+z)$

(b) $T(x,y,z) = (2x-7y-4z,\ 3x+y+4z,\ 6x-8y+z)$

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Q.6 Let V be the vector space of functions with $S = \{\sin t, \cos t, e^{3t}\}$, and let $D: V \to V$ be the differential operator by $D(f(t)) = \frac{d}{dt}(f(t))$. Find the matrix D in the basis S.

Q7 A linear transformation T on R^2 defined by $T(x, y) = (5x - y, 2x + y)$ and the following bases of R^2: P-225

$E = \{e_1, e_2\} = \{(1,0), (0,1)\}$

$S = \{u_1, u_2\} = \{(1,4), (2,7)\}$

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Q8 The vectors $u_1 = (1,1,0)$, $u_2 = (0,1,1)$, $u_3 = (1,2,2)$. Find the coordinates of an arbitrary vector $v = (a,b,c)$ relative to the basis S.

Q9. let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & -4 \\ 1 & -2 & 2 \end{bmatrix}$. Find the matrix B 226 that represents the linear operator A relative to the basis

$S = \{u_1, u_2, u_3\} = \{(1,1,0), (0,1,1), (1,2,2)\}$

Q10 Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$. Find the matrix B that represents the linear operator A relative to the basis $S = \{u_1, u_2\} = \{(1,3), (2,5)\}$ 225

Q11 Consider the following bases of R^2 222

$S = \{u_1, u_2\} = \{(1,-2), (3,-4)\}$ and

$S' = \{v_1, v_2\} = \{(1,3), (3,8)\}$

(a) Find the coordinate of $v = (a, b)$ relative to the basis S.

(b) Find the change-of-basis matrix P from S to S'.

(c) Find the coordinates of $v = (a, b)$ relative to the basis S'

(d) Find the change-of-basis matrix Q from S' to S.

Answers

A-4 $\quad B = \begin{bmatrix} -6 & -28 \\ 4 & 15 \end{bmatrix}$

A-5(a) $\quad T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

A 5(b) $\quad T = \begin{bmatrix} 2 & -7 & -4 \\ 3 & 1 & 4 \\ 6 & -8 & 1 \end{bmatrix}$

A-6 $\quad D = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

A-7 $\quad A = \begin{bmatrix} 5 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 \\ -2 & 1 \end{bmatrix}$

A-8 $\quad [v]_S = [\,b-c,\ -2a+2b-c,\ a-b+c\,]$

A-9 $\quad B = \begin{bmatrix} 8 & 1 & 3 \\ 7 & -6 & -11 \\ -5 & 3 & 6 \end{bmatrix}$

A-10 $\quad B = \begin{bmatrix} -53 & -89 \\ 32 & 54 \end{bmatrix}$

A-11 (a) $(a,b)_S = [-2a-\frac{3}{2}b,\ a+\frac{1}{2}b]$ $\quad$ (c) $[a,b]_{S'} = [-8a+3b,\ 3a-b]$

(b) $P = \begin{bmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{bmatrix}$ $\quad$ (d) $Q = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$