EX transformada Z

Example 10.25

Consider an LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]. \tag{10.102}$$

Applying the z-transform to both sides of eq. (10.102), and using the linearity property set forth in Section 10.5.1 and the time-shifting property presented in Section 10.5.2, we obtain

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z),$$

or

$$Y(z) = X(z)\left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]. \tag{10.103}$$

From eq. (10.96), then,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}. \tag{10.104}$$

This provides the algebraic expression for $H(z)$, but not the region of convergence. In fact, there are two distinct impulse responses that are consistent with the difference equation (10.102), one right sided and the other left sided. Correspondingly, there are two different choices for the ROC associated with the algebraic expression (10.104). One, $|z| > 1/2$, is associated with the assumption that $h[n]$ is right sided, and the other, $|z| < 1/2$, is associated with the assumption that $h[n]$ is left sided.

Consider first the choice of ROC equal to $|z| > 1$. Writing

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right)\frac{1}{1 - \frac{1}{2}z^{-1}},$$

we can use transform pair 5 in Table 10.2, together with the linearity and time-shifting properties, to find the corresponding impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

For the other choice of ROC, namely, $|z| < 1$, we can use transform pair 6 in Table 10.2 and the linearity and time-shifting properties, yielding

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n].$$

In this case, the system is anticausal ($h[n] = 0$ for $n > 0$) and unstable.

P22.3

Shown in Figure P22.3 is the pole-zero plot for the z-transform $X(z)$ of a sequence $x[n]$.

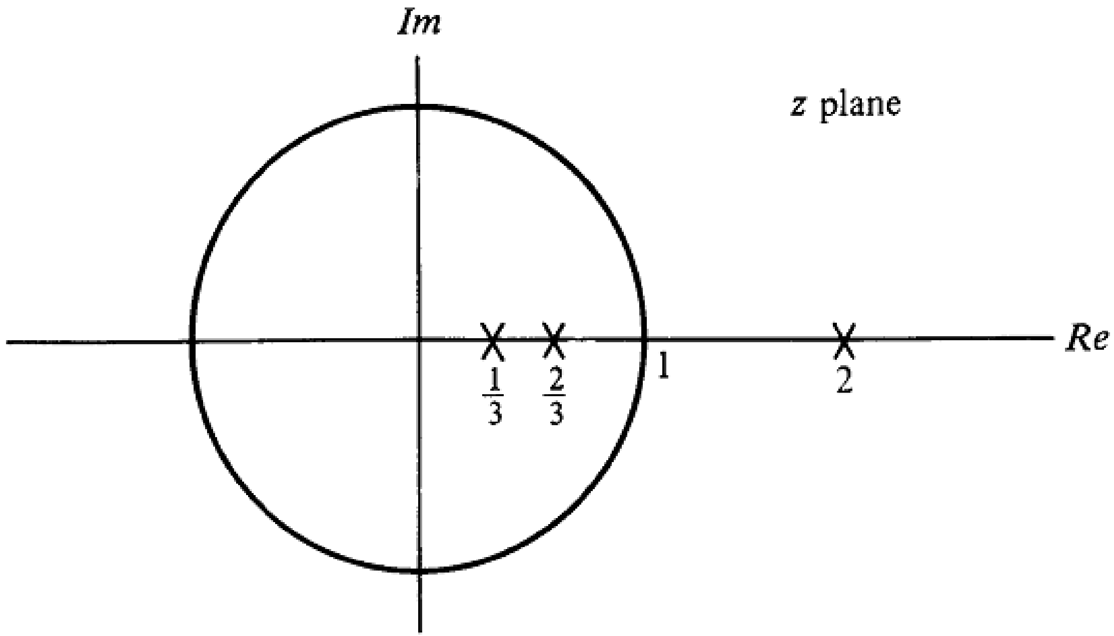


Figure P22.3

Determine what can be inferred about the associated region of convergence from each of the following statements.

(a) $x[n]$ is right-sided.

(b) The Fourier transform of $x[n]$ converges.

(c) The Fourier transform of $x[n]$ does not converge.

(d) $x[n]$ is left-sided.

S22.3

(a) Since $x[n]$ is right-sided, the ROC is given by $|z| > \alpha$. Since the ROC cannot include poles, for this case the ROC is given by $|z| > 2$.

(b) The statement implies that the ROC includes the unit circle $|z| = 1$. Since the ROC is a connected region and bounded by poles, the ROC must be

$$\tfrac{2}{3} < |z| < 2$$

(c) For this situation there are three possibilities:

(i) $|z| < \frac{1}{3}$

(ii) $\frac{1}{3} < |z| < \frac{2}{3}$

(iii) $|z| > 2$

(d) This statement implies that the ROC is given by $|z| < \frac{1}{3}$.

P22.5

Consider the pole-zero plot of $H(z)$ given in Figure P22.5, where $H(a/2) = 1$.

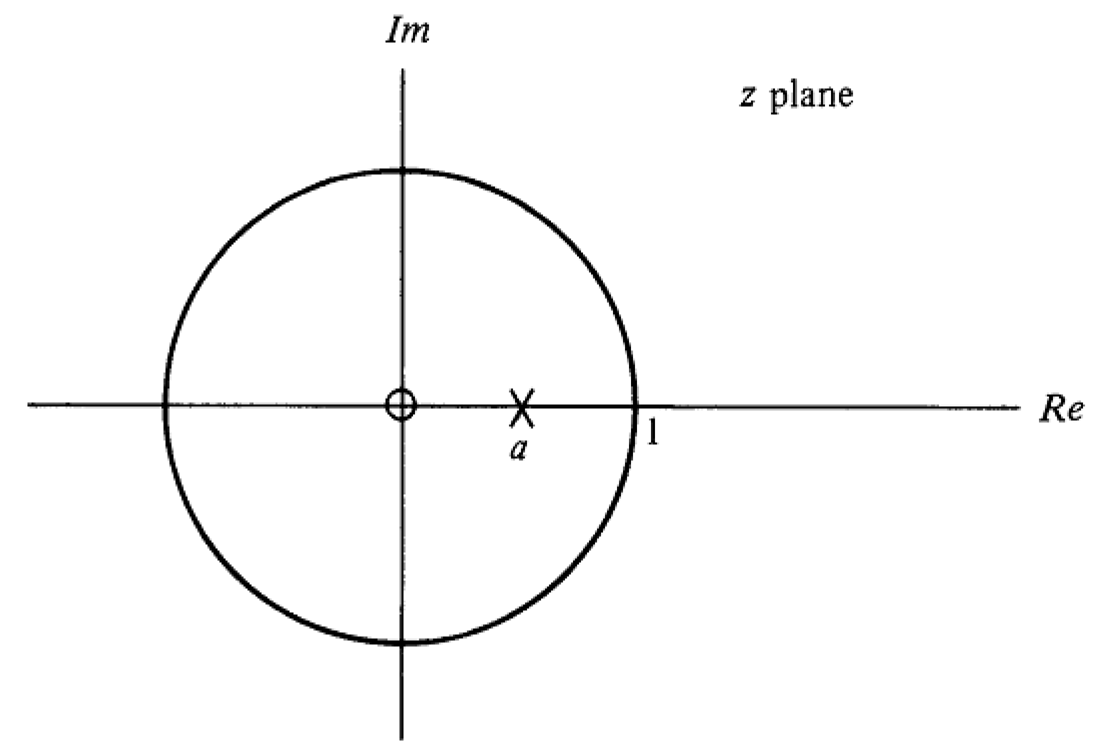


Figure P22.5

(a) Sketch $|H(e^{j\Omega})|$ as the number of zeros at $z = 0$ increases from 1 to 5.

(b) Does the number of zeros affect $\sphericalangle H(e^{j\Omega})$? If so, specifically in what way?

(c) Find the region of the z plane where $|H(z)| = 1$.

S22.5

Consider the pole-zero plot of $H(z)$ given in Figure S22.5-1, where $H(a/2)$

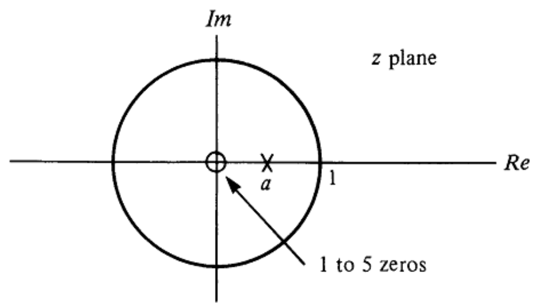


Figure S22.5-1

(a) When $H(z) = z/(z - a)$, i.e., the number of zeros is 1, we have

$$H(e^{j\Omega}) = \frac{\cos \Omega + j \sin \Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega},$$

and we can plot $|H(e^{j\Omega})|$ as in Figure S22.5-2.

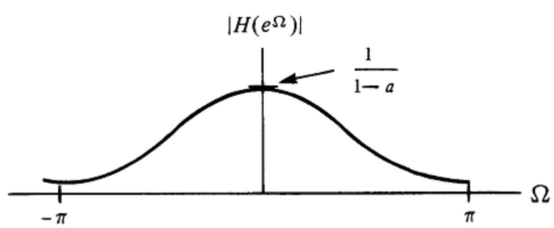


Figure S22.5-2

When $H(z) = z^2/(z - a)$, i.e., the number of zeros is 2, we have

$$H(e^{j\Omega}) = \frac{\cos 2\Omega + j \sin 2\Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega}$$

Hence, we see that the magnitude of $H(e^{j\Omega})$ does not change as the number of zeros increases.

(b) For one zero at $z = 0$, we have

$$H(z) = \frac{z}{z - a},$$

$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

We can calculate the phase of $H(e^{j\Omega})$ by [$\Omega - \sphericalangle$ (denominator)]. For two zeros at 0, the phase of $H(e^{j\Omega})$ is [$2\Omega - \sphericalangle$ (denominator)]. Hence, the phase changes by a linear factor with the number of zeros.

(c) The region of the z plane where $|H(z)| = 1$ is indicated in Figure S22.5-3.

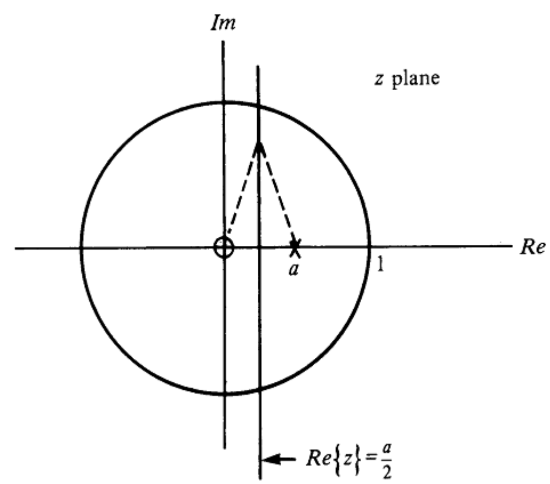


Figure S22.5-3

• EXERCÍCIO 0.4 (Op. 10.7)

Suponha que a transformada Z de x[n] seja $X(z) = \frac{1-\frac{1}{4}z^{-2}}{(1+\frac{1}{4}z^{-2})(1+\frac{5}{4}z^{-1}+\frac{3}{8}z^{-2})}$. Quantas ROCs diferentes correspondem a X(z).

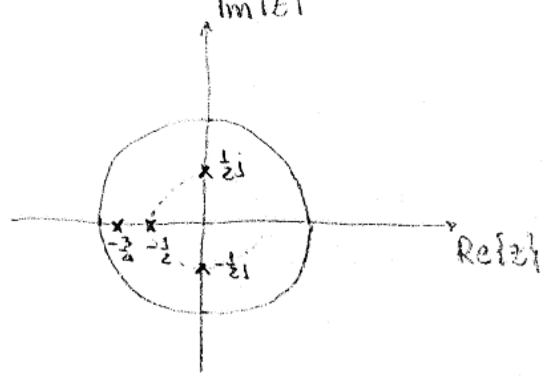
Ex 10.7

4. (Exp 10.7)

$$X(z) = \mathcal{Z}\{x[n]\} = \frac{(1-\frac{1}{4}z^{-2})}{(1+\frac{1}{4}z^{-2})(1+\frac{5}{4}z^{-1}+\frac{3}{8}z^{-2})}$$

→ polos de $X(z)$:

$$z_{p1} = -\frac{1}{2}j, \quad z_{p2} = \frac{1}{2}j, \quad z_{p3} = -\frac{1}{2} \text{ e } z_{p4} = -\frac{3}{4}$$



Uma RDC não pode incluir um pólo ⇒

RDC$_1$: $|z| < 1/2$ sinal aberto à esquerda

RDC$_2$: $1/2 < |z| < 3/4$ sinal bilateral

RDC$_3$: $|z| > 3/4$ sinal aberto à direita

10.35. Consider an LTI system with input $x[n]$ and output $y[n]$ for which

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n].$$

The system may or may not be stable or causal.

By considering the pole-zero pattern associated with the preceding difference equation, determine three possible choices for the unit sample response of the system. Show that each choice satisfies the difference equation.

10.20. Consider a system whose input $x[n]$ and output $y[n]$ are related by

$$y[n-1] + 2y[n] = x[n].$$

(a) Determine the zero-input response of this system if $y[-1] = 2$.
(b) Determine the zero-state response of the system to the input $x[n] = (1/4)^n u[n]$.
(c) Determine the output of the system for $n \geq 0$ when $x[n] = (1/4)^n u[n]$ and $y[-1] = 2$.

10.20. Applying the unilateral z-transform to the given difference equation, we have

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = \mathcal{X}(z).$$

(a) For the zero-input response, assume that $x[n] = 0$. Since we are given that $y[-1] = 2$,

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = 0 \Rightarrow \mathcal{Y}(z) = \frac{-1}{1 + (1/2)z^{-1}}.$$

Taking the inverse unilateral z-transform,

$$y[n] = -\left(-\frac{1}{2}\right)^n u[n].$$

(b) For the zero-state response, set $y[-1] = 0$. Also, we have

$$\mathcal{X}(z) = \mathcal{UZ}\{(1/2)^n u[n]\} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}.$$

Therefore,

$$\mathcal{Y}(z) = \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)\left(\frac{2}{2 + z^{-1}}\right).$$

We use partial fraction expansion followed by the inverse unilateral z-transform to obtain

$$y[n] = \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{6}\left(\frac{1}{4}\right)^n u[n].$$

(c) The total response is the sum of the zero-state and zero-input responses. This is

$$y[n] = -\frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{6}\left(\frac{1}{4}\right)^n u[n].$$

10.35. Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z - \frac{5}{2} + z^{-1}} = \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}}.$$

The partial fraction expansion of $H(z)$ is

$$H(z) = \frac{-2/3}{1 - \frac{1}{2}z^{-1}} + \frac{2/3}{1 - 2z^{-1}}.$$

If the ROC is $|z| > 2$, then

$$h_1[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(2)^n u[n].$$

If the ROC is $1/2 < |z| < 2$, then

$$h_2[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{3}(2)^n u[-n-1].$$

If the ROC is $|z| < 1/2$, then

$$h_3[n] = \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{2}{3}(2)^n u[-n-1].$$

For each $h_i[n]$, we now need to show that if $y[n] = h_i[n]$ in the difference equation, then $x[n] = \delta[n]$. Consider substituting $h_1[n]$ into the difference equation. This yields

$$\begin{aligned} \frac{2}{3} \quad & \left(\tfrac{1}{2}\right)^{n-1} u[n-1] - \tfrac{2}{3}(2)^{n-1}u[n-1] - \tfrac{5}{3}\left(\tfrac{1}{2}\right)^n u[n] \\ & + \tfrac{5}{3}(2)^n u[n] + \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n+1} u[n+1] - \tfrac{2}{3}(2)^{n+1}u[n+1] = x[n] \end{aligned}$$

Then,

$$\begin{aligned} x[n] &= 0, \qquad \text{for } n < -1, \\ x[-1] &= 2/3 - 2/3 = 0, \\ x[n] &= 0, \qquad \text{for } n > 0. \end{aligned}$$

It follows that $x[n] = \delta[n]$. It can similarly be shown that $h_2[n]$ and $h_3[n]$ satisfy the difference equation.

10.36. Consider the linear, discrete-time, shift-invariant system with input $x[n]$ and output $y[n]$ for which

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n].$$

The system is stable. Determine the unit sample response.

Ep 10.36

linear, DT, invariante no deslocamento

$y[n-1] - \frac{10}{3} y[n] + y[n+1] = x[n]$

sistema estável.

resposta ao impulso?

$\Rightarrow \mathcal{Z}\left\{ y[n-1] - \frac{10}{3} y[n] + y[n+1] \right\} = \mathcal{Z}\{ x[n] \}$

$z^{-1} Y(z) - \frac{10}{3} Y(z) + z\, Y(z) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z}$$

$$\frac{100 - 36}{9} \qquad \frac{64}{9}$$

$$\frac{+\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4}}{2} \qquad \frac{+\frac{10}{3} \pm \frac{8}{3}}{2}$$

$$= \frac{z^{-1}}{1 - \frac{10}{3} z^{-1} + z^{-2}} = \frac{A}{1 - 3z^{-1}} + \frac{B}{1 - \frac{1}{3} z^{-1}}$$

$$\frac{18}{3} \cdot \frac{1}{2} = 3 \qquad \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

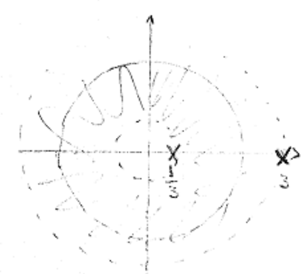
em que $A = (1 - 3z^{-1}) \cdot H(z) \Big|_{z=3} = \frac{z^{-1}}{1 - \frac{1}{3} z^{-1}} \Big|_{z=3} = \frac{1/3}{1 - 1/9} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{9}^{3}}{8} = \frac{3}{8}$

$B = (1 - 1/3\, z^{-1}) \cdot H(z) \Big|_{z=1/3} = \frac{z^{-1}}{1 - 3z^{-1}} \Big|_{z=1/3} = \frac{3}{1 - 3 \cdot 3} = \frac{3}{-8} = -\frac{3}{8}$

Como $H(z)$ é estável $\Rightarrow$ RDC: $1/3 < |z| < 3$

$\Downarrow$

RDC inclui $z = 1$



$$\Rightarrow h[n] = -\frac{3}{8} \left(\frac{1}{3}\right)^n u[n] - \frac{3}{8} (3)^n u[-n-1]$$

10.37. The input $x[n]$ and output $y[n]$ of a causal LTI system are related through the block-diagram representation shown in Figure P10.37.

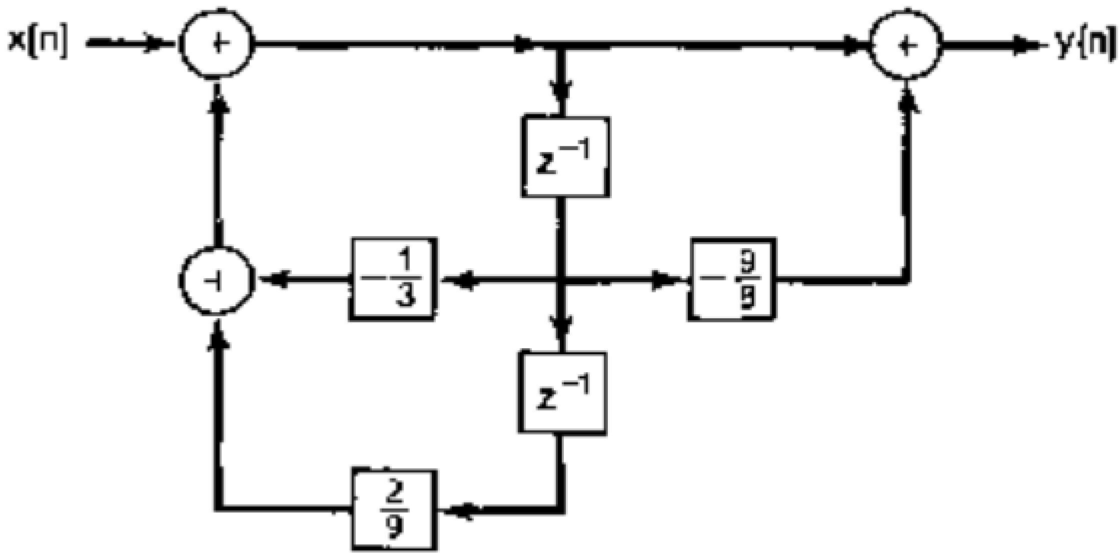
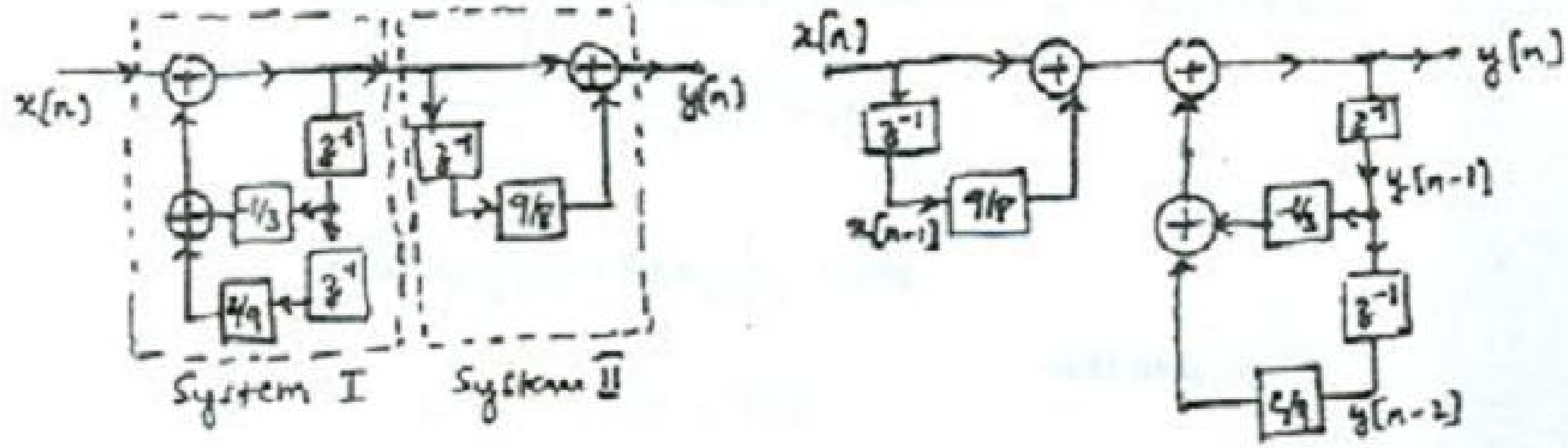


Figure P10.37

(a) Determine a difference equation relating $y[n]$ and $x[n]$.
(b) Is this system stable?

10.37. (a) The block-diagram may be redrawn as shown in part (a) of the figure below. This may be treated as a cascade of the two systems shown within the dotted lines in Figure S10.37. These two systems may be interchanged as shown in part (b) of the figure Figure S10.37 without changing the system function of the overall system. From the figure below, it is clear that

$$y[n] = x[n] + \frac{9}{8}x[n-1] - \frac{1}{3}y[n-1] + \frac{2}{9}y[n-2].$$



(b) Taking the z-transform of the above difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{9}{8}z^{-1}}{1 + \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2}} = \frac{1 + \frac{9}{8}z^{-1}}{(1 + \frac{2}{3}z^{-1})(1 - \frac{1}{3}z^{-1})}.$$

$H(z)$ has poles at $z = 1/3$ and $z = -2/3$. Since the system is causal, the ROC has to be $|z| > 2/3$. The ROC includes the unit circle and hence the system is stable.

10.45. Determine which of the following z-transforms could be the transfer function of a discrete-time linear system that is not necessarily stable, but for which the unit sample response is zero for $n < 0$. State your reasons clearly.

(a) $\dfrac{(1 - z^{-1})^2}{1 - \frac{1}{2}z^{-1}}$ **(b)** $\dfrac{(z - 1)^2}{z - \frac{1}{2}}$

(c) $\dfrac{(z - \frac{1}{4})^5}{(z - \frac{1}{2})^6}$ **(d)** $\dfrac{(z - \frac{1}{4})^6}{(z - \frac{1}{2})^5}$

Ex 10.45

Determinar quais das seguintes TZ pode ser a função de transf de um sist linear (não necessariamente estável) mas com $h(n)=0 \ \forall n<0$

a) $\dfrac{(1-z^{-1})^2}{1-\frac{1}{2}z^{-1}}$ b) $\dfrac{(z-1)^2}{z-\frac{1}{2}}$

c) $\dfrac{(z-\frac{1}{4})^5}{(z-\frac{1}{2})^6}$ d) $\dfrac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5}$

SOL

Nota: para qualquer sinal $x[n]$ eu posso escrever $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k] = \sum_{k=-\infty}^{+\infty} a_k \delta$

$\Rightarrow X[z] = \sum_{k=-\infty}^{+\infty} a_k z^k$

Se eu consigo escrever $H(z)$ dessa forma, é somente verificar que $a_k = 0 \ \forall k>0$

a) $H(z) = \dfrac{1 - 2z^{-1} + z^{-2}}{1-\frac{1}{2}z^{-1}} \Rightarrow$

$$\begin{array}{l|l} 1 - 2z^{-1} + z^{-2} & 1-\frac{1}{2}z^{-1} \\ \hline -1 + \frac{1}{2}z^{-1} & 1 - \frac{3}{2}z^{-1} + \frac{1}{4}z^{-2} \\ \hline -\frac{3}{2}z^{-1} + z^{-2} & \\ +\frac{3}{2}z^{-1} - \frac{3}{4}z^{-2} & \\ \hline +\frac{1}{4}z^{-2} & \end{array}$$

$\Rightarrow H(z) = 1 - \frac{3}{2}z^{-1} + \frac{1}{4}z^{-2} +$

$H(z) = \sum_{k=0}^{+\infty} a_k z^{-k}$

$a_0 = 1$
$a_1 = -\frac{3}{2}$
$a_2 = \frac{1}{4}$

$\Rightarrow x[n] = \delta[n] - \frac{3}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$ ✓

b) $H(z) = \dfrac{(z-1)^2}{z-\frac{1}{2}} = \dfrac{z^2(1-z^{-1})^2}{z(1-\frac{1}{2}z^{-1})} = z \cdot \dfrac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})}$

$\Rightarrow$ Olhando a solução precedente $\Rightarrow H(z) = z - \frac{3}{2} + \frac{1}{4}z^{-1}$

$\Rightarrow x[n] = \delta[n+1] - \frac{3}{2}\delta[n] + \frac{1}{4}\delta[n-1]$ ✗

c) $H(z) = \dfrac{(z-\frac{1}{2})^5}{(z-\frac{1}{4})^6} = \dfrac{z^{-1}(1-\frac{1}{2}z^{-1})^5}{(1-\frac{1}{4}z^{-1})^6} = \dfrac{\alpha_1 z^{-1} + \alpha_2 z^{-2} + \quad + \alpha_6 z^{-6}}{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \quad + \beta_6 z^{-6}} = \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3$

$\Rightarrow x[n] = \gamma_1 \delta[n-1] + \gamma_2 \delta[n-2] + \dots$ ✓

d) ✗ por motivos especulares

Op 10.46 (MODIFICADO, somente (a) e (b))
LTI system

$$x[n] = \delta[n] - e^{-4\alpha}\delta[n-4] \qquad 0<\alpha<1$$

a) Find $H_1(z) = \dfrac{X(z)}{S(z)}$ + polos e zeros + RDC

b) Find the LTI system que recupera S a partir de X

⇒ seja $H_2(z) = \dfrac{Y(z)}{X(z)}$ onde $y[n] = s[n]$

⇒ encontrar polos + zeros e ~~estabilidade~~ as possíveis RDC e discutir convergência

SOL

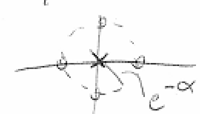
a)

$$X(z) = S(z) - e^{-4\alpha} S(z) z^{-4} \Rightarrow$$

$$H(z) = \frac{X(z)}{S(z)} = 1 - e^{-4\alpha} z^{-4} = \frac{z^4 - e^{-4\alpha}}{z^4}$$

⇒ polos: 4 em $|z| = 0$

zeros: 4 em: $e^{-\alpha}, -e^{-\alpha}, je^{-\alpha}, -je^{-\alpha}$



RDC: $|z| > 0$

b)

Note que, por definição $H_2(z) = H_1(z)^{-1} = \dfrac{z^4}{z^4 - e^{-4\alpha}}$

⇒ polos: 4 em $e^{-\alpha}, -e^{-\alpha}, je^{-\alpha}, -je^{-\alpha}$

zeros: 4 em $z = 0$

RDC:
- $|z| > e^{-\alpha}$ ⇒ causal (direita) estável (porque $e^{-\alpha} < 1$)
- $|z| < e^{-\alpha}$ ⇒ não causal (esquerda) instável (" ")

10.59. Consider the digital filter structure shown in Figure P10.59.

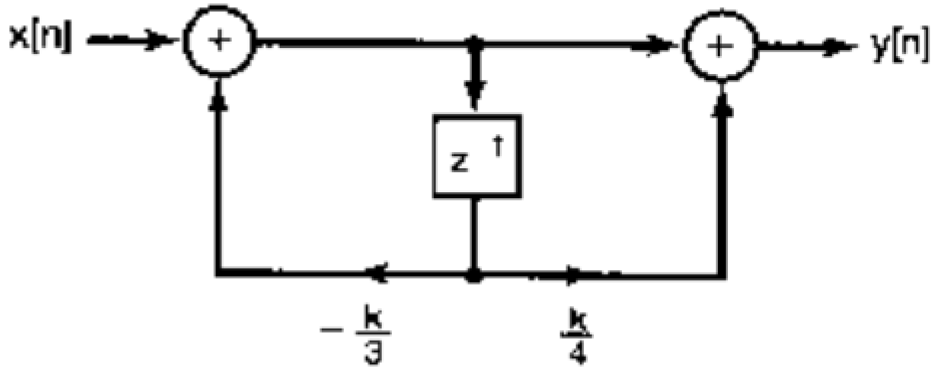


Figure P10.59

(a) Find $H(z)$ for this causal filter. Plot the pole-zero pattern and indicate the region of convergence.
(b) For what values of the k is the system stable?
(c) Determine $y[n]$ if $k = 1$ and $x[n] = (2/3)^n$ for all n.

0.59. (a) From Figure S10.59, we have

$$W_1(z) = X(z) - \frac{k}{3}z^{-1}W_1(z) \quad \Rightarrow \quad W_1(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}}.$$

Also,

$$W_2(z) = -\frac{k}{4}z^{-1}W_1(z) = -X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Therefore, $Y(z) = W_1(z) + W_2(z)$ will be

$$Y(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}} - X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Finally,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Since $H(z)$ corresponds to a causal filter, the ROC will be $|z| > |k|/3$.

(b) For the system to be stable, the ROC of $H(z)$ must include the unit circle. This is possible only if $|k|/3 < 1$. This implies that $|k|$ has to be less than 3.

(c) If $k = 1$, then

$$H(z) = \frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{3}z^{-1}}.$$

The response to $x[n] = (2/3)^n$ will be of the form

$$y[n] = x[n]H(2/3) = \frac{5}{12}(2/3)^n.$$