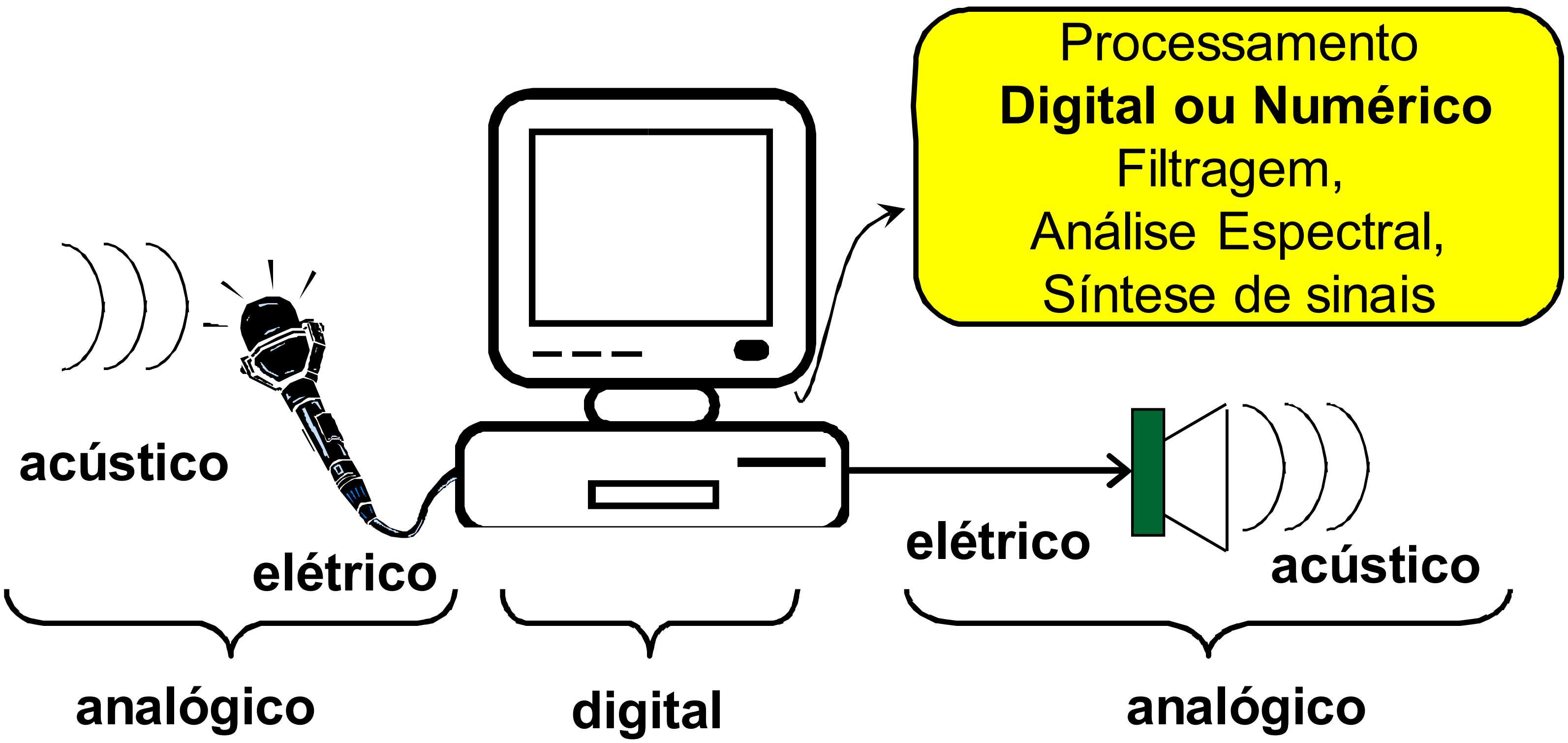
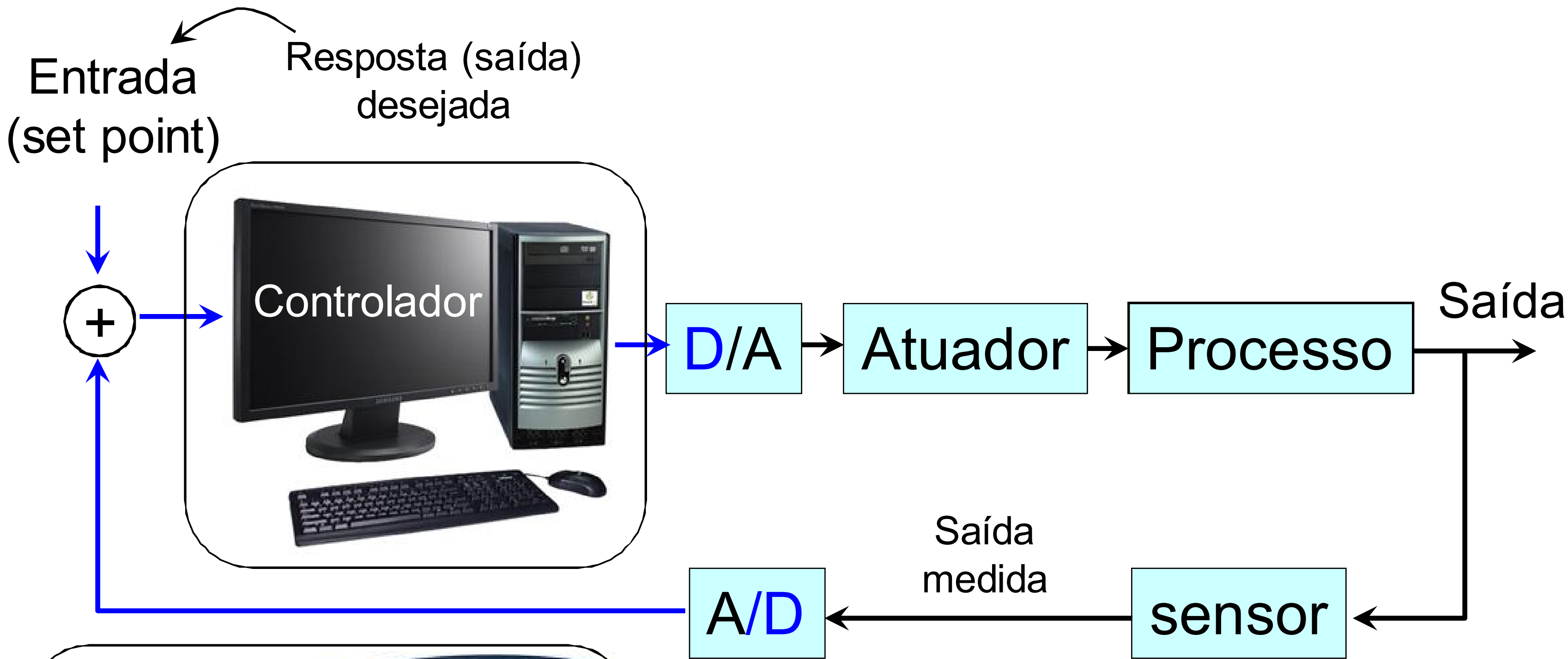
Modulo 06
Amostragem

Processamento Digital de Sinais analógicos



Exemplo comum do processamento de áudio em computadores

Controle digital

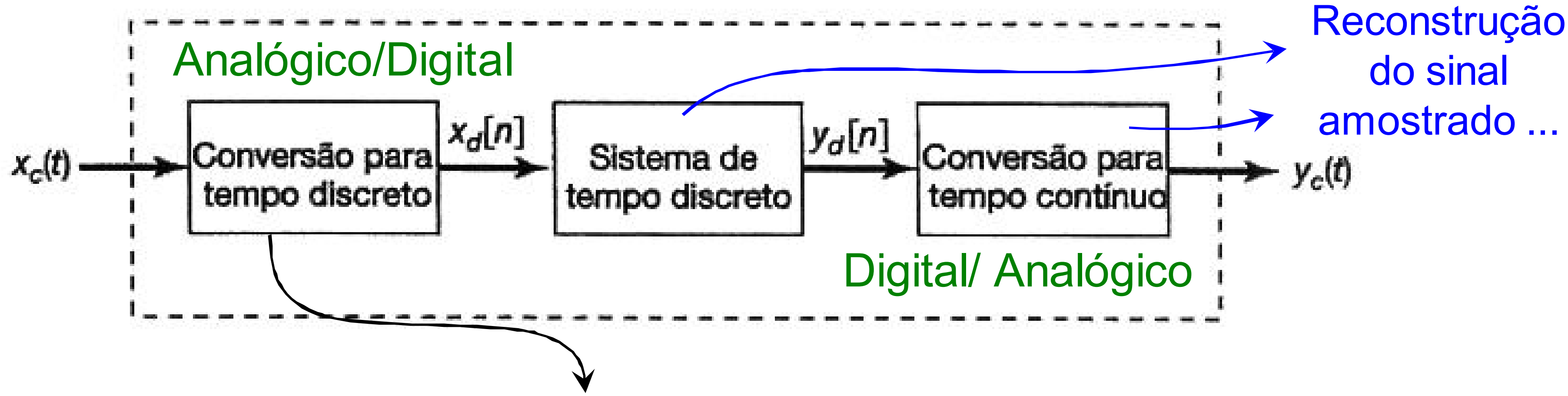


Tipo de sinal:

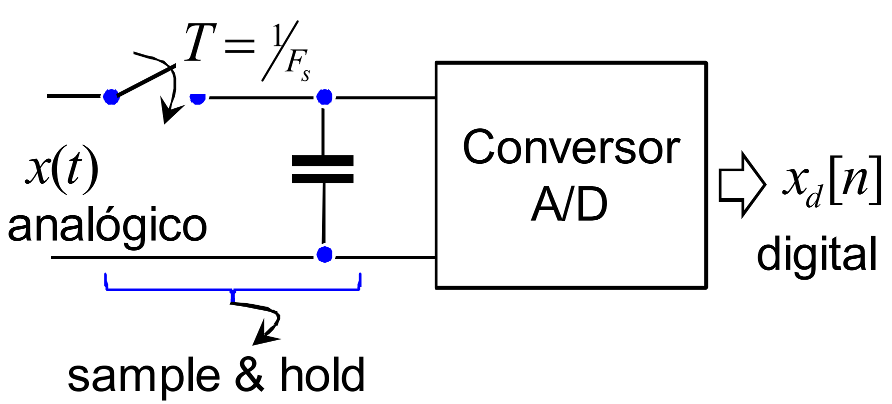
D = digital

A = Analógico

Amostragem de sinais de tempo contínuo



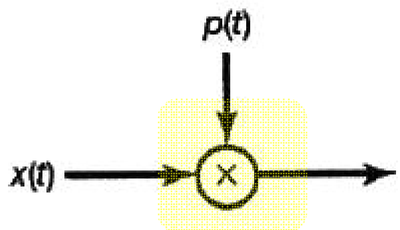
Discretização no tempo	*Quantização da amplitude*



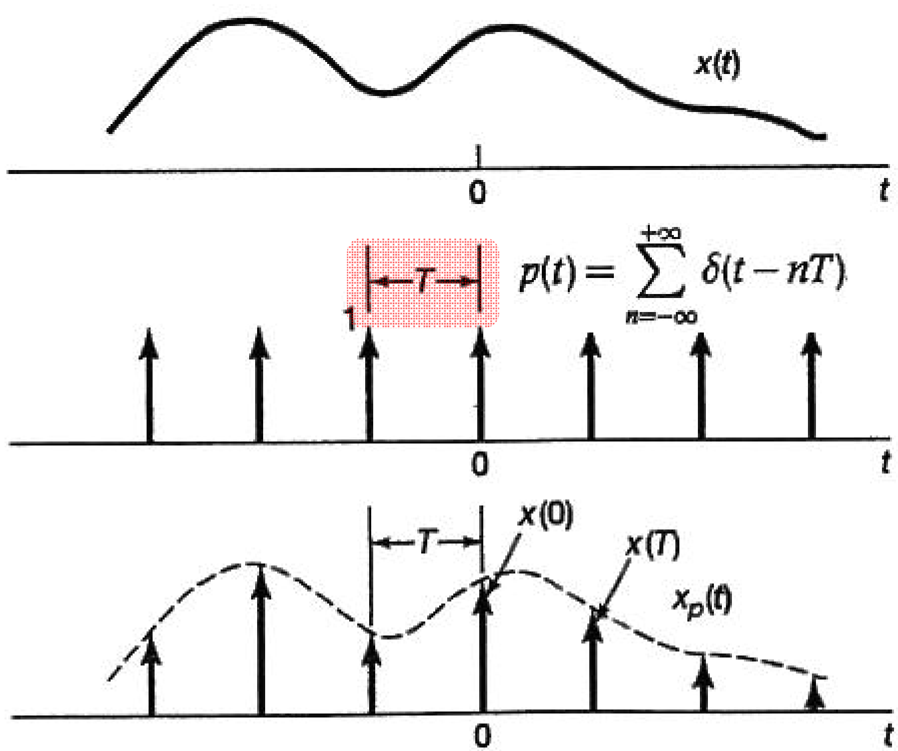
*16 bits = 2^{16} = 64*1024 níveis*

Placas de som, interfaces USB, etc.

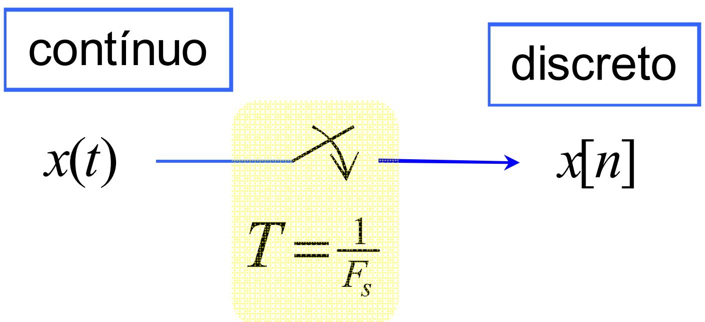
7.1.1 Amostragem de sinais de tempo contínuo



$$x_p(t) = x(t) \cdot p(t)$$



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

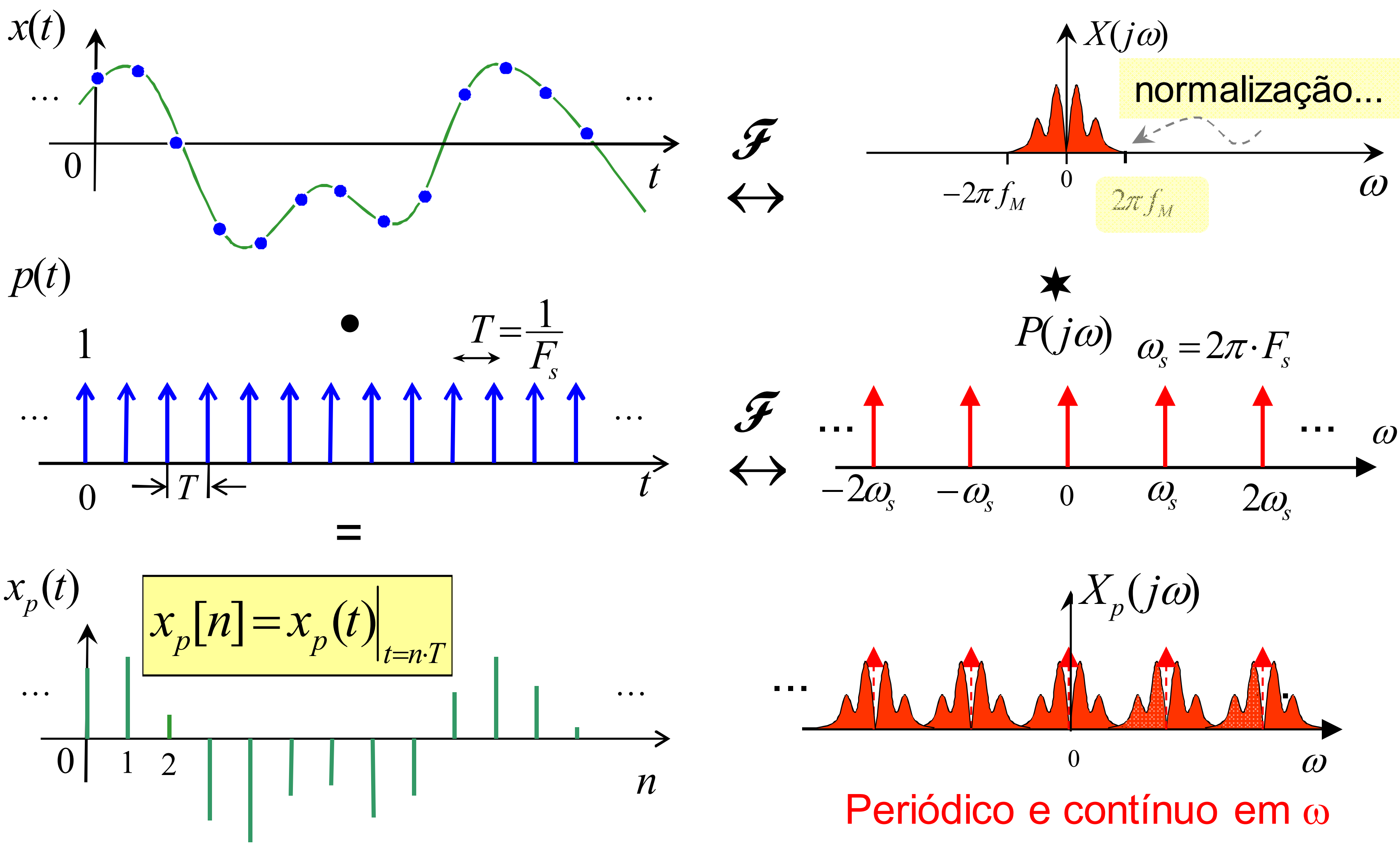


$$T = \frac{1}{F_s}$$

T = Taxa de amostragem

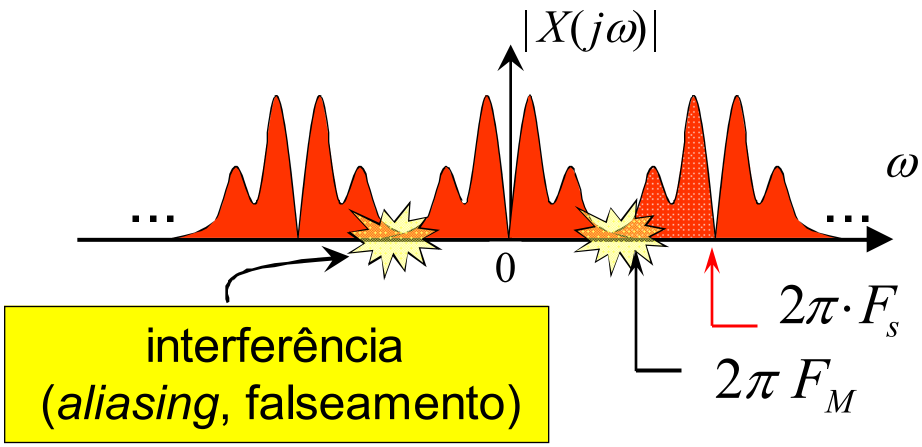
$$x[n] = x_p(t)\Big|_{t=n \cdot T}$$

Amostragem por trem de impulsos



$X(e^{j\omega})$: Réplicas do espectro original deslocadas em freqüência e superpostas

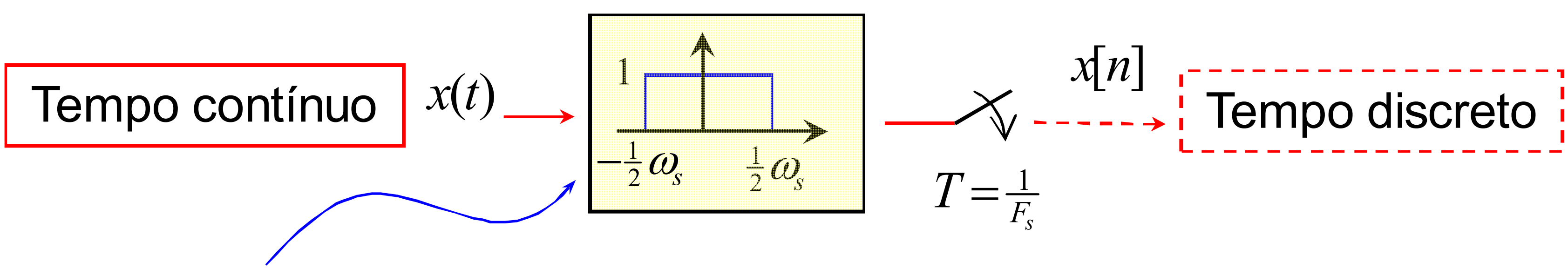
Teorema da amostragem (Nyquist)



Espectro original preservado se $F_s > 2F_M$

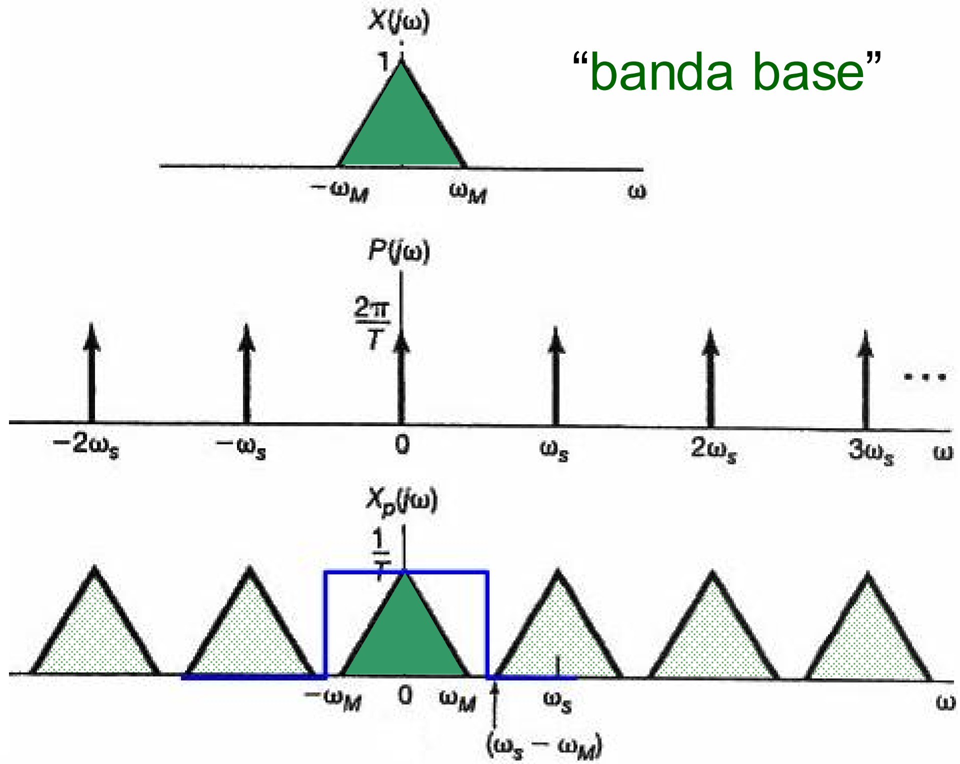
Teorema da amostragem

Pré-filtragem (analógica) anti-aliasing



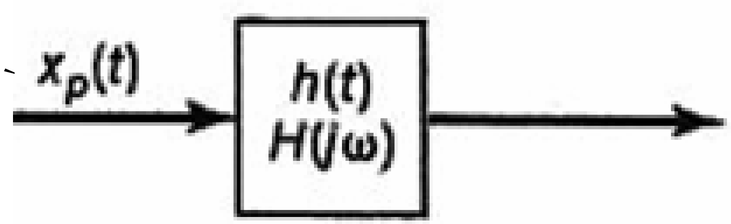
Filtro passa-baixas: garante que o sinal amostrado é limitado em faixa

Recuperando $X(j\omega)$

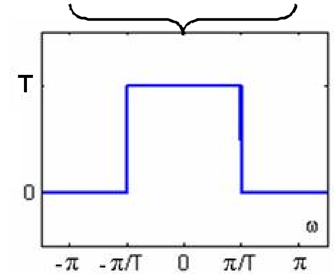


"banda base"

- $X(j\omega)$ amostrado por impulsos:→ recuperação com filtragem passa-baixas ideal (interpolação)



$$X_r(j\omega) = \underbrace{H(j\omega)\cdot P(j\omega)}$$

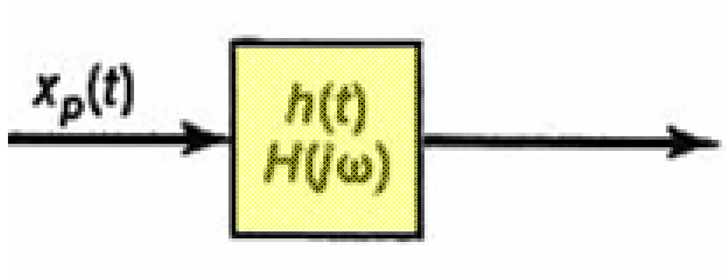


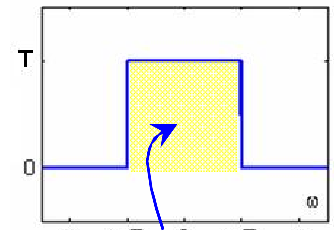
contém réplicas da banda base

Apenas "banda base"

Idealmente, $X_r(j\omega) = X(j\omega)$

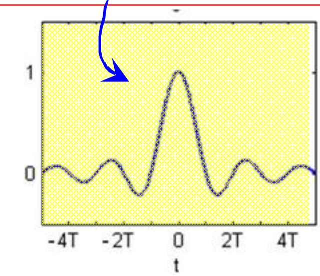
Recuperando *x(t)*

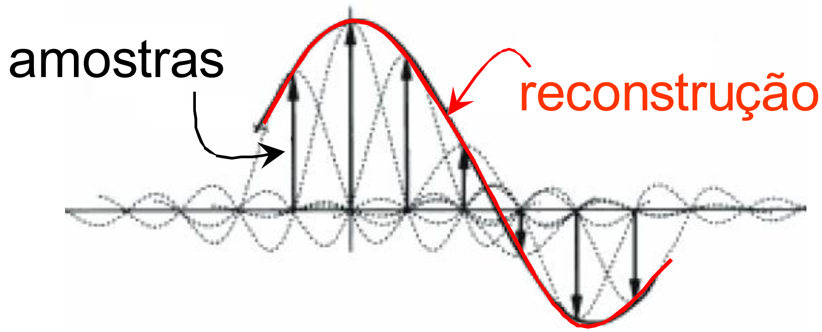




$$X_r(j\omega) = H(j\omega) \cdot X_p(j\omega)$$

$$\boxed{x_r(t) = h(t) * x_p(t)}$$



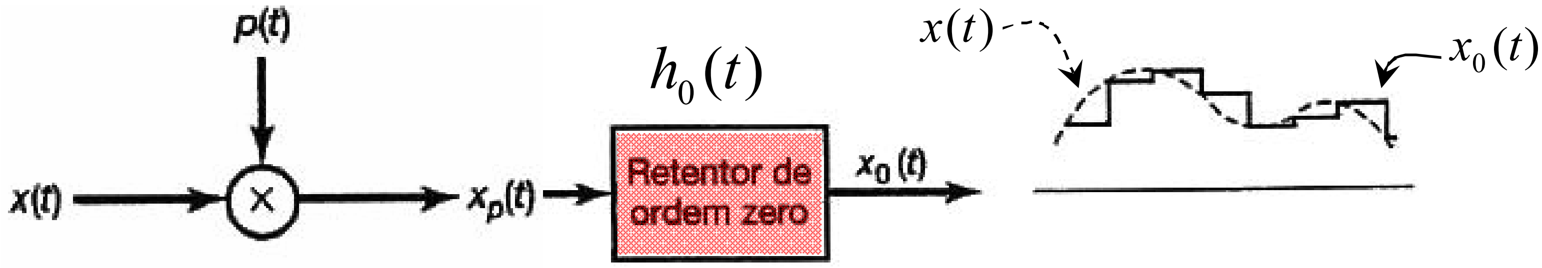


$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\operatorname{sen}(\omega_c (t - nT))}{\omega_c (t - nT)}$$

$h(t)$: não causal e com duração infinita (não realizável na prática) $\rightarrow$ truncamento (aproximação)

7.1.2 Amostragem com pulsos retangulares

- Impulsos podem ser difíceis de ser aproximados/transmitidos
- Utilização do segurador (retentor) de ordem zero (reta horizontal)



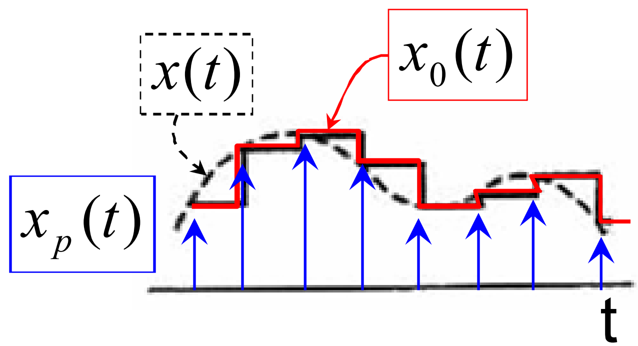
- Interpretação:

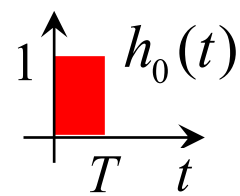
$$x_o(t) = \boxed{x_p(t)} * \boxed{h_0(t)}$$

$$\Updownarrow \mathcal{F}$$

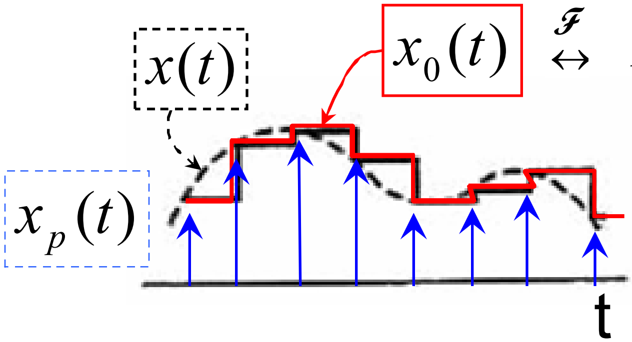
$$\underbrace{X_o(j\omega)} = X_p(j\omega) \bullet \underbrace{H_0(j\omega)}$$

$$|H_0(j\omega)| = T \cdot \mathrm{sinc}(\omega T / 2)$$

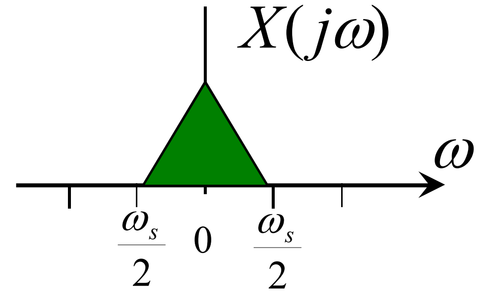


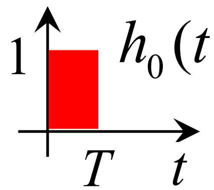


Espectro amostrado é multiplicado (distorcido) por H_0 $(j\omega)$...

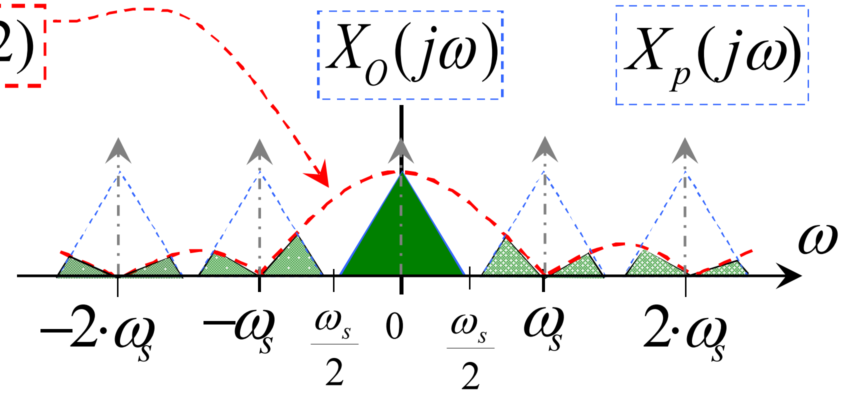


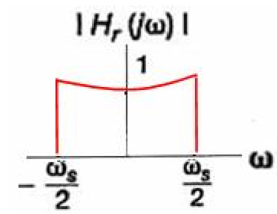
$$x_0(t) \overset{\mathcal{F}}{\leftrightarrow} X_o(j\omega) = X_p(j\omega) \bullet H_0(j\omega)$$





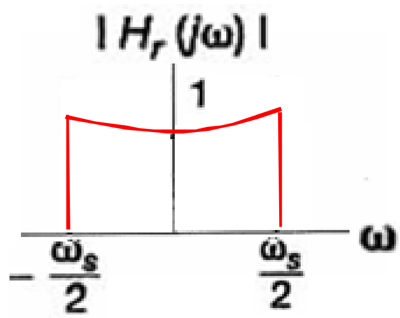
$$\overset{\mathcal{F}}{\leftrightarrow} \; T \cdot \mathrm{sinc}(\omega T / 2)^{*}$$





- Sinal é recuperado com um filtro reverso $H_r(j\omega)$ que compensa a distorção causada pelo segurador de ordem zero

$$X_r(j\omega) = X_0(j\omega) \cdot H_r(j\omega)$$



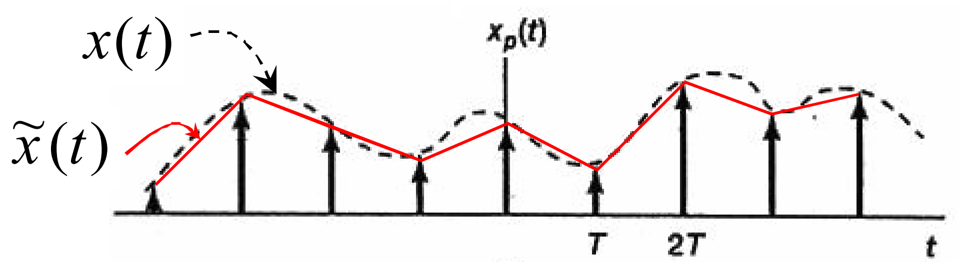
$${}^{*}zeros \;:\; \frac{\omega T}{2} = k\pi = \frac{\omega}{2 \cdot F_s} \rightarrow \omega = k \cdot \omega_s$$

Segurador de ordem zero: importante em sistemas de controle digital

Interpolação linear

- Sinal $x(t)$ aproximado por segmentos de reta:
- Precisão melhora com aumento da taxa de amostragem

$$\tilde{x}(t) \approx x(t)$$



- Compromissos: tempo de cálculo na interpolação x memória para armazenar amostras, etc.
- É possível fazer uma análise em freqüência semelhante à feita para o segurador de ordem zero (Ver fig. 7.4 do livro-texto).

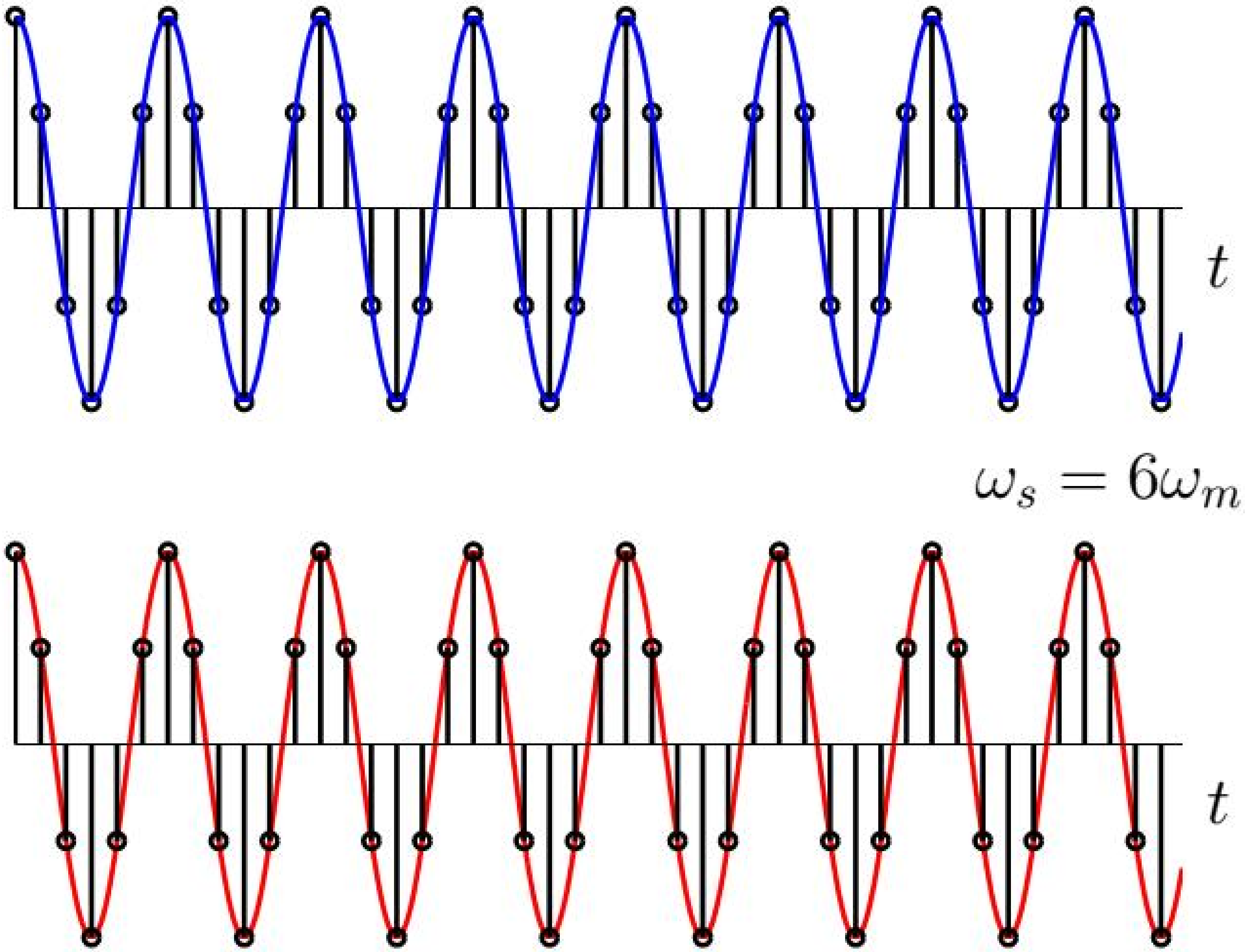
7.3 Efeito da subamostragem: *aliasing*

Aliasing = “falseamento” de freqüência:

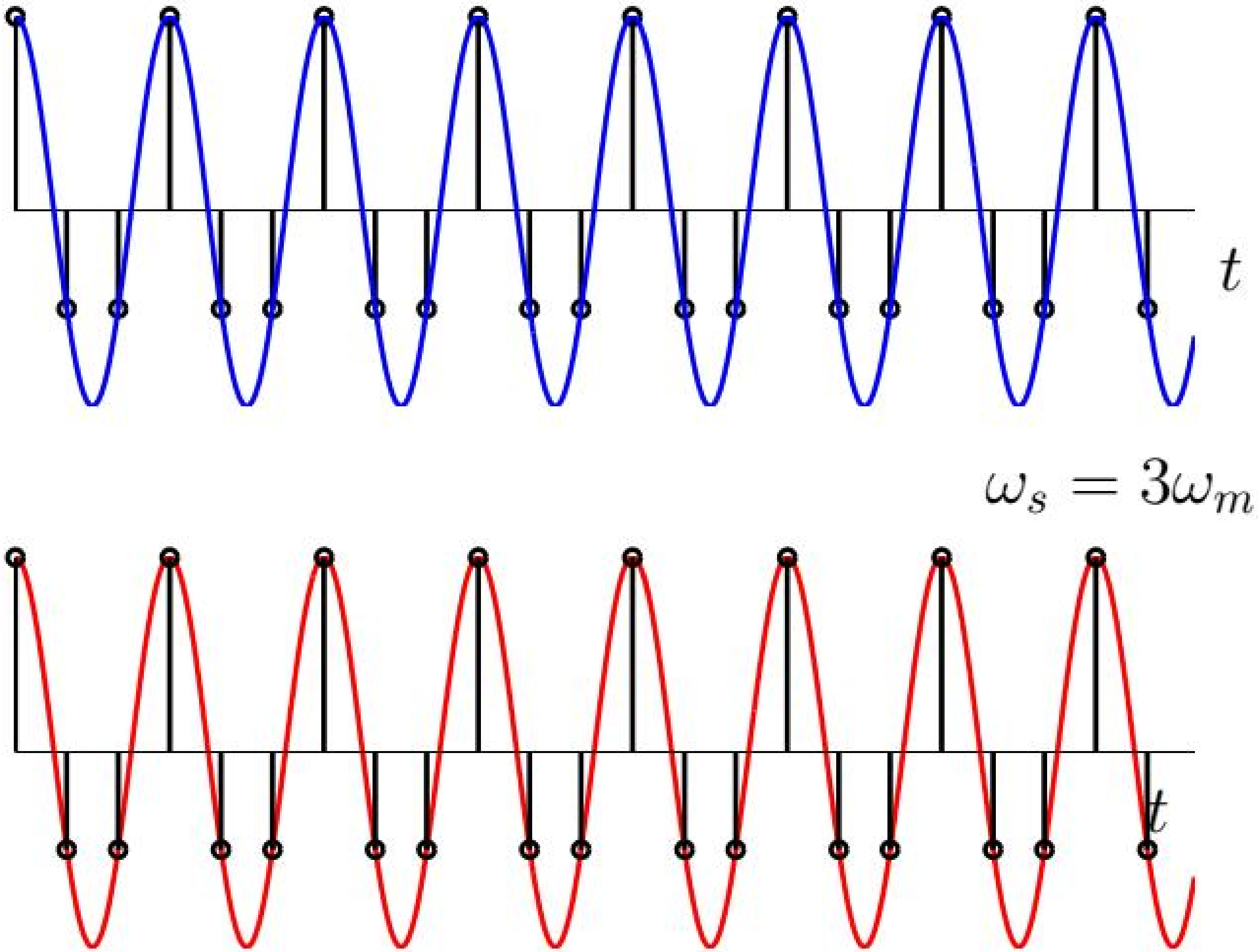
sinal reconstruído com freqüência incorreta se o sinal original é subamostrado

Teorema da Amostragem requer $\omega_s > 2 \cdot \omega_0$

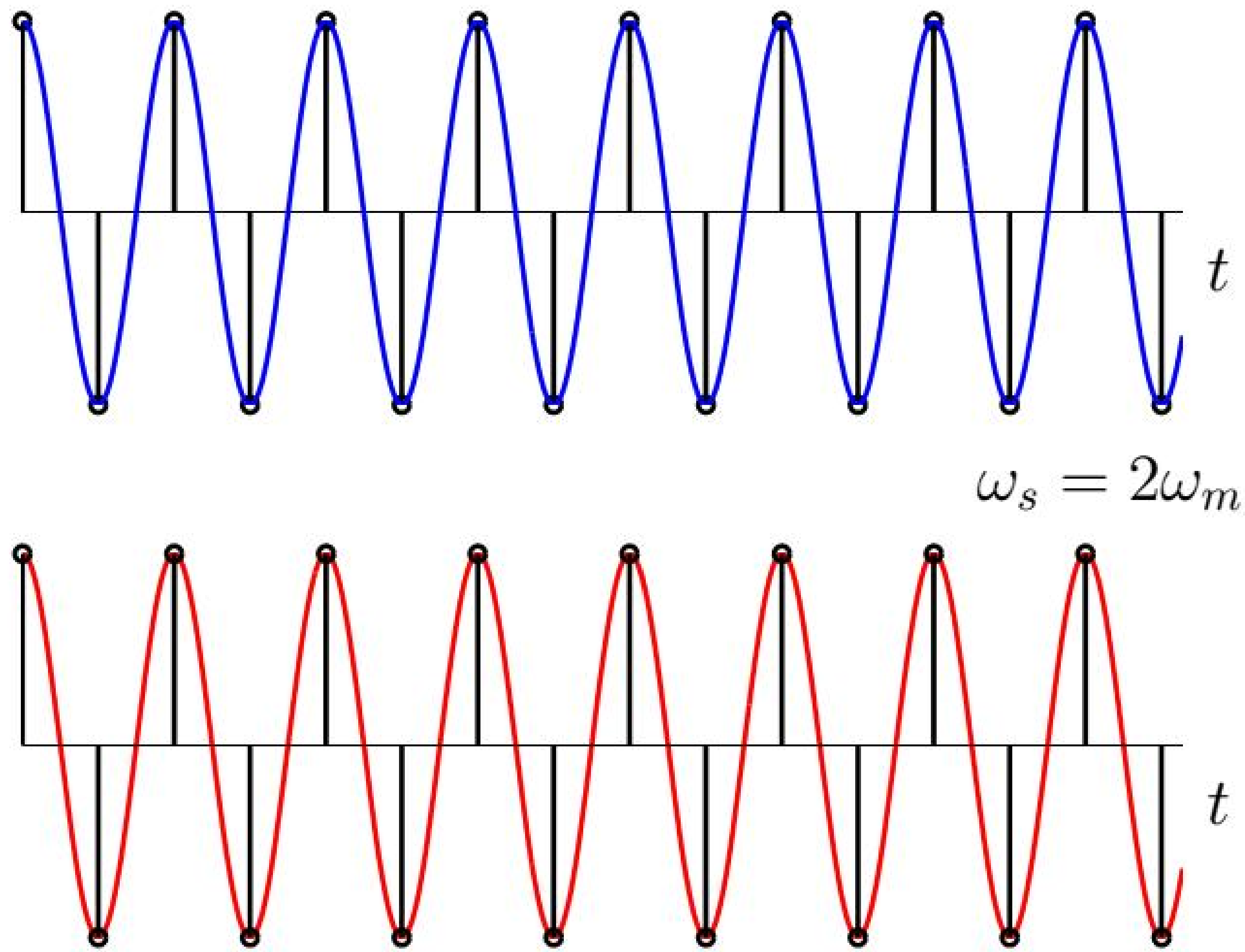
Domínio do tempo: Aliasing



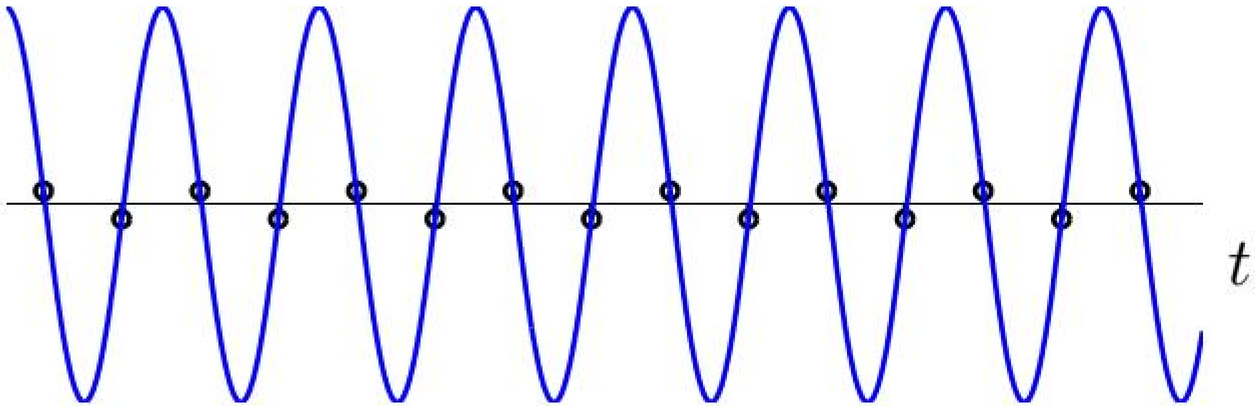
Domínio do tempo: Aliasing



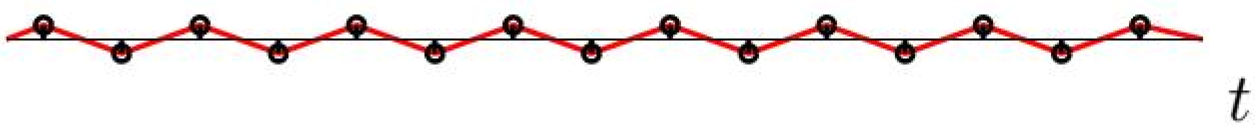
Domínio do tempo: Aliasing



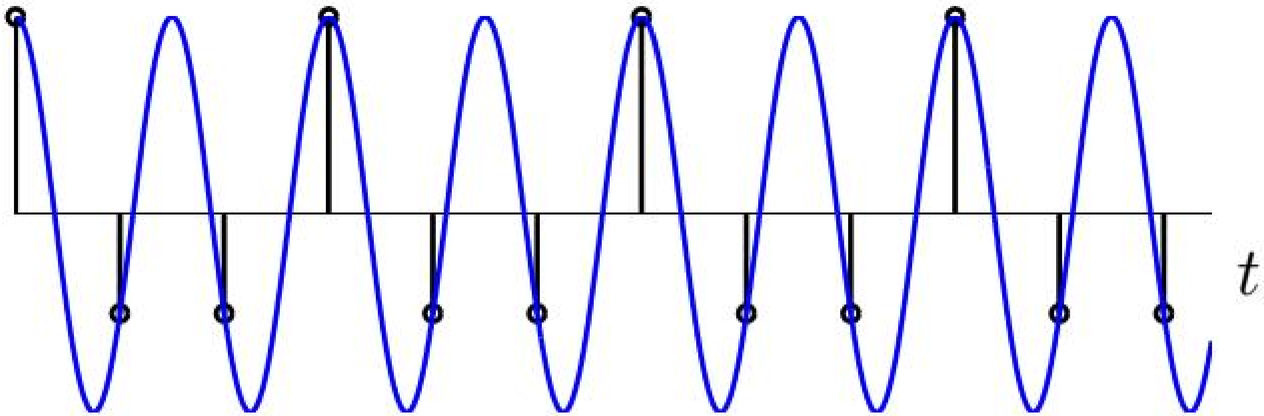
Domínio do tempo: Aliasing



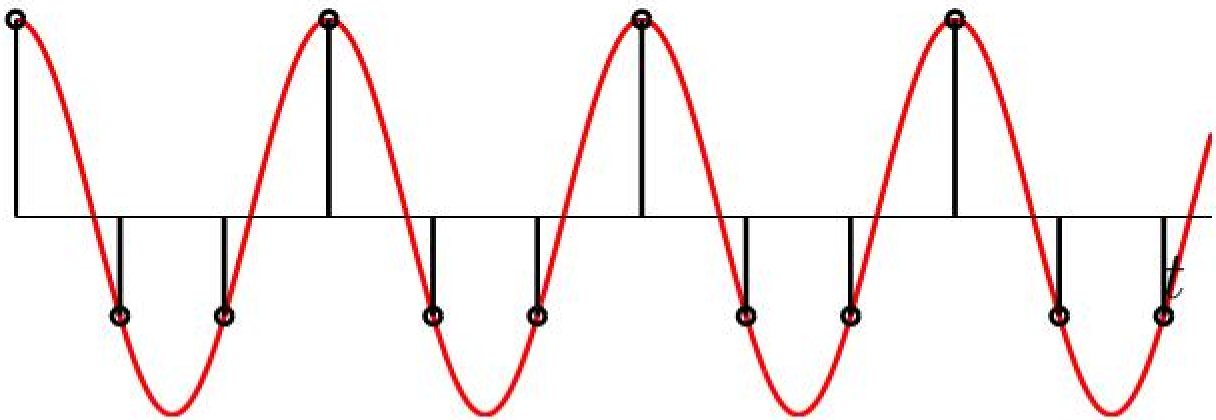
$\omega_s = 2\omega_m$



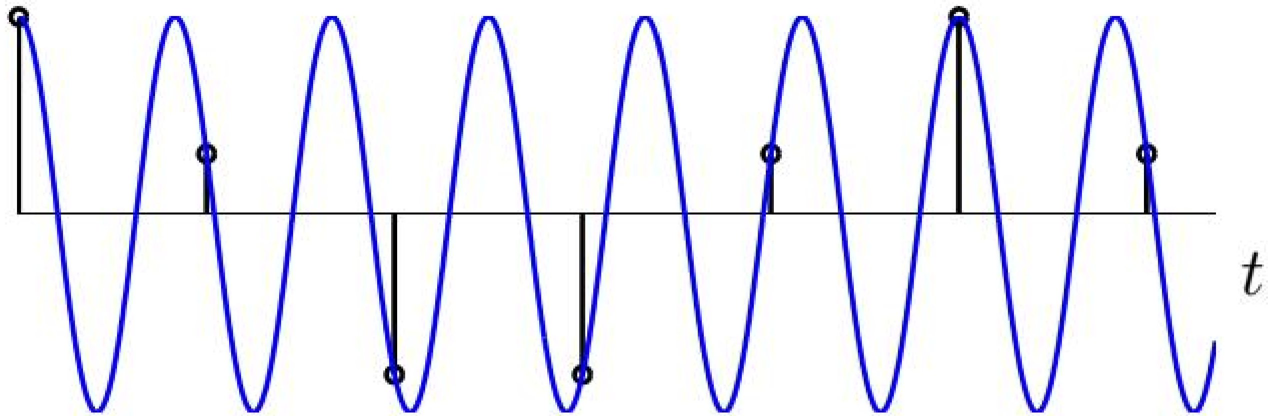
Domínio do tempo: Aliasing



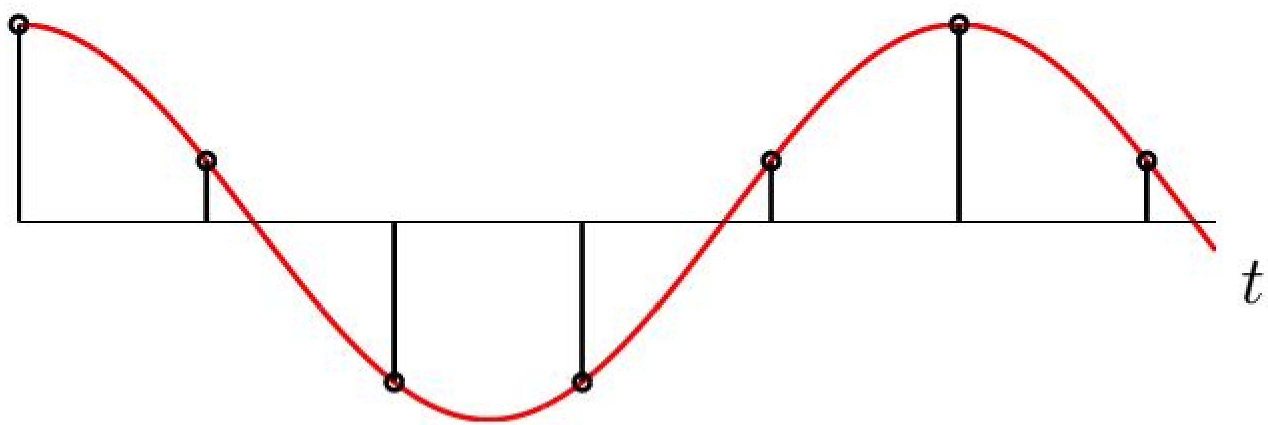
$$\omega_s = 6/4\omega_m$$



Domínio do tempo: Aliasing

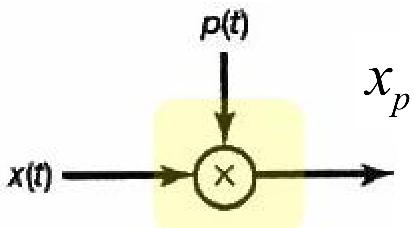


$\omega_s = 6/5\omega_m$



Aliasing: interpretação no espectro (1/3)

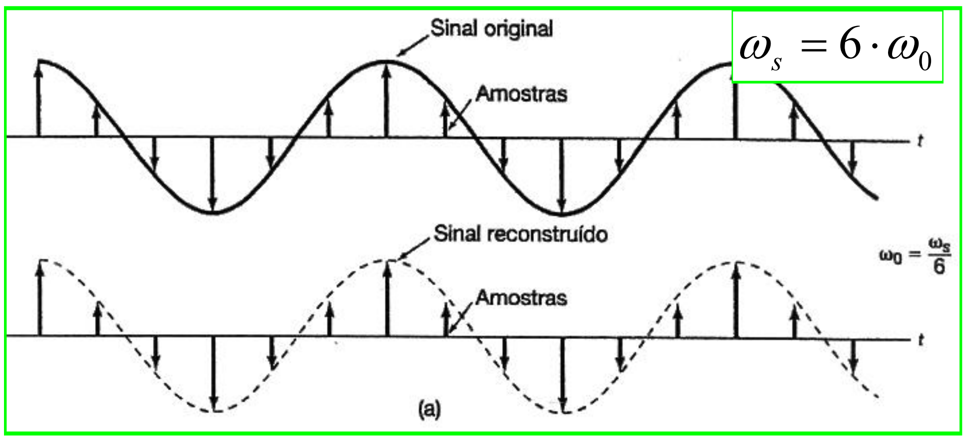
Relembrando:

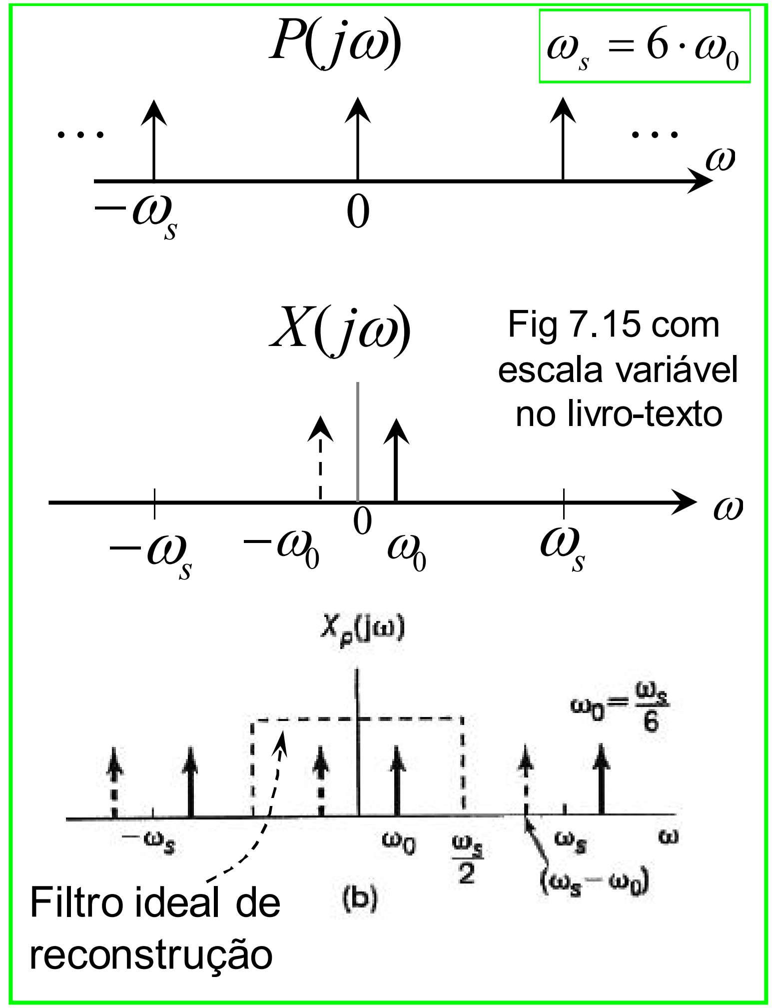


$$x_p(t) = x(t) \cdot p(t)$$

$$\updownarrow \mathcal{F}$$

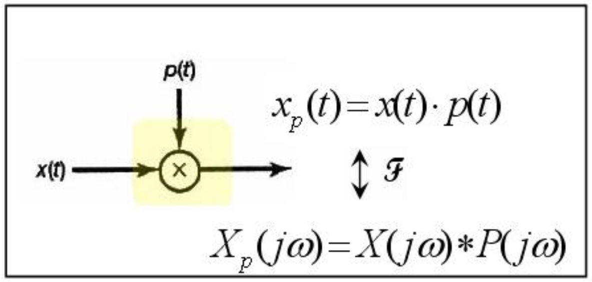
$$X_p(j\omega) = X(j\omega) * P(j\omega)$$

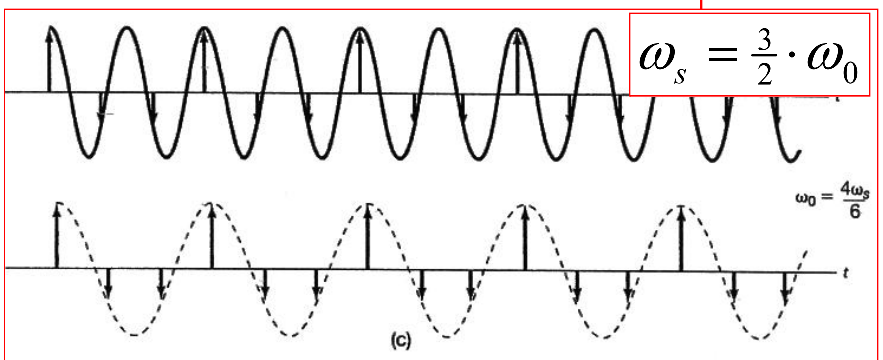


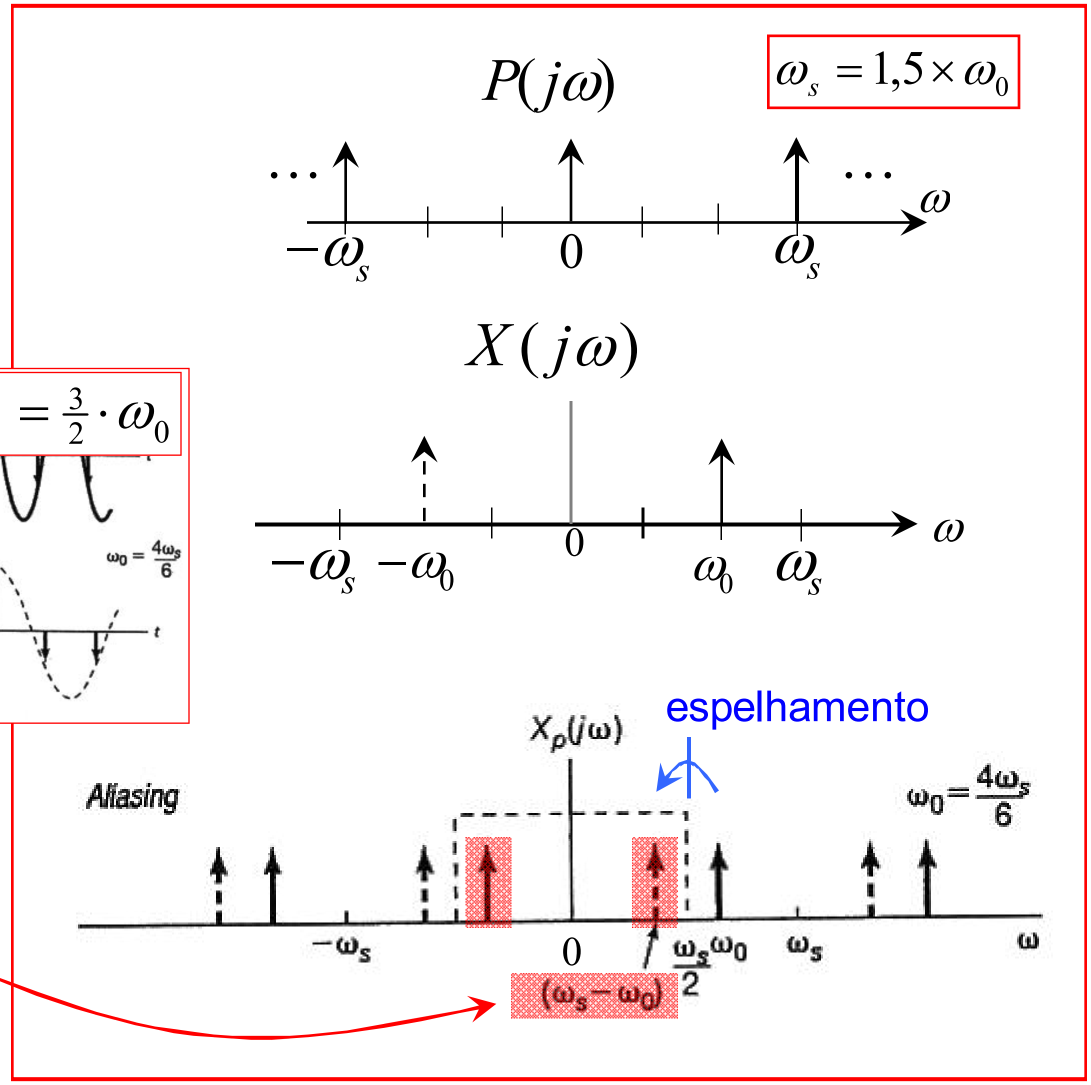


Teorema da Amostragem $\omega_s > 2\omega_0$ $\therefore$ para não haver *aliasing*

Aliasing: interpretação no espectro (2/3)







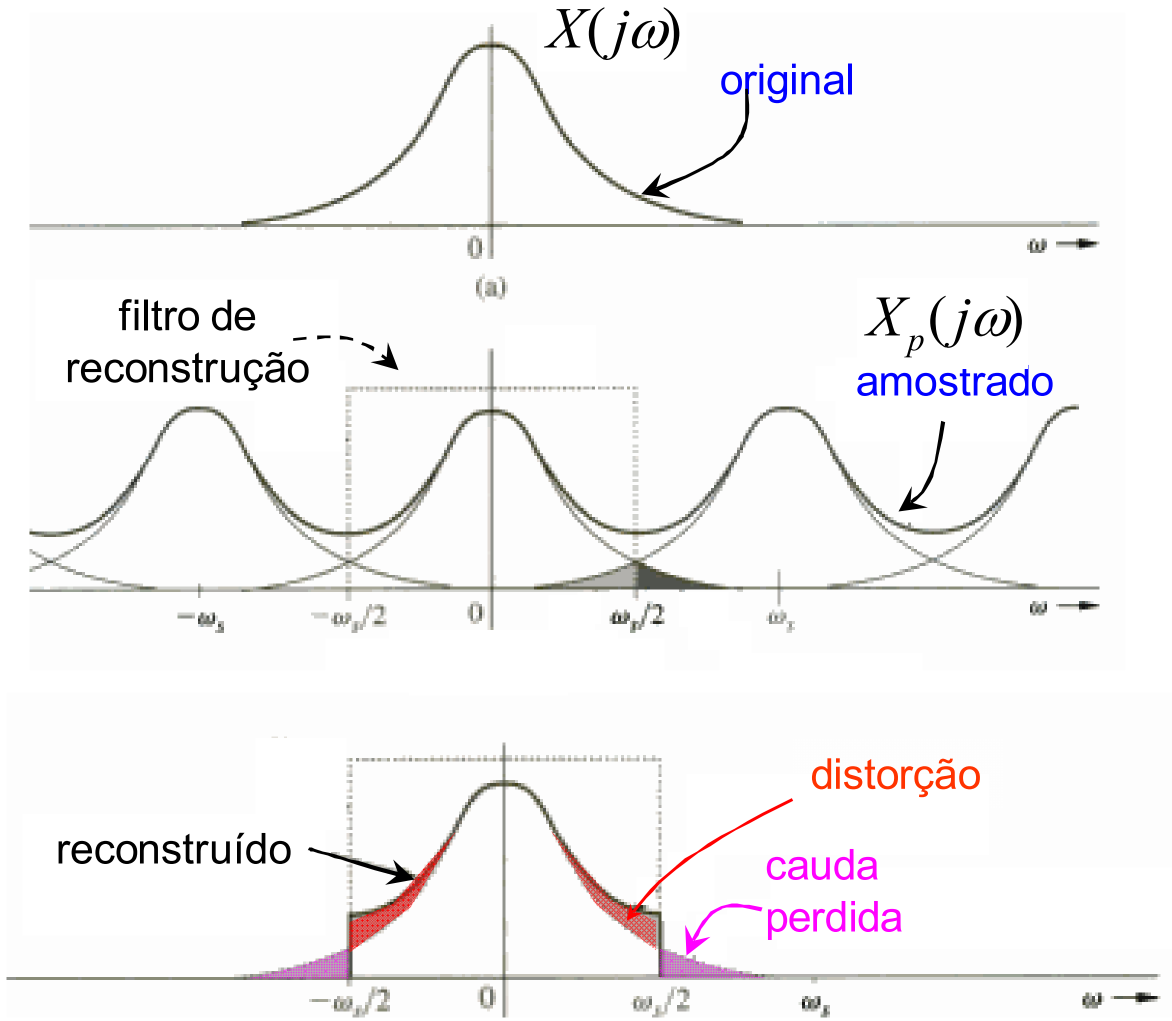
Freqüência no sinal reconstruído é diferente (menor) que no sinal original

Teorema da Amostragem requer $\omega_s > 2\omega_0$: há *aliasing*

Subamostragem de sinais de banda larga 3/3

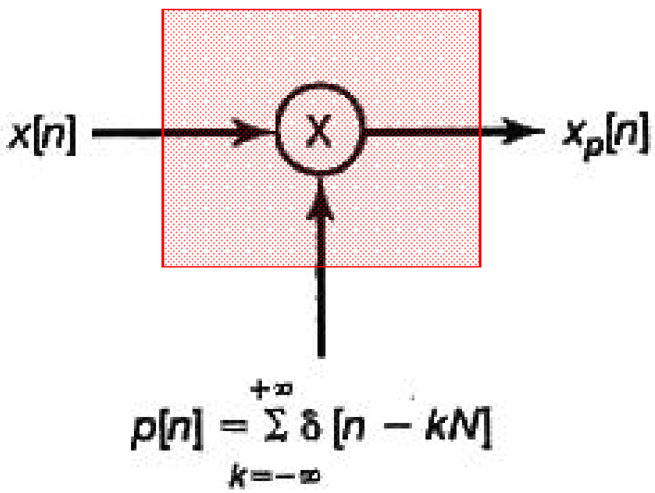
Subamostragem pode causar:

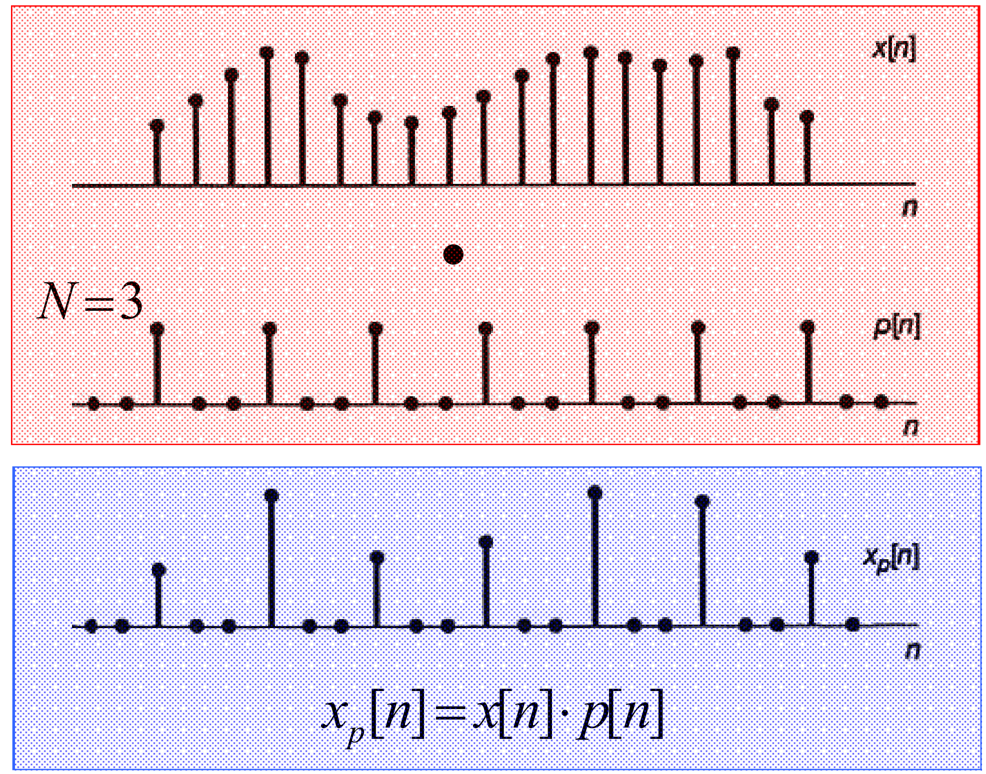
- Distorção pela sobreposição de partes das réplicas do espectro original
- Perdas em altas freqüências



7.5 Amostragem de sinais de tempo discreto 1/2

- Análogo à amostragem de sinais de tempo contínuo

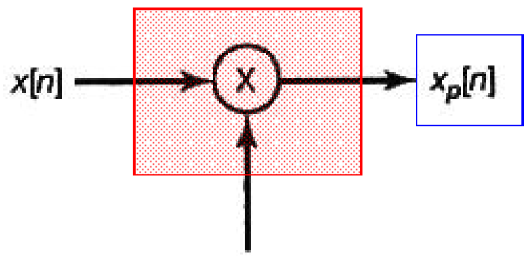




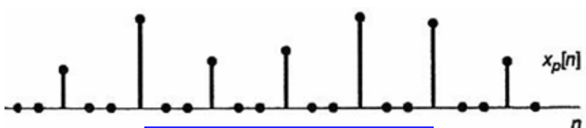
$$x_p[n]=\begin{cases} x[n], & \text{se } n = \text{ um inteiro múltiplo de } N \\ 0, & \text{caso contrário} \end{cases}$$

Amostragem de sinais de tempo discreto

$$x_p[n] = x[n] \cdot p[n]$$

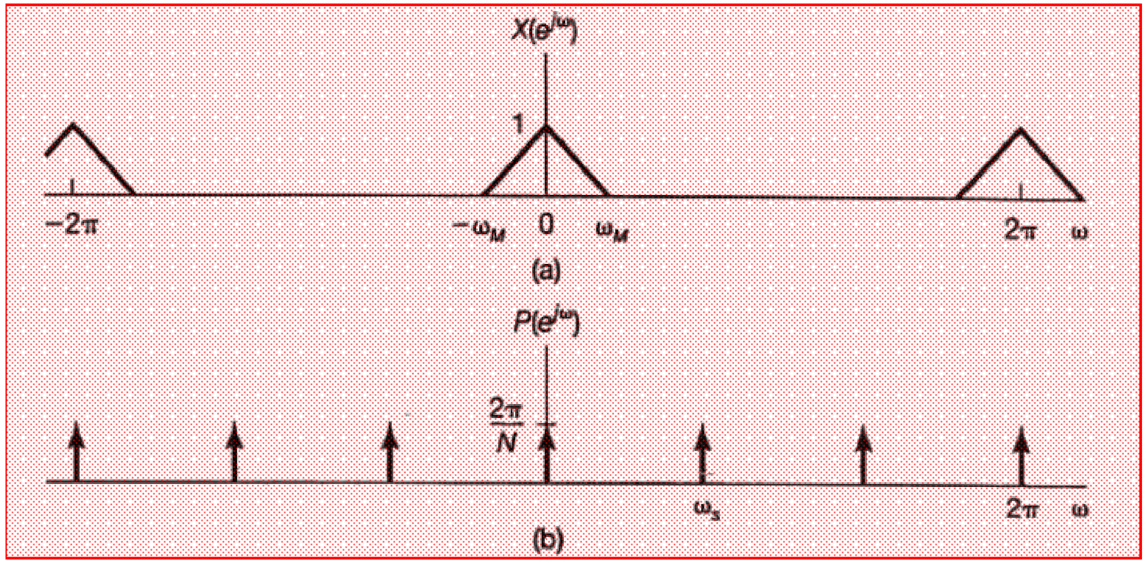


$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

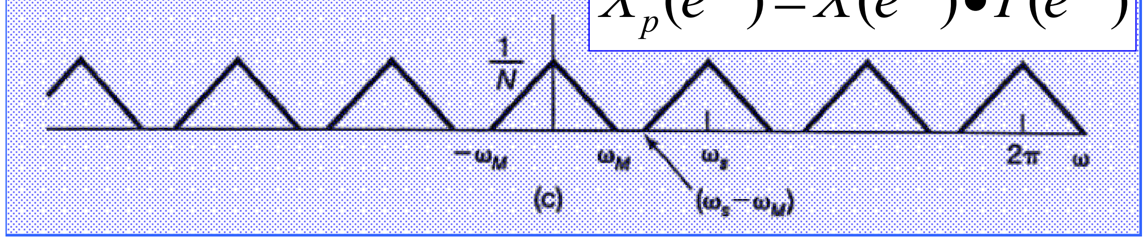


$$\boxed{x_p[n] = x[n] \cdot p[n]}$$

Interpretação espectral



$$X_p(e^{j\omega}) = X(e^{j\omega}) \bullet P(e^{j\omega})$$



Teorema da amostragem:

$\omega_s > 2\omega_M$ para não ocorrer *aliasing*

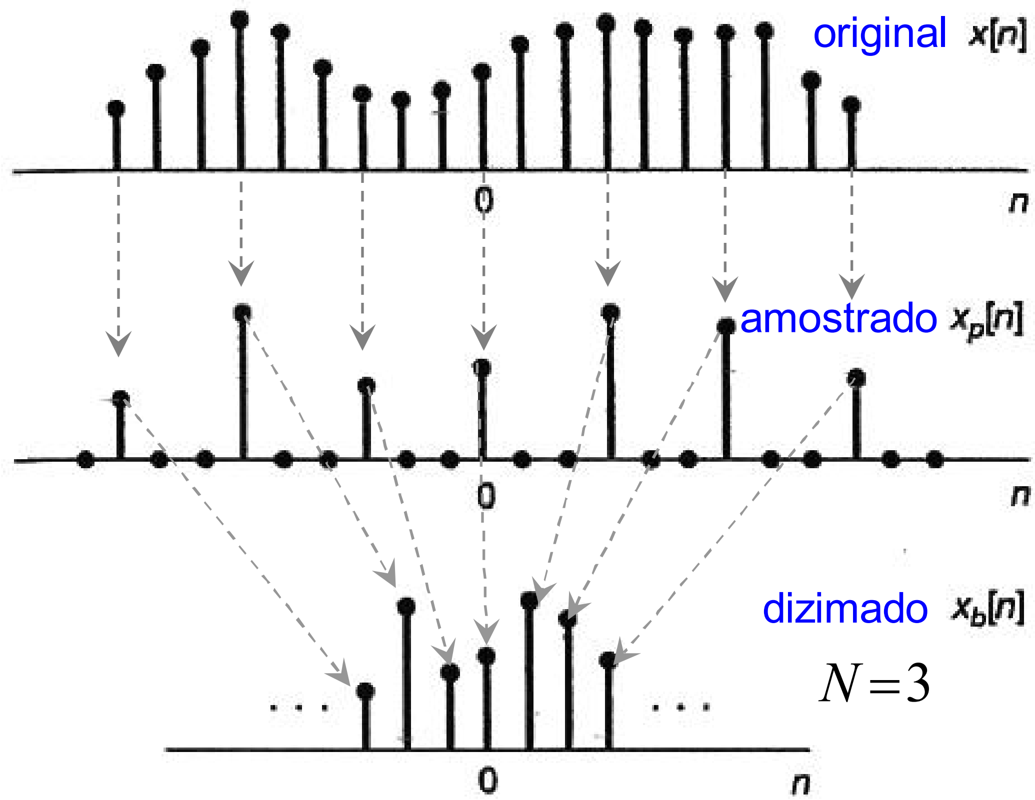
$x(t)$ deve estar sobreamostrado

Dizimação (decimação)

- Amostragem de tempo discreto → amostras nulas
- Armazenamento e/ou transmissão ineficientes
- “Descarte” de amostras nulas: Dizimação

$$x[n] \longrightarrow \boxed{\downarrow N} \longrightarrow x_b[n]$$

Redução de taxa (downsampling)



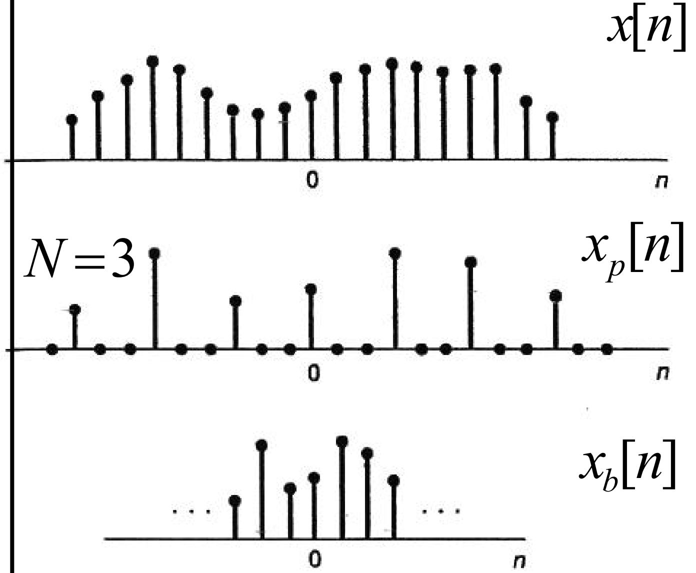
$$x_b[n] = x_p[nN] = x[nN]$$

Qual a relação entre $X_b(e^{j\omega})$ e $X(e^{j\omega})$?

Dizimação – Interpretação espectral

$$x[n] \longrightarrow \boxed{\downarrow N} \longrightarrow x_b[n]$$

$$x_b[n] = x_p[nN] = x[nN]$$

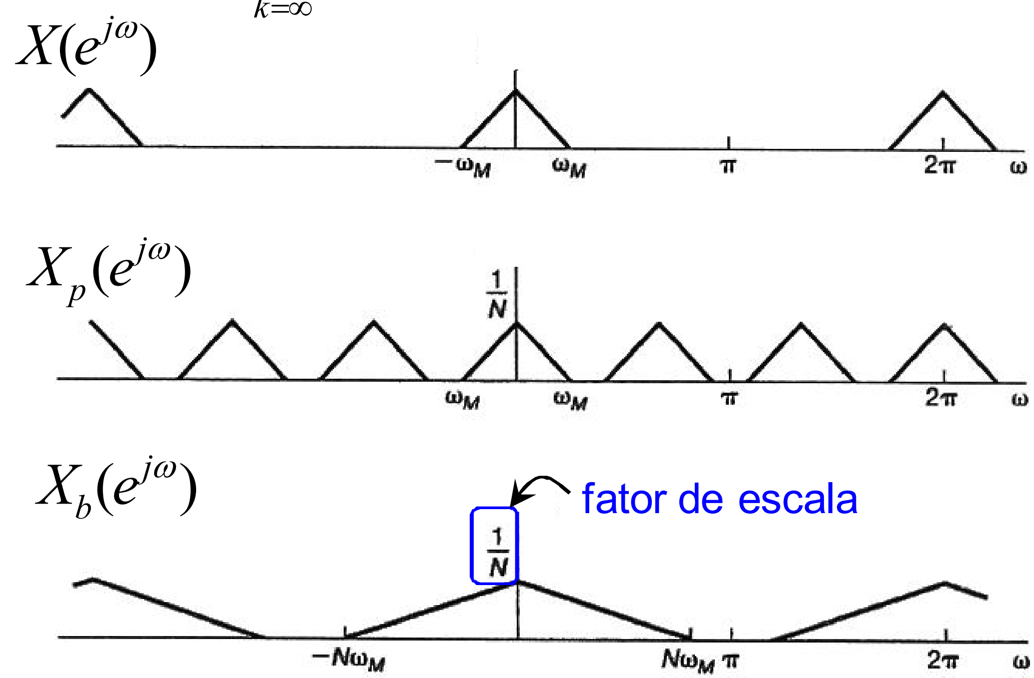


Qual a relação entre $X_b(e^{j\omega})$ e $X(e^{j\omega})$?

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_b[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_p[nN]e^{-j\omega n}$$

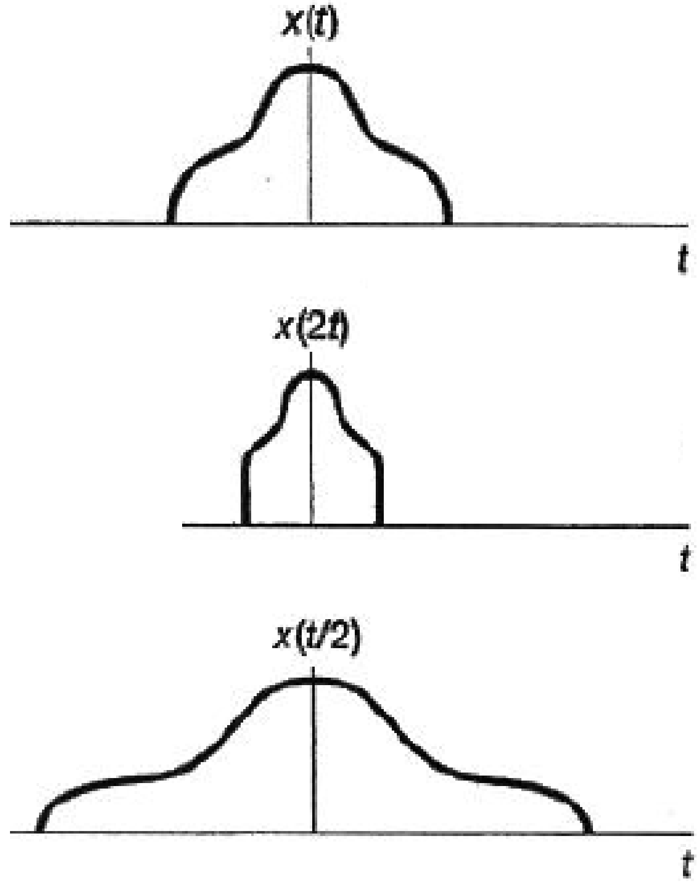
$nN=k$

$$= \sum_{k=\infty}^{\infty} x_p[k]e^{-j\omega k/N} = X_p(e^{j\omega/N})$$



Dizimação: espalha o espectro do sinal

Lembrete: mudanças de escala (cap. 1)



Exercícios (básicos) sugeridos

- 7.1, 7.2, 7.3, 7.4

P16.2

Consider the system in Figure P16.2.

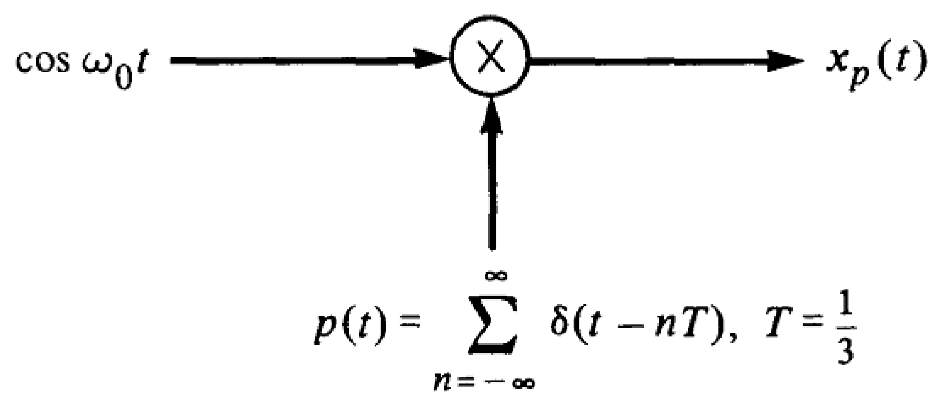


Figure P16.2

(a) Sketch $X_p(\omega)$ for $-9\pi \le \omega \le 9\pi$ for the following values of ω_0.

(i) $\omega_0 = \pi$

(ii) $\omega_0 = 2\pi$

(iii) $\omega_0 = 3\pi$

(iv) $\omega_0 = 5\pi$

(b) For which of the preceding values of ω_0 is $x_p(t)$ identical?

S16.2

The sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad T = \tfrac{1}{3},$$

has a spectrum given by

$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$
$$= 6\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 6\pi k),$$

shown in Figure S16.2-1.

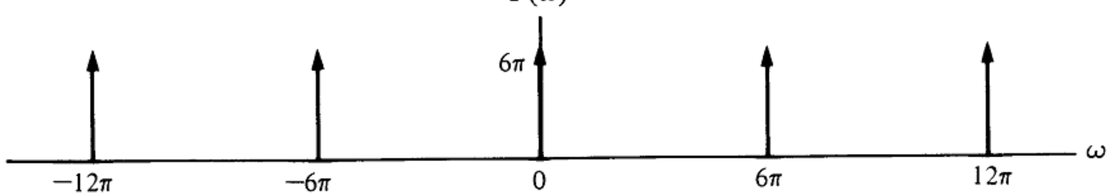


Figure S16.2-1

$\cos(\omega_0 t)$ has a spectrum given by $\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$, shown in Figure S16.2-2.

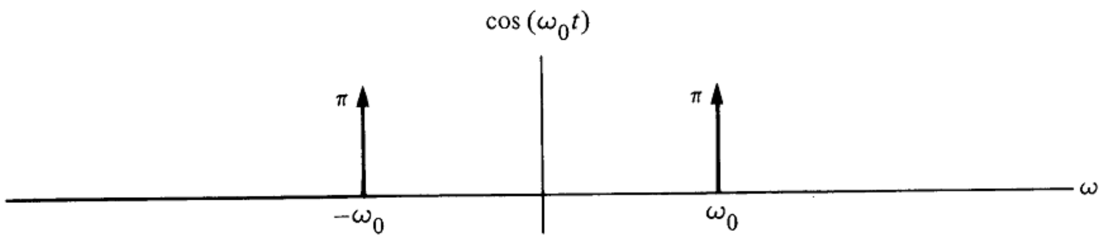


Figure S16.2-2

From the convolution theorem

$$X_p(\omega) = \frac{1}{2\pi} P(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

Hence, it is straightforward to find $X_p(\omega)$.

(a) (i) For $\omega_0 = \pi$:

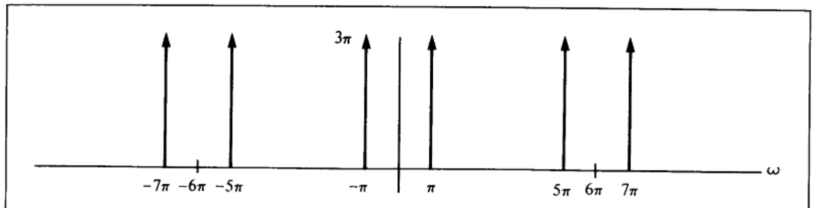


Figure S16.2-3

(ii) For $\omega_0 = 2\pi$:

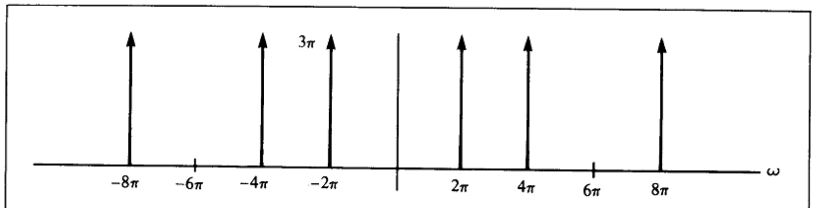


Figure S16.2-4

(iii) For $\omega_0 = 3\pi$:

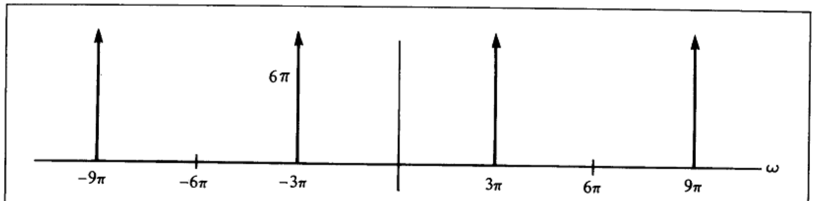


Figure S16.2-5

(iv) For $\omega_0 = 5\pi$:

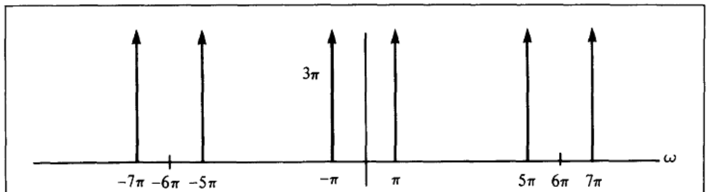


Figure S16.2-6

Consider the system in Figure P16.5-1.

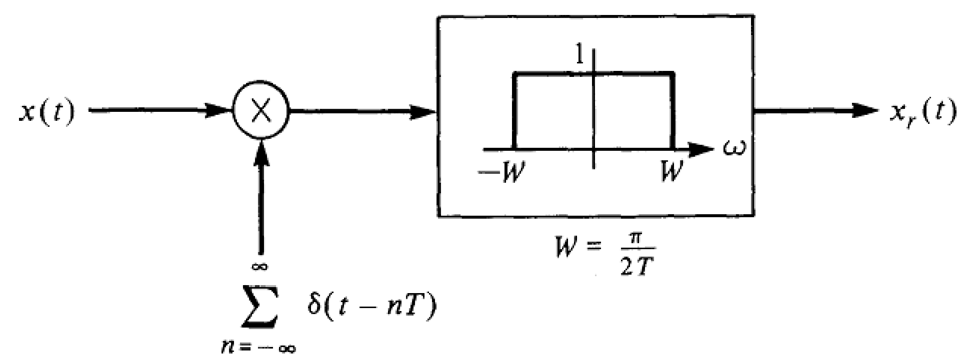


Figure P16.5-1

Figures P16.5-2 and P16.5-3 contain several Fourier transforms of $x(t)$ and $x_r(t)$. For each input spectrum $X(\omega)$ in Figure P16.5-2, identify the correct output spectrum $X_r(\omega)$ from Figure P16.5-3.

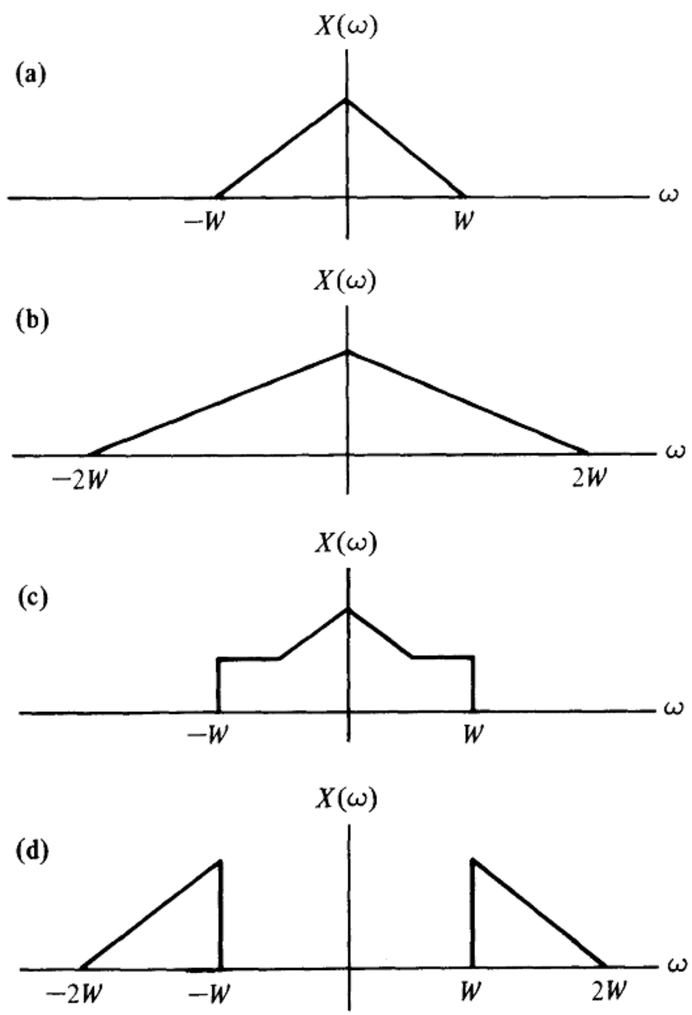


Figure P16.5-2

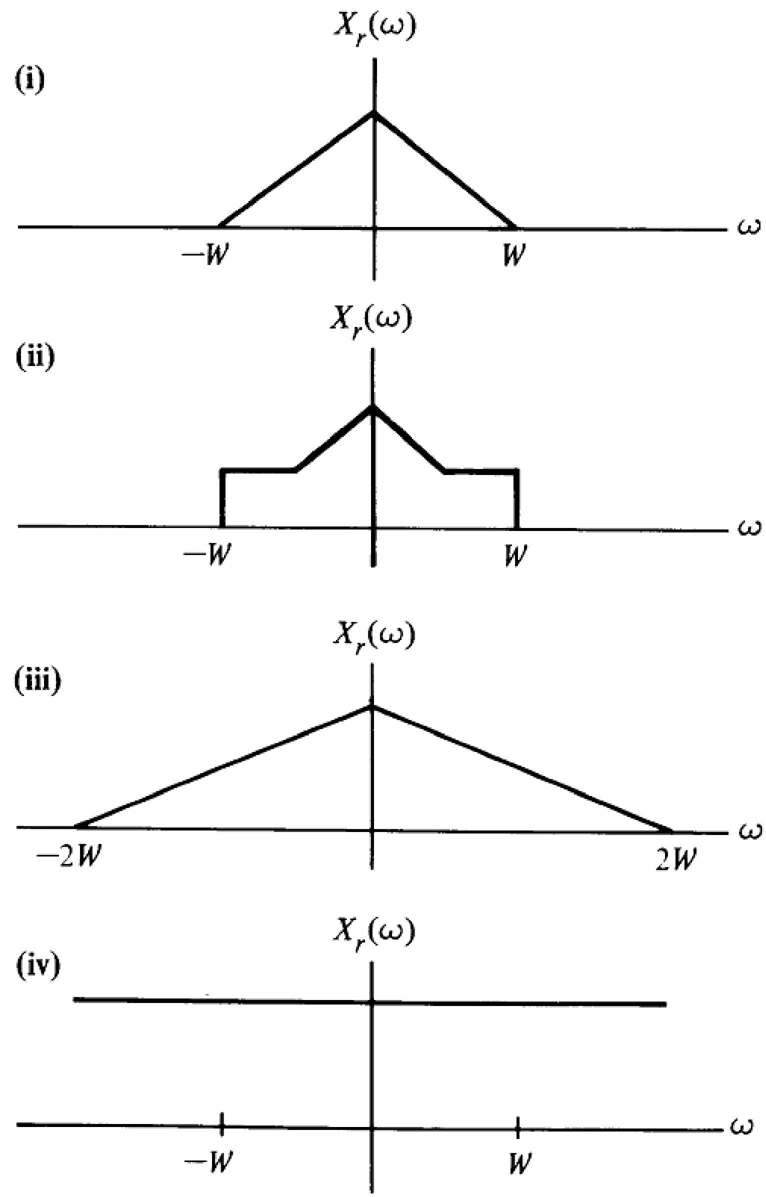


Figure P16.5-3

S16.5

(a) The transform of the sampled function appears as in Figure S16.5-1.

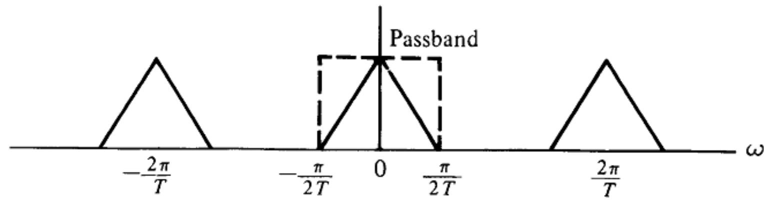


Figure S16.5-1

Hence, (a) matches (i).

(b)

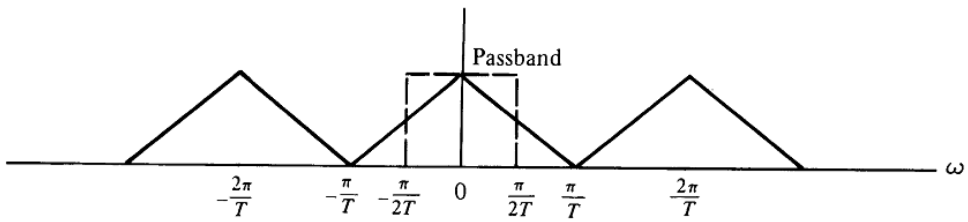


Figure S16.5-2

Hence, (b) does not match any.

(c) Matches (ii).

(d)

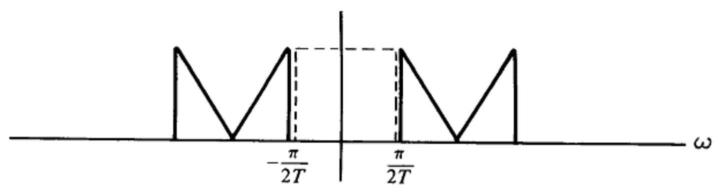


Figure S16.5-3

Hence, (d) does not match any.

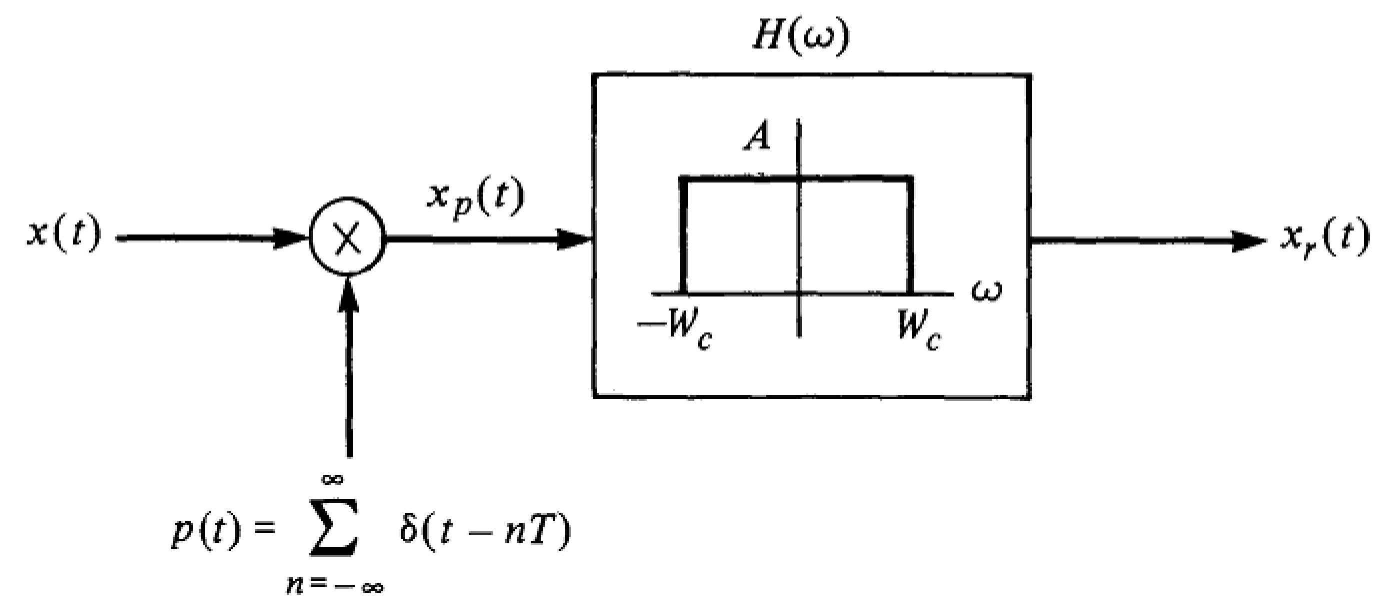


Figure P16.11

(a) If $X(\omega) = 0$ for $|\omega| > W$, find the maximum value of T, W_c, and A such that $x_r(t) = x(t)$.

(a) From the sampling theorem, $2\pi/T \geq 2W$. Hence,

$$T \leq \frac{\pi}{W} \rightarrow T_{\max} = \frac{\pi}{W}$$

Since

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T}\right),$$

we require $A = T$ for $x_r(t) = x(t)$.

The minimum value of W_c is W so that we do not lose any information, and the maximum value of W_c is $(2\pi/T) - W$ to avoid periodic spectral contribution.

P16.3

In the system in Figure P16.3, $x(t)$ is sampled with a periodic impulse train, and a reconstructed signal $x_r(t)$ is obtained from the samples by lowpass filtering.

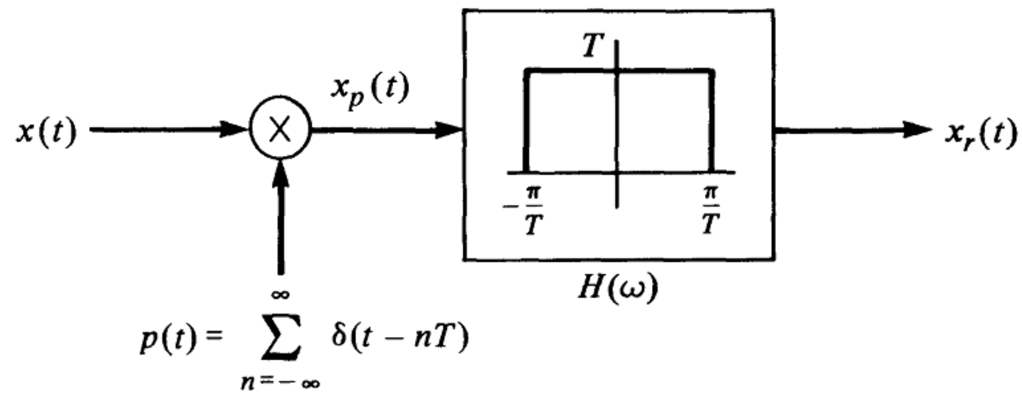


Figure P16.3

The sampling period T is 1 ms, and $x(t)$ is a sinusoidal signal of the form $x(t) = \cos(2\pi f_0 t + \theta)$. For each of the following choices of f_0 and θ, determine $x_r(t)$.

(a) $f_0 = 250$ Hz, $\theta = \pi/4$

(b) $f_0 = 750$ Hz, $\theta = \pi/2$

(c) $f_0 = 500$ Hz, $\theta = \pi/2$

The signal $x(t) = \cos(\omega_0 t + \theta)$, where $\omega_0 = 2\pi f_0$, can be written as

$$x(t) = \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

and the spectrum of $x(t)$ is given by

$$X(\omega) = \pi e^{j\theta}\delta(\omega - \omega_0) + \pi e^{-j\theta}\delta(\omega + \omega_0)$$

The spectrum of $p(t)$ is given by

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore, the spectrum of $x_p(t)$ is

$$X_p(\omega) = \frac{1}{2\pi}\left(\frac{2\pi^2}{T}\right)\left[\sum_{k=-\infty}^{\infty} e^{j\theta}\delta\left(\omega - \frac{2\pi k}{T} - \omega_0\right) + e^{-j\theta}\delta\left(\omega - \frac{2\pi k}{T} + \omega_0\right)\right]$$

and the spectrum of $X_r(\omega)$ is given by

$$X_r(\omega) = H(\omega)X_p(\omega)$$

(a) $\omega_0 = 2\pi \times 250, \quad \theta = \frac{\pi}{4}, \quad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 250)]$$

Hence, only the $k = 0$ term is passed by the filter:

$$X_r(\omega) = \pi[e^{j\theta}\delta(\omega - 2\pi \times 250) + e^{-j\theta}\delta(\omega + 2\pi \times 250)]$$

and

$$\begin{aligned} x_r(t) &= \frac{1}{2}e^{j\theta}e^{j2\pi \times 250t} + \frac{1}{2}e^{-j\theta}e^{-j2\pi \times 250t} \\ &= \cos(2\pi \times 250t + \theta) \\ &= \cos\left(2\pi \times 250t + \frac{\pi}{4}\right) \end{aligned}$$

(b) $\omega_0 = 2\pi \times 750 \text{ Hz}, \quad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 750) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 750)]$$

Only the $k = \pm 1$ term has nonzero contribution:

$$X_r(\omega) = \frac{\pi}{T}[e^{j\theta}\delta(\omega + 2\pi \times 250) + e^{-j\theta}\delta(\omega - 2\pi \times 250)]$$

Hence,

$$\begin{aligned} x_r(t) &= \cos(2\pi \times 250t - \theta) \\ &= \cos\left(2\pi \times 250t - \frac{\pi}{2}\right) \end{aligned}$$

(c) $\omega_0 = 2\pi \times 500, \quad \theta = \frac{\pi}{2}, \quad T = 10^{-3},$

$$X_p(\omega) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}[e^{j\theta}\delta(\omega - 2\pi \times 10^3 k - 2\pi \times 500) + e^{-j\theta}\delta(\omega - 2\pi \times 10^3 k + 2\pi \times 500)]$$

Since $H(\omega) = 0$ at $\omega = 2\pi \times 500$, the output is zero: $x_r(t) = 0$.