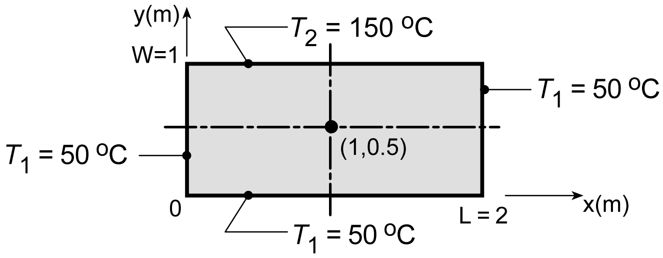
PROBLEM 4.2

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperature at the mid-point using the exact solution considering the first five non-zero terms; assess error resulting from using only first three terms. Plot the temperature distributions T(x,0.5) and T(1,y).

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(x,y) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi x}{L}\right)\cdot\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \quad (1,4.19)$$

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\theta} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi}{2}\right)\cdot\frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}.$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{2\sin\left(\frac{\pi}{2}\right)\frac{\sinh(\pi/4)}{\sinh(\pi/2)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh(3\pi/4)}{\sinh(3\pi/2)} + \right.$$

$$\left.\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh(5\pi/4)}{\sinh(5\pi/2)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh(7\pi/4)}{\sinh(7\pi/2)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh(9\pi/4)}{\sinh(9\pi/2)}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}[0.755 - 0.063 + 0.008 - 0.001 + 0.000] = 0.445 \quad (2)$$

$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150-50) + 50 = 94.5^\circ C. \quad <$$

If only the first three terms of the series, Eq. (2), are considered, the result will be θ(1,0.5) = 0.46; that is, there is less than a 0.2% effect.

Using Eq. (1), and writing out the first five terms of the series, expressions for θ(x,0.5) or T(x,0.5) and θ(1,y) or T(1,y) were keyboarded into the IHT workspace and evaluated for sweeps over the x or y variable. Note that for T(1,y), that as y → 1, the upper boundary, T(1,1) is greater than 150°C. Upon examination of the magnitudes of terms, it becomes evident that more than 5 terms are required to provide an accurate solution.