

Topological Consequences of STDP in Neuronal Networks

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Abstract—

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I. INTRODUCTION

II. EXTERNALLY EXCITED PLASTIC NEURONAL NETWORKS

Definition. A *directed graph* is a set V of vertices together with a set $E \subset V \times V$ of edges. To ease future notation, write

- 1) $i \leftarrow_E j$ (or, equivalently, $j \rightarrow_E i$), if $(i, j) \in E$.
- 2) $i \leftrightarrow_E j$ if $(i, j) \in E$ and $(j, i) \in E$.
- 3) $i -_E j$ if $(i, j) \in E$ or $(j, i) \in E$.

and take

- 1) $\{i \leftarrow j\} = \{(i, j)\}$
- 2) $\{i \leftrightarrow j\} = \{(i, j), (j, i)\}$
- 3) $\{i - j\} = \{\{(i, j)\}, \{(j, i)\}, \{(i, j), (j, i)\}\}$

Definition. An *externally excited plastic neuronal network (EPN)* is a directed graph together with an **activity sequence** $a : \mathbb{R}_{\geq 0} \rightarrow \{0, \delta(0)\}^N$ defined component-wise inductively by

$$a_i(0) = 0$$

$$a_i(t) = \begin{cases} \delta(0) & \text{if } a_i[t - r, t) = 0 \\ & \text{and } \begin{cases} \exists j \in V : w_{ij}(t - \tau)a_j(t - \tau) > 0 \\ \text{or } t \bmod p_i = 0 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

where $r \in \mathbb{R}_+$ is the **refractory period**, $p_i \in \mathbb{R}_{\geq 0}$ is the **excitation period** of neuron i , $\tau \in \mathbb{R}_+$ is the **spike propagation latency**, and $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}^{N \times N}$ is the **weight sequence** defined component-wise inductively by

$$w_{ij}(0) = \begin{cases} 1 & \text{if } i \leftarrow_E j \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{d}{dt}w_{ij}(t) = [P_{ij}(t) - \delta w_{ij}(t)]w_{ij}(t)$$

where $\delta \in (0, 1)$ is the decay rate and

$$P_{ij}(t) = \begin{cases} 1 + f(\min_{t' \leq t} \{t - t' | a_j(t') = 1\}) & \begin{cases} a_i(t) = \delta(0) \\ a_j(t) = 0 \end{cases} \\ 1 - f(\min_{t' \leq t} \{t - t' | a_i(t') = 1\}) & \begin{cases} a_j(t) = \delta(0) \\ a_i(t) = 0 \end{cases} \\ 1 & \text{otherwise} \end{cases}$$

is the plasticity function with STDP rule f . We call the vertices **neurons**. Neuron i is said to be **active** at time t if $a_i(t) = \delta(0)$ and **inactive** at time t if $a_i(t) = 0$.

Elaborate: STDP (and choice of STDP function) and weight decay - explanation justification. The STDP learning rule is commonly chosen as $f(x) = \gamma e^{-x/\lambda}$.

III. EPN EVOLUTION

From now on, whenever we say "weight," we are referring to a nonzero weight.

Definition. A weight of an EPN **persists** if it is asymptotically nonzero and **breaks** if it is asymptotically zero. We say that an EPN **persists entirely** if each of its weights persists, and we say that an EPN **breaks entirely** if each of its weights breaks.

Proposition. Suppose there is no path to neuron q from any neuron i . Then for each $i \in V$, the weights w_{qi}, w_{iq} break.

Proof: In this case, a_q is identically zero, so P_{iq} and P_{qi} are identically 1 for all $i \in V$. Then w_{iq} and w_{qi} experience only decay. ■

Definition. We write $E_0 \rightarrow E_1$ if $E_1 = \{(i, j) \in E_0 | w_{1,ij} \text{ persists}\}$, and we say that E_n is an **evolution** (precisely, the n th **evolution**) of E_0 if $E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n$ for distinct E_0, \dots, E_n . The **final evolution** of E_0 is E_∞ such that E_0 eventually evolves into E_∞ and E_∞ persists entirely; symbolically, we write $E_0 \Rightarrow E_\infty$.

A. Chain Evolution

Proposition. The chain $E = \{1 \rightarrow \dots \rightarrow N\}$ persists if

$$\frac{1}{(1 - \delta)^\rho} \leq [1 + f(\tau)][1 - f(p - \tau)]. \quad (1)$$

Otherwise, it breaks entirely.

Proof: In this case we have $P_{i-1,i}(t) = f(\tau)$ whenever $a_i(t) = \delta(0)$ and $i > 1$. Each neuron is active exactly once every p timesteps, and i activates τ timesteps after $i - 1$ activates, so i activates $p - \tau$ timesteps before neuron $i - 1$ activates. Then $P_{i,i-1}(t) = f(p - \tau)$ whenever $a_i(t) = \delta(0)$.

Since each neuron is active exactly once every p timesteps, $P_{i,i-1}$ takes on each value $1 + f(\tau)$ and $1 - f(p - \tau)$ once every p timesteps. Then $\lim_{t \rightarrow \infty} w_{i,i-1}(t) = \lim_{t \rightarrow \infty} (1 - \delta)^t [1 + f(\tau)]^{t/p} [1 - f(p - \tau)]^{t/p}$. Define $\alpha = (1 - \delta)^{1/p} \sqrt[p]{[1 + f(\tau)][1 - f(p - \tau)]}$ so that $\lim_{t \rightarrow \infty} w_{i,i-1}(t) = \lim_{t \rightarrow \infty} \alpha^t$. We see that $\lim_{t \rightarrow \infty} w_{i,i-1}(t) > 0 \Leftrightarrow \alpha \geq 1 \Leftrightarrow \frac{1}{(1 - \delta)^p} \leq [1 + f(\tau)][1 - f(p - \tau)]$. ■

Proposition. If $f(x) = \gamma e^{-x/\lambda}$ and $\delta, \gamma \approx 0$, (1) holds when $p \geq 2\tau$.

Proof: As $\delta \rightarrow 0$, (1) becomes $1 \leq [1 + f(\tau)][1 - f(p - \tau)]$, so $f(p - \tau)[1 + f(\tau)] \leq f(\tau)$. Then $\frac{1}{f(p - \tau)} \geq \frac{1 + f(\tau)}{f(\tau)} = \frac{1}{f(\tau)} + 1$, and hence $\frac{1}{f(p - \tau)} - \frac{1}{f(\tau)} \geq 1$. Substituting for f , we find $e^{(p - \tau)/\lambda} - e^{\tau/\lambda} \geq \gamma$. As $\gamma \rightarrow 0$, we reach $e^{p - \tau} \geq e^\tau$, so $p \geq 2\tau$. ■

(Note that if $r > 2\tau$, then we necessarily have $p > 2\tau$ since $p > r$. This assumption is plausible; for example, it is satisfied when $r = 5\text{ms}$ and $\tau = 2\text{ms}$.)

The final evolution of noncyclic intersecting chains can be obtained via successive evolution of daughter chains, starting with a chain containing neuron 1. Each daughter chain evolves as though it were excited with period p at its point of intersection with its parent chain, provided its connection to its parent chain still exists after evolving the parent chain. Otherwise, the daughter chain (and consequently, all of its daughter chains) breaks entirely.

If a parent chain eventually breaks its connection to the daughter chain, then the daughter chain will not receive any excitation and will consequently break. However, if a parent chain maintains its connection to the daughter chain, then the daughter chain will experience excitation at its intersection with its parent chain. This excitation will occur with some lag after 1 is excited, but it will occur with the same period p .

B. Cycle Evolution: Analytical Treatment

Proposition. The "small" ($N\tau < r$) cycle $E = \{1 \rightarrow \dots \rightarrow N \rightarrow 1\}$ persists entirely if

$$\frac{1}{(1 - \delta)^\rho} \leq [1 + f(\rho - (N - 1)\tau)][1 - f((N - 1)\tau)]. \quad (2)$$

Otherwise, it evolves into the chain $E_\infty = \{1 \rightarrow \dots \rightarrow N\}$.

Proof: We know that $\{1 \rightarrow \dots \rightarrow N\}$ persists entirely if (1) is satisfied but breaks entirely if (1) is not satisfied. Hence, we need only consider w_{1N} .

Observe that $a_N(n) = 1 \Leftrightarrow a_{N-1}(n - \tau) = 1 \Leftrightarrow \dots \Leftrightarrow a_1(n - (N - 1)\tau) = 1$, so $(N - 1)\tau \in \{n - t | a_1(t) = 1\}$ whenever $a_N(n) = 1$. Because $N\tau < r$, we have $(N - 1)\tau = \min_{t \leq n} \{n - t | a_1(t) = 1\}$. Using similar argument as in previous proof, we have that $\rho - (N - 1)\tau = \min_{t \leq n} \{n - t | a_N(t) = 1\}$ whenever $a_1(n) = 1$, and consequently w_{1N} persists iff $\frac{1}{(1 - \delta)^\rho} \leq [1 + f(\rho - (N - 1)\tau)][1 - f((N - 1)\tau)]$. ■

Proposition. When $f(x) = \gamma e^{-x/\lambda}$ and $\delta, \gamma \approx 0$, (2) holds iff $\rho \leq 2(N - 1)\tau$.

Proof: Proceeding as before we reach $e^{(N-1)\tau/\lambda} - e^{[\rho - (N-1)\tau]/\lambda} \geq \gamma$. As $\gamma \rightarrow 0$, we reach $e^{(N-1)\tau} \geq e^{\rho - (N-1)\tau}$, so $\rho \leq 2(N - 1)\tau$. ■

Our attempts to analytically solve the spike distribution in cycles of arbitrary size have been thwarted by iterated mods. Small cycles were a special case; we will present analytic

treatments of two more special cases, and then we will turn to computational methods to explore the general case.

C. Cycle Evolution: Computational Treatment

It is of interest to answer, in general,

- 1) How many pulses exist on the cycle at a given time?
- 2) What are the mean and variance of the pulse spacing distribution?

The answers to the questions above may have implications for the STDP evolution of the cycle.

D. Network Evolution

IV. CONCLUSION

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REFERENCES