ME 7247: Advanced Control Systems	**Fall 2021–22**

Homework 2

Due Date: Thursday October 14, 2021

1. *Subspace properties.* Suppose $A \in \mathbb{R}^{m\times n}$. Use the SVD of A and its associated properties to verify the following facts:

   (a) $\operatorname{rank}(A) + \dim(\operatorname{null}(A)) = n$. This is known as the "rank-nullity theorem".

   (b) $\operatorname{range}(A^\mathsf{T}) = \operatorname{null}(A)^\perp$.

   (c) $\operatorname{range}(A) = \operatorname{range}(AA^\mathsf{T})$.

   (d) $\operatorname{null}(A) = \operatorname{null}(A^\mathsf{T}A)$.

2. *Ellipsoid visualization.* Write a Matlab function `plot_ellipse(A)` that takes as its argument a matrix $A \in \mathbb{R}^{2\times 2}$ and plots the set of points $\left\{x \in \mathbb{R}^2 \;\middle|\; \|Ax\| \le 1\right\}$.

   Hint. A simple way to plot a circle in Matlab is parametrically, for example:

```
t = linspace(0, 2*pi, 200);   % parameter t (angle)
plot(cos(t), sin(t))          % makes a circle of radius 1
axis equal                    % makes both axes use the same scale
```

   Test your script on the matrix $A = \left[\begin{smallmatrix}3 & 2\\ 1 & 4\end{smallmatrix}\right]$. On the same figure, draw the semi-major and semi-minor axes (farthest and nearest point to the origin on the boundary of the ellipse), and show how to compute them in terms of the singular values and singular vectors of A.

3. *Frobenius norm of a matrix.* The Frobenius norm of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $\|A\|_\mathrm{F} = \sqrt{\operatorname{trace}(A^\mathsf{T}A)}$. (Recall the trace of a matrix is the sum of the diagonal entries.)

   (a) Show that

$$\|A\|_\mathrm{F} = \left(\sum_{i,j} |A_{ij}|^2\right)^{1/2}.$$

   Thus the Frobenius norm is simply the Euclidean norm of the matrix when it is considered as an element of $\mathbb{R}^{n^2}$. Note also that it is much easier to compute the Frobenius norm of a matrix than the (spectral) norm (i.e., maximum singular value).

   (b) Show that if U and V are orthogonal, then $\|UA\|_\mathrm{F} = \|AV\|_\mathrm{F} = \|A\|_\mathrm{F}$. Thus the Frobenius norm is not changed by a pre- or post- orthogonal transformation.

   (c) Show that $\|A\|_\mathrm{F} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where $\sigma_1, \dots, \sigma_r$ are the singular values of A. Then show that $\sigma_{\max}(A) \le \|A\|_\mathrm{F} \le \sqrt{r}\sigma_{\max}(A)$. In particular, $\|Ax\| \le \|A\|_\mathrm{F}\|x\|$ for all x.

4. *Properties of symmetric matrices.* In this problem P and Q are symmetric matrices. For each statement below, either give a proof or a specific counterexample.

   (a) If $P \geq 0$ then $P + Q \succeq Q$.

   (b) If $P \succeq Q$ then $-P \preceq -Q$.

   (c) If $P \succ 0$ then $P^{-1} \succ 0$.

   (d) If $P \succeq Q \succ 0$ then $P^{-1} \preceq Q^{-1}$.

   (e) If $P \succeq Q$ then $P^2 \succeq Q^2$.

   Hint: you might find it useful for part (d) to prove $Z \succeq I$ implies $Z^{-1} \preceq I$.

5. *Quadratic form for smoothing.* Express $\sum_{i=1}^{n-1}(x_{i+1} - x_i)^2$ in the form $x^\mathsf{T} P x$ with $P = P^\mathsf{T}$. Is it true that $P \succeq 0$? Is it true that $P \succ 0$? Explain.

6. *Image compression via SVD.* Consider a matrix $X \in \mathbb{R}^{m \times n}$ that represents an image, so X_{ij} is the intensity of the pixel at coordinate (i, j). One way to approximate the matrix X is to take the SVD of X, but to only keep the k largest singular values. In other words, write

$$X = \sum_{i=1}^{r} \sigma_i u_i v_i^\mathsf{T} \approx \sum_{i=1}^{k} \sigma_i u_i v_i^\mathsf{T}.$$

   Storing the full matrix X would require storing mn numbers. But if we store the vectors $\{\sigma_1 u_1, \ldots, \sigma_k u_k\}$ and $\{v_1, \ldots, v_k\}$, we only need to store $k(m + n)$ numbers. Therefore, we only require a fraction

$$\eta = \frac{k(m + n)}{mn}$$

   of the storage needed compared to storing the entire X matrix. We will use this procedure to compress an image and see how much the quality degrades.

   Run the following Matlab code to load and display an image of a mandrill:

```
load mandrill    % load image of mandrill in the matrix X
image(X)         % plot the matrix X as an image
colormap gray    % make it grayscale
axis image       % scale axes equally
axis off         % remove axes
```

   For a compression ratio of $\eta = 0.5$, find the corresponding k and plot the compressed image. Repeat the procedure for $\eta = 0.2$ and $\eta = 0.1$. How much compression can be tolerated before you notice an appreciable loss in image quality? (this is a subjective question!)