The Modeling and Simulation of Three-link Biped Using Hybrid Zero Dynamics

Andrew Lessieur
Northeastern University
360 Huntington Ave.
Boston, MA 02115
lessieur.a@husky.neu.edu

Jagatpreet Singh Nir
Northeastern University
360 Huntington Ave.
Boston, MA 02115
nir.j@northeastern.edu

Edward P. Willey
Northeastern University
360 Huntington Ave.
Boston, MA 02115
willey.e@husky.neu.edu

Stephen T. Hagen Jr
Northeastern University
360 Huntington Ave.
Boston, MA 02115
hagen.s@husky.neu.edu

Abstract

The control of a bipedal system is a challenging and complex problem that requires a powerful solution. The method of Hybrid Zero Dynamics developed at the University of Michigan is one such solution. This paper details the complete implementation of the hybrid zero dynamics of a three-link biped through the development of the kinematics, dynamic modeling, and controller design. Using the zero dynamics of our derived model, we find an energy efficient gait design, and integrate the full dynamics with a PD controller. These results are then verified with a step simulation.

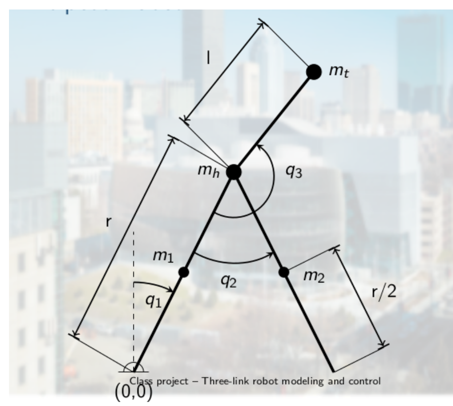


Figure 1: Three-link biped model

1. Mini Project 1

Throughout this semester these projects will cover the development for modeling and controlling a three-link robot design. It is given that the robot has no slip condition, uses one contact point at a time, and all impacts are instantaneous. This first project covers the kinematic derivations of three-link biped shown in Figures 1 and 2. The three-link model contains the parameters shown in Figure 3 for deriving the kinematics and dynamics.

1.1. Forward Kinematic Analysis

We used homogeneous transformations to define the spatial relationship between each of the coordinate frames labelled in Figure 2. Starting from the lower left joint, which we connect to the 0 world frame (0,0), we can find the connecting base transformation to joint 1. The coordinate systems are chosen so that an anticlockwise rotation points towards positive z-direction. Note that the Z-axis is perpendicular to the plane of the diagram for all of the joints. From there, we connect the hip denoted as joint 1 and find the relevant transformation. We did the following transformations

$$T_1^0 = rot_z(q_1) \times trans_y(r)$$

where $q_1 > 0$ for anticlockwise rotation.

$$T_1^0 = \begin{pmatrix} \cos(q_1) & -\sin(q_1) & 0 & -r\sin(q_1) \\ \sin(q_1) & \cos(q_1) & 0 & r\cos(q_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From the hip, the transformation of frame 2 with respect

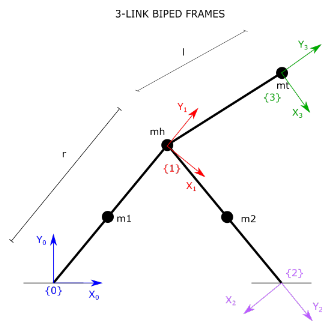


Figure 2: Three-link frame model joints with their local coordinate frames attached

to frame 1 is defined by

$$T_2^1 = rot_z(\pi) \times rot_z(q_2) \times trans_y(r)$$

$$T_2^1 = \begin{pmatrix} -\cos(q_2) & \sin(q_2) & 0 & r\sin(q_2) \\ -\sin(q_2) & -\cos(q_2) & 0 & -r\cos(q_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation of frame 3 at torso with respect to frame 1 is.

$$T_3^1 = rot_z(\pi) \times rot_z(q_3) \times trans_y(l)$$

$$T_3^1 = \begin{pmatrix} -\cos(q_3) & \sin(q_3) & 0 & l\sin(q_3) \\ -\sin(q_3) & -\cos(q_3) & 0 & -l\cos(q_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using the coordinate transformations, we can derive the transformation of each of the local frames with respect to the global frame.

$$T_2^0 = T_1^0 \times T_2^1 \quad (1)$$

$$T_3^0 = T_1^0 \times T_3^1 \quad (2)$$

We then can define the coordinates of points using the transformation matrices given as

$$P_{m_1}^0 = \begin{pmatrix} -\frac{r\sin(q_1)}{2} \\ \frac{r\cos(q_1)}{2} \end{pmatrix}$$

$$P_{m_2}^0 = \begin{pmatrix} \frac{r\,(\sin(q_1+q_2)-2\sin(q_1))}{2} \\ -\frac{r\,(\cos(q_1+q_2)-2\cos(q_1))}{2} \end{pmatrix}$$

$$P_{m_H}^0 = \begin{pmatrix} -r\sin(q_1) \\ r\cos(q_1) \end{pmatrix}$$

$$P_{m_T}^0 = \begin{pmatrix} l\sin(q_1+q_3) - r\sin(q_1) \\ r\cos(q_1) - l\cos(q_1+q_3) \end{pmatrix}$$

2. Mini Project 2

Parameter	Value	Description
m1	5 kg	mass of swing leg
m2	5 kg	mass of stance leg
mh	15 kg	mass of hip
mt	10 kg	mass of torso
l	0.5 m	length of torso
r	1 m	length of each leg

Figure 3: Link lengths and masses of the biped model in Figure 1

2.1. Energy Analysis

Using the kinematic equations from mini project 1, we derived the Lagrangian functional for energy of the system. This provides the dynamics of our planar three-link biped. The system assumes point masses at all the joints. The assumption that all links are point masses simplifies the dynamics by making all the rotational kinetic energy terms of the links to be zero. This results in a kinetic energy equation of:

$$\begin{aligned} \mathbf{K} = & \frac{\mathrm{M_t}\left(\dot{q}_1^2\, l^2 - 2\cos(q_3)\,\dot{q}_1^2\, l\, r + \dot{q}_1^2 r^2\right)}{2} \\ & + \frac{\mathrm{M_t}\ \left(2\,\dot{q}_1\,\dot{q}_3\, l^2 - 2\cos(q_3)\,\dot{q}_1\,\dot{q}_3\, l\, r + \dot{q}_3^2\, l^2\right)}{2} \\ & + \frac{m\, r^2\left(2\,\dot{q}_1\,\dot{q}_2 + 5\,\dot{q}_1^2 + \dot{q}_2^2 - 4\,\dot{q}_1^2\cos(q_2) - 4\,\dot{q}_1\,\dot{q}_2\cos(q_2)\right)}{8} \\ & + \frac{\mathrm{M_h}\,\dot{q}_1^2\, r^2}{2} + \frac{m{\dot{q}_1}^2\, r^2}{8} \end{aligned}$$

The potential energy of the complete model is

$$\begin{aligned} \mathbf{V} = & \, g\,(\frac{3\,m\,r\cos(q_1)}{2} - \mathrm{M_t}\, l\cos(q_1+q_3) \\ & -\frac{m\,r\cos(q_1+q_2)}{2} + \mathrm{M_h}\, r\cos(q_1) + \mathrm{M_t}\, r\cos(q_1)) \end{aligned}$$

This gives a complete picture of the biped's energy that can be used to find the proper Lagrange formulation using the following set of equations,

$$L = K - V \quad (3)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \quad (4)$$

2.2. Dynamic Model of the Swing Phase of a Three-Link Biped

Using (4), we derive the dynamics of the three link biped during single support phase, referred as swing phase, in the form

$$D(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = B\,u \tag{5}$$

The D matrix is represented by its components:

$$D = \begin{bmatrix} D_{1,1} & D_{1,2} & D_{1,3} \\ D_{1,2} & D_{2,2} & 0 \\ D_{1,3} & 0 & D_{3,3} \end{bmatrix} \tag{6}$$

$$D_{1,1} = A + \mathrm{M_h}\,r^2 + \frac{m\,r^2}{4} - \frac{m\,r^2\,(8\cos(q_2) - 10)}{8}$$

where $A = \mathrm{M_t}\,l^2 + \mathrm{M_h}\,r^2 + \mathrm{M_t}\,r^2 + \dfrac{3\,m\,r^2}{2}$

$$D_{1,2} = -\frac{m\,r^2\,(4\cos(q_2) - 2)}{8}$$

$$D_{1,3} = \frac{\mathrm{M_t}\left(2\,l^2 - 2\,l\,r\cos(q_3)\right)}{2}$$

$$D_{2,1} = -\frac{m\,r^2\,(4\cos(q_2) - 2)}{8}$$

$$D_{2,2} = \frac{m\,r^2}{4}$$

$$D_{3,1} = \frac{\mathrm{M_t}\left(2\,l^2 - 2\,l\,r\cos(q_3)\right)}{2}$$

$$D_{3,3} = \mathrm{M_t}\,l^2$$

The C matrix is represented by its components:

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & 0 & 0 \\ C_{3,1} & 0 & 0 \end{bmatrix} \tag{7}$$

$$C_{1,1} = \frac{\dot{q}_2\,m\sin(q_2)\ r^2}{2} + \mathrm{M_t}\,\dot{q}_3\,l\sin(q_3)\ r$$

$$C_{1,2} = \frac{m\,r^2\sin(q_2)\ (\dot{q}_1 + \dot{q}_2)}{2}$$

$$C_{1,3} = \mathrm{M_t}\,l\,r\sin(q_3)\ (\dot{q}_1 + \dot{q}_3)$$

$$C_{2,1} = -\frac{\dot{q}_1\,m\,r^2\sin(q_2)}{2}$$

$$C_{3,1} = -\mathrm{M_t}\,\dot{q}_1\,l\,r\sin(q_3)$$

The gravity, G matrix is defined as

$$G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \tag{8}$$

$$\begin{aligned} G_1 = {} & -g\,(\mathrm{M_h}\,r\sin(q_1) + \mathrm{M_t}\,r\sin(q_1)) \\ & -g\frac{3\,m\,r\sin(q_1)}{2} + g\mathrm{M_t}\,l\sin(q_1 + q_3) \\ & g\frac{m\,r\sin(q_1 + q_2)}{2} \end{aligned}$$

$$G_2 = \frac{g\,m\,r\sin(q_1 + q_2)}{2}$$

$$G_3 = \mathrm{M_t}\,g\,l\sin(q_1 + q_3)$$

Since we have two control inputs at the actuators for one of the legs and the torso, therefore the control matrix B is

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

The state space equation of the full dynamics becomes

$$\dot{x} = f(x) + g(x)u \tag{10}$$

where $x = [q, \dot{q}]$ and u is the control effort applied by the actuators.

$$f(x) = \begin{bmatrix} \dot{q} \\ -D^{-1}\,C(q,\dot{q})\,\dot{q} - D^{-1}G \end{bmatrix} \tag{11}$$

$$g(x) = \begin{bmatrix} 0 \\ -D^{-1}\,B \end{bmatrix} \tag{12}$$

2.3. The Unpinned Dynamic Model

For the unpinned model, we augment the set of generalized coordinates q with the position of the base frame $\{0\}$, which is the point of contact of the stance foot of the whole system in a 2D plane as $p_e = (p_e^h, p_e^v)$. Therefore, the new generalized coordinates become $q_e = (q, p_e)$. With these choice of generalized coordinates, the augmented inertial matrix is defined as

$$D_e = \sum_{i=1}^{N} m_i \frac{\partial p_{i_e}}{\partial q_e}^T \frac{\partial p_{i_e}}{\partial q_e} \tag{13}$$

3. Mini Project 3

3.1. Zero Dynamic Model of the Biped

The goal is to find an analytic expression of the zero dynamics and do gait planning using the constraints imposed by zero dynamics. The first step is to partition the whole dynamic model in (5) in the following way

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{q_n} \\ \ddot{q_b} \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = Bu \tag{14}$$

where

$$q_b = [q_2 \quad q_3]^T,\ q_n = q_1$$

$$D_{11} = D_{1,1}, D_{12} = \begin{bmatrix} D_{1,2} & D_{1,3} \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} D_{2,1} \\ D_{3,1} \end{bmatrix},\ D_{22} = \begin{bmatrix} D_{2,2} & D_{2,3} \\ D_{3,2} & D_{3,3} \end{bmatrix}$$

and let P be defined as

$$P = C\,\dot{q} + G = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

from which we have

$$H_1 = P_3,\ H_2 = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

In our partitioned dynamics, (15) represents the zero dynamics

$$D_{11}\,\ddot{q_n} + D_{12}\,\ddot{q_b} + H_1 = 0 \tag{15}$$

We will proceed by representing equation (15) in terms of q_n which is our gait design variable for this biped model. The complete zero dynamics will be expressed in terms of q_n. For gait-planning, we define a normalized variable s in terms of the gait stance variable q_n. For our specific case we have q_n as q_1.

$$s = \frac{q_1 - q^+}{\Delta\, q_1^{max}} \tag{16}$$

where $\Delta\, q_1^{max} = q_1^- - q_1^+$. A complete swing phase cycle with continuous dynamics model is defined as the leg stance after impact and ends just before impact. In one complete cycle, q_1^+ represents the angle of the stance leg which is in contact with ground just after impact. q_1^- represents the angle of the same leg which was in contact with ground just before impact. After this time period, the other leg will come in contact. Therefore, the gaits are cyclic. We have defined our output function $y = [y_1, y_2]$ in terms of holonomic constraints as

$$y_1 = q_2 - h_1(s) \tag{17}$$

$$y_2 = q_3 - h_2(s) \tag{18}$$

Here $h_i(s)$, for $i \in 1, 2$ represents the Bézier curve polynomials for gait planning in terms of the stance variable. While on the zero dynamics manifold, our output function and it's first derivative will be zero, $y = 0, \dot{y} = 0$. Therefore, the following relations hold on this manifold

$$q_2 = h_1(s)$$

$$q_3 = h_2(s)$$

From the above two equations, we also have $\dot{q_j} = 0,\ \ddot{q_j} = 0$, for $j \in 2, 3$. We have $\ddot{q_b} = [\ddot{q_2}, \ddot{q_3}]$, and substituting $\ddot{q_b}$ in (15), we get the following two equations of the form:

$$\ddot{q_b} = \beta_1 + \beta_2\,\ddot{q_n} \tag{19}$$

$$(D_{11} + D_{12}\beta_2)\,\ddot{q_n} + D_{12}\,\beta_1 + H_1 = 0 \tag{20}$$

In our case, $q_n = q_1$, therefore we have

$$(D_{11} + D_{12}\beta_2)\,\ddot{q_1} + D_{12}\,\beta_1 + H_1 = 0$$

3.2. Impact Map

We assumed the following for impact map modeling:

1. At impact, the swing leg does not slip or rebound, while the stance leg releases without interaction with the ground.
2. The double support phase, where both stance and swing leg are in contact, is instantaneous and the impact can be modeled as a rigid contact.
3. The design of the gait has to be symmetric. The stance leg and the swing leg swaps the role after the impact.

Using the kinematics of the three- link biped, we can determine the generalized position $q_e$of the unpinned dynamic model in section 2.3 just before the impact.

$$q_e^- = \begin{bmatrix} q^- \\ \Upsilon_e(q^-) \end{bmatrix} \tag{21}$$

With the above assumptions, the velocity just before the impact is determined from the single support model. We obtain from here,

$$\dot{q_e^-} = \begin{bmatrix} I_{N\times N} \\ \frac{\partial}{\partial q}\Upsilon_e(q^-) \end{bmatrix} \dot{q^-} \tag{22}$$

The impact model maps the $(q_e^-, \dot{q_e}^-)$ to $(q_e^+, \dot{q_e}^+)$ and is given by

$$\begin{bmatrix} q^+ \\ \dot{q^+} \end{bmatrix} = \begin{bmatrix} Rq^- \\ [R \quad 0_{N\times 2}]\Delta_{\dot{q}_e}(q^-) \end{bmatrix} \tag{23}$$

where

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Delta_{\dot{q}_e} = D_e^{-1} E_2' \Delta_{F_2} + \begin{bmatrix} I_{N\times N} \\ \frac{\partial}{\partial q}\Upsilon_e \end{bmatrix}$$

$$\Delta_{F_2} = -(E_2 D_e^{-1} E_2^T)^{-1} E_2 \begin{bmatrix} I_{N\times N} \\ \frac{\partial}{\partial q}\Upsilon_e \end{bmatrix}$$

$$E_2 = \frac{\partial p_2(q_e)}{\partial q_e}$$

where $p_2(q_e)$ is the position of the end of the swing leg with respect to the inertial frame. The matrix R represents the change of coordinate system so that the base coordinate system switches from the current stance leg to swing leg as the contact switches. We express (23) in shorthand as

$$x^+ = \Delta(x^-) \tag{24}$$

The complete model during swing phase and impact phase can be represented as a combination of (13) and (24).

3.3. Bézier curve and Gait Optimization

A 4^{th} order Bézier curve and its derivative goes in the following equations

$$b_1(s) = \sum_{k=0}^{4} \alpha_k \frac{4!}{k!(4-k)!} s^k \, (1-s)^{(4-k)}$$
$$b_2(s) = \sum_{k=0}^{4} \gamma_k \frac{4!}{k!(4-k)!} s^k \, (1-s)^{(4-k)}$$

The derivative of the above Bézier curve is defined as

$$\frac{\partial b_1(s)}{\partial s} = \sum_{k=0}^{3} (\alpha_{k+1} - \alpha_k) \frac{4!}{k!(4-k-1)!} s^k \, (1-s)^{(4-k-1)}$$
$$\frac{\partial b_2(s)}{\partial s} = \sum_{k=0}^{3} (\gamma_{k+1} - \gamma_k) \frac{4!}{k!(4-k-1)!} s^k \, (1-s)^{(4-k-1)}$$

Now, using the definition of the Bézier curves and their partial derivatives, we can represent the joint variables in (17) and (18) (q_2 and q_3) as a function that depends on the selection of the variables α and γ. The final aim is to optimize over the parameter space of α_k and γ_k where $k \in \{0, 1, 2, 3, 4\}$ to get an energy efficient gait. We minimize the cost of mechanical transport to obtain our trajectory variables. We define the problem more formally here as minimizing the mechanical transport cost

$$J = \int_{t^+}^{t^-} ||u||^2 dt \tag{25}$$

to obtain a set of parameters α and γ while obeying the following constraints:

$$(D_{11} + D_{12}\beta_2)\, \ddot{q}_1 + D_{12}\, \beta_1 + H_1 = 0$$
$$x^+ = \Delta(x^-)$$

3.3.1 Bézier Curve Intuition

When optimizing the trajectories for the body angles, q_2 and q_3, we use 4^{th} order Bézier curves. For each curve, there are a total of five coefficients that influence the shape of the curve. However, we do not have access to two of these coefficients, α_1 and α_2, as these are dependent on the post-impact states for q_1 and $\dot{q}_1$. This reduces the complexity of the optimization problem, though, as there are two less parameters per body angle to optimize.

By definition, the final coefficient, α_5, is the value of the Bézier curve at $s = 1$. Therefore, on the zero dynamics manifold, this is the value that q_b will take at the end of the swing phase. The slope of the Bézier curve is related to the derivative of the body angles, q_b. This slope can be approximated with the coefficients, α_4 and α_5.

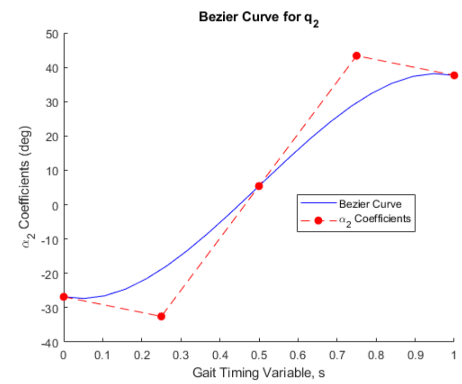


Figure 4: The plot describes the optimal trajectory of q_2 for minimum mechanical transport as the biped achieves a stable limit cycle shown in Figure 6. The parameters have been obtained by running an optimization routine to minimize the mechanical transport cost function in equation 26 and the zero dynamic constraints are obeyed.

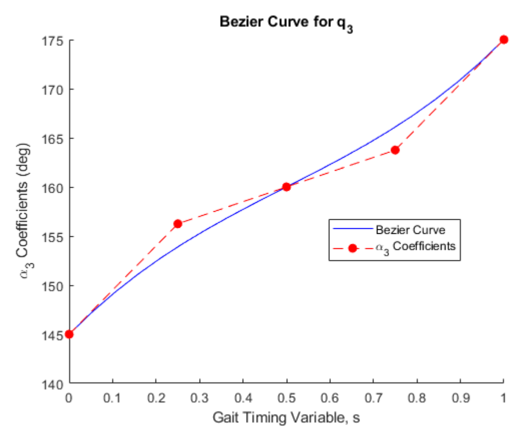


Figure 5: The plot describes the optimal trajectory of q_3 for minimum mechanical transport as the biped achieves a stable limit cycle shown in Figure 6

Given this intuition, an initial choice for Bézier coefficients, which is influenced by the desired body angle trajectories, is passed to the optimizer.

3.3.2 Step Length Parameterization

An alternative approach we implemented was parameterizing the Bézier Curves by a step length, a. We parameterize our pre-impact states in terms of a by assuming a symmetric gait and using trigonometric identities related to leg length, r.

$$\theta = \arccos\left(\frac{a}{2r}\right) \tag{26}$$

$$\beta = \pi - 2\theta \tag{27}$$

From Eqns. 26 and 27, we have

$$\begin{aligned} q_1^- &= \theta - \frac{\pi}{2} \\ q_2^- &= \beta \\ \dot{q}_2^- &= \pi - abs(\dot{q}_1) \\ q_3^- &= \delta_1, \text{ a fixed constant} \end{aligned}$$

The Bézier coefficients are then defined as

$$\alpha_5 = q_b^-$$

$$\alpha_4 = \alpha_5 - \frac{1}{M}\dot{q}_b^-$$

This leaves the choice of optimization parameters as a, $\dot{q}_1^-$, q_3^-, $\dot{q}_3^-$, α_3, and γ_3. Rather than iterating over Bézier coefficients, this parameterization of the Bézier coefficients provides a physical intuition for values selected as an initial guess. It can be assumed that $q_3^- \approx \pi$, $\dot{q}_3^- \approx 0$, while α_3, and γ_3 can be chosen to be approximately what value we imagine q_2 and q_3 will take around the middle of the gait.

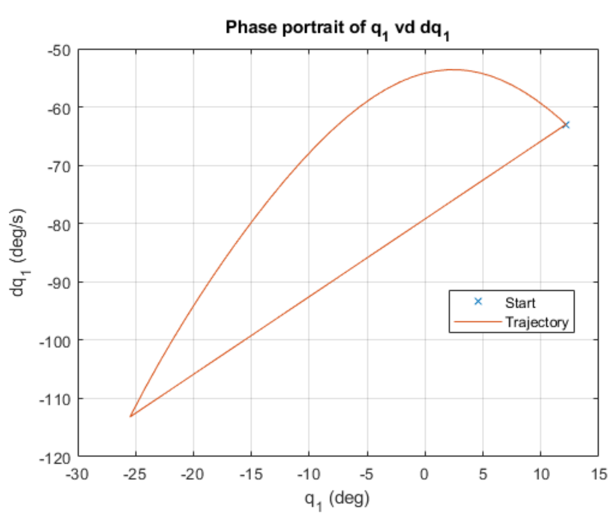


Figure 6: The phase portrait exhibits the repetitive behavior of cyclic variable q_1 for the input trajectory provided by the Bézier curves shown in Figure 4 and 5

4. Mini-project 4

4.1. Simulation of Full Dynamics

From the optimization function, the parameters denoted as $f = [q_1, \dot{q}_1, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3]$ are imported into the full dynamics simulation. The values from the optimization did not provide better results than the initial guess, so the parameters we used in simulation were:

$f = [-0.3478,\ -3.0765,\ 0.1520,\ 0.3604,\ 0.5711,\ 2.8535$
$2.8806,\ 3.2398]$

The simulation uses the model parameters and initial conditions from a designated configuration to assign values to the matrices of D,C,G and B in section 2.2. The state space model is then used from equation 10 to get $f(x)$ and $g(x)$.

4.2. Function Feedback

The controller for this system is a basic PD controller featuring feedback linearization. Figure 7 shows a simplified view of the controller.

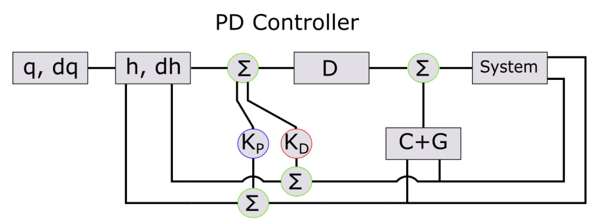


Figure 7: Simplified PD + Feedback linearizing Controller Overview

The state space equation Y is nonlinear as its derives $\dot{Y}$ and $\ddot{Y}$. These time derivatives of Y can be defined as

$$\dot{Y} = \frac{\partial h}{\partial q} \tag{28}$$

$$\ddot{Y} = \frac{\partial}{\partial q}\left[\frac{\partial h}{\partial q}\,\ddot{q}\right] + \frac{\partial h}{\partial q}\left[D^{-1}[-C\,\dot{q} - G] + D^{-1}\,B\,u_2\right] \tag{29}$$

In order to linearize this system for the controller, Lie derivatives are used. From this, $\dot{Y}$ and $\ddot{Y}$ can be interpreted as

$$\dot{Y} = L_f h \tag{30}$$

$$\ddot{Y} = L_f^2 h + L_g L_f h\, u \tag{31}$$

where

$$L_f h = \frac{\partial h}{\partial x} f(x)$$
$$L_g h = \frac{\partial h}{\partial x} g(x)$$
$$L_f^2 h = \frac{\partial h \, L_f h}{\partial x} \, f(x)$$
$$L_g L_f h = \frac{\partial h \, L_f h}{\partial x} \, g(x)$$

In this system, $L_f^2 h$ is the feed-forward term, and $L_g L_f h \, u$ is the feedback component. The control action u component from $\ddot{Y}$ is defined

$$u = (u^* - v) \tag{32}$$

where u^* and v are defined in terms of Lie derives as

$$u^* = -(L_g L_f h)^{-1} L_f^2 h \tag{33}$$
$$v = K_p \, h + K_d \, L_f h \tag{34}$$

The component of feedback u^* is responsible for cancelling the internal systems dynamics caused by the feed-forward term and the decoupling matrix. Removing these dynamics allow the system dynamics to evolve strictly on the stabilizing feedback controller v.

This process of feedback linearization is particularly advantageous because it is control invariant. Using u^* to cancel out the internal dynamics allows the designer to influence the convergence onto the zero dynamics manifold based on preference for any controller, even one such as the PD controller implemented in this project.

The proportional and derivative constant values to be applied to v were estimated to be:

$$K_p = \begin{bmatrix} 250 & 0 \\ 0 & 250 \end{bmatrix}$$
$$K_d = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

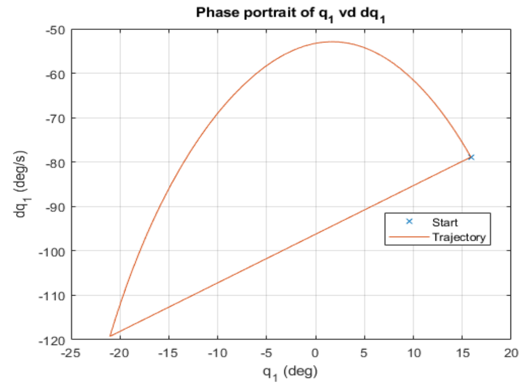
These values came from an iterative process of maximizing K_p while holding K_d to zero. Once this was done, we increased damping to improve results of phase portrait.

4.3. Phase Portrait Analysis

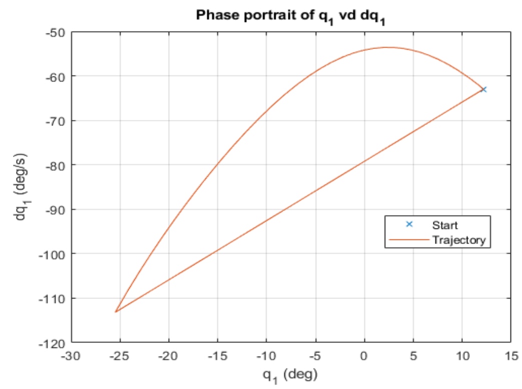
The objective of the controller is force the system to converge onto the zero dynamics manifold. In our simulations, we used two methods of selecting Bézier coefficients:

1. Using a provided initial guess for our optimizer, then iterating through nearby solutions
2. Using step length as primary parameter and basing coefficients off physical intuition for initial guess

Taking this approach gives us the opportunity to design a gait and controller then compare performance with a predesigned gait.



(a)



(b)

Figure 8: Zero Dynamics Limit Cycles:
(a) Provided Guess; (b) Step Length Parameterization

4.3.1 Zero Dynamics Limit Cycle

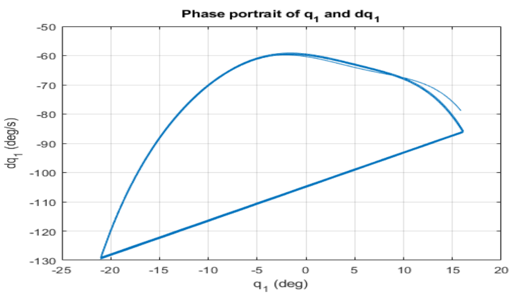
During gait design, the objective is to choose Bézier coefficients and initial conditions that result in a stable limit cycle. This will become the desired limit cycle when designing a controller for the full dynamics simulation.

The provided initial guess was passed to our optimization script, as previously described. Then, using our step length parameterization, configurations were chosen to result in similar limit cycles. The resulting zero dynamics phase portraits are shown in Figure 8.

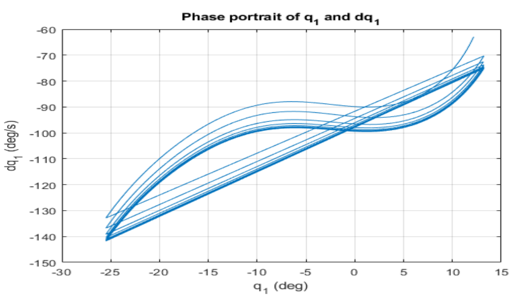
4.3.2 Controller Design

Choosing appropriate values for the PD gains was an iterative process described in Section 4.2. The qualities desired during this iterative process were strong convergence to a stable limit cycle and resemblance of the Zero Dynamics limit cycle.

We quickly noticed that the limit cycle with Bézier Curves parameterized by step length did not converge to the Zero Dynamics limit cycle. The behavior shown in Figure



(a)



(b)

Figure 9: Phase portrait of the cyclic variable $q_1$obtained when the stabilizing control input from (32) was provided to the biped. (a) Provided Initial Guess; (b) Step Length Parameter

9b indicates a change in kinematic and dynamic behavior. This is further supported when comparing the joint angle trajectories shown in Figure 10. The strong convergence to a stable limit cycle was preferred over other trials during the controller design, but no trials with this parameterization converged to a limit cycle that resembled the Zero Dynamics limit cycle.

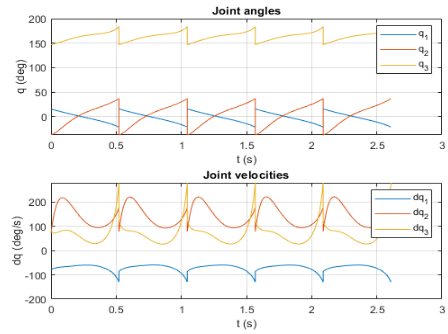
Using the same controller design on the provided parameters resulted in a limit cycle that satisfied both desires. Therefore, moving onto Poincaré analysis, we only considered the provided initial guess.

4.3.3 Poincaré Analysis

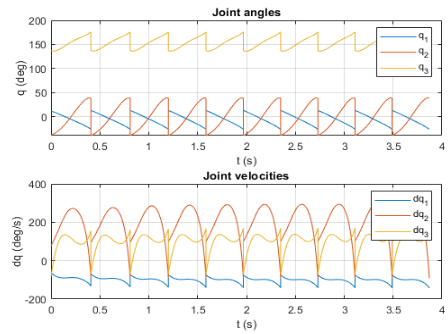
To stress test the controller design, we kept the PD gains constant but changed the initial point for the full dynamics simulation. For a given initial condition, we assessed how well the system would converge back to the zero dynamics manifold. This can be seen with a representative new initial condition shown in Figure 11. The system will converge back to the Zero Dynamics manifold after a few steps.

4.4. Limitations

The current controller faces a few issues when it comes to the impact piece of the model. As the robot steps in simulation it 'sinks' into the floor. This is likely due to the simulation event handler incorporated with *ode45*. The param-



(a)



(b)

Figure 10: Joint trajectories for n steps for different parameters: (a) Provided Initial Guess; (b) Step Length Parameter

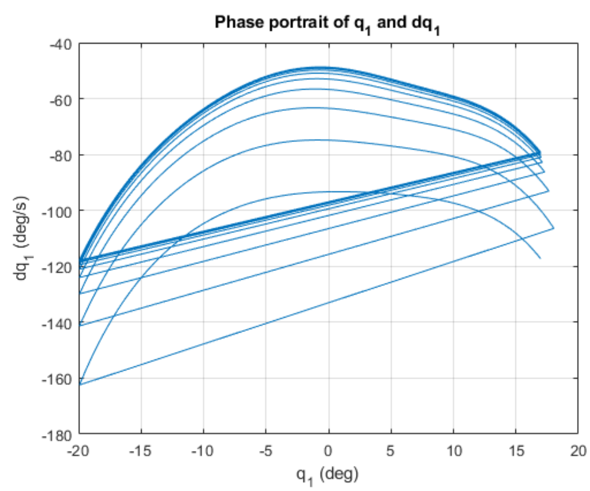


Figure 11: Given parameter dynamics under feedback control

eter that determines the end of a step is characterized by the gait timing parameter, s, rather than the swing foot location, p_2. Because of this, there is no guarantee that the swing foot is on the ground when a simulated step is ended. However, it is difficult to correct for this with a two- or three-link biped because they lack knee joints.

Additionally, the model seems not to be impact invariant. It is likely that when parameterizing the gait via step length, some assumptions and constraints were violated. This led to the system not being impact invariant, allowing impulsive effects to strongly alter the shape of the limit cycle.

5. Conclusions and Future work

The report presents complete methodology of modeling a biped. Using the hybrid zero dynamics as a constraint, we optimised the desired joint trajectories to reduce mechanical transport. Further, a full dynamic controller was implemented with a stabilizing control law. The project reveals interesting insights as mentioned in section 4.4. In the simulation of the full dynamics, the system does not converge exactly to the zero dynamics manifold. Through this project, the hybrid zero dynamics of a three link biped was utilized to produce a walking gait and control this system to walk several steps. The results did not come without a lot of time spent debugging and iterating, and the final results themselves are not perfect.

Future work requires to build a concrete way to describe the differences seen between the phase portraits obtained when the initial conditions are changed. In addition, constraints need to be imposed so that the walking biped does not descend along a slant as seen in some simulations.