may just simply terminate !) ... infinity — at which point all *common sense* intuitions and reckonings

All the difficulties that we have found with ZF Set Theory would be obviated if the Replacement and Selection schemata in Equations (13) and (14) were revised to additionally contain a predicate requiring that their base sets z had countable size. (Indeed instead, it might feasible to merely require that the cardinality of z be strictly less than the cardinality of P_1 in these axioms.)

Let us use the term Low-ZF to refer to such a weaker version of ZF. The formalism Low-ZF is probably too weak for it to be thought of as a serious variant of set theory. For example, common set manipulation operations such as set-subtraction, set-intersection, function-composition and the construction of the cross product of two sets require usage of a version of the Replacement and/or Selection schemata that are not available in Low-ZF. (Also, the definition of the set of non-negative integers as the minimal-sized infinite set satisfying Axiom I's "infinity" condition requires usage of the Selection axiom.) One would therefore wish to add to Low-ZF a series of other proper axioms so that these functional operations and objects would be retained.

Our intuition is that some starting construct, similar Low-ZF, is needed as a base for repairing ZF's inconsistencies. However, a further difficulty is that Low-ZF may also be too strong (especially when it is united with new axioms for defining the set of integers and the operations of intersection, cross-product, function-composition and set-subtraction). The difficulty is that the unlimited usage of arbitrary quantified variables within the formulae of $\Psi(x,y)$ and $\Phi(u)$ in the Replacement and/or Selection axiom schemata is potentially troublesome (and could potentially cause Low-ZF to be inconsistent).

After all, Russel's paradox concerning the conception of the notion of the *"set of all sets such that ..."* has a framework that has an uncomfortable analogy with the arbitrary use of quantifiers within the formulae of $\Psi(x,y)$ and $\Phi(u)$ in the Replacement and/or Selection axiom shemeta.... *Should this be permitted ?* .... Or alternatively should each set quantified variable in $\Psi(x,y)$ and $\Phi(u)$ be a "bounded set quantifier" of the approximate form of " $\forall\, p \in q$ " and " $\exists\, p \in q$ ", where q is some pre-specified fixed set?

One is tempted to follow the analogy of Russel's Paradox and to wonder whether some type of bounded quantifiers should be required to appear in $\Psi(x,y)$ and $\Phi(u)$ .... But it is not 100% clear?

The confusing aspect is that we were already implicitly using bounded quantifiers in Section 's definition of Support-ZF because all the intermediate variables used to define the notions of completeness, consistency and maximality can be formalized by quantifiers ranging over the set of positive integers (or their collection of associated Gödel numbers). *Thus even with this added constraint in place about quantifier ranges* ..... , Theorem 3's vexing inconsistency result continues to remain in force *unless one forbids* the base sets, called z in the Replacement and Selection schemata of Equations (13) and (14), from having a cardinality as large as that of P_1.

In a context where P_1 is a set whose infinite cardinality has a well-known "uncountable" dizzying nature, this restriction will hopefully affect not many results in Applied Mathematics, Computer Science or in most of the concrete facets of Theoretical Mathematics.

NEW SECTION BREAK : Concluding Remarks

The prior chapter of this paper had clearly indicated that one possible method to repair ZF Set Theory would be to simply drop its power set axiom. Then the P_1 power set would no longer be available to interact with the Replacement axioms. In our opinion, this option would be too radical — in that it would force one to depart from the majestic foundational formalism that Hilbert had called "Cantor's Paradise".

A better solution is to weaken ZF's Replacement axiom schemata (which was not part of the initial somewhat informally specified Cantor scheme). In that case, ZF's Replacement axiom schemata would still retain an infinite number of instances, but it would not be as broad as the current schemata.

Our hope and anticipation is that most of the renown beauty of Set Theory to conceptualize highly abstract objects would be retained within such a revised framework — while the inconsistencies that arise from the excesses of the current version of ZF would be singularly removed. In such a context, a new revised version of ZF Set Theory would presumably support all the predictions of Applied Mathematics and most of the formalisms of Theoretical Mathematics — while being protected from inconsistencies.

The author of this article plans to accompany this paper with a second article, which outlines our proposals for revising ZF's formalism. We deliberately do not include those proposals in this paper. This is because any effort to revise Set Theory at a short notice would be speculative (because if constrcuted hastily it could either be inconsistent on account of its undue strength ..... or alternatively it might fail to be sufficiently far-reaching on account of its being excessively conservative). We would thus prefer our proposals on how to reconstruct Set Theory to appear in a separate manuscript — so that the community of readers could not possibly confuse the speculative part of of our research project from the firmly derived results concerning the inconsistency of ZF, given in this paper.

Some partial speculations about the likely shape of a new version of set theory had appeared in Section . Our suspicion is that such a revised formalism is unlikely to cause major changes in Discrete Mathematics, in Computer Science, in the concrete facets of Theoretical Mathematics or in Applied Mathematics because the major part of these formalism can be presumably couched in terms that do not require ascending very far into the heights of uncountably large infinite sets.

The author of this article is not adequately familiar with the literature about large cardinals to make any firm comments about it. It is plausible that large cardinal numbers may somehow play a very significant rôle in some new type of axiomization for set theory. This is because if the Replacement Schemata is weakened then theorems about large cardinals may play useful as intermediate results for understanding the property of smaller sets (with conventional cardinalities) that occur in every-day reality. (In particular in Number Theory and Combinatorics, one often constructs very large integers during the intermediate steps of a proof to understand better the properties of smaller more regularly sized objects. It would not be surprising if an analogous framework in a new version of Set Theory, that no longer has the luxury of relying upon a strong version of the Replacement Schema, analogously employs large cardinals as an intermediate step within a proof to gain a better understanding of smaller sets with more compressed cardinalities.)

The essential point is that ZF Set Theory has had a remarkable success record over the last 100 years despite its technical inconsistency. Otherwise, it would not have been the subject of so much research attention during the last century and defied prior efforts to find any inconsistency embedded within it. In such a context, we suspect that a revised form of Set Theory is feasible and will continue the wonderful magic that Hilbert had called "Cantor's Paradise".

In essence, some type of revised form of Set Theory is needed to explain how a language of logic can conceptualize and formalize the many non-recursively defined entities that appear in the world of mathematics, computer science, philosophy and every-day reality in a formally consistent and helpful manner.

nemo 775 746 3630 office home 746 2154 12pm

1-800 471 1805

pin 149 818 1487

The future utility of set theory is likely to belong as much to philosophy and to general models of every-day reality, as it is to the formalization of the ultimate foundations of mathematics and to computer science.

NEW SECTION BREAK : Appendix: Summary of the Proof of the Statement ++

This appendix will briefly summarize how one can use the Gentzen Sequent Calculus formalism, as was summarized in Chapter 1 of Takeuti's textbook in [18], to extrapolate a proof of the following statement:

++ For each integer k there exists a $n > k$ such that ZF_n can prove the consistency of ZF_k.

We need one definition in order to summarize how ++ may be proven.

Definition 2. Given any axiom system α containing a *strictly finite number* of proper axioms, the following notation will be used

- **a.** The symbol Empty(α) will denote the Gentzen style sequent which (using Takeuti's notation [18]) enumerates all of α's axioms on the left side of its turn-style symbol and which contains the empty set on its right side.
- **b.** The symbol InconsCF(α) will denote that there is a cut-free sequent calculus proof of Empty(α) using the Gentzen sequent calculus formalism called **LK** in Chapter 1.2 of Takeuti's textbook.
- **c.** The symbol InconsHilb(α) will denote that there is a Hilbert-style proof of α's inconsistency.

d. The symbol ConsCF(α) will denote the negation of the sentence InconsCF(α). Thus it will designate that there exists no cut-free sequent calculus proof of Empty(α) using the Gentzen Cut-Free sequent calculus formalism.

In order to formally prove $++$, one needs to employ the following facts:

1. Gentzen's Cut Elimination Theorem (which has a very nice proof in Takeuti's textbook [18]) implies that InconsCF(α) and InconsHilb(α) are logically equivalent to each other. (Moreover for some fixed constant c_0 this proof can be carried out within ZF_m for all $m > c_0$. Thus if we choose n to be large enough in the statement $++$, then the knowledge of this effect will be available to ZF_n .)

2. For any fixed k , it is easy to choose a large enough $n > k$ such that ZF_n can prove the statement ConsCF(ZF_k). This is because each formula in a proof of Empty(ZF_n) will contain no more than than a fixed number of quantifiers, denoted by some number L_k , where the value of the constant L_k depends only on k . Thus if we choose n to be large enough, ZF_n will be capable of constructing a model that houses all the proper axioms of ZF_k and thereby shows that it is impossible to construct a cut-free proof of Empty(ZF_n).

The combination of Items (1) and (2) imply the validity of $++$. This is because Item (2) implies ZF_n can prove ConsCF(ZF_k) (when n is large enough), and Item (1) implies that ZF_n knows the latter to be equivalent to the Hilbert consistency of ZF_k. (See footnote [4] to explore one significant point that may otherwise potentially confuse some readers.) □

It was by deliberate intention that I put the proof of $++$ in an appendix section of this paper, rather than in one of the five main chapters of this article. This is because I am quite convinced that $++$ is already known in the literature.

Thus, I conceived of this theorem 15 years ago when reading Chapter 1 of Takeuti's textbook [18]. My somewhat hazy memory is that after proving $++$ during my reading of Takeuti's textbook in 1993, I became convinced that someone else had proved $++$ earlier. The first author who proved an analog of this result was probably Mostowski in connection with what is called "reflective" axiom systems, but I am not 100 % sure who did what and when?

It was for this reason that I thought it was safest to put the proof of $++$ in an appendix section of this article. The result is relatively easy to prove, and the correct citations about who proved it first can can be inserted into a later draft of this paper before publication takes place.

Don't forget to check section numbes and theorem numbers. One mistake found on Mar 17 and others may exist!

NEW SECTION BREAK: Old Section 5 to be removed

The preceding difficulties could be avoided by a new system. WZF (with the W for Willard) where the Replacement Axiom's base formalae are required to have "bounded set quantifiers". These would force the $\forall$ and $\exists$ quantifiers to select elements from a prespecfied sets defined by earlier stages of a proof, called say $S_1, S_2, \ldots$ where each set S_i is a countable set. It may be necessary to add some other set algebraic operatons to WZF. For example, if set subtraction and intersection cannot be encoded in this theory, then they should probably be added as new operations.

Under the formalism I envision, a proof would construct a series of new sets in its first k stages that could be called say C_1 , C_2 , C_3 , ... The new replacement axiom could use any of these sets when it contructs a new set called say C_k . However, the only quantifiers that are allowed to vary over an infinite number of sets would have their range restricted to either the set of positive integers or a countable set. (These two notions are equivalent because a countable set is generated by I and a pre-constructed function.)

It is possible that the above may even be too much. Perhaps one should allow only one universal or existential quantified variable (called X) to formally range over the set of integers. All other quantifiers ranging over the set of intgers should be bounded quantifiers of the forms $\forall\, v \leq X$ or $\exists\, v \leq X$ where X is the integer generated by the initial unbounded quantifier. Somehow, I find this idea especially appealing.

Other Remarks Gödel's Completenss and Compactness Theorem and the Lindenbaum Lemma are likely to be invalid under WZF. It may (?) be possible to add other rules to WZF where one defines the notion of quasi-set, which can be perhaps be defined by stronger versions of the Replacement axiom but whose application in the logic is somehow limited. In such a respect (which is currently ambiguous ?), one might be able to prove a theorem that is analogous to Gödel's Completeness Theorem (but involves quasi sets rather than sets being models of consistent axiom systems).

My guess is that Gödel's Completeness Theorem can be partially reconstructed in such a diluted form. For example, the way one can partially escape this whole dilemma is that the quasi-sets might be defined so that they are not a subset of any power set. Then one might be able to define a quasi-set Q, all of whose members are positive integers, but which is not an element of the Section 's power set P_1.

References

[1] R. Carnap, *Die Logische Syntax der Sprache* Springer 1934. English translation as: *The Logical Syntax of Language* published by Routeledge & Keegan Paul (1937).

[2] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin Press 1966.

[3] A. A. Fraenkel, "Zu den Grundlagen der Cantor-Zermelo Mengenlehre", *Math Annalen* 86 (1922) pp. 230-237.

[4] G. Frege, *Grundegsetzs der Arithmetik, Begriffschtflich* (1893) Partial Translation in *The Basic Laws of Arithmetic: Exposition of the System*, University of California Press, 1964.

[5] K. Gödel, " Die Vollstäandigkeit der Axiome des logische Funktionkaiklüls *Monatshefte für Math. Phys.* 37 (1930) pp. 349-360. (English traslation by Van Heijenoort on pp. 582-591 of reference [10].)

[6] K. Gödel, " Über formal unentscheidbare Sätse der Principia Mathematica und Verwandte Systeme I", *Monatshefte für Math. Phys.* 37 (1931) pp. 349-360. (Translated by Meltzer for Dover Press 1962.)

[7] K. Gödel, "The consistency of the Axiom of Choice and Generalized Continuum Hypothesis", *Proceedings of National Academy of Sciences (USA)* 24 (1939) pp. 556-557 with a formal proof in 25 (1940) pp. 220-226.

[8] P. Hájek and P. Pudlák, *Metamathematics of First Order Arithmetic*, Springer Verlag 1991.

[9] S. Hafden and J F. Kennison, *Zermello-Fraenkel Set Theory*, Charles E.Merril Publishing Company, 1968.

[10] J van Heijenoort, *From Frege to Gödel, A Source Book in Mathematical Logic.* Harvard University Press (1967).

[11] D. Hilbert and P. Bernays, *Grundlagen der Mathematic*, Springer 1939.

[12] A. Lindenbaum and A Mostowski, "Über die Unabhängigkeit des Auswahl-axioms und eineger seiner Folgerungen", *Compes Rendes Sciences Varsovic* III,31 (1938) pp. 27-32.

[13] E. Mendelson, *Introduction to Logic*, Chapman and Hall, 1997

[14] G. H. Moore, *Zermelo's Axiom of Choice: Its Origins, Development and Influnce*, Springer 1982.

[15] A. A. Mostowski, I don't remember which paper of Mostowski had this result, and I will have to fix this citation later. (I know that the the mentioned theorem about the relationship between ZF_k and ZF_n is true because I reconstructed the proof.) I have not yet had time to check, but Mostowski's result was probably in "A Generalization of the Incompleteness Theorem", *Fund Math* 49 (1961) pp. 205-232.

[16] B. Russel, "On Some Difficulties in the Theory of Transfinite Number and Order Types", *Proceedings of the London Mathematical Society (2nd Sereis)* 4 (1906) pp. 29-53.

[17] B. Russel, "Mathematical Logic is Based on the Theory of Types", *Am J. Math* 30 (1908) pp. 222-262.

[18] G. Takeuti *Proof Theory* Springer 1987.

[19] G. Takeuti and W. Zaring, *Introduction to Axiomatic Set Theory*, Springer 1980.

[20] A. Tarski, "Der Wahrheitsbegriff in den Sprachen", *Studia Phil.* 1 (1936) pp.261-405.

[21] E. Zermelo, *Beweiss, das jede Mengew ohlgeorder weiden kann,* Math Annalen 59 (1904) pp. 514-516. (English traslation by Van Heijenoort on pp. 199-215 of reference [10].)

[22] E. Zermelo, *Untersuchungen üger die Grundlagen der Menglehre weiden kann,* Math Annalen 65 (1908) pp. 262-281.

[4] This construction does not imply that ZF can prove its own consistency because the fact that it can prove every finite subset of it to be consistent does not imply a similar argument applies to their infinite union. For example, it is known that Peano Arithmetic can prove every finite subset of its axioms is consistent, but it cannot prove their infinite union is consistent (assuming as we do that Peano Arithmetic is consistent).

Dan E Willard

Sworn before me this 21 Day of march 2008

SCOTT R. BECHARD
Notary Public, State of New York
No. 01BE6141598
Qualified in Saratoga County
Commission Expires Feb. 27, 2010

Scott R Bechard
NOTARY PUBLIC