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More and ~~other~~ supporting details

On a 3-Part "Tripod" Styled Reply to Hilbert's Mysterious Second Problem

and its Further ~~Subtle Subtle Albert~~ Implications

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Abstract

Hilbert's mysterious year-1900 Second Problem asked mathematicians to devise a methodology whereby Peano Arithmetic can confirm its own consistency. Gödel's famous 1931 paper showed that a fully positive reply can never be made to Hilbert's question. This article will explain how Hilbert's question is such a complicated issue that it can be better receive a 3-way styled "Tripod" reply.

We also provide substantial evidence that Gödel would likely agree with the main opinions expressed in this article.

Keywords and Phrases: Gödel's Second Incompleteness Theorem, Hilbert's Second Problem, Consistency, Smullyan-Fitting Semantic Tableau Deduction, Hilbert-Frege Deduction.

and About the Grey Area in ~~of~~ Symbolic Logic and Philosophy

8AM–3PM Wednesday

1 Introduction

This article is a continuation of a series of papers that began with the 1993 article [38] and continued until and through the year-2021 article [47]. This series, which included six papers appearing in the JSL and APAL, had focused on discussing generalizations and boundary case exceptions for the Second Incompleteness Theorem. The two goals of this paper will be to explore the underlying philosophy that motivated this series and to explain how it is related to several generalizations and boundary-case exceptions to the Second Incompleteness Theorem , including some important new results formalized by Artemov [3, 4].

Our general theme will be that Hilbert's year-1900 Second Problem is too complex an issue to receive a 1-directional or even 2-directional answer. Instead, it will require a 3-directional answer, called the "Tripod Reply" to Hilbert's question.

The first leg of this 3-part response to Hilbert's Second Problem will rest on the combination of Gödel's initial version of his Second Incompleteness Theorem and its numerous generalizations. They collectively establish that many logical formalisms, besides Peano Arithmetic (PA), are unable to corroborate their own consistency in a fully extensive respect. While these generalizations of Gödel's Second Incompleteness paradigm are indisputably important, the second section of this article will explain Hilbert and Gödel strongly doubted they constituted a full answer to Hilbert's year-1900 problem.

A second leg of a 3-part reply to Hilbert's Second Problem was formalized by Artemov recently in [3, 4]. He noted Peano Arithmetic (PA) can prove an infinite schema of theorems, whose collective union confirms PA's own consistency. Let us call Artemov's method a Step-By-Step Infinite-Schema approach (**SBSIS**). This technique is an extension of the Justification Logics explored by Artemov and Beklemishev [2, 5, 6]. It was motivated by Artemov's observation that the year-1900 logic community (including Hilbert) were not aware that PA would require an infinite number of proper axioms. Thus, an SBSIS schema-driven logic approach is a valid reply to Hilbert's year-1900 second problem, although its potential significance was not well appreciated during the era when Hilbert posed his Second Problem.

Artemov's analysis [3, 4] uses Tarski's partial definitions of truth for sentences with a bounded number of quantifiers as an intermediate step. It essentially constructs a sequence of finite subsets of Peano Arithmetic $S_1 \subset S_2 \subset S_3 \subset \ldots$ where

1. PA $= S_1 \cup S_2 \cup S_3 \cup \ldots$

2. Each S_{j+1} can prove a Π_1 theorem asserting there exists no proof from S_j of 0=1, in a context where all logical axioms in the concerned proof from S_j are no more complex than Π_j or Σ_j statements. (We will henceforth call this theorem T_{j+1}.)

While Artemov's SBSIS-style response to Hilbert's year-1900 second question is intriguing, one would ideally still like access to a system that does not rely upon his infinite series $S_1\, S_2\, S_3 \ldots$.

A third possible leg of a proposed Tripod reply to Hilbert's second problem involves formal systems using Fixed Point axiomatic sentences that confirm their own consistency. For example, [38, 39, 41, 42, 47] examined systems strictly weaker than PA, that verified their own self-consistency, under mostly semantic tableau deduction [13, 32].

This third leg, called the **Declarative Approach**, rests on using a self-referencing *"I am consistent"* axiomatic declaration, so that a formalism can confirm its own consistency. This approach, studied in [38, 39, 41, 42, 47], will essentially be defined by the current paper's statement $\oplus$. Its advantage is that its declaration of self-consistency is compressed into one single sentence, while its **non-trivial drawback** is that its particular statement $\oplus$ (defined axiom system, whose relationship to the ISCE(β) system is as follows:

i. The Group-1 and Group-2 schemes for $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ and ISCE(β) are essentially identical.

ii. The Group-Zero scheme for $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ is stronger than that of ISCE(β) because it replaces the Additive Naming Convention with a stronger more compact "Type-A" statement declaring that the operations of Addition and Double(x) $= x + x$ are total functions.

iii. The Group-3 scheme for $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ is, however, weaker that its analog under ISCE(β) because it only recognizes its self-consistency under a semantic tableau form of deduction.

In essence, $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ is a self-justifying formalism that avoids being inconsistent by using a different type of trade-off than ISCE(β), where its Group-Zero schema is stronger while its Group-3 statement uses a weaker deductive methodology. (A more detailed more formalistic definition for $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ can be found in our ~~2021~~ TCPS-2020 article) Not Indy

Also, we will employ the Example 5.1 from [47], where $\mathrm{IS}^{M}_{\mathrm{Tab}}(\beta)$ denotes the natural modification of $\mathrm{IS}_{\mathrm{Tab}}(\beta)$ wherein:

1. The Group-Zero axiom further recognizes integer-multiplication as a total function.

2. The Group-3 axiom is the same as that described in (iii) except that its Group-3 *"I am*

one single theorem, asserting a formalism's overall consistency. (Instead, it produces a sequence of theorems T_1, T_2, T_3, ... , whose approximate union comprises the desired *"I am consistent"* statement.)

Robert said remark

b. The comparable drawback in Willard's self-justifying formalisms in [38]-[47]'s ~~28 year~~ long series of articles is that none of these papers involve a system as powerful as Peano Arithmetic declaring its own self-consistency.

The nice aspect of Remark 13's hybridization of these two methods is that its methodology views the two formalisms from Propositions 11.a and 12.a as declaring their self-consistencies under certain unifying perspectives, while owning a formalized awareness about Artemov's broader expanding series of theorems T_1, T_2, T_3, ... This paradigm overcomes the main challenges that were posed by Items (a) and (b), and it allows us to construct hybrid of Artemov's and Willard's methodology

For the sake of clarity, Remark 13's observations certainly should not be viewed as being a full panacea. Thus, the Second Incompleteness Theorem certainly imposes severe restrictions upon any attempts to evade it. Thus while not fully embracing the philosophical positions of Hilbert's and Gödel's statements of $*$ and $**$, our perspective in Remark 13 certainly conveys compatible with self-justifying logics using semantic tableau deduction.

7 Overall Perspective

Make Herbrand passing and maybe usf

The preceding discussion has focused mainly on only one of the three legs of a more elaborate Tripod Response to Hilbert's Second Problem. The other two legs, involving generalizations of Gödel's Second Incompleteness Theorem and Artemov's step-by-step SBSIS-like approach

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[3, 4], have been surveyed only ~~very much~~ more briefly. Together, these three legs formulate a

substantially

triple reply to Hilbert's Second Problem, which is significantly more far-reaching than a more isolated type of one-or-two leg response.

Collectively, these three legs clarify the nature of the statements * and **, which Hilbert and Gödel had articulated. Moreover, the Question +++ (in §6) is quite tantalizing,

as are the additional remarks appearing in §7.

In essence, Hilbert's Second Problem is so complex that an isolated one-or-two legged reply is insufficient to answer Hilbert's fascinating question. Moreover, Gerald Sacks has recalled Gödel expressing opinions that were *"almost the opposite of what every one else would have expected"* him to make (see again §2). Thus, Gödel would presumably approve the philosophical approach that Remark 13 and Section 6 had formulated, since Gödel repeated several analogs of his often-quoted controversial 1931 remark ** during numerous private conversations ~~Gödel~~ Use

shared with Gerald Sacks [31].

Added Sentence about Further ~~Paragraph~~ Paper Maybe?

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and Maybe Not,

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