An overview of Self-Justifying Axiom Systems, and related questions

Contents

What and Why

Goedel's Second Incompleteness Theorem (G2):

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What and Why

Self-Justifying Axiom System (SJAS):

i. A theory S includes a statement of the consistency of S.

ii. S is consistent relative to a known theory T.

Parametrizing Expressivity

Generalized Arithmetic Configuration:

i. Axiom Basis (language, consistency statement, et al.)

ii. Deductive Apparatus
- Logical Axioms
- Rules of Inference

Parametrizing Consistency

"An axiom system α owns a Level-1 appreciation of its own self- consistency (under a deductive apparatus D) iff it can verify that D produces no two simultaneous proofs for a Π_1 sentence and its negation."

Axiom Basis

First Order Language without Induction:

Integer Subtraction(x, y)
Integer Division(x, y)
Maximum(x, y)
Length(x) = |x|
Root(x, y) = ⌈x^(1/y)⌉
Count(x, j) := the number of "1"s in x's rightmost j bits

Mult(x, y, z):=
$[(x = 0 \vee y = 0) \Rightarrow z = 0] \wedge [(x \mathrel{!=} 0 \wedge y \mathrel{!=} 0) \Rightarrow (z/x = y \wedge ((z - 1)/x) < y)]$

Construct Δ_k, Π_k, Σ_k sentences as per normal.

Axiom Basis

Group 0: Constants 0, 1, Addition(x, y), Double(x)

Group 1: a finite set of Π_1 sentences, proving any Δ_0 sentence that holds true under the standard model

Group 2: For each Π_1 sentence Φ, $\forall p \, \{\mathrm{HilbPrf}\,(<\Phi>, p) \Rightarrow \Phi\,\}$

Group 3: Kleene Fixed Point encoding of consistency:

$\forall x\ \forall y\ \forall p\ \forall q\ \neg[\mathrm{Pair}(x, y) \wedge \mathrm{Prf}(x, p) \wedge \mathrm{Prf}(y, q)]$

Deductive Apparatus

HilbPrf – Hilbert style proofs, using axiom schemae and modus ponens

Tab – standard semantic tableaux

X-Tab – semantic tableaux with an axiom scheme for LEM

Tab-1 – allows application of modus ponens for previously derived sentences.

Consistency Preservation

"An operation I(•) that maps an initial axiom system A onto an alternate system I(A) will be called Consistency Preserving iff I(A) is consistent whenever all of A's axioms hold true under the standard model of the natural numbers.

Suppose the symbol D denotes either semantic tableaux deduction or its Tab−1 generalization. Then the I_D(•) mapping operation is consistency preserving (e.g. I_D(A) will be consistent whenever all of A's axioms hold true under the standard model of the natural numbers)."

Trade-Offs

(1) $\forall x\ \exists z\ Add(x, 1, z)$
(2) $\forall x\ \forall y\ \exists z\ Add(x, y, z)$
(3) $\forall x\ \forall y\ \exists z\ Mult(x, y, z)$

NS: None
S: (1)
A: (1), (2)
M: (1), (2), (3)

Tab-1 + A = Hilb + θ = SJAS

The Local Cluster

Artemov – proves each of an infinite tower of consistency statements for subsets of PA

Beklemishev – unpublished simplification of Willard’s SJAS

Ganea – SJAS does not define an intermediate r.e. degree

Niebergall – non-arithmetically definable, consistency proving theories

Pakhomov – “A weak set theory that proves its own consistency”

Further Questions

Is it correct?

Other parameter values

- Deduction systems: cirquent calculus, resolution

Executable realization

- Relational logic programming

Autarkic Formal Systems

To what extent are the aspects of a formal system determined by its own powers?

- Consistency
- Definability
- Decidability
- Interpretation
  Brown and Palsberg, self-interpretation in System F
- Replication
  Von Neumann automata

Autarkic Formal Systems

To what extent are the aspects of a formal system determined by its own powers?

- Consistency
- Definability <= Lawvere's Fixed Point Theorm (per Yanofsky)
- Decidability

Further motivation for intensional exploration.

Selected References

Willard, D., 2005, An exploration of the partial respects in which an axiom system recognizing solely addition as a total function can verify its own consistency, DOI: 10.2178/jsl/1129642122

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Additional information is available here: https://github.com/jpt4/sjas