Why complexity theorists should care about philosophy

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Computational Complexity

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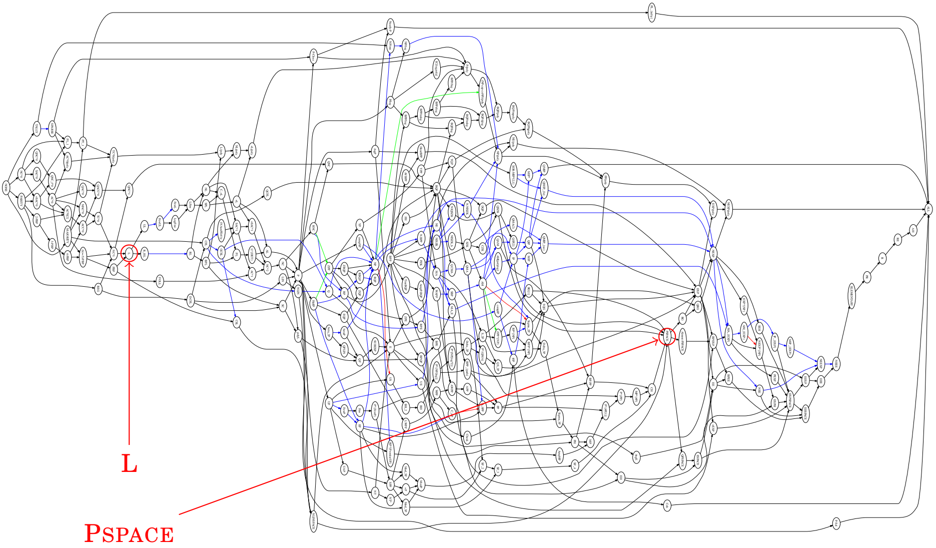
And now let's fastforward.

Complexity Theory, Today (well, in 2006)



Complexity Theory, Today

- A number of separation results were obtained, most of them in the 70s. But a lot of questions remain open. For instance: we know $\mathrm{L} \subsetneq \mathrm{PS}\textsc{pace}$, but we don't know which of these inclusions are strict: $\mathrm{L} \subset \mathrm{P} \subset \mathrm{NP} \subset \mathrm{PS}\textsc{pace}$.

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- **Thus: no proof methods for (new) separation results exist today.**
- (Proviso) One research program (but one only) is considered as viable for obtaining new results: Mulmuley's *Geometric Complexity Theory* (GCT). However, according to Mulmuley, **if** GCT produces results, it will not be during our lifetimes (and maybe not our childrens' lifetime either*), since it requires the development of much involved new techniques in algebraic geometry.

Barriers in Computational Complexity.

Morally, there are two barriers (here for P vs. NP):

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Conclusion: Lack of proof methods for separation. But why?

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State of the Art in Complexity (Separation Problem): Barriers.

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Thus arguably due to the following:

(Note Moschovakis already argues along these lines, but does not discuss barriers)

"*What is a computable function?*" ← Solved (at least for nat → nat)

"*What is a program / algorithm?*" ← Not Solved (Attempts exist though)

Digression: How to overcome barriers

- Mulmuley's program breaks *Largness*, i.e. it aims at developing techniques to prove lower bounds for a specific problem, i.e. the techniques are problem-dependent. (Mainly, GCT works with the determinant vs permanent problem in arbitrary characteristic and advocates for the use of techniques from algebraic geometry.)
- I want to break *constructivity*. (keeping the possibility of breaking largeness as well).

A more general issue

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RWPR, fold only	PRIM.REC	PRIM^1REC	PRIM^2REC	PRIM^3REC	$\text{PRIM}^\omega\text{REC}$
RO, unrestricted	PTIME	EXPTIME	EXP^2TIME	EXP^3TIME	ELEMENTARY
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- More generally, complexity is a bad measure of the expressivity. Somehow, it is erroneous to think that characterising a specific class of functions, e.g. Ptime, means we understood something about complexity.
- This functional point of view can explain why we are not able to generalise the notion of complexity to higher-order functions / concurrent computation.

Better foundations

- Hypothesis: Current lack of proof methods for separation is due to a lack of adequate **mathematical** foundations.
- Suppose there exists adequate mathematical foundations.
  I.e. (this is objectively very fuzzy) for every computation $\mathscr{C}$ there exists a mathematical object $\|\mathscr{C}\|$ with $\|\cdot\|$ injective.

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There are no barriers for the set of proof techniques based on such foundations.

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What is a computation/algorithm?

Several proposals.

- Turing
- Kolmogorov
- Gandy
- Moschovakis
- Gurevich.

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- From the point of view of our project: ad-hoc objects, not based on well-founded mathematical theory. In fact, ASM may be described as generalised pseudo-code.

What is a computation/program/algorithm?

From a philosophical point of view, very few work tackle this question (at least, I could not find many). It is even more actual, with the development of new models of computation (e.g. quantum, biological).

As a starting point for the reflexion, let us consider the following questions:

- Is the universe just a big computation?
- If I let a rock fall from the top of a tower, is this a computation? If not, why?
- What about if I let a rock fall from the same tower, but depending on the initial height it activates a number n of mechanical apparatus that release a number m of balls? (e.g. the rock activates levers every meter, with the lever at height k releasing 2k+1 balls)
- What about a similar experiment where flowing water activates some mill equipped with a similar apparatus? (Is this a computation on streams?)

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  - You fix a mass to a spring, let go, and write down the oscillations. Is this a computation?

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[Complexity] Implicit Computational Complexity.
Size-change termination (Lee, Jones, Ben-Amram), mwp-polynomials (Jones, Kristiansen), Loop peeling (Moyen, Rubiano, Seiller).

[Semantics] Dynamic Semantics
Geometry of Interaction (Girard), Game Semantics (Abramsky/Jagadeesan/Malacaria, Hyland/Ong), Interaction Graphs (Seiller).

[Compilation] Compilation techniques.
Work by U. Schöpp (cf. Habilitation thesis), Loop peeling (Moyen, Rubiano, Seiller)

[VLSI design] Synthesis methods for VLSI design.
Geometry of Synthesis programme (Ghica).

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> *This paper is the main piece in a general program of mathematatisation of algorithmics, called geometry of interaction. We would like to define independently of any concrete machine, any extant language, the mathematical notion of an algorithm (maybe with some proviso, e.g. deterministic algorithms), so that it would be possible to establish general results which hold once for all.*
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> *Girard, Geometry of Interaction II (1988)*

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- At first (technically) limited to sequential, deterministic, computation (may explain why it was somehow forgotten/discarded);
- New approach – Interaction Graphs – bypasses these limitations and allows for modelling many aspects of computation. Technically, we replace operators (i.e. bounded linear operators acting on Hilbert spaces) by *graphings*, obtaining a model which is both more general and more tractable.

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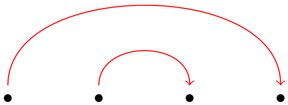
- Pick a directed graph.



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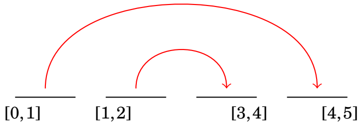

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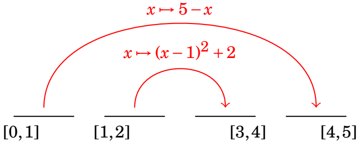

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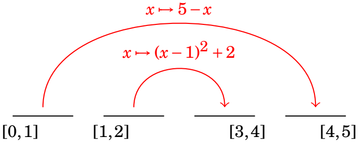

The parameters of the construction:

- A measure space $(X, \mathscr{B}, \mu)$;
- A monoid $\mathfrak{m}$ of measurable maps $X \to X$ – called a **microcosm**;
- A monoid Ω;
- A type of graphing (e.g. deterministic, probabilistic);
- A measurable map $g : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$.

Models of computation and logic

Logic	Lambda-calculus	Interaction Graphs

Proofs "Pararoofs"	Terms "Paraterms"	Winning Graphings Graphings
Cut rule	Application	Feedback
Normalisation cut-elimination	Reduction β-rule	Execution Compute paths

"Proofness" Correctness criterion	Orthogonality $t \perp E(\cdot)$ iff $E(t)$ SN	Orthogonality Complicated measurement
Formulas $\text{Proofs}(A^{\perp}) = \text{Tests}(A)$	Types Realisability constr.	"Conducts" $C = T^{\perp}$, (iff $C = C^{\perp\perp}$)

Hierarchies of models

Theorem (Seiller, APAL 2017)

For every monoid of measurable maps $\mathfrak{m}$ *(and every monoid* Ω*, and every measurable map* $g : \Omega \to \mathbf{R}_{\geqslant 0} \cup \{\infty\}$ *), the set of* $\mathfrak{m}$*-graphings defines a non-degenerate model of Multiplicative-Additive Linear Logic.*

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Quantitative Aspects
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Geometric Measurement
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Constraints on Graphings
(e.g. deterministic: (partial) measured dynamical systems,
probabilistic: (discrete time) Markov processes)

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All Geometry of Interaction constructions are recovered as specific cases

Operators in C* / von Neumann algebras (1989,1990,2011)

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Microcosms: Geometric Aspect of Complexity

We can define microcosms

$$\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \cdots \subset \mathfrak{m}_\infty \subset \mathfrak{n} \subset \mathfrak{p}$$

in order to obtain the following characterisations (as the type $\texttt{nat} \rightarrow \texttt{nbool}$).

Microcosm	$\mathbb{M}_m^{\text{det}}$	$\mathbb{M}_m^{\text{ndet}}$		$\mathbb{M}_m^{\text{prob}}$	Logic	Machines
$\mathfrak{m}_1$	REG	REG	REG	STOC	MALL	2-way Automata (2FA)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathfrak{m}_k$	D_k	N_k	coN_k	P_k	$(\dots)$	k-heads 2FA
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathfrak{m}_\infty$	L	NL	coNL	PL	$(\dots)$	multihead-head 2FA (2MHFA)
$\mathfrak{n}$	P	P	P	PP	$(\dots)$	2MHFA + Pushdown Stack

Refines and generalises both:

- a series of characterisations of complexity classes (e.g. L, P) with operators (with Aubert) and logic programs (with Aubert, Bagnol and Pistone);
- an independent result where I relate the expressivity of GoI models with a classification of *inclusions of maximal abelian sub-algebras*:

$$\ell^\infty(\mathbf{X}) \subseteq \ell^\infty(\mathbf{X}) \rtimes \mathfrak{m} \quad \left(\subseteq \mathscr{B}(\ell^2(\mathbf{X}))\right) \quad \text{[Feldman-Moore 1977]}$$

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- Only known correspondence between infinite hierarchies of mathematical objects and complexity classes.
- Indicates a strong connection between *geometry* and complexity: cf. microcosms generalise *group actions*, use of (generalised) Zeta functions, (homotopy) equivalence between microcosms implies equality of the classes.

A Geometric Theory of Complexity

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Conjecture

(Equivalence classes of) microcosms correspond to complexity constraints.

Conjecture

$$\mathfrak{m} \equiv \mathfrak{n} \Leftrightarrow \mathtt{Pred}(\mathfrak{m}) = \mathtt{Pred}(\mathfrak{n})$$

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Enable (co)homological invariants to prove separation , e.g. $\ell^{(2)}$-Betti numbers:

$$\texttt{Pred}(\mathfrak{m}) = \texttt{Pred}(\mathfrak{n}) \Rightarrow \mathfrak{m} \equiv \mathfrak{n} \Rightarrow \mathscr{P}(\mathfrak{m}) \simeq \mathscr{P}(\mathfrak{n}) \overset{!}{\Rightarrow} \ell^{(2)}(\mathscr{P}(\mathfrak{m})) = \ell^{(2)}(\mathscr{P}(\mathfrak{n}))$$

($\mathscr{P}(\mathfrak{m}) = \{(x,y) \mid \exists h \in \mathfrak{m}, h(x) = y\}$ is a *measurable preorder*)

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Summary

- Understand the first part as a *manifesto* to start a collaborative reflexion on the question: "What is a program", in the same way researchers once tackled the question "What is a computable function?".
- The second part is my own proposition for an answer. While I believe it is a (good starting point for finding a) satisfying solution, I expect it to be challenged.
- The last part shows (well, very quickly mentions) how this proposition reveals some geometric nature of computation/complexity which could be exploited for developing separation methods.
- In particular, the approach defines the complexity of a *program* intrinsically (i.e. as an equivalence class of group/monoid actions/acts), i.e. a definition which is not based on an arbitrary input/output behaviour.
- While I insisted on complexity issues, the whole framework comes from logic, and raises numerous questions as to which logical systems arise from these abstract models of computation.
- Although very abstract, this lead to an automatic optimisation tool (prototype) in the LLVM compiler.

Why complexity theorists should care about philosophy

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