Interaction Graphs and Quantitative Semantics

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Interaction Graphs (IG) models were introduced [4, 6] as a generalisation of Girard's Geometry of Interaction (GoI) constructions based on the interpretation of proofs as (finite, weigthed) graphs. Recent results [5] use IG models to bring into vision a new relation between dynamic and denotational semantics.

The first contribution of this work is the definition of categories of *triskells*, which generalises both the bicategory of spans and the categories of matrices and arrays over a semiring (also known as categories of weighted relations [3]). Secondly, it sheds light onto a new relationship between dynamic and quantitative denotational semantics for multiplicative linear logic (MLL), providing formal grounds to the claim that IG models are a quantitative generalisation of dynamic semantics, i.e. GoI and game semantics. Finally, this functor is shown to preserve not only the interpretation of proofs but also induced double-glueing refinements [2]: it is shown to lift to a map from the double-glueing construction defining IG models to the double-glueing construction yielding coherence spaces. For this purpose, a very general notion of *quantitative coherence spaces* is introduced, and shown to model full linear logic.

Interaction Graphs Categorically: Triskells. As the objects used to interpret proofs in IG models are (edge-)weighted graphs, we introduce the natural notion[1] of *triskell*, generalising spans [1]. A triskell $\mathscr{T}$ in a category $\mathfrak{C}$ is a triple of morphisms s,t,w sharing the same domain E – the *edges object* –, respectively called the *source*, *target* and *weight* maps. Their respective codomains are called the *source*, *target*, and *weight* objects, and the pair of maps (s,t) defines the *underlying span* of $\mathscr{T}$.

Provided the underlying category is finitely complete, and fixing a monoid object Ω in $\mathfrak{C}$ (for the cartesian monoidal structure), one defines the category of triskells with weight object Ω, denoted $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathfrak{C})$. Composition is defined by the composition of the underlying spans (i.e. by pullbacks) and the pointwise product of the weight maps. If $\mathfrak{C}$ has coproducts, $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathfrak{C})$ has two monoidal products $\otimes$ and $\oplus$, respectively defined by taking the product and the coproduct of the edge objects. Moreover, if $\mathfrak{C}$ has all countable coproducts $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathfrak{C})$ is a traced monoidal category w.r.t. the monoidal product $\oplus$. For our purpose, we now fix the base category $\mathfrak{C}$ to be the category of sets smaller than a given cardinal $\kappa > \aleph_0$.

Triskells as Generalized Weighted Relations. Triskells may possess several edges with same source and target, i.e. the pairing $[s,t]: E \to S \times T$ need not be a monomorphism. If the weight object Ω has the structure of *complete semiring*, a triskell $\mathscr{T}$ can however be "contracted" to a *simple triskell* $\mathfrak{C}(\mathscr{T})$. On the underlying spans, this corresponds to replacing the edge object E by $\bar{E}$, where $E \xrightarrow{e} \bar{E} \xrightarrow{m} S \times T$ is the epi-mono decomposition of the pairing $[s,t]$. This operation in fact defines a monoidal functor (for both monoidal structures) from the category of triskells $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathfrak{C})$ to the category of weighted relations $\mathbb{R}\mathrm{el}_{\Omega}(\mathfrak{C})$ [3]. This settles triskells as generalised weighted relations: they allow for a wider notion of weights, i.e. monoids vs. complete semirings, and their definition does not refer to external objects.

Interaction Graphs and Quantitative Coherence Spaces. The category of triskells can be used to define both (weighted) dynamic $\|\pi\|_{\mathrm{dyn}}$ and (weighted) denotational $\|\pi\|_{\mathrm{den}}$ interpretations of a MLL proof π, though these interpretations of proofs differ: linear logic's *tensor* is interpreted by $\oplus$ in the former case and by $\otimes$ in the latter. Nonetheless, we exhibit a functor $\mathfrak{F}$ mapping one to the other, i.e. $\mathfrak{F}(\|\pi\|_{\mathrm{dyn}}) = \|\pi\|_{\mathrm{den}}$. Furthermore, this functor is shown to map (double-glueing) orthogonalities, thus relating interaction graphs models to a very general notion of *quantitative coherence space*.

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[1]Although graphs are usually defined as a pair of parallel arrows, one can equivalently use spans in the presence of coproducts.

References

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