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A SURVEY OF PROOF THEORY

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§1. Introduction. One might fairly say that the very meaning of our subject has changed since Hilbert introduced it under the name *Beweistheorie* (it was meant to be the principal tool for formulating Hilbert's general conception of how to analyze mathematical reasoning). Specifically, the roles of the two principal elements of proof theory, namely the intuitive proofs accepted and the formal proofs (or derivations) studied, have turned out to be quite different from what Hilbert thought. In his view the hard work had been done in the discovery of formalization, and what remained was the study of certain given formal systems. But, as knowledge accumulated, it turned out that the analysis of the intuitive proofs considered and the choice of formal systems needed most attention. In short, for Hilbert *Beweis* (in *Beweistheorie*) referred to *formal* derivations; for *proof theory* to be viable at the present time, it has to concern itself with the *intuitive proofs* too.[1]

I think we shall get a clear and correct perspective, both on what has been done in proof theory and on what to do next, if we first recall how Hilbert himself regarded the matter.

Separating the foundations of mathematics from philosophy (epistemology). Hilbert wanted to accept only that part $\mathscr{P}_0$ of mathematical reasoning, of which school mathematics is typical, and without which there could be no science at all; for more detail, see Note Ic(iii). He called $\mathscr{P}_0$ 'finitist' because he believed that *finiteness* was essential to its elementary character; but this is in itself a problematic assumption because one works with *variables* (other names for $\mathscr{P}_0$ reflecting a different philosophical analysis of what is essential to $\mathscr{P}_0$ are (1) combinatorial or (2) concrete). One knows fairly well what $\mathscr{P}_0$ is, but giving a precise definition is the object, not the starting point of research here.

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[1] The present paper, which deals mainly with subsystems of classical analysis, is complementary to three other publications: (i) the survey article [31], the contributions by (ii) Feferman and (iii) myself to the Congress for Logic, etc., at Amsterdam, August, 1967. The *basic* theorems on which §1–12 depend, are also contained in [31]; the present exposition is meant for readers not familiar with non-classical systems, while [31] considers proof theory for intuitionistic formal systems, for the technical reasons given in [31, p. 156, 3.242]. However, I give here several new *formulations*, both of systems and results; Feferman's paper (ii) describes the use of autonomous progressions for the characterization of intuitive conceptions $\mathscr{P}$, its history and its problems; this important part of proof theory is neglected here altogether. The paper (iii) brings [31] up to date as far as specifically intuitionistic questions are concerned. Several results first stated in [31] appear with full proofs and, often, technical refinements in [36].

Hilbert had two discoveries available: First, the empirical discovery of formalization of (existing) mathematics in formal systems S (Principia Mathematica); that means, *if* one accepts the abstract notions of mathematical practice then mathematical reasoning is correctly expressed by S. Since most of these notions have nothing to do with $\mathscr{P}_0$, there has to be a link with $\mathscr{P}_0$. This was provided by Hilbert's own discovery of how to formulate in $\mathscr{P}_0$ itself suitable *adequacy conditions* on a formalization.

We have a certain class $\mathscr{A}$ of assertions in $\mathscr{P}_0$, a translation T which maps assertions $A \in \mathscr{A}$ into formulae A_T of S. Recall that every assertion of $\mathscr{P}_0$ is decidable. The first requirement on a formalization of intuitive $\mathscr{A}$ in S is this: a function π from $\mathscr{A}$ into formal derivations of S and a proof in $\mathscr{P}_0$ of

(i) $A \rightarrow \mathrm{Prov}_S\,(\pi A, A_T)$, for $A \in \mathscr{A}$,

where Prov_S is the proof predicate for S. Second, again for $A \in \mathscr{A}$, and variable p,

(ii) $\mathrm{Prov}_S\,(p, A_T) \rightarrow A$

is to be established in $\mathscr{P}_0$. It is a familiar technical matter (e.g. [14, p. 304]) that, under quite general conditions on S, (ii) is equivalent to the *consistency problem*, and further that (i) and (ii) are proper *adequacy conditions*; cf., e.g. [35, p. 209].

Thus he had established the remarkable fact, due to formalization, that adequacy conditions, i.e., (i) and (ii), can be *formulated* in $\mathscr{P}_0$; more precisely, Prov_S being rudimentary in the sense of Smullyan, *the only assumption* on $\mathscr{P}_0$ *needed is that rudimentary relations* $\in \mathscr{P}_0$.

The so-called separation of foundations from philosophy may be expressed as follows:

(i) *and* (ii) *should be established by use of very simple assumptions on* $\mathscr{P}_0$ (but possibly by means of complicated mathematical constructions); in other words, Hilbert thought that no detailed philosophical analysis of $\mathscr{P}_0$ was necessary (as, sometimes, one succeeds in answering a physical problem purely mathematically). This is Hilbert's programme, stripped of some unfortunate formulations: see Note I(c).

What made this programme plausible was the tacit conviction that assertions as elementary as (i) and (ii) could be *decided* in $\mathscr{P}_0$. Strictly, it would be sufficient to assume this decidability for those S which are accepted in mathematical practice. But then there is no convincing reason why (ii) should be provable in $\mathscr{P}_0$ for the next system that presents itself in practice. And if one had to make a fresh start one would not have the definitive separation which Hilbert wanted. (For even stronger assumptions, cf. §4, Remark p. 326.)

Consequences of Gödel's Theorems for proof theory. The following two consequences of the *second* incompleteness theorem are well known.

1. There will be a need for a philosophical analysis of conceptions $\mathscr{P}$, other than $\mathscr{P}_0$, and the interest will depend on the intrinsic significance of $\mathscr{P}$. Thus we do not have the separation which Hilbert wanted.

2. For a given S we cannot in general expect to establish (i) and (ii) in some *given* $\mathscr{P}$, but we shall have to search for an *appropriate* $\mathscr{P}$. Note that if $\mathscr{P}$ is changed, also the appropriate class $\mathscr{A}$ will in general change. Further (this is not used in the present paper), it should not be assumed that S is necessarily a formal system, if $\mathscr{P}$ is not combinatorial, since nonrecursive descriptions may be accepted in such a $\mathscr{P}$.

But also the *first* incompleteness theorem affects Hilbert's scheme.

3. The choice of S becomes a *central* problem. With Hilbert's assumptions (as listed above), it did not really matter what systems S, among those accepted in practice, were chosen, because $\mathscr{P}_0$ was supposed to establish (i) and (ii) for all such S. Moreover, and independently of their relation to $\mathscr{P}_0$, the systems S for arithmetic and analysis, which Hilbert had introduced, were thought to be *complete*, i.e., the rules were believed to decide every proposition formulated in the language of S. So, naturally, on the one hand he gave no attention to *subsystems* of his formal systems (e.g., arithmetic with induction restricted to free variable formulae in §4); they would have been regarded as artificial! On the other hand, he did not look for *extensions* of his systems S; for, by completeness, no new rules are required when formulae in the language of S are considered, and, by the empirical discovery (p. 322), the concepts of mathematical practice were known to be expressible in this language. (As is well known, the discovery of extensions of the usual formal systems is one of the major problems of foundations; but not for Hilbert's own conception of proof theory since the usual systems gave enough trouble!)

In view of Gödel's incompleteness theorems, there are 2 alternatives:

1. Since all formal systems are incomplete, i.e., are subsystems, they are artificial: that, roughly, is the end of the kind of proof theory here considered.

2. The *choice* of system becomes fundamental, e.g., if we are able to establish, in a given $\mathscr{P}$, (i) and (ii) for a subsystem S' of S, but not for S itself. Clearly, if a reduction to $\mathscr{P}$ has any intrinsic significance at all, the question: Can existing mathematics by chance be formalized in S'? is just as significant as whether it can be formalized in the 'obvious' system S; and, in general, more significant than whether existing mathematics can be formalized in *some* (consistent) system.

This conclusion conflicts with a widespread opinion dating from the beginning of this century and before; naturally, since the conclusion is derived from Gödel's (later) discoveries. It is likely that the progress of foundational research was held up by the mathematician's habit of concentrating on the new *methods* introduced in a proof, such as Gödel's, instead of re-examining the significance of old problems in the light of the new *result*.

§2. Summary (philosophical side). The conceptions $\mathscr{P}$ involved in the present lecture are combinatorial, nonconstructive predicative (relative to the notion of natural number) and, to some extent, intuitionistic proofs. It is often useful, if not necessary, to formulate explicitly which *properties* of these conceptions are needed in any particular argument, just as formal axioms of set theory formulate some of the mathematically most useful properties of the notion of set. There is one important difference: since *proofs* are involved in $\mathscr{P}$, not only axioms, but also rules of inference express significant properties of these conceptions $\mathscr{P}$: an *interpretation* of a formalism will not only assign a universe and objects to relation and function symbols, but will assign intuitive proofs to formal derivations. In other words, not only the set of theorems, but the set of formal derivations is important.

Broadly speaking, the properties of $\mathscr{P}$ discussed will concern: principles of proof by induction and definition by recursion; definition principles for functions of type $N^N \to N$; and definition principles for functions of all finite types.[2]

The philosophical interest of these principles is explained in §6(d), §10, Notes IV and VII. But the novice, coming from mathematics, will feel ill at ease unless the following objection is answered:

The principles $\mathscr{P}$ considered are more or less traditional. Why should we *restrict* ourselves to them? Would it not be better to approach the consistency problem by the *light of nature*? and search for a consistency proof which is mathematically as informative as possible? When we have found it we can then analyze its philosophical significance. In other words, once again, we try to separate the foundations of mathematics from epistemology, only in a somewhat weaker sense than originally intended by Hilbert.

Note I(b) is devoted to this matter.

§3. Summary contd. (mathematical side). §§6–12 contain mathematical results involving various formal systems for branches of current (nonconstructive) mathematics or for the conceptions $\mathscr{P}$ mentioned in the last section.

Since the analysis of intuitive conceptions $\mathscr{P}$ has acquired such importance, proof theory has the general character of *applied* mathematics. To isolate what is mathematically essential it is natural to look at the methods used, and to see how they *generalize*, as, e.g., a proof concerning 3-dimensional space may generalize. One would expect such generalizations of proof theory to have similar functions. Thus, technically, the general problem may force us to simplify the argument, and so enable us to solve more complicated problems about $\mathscr{P}$ *itself*. Also distinctions that are 'subtle' in the particular case, may become quite easy in the general case. Finally, the notions developed in such a generalization may be used in analyzing a new situation (here: new conception) for which the existing theory is inadequate (e.g., by use of hierarchies of systems instead of a single one, cf., footnote 1).

Both these functions seem to me well illustrated by the extension of proof theory to *infinite formulae*, or, more precisely, to suitably chosen infinite formulae. I shall give a short progress report in §13. The technical use will be evident. Personally I believe the subject to be *philosophically* important, roughly because thoughts seem to me much better represented by infinite objects than by the words we use to communicate them. The idea is, of course, not new, but one would have less confidence in it if one did not have a well working theory of infinitely long formulae.[3]

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[2] Naturally, one will try to decide a specific question by using properties of the conception $\mathscr{P}$ which are evident on quite a superficial understanding of $\mathscr{P}$. Specifically, one needs approximations to $\mathscr{P}$ from *below* for positive results, and from *above* for negative results; based on such work, the approximation, one hopes, can then be improved. In other words, *we do not start with a definition of* $\mathscr{P}$: this side of foundational research is often strange for the *pure* mathematician. People familiar with model theory may find a comparison with the so-called *mathematical* or *algebraic characterization of syntactically defined classes of models* helpful; nobody has ever explained, except by a few examples, what is here meant by 'mathematical'. But one recognizes a good solution when one sees it. More important: it *is*, I think, an interesting question to explain the programme precisely, but this definition may only be possible at quite an advanced stage of the programme.

[3] A more elaborate discussion of the representation of thoughts by infinite configurations is in [32], written specifically for Bertrand Russell: the present paper, it is hoped, is suitable for a wider audience.

Theory of formal systems and proof theory; a distinction. Evidently there are many questions about formal systems which do not come under the heading of proof theory in our sense at all, e.g., a *completeness theorem* with respect to an intuitive notion which does not involve any notion of proof. Thus for closed formulae A of first order predicate calculus, we have a set theoretic formula V(A) expressing that A holds in all structures, and a completeness theorem of the form $\forall A[V(A) \rightarrow \exists p \, \mathrm{Prov}_1 (p, A)]$, where Prov_1 is the proof relation for one of the usual formulations of predicate logic. The very statement of completeness does not make sense in $\mathscr{P}$, if V is not defined in $\mathscr{P}$, e.g., for $\mathscr{P}_0$.

Another important example are *independence proofs*, i.e., assertions that for a specific A_0, and for all n, $\neg\mathrm{Prov}(n, A_0)$, when one simply does not know the answer; e.g., for Quine's set theory and $A_0 \colon 0 = 1$. It is then not only worthwhile to use model theoretic methods, but ridiculous not to do so. (Incidentally, quite often it is rather easy and not specially interesting to convert such arguments into *relative* independence proofs which use only combinatorial methods, cf., Note II.)

Occasionally it may happen that proof theory in the present sense provides the easiest solution of a *technical* problem, i.e., one which is formulated in ordinary mathematical terms; in particular which does not refer to any conception $\mathscr{P}$. An instance are questions about finite axiomatizability and related matters. We shall call such technical uses of proof theory: *applications*, to distinguish them from philosophical 'implications'.

§4. Choice of subsystems. For Hilbert's programme a formal system *S* merely functions as a compact description of mathematical practice. But one certainly does not discover such *S* by making a statistical analysis of mathematical texts! and, if they had been so found, they wouldn't have been convincing! (equally the *rules* of predicate logic wouldn't have been found if one had not looked at the intuitive notion of logical consequence either in the now usual set theoretic semantic sense or in the sense of intuitive logical consequence as understood by Frege; for details on the distinction see, e.g., §2 of [33].)

The general principle, which distinguishes the discovery of the basic axiomatic systems here considered from technical ones like the axioms of group theory, is this:

We have a second order (categorical) *axiomatization of a mathematical structure, and then pass to a first order system either* (i) *by using a schema or* (ii) *by using a many sorted calculus and giving explicit closure conditions for the "sets".*[4]

Example. Consider Peano's axioms for arithmetic, i.e., the structure $\langle N, 0, S\rangle$ of the set of natural numbers with a distinguished first element 0 and the successor function; besides the first order axioms for 0 and S, we have the second order *induction principle*

$$\forall \mathbf{X}(\forall x[\mathbf{X}(x) \rightarrow \mathbf{X}(Sx)] \rightarrow \forall x[\mathbf{X}(0) \rightarrow \mathbf{X}(x)])$$

where $\mathbf{X}$ is a second order variable.

[4] The structures here considered concern arithmetic and the continuum, and, implicitly, the cumulative hierarchy of types up to the first inaccessible cardinal. All these structures possess second order axiomatizations.

Case (i). One replaces this by a schema, leading to the familiar system of classical first order arithmetic (cf., Note III for some formal details). Note that this step is *ambiguous*: thus, Peano's axioms mention explicitly only the function symbols 0 and S (of zero, resp. one argument), but the familiar system contains + and × too! In technical language: the existence of addition and multiplication are second order consequences of Peano's axioms, and the choice of first order language takes into account at least *some* such consequences.[5] It is well known that the corresponding schema in the first order language of (0, S) is quite inadequate for the formulation of arithmetic reasoning.

REMARK. The formal system (with the induction schema) restricted to the language (0, S) is *complete*: did Hilbert think that, for any primitive recursive function f defined by use of auxiliary functions $f_1, \cdots, f_k$, the recursion equations for $(f, f_1, \cdots, f_k)$ together with the successor axioms and the schema for the language $(0, S, f, f_1, \cdots, f_k)$ was similarly complete?

Case (ii). One considers a two sorted predicate calculus, adds membership, and writes down existential axioms for the X corresponding to the formation rules of first order formulae, roughly as in the familiar finite axiomatization of set theory by means of classes (for a general description, cf. [33, App. A]), and writes induction in the form

$$\forall X[\forall x(x \in X \to Sx \in X) \to \forall x(0 \in X \to x \in X)].$$

Here, we have an evident ambiguity in the choice of existential axioms (among all those that hold in the given structure).

REMARK. Philosophically, the second form is more attractive because one *understands* the infinite set of axioms in (i) merely because they are instances of (ii)! But for a good formalization of analysis it will be necessary to separate induction from set existence axioms.

Thus in both cases (i) and (ii) we have an ambiguity in the choice of first order axiomatization. *Therefore special interest attaches to results that are as independent of these choices as possible.* Below I shall try to give such formulations, e.g., in connection with the role of ϵ_0 in arithmetic.

Analysis. The structure considered is $\langle N, \mathfrak{P}(N), 0, S, \in\rangle$. The familiar second order characterization adds to Peano's axioms: extensionality for $\in$ and the so called *comprehension axiom*

$$\text{(CA)} \quad \forall \mathbf{X}\exists X\forall x[x \in X \leftrightarrow \mathbf{X}(x)].$$

A more convenient formulation uses the structure

$$\langle N, \mathfrak{P}(N), N^N, 0, S, \in, \circ\rangle,$$

where $\circ$ denotes application, sometimes called function evaluation: $N^N \times N \to N$, and f, g $\cdots$ are function variables (of lowest type), and second order binary relations **X** on $N \times N$, $N \times N^N$, resp. $N^N \times N^N$. Besides the usual axioms relating functions and their graphs, we have the *axiom of choice* (AC) in one of the forms

$$(AC_{00}) \quad \forall \mathbf{X}\exists f[\forall x\exists y\mathbf{X}(x, y) \to \forall x\mathbf{X}(x, fx)]$$
$$(AC_{01}) \quad \forall \mathbf{X}\exists f[\forall x\exists g\mathbf{X}(x, g) \to \forall x\mathbf{X}(x, f_x)]$$
$$(DC_{11}) \quad \forall \mathbf{X}\forall h\exists k[\forall f\exists g\mathbf{X}(f, g) \to \forall x(k_0x = hx \wedge \mathbf{X}(k_x, k_{x+1})]$$

[5] + and × are particularly elementary: they are implicitly definable in $\langle N, 0, S\rangle$.

where f_x is defined by: $f_x(y) = f(\langle x, y\rangle)$, $\langle x, y\rangle$ being a term that defines a pairing function.

It is well known that $DC_{11} \to AC_{01} \to AC_{00} \to CA$. By means of quite elementary (first order) closure conditions we also have $CA \to AC_{00}$. But since DC_{11} is true in the unique structure satisfying the axioms for analysis (CA), also DC_{11} is a second order consequence of CA (together with the elementary axioms)! For some comments on interpretations for which DC_{11} or even AC_{01} is not plausible, cf., Note IV(b).

First order systems, where X is replaced by explicitly defined relations Ξ in the second order axioms (of induction, comprehension, choice respectively). One considers subsystems for reasons explained in the introduction; but also if one simply wants a faithful picture of the arguments *actually* used in existing analysis. *One* way of obtaining such subsystems is by *restricting the syntactic form* of Ξ; e.g., by the number of alternating quantifiers in its prefix (in prenex form) familiar from Kleene's hierarchy. Two points should be noted: (i) Inspection of informal analytic arguments suggests that the induction schema and the other schemata should be treated asymmetrically; one applies freely induction to arbitrary formulae (containing variables of all sorts in the formalism considered), but it turns out that various 'branches' of analysis only use (AC) or (DC) in quite restricted form. A more theoretical reason for this asymmetry can be given in terms of ω-models in Note IV(a): if a formula A is not derivable 'because' induction is restricted, a model of $\neg A$ must contain nonstandard integers, i.e., it cannot be an ω-model. (ii) Though certain levels of (the syntactic analogue to) Kleene's hierarchy lead to significant subsystems, e.g., for $\Xi \in \Sigma_1^1$ (cf., Note V) and almost certainly $\Xi \in \Sigma_2^1$, it should not be assumed that this necessarily remains interesting for all Σ_n^1.

Actually it will turn out that, for present day proof theory, a completely different classification has so far been much more successful.

This will be considered in §9 below. Roughly speaking, one there avoids a *fundamental defect*, formulated and established in §8, of the subsystems above. There are canonical definitions R of quite familiar well orderings of the natural numbers, e.g., of ordinal ϵ_0, such that the 'least element' principle

(*): $$\exists x\Xi(x) \to \exists u[\Xi(u) \land \forall v\{R(v, u) \to \neg\Xi(v)\}],$$

cannot be derived in the subsystem for sufficiently complicated Ξ. Since these canonical definitions define the well orderings considered in all ω-models, once again as in the case of restricted induction, a model negating (*) is bound to be nonstandard (even with respect to the natural numbers).

A word on *significance* of subsystems. Philosophically, the criterion is clear: given $\mathscr{P}$ one wants to find a subsystem which can be reduced to $\mathscr{P}$ (in the sense made precise in this introduction) and, at the same time, permits a convenient formulation of mathematical practice without *ad hoc* tricks. Mathematically, the question is less clear cut, and, often, a bit subjective, being relative to existing knowledge. (That is one reason why mathematical logic is so often frustrating without philosophical interests!) A subsystem is clearly interesting if it is satisfied by a *familiar class of* (sets and) *functions which does not satisfy the full system*: e.g., in the case of Σ_1^1-AC_{01} or Σ_1^1-DC_{11}, the class of hyperarithmetic functions. (Evidently, to appreciate this example one has to know the notion of hyperarithmetic function, *and* to recognize its value.) In particular, most results of hyperarithmetic (and also of so called recursive) analysis are immediate corollaries of the fact that these results can be derived in suitable subsystems of classical analysis: people working in this subject seem to enjoy rewriting classical arguments, having forgotten the *axiomatic method* altogether! for specific examples, see, e.g., the review of [4]. (Naturally, independence results require ad hoc constructions [25].) The crucial problem in the use of subsystems is almost always the proper *choice of definition* for a concept among those that are equivalent in full analysis or set theory. For instance, different 'definitions' of the class of hyperarithmetic functions, i.e., different properties which single out these functions in N^N, are, in general, not provably equivalent in Σ_1^1-AC_{01}. [Interestingly, it is often the earlier 'clumsier' definitions for which more can be proved in weak subsystems; for striking cases, cf., Note V(a).] People are frightened of asking: which definition is correct? or which is fundamental? and so investigate a dreary list of alternatives. Here is an obvious proposal: One chooses that definition of the concept studied for which the principal properties used in the informal theory can be proved in a weak subsystem. (This suggestion is heuristic and research is needed to verify whether it holds in a particular case, i.e., for a given concept and a proposed subsystem.) Of course this suggestion applies not only to an axiomatic theory of the hyperarithmetic hierarchy, but, even more, to definitions of more intuitive concepts such as the geometric concepts of open or closed sets; now one requires of the definition that the *intuitively evident* properties of the concepts be provable (not merely the 'principal' properties that happen to have been found useful.) Another useful but less decisive criterion concerns the following kind of *stability*; if one goes over to a different language and writes down the axioms that strike one as 'analogous', the resulting system should be 'closely' related to the original system: for more precise information, cf., Note VI(c).

§5. Classical first order predicate logic. This is, of course, the most familiar part of proof theory and the area where many of its fruitful ideas were first applied. But, though the philosophical interest (for $\mathscr{P}_0$) of the work of e.g., Gentzen or Herbrand is quite easy to state, the mathematical side is confusing: one feels that something is achieved, but many of the results seem to be got more easily by non-constructive methods, in particular, so-called semantic *completeness* and *soundness* of the rules studied. (As said before, these are paradigms for properties of formal systems that have nothing to do with proof theory.)

EXAMPLE 1. Consider systems of arithmetic with a purely universal axiom A. Then, if $\forall x\exists y B(x, y)$ is a consequence of A, there are terms $t_1, \cdots, t_n$ built up from the function symbols in A such that $B(x, t_1) \vee \cdots \vee B(x, t_n)$ is a consequence of $A_1 \wedge \cdots \wedge A_n$ where each A_i is a substitution instance of A. Furthermore there is an operation f in $\mathscr{P}_0$ giving a purely propositional derivation from a formal derivation d of $A \rightarrow \forall x\exists y B(x, y)$ (and hence a bound on the complexity of the t_i in terms of d).

Philosophically, this is important because in the derivation d there may occur formulae which have no sense in $\mathscr{P}_0$ (if no interpretation is given to quantified formula) or are simply false (for the natural interpretation of quantifiers in $\mathscr{P}_0$);

but the result shows that if A *is* valid, for the natural interpretation, in $\mathscr{P}_0$, so is $\forall x \exists y B(x, y)$.

Mathematically, this *realization of existential quantifiers by explicitly defined combinatorial functions* has some algebraic applications, discussed in [22]; see however Note I(d). If, from a given *d*, one actually wants to get these realizations t one does have to go back to the combinatorial proof. But if one merely wants the existence of such t, or of a recursive function f, one has a very simple model theoretic argument. (So if one is neither thinking of actual computations nor has a grasp of $\mathscr{P}_0$ one will be ill at ease with the old arguments.)

EXAMPLE 2. One of the proofs of the preceding result uses Gentzen's so called cut elimination. Consider, for the moment, any of the systems that you know as 'cut free'. Gentzen gave a proof in $\mathscr{P}_0$ how to obtain a cut free derivation of a formula A from a derivation in the usual systems. But the existence of the cut free derivation is an immediate consequence of (i) *completeness of the cut free rules*[6] and (ii) soundness of the usual rules. Since, for many algebraic applications only the existence of a cut free derivation is needed, we have the same situation as in Example 1. For refinements, see Note IIb(i).

Of course, with a little sophistication one sees that *cut elimination is quite a different theorem from* (i), since the method of proving cut elimination was applied directly by Gentzen to intuitionistic systems which are patently not complete for ordinary semantic validity.

Towards the end of this survey we shall find it necessary to analyze *what is essential about cut free systems*. For the present (i.e., in the next 3 sections) the mathematical applications will turn on the *subformula property*, i.e., the fact that in a 'cut free' derivation of A only 'subformulae' of A appear; the usefulness of this property, in turn, depends on the *existence of partial truth definitions* for the subformulae of any fixed A, for a natural meaning of 'subformula'. (As observed by Tarski, we do not have unrestricted truth definitions.) Applications of this property are spelt out *ad nauseam* in [36].

For differences between the 'usual' rules and 'cut free' rules in relation to Gödel's second incompleteness theorem, cf., footnotes 8 and 16.

From the point of view of *length* of derivations the situation is less clear cut. Thus, the addition of *cut* (*modus ponens*) shortens calculations; on the other hand the use of truth tables (in propositional calculus), e.g., formalized by the construction of Post's normal forms, is generally longer.

Note in passing that Gentzen [8] attributed a much more central significance to cut free rules; roughly that *logical* operations are *defined* by rules of deduction, and that the cut constitutes a certain impredicativity (which he wished to eliminate; Lorenzen's operative logic [45] and game theoretic analysis [44] are attempts in the same direction). There may be something to this: but the two best known logical operations, truth functional and intuitionistic ones, are certainly *not* defined by rules of deduction; on the contrary, one *looks* for rules that are valid for these operations. Also in analysis or set theory, the logical operations are used as princi-

[6] With respect to validity, not consequence! Since, e.g., from $A \wedge B$ one cannot derive A by a system of rules that possesses the subformula property.

pal means of (impredicative) definitions in the comprehension axioms. I shall return to this point below.

As so often, elaborate mathematics goes some way towards replacing a firm grasp of an intuitive conception such as $\mathscr{P}_0$. The perhaps 'subtle' distinctions made above become *much clearer when applied to the case of infinitary languages*, where we have natural (classical!) examples of cut elimination without completeness; also what I said about length of derivations (which many logicians would regard as 'piddling') is magnified enormously: for countable languages we don't even have (countable) Post normal forms even when we do have cut free rules of proof [54]![7]

Further, as Tait has pointed out [61], the study of infinitary propositional language (negation, infinite disjunction) is very suitable for a proof theoretic analysis of arithmetic *provided only that one pays due attention* (i) *to the principles of definition employed in describing the* (*infinite*) *proof trees and* (ii) *to the methods of proof for showing that the proof trees so described are well founded.* (For the proof theoretic study of subsystems of analysis in place of arithmetic, one has a choice between using infinitary languages with finite strings of quantifiers [3] or a more or less direct reduction to the propositional case [61].)

The next sections will explain just what is involved in (i) and (ii). It should be noted that precisely analogous questions had to be considered before when one used infinite proof trees but only finite (quantified) formulae as in Schütte [50], or interpretations such as the no-counterexample-interpretation for arithmetic.

§6. Classical first order arithmetic Z (formulated by means of the *schema* of induction). The proof theoretic results in (a) and (b) below establish formal relations between **Z** and certain systems which formulate so called ϵ_0-induction. More precisely, this induction principle involves a definition $\prec$, i.e., a formula with two free variables, of a binary relation on the natural numbers which strikes one as a *natural ordering of ordinal* ϵ_0.

The notion of 'natural ϵ_0-ordering' will be analyzed in (d) below and shown to be unique up to isomorphism (in a suitable class of mappings). Among all definitions of this ordering one picks out a *canonical* definition (unique up to 'provable' isomorphism) as in [31, p. 154, 3.222].—NB. The formal results are quite independent of this analysis since the definitions one naturally thinks of (or which have been thought of in the literature, e.g., Gentzen [10], Hilbert-Bernays, Schütte [50], Tait [59]) all satisfy the criterion for a canonical natural ϵ_0-ordering. However, the significance of these results for $\mathscr{P}_0$ depends on this analysis: according to taste, the reader may read (d) before or after (a)–(c) (or skip it altogether if he finds the discussion too discursive).

(a) *Main result.* We consider **Z**$'$, obtained, roughly speaking, from **Z** by *restricting* the rules of logic to be cut free, *extending* the schema of induction to the rule:

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[7] Sometimes the other extreme (*fragments* of predicate calculus) also illustrates the same point; but again, fragments would generally (I think, wrongly) be considered piddling. Incidentally, the advantages of Gentzen's cut free rules over methods based on the intuitively much clearer no-counterexample interpretation (see the remark in Note VII) also show up in the infinitary case or with fragments that do not possess prenex normal forms; cf. Note VIII.

for each numeral $\bar{n}$, infer $(\forall x \lessdot \bar{n})Ax$ from $(\forall x \lessdot \bar{n})[(\forall y \lessdot x)Ay \rightarrow Ax]$, and *adding* the schema of definition by recursion $\lessdot\bar{n}$ (for formal details of this schema, see, e.g., [59]). *Then* **Z** *and* **Z**′ *have the same set of theorems in the language of* **Z**.

Though this simple formulation does not seem to occur in the literature, *some* version of this sort follows directly from any of the familiar proof theoretic analyses of **Z** given in (b); we shall sketch an argument using Schütte's [50].—NB. The rules of **Z**′ will be indicated at the end of the sketch. They are 'cut free' in the sense of p. 9, i.e., they have the subformula property, but are not elegant: specifically, they permit certain cuts, mentioned below, and substitution of numerals for *free* variables. It might be interesting to see whether these modifications of the usual cut free rules are essential. I am indebted to C. D. Parsons and L. H. Tharp for constructive criticism of an earlier formulation.

Note that some extension of the schema of ordinary induction *is* necessary, e.g., by [31, p. 163, 3.33].

First step. Given a formal derivation of the formula A in **Z**, and using the informal instructions in [50] we get a *description* of an infinite tree of (finite) formulae and a *purely quantifier free derivation* in **Z**′ establishing the following results (cf. [31, pp. 163–164, 3.33]). (i) The tree is a *locally correct* proof figure, i.e., for each node $\bar{N}$, the formula A_N at $\bar{N}$ is related to the formulae at the immediate neighbours of $\bar{N}$ according to the rules of inference in [50], (ii) the tree is well founded in the (strong) sense that we give an explicit order preserving mapping of the tree into a segment of the canonical ϵ_0-ordering. This step can be formalized in primitive recursive arithmetic: the explicit mapping is necessary to avoid quantifiers in the definition of well foundedness.

Second step. We use a coding of our infinite proof trees, and the proof predicate $\mathrm{Prov_I}$ for them (as in [31, p. 164]). By the fundamental subformula property, we have a formula A(N) with variable N that enumerates the subformulae of A, i.e., $A \leftrightarrow A(\langle\ \rangle)$ where $\langle\ \rangle$ denotes the top node of the tree, and, for each node $\bar{N}$, $A_N \leftrightarrow A(\bar{N})$ are formally derivable. Let $\pi(N)$ be the (proof) tree below $\bar{N}$, and let R(N) be the formula

$$\mathrm{Prov_I}\,[\pi(N), s_A(N)] \rightarrow A(N)$$

where $s_A(N)$ defines the number of the formula at the node $\bar{N}$.

Third step. Let $\prec$ be the partial ordering between nodes already used in (ii) of the first step above. We get an *elementary cut free derivation* of the implication $(\forall N' \prec N)R(N') \rightarrow R(N)$, and hence, via the mapping l.c., a derivation in **Z**′ of

$$\mathrm{Prov_I}[\pi(N), s_A(N)] \rightarrow A(N)$$

and so of $\mathrm{Prov_I}\,[\pi(\langle\ \rangle), s_A(\langle\ \rangle)] \rightarrow A(\langle\ \rangle)$. Since we already have a (quantifier free) proof in **Z**′ of $\mathrm{Prov_I}\,[\pi(\langle\ \rangle), s_A(\langle\ \rangle)]$, applying cut to this formula, we get $A(\langle\ \rangle)$. This gives a derivation of A itself modulo the step $A(\langle\ \rangle) \leftrightarrow A$; the enumeration A(N) can certainly be so chosen that cuts need only be applied to formulae of complexity less than A itself.

Note that the reduction can be given by free variable ϵ_0-recursion.[8]

[8] I have not checked whether the equivalence proof can be formalized in **Z** itself, or even in primitive recursive arithmetic. Philosophically, the question is not particularly interesting, as elaborated in Note IIb. However, technically, it is of interest in connection with *Gödel's second*

The reduction of **Z**′ to **Z** is well known; it can evidently be established in primitive recursive arithmetic, e.g., Schütte [50, pp. 202–209]. More important, he analyzes conditions, namely the axioms A1–A12, pp. 202–203, on $\prec$ *together* with certain (ordinal) functions on $\prec$, which ensure that the rule of induction $\prec\bar{n}$ can be formally derived in **Z** for each n. Though in Schütte's exposition this analysis played a purely formal role, we shall be able to use it in (d)(vi) below in a more significant way.

Corollaries to the main result. (i) Analysis of the cut free rules shows that, if a universal formula $\forall x A$ (A quantifier free) is derivable in **Z** then it is derivable by α-induction for some $\alpha < \epsilon_0$. And if $\forall x \exists y A(x, y)$ is derivable in **Z** there is a term tx representing a definition by α-recursion for which A(x, tx) is derivable by α-induction. (For a precise description of free variable α-induction and α-recursion, see, e.g. [59].)

(ii) Let *Prov* be a canonical definition of the proof relation for **Z** and let $s_A x$ be the canonical definition of the Gödel number of $A(\bar{x})$. Then (cf. [31, p. 165, 3.3322]), adding to **Z** the reflection principle, for all A:

$$\forall x[\exists y \operatorname{Prov}(y, s_A x) \to A]$$

is equivalent to adding, again for all A

$$\forall x[(\forall y \prec x) A y \to Ax] \to \forall x Ax.$$

Note in passing that the reflection principle is equivalent over **Z** to the rule: infer $\forall x A$ from $\forall x \exists y \operatorname{Prov}(y, s_A x)$.

(b) How does ϵ_0 come into the *proofs* of the main result? (i.e., of the reduction of **Z** to **Z**′). Gentzen assigned ordinals to formal derivations, i.e., to finite syntactic structures. I like to see *ordered structures to which we simply assign their order type*. I do not believe that one has any hope of getting a significant ordering of formal derivations if one only looks at their syntactic structure. What is needed is to look at the intuitive proofs described by formal derivations, and to use properties of these proofs to *discover* a useful ordering of the formal derivations.

Cut elimination (either infinitary propositional calculus or [50], already used in (a) above). If we start with a formal derivation corresponding to a derivation in **Z**, and apply cut elimination in a natural way, we finish up with a cut free derivation of ordinal $<\epsilon_0$. This bound is minimal in the sense that for each $\alpha < \epsilon_0$ there is a derivation in **Z** which *this* method transforms into a cut free derivation of ordinal $>\alpha$. (This is independent of whether we consider the ordinal of the total ordering of the branches of a proof tree ordered from left to right, or the partial ordering by means of the ancestral of the nodes.)

No counterexample interpretation. Here one associates with each arithmetic A its nci: $\exists F \forall f A'(F, f)$, A′ quantifier free, f of type N^N, F of type $N^N \to N$ (F in-

incompleteness theorem applied to cut free systems. For the 'usual' systems S (see, e.g. [31, p. 155, 3.233] and also footnote 16 below) for which, besides the two other 'derivability' conditions of Hilbert-Bernays, closure under cut can be formally proved in S itself, Gödel's theorem simply implies that the consistency of S is not derivable in S. For cut free systems we have the further possibilities that (i) consistency *is* derivable, but *not* closure under cut and (ii) neither is derivable. Clearly, if the equivalence proof under discussion *can* be formalized in **Z**, the consistency of the cut free system **Z**′ *cannot* be proved in **Z**′, and closure under cut can; at present I do not know which of the three possibilities applies to **Z**′.

tended to range over constructive operations continuous for the product topology). With each F is associated the ordering of its unsecured sequences. If, for A formally derived in Z, one associates natural definitions of F_A: $\forall f A'(F_A, f)$, it turns out again that the corresponding order types fill up the segment $< \epsilon_0$. (For explicit analysis, cf. [60].)[9]

REMARK. Both methods of proof generalize and explain 'how ϵ_0 is connected with **Z**' in the following *formal* sense: if we add to **Z** the schema of induction for (a formula $<_\alpha$ instead of $<$), the definition of ϵ_0, i.e., the first ϵ-number after ω, is replaced by the corresponding definition of the first ϵ-number after α (and not, e.g., by α^{ϵ_0} which would also fit in with the result above for $\alpha = \omega$). More precisely, the question we consider here is how ϵ_0 is connected with the formal system **Z**, not how it is connected with $\mathscr{P}_0$! For, since the generalization above is quite independent of how definition by recursion on $<_\alpha$ is *recognized*, it has no more interest for $\mathscr{P}_0$ than, e.g., the generalization of a geometric theorem to 25 dimensions has for physical or visual space.

A better formulation of the question 'how ϵ_0 is connected with **Z**' will be given in the next section, where the alternative formulation of induction (by means of a second sort of variable) is treated: a strong *negative* result on underivability of induction up to and including ϵ_0. The obstacle to a good formulation at the present stage is this:

The schema of induction (for arithmetic A) can be proved in **Z** *for* $<$ *which are not well orderings at all* (i.e., do not define a well ordering in the structure of arithmetic).

(Take the obvious implicit definition of the truth predicate T of arithmetic, which definition has the form $\forall x \exists y A(T, x, y)$, A quantifier free. Put it in the form $\exists f \forall x A(T, x, fx)$, and consider the ordering of the unsecured sequences of $\forall x A[\lambda y g(2y), x, g(2x + 1)]$.)

(c) *How not to talk of ordinals in proof theory.* The next section (d) will discuss the significance for $\mathscr{P}_0$ of ϵ_0, more precisely of the natural orderings of ordinal ϵ_0. But we can point out here the shortcomings of some attempts in the literature, e.g. [53], to give more 'simple minded' conditions on the ϵ_0-orderings to be used in

[9] There is a third analysis which leads to the ordinal ϵ_0, using Gödel's interpretation by means of functions of finite type [12] and an assignment of ordinals to the terms defining the functions involved (the ordinals reflect the structure of the computations). The interpretation is similar to the no-counterexample interpretation in that (i) to each formula A is associated an interpretation of the form $\exists s \forall t A_1(s, t)$, A_1 quantifier-free, and (ii) the laws of (classical) logic are valid for this interpretation when applied to suitably restricted classes of formulae, provided the closure conditions formulated in [12] are satisfied by the functions considered (for negations of prenex formulae in both interpretations, for formulae built up from $\neg$, $\wedge$, $\forall$ in [12]). For a brief comparison, see Note VII. Although [12] was published in 1958, (as Professor Gödel tells me) the formal details of the interpretation were presented by him in lectures at Princeton as early as 1941. Despite the elegance of the interpretation, not even rumours of its existence became known; I heard of it first in 1955 (from Professor Gödel) and spoke about it at Cornell in 1957. It should be remarked that Kleene proposed his recursive realizability interpretation (cf. [20]) in the early forties; for the present purpose it is essentially different from the two interpretations above because the relation: *e* realizes the formula A, is itself expressed by a formula in which $\neg$, $\wedge$, $\forall$ are applied to *undecided* expressions: in other words, *these* logical operations are not eliminated.

the proof theory of Z, such as primitive recursiveness of the ordering relation and of the set of limit elements.

To spell this out more formally, observe:

(i) For every true arithmetic A there is a primitive recursively described cut free proof of ordinal ω^ω (use the primitive recursive ω-rule instead of the recursive ω-rule [56]).

(ii) For every true arithmetic A there is a functional F_A with primitive recursive neighborhood function such that $\forall f A'(F_A, f)$ and the set of unsecured sequences of F_A has ordinal ω^{ω^2} (cf. [37, p. 59]).

(iii) Suppose S includes Z, and, like many usual systems, is such that the consistency of each finite subsystem of S can be formally proved in S [and that this fact, i.e., $\forall n \exists y$ Prov (y, $\ulcorner$Con $S_n\urcorner$) can also be proved in S]. Then there is a primitive recursive ordering $<$ of type $<\omega^2$ with the following properties:

For each segment of $<$ the schema of induction can be formally proved in S.

The consistency of S can be proved by induction on $<$.

(Construct $<_n$ 'equivalent' to Con S_n in the sense of [37, §7]; $<_n$ is of ordinal ω, since Con S_n is purely universal. Put $< = \bigcup_n <_n$.)

Since the following result belongs here, it is included although its formulation uses free function or predicate variables.

(iv) For systems S as in (iii), but containing function variables there is a primitive recursive ordering $<_S$ with the following properties:

Every segment of $<_S$ can be proved to be a well ordering in S, and hence every Π_1^1-theorem of S can be proved by induction on $<_S$.

Simple consistency (etc.) of S can be proved by quantifier free induction on $<_S$.

(Take for $<_S$ the union of all primitive recursive orderings that can be formally proved in S to be well orderings.)

As always when one has neglected the essential point involved, one can extend this dreary list of 'pathological' properties indefinitely.

(Intuitively these conditions are inadequate because they say nothing about how ϵ_0 is built up, which will be central in (d). Put differently: it is the complexity, not the size of the orderings that matters. Just how far off the mark these conditions are will appear in §7.)

Warning. There is another 'brutal' use of ordinals in mathematical logic, namely ordinals of the ramified analytic hierarchy, which also seem to have very limited interest for proof theory (cf. Note V).

(d) *ϵ_0 and $\mathscr{P}_0$.* Let us accept that $\mathscr{P}_0$ is concerned with combinatorial operations on the natural numbers, i.e., number theoretic functions defined by rules which can be recognized, by methods of $\mathscr{P}_0$, to be well defined. We call the latter $\mathscr{P}_0$-rules. From a formal point of view: $\mathscr{P}_0$-proofs will be described by means of *quantifier-free* formalisms, with variables for natural numbers and function constants. Let us further assume

(I) All primitive recursive definitions, i.e., the usual definition schemata for primitive recursive functions, are accepted in $\mathscr{P}_0$.

(II) All $\mathscr{P}_0$-acceptable definitions can be reduced (i.e., seen by methods of $\mathscr{P}_0$ to be equivalent) to *recursion equations*; thus all $\mathscr{P}_0$-definitions define recursive functions.

(i) **Z** and $\mathscr{P}_0$. Recall the corollaries to the main result of (a), which imply that **Z** can be reduced to $\mathscr{P}_0$ provided the principles of proof by induction with respect to $\prec$, respectively definition by recursion with respect to $\prec$, are valid in $\mathscr{P}_0$ for each proper segment of $\prec$.

Recall the precise formal description of these principles in [59]; note that recursion implies induction, and hence, loosely, the existence of a function defined by recursion implies uniqueness; l.c. p. 161. Roughly, the definition principle is this: if g and h have been introduced in $\mathscr{P}_0$, we also introduce f by the rule

$$\begin{aligned} \mathrm{fn} &= \mathrm{c} \text{ if } \mathrm{n} = 0 \quad \text{or} \quad \neg \mathrm{hn} \prec \mathrm{n} \quad \text{or} \quad \neg \mathrm{n} \prec \mathrm{p} \\ &= \mathrm{g}(\mathrm{n}, \mathrm{fhn}) \text{ otherwise.} \end{aligned}$$

We have two questions: Are all such definitions, for *given* $\bar{\mathrm{p}}$, acceptable in $\mathscr{P}_0$? Is some such definition (with primitive recursive g and h) not valid *without* the restriction $\mathrm{n} \prec \bar{\mathrm{p}}$? in short, is α-recursion valid for each $\alpha < \epsilon_0$, and ϵ_0-recursion not valid in $\mathscr{P}_0$? It is, of course, not assumed that *these* questions will be settled by methods of $\mathscr{P}_0$.

Comments. (a) By footnote 2 one must expect that different (though compatible!) assumptions about (or: properties of) $\mathscr{P}_0$ will be used in these answers. What one would like, of course, is a set of evident properties of $\mathscr{P}_0$ sufficient to determine completely the set of formulae expressing assertions which are provable in $\mathscr{P}_0$ and are expressible in some given language, e.g., of primitive recursive arithmetic. (For a very simple paradigm, when 'provable in $\mathscr{P}_0$' is replaced by 'intuitively valid', and 'primitive recursive arithmetic' by 'predicate calculus of first order' see axioms II and III for Val on p. 190 of [35].)

(b) Note that what we have to justify in $\mathscr{P}_0$ is the assertion that the equations for α-recursion are well defined rules. The notion of well ordering or ordinal does not appear explicitly. But the equations are plausible because we think of them as definitions by transfinite recursion.[10]

(ii) $\mathscr{P}_0$ and the notion of well ordering; a fallacy. To justify α-recursion we have to see that, for $\mathrm{n} \prec \bar{\mathrm{p}}$ and $\mathrm{h}_1 0 = \mathrm{n}$, $\mathrm{h}_1(\mathrm{m}+1) = \mathrm{h}(\mathrm{h}_1\mathrm{m})$, the sequence $\mathrm{h}_1 0, \mathrm{h}_1 1, \mathrm{h}_1 2, \cdots$ 'turns back', i.e., for some m, $\neg \mathrm{h}_1(\mathrm{m}+1) \prec \mathrm{h}_1\mathrm{m}$. Since $\mathrm{h}_1 \in \mathscr{P}_0$, $\prec$ *need not define a well-ordering* (containing *no* infinite descending sequence), *only a*

[10] If I interpret the early literature at the time of Fermat correctly, the principle of proof by *transfinite* induction applied to *quantifier-free* expressions (equations), was accepted quite early and called *method of infinite descent*. Thus, e.g., in the proof of $x^3 + y^3 \neq z^3$ (for positive integers x, y, z), we have *prima facie* a choice between (i) *ordinary* induction applied to the *quantified* formula Fu: $\forall x \forall y (\forall z < u)(x^3 + y^3 \neq z^3)$ and (ii) *transfinite* induction applied to the equation $E(x, y, z)$: $x^3 + y^3 \neq z^3$ in the lexicographic ordering of $\langle z, x, y\rangle$, say. It is true that, since only pairs $\langle x, y\rangle$ with $x < z$, $y < z$ need be considered, we get back to an ω-ordering contrary to first impressions. [The essential difference turns out to be that $F(u+1)$ is inferred from Fu, while $E(x, y, z+1)$ is inferred from $E[f(x, y, z), g(x, y, z), h(x, y, z)]$ with $h(x, y, z+1) \leq z$, and, roughly speaking, the functions f, g, h cannot be too 'simple'; for a definitive discussion, see Shepherdson's [55].] It seems clear that at the time of Fermat one neither thought of applying induction to quantified formulae nor did people realize that the lexicographic ordering of $\{\langle z, x, y\rangle : x < z, y < z\}$ is an ω-ordering. Gentzen's result shows that the uses of induction in **Z** for proving arithmetic identities can be replaced by transfinite induction applied to *equations* only (first impressions were right in the sense that induction on the ordinary ordering is not enough!).

quasi-well-ordering, i.e., one containing no infinite descending $\mathscr{P}_0$-sequence, which, by principle II, only requires that $\prec$ do not contain any descending recursive sequence.

Fallacy: it should be easier to establish that $\prec$ is a quasi-well-ordering than that it is a well ordering!

What is overlooked here is that the notion of quasi-well-ordering is not even formulated in $\mathscr{P}_0$; and *whatever is to be established will have to be done by the methods* of $\mathscr{P}_0$.—NB. My formulation, p. 295 of [23] is unsatisfactory as it stands, because these points were not considered properly.

EXAMPLE. In the theory of ordinals we think of ϵ_0 as the limit, or, equivalently, the sum of ω, ω^ω, ω^{ω^ω}, $\cdots$ and use the results: (a) if each order type w_1, w_2, w_3, $\cdots$ is well founded, so is their union, (b) if w_1 and w_2 are well founded, so is $w_1{}^{w_2}$, for the usual definition of exponentiation of (not necessarily well founded) order types. *This* way of reasoning, in particular (b), is patently false if quasi-well-orderings are meant; we use (I) and (II). For Parikh [48] has constructed a primitive recursive ordering w_2 which has no recursive descending sequence such that for a two element order w_1, $w_1{}^{w_2}$ has a primitive recursive descending sequence. This w_2 is a quasi-well-ordering, so is, of course, w_1, but not $w_1{}^{w_2}$.

Note that, if we replace (II) by the assumption that the $\mathscr{P}_0$-acceptable functions all belong to a recursively enumerable collection of recursive functions, by Theorem 5 of [48], not even multiplication preserves quasi-well-ordering. Even addition needs care! For instance, the proof of Theorem 4 (a) of [48] is nonconstructive in the following sense: suppose we know that the sequence h descends in $w_1 + w_2$; we do not necessarily know whether it stays in w_2 or goes ultimately into w_1, and so, given such an h we do not necessarily know how to find i and an h_i which descends in w_i (for $i = 1$ or $i = 2$).

Digression. The use of the example above is typical of a very *general principle that can be applied to a wide class of informal notions of proof* $\mathscr{P}$ which do not admit the general notion of function. Suppose it is evident that all functions which are admitted in $\mathscr{P}$, belong to $\mathscr{F}$ and all functions in $\mathscr{F}_1$ are admitted in $\mathscr{P}$ (e.g., $\mathscr{P}$ = predicative mathematics in the sense of [5], $\mathscr{F}$ = class of hyperarithmetic functions, $\mathscr{F}_1$ = class of arithmetic functions). Then a function F defined on orderings w, does not preserve quasi-well-orderings with respect to $\mathscr{P}$, if there is a w with characteristic function in $\mathscr{F}_1$ such that w contains no infinite descending sequence in $\mathscr{F}$, but Fw contains an infinite descending sequence even in $\mathscr{F}_1$. (Such an F corresponds to exponentiation to the base 2 in the example above.)

(iii) $\mathscr{P}_0$ and the notion of $\mathscr{P}_0$-well ordering; a fresh start. Instead of trying to exploit the fact that we need 'only' quasi-well-orderings, as in (ii) above, let us do the opposite:

Formulate a stronger notion of $\mathscr{P}_0$-ordinal, which implies quasi-well-ordering trivially, and then see whether such operations as addition or multiplication preserve $\mathscr{P}_0$-ordinals. (Specifically, the property of being a $\mathscr{P}_0$-ordinal applies to orderings defined in $\mathscr{P}_0$, and we consider operations on orderings.)

The general idea, mentioned already, is that a $\mathscr{P}_0$-ordinal is *built up* by combinatorial operations which i.a. preserve quasi-well-orderings. To *express* this idea one needs not only the ordering, but *functions* defined on it corresponding to these operations. In other words one will deal with *algebraic structures* consisting of an *ordering* **together** *with ordinal functions and their inverses*, the latter being needed to retrace the way in which the structure is built up; we shall call the use of such structures: a *functorial analysis*.

Two remarks. First, the present idea is closer to Gentzen's analysis of $\mathscr{P}_0$ in [9, p. 559, 1.18–21] than to Gödel's [12, p. 281, 1.4–7], the latter being closer to the quasi-well-orderings of (ii) above. (A formulation corresponding to the present idea, and superseding [23, p. 295] is in [31, 3.42, pp. 171–172].) Second, we shall return in §12 to the formulation of the general abstract principles behind this idea of $\mathscr{P}_0$-ordinal, perhaps related to the set theoretic notion of well-ordering as the algebraic notion of polynomial is related to the set theoretic notion of function. But at the present time it may be helpful to illustrate the idea and the problems it raises by discussing the simplest application and showing how a *functorial analysis* arises in a natural way.

(iv) *Finite orderings*; an example. The notion of finite ordering illustrates the idea in as much as these orderings are trivially (quasi-) well-orderings. Consider how we *see* that an ordering, say the segment $\prec\bar{p}$, is finite, and how we can *express* this knowledge. Clearly it is not enough that $\forall x[x \preceq \bar{p} \leftrightarrow (x = \bar{n}_0 \lor x = \bar{n}_1 \lor \cdots \lor x = \bar{n}_k)]$ be *true* for some list $\{\bar{n}_0, \cdots, \bar{n}_k\}$ since we may never know it! Even if we can prove it, this would not show that $\preceq p$ is an ordering; and even if we could prove the latter, *many additional steps* would be needed to arrange $\{\bar{n}_0, \cdots, \bar{n}_k\}$ in this order. The natural thing to say is that the segment $\prec\bar{p}$ is *built up from an initial element, say* n_0, *by adding one element at the end, and iterating this process.*

Formally, suppose we have *successor* and *predecessor* functions σ and π on $\prec$, for which we have proved, with free variables x and y:

$$x \prec \bar{p} \to [y \prec \sigma x \leftrightarrow (y \prec x \lor y = x)], \quad x \prec \bar{p} \to x \prec \sigma x, \quad \bar{n}_0 \neq \sigma x$$

and
$$\pi\bar{n}_0 = \bar{n}_0, \quad \bar{n}_0 \prec x \prec \bar{p} \to x = \sigma\pi x.$$

Then if $\pi^{(k)}$ denotes the kth iterate of π (for fixed k)

$$\pi^{(k)}\bar{p} = \bar{n}_0$$

expresses that the segment $\preceq\bar{p}$ has been built up starting at n_0 and iterating, at most k times, the operation of adding an element at the end.—NB. It would be interesting to analyze more fully the informal step of recognizing that our formulation expresses the intended idea, perhaps by use of combinators.—This functorial analysis allows one to state further facts about the ordering.

Given $\bar{n}_0$, σ, π, we can *reconstruct* the relation $\prec$ restricted to $\preceq p$ just in terms of $=$. Put differently, we no longer refer to the 'nature' of the objects in the field of the ordering (in our case, they are, 'accidentally', natural numbers), but only to the structure.

Given another ordering $\preceq'\bar{p}'$ which 'happens' to be isomorphic to $\preceq\bar{p}$, together with the corresponding operations $\bar{n}'_0$, σ', π' (the constant $\bar{n}$ is an 'operation' of zero arguments); then we can *explicitly* define the isomorphism by means of boolean operations from the given structures.

Finally, observe the *closure condition*: suppose **F** is any operation which maps finite orderings into extensions of themselves; if **F** is iterated along any finite ordering, i.e., iterated finitely often, and if the iterate is applied to a finite ordering, the result is again a finite ordering.—NB. Of course, for any given conception $\mathscr{P}$ it is not necessary that *arbitrary* F be considered, but only those accepted in $\mathscr{P}$ itself; so the closure condition above is particularly strong. One cannot expect an equally strong closure property when 'finite order' is replaced by '$\mathscr{P}_0$-well-order'; the need for care in the generalization of *finite* is familiar from [28] and, particularly, [30], and will be taken up again in §13.

(v) $\mathscr{P}_0$: *iterating the* **process** *of finite iteration*. Perhaps only the study of *specific* finite configurations as in (iv) deserves the name *finitist*. Be this as it may, the reasoning in $\mathscr{P}_0$ certainly is not confined to such configurations since one uses variables over such configurations (cf., p. 1); more precisely, $\mathscr{P}_0$ concerns the *process* of building up finite configurations, in the first place, in *ω-order*. Once this process is grasped (or accepted) it can itself be iterated, as, e.g., in building up an $\omega + \omega + \cdots$, i.e., an $\omega \cdot \omega$, order. The problem of analyzing $\mathscr{P}_0$ is to give a theoretical analysis of *what is implicit in accepting ω iterations*. Applied to the case of ordinals, we ask: *what orderings can be built up from ω by iterating the process of ω-iteration*? These orderings are then, by definition, the $\mathscr{P}_0$-ordinals.

I believe the general type of analysis required here is quite well illustrated by the discussion of finite orderings in (iv); but I do not know how to say it correctly. Roughly speaking, it is clear that the *sort* of detail needed is given in the proof theoretic study of subsystems of first order arithmetic **Z**; but the interesting question is to *formulate* the significance of such detail. The autonomous progression described in [31, 3.42, pp. 171–172], seems to be directly relevant here, in particular to making clear that $\mathscr{P}_0$-ordinals are quasi-well-orderings.

But leaving aside the exact philosophical, or perhaps, psychological analysis of the ω-iteration process, we can use the *general* ideas above to give precise *mathematical* results as follows.

(vi) *Natural α-orderings for $\alpha < \epsilon_0$*. We shall extend the functorial analysis of finite orderings (with first element, successor and predecessor) in (iv). The ordinal functions used now will be the constants 1 (first element) and ω; addition (and, in some formal results, exponentiation to the base 2), together with their inverses, the latter, as always, being needed to avoid alternating quantifiers. In the first place we shall assume the *existence of such ordinal functions*, i.e., of functions satisfying the recursion equations given, e.g., in [50, pp. 202–203]; their *uniqueness*, which depends on the fact that the orderings actually considered have no automorphisms, will be established later.

An *isomorphism* result (up to *recursive*, not necessarily $\mathscr{P}_0$-mappings). Suppose $\lessdot'\overline{\mathrm{p}}'$ is an ordering of the natural numbers of ordinal α ($\alpha < \epsilon_0$) together with the ordinal functions mentioned above, including exponentiation (we here suppose the ordering to be given together with an enumeration of all its elements: so the least number operator can be applied and the inverse functions are not needed). Then there are order preserving mappings, *recursive* in these functions and in $\lessdot'$, between $\lessdot'\overline{\mathrm{p}}'$ and the corresponding segment $\lessdot\overline{\mathrm{p}}$ of the canonical ϵ_0 ordering.

The proof is quite straightforward; the given functions allow one to construct Cantor's normal form t(n) for $\mathrm{n} \lessdot \overline{\mathrm{p}}$, recursively in the given functions on $\lessdot$; and the functions on $\lessdot'$ provide the element in $\lessdot'$ defined by the form t.

As a corollary we get, of course, an isomorphism between any two algebraic structures $\lessdot'\bar{p}'$, $\lessdot''\bar{p}''$ and the corresponding ordinal functions, by mapping each on $\lessdot\bar{p}$. For the given functions this is possible only for $\alpha < \epsilon_0$.[11]

Since not all recursive functions are accepted in $\mathscr{P}_0$, a somewhat different isomorphism result is more pertinent. For $\alpha = 2^\beta$, we consider the algebraic structures above and the inverse functions; then we can define mappings which are not only recursive in these structures, but defined by β-recursion.

Finally, a *closure property* of ϵ_0. We call a collection $\mathcal{O}$ of orderings *closed* for an operation F on orderings if, for $w \in \mathcal{O}$, also $Fw \in \mathcal{O}$. Let $\mathcal{O}$ be the collection of orderings $\lessdot\bar{p}$ ($\bar{p} = 0, 1, \cdots$) for the natural ϵ_0-ordering defined above. Then, up to the class of mappings introduced above, $\mathcal{O}$ is the least collection of orderings closed for the following class $\mathscr{F}$ of operations on orderings: (i) The constant operations 1 (single element) and $\omega \in \mathscr{F}$; (ii) addition and composition $\in \mathscr{F}$ (where composition, applied to a constant function, includes substitution); (iii) if $F \in \mathscr{F}$ and the constant (ordering) $w \in \mathcal{O}$, then $Fw \in \mathcal{O}$ and the iterate $F^w \in \mathscr{F}$, where the iterate 'along' w is defined in the usual way.

Discussion. The *significance* of this closure property depends on verifying in $\mathscr{P}_0$ that taking the iterate of operations in $\mathscr{F}$ with respect to $\mathscr{P}_0$-orderings is well defined. Clearly, more is needed here than quasi-well-ordering.

§7. Theory of arithmetic properties. (Induction formulated in a two sorted formalism by a single axiom, i.e., (ii) on p. 6). Here the *existential* axioms on arithmetic properties are replaced by *functors* ('third order' constants); typical examples: F_c, F_i, F_p with the axioms

$$\forall X \forall x[x \in F_c X \leftrightarrow \neg\, x \in X), \quad \forall XY \forall x[x \in F_i(X, Y) \leftrightarrow (x \in X \wedge x \in Y)],$$
$$\forall X \forall x[x \in F_p X \leftrightarrow \exists y(\langle x, y\rangle \in X)],$$

for a full list, see [33, App. A]. In this way one separates, at least formally, the use

[11] To avoid misunderstanding: here we define the isomorphism *uniformly* for pairs of structures of ordinal α (together with the ordinal functions mentioned), and this is possible for $\alpha < \epsilon_0$ only. In the literature, e.g. [4], one also considers isomorphisms defined by use of (a finite number of) *constants*, specifically for elements e_i, e_i' ($0 \leq i \leq n$) in $\lessdot$, $\lessdot'$ respectively corresponding to ϵ_i; for each $\alpha < \epsilon_\omega$ and each pair of structures of ordinal α there *exists* such an isomorphism, but in general we wouldn't know how to find it. Evidently, the essential property of ordinal functions involved in the uniform definition of the isomorphism is their *completeness*: the functions generate a segment of the ordinals starting from 1. Feferman, in [6], analyzes the much less obvious problem of *extending* complete systems to larger ones. For any two orderings, let [a], [a′] denote the segments preceding *a*, *a′* respectively. One of Feferman's conditions is *repleteness*: this ensures, if (i) the functions generate the segment of the ordinals $<\beta$ from the segment $\leq \alpha$, and (ii) a, a′ correspond to α; b, b′ to β; then an isomorphism between [b], [b′] can be defined uniformly from any given isomorphism between [a], [a′]. The second more recondite condition is *relative categoricity*, i.e., relative to the ordering of the inaccessibles of the given system of functions. This purely algebraic condition has two important properties: (i) it is preserved when the derived function of the given system is added, i.e., the function enumerating the inaccessibles (even for suitable *transfinite* iteration of this process), (ii) if $a_1, \cdots, a_{n+1}$ are the first n + 1 inaccessibles, the ordering of $[a_{n+1}]$ can be defined uniformly from the ordering of $[a_1]$ by use of $a_1, \cdots, a_n$. Property (i) permits application to Veblen's hierarchies, property (ii) ensures the existence of recursive orderings on which the given ordinal functions are also recursive. Note particularly that the basic properties are established in [6] without using the full classical theory of well-ordering: roughly, the operations can be extended to wider classes of orderings.

of logical operations in (i) constructing assertions, and (ii) defining objects; cf., top of p. 331.

For the formulation of some results it is convenient to introduce a third sort of variable of the type of functions (f, g, h, $\cdots$) and the usual axioms relating functions and (the sets which are) their graphs.

This system is equivalent to the version of formal ramified analysis of level one, where *induction* is applied only to first order formulae (not containing set nor function quantifiers); in contrast, e.g., to [50] where induction is not so restricted.

Our main interest here is to give the improved formulations of the role of ϵ_0, promised at the end of §6c, which need function or set variables. It might be of interest to extend the work of §6a, i.e., to give an elegant cut free formulation of the present system using only *finite* formulae and finite derivations with a suitable *rule* of $\prec\bar{p}$ induction, for $\bar{p} = 1, 2, \cdots$.

NB. The interest of a cut free formulation is limited if *for an arbitrary term* T *built up from the functors* F, *the formula* A(T) *is considered to be a subformula of* $\forall X A(X)$ or $\exists X A(X)$. For, in that case, there is no formula T_A which can be *proved* to be a truth definition for all subformulae of A (unless all the quantifiers of A are numerical).

Well foundedness and proof by transfinite induction (two senses of: well-foundedness). Let R be a formula containing the two variables x and y (i.e., R defines a binary relation)

(i) WF(R): $\forall X[\exists x(x \in X) \to \exists x(x \in X \wedge \forall y[R(y, x) \to \neg y \in X])]$.

In terms of function variables we have, more simply, for arithmetic R $\forall f \exists x \neg R[f(x + 1), f(x)]$. (This is used in [17].)

(ii) The *schema*, for any formula A, containing the variable x, but not y:

$$\mathrm{TI}(R, A): \qquad \forall x(\forall y[R(y, x) \to Ay] \to Ax) \to \forall x Ax.$$

Note that TI is the contrapositive of the least-element-principle on p. 327; the *second* order formulation, from which the schema is derived, will be discussed in §9.

MAIN RESULTS. 1. If WF(R) is a theorem, so is TI(R, A) for each formula A containing only numerical quantifiers. (The proof is immediate from the fact that for such A: $\exists X \forall x(x \in X \leftrightarrow Ax)$.)

2. Even for the ordinary definition R_0 of the natural ordering of the integers, there is an A_0 for which $\mathrm{TI}(R_0, A_0)$ cannot be derived.

(One way of seeing this is the consistency proof for arithmetic given in [15, pp. 366–367]; this consists essentially in deriving $\mathrm{WF}(R) \to \mathrm{WF}(R^{\#})$, and then applying ordinary induction to the Π_1^1-formula $\mathrm{WF}(W_n)$ where W_0 is the ordering of $\omega + \omega$, $W_{n+1} = W_n{}^{\#}$ and, generally, $R^{\#}$ is the canonical definition of 2^R.)

The asymmetry between WF(R) *and the schema* TI(R, A) *will play an important role in the next section.*

3. Let R be an essentially Σ_1^1 formula, i.e. (in prenex form) containing only existential quantifiers, numerical quantifiers not being counted, and suppose that the free variables of R are x and y: if WF(R) can be proved, then R defines (in the standard model) a well founded relation of type $< \epsilon_0$ [26]. If, further, R is quantifier free, then there is a term tx for which (i)

$$R(x, y) \to t(x) \prec t(y) \prec \bar{n}_R$$

can be derived, where, as above, $\lessdot$ is the canonical ordering of ϵ_0; (ii) it is a theorem that tx is formally computable.

Here we have a negative result which expresses that *ϵ_0 is the limit of provable well orderings without assumptions on the existence of definable ordinal functions on the ordering considered.* (Only the purely syntactic, and, obviously necessary, Σ_1^1-condition is imposed.)

Note that the bound ϵ_0 is not changed if any true Σ_1^1-sentence is added, in particular one asserting the existence of a function satisfying a recursion equation; for the significance of this, cf., p. 326.

Reflection principles and consistency statement. Now that we have *function variables* available (or set variables satisfying a functionality condition) the advantages of a reflection principle over mere consistency can be stated as follows. (NB. We need the reflection principle only for first order formulae with set parameters.)

Let S be a system including arithmetic and let Prov_S *be natural, in particular, suppose it satisfies the conditions for Löb's theorem* [42]. *Then if*

$$\forall n[\exists y\, \mathrm{Prov}_S\,(y, s_A n) \to An]$$

(s_A a canonical definition of the Gödel number of $A\bar{n}$ where $\bar{n}$ is the nth numeral) *can be derived by adding* WF(R) *to S for an arithmetic R and* WF(R′) *can be derived in S then*

$$(*) \qquad \neg\exists f \forall xy[R(x, y) \leftrightarrow R'(fx, fy)]$$

is derivable in *S*, i.e., there is no order preserving mapping of *R* into *R′*; in other words, if R′ defines a well-ordering its ordinal is less than the ordinal of R.

(The proof is in [31, 3.3421, pp. 166–167]. For, if $\neg(*)$ is added to *S*, $\mathrm{WF}(R') \to \mathrm{WF}(R)$; so, taking $(*)$ for A and s_* for s_A, we have $\neg(*) \to [\exists y\, \mathrm{Prov}_s\,(y, s_*) \to (*)]$, hence $\exists y\, \mathrm{Prov}_s\,(y, s_*) \to (*)$ are both theorems of *S*. By Löb's theorem, $(*)$ itself is derivable in *S*.)

Thus in contrast to the consistency statement [cf., §6c(iii)] which can be formally derived by induction on *some* ordering of ordinal ω, *there is a nontrivial lower bound for the ordinals* (of recursive well orderings) *which permit a proof of the reflection principle by means of transfinite induction applied to first order predicates.* This bound is exact, e.g., if *S* contains the arithmetic comprehension axiom (with parameters).

Note in passing that Gödel's original proof of his *first* incompleteness theorem for systems *S* of arithmetic shows underivability of the reflection principle applied to the formula expressing 'I am not derivable'. Underivability of the consistency statement requires significantly stronger conditions on *S* and on the representation of its proof predicate; cf. [31, p. 154, 3.221], or §11b(i) below.

§8. Elementary analysis $\mathscr{E}$. This is a two sorted formalism with variables for individuals and functions, and induction formulated as a schema applied to *all* formulae of the formalism; the main existential axioms, which are weak, assert the existence of the successor function, pairing and projection functions and closure under primitive recursive operations. For a precise description, see [17, App. 1 (p. 350)], where this system is called Z_1.

It is evident that $\mathscr{E}$ is a conservative extension of first-order arithmetic since its axioms can be proved in **Z** to be satisfied if the function variables range over the (primitive) recursive functions.

MAIN RESULT [cf., §6(a)]. For the ordering $\prec$ of the natural numbers of ordinal ϵ_0 considered in §6, $\mathscr{E}$ is equivalent to a (two sorted) system obtained by restricting the rules of logic to be cut free and by extending the schema of induction to the rule:

For each numeral $\bar{n}$, derive $(\forall x \prec \bar{n})Ax$ from $(\forall x \prec \bar{n})[\forall y \prec x)Ay \to Ax]$, for all formulae A of the language considered.

The natural proof uses an *infinitary* formulation of $\mathscr{E}$ whose rules are cut free in the usual sense.

Corollaries using the notation of §7 (and canonical representations *R* of primitive recursive binary relations).

1. WF(R) is derivable in $\mathscr{E}$ if and only if, for all A, TI(R, A) is so derivable. Thus $\mathscr{E}$ is 'balanced' for the two meanings of well foundedness.

As in §7, if WF(R) is derivable, R defines (in the standard model) a partial ordering of ordinal $< \epsilon_0$.

2. For each formula A with, say, a single free variable x, let $s_A x$ define canonically the Gödel number of the formula $A\bar{x}$ for the x-th numeral. Then, as in [36], the *reflection principle* (schema)

$$\forall x[\exists y \,\mathrm{Prov}_{\mathscr{E}}(y, s_A x) \to Ax]$$

where $\mathrm{Prov}_{\mathscr{E}}$ denotes the proof relation for $\mathscr{E}$, is equivalent over $\mathscr{E}$ to the schema, again for each A,

$$\forall x[(\forall y \prec x)Ay \to Ax] \to \forall x Ax.$$

3. A direct consequence of the reflection principle is this. *For any finite extension* F *of* $\mathscr{E}$, in the language of $\mathscr{E}$, *the consistency of* $\mathscr{E} \cup \{F\}$ *can be derived by adding to* $\mathscr{E} \cup \{F\}$ *an instance* TI($\prec$, A_F) *of* ϵ_0*-induction*, where A_F depends on F. Consequently, *the schema* TI($\prec$, A) *is not derivable in any finite extension of* $\mathscr{E}$.

The proof is routine. $\forall y \neg \mathrm{Prov}_{\mathscr{E}}(y, \ulcorner F \to 0 = 1 \urcorner)$ expresses the consistency of $\mathscr{E} \cup \{F\}$; by above, $\forall y[\mathrm{Prov}_{\mathscr{E}}(y, \ulcorner F \to 0 = 1 \urcorner) \to \neg F]$ is derivable in $\mathscr{E}$ from a suitable instance of ϵ_0-induction, and, since F is a theorem of $\mathscr{E} \cup \{F\}$, we should have $\forall y \neg \mathrm{Prov}_{\mathscr{E}}(y, \ulcorner F \to 0 = 1 \urcorner)$.

Examples of extensions of $\mathscr{E}$ which are given by schemata, but are equivalent to a finite number of instances, cf. [36]:

(i) The 'arithmetic' comprehension axiom can be replaced by the single axiom

$$\forall f \exists g \forall x(gx = 0 \leftrightarrow \exists y[f(\langle x, y \rangle) = 0]).$$

The system so obtained is formally identical (except for the use of function variables instead of set variables) with so called ramified analysis of level one as formulated in [50]; cf. also Note IV.

(ii) In the notation of p. 327, (AC) for formulae Ξ, containing parameters, that are essentially Σ_1^1 or Σ_2^1.

Ad COROLLARY 1. Note that the *implication* WF(R) $\to$ TI(R, A) is in general not provable in $\mathscr{E}$. For instance, WF($\prec$) is derivable in the extensions (i) or (ii) above (e.g., by [50], well foundedness in the sense WF is derivable in (i) for all canonical orderings $<\epsilon_{\epsilon_0}$). We shall return to the above in §10. Trivially, for every definable R, well foundedness in sense WF is expressed by a *single* formula.

By combining these results with very pretty model theoretic constructions, Friedman has obtained interesting results on Σ_1^1-AC: cf. Note V. The situation reminds one of proofs of recursive undecidability in the style of [65]: there is a basic proof theoretic result (corresponding to their basic result from recursion theory) and the rest of the argument is model theoretic and easy to follow.

§9. **A formulation of full analysis** (in terms of well foundedness). We interrupt here proof theoretic analysis to give a straightforward formal equivalence, straightforward in the sense that we have *implications* (which are valid for arbitrary extensions of the systems considered) and not only *proof theoretic rules* of the form: if *A* can be *derived* by means of the given rules then so can *B*.

(a) For a variable *Y*, write $Y(x, y)$ for $\langle x, y\rangle \in Y$.

$$\forall X\forall Y[(\exists x(x \in X) \wedge \forall f\exists x \neg Y[f(x+1), f(x)]) \to \exists x(x \in X \wedge \forall y[y \in X \to \neg Y(x, y)])]$$

is equivalent to AC_{00} when the latter is formulated as a single axiom in the first order (two-sorted) theory of arithmetic properties in §7.

(The proof in [17, p. 352], applies unchanged although it is formulated there for schemata instead of axioms.)

Corollaries. The *schema* $WF(R) \to TI(R, A)$, for all explicitly definable relations *R* and properties *A*, is equivalent to the *schema* of AC_{00}, because the class of formulae considered in the schemata is closed under arithmetic quantification. (For refinements concerning relations between this schema and the corresponding rule, see Note III.) More interesting:

The *second order* axiom AC_{00} of p. 326 is equivalent to

$$\forall XY([\forall f\exists x \neg Y[f(x+1), f(x)] \wedge \exists xX(x)] \to \exists x[X(x) \wedge \forall y\{X(y) \to \neg Y(x, y)\}]),$$

because of course the class of *all* properties of natural numbers satisfies the closure conditions in the theory of arithmetically (definable) properties.

(b) Extend the theory of arithmetic properties by adding a new sort of variable $\dot{X}$ for properties of functions, extend the use of $\in$, but use only axioms for closure under *first order* operations applied to the new sort of variable.

For a variable f, write f^* for the sequence of functions coded by f, i.e., $f^*(x) = \lambda y f(\langle x, y\rangle)$.

$$\forall\dot{X}\forall\dot{Y}[(\exists f(f \in \dot{X}) \wedge \forall f\exists x \neg \dot{Y}[f^*(x+1), f^*(x)]) \to \exists f(f \in \dot{X} \wedge \forall g[g \in \dot{X} \to \neg\dot{Y}(f, g)])]$$

is equivalent to DC_{11}.

(The proof is also in [17, p. 352], and the remarks and corollaries above apply.)

Applications of this reformulation will be made in §§10 and 11.

Subsystems. If we consider the *schemata* in the notation of elementary analysis corresponding to (a) and (b), we can choose *syntactic restrictions on* R *and* A *independently*; analogously to the familiar asymmetric treatment of induction and comprehension axioms. The next section will show that, at the present time, the subsystems obtained by such restrictions are more appropriate than the conventional ones mentioned p. 327.

Discussion. Philosophically, the reformulation above is not significant at all. For the intuitive conception of the structure of analysis, (CA) is evident and the equivalence above is derived.

Technically, there is something to be said for it at least *as long as the proof theoretic analysis involves a reduction to intuitionistic methods of proof.* The following formal results, which will be briefly discussed below, summarize the main facts about the systems obtained from $\mathscr{E}$ by replacing the classical rules by intuitionistic rules and adding (i) (AC) and (ii) WF(R) $\rightarrow$ TI(R, A) (for arbitrary R and A in the language of $\mathscr{E}$). Let $\mathscr{E}_1$ denote the intuitionistic version of $\mathscr{E}$, which is called **H** in [17].

(i) (AC) in *any* of its forms is reducible to arithmetic (e.g. [12] applies without modification).

(ii) The schema above (for intuitionistic logic) reduces, via continuity axioms, to the schema restricted to *quantifier-free* R (by [21] or [17]): it is satisfied if f, g, $\cdots$ range over free choice sequences.

Note that if we also add set variables to $\mathscr{E}_1$ then (CA) together with the axiom $\forall X\exists f\forall x(fx = 0 \leftrightarrow x \in X)$ is certainly not valid, when the variables X, Y, $\cdots$ range over species and f, g, $\cdots$ over either constructive functions or over free choice sequences: for (CA) requires the existence of undecidable species.

(iii) If set variables *replace* function variables in $\mathscr{E}_1$ and (CA) is added, the resulting system is satisfied by the most general notion of species of natural numbers (as will be elaborated in my paper at Amsterdam cited in footnote 1). Now a straightforward extension of Gödel's old translation (see, e.g. [20]) of $\mathscr{E} \cup$ (CA) to this system *provides an intuitionistic consistency proof of classical analysis.*

Discussion. Quite naively, this easy proof in no way reduces the interest of a more detailed proof theoretic reduction in the next section; just as Gödel's original intuitionistic consistency proof for classical arithmetic **Z** did not make Gentzen's reduction superfluous.

For the logician a *principal* problem is to formulate the reasons for this naive impression; e.g., in the case of $\mathscr{P}_0$ this was done by Gödel [12]: $\mathscr{P}_0$ only admits constructions on combinatorial objects while the laws of intuitionistic logic are *immediately* evident only if constructions on abstract objects (such as constructions on functions or even on proofs) are accepted.

The next section gives a reduction to certain *definition principles* for operations on sequences of natural numbers, in particular, higher types than this will be *reduced.* For further discussions including some open problems, see end of §11 below, and, particularly, my paper at Amsterdam l.c.

§10. Well foundedness of elementary relations. We consider now elementary analysis $\mathscr{E}$ extended by the schema WF(R) $\rightarrow$ TI(R, A) for arbitrary formulae A and 'elementary' R: specifically canonical definitions R of primitive recursive relations [for the (obvious) precise formulation of canonical definitions, unique up to provable equivalence, cf. [31, p. 154, 3.222]]. For details on the effect of parameters, see Note V.

Properties of the system. (i) The schema implies Σ_1^1-AC_{01} (for number theoretic relations R) and Σ_1^1-DC_{11} (for relations R between functions). This is seen by inspection of the equivalence in §9 above; cf. [16].

(ii) The converse of (i) fails in the strong sense that the collection of hyperarithmetic functions is an ω-model of Σ_1^1-AC_{01} by [29] (and also of Σ_1^1-DC_{11} by the same method), but not of the schema. In fact there are parts of 'practical' analysis that can be developed in the present system, but not from Σ_1^1-DC_{11}, e.g., the theorem of Cantor Bendixson in the theory of sets of points [25].

(iii) A simple *model* satisfying the schema is given in Note V: the properties needed are proved by use of Σ_2^1-AC_{01} (or, equivalently, Π_1^1-AC_{01}, or again Δ_2^1-CA). But a more delicate proof theoretic argument, via reduction to intuitionistic systems, shows that the *consistency* of the schema can actually be proved in $\mathscr{E} \cup \{\Sigma_1^1\text{-CA}\}$. In contrast:

(iv) The schema is not *included* in any finite extension of elementary analysis. (One first shows WF($\lessdot$) directly, and then applies p. 342 §8(3).)

Thus (iii) shows that the assumptions required for establishing the consistency of the schema are relatively weak, and (iv) indicates why certain consequences of the schema *seem* to assume strong existential axioms, namely: if we *insist* on deriving the schema from comprehension axioms we *do* need essentially all instances! Incidentally, this is the kind of situation in which new axiomatizations of an informal branch of mathematics can be foundationally really efficient (cf., e.g. [29, top of p. 328]). And, it may be added, this possibility is not nearly as often exploited as it could be. This failure of contemporary logic to use the axiomatic method is perhaps related to the situation described on p. 328, and ultimately to the philosophical problems presented by the choice of subsystems (p. 323).

Proof theoretic analysis of the system. The three principal results are, roughly, these:

(i) The system is interpreted, by use of Gödel's method in footnote 9, in a quantifier free formal system, say $\mathscr{T}$, for functions of finite type containing besides elementary (primitive recursive) schemata, the following so called *bar recursion schema.*

For G, H given, define the constant $\phi_{G,H}$ whose arguments are functionals F of lowest type, and finite sequences c of natural numbers, by:

$$\phi_{G,H}(F, c) = G(F, c) \quad \text{if } F([c]) < lc$$
$$\phi_{G,H}(F, c) = H[F, c, \lambda x \phi_{G,H}(F, c * x)], \quad \text{otherwise,}$$

where lc is the length of c, $c = \langle c_0, \cdots, c_i, \cdots \rangle$ for $i < lc$, $[c]$ is the function defined by $[c](x) = c_x$ for $x < lc$, $[c](x) = 0$ beyond, and G, H have types for which the substitutions make sense.

(This schema is a particular case of a general schema formulated explicitly by Spector [58].)

(ii) The next step is to define a model of $\mathscr{T}$ by interpreting the functions of higher types as *neighborhood functions* of continuous functions in the sense of [24]. In this way the axioms of $\mathscr{T}$ are translated into statements in the language of analysis. The main principle needed to prove these statements is the schema WF(R) $\rightarrow$ TI(R, A) itself, but using *intuitionistic* logic only.

(iii) This latter system, say $\mathscr{I}$, has been interpreted ([31, p. 140, 2.621]) in the purely *arithmetic* system (only variables for natural numbers) consisting of intuitionistic first order arithmetic and a monadic predicate constant for the notion of recursive ordinal, i.e., Church-Kleene's set O of recursive ordinal notations, together with the corresponding principle of proof by induction, namely:

Let D be the quantifier free formula such that the 'inductive definition' of O takes the form

$$(*) \qquad \forall x[\forall u D(x, u, O) \to Ox];$$

we take as axioms (*) itself and the schema, for all formulae A (the predicate letter O being included)

$$\forall x[\forall u D(x, u, \lambda y A y) \to Ax] \to \forall x(Ox \to Ax).$$

(Conversely this system can be interpreted in $\mathscr{I}$ by using an explicit definition for O; note that in [31, 2.621], the set K replaces O.)

(iv) By analysis of $\mathscr{T}$ Howard has shown that Bachmann's ordinal $\varphi_{\epsilon_{\Omega+1}}(1)$, which is recursive, is a bound for all provable well orderings, and his student Gerber has verified that it is exact. (For a detailed description, see [11].) Roughly speaking, one starts with Bachmann's definitions [1] involving the cardinal Ω (= the first uncountable ordinal, $= \epsilon_\Omega$), the first ϵ-number $> \Omega$, i.e., $\epsilon_{\Omega+1}$, etc. One then considers the terms used for these definitions with *names* of the cardinals involved, and considers the ordering of these *terms t* defined by the order of magnitude of the *values* of t. It turns out that this ordering is recursive. (For a systematic formulation, see footnote 11 and, of course, [6].) Since Bachmann's notations involve normal functions, unique characterizations follow automatically.

A consequence of any one of the results (i)–(iii) is that the consistency of the schema *cannot* be proved in Feferman's (IR) [5], and hence *not* by *predicative methods*.[12]

Discussion. In (i) the system is reduced to *currently formulated* intuitionistic principles. The use made of variables of type higher than that of F is quite elementary. As far as *actual* understanding is concerned this reduction is significant simply because functionals of lowest type have been studied more. But there is also a more theoretical reason: Suppose given the elementary notion of function of finite type as described in [12], i.e., constructive arithmetic functions (given by rules), functions whose arguments and values are such functions, etc. We now *pick out* from them, first the constructive arithmetic functions ($\in \mathscr{F}_1$), then those F for which, roughly, recursion on their unsecured sequences is valid ($\in \mathscr{F}_2$), and after that we only admit the functions obtained by applying primitive recursive operations and $\varphi_{G,H}$ to $\mathscr{F}_1 \cup \mathscr{F}_2$: all that has to be verified then is that the operations of type $\mathscr{F}_2$ so obtained are again in $\mathscr{F}_2$. (Of course, the verification should use only elementary facts about the notion in [12].) The appropriate technical means for picking out $\mathscr{F}_2$ is by use of constructive ordinals. (The reduction to arithmetic involves the further restriction to *recursive* ordinals.)

The reduction to (ii) is *prima facie* significant in that higher type variables are here avoided altogether; in particular there is no appeal to the (allegedly specially

[12] I proposed the characterization of *predicative proof* here employed in [23] and then again in [27]. But my specific technical conjecture on the actual limit given in the review of [50, bottom of p. 246], was false.

problematic) notion of free choice sequence.[13] Concerning a coherent notion of ordinal which is independent of the more abstract constructions involved in the basic intuitionistic notions, cf. §12.

The full significance of result (iii), and, in particular, of the ordinal $\varphi_{\epsilon_\Omega+1}(1)$ has not been established. Specifically, granted the notion of ordinal, formally stronger principles than those of (iii) are justified, in particular *definition* by recursion on O.

It should be mentioned that the systems here discussed cover the bulk of actual mathematics that can be formulated in full analysis *at all* (of course, not the theory of larger cardinals). So, at the present stage of knowledge, the reduction of analysis to familiar intuitionistic principles, even excluding the theory of species, is problematic only for possible, not for actual uses of classical analysis: cf. Gentzen's remark on the situation in number theory in 11.4 on pp. 532–533 of [9].

§11. Partial results on full analysis. Consider first the extension of the preceding paragraph to full analysis: actually, it was the extension that was first treated (by Spector in [58]) from which the special case above was isolated!

(a) Well founded relations between objects of finite type. As pointed out in [24], Gödel's interpretation [12] extends formally to full analysis; furthermore, the equivalence $A \leftrightarrow \exists s \forall t A_1(s, t)$ (in the notation of footnote 9) is proved by apparently quite elementary instances of the axiom of choice (this is the 'principal result' of [24], cf. Note VII).

Spector discovered that, for *theorems* A of full analysis (based on AC_{01}), the functionals *s* needed can be generated by extending the *principle of definition* (in §10) *of recursion on the unsecured sequences of F to all finite types*: specifically, while *F* maps sequences of natural numbers into natural numbers, we now consider F^τ mapping sequences of type τ objects into natural numbers. Put differently, we consider elementary well-founded relations between objects of type τ (and not only between numbers or functions). Spector's proof was a bit difficult, because he considered the interpretation of AC_{01} directly; the matter becomes routine if one considers the interpretation of $WF(R) \rightarrow TI(R, A)$ instead! and by §9, one even gets DC_{11}.[14] The higher types come in because one now considers *R* defined by use of function quantifiers.

No notion of constructive *function* of finite type is known for which the extended schema of bar recursion can be established. Of course, the schema can be shown to be consistent by use of the theory of species, but then it is unnecessary (for a consistency proof of classical analysis) since, by p. 344, Gödel's old method is here available. Its *formal* similarity to the restricted schema of §10 is certainly no guarantee for its constructive validity: just as formal similarity is not a reliable index of proof theoretic strength.

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[13] More precisely, it is quite correct that, after the elimination of higher types, all the objects considered can be *named* or *listed*; but, contrary to some traditional assumptions, such listing is not essential for constructive reasoning; see especially the notion of constructive function of finite type in [12].

[14] It is quite usual that interpretations are much easier to verify for one of several formally equivalent axiomatizations.

Remark. When I originally considered the extension of [12] to analysis (in [24]) I believed that the *particular* notion of functional of finite type there described could be proved by intuitionistic methods to satisfy $\exists s \forall t A_1(s, t)$ for theorems A, having shown that this could be done nonconstructively. Put differently, I thought that the *existing* intuitionistic theory of free choice sequences, especially if one used the formally powerful continuity axioms, was of essentially the same proof theoretic strength as full classical analysis! In other words, I thought the relation was analogous to that between formal intuitionistic arithmetic and first-order classical arithmetic mentioned at the end of §9, or, as pointed out in the discussion of §9, between formal classical analysis and the intuitionistic theory of species. In terms of §9 (which, of course, was not known then) I had not realized the result (ii) on p. 344.

(b) *Notions of 'subformula' in higher order logic.* (Recall §7.) It has been observed (e.g. [31, p. 167, last paragraph], without proof) that a formal system S of analysis containing the schema $\mathrm{WF}(R) \to \mathrm{TI}(R, A)$, does not allow a cut free formulation using recursive derivations in the sense of p. 329, even if R is restricted to be primitive recursive.

(The proof is straightforward, as soon as the sense of p. 329 is analyzed. Specifically, we suppose the cut free, possibly infinitary syntactic rules to be *defined* in S, and for each formula A we suppose that the *validity* or *soundness* of the rules, when applied to subformulae of A, can be formally proved in S; here the partial truth definitions of p. 329 are needed. But then our schema permits a proof of the reflection principle for the cut free derivations and any finite extension. Take the extension obtained by adding the sentence that the given system S is formally equivalent to the cut free system, and apply Gödel's first incompleteness theorem as stated at the end of §7.)

Takeuti [63] considers the 'pure' theory of types, which can be described (for types ≤ 3) as follows: instead of using a two-sorted formalism as in §7, one has also a third sort $\mathring{X}$ of variable for properties of properties, and functors for the corresponding second-order closure operations, in particular for 'second-order' projections:

$$\forall \mathring{X} \forall X[X \in F_P \mathring{X} \leftrightarrow \exists Y(\langle X, Y \rangle \in \mathring{X})],$$

(extensions to all finite types are clear).

Takeuti's notion of subformula is as in §7; for an arbitrary term T built up from the functors considered, $A(T)$ is regarded as a subformula of $\forall X A(X)$ or $\exists X A(X)$. The rules of proof are then essentially those of ordinary predicate logic. Evidently, when we apply the system (e.g., to the type structure on the natural numbers) we do not have partial truth definitions for the subformulae of a formula A unless A is (equivalent to) a formula without higher order quantifiers.

(i) *'Completeness' of cut free rules,* in the sense that any theorem of the full system can also be derived without cut (not, of course, in the sense of completeness for validity in all principal models). Tait [62], by essential use of earlier work of Schütte [51], gave a very short proof of completeness using the principles of third-order arithmetic in which the existence of a standard model of analysis can be formally proved. (Recall the related situation in §5.) (There is even an unpublished manuscript of Takahashi: *A proof of cut-elimination theorem in a simple type* *theory*, and of Prawitz: *Hauptsatz for higher order logic*, where the theory of *all* finite types is considered.)[15]

As Takeuti observed [63], the consistency of his 'cut free' system of second-order logic together with a certain first-order axiom (asserting the existence of the successor function) can be proved in the system itself. So one of the 'derivability conditions' in Gödel's second incompleteness theorems is violated, in particular *closure under cut of the system cannot be proved in the system* (even if cut were added). So *within* the general framework of Tait's proof, the use of third-order existential axioms is necessary.[16]

(ii) *Cut elimination*. Leaving aside the *fact* of closure under cut of Takeuti's rules, the *problem* of cut elimination has had heuristic value, at least for him. He was led to longer, but more explicit proofs of cut elimination for a series of subsystems of analysis, the last of which having a simple description in familiar terms, namely Σ_1^1-CA (or, equivalently, Π_1^1-CA) added to elementary analysis (and thus containing the unrestricted schema of induction) [64]. The cut elimination is described by means of definition by recursion on certain primitive recursive orderings which he calls, somewhat colorlessly, ordinal diagrams. (One certainly has the impression that they are more 'natural' than the pointless orderings of §6c, but I do not know what property makes them specially interesting; if I did I could propose a better name for them!)

Warnings. 1. Tait's proof [62] does not establish cut elimination for *subsystems* of analysis (even for Takeuti's notion of subformula). I have not checked whether, in [64], Takeuti gives cut elimination for derivations of *arbitrary* formulae and I do not know if full cut elimination holds for the subsystem considered.

2. It has been suggested that cut elimination may be connected with *explicit realizations of existential quantifiers* in the sense that there are explicitly defined $X_0, \cdots, X_k$ such that $A(X_0) \vee \cdots \vee A(X_k)$ is derivable if $\exists X A(X)$ is derivable. This is false. (Essentially, the same type of counterexample as to the analogue in

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[15] Tait's result was asserted in [43], but the idea of [43] is certainly not correct. The suggestion is that the formulae, in particular the terms of the theory of simple types should be interpreted with quantifiers ranging over the constructible sets, and the elimination of cuts should proceed according to the order (in the constructible hierarchy) of the sets so defined. It is quite correct that in this way one will get a proof figure with atomic formulae $T \in T'$ as axioms. But there is no reason to suppose that these formulae are theorems of analysis (or even true in the universe of *all* sets of finite type, since the reduction only ensures that $T \in T'$ holds for the sets defined by T and T′ in the collection of constructible sets). Taken literally, the statement of [43] was in any case suspect: if cut elimination can be proved under the assumption $V = L$, it can be proved without, since the statement of cut elimination is purely arithmetic. The following possibility has not yet been excluded. We may find a 'natural' reduction procedure on formal derivations and discover that it reduces the ordinal associated to the derivation roughly as follows: take the maximum of the orders in the ramified analytic hierarchy of the sets defined (in this hierarchy) by the abstraction terms occurring in the derivation. Because of the absoluteness (invariance) of Δ_2^1-definitions, this kind of thing is particularly promising for the subsystem Δ_2^1-CA of classical analysis.

[16] This situation is sometimes formulated in the literature by saying: Takeuti's conjecture (closure under cut of his system) implies finitistically the consistency of analysis. In these terms, i.e., in terms of consistency proofs, Tait's argument would only have proved the consistency of classical analysis in third order arithmetic!

first-order logic: let $A(X)$ express that X is not constructible; we have of course $\exists X \forall Y[A(X) \vee \neg A(Y)]$; but there can be no sequence $X_0, \cdots, X_k$ of the type required because there is a model [41], even of set theory, in which every *definable* set is constructible, but not all.)

(c) *Conjectures and Problems.* Of course, it would be interesting to use the results in (a) and (b) above in a more constructive proof theory of classical analysis. At the present time the principal difficulty seems to be philosophical; cf. the remarks at the end of §9 and in Note Ib. Meanwhile, here are some formal *mathematical problems*, suggested by the two remarks at the end of (b) above.

1. Let Prov_{CA} and Prov_{CF} denote the proof predicates for classical analysis with and without the rule of cut. Let us assign a measure of *complexity* to formulae of analysis; for examples of such measures, see [36]; the complexity of a derivation will be, by definition, the maximum complexity of the formulae occurring in it.

We know that

$$(*) \qquad \forall p \forall a[\mathrm{Prov}_{CA}(p, a) \rightarrow \exists q\, \mathrm{Prov}_{CF}(q, a)]$$

is true, hence there is a recursive function ψ such that

$$(**) \qquad \forall p \forall a[\mathrm{Prov}_{CA}(p, a) \rightarrow \mathrm{Prov}_{CF}(\psi p, a)]$$

(since a is determined by p, ψ may be taken to depend only on p, and ψ is determined uniquely if we take the minimum); further, we know that (*) cannot be formally derived in analysis.

Question 1. For each numeral $\bar{c}$, let '$p < \bar{c}$' mean that p is of complexity $< \bar{c}$: can

$$(*)_{\bar{c}}: \qquad (\forall p < \bar{c})\forall a[\mathrm{Prov}_{CA}(p, a) \rightarrow \exists q\, \mathrm{Prov}_{CF}(q, a)]$$

be formally proved in analysis? (for the usual measure of complexity).

Since the proofs in (b) only use the existence of ω-models, and since the existence of such models for subsystems of bounded complexity can be established in analysis [36], one expects a positive answer to Question 1.

Question 2. Can we formally prove in analysis *that*, for variable c, $(*)_{\bar{c}}$ can be formally proved in analysis?

If so, we should have a formal proof of (*) itself in the system consisting of analysis together with the *reflection principle* for analysis applied to Π_2^0-formulae, i.e., by [36]:

The ω-consistency of analysis implies (*) (in arithmetic).

Question 3. Returning to the function ψ above, we should like to decide between the alternative:

(i) There is no recursion equation e, which can be proved in analysis to define a total function, i.e., no 'provably recursive' function $\{e\}$, such that (**) is *true* with $\psi \equiv \{e\}$;

(ii) There is a provably, perhaps even primitive, recursive ψ such that (**) is true, only it is not formally derivable in analysis.

To exclude (ii), it is enough to verify that a cut free derivation of $\exists x A x$ for free-variable A contains a numeral $\bar{n}$ such that $A\bar{n}$: for then ψ would provide an enumeration of all provably recursive functions of analysis.

On the other hand, a formal proof of (ii) would strengthen the corollary to Question 2; replacing 'ω-consistency' by '*simple consistency*'.

2. Let us consider, instead of classical analysis, the theory of species, i.e., the comprehension axiom together with intuitionistic logic, as on p. 344. (I owe this suggestion to a conversation with D. Prawitz.)

Prawitz suggested that a suitable extension of his work would lead to the result: if $\exists X A(X)$ is formally derivable then there is an explicitly defined X_0 such that $A(X_0)$ is derivable; and hence if $A \vee B$ is a closed theorem, so is either A or B; if $\exists x Ax$ is a closed theorem, there is a numeral $\bar{n}$ such that $A\bar{n}$ is a theorem. Finally (cf. [31, p. 160, 3.322]) one expects: if $\forall x(Ax \vee \neg Ax)$ and $\neg\neg\exists x Ax$ are, not necessarily closed, theorems, so is $\exists x Ax$; in particular, exactly the same equations can be proved to be recursion equations (for total functions) in classical analysis and in the theory of species.

It might be added that a *formalization* in analysis of the proofs suggested by Prawitz would have the consequence:

If $\forall x \exists y A(x, y)$ is a closed theorem of the theory of species there is a numeral $\bar{e}$ such that $\forall x A[x, \{\bar{e}\}(x)]$ is also a theorem; cf., Church's thesis.

For, any given derivation of $\forall x \exists y A(x, y)$ is of bounded complexity $\bar{c}$, hence (for the usual measure of complexity) we have derivations $\leq \bar{c}$ of $\exists y A(\bar{n}, y)$ for each $\bar{n}$. Formalization of the metamathematical proofs used would yield a formal proof of $\forall x \exists y T(\bar{e}, x, y)$ in analysis for the obvious recursion $\bar{e}$, giving y as a function of n.

Perhaps we are to look at the situation as in [31, p. 156, 3.242]. We first have the easy reduction of classical analysis to the theory of species with intuitionistic logic. And then the hard work of a more detailed explicit treatment of the intuitionistic system starts. In short, not proof theory *by* intuitionistic means, but proof theory *of* intuitionistic systems.

§12. Existing proof theory. Developing this last remark we can say that the bulk of the work described in this survey lies between Hilbert's original, strictly combinatorial $\mathscr{P}_0$ and the most general, abstract or intuitionistic, notion of constructivity. We have various reduction procedures, as in Gentzen's [9] or [10], functionals, or operations on well-founded trees. All these superficially different operations are closely related, the common mathematical element being the use of *ordinals* or, more precisely, of *transfinite iteration* of processes, including transfinite types. Consciously or unconsciously, the idea of such *iterations* is accepted in the literature to be used as a *means* of analysis. Thus, in contrast to §6(d), here the use of ordinals is not in need of analysis; nor is the notion of ordinal *defined*, e.g., in the Frege-Dedekind manner, as would be natural if the general intuitionistic notion of species (§11) is accepted. In other words, *ordinal is treated as a primitive notion.*

To complete, so to speak, this current direction in proof theory, it is necessary (a) to analyze this idea of iteration more closely, and also (b) to look for the proper mathematical tools. It should not be expected that (a) would in turn use ordinals (though the use of *set-theoretic* ordinals is not excluded); perhaps comparison with familiar subsystems of analysis will be better. One would hope (b) to help one with the problems of §6(d).

(a) The most natural starting point for a philosophical analysis is the old conception of *inductive definition*, say $\mathscr{P}_{\mathrm{ID}}$, which, roughly, allows definitions from 'below' referring to previously introduced notions, and proofs by 'reflecting' on such definitions. The definition by Church-Kleene of ordinal notations in §10(iii) is an example of such definitions, and the schema on p. 26 expresses this kind of proof. If $\mathscr{P}_{\mathrm{ID}}$ is to have any *foundational* interest at all, it is pointless to regard its definitions as implicit definitions to be converted into explicit ones: for then the *basic* notions are those used in the explicit definition. What makes $\mathscr{P}_{\mathrm{ID}}$ difficult for most (and interesting to others) is that *neither the usual set theoretic semantic nor the intuitionistic meaning of the logical operations applies* ([31, p. 139, footnote 26]). This is clear from the most elementary case (iteration of purely mechanical steps) in the definition of recursively enumerable sets: if this class of objects is to have an independent interest, this is so because one can verify that a number *is* in such a set in a purely mechanical way: but a proof (i.e., not a mere calculation) is needed to establish the opposite. Whatever else this may imply, it requires that the meaning of *negation* be reformulated for $\mathscr{P}_{\mathrm{ID}}$, if it is to be used at all. The same applies to implication, except in the special case when implication is not iterated; then the premise can be absorbed in a quantifier restricted to an already defined set. (Here we have only spoken of logical operations applied to assertions; in the case of *generalized* inductive definitions a new problem arises in the interpretation of logical operations occurring in the definitions themselves.) Note that, in contrast, the truth functional meaning of the propositional operations applies both in combinatorial and in predicative mathematics: the former is anyway 'logic free'; for the latter this follows from Poincaré's fundamental principle, analyzed in [27] and [5], that a predicative definition be invariant for extensions of the universe.[17] As an immediate objective for the analysis of $\mathscr{P}_{\mathrm{ID}}$ it seems reasonable to demand:

To find enough evident properties of $\mathscr{P}_{\mathrm{ID}}$ to decide whether $\mathscr{P}_{\mathrm{ID}}$ allows a solution of Hilbert's problem for the subsystem of analysis obtained from elementary analysis in §8 by adding Δ^1_2-CA or, equivalently, Σ^1_2-AC.

At present we do not know the relation between Δ^1_2-CA and the formal theory of definition by recursion on O of species of natural numbers, mentioned at the end of §10; it seems plausible that the latter theory is closely related to $\mathscr{P}_{\mathrm{ID}}$.

(b) The discussion of §6 stressed the problem of analyzing the idea of a natural well-ordering and pushed it back to the problem of finding natural ordinal functions. *Is there a mathematical way of characterizing natural structures* which generalizes the analysis of §6?

What I have in mind is the answer of category theory to the question: what is a natural embedding? Or, more specifically, this: in §6(d) we looked at ϵ_0 as the limit of what is *implicit* in accepting 1, ω, and addition. Is this not reminiscent of the magic of adjoint functors? Given the Cartesian product $X \times Y$, the adjoint extracts exponentiation, i.e., the operation which *iterates*, for given X and Y, the product operation $(X \times X) \times \cdots$, Y 'times'.

[17] One can probably get a rough idea of the situation in model theoretic terms by reinterpreting the logical operations along the lines of [28, pp. 136–137]; except that extensionally definite (= invariant) should be replaced by persistent (in the sense of [49]). This is a far cry from the naive (but common) idea that inductive definitions 'create' or 'generate' mathematical objects, instead of picking them out from among given ones.

§13. **Infinitely long expressions.** The principal problem here is to choose suitable classes of such expressions (infinitary languages). For model theory one wants a compromise: a wide class in order to define lots of structures, yet regular enough to ensure simple laws. The need for infinite formulae from the point of view of the present survey (choice of subsystems of analysis) will be described at the end of this section.

As usual we take first order predicate calculus PC as a typical example of a formal language. We use L_α to denote the ramified hierarchy of sets of levels $< \alpha$ and, for initial ordinals (cardinals) α, we use H_α for the collection of sets of hereditary cardinal $< \alpha$. Thus $H_\omega = L_\omega$ = collection of hereditarily finite sets, $H_{\aleph_1}$ = collection of hereditarily countable sets.

The principal ideas of (a) and (b) below were introduced in [30], developed and applied by Kunen [39] and particularly Barwise [2]. The principal interest of the present section is the discussion of open problems and of new fields of application.

(a) *Analysis of the problem: syntax and semantics.*

(i) General syntax concerns the combination of symbols; thus the syntactic objects of PC are elements of H_ω or finite sequences or simply finite ordinals; the latter when, e.g., one identifies formulae and their Gödel numbers.

The difference between sets and ordinals (or constructible sets) is minor in the finite case because $H_\omega = L_\omega$ and the natural structures on H_ω and ω are interdefinable. But it will matter in the infinite case, e.g., for $H_{\aleph_1}$ and $L_{\aleph_1}$ respectively.

(ii) Semantics assigns a realization to the basic symbols used and thereby determines the two familiar properties of syntactic objects: being a *meaningful*[18] expression, and: being *valid*. Further there is the *consequence* relation (between a formula A and a *class* $\mathscr{A}$ of formulae).

In PC the usual choice of propositional operations ($\neg$, $\wedge$) is distinguished by their functional completeness; Mostowski [46] has given an interesting, but as yet inconclusive, criterion of characterizing the quantifiers ($\forall$, $\exists$) in terms of the form of their validity predicate. The logical operations in infinitary languages will have to be chosen by more delicate considerations than functional completeness, cf. the choice of subsystems in §1 above.

In PC the set of consequences of an arbitrary $\mathscr{A}$ is defined from the validity predicate by use of the *finiteness* theorem and the fact that every finite set of formulae is equivalent to a single one. Note that in the early days of logic one did not consider arbitrary $\mathscr{A}$ but only formal systems; their sets of consequences are r.e.: it was a (surprising) *discovery* that the main theorems in model theory of PC apply to arbitrary, and not only r.e., sets $\mathscr{A}$. Naturally, one exploited this discovery, and alternative proofs not using it were often found surprisingly late, cf. [7] or Chapter 6 of [35]. Finally, note that, for those earlier proofs, PC with

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[18] Two senses of *meaningful* are to be distinguished: (i) for all realizations of the language or only (ii) for models of given axioms. For example, different treatment of ι-terms are suitable in the two cases: in (i) one would like every ι-term to have a meaning, in (ii) only those $\iota_x Ax$ for which $\exists! x Ax$ is a consequence of the axioms. In the case of ι-terms, requirement (i) leads to a recursive set of terms, (ii) only to an r.e. set: I do not know if this case is typical of the general situation.

nondenumerably many symbols was needed, but not for the later ones. This difference will also be important for the generalization.

(b) *Principal proposals.* The older work on infinitary languages as presented, e.g., in [18] fixed the syntax by cardinality conditions, and chose the logical operations by 'straight' analogy: $\neg$, $\wedge$, $\bigwedge$, arbitrary strings of quantifiers; or by pragmatic considerations: $\neg$, $\wedge$, $\bigwedge$, finite strings of quantifiers [54].

In contrast, we choose the syntax and the logical operations by *definability*, more precisely invariant definability considerations (which coincide trivially with cardinality conditions in PC). Our conditions apply, in the first place, to finite formulae used to define the infinite expressions: afterwards it is shown that by use of our infinite formulae no new ones can be defined (closure conditions). Once definability criteria are accepted in the choice of language, it is cogent to use them also, e.g., in the generalization of the finiteness theorem, instead of the cardinality conditions in the familiar generalization of the so-called compactness theorem.

Basic principles We separate our problem into first generalizing the *notions* used in syntax and semantics (to a wide class of sets A instead of H_ω used in PC), and then investigating what additional conditions on A are needed in order that the basic *results* on PC hold for our generalization.

(i) Recursion theoretic notions.

A-finite = absolutely invariantly definable,
A-rec = invariantly definable *on* A,
A-r.e. = semi-invariantly definable *on* A.[19]

An elaborate discussion is in [30]; in any case, recursive definitions are the most familiar example of implicit invariant definitions. Regarding finiteness as a recursion theoretic rather than cardinality property in this context is the problematic and fruitful step.

(ii) Logical notions. The syntactic objects themselves such as terms and formulae are to be *elements* of A. Formal derivations, if used at all, are to be A-finite. The set of valid formulae is to be A-r.e., and the set of meaningful ones, possibly, A-rec. The finiteness theorem takes the form:

A consequence of an A-r.e. set $\mathscr{A}$ of formulae is also consequence of an A-finite subset of $\mathscr{A}$.

For any given A, and $X \subset A$, all the definability notions above can be relativized to: invariant definability *from* X. Note that the ordinary finiteness theorem for arbitrary A is included in our version since, for all X, H_ω-finite *in* $X \leftrightarrow$ finite.

(c) *Choice of definitions.* To apply the principles of (b), one has to specify two things: what (finite) language is to be used for the invariant definitions or, equivalently, what *structure* is put on A; and also for what *class* of realizations of this language the definitions are to be invariant. Our aim is to make the notions insensitive to the choice of language: in general, restricting the class of realizations helps

[19] The terminology of [39] differs slightly from [30], e.g., 'strongly' for 'absolutely' or 'A is self-definable' for 'A is absolutely invariantly definable on (A, $\in$)'. The notion A-r.e. was introduced in [39], under the name: semi-invariantly implicitly definable, following Mostowski [47]; the other notions are in [30]. Note that, in [30], A-r.e. was *defined* to mean the range of an A-rec. function: this definition is not reasonable for general A, e.g., for $A = N^N$, since an r.e. set of functions is Σ^0_1 and not Σ^1_1.

because more relations become invariantly definable and, by transitivity, adding them to the structure on A does not alter our notions. Similarly, the use of *implicit* rather than explicit definitions makes for stability (see [34] for comparison with earlier work in [72]).

For collections A of sets or ordinals, the natural structures are $\in$ and $<$ respectively. Beyond this the choice seems, at first, sheer guesswork. The latter is reduced by use of the notion of *transitive* or *end extension*, for (binary) $R = \in$ or $<$ restricted to $A \times A$:

$$(A', R') \text{ is an end extension of } (A, R) \text{ means}$$
$$A \subset A' \quad \text{and} \quad \forall x \forall y[(y \in A \wedge xR'y) \leftrightarrow xRy].$$

(If (A', R') is *not* an end extension, even if $A \subset A'$ the object $y \in A \cap A'$ is not the same abstract object in A as in A′.)

Note in passing that the familiar Rosser condition in arithmetic

$$\forall x[x < 0^{(n+1)} \leftrightarrow (x = 0 \vee x = 0' \vee \cdots \vee x = 0^{(n)})]$$

ensures that all models are end extensions of $(\omega, <)$.

Principle. In the case of collections A of sets we use the structure $(A, \in)$. The class of realizations considered in the first place are *finitely axiomatizable end extensions*, possibly with additional structure.

It turns out that the usual set theoretic relations are indeed A-invariantly definable. Furthermore, for the infinitary languages associated with A below, no new relations are so definable even if one uses A-r.e. sets of infinite formulae.

As far as I know, the situation is less clear cut for collections A of *ordinals* and the structure $(A, <)$.

(d) *Simultaneous choice of* A *and an associated language* $\subset$ A. All the languages considered will include

$\mathscr{L}_A$: its operations are $\neg$, $\wedge$, $\bigwedge$; *finite* strings of quantifiers $\forall$, $\exists$.

All the A considered will satisfy elementary *closure* conditions (needed to ensure that $\mathscr{L}_A \subset A$); for an explicit list of axioms for such a 'rudimentary' set theory, see [2, p. 21].

PROBLEM. What further conditions must A satisfy in order that the basic properties of PC generalize to $\mathscr{L}_A$ for the translation given in (b)? For what extensions of $\mathscr{L}_A$ do these properties persist?

To avoid a circle in the choice of A, one uses properties **P** (of a language) which are *stable for restrictions* of the language: if $\mathscr{L}_A$ doesn't have **P**, neither does any extension of $\mathscr{L}_A$; e.g., if **P** = the validity predicate is A-r.e. In contrast, e.g., the interpolation property is *not* stable for restrictions. Trivially, consider ordinary propositional formulae A and B without common variables, such that $A \rightarrow B$ holds: the fragment $(\neg, \wedge, \top)$ contains an interpolation formula, but $(\neg, \wedge)$ does not.

It is easy to guess reasonably good additional conditions on A: e.g., in [30, p. 194(b)] for a recursion theory on L_{ω_1}, where ω_1 is the first nonrecursive ordinal, and for ω-logic; independently, in [38], equivalent conditions for a recursion theory on ordinals. Platek considered a modification of our axioms, more suited to the case when A does not have an invariantly definable well ordering, and called sets satisfying his axioms *admissible*.

The work below (for details see [2] and [39]) bears on the question: admissible for what?

(i) Validity and consequence. In one direction the main results are:

For *all* admissible A, an A-r.e. set of formulae in $\mathscr{L}_A$ has an A-r.e. set of consequences [39].

For *countable* admissible A, A-r.e. $= \Sigma_1$ (over A, in the sense of [41]), A-rec $= \Delta_1$, A-finite $=$ being element of A ([39] and [2]).

But there are uncountable A for which these last results fail, e.g., $A = H_{\aleph_1}$ or $A = H_{\kappa_1}$ where κ_1 is the first inaccessible; so to speak, if one accepts uncountable cardinals at all, $\aleph_1$ and κ_1 play the role of specific natural numbers for the 'finite mind'. This property of κ_1 is not surprising; see footnote 6 of [31, p. 104].

In the opposite direction [2] shows, for countable rudimentary A:

If, uniformly for all sets $X \subset \mathscr{L}_A$, the set of consequences of X is Σ_1 in X, then A is admissible.

There are countable nonadmissible unions A of admissible sets, for which the validity predicate is A-r.e., with refinements when A is of the form L_α and $\alpha < \aleph_1$.

So one properly distinguishes between *validity-admissible* and *consequence-admissible* A.

Incidentally, it would be more satisfactory to prove the

CONJECTURE. For all rudimentary A, if all A-r.e. sets of formulae $\in \mathscr{L}_A$ have an A-r.e. set of consequences, then A is admissible, i.e., the result above with A-r.e. in place of Σ_1.

(ii) Choice of language. First of all, for many (uncountable) admissible A, e.g., $A = H_\kappa$ where κ is inaccessible, the validity predicate is A-r.e. even if *infinite* strings of quantifiers are allowed [39]. So the *restriction to finite strings is not justified generally.*

For countable A of the form L_α, by [2]:

For recursively accessible α and $A = L_\alpha$, the validity predicate for $\mathscr{L}_A$ is a *complete* A-r.e. predicate; for rec. inaccessible (admissible) α it is A-rec.

It seems plausible that this result can be improved to establish the

CONJECTURE. For rec. accessible α, suppose the extension of $\mathscr{L}_A$ by a new propositional operation π has an A-r.e. validity predicate; then π is *definable* in $\mathscr{L}_A$. (For $A = L_{\omega_1}$, this is true.)

PROBLEM. For rec. inaccessible α: what 'natural' propositional operations and/or quantifiers have to be added to $\mathscr{L}_A$ to get a complete A-r.e. validity predicate?

Other problems of this type will occur to the reader automatically if he goes into the subject.

(iii) A-finiteness: a controversial issue. Barwise [2] makes a strong case for imposing the following closure condition on A:

$$\forall X(X \text{ is A-finite} \leftrightarrow X \in A)$$

(satisfied by all *countable* admissible A). In general terms his idea is this: Let us look for a simple condition $\mathfrak{F}$ on sets A such that, for countable A, $\mathfrak{F}$ is equivalent to admissibility and, generally, $\mathfrak{F}$ implies the closure condition above, the equation A-r.e. $= \Sigma_1$, and the generalized finiteness theorem. Note that these consequences of $\mathfrak{F}$ are known in the special case when A is of the form H_α. (In [2] Barwise considers a certain 'indescribability' condition $\mathfrak{F}$.)

The closure condition has evident consequences for a proof theoretic (inductive) analysis of the validity predicate for $\mathscr{L}_A$. But I should like to know if the condition is *necessary*; e.g., by finding a counterexample to the generalized finiteness theorem for some, necessarily uncountable, admissible A (which does not satisfy the closure condition—and hence not $\mathfrak{F}$).

Or, if there is none, to show that every admissible A can be regarded as a *fragment* of a set A′ satisfying $\mathfrak{F}$; 'fragment' in the sense that

$$\begin{aligned}&\text{A}' \text{ is a transitive extension of A},\\ &\forall X[(X \subset A \wedge X \text{ is A}'\text{-finite}) \leftrightarrow X \text{ is A-finite}],\\ &\forall X[(X \subset A \wedge X \text{ is A}'\text{-r.e.}) \leftrightarrow X \text{ is A-r.e.}].\end{aligned}$$

This situation would be analogous to the use of a finite language for ω-logic where, roughly, $A = L_\omega \cup \{L_\omega\}$, and $A' = L_{\omega_1}$: 'roughly' because this A is, of course, not admissible.

(e) *Testing the framework* (b). While the work reported in (d) *accepted* the basic principles of (b), we now wish to consider *alternatives* to (b).

(i) Old cardinality conditions. Here the languages $\mathscr{L}_A$ for countable A, though eminently suitable for defining interesting algebraic structures, are excluded from the start. By [13], strong compactness cuts out many A of the form H_κ which even satisfy Barwise's Π_1^1-indescribability criterion. As pointed out at the end of (a), there is at present little evidence that strong compactness is actually needed for generalizing basic properties of PC.

(ii) Alternative recursion theories on sets (Platek). For all admissible $\alpha > \aleph_1$, the set of valid propositional formulae of $\mathscr{L}_{L_\alpha}$ is not constructible, hence not Σ_1 on L_α, and so not generalized r.e. in the sense of Platek, though even L_α-rec by [39]. This result, due to Kunen (see [2, p. 52]), uses Ramsey cardinals; it is of course not surprising that large cardinals should have consequences for logical questions: see the lecture (Jan. 1964) [31, bottom of p. 116].

(iii) Restriction to constructible structures. Once again we assume large cardinals and consider $A = L_\alpha$ for α strictly between $\aleph_1^{(L)}$ (the first constructibly uncountable ordinal) and $\aleph_1$. By [2], the set of formulae of $\mathscr{L}_A$ valid in all models is L_α-r.e. and hence Σ_1 on L_α; but, relativizing [19], there are many such α for which the set of formulae valid in all *constructible* models is *not* L_α-r.e.

Here we have a striking parallel to Vaught's result [66] for PC, if we compare *constructive* (in the sense of [66]) for PC, with *constructible* (in the sense of Gödel) for $\mathscr{L}_A$, when $A = L_\alpha$ and $\aleph_1^{(L)} < \alpha < \aleph_1$.

(iv) Interpolation lemma. As pointed out in (d), the interpolation property is not stable for restrictions. Therefore 'counterexamples' have to be judged by whether a suitable *choice of language* was made. I have the impression that Maltiz' ingenious counterexamples [90] would be avoided by adding to the $\mathscr{L}_A$ considered certain quantifiers with zero arguments (expressing something about the cardinality of the universe) in such a way that the extended language still has an A-r.e. validity predicate.

Another 'counterexample' to the interpolation lemma is given in [3] for the case of finitary *higher order languages*: defects are discussed in the review of [3]. In my opinion, higher order languages present the best potential test for the ideas of (b).

Clearly the *notions* of invariant definability make perfect sense also for higher order languages. Now, as is well known, finite higher (even second) order formulae permit the absolute invariant definition of *large* structures. Suppose one accepts the generalization of finiteness in (b) and Barwise' condition in (d)(iii) on the class A from which our formulae are taken. Then a second order language, and, *a fortiori*, higher order languages, should contain 'huge' formulae. Put differently, the class of finite second order formulae cannot be expected to satisfy anything like the closure conditions needed for a smooth theory.[20]

(f) *Proof theory: applications of infinitary languages.* Several technical uses of infinitary languages were mentioned in earlier sections. [2, p. 29] finds A which are *not* validity admissible, but the class of those formal derivations which $\in$ A, is closed under cut; this illustrates clearly the discussion at the end of § 5.

But also, infinitary languages seem to provide a good criterion for the choice of subsystems (discussed in §4). Consider, for example, Δ_1^1-CA and Π_1^1-CA; the results of Note V below suggest that the former is 'nicer' than the latter. From the point of view of the present section, we have a neat distinction: The class Π_1^1 is *not* closed under the simplest kind of infinite disjunctions, i.e., there are such disjunctions of (finite) Π_1^1-formulae which are not equivalent to any Π_1^1-formula, e.g., the truth definition for Π_1^1-formulae. In contrast, the class Δ_1^1 *is* closed for the natural associated infinitary language (hyperarithmetic formulae) whose 'atoms' are Δ_1^1-formulae. These uses have not yet been explored systematically at all.

Technical Notes

§I. Mathematics and foundations. The principal purpose of this note is to elaborate on the view of (proof theoretic) foundations expressed in the introduction to the present survey. In particular, I shall formulate and criticize a widely held opposing view.

This view agrees with Hilbert's in believing that, roughly speaking, 'mathematics can, or even must, look after its own foundations'. It differs from Hilbert's in that it appeals to 'common sense' without expressing a sharp philosophic position on the nature of mathematical reasoning. It differs radically from the present survey in that it draws the following conclusion from Gödel's incompleteness theorems:

"True, Hilbert's programme is refuted if one interprets 'elementary' or 'finitist' proof as Hilbert originally meant it, or even in the sense of combinatorial proof $\mathscr{P}_0$ of §6 or predicative proof of [5]. This is no reason for getting involved in

[20] This point is, perhaps, relevant to an analysis of the common *malaise* about second order languages, despite the fact that the *meaning* of a second order formula is formulated in ordinary set theoretic terms (also used in defining the meaning of first order formulae). A familiar reason for the malaise is that the *theory* of second order validity depends sensitively on the existence of large cardinals ([33, p. 157, paragraph 2]), while first order validity is reduced, by Skolem-Loewenheim, to validity in countable domains. Now, if my view is right, we have an additional reason in that the class of finite second order formulae is not even a passable approximation to the 'full' second order 'language'.

philosophical analysis and the ambiguities involved in informal conceptions. Let's rely instead on the judgement developed by mathematical experience: let's take formal systems such as current set theory which continue to appeal to mathematicians and let's give a mathematically informative analysis of the structure of formal derivations. The rest will look after itself: not only (formal) consistency and independence results will follow but also the philosophical significance of the analysis will be apparent (if one is interested in this)."

This view will now be considered in several steps.

(a) *Mathematical and logical foundations:* a useful distinction. Logical foundations analyze the *validity* of mathematical principles (axioms, rules): this analysis usually refers to one's conception of the nature of mathematics. In so-called mathematical 'foundations', validity is taken for granted, and one tries to make arguments more systematic and *intelligible*: this involves reorganization, often by a good choice of language. The aims of the two kinds of foundation are occasionally in conflict.

(i) Logical foundations require us to analyze what we are talking about: and, in general, the more *specific* the subject matter, the easier it is to decide the validity of assertions about it. For instance, in set theoretic foundations one considers particular *segments* of the hierarchy of sets; cf. [31, p. 101, 1.17–22] or [35, pp. 174–176].

(ii) In contrast, once the validity of the principles used is recognized, particular proofs may become more intelligible by eliminating axioms, so to speak, by making the subject matter less specific. Again a change of concepts may make for clarity even if they are defined in terms of the original concepts, i.e., if they are logically dependent on the latter. For instance in category theory, the notions of function and composition are more suitable than set and membership.

Thus, according to the view under discussion, mathematics not only looks after its mathematical 'foundations' (as everybody admits), but also after its logical foundations.

(b) *Putting the evidence into perspective*, or: a little learning is a dangerous thing. The view considered has, in an obvious sense, (i) a positive and (ii) a negative ('antiphilosophical') side to it.

(i) Early results in logic support the view. Gödel's completeness theorem and Gentzen's cut elimination provide clear examples where 'little' has to be added to the mathematical constructions; e.g. [35, p. 190, Axioms II, III] concerning intuitive logical validity, respectively §5 above.[21]

But in the case of formal systems for arithmetic or analysis, the evidence *against* the view is impressive. It may fairly be said that, whenever the significance of 'formal consistency proofs' was made clear at all, this analysis was more demanding than the original argument. (I think this applies even to Spector's excellent [58] and the subsequent analysis of the kinds of functionals for which various types of

[21] Though little need be added, *some* analysis of the conceptions involved was needed. What else prevented Herbrand and Skolem from pulling their work together to obtain Gödel's completeness theorem? Herbrand did not believe that validity made sense, and Skolem did not distinguish at all between validity and formal derivability.

Spector's bar recursion are valid.) And outside proof theory, the beautiful work on forcing with its many formal independence results has left untouched not only the truth of the formally undecided propositions, but even what *kind* of evidence may be involved in their decision.

(ii) Unlike 'crude' formalism (see, e.g., [35, pp. 120–121]) the view considered would grant that, 'behind' the formal system are intuitive principles of proof and the conviction they carry. But it holds that such psychological elements do not permit a precise analysis; so, to get anything definite one must *replace* them by formal rules and study the latter.

Now, as long as one (believes one) has complete formal systems, it is perfectly sound to exploit this fact, i.e., that the formal rules *do* replace intuitive principles as far as the set of true statements is concerned. Putting first things first, one begins by ignoring more delicate questions of the *kind* of knowledge or conviction involved; these can, perhaps, never have quite as definite answers as purely 'extensional' questions on truth. (Naturally one considered the latter questions in the early days of logic.)

Once one deals with incomplete systems, there is anyway no longer any possibility of this kind of definiteness. But—and this is important—when one looks at the more delicate questions closely, the situation is not at all discouraging: cf. Gödel's clear distinction in [12] between combinatorial and intuitionistic proof, the stability of various characterizations of informal notions of proof (e.g. [5] or [31, pp. 169–179]). Incidentally, the subsystems of analysis which have attractive *formal* properties were found not by formal syntactic considerations, but by looking for systems corresponding to suitable informal principles of proof.

(c) *Hilbert's analysis of the significance of combinatorial foundations:* a source of current 'common sense'. Besides giving the tools for reducing abstract mathematics to $\mathscr{P}_0$ described in the introduction to the present paper, Hilbert also tried to explain the significance of such a reduction; specifically by reference to very general properties of $\mathscr{P}_0$, and not to a detailed description of $\mathscr{P}_0$, i.e., of the possibilities of the combinatorial imagination.

Warning. Some of Hilbert's terminology is loaded! It makes sense only in situations where his programme can be carried out; so if one uses the terminology one is liable to believe in the programme!

(i) $\mathscr{P}_0$ and mathematical proof *tout court*. Hilbert believed that the use of abstract concepts could be shown to be a mere *façon de parler* which brings no more knowledge by itself than playing a game (with symbols). He coined the word 'metamathematics' to distinguish arguments in $\mathscr{P}_0$ about such a game from the 'abstract nonsense'. The significance of (a reduction to) $\mathscr{P}_0$ was clear enough: $\mathscr{P}_0$ is the *whole* of meaningful mathematics!

Note that his distinction makes good sense *only* if one accepts his view! For, if we accept abstract concepts we look at the matter quite differently; e.g., an abstract proof of an arithmetic identity $fn = 0$ provides just as *much* conviction that, e.g., the computation of $f0$ will have 0 as an answer, as if we had a combinatorial proof. Only the *manner* of stepping from the proofs of $fn = 0$ to the computation will be different in the two cases.

Hilbert's way of describing abstract proofs as formal manipulations was soon taken, by him and others, as a *definition* of the essence of proof! But, by the last paragraph, this needs justification! Taken seriously, this leads to *completeness problems*. It is perhaps, instructive to recall that these problems were solved by Gödel, *outside* the Hilbert school, positively for logical reasoning in the narrow sense (about truth functions and quantifiers) and negatively for reasoning about specifically mathematical notions such as the natural numbers.

(ii) $\mathscr{P}_0$ and reliability. Since, for Hilbert, $\mathscr{P}_0$ was the whole of mathematics, he did not connect reasoning in $\mathscr{P}_0$ with a particular *kind* of conviction, but with reliability (without further qualification).

Yet he stressed the practical reliability of ordinary mathematics [78, p. 158] and was, I am told, quite aware of the practical unreliability of computations (which make up a small part of $\mathscr{P}_0$). To prove Fermat's conjecture he suggested: so lange herumrechnen bis man sich endlich mal verrechnet.

Here it might be remarked that the widespread idea of connecting formalization and reliability is equally incoherent. When we really want to make sure that a mathematical result is correct we do not formalize its proof, i.e., compare the steps with given formal rules, but we try to make it intelligible. We rely on mathematical, not logical foundations!

Evidently, one reason for introducing the fiction of reliability is to avoid a more searching analysis of $\mathscr{P}_0$.

(iii) $\mathscr{P}_0$ and finiteness. Perhaps the name 'finitist' which Hilbert gave to $\mathscr{P}_0$, was meant as a step towards such an analysis: he certainly thought that 'the infinite' would be eliminated in $\mathscr{P}_0$, which was to be about (hereditarily) finite configurations or concepts, i.e., those that can be realized concretely in space and time; cf. [12].

But this is unconvincing since even the simplest kind of *rule* cannot be so realized. And once this is recognized the restriction to a finite number of rules seems irrelevant and one uses rules for enumerating rules; cf. [35, p. 214(d)].

(d) *Mathematical significance of combinatorial foundations:* compare the title of [22], where I tried to convince myself of the positive side of b(i) above (and, quite explicitly, left the *logical* problem of analyzing $\mathscr{P}_0$ alone). Specifically, I found Herbrand's theorem 'genuinely interesting' and expected, in accordance with the view considered in (b), mathematically interesting by-products.

Perhaps the case is to be compared to a certain stage in *abstract algebra* where one did not rely on the intrinsic interest of the structures that arose, but looked for applications to familiar problems. In fact I expected the comparison to be quite 'close': to the elimination of unnecessary *axioms* in abstract algebra was to correspond the elimination of unnecessary (nonconstructive) *rules* of inference.

Amusingly, the two main applications in [22] of this line of thought are largely superseded by ordinary methods!

(i) Let f be the general polynomial of degree d in the variables $x_1, \cdots, x_n$. The analysis in [22] of Artin's proof was applied by Daykin (unpublished) to give expressions

$$(q^{(j)})^2 f = \sum_{i \le N} a_i^{(j)} (p_i^{(j)})^2 \quad \text{for } j = 1, \cdots, M,$$

where N and M depend on d and n, with the following properties:

$a_i^{(j)}$ are polynomials in the coefficients of f;

$p_i^{(j)}$ and $q^{(j)}$ are polynomials in the coefficients of f *and* $x_1, \cdots, x_n$;

if the field K generated by the coefficients of f is ordered, and K′ is a real closed extension of K and

$$(\forall x_1 \cdots \forall x_n \in K')(f \geq 0)$$

then, for some $j \leq M$, $a_i^{(j)} \geq 0$ for all $i \leq N$, and $q^{(j)}$ not identically zero.

If every positive element of K is a sum of $\leq$ P squares, one can contract, i.e., take $M = 1$, at the expense of introducing functions $\sigma_1, \cdots, \sigma_P$ defined on K such that, for $a \in K$,

$$a \geq 0 \rightarrow a = (\sigma_1 a)^2 + \cdots + (\sigma_P a)^2, \quad a \leq 0 \rightarrow \sigma_i a = 0.$$

But, at least for real *closed* K, Ax in [68] for $n \leq 3$ (and, I understand, Pfister for all n) has given much better bounds for N. Thus, for $n = 3$, [22] gives essentially $N \leq \exp_2 \exp_2 \exp_2 d$, while [68] gives the bound $N \leq 2^n$ *independent* of d!

(ii) The second application concerned a bound for the first zero of $\pi x - \operatorname{li} x$; [22] gives roughly

$$\exp_e \exp_e \exp_e \exp_e 8,$$

while [89] gives $(1.65)\cdot 10^{1165}$.

In view of these improvements, the present value of [22], for the two problems above, consists not in the bounds themselves, but only in analyzing the *general nature* of these problems; it separates what bounds are got from quite general considerations and what improvements need special study. This type of analysis is a typical logical contribution, cf. 1.-9 to 1.-7 on p. 155 of [35] in connection with model theory.

Conclusion. Ultimately, I think, the interest of (proof theoretic) foundations is as a tool in empirical research: does a proof, as I understand it at a given moment, belong to $\mathscr{P}$? [32, p. 238]. Proof theory provides the concepts in terms of which such empirical facts are stated. And its interest depends on whether one wants to know such facts.

This conclusion is of course quite consistent with the idea of logical foundations as *applied* mathematics; cf. §3. Early results in logic correspond to early work in mechanics where the physics (*finding* the differential equations) used only generally known facts, while the mathematics (*solving* the equations) was recondite. Nowadays one usually believes that one also has to look at recondite physical facts. The counterpart to this in proof theory is the philosophical or, more specifically, phenomenological analysis of different kinds of mathematical reasoning.

The bearing of the considerations (a)–(c) on the view described in (b) may be summarized as follows. The view errs on both its principal counts because it overestimates the value, at the present time, of formal work and underestimates the possibility of rigorous informal work. Probably, holding the view makes it difficult to appreciate what has already been done in proof theory. What makes the view plausible is, at least partly, a confusion between mathematical and logical foundations and, probably mainly, the unconscious acceptance of formalist views which have gone into current 'common sense' by way of (Hilbert's) formalist terminology.

§II. Relations between formal systems. The purpose of this note is to point out which, among such relations in the literature, have turned out to be most useful, and to establish an *order* between them. The latter will correct some common misconceptions about the relative status of model theoretic and elementary independence or consistency proofs.

(a) Consider first *syntactic properties*. Though most of them are most naturally defined for arbitrary formal systems, we confine ourselves to systems formulated in many sorted predicate calculus, containing arithmetic. These may be thought of as subsystems of analysis.

(i) As far as a *single* system is concerned, the best known syntactic properties are completeness (saturation) and consistency. But they are not particularly useful for our systems: none of them is complete, and, generally, a more useful property than consistency is some form of *reflection principle* (of which consistency is a special case): $\exists y \operatorname{Prov}(y, \ulcorner A \urcorner) \to A$; see [36].

Under well-known conditions, Gödel's first incompleteness theorem states that *some* instance of the reflection principle for a system S is not derivable in S; and Löb's extension [42] of the second theorem shows that *no* instance is derivable in S except, in the trivial case, when A itself is derivable in S.

Stronger conditions on S, such as those in [42], are needed for the second theorem since, for example, by [61], Takeuti's 'cut-free' analysis formally proves its own consistency, but Gödel's first theorem applies to Takeuti's system.

(ii) Two familiar syntactic relations between *pairs* S, S′ of formal systems are, first relative consistency and second, if $S \subset S'$, the relation of conservative extension, i.e., a theorem of S′ in the language of S is also a theorem of S.

A formal proof in some system S″ of: S′ is a conservative extension of S, yields also a relative consistency proof in S″, but the converse is not generally true:

EXAMPLE (END OF [86]). Let S be ordinary arithmetic, R = Rosser's undecided formula, S″ (even) primitive recursive arithmetic; thus $S'' \subset \mathscr{P}_0$. Then

$$S'' \vdash \operatorname{Con} S \to \operatorname{Con}(S \cup \{R\}), \quad S'' \vdash \operatorname{Con} S \to \operatorname{Con}(S \cup \{\neg R\})$$

but $S' = S \cup \{R\}$ is not a conservative extension of S, not even for universal formulae; $S' = S \cup \{\neg R\}$ not even for existential ones.

The moral is to state, when possible, conservative extension rather than relative consistency results. Note also the *extensional* character of the former type of result since it involves only the *sets* of theorems of S and S′. In contrast (relative) consistency problems involve the *rules* (generating the theorems) of S and S′, and so these problems have to be formulated in terms of *canonical* definitions ([31, p. 154, 3.222]). Incidentally, for many familiar pairs of systems (S, S′) one simply does not know whether S′ is a conservative extension of S, while relative consistency is often problematic only if S″ is restricted. (More explicitly, if both S and S′ are *accepted* systems, one has *some* argument for the consistency of both.)

We do not discuss here the relations most useful for proofs of recursive undecidability or inseparability since this topic has been much explored.

Note in passing that various, apparently, intermediate relations mentioned in the literature, such as 'translatability' [83], are of little interest for our systems because they are satisfied automatically; cf. [84] and, particularly, [88].

(b) *Model theoretically meaningful relations.* Undoubtedly the best known relation is that of *interpretation of* S *in* S′ [65] which we shall call, more descriptively, *uniform* S′-model of S. This is given by a *definition* which, in each model of S′, defines a structure satisfying the axioms of S.

(i) A model theoretically more attractive relation between S and S′ is given in [91]: for *each* model M′ of S′ there is a definition $\delta_{M'}$ of a model for S. It is then a *theorem* (by routine use of the finiteness theorem) that there is a *uniform* S′-model of S.

Gödel's completeness theorem yields (by [15]), for systems S′ including arithmetic; if $S' \vdash \mathrm{Con}\, S$ then S has a uniform S′-model (for precise conditions see [69]).

A less well-known *strengthening* is this: if $S' \vdash \mathrm{Con}\, S$ then there is a definition of an S′-model of S *for which the truth predicate can be defined and proved in* S′. (Use Henkin's proof of the completeness theorem.) In fact, the result is *uniform* for variable, finitely axiomatized, S, and can be stated as follows, with a variable *s* for the Gödel number of the formula S: There is a predicate T(s, n), in Δ_2^0 form, such that $\forall s(\mathrm{Con}(s) \to M[s, \lambda n T(s, n)])$ is derivable in first order arithmetic, where $M[s, \lambda n P(n)]$ expresses that P is a truth predicate for a model of the formula with number s. (As Kripke has pointed out, the present version permits an immediate formalization, in first-order arithmetic, of the obvious model-theoretic proof of cut elimination, mentioned in Example 2 of §5.)

The strengthened version admits a converse, but not the weaker version, as can be seen by taking S = S′. As for relative consistency proofs: $\mathrm{Con}\, S \to \mathrm{Con}\, S'$; given an S-model of S′, we get such a proof automatically, even in $\mathscr{P}_0$, when S′ is *finite*. The converse does not hold: take the Gödel-Bernays theory of classes for S′ and set theory for S [87].

This last example also shows that a conservative extension result (for $S \subset S'$), does not imply the existence of an S-model for S′.

A more symmetric relation is obtained for an interesting special class of systems S′, containing many familiar ones: Suppose, for each finite subsystem S'_n of S′, we have $S' \vdash \mathrm{Con}\, S'_n$. Then (Orey's compactness theorem [69]), for r.e. systems S and S′:

There is an S′-model of S if and only if, for each *m*, there is an S′-model of S_m or, equivalently, if $\forall n(S' \vdash \mathrm{Con}\, S_n)$ or again, $\forall n \exists m(S'_m \vdash \mathrm{Con}\, S_n)$.[22]

[22] More precisely, given a *description* of S in S′ (for precise conditions see [69]) one gets effectively a definition δ of a realization of the language of S such that, *if* $\forall n(S' \vdash \mathrm{Con}\, S_n)$, then δ is an S′-model of S; δ is *uniform* for all S considered. This has an interesting application, noted by Feferman in conversation if we take e.g. set theory ZF for S′ and for S, say, ZF together with the axiom of choice (AC) and the negation of the continuum hypothesis (CH). Orey's construction provides explicit formulae M(x) and E(x, y) of ZF which determine a ZF-model of $ZF \cup (AC) \cup \{\neg CH\}$ provided only $ZF \vdash \mathrm{Con}(ZF_n \cup (AC) \cup \{\neg CH\})$ is true for each *n*. Once this last fact is *known* (quite independently of how it is proved) the explicit construction of the ZF-model M(x) and E(x, y) is not problematic. What is left open is, e.g., whether *this* model is faithful (say for arithmetic propositions) while, in fact, $ZF \cup (AC) \cup \{\neg CH\}$ is a conservative extension of ZF even for an important class of analytic propositions; cf. Note VIb(ii).

(ii) *Commuting* models for (S, S′): Suppose δ defines uniformly an S′-model of S, δ' an S-model of S′ and, for each model M′ of S′, *the structure defined by* ($\delta'\delta$(M′) *is isomorphic to* M′ *by means of a mapping defined in* S′.

EXAMPLE. Let S′ be first order arithmetic, S general set theory [67], δ the S′-model of S given in [67], δ' the usual development of arithmetic in set theory without use of the axiom of infinity.

The existence of a commuting pair of models implies the syntactic relation of *faithfulness* ([72] for all A in the language of S): $S \vdash (A \leftrightarrow \delta'\delta A)$; but again the converse is not generally true.

(c) *Discussion of some relative consistency proofs.* One of the hangovers from Hilbert's programme (see Note I) is the requirement that relative consistency proofs (Con S $\rightarrow$ Con S′) be given in $\mathscr{P}_0$. This depends on the tacit conviction that, some day, Con S itself would be proved in $\mathscr{P}_0$. But, a chain being as weak as its weakest link, a philosophically more meaningful requirement is to use any of the principles that one expects to need for proving Con S.

A posteriori some 'justification' can be given for the old requirement by showing that it is automatically fulfilled under more reasonable conditions:

One condition, already mentioned in b(i), applies to finitely axiomatized systems S′ which have an S-model; then $\mathscr{P}_0 \vdash \text{Con S} \rightarrow \text{Con S}'$. A large class of 'model theoretic consistency proofs' of systems S′ come under this case.

Next, suppose we have reason to believe that the methods $\mathscr{P}$ proposed for a proof of Con S would also establish Con(S $\cup$ {Con S}); all the $\mathscr{P}$ discussed in the main text satisfy this condition (and more; e.g., they establish reflection principles). Then nothing is lost by accepting a relative consistency proof in S itself! For

$$\text{if } S \vdash \text{Con S} \rightarrow \text{Con S}', \quad \text{then } \mathscr{P}_0 \vdash [\text{Con}(S \cup \{\text{Con S}\}) \rightarrow \text{Con S}'],$$

by [85]. (Note that, theoretically, *some* restriction on S is needed: suppose S_1 is consistent, $S_1 \cup \{\text{Con } S_1\}$ is not, and take $S = S_1 \cup \{\neg\text{Con } S_1\}$.)

Finally, let us look at two familiar examples of 'model theoretic' proofs of Con S $\rightarrow$ Con S′ which use methods beyond $\mathscr{P}_0$.

First, there is the *obvious* way of extending an arbitrary model M of set theory to the model M′ of the theory of classes by adding (as classes) all *definable* subcollections of M. *This* proof uses ramified analysis of level 1, i.e., the system of §8 (3)(i). A further *trick* is needed to give an elementary proof, as in Shoenfield [96] or my [82]. But I know of no advantage of the latter proofs.

Next, consider the so-called impredicative theory of classes S and let S′ = S $\cup$ {V = L}. The obvious relative consistency proof uses the constructible sets of rank $\leq$ first inaccessible κ and order $< \kappa^+$ say (sets are of rank $< \kappa$, classes of rank $\leq \kappa$). An elementary relative consistency proof is much more delicate. Now the advantage of the latter proof is that it yields *conservative extension* properties, and undoubtedly the proper formulation of the facts should refer to these properties.

It is fair to say that, both mathematically and philosophically, the importance of *elementary* relative consistency proofs has been much exaggerated. The same applies of course to formal independence proofs.

§III. Axioms, rules, parameters. The purpose of this note is to set out some points of detail concerning the step from second order axioms to first order schemata considered in §4. The second order axioms will have the form $\forall \mathbf{X}(\Phi \to \Psi)$ where $\mathbf{X}$ is the only second order variable of $\Phi \to \Psi$. Without loss of generality we take the variables of the first order formulae A to be distinct from those of $\Phi \to \Psi$; further, we suppose that the free variables of A are among $u, u_1, \cdots, u_n$ and that they have no bound occurrences (by hypothesis, $u, u_1, \cdots, u_n$ do not occur in $\Phi \to \Psi$). $\Phi(A)$, $\Psi(A)$ will denote the formulae obtained from Φ and Ψ respectively by replacing each occurrence $\mathbf{X}(t)$ by $A(t)$, the latter being the result of substituting the term t for u.

The rules corresponding to $\forall \mathbf{X}(\Phi \to \Psi)$ are, by definition, the schemata (of rules):

For formulae A with the *single* free variable u, derive $\Psi(A)$ from $\Phi(A)$ (parameterless rule);

For formulae A, whose free variables are in the list $u, u_1, \cdots, u_n$, derive $\forall u_1 \cdots \forall u_n \Psi(A)$ from $\forall u_1 \cdots \forall u_n \Phi(A)$ (rule with parameters).

The results below illustrate the logical relations between the axiom schemata and rules corresponding to $\forall \mathbf{X}(\Phi \to \Psi)$. Evidently, a schema with parameters implies the parameterless schema, and the *axiom* schema implies the corresponding schema of rules.

Warning. Just because the selection of subsystems is a *central* problem, one cannot expect too much help in this from such simple minded criteria as the presence or absence of parameters. However, if one is convinced of the significance of something *like* a given axiom schema, it is natural to study details, such as the effect of parameters. This may be compared to the study of minor variants of important algebraic notions. As a general rule, stability results are interesting, if they can be interpreted as supporting the choice of the schema.

(a) *Equivalence between axioms and rules.* Most of these results apply to the case of a *full* schema, where the A are not restricted, and $\Phi(A) \to \Psi(A)$ is derived by applying the rule: from $\Phi(A_1)$ infer $\Psi(A_1)$, to a formula A_1 which is more complicated than A. A notable exception is Shepherdson's result in (i) below.

(i) *Induction* or transfinite induction. Here we have the strongest kind of equivalence: the axiom schema with parameters follows from the rule without parameters:

$$\Phi \text{ is } \mathbf{X}(0) \wedge \forall x[\mathbf{X}(x) \to \mathbf{X}(x')], \quad \Psi \text{ is } \forall y \mathbf{X}(y).$$

Put $A_1 u = \forall u_1 \cdots \forall u_n([A0 \wedge \forall v(Av \to Av')] \to Au)$. Then $\Phi(A_1)$ is logically valid, and $\Psi(A_1) \leftrightarrow [\Phi(A) \to \Psi(A)]$. The extension to transfinite induction is immediate.

Shepherdson [55] showed, by a special argument, that *quantifier-free* consequences of the axiom applied to quantifier-free A, are also consequences of the rule applied to *quantifier-free* A_1 (while the A_1 above is logically more complicated than A). The philosphical significance, for $\mathscr{P}_0$, of Shepherdson's result was stressed in §5.

Other results on A of simple syntactic structure are given in [92]. It might be interesting to study their extension to transfinite induction.

(ii) *Axioms* and *rules of choice.* We consider axioms and rules where either both allow parameters or neither. Consider AC_{00}; AC_{01}, DC_{11}, DC_{00}[23] are similar.

$$\Phi \text{ is } \forall x \exists y X(x, y), \quad \text{and} \quad \Psi \text{ is } \exists f \forall x X(x, fx).$$

Put $A_1(u, v) = \forall u_1 \cdots \forall u_n[\forall z \exists w A(z, w) \to A(u, v)]$. Then $\Phi(A_1)$ is logically valid, and

$$\Psi(A_1) \leftrightarrow \forall u_1 \cdots \forall u_n[\forall z \exists w A(z, w) \to \exists f \forall x A(x, fx)].$$

Concerning the effect of parameters, see (c) below.

(iii) *Well foundedness*, cf. §9. The proof in [17, p. 352] is easily modified to derive DC_{00} and DC_{11} from the *rule*: from WF(R) infer TI(R, A).

(b) *Nonequivalence results.* Consider syntactic restrictions on the formulae A used in the schemata. Then, for instance

(i) The *rule* AC_{01} applied to (essentially) Σ^1_1-formulae is proof theoretically equivalent to ramified analysis of level ω^ω by [5],

(ii) The *axiom* (Σ^1_1-AC_{01}) is proof theoretically equivalent to ramified analysis of level ϵ_0 by [74], where both (i) and (ii) are added to elementary analysis $\mathscr{E}$ of §8.

(c) *Role of parameters.* Friedman [74] has gone into this matter in great detail. The following well-known cases may serve as an introduction. We consider the language of $\mathscr{E}$ and hence must distinguish between number and function parameters. We treat the latter.

(i) $\mathscr{E} \cup \{\Pi^1_1\text{-CA}\}$: a case of nonequivalence. Consider (Π^1_1-CA) without parameters, i.e., the schema of axioms $\exists f \forall x(fx = 0 \leftrightarrow Ax)$, where $A \in \Pi^1_1$ and A has no free function variables. (We use 'Π^1_1' to refer both to a certain class of formulae and to the class of sets defined by such formulas in the principal model.) The first model we think of consists of the collection $\mathscr{C}$ of functions which are recursive in some *complete* Π^1_1-set, e.g., Kleene's O. This *is* a model, with the natural numbers as individuals, because by Kleene's basis theorem,

$$\{n = An \text{ is true in } \mathscr{C}\} = \{n : An \text{ is true in the principal model}\},$$

and so the characteristic function of the set (of natural numbers) defined by the formula A in the model $\mathscr{C}$, belongs to $\mathscr{C}$.

This model satisfies also Π^1_1-CA with *numerical* parameters, but *not* with function parameters: take the usual definition, say, $\forall f A_0(X, f, n)$ of a complete Π^1_1-predicate for sets of natural numbers, and note that Kleene's O has a characteristic function

[23] In the list of *axioms of choice* in §4 the very convenient DC_{00} (or simply DC_0), i.e.,

$$\forall x \exists y X(x, y) \to \forall z \exists f \forall x(f0 = z \wedge X[f(x), f(x+1)])$$

was overlooked. We have $DC_{00} \leftrightarrow AC_{00}$:

To derive DC_{00} from AC_{00}, which provides $g: \forall x X(x, gx)$, define $f0 = z$, $f(x + 1) = g[f(x)]$.

To derive AC_{00} from DC_{00}, apply the latter to the relation X_1 where $X_1(x, y)$ is defined by

$$\forall u \forall v \forall w(x = 2^u 3^v \to [y = 2^{u+1} 3^w \wedge X(u + 1, w)]).$$

Then $\forall x \exists y X(x, y) \to \forall x \exists y X_1(x, y)$. Applying DC_{00} with $z = 3^{w_0}$ where $X(0, w_0)$, we get f such that

$$\forall x X_1[f(x), f(x + 1)] \wedge f0 = 3^{w_0},$$

and

$$\forall x \exists v \exists w[fx = 2^x 3^v \wedge f(x + 1) = 2^{x+1} 3^w \wedge X(x + 1, w)]$$

AC_{00} is satisfied by taking for gx the exponent of 3 in the value of fx.

in $\mathscr{C}$, but $\{n: (\forall f \in \mathscr{C})A_0(O, f, n\}$ has not. Actually, an even stronger result is true: our collection $\mathscr{C}$ can be formally defined in the language of analysis, and, for the natural definition, $\mathscr{E}$ extended by (Π^1_1-CA) *with* function parameters formally implies the existence of a function which enumerates $\mathscr{C}$. Hence the consistency of the system obtained from $\mathscr{E}$ by adding the axiom Π^1_1-CA *without* parameters, can be proved in $\{\Pi^1_1\text{-CA}\} \cup \mathscr{E}$; so the latter is not even a conservative extension of the former for arithmetic statements.

(ii) $\mathscr{E} \cup \{\Delta^1_1\text{-CA}\}$: a case of equivalence with respect to e.g. Δ^1_1-statements. The first model one thinks of, for the system without parameters, is its minimum ω-model which, by [28], consists of the hyperarithmetic sets: this model also satisfies $\mathscr{E} \cup \{\Delta^1_1\text{-CA}\}$ with parameters.

To see this one goes back to the familiar result that the Δ^1_1-sets are closed under Δ^1_1-operations (while a set which is Π^1_1 in a Π^1_1-set is not necessarily Π^1_1). One then verifies that the result relativizes to the minimum ω-model above (that is, the function variables in the Δ^1_1-definitions considered range over the class of hyperarithmetic functions). But more delicate considerations seem to be needed for the conservative extension results of [74].

The next two notes concern the important, but elusive, matter of *interesting* systems. The systems considered in the present Note provide examples! On the one hand the difference between the rule in b(i) and the corresponding axiom b(ii) has a clear interest for predicativity; see Note Va. On the other hand I know of no conception of proof for which the Axiom (Π^1_1-CA) without parameters is immediately evident, and for which (Π^1_1-CA) with parameters is not, and the same applies to (Δ^1_1-CA). So, without some novel interpretation, the difference between these schemata and their parameterless version is of purely technical interest, for instance for an axiomatic analysis of the theory of sets recursive in O, and of sets Π^1_1 in O respectively; cf. Note V.

§IV. Philosophically interesting models of the language of analysis: definability. This note contains concrete examples of 'interesting' distinctions, so often contrasted in this article with 'purely formal' ones. (Since we speak of 'models' we are principally concerned with the usual classical interpretation of the logical operations.)

When all is said and done, the most basic model of the language of analysis described in §7 or §8, is the structure $\langle N, \mathfrak{P}(N), 0, S, \in\rangle$ of §4 (or, if preferred, $\langle N, \mathfrak{P}(N), N^N, 0, \in, \circ\rangle$). We shall replace $\mathfrak{P}(N)$ by two of its subclasses consisting, respectively, of the sets $\subset N$ which are definable *tout court* and of those definable by means of purely *arithmetic* methods. The latter is *elementary* in that it reduces existential assumptions about sets; the former is of interest because it is *problematic* and thus in need of analysis. Both examples turn out to be relevant to the discussion of the axiom of choice.

(a) *Ramified analytic hierarchy:* segments and extensions of the predicative hierarchy [5]. Let S be an ω-realization[24] of the language of analysis, and **D**(S) the

[24] An ω-realization (of our language) has N as domain of its l.c. variables, and hence a subset of $\mathfrak{P}(N)$ as domain of cap. its variables, also called 'set' variables. Thus intersections,

collection of sets $\subset$ N definable (by formulae of this language) in S. Let A_0 be the collection of sets definable in first order arithmetic,[25] and, for ordinals $\alpha > 0$, $A_\alpha = \bigcup_{\beta<\alpha} DS_\beta$, where S_β is the ω-realization in which A_β is the domain of the set variables.

(i) A_1 is ramified analysis in the sense of H. Weyl. It is the minimum ω-model of (Σ^0_∞-CA) or, equivalently, (Σ^0_∞-DC) with (and also, incidentally, without) parameters.

(ii) A_{ω^ω} is the minimum ω-model of the (Δ^1_1-CA)-*rule* or, equivalently [5], the (Σ^1_1-DC_{11})-*rule* with parameters. These rules are distinguished by the fact that the sets introduced by these rules are ***invariantly*** definable in all ω-models containing A_{ω^ω}.[26]

(iii) A_{Γ_0} is predicative analysis according to [5] and [52], where Γ_0 is the first strongly critical ordinal. The segment A_{Γ_0} satisfies very strong closure conditions.

(iv) A_{ω_1}, where ω_1 is the first nonrecursive ordinal, is the minimum ω-model of the (Δ^1_1-CA) *axiom* and also of the (Σ^1_1-DC)-*axiom*. The definition of A_{ω_1} is no longer arithmetic because, for ordinals $\alpha \geq \Gamma_0$, the notion of *iteration-through-α-steps* is not reducible to *arithmetic* concepts.

See Note V for additional *technical* information.

(b) *Absolutely definable sets of natural numbers.* Let us suppose that our concept of definable set of natural numbers is clear (enough to justify the assertions made about it below) and denote it by $\mathscr{D}$. When there is no ambiguity we shall also denote the *collection* of definable sets by $\mathscr{D}$.

Let $\mathscr{D}_A$, $\mathscr{D}_E$, denote the sets $\subset$ N which are definable in the language of analysis and of set theory respectively (both applied to the principal models) and let $\mathscr{D}_L$ be the collection of constructible sets $\subset$ N in the sense of Gödel. The principal assumptions are:

I. $\mathscr{D} \supset \mathscr{D}_A$ and, more important,

II. $\mathscr{D}$ is well defined so that quantification over $\mathscr{D}$ satisfies the rules of classical logic.

Assumption I is not problematic, since if $\mathscr{D}$ and $\mathscr{D}_A$ are accepted at all, $\mathscr{D}$ contains a truth definition for $\mathscr{D}_A$.

Assumption II is certainly delicate; for instance, if we considered definitions of ordinals instead of definitions of sets of natural numbers, Gödel's observations [76] would imply that $\mathscr{D}$ is uncountable.

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unions, differences of ω-realizations are again ω-realizations. It may be remarked that another well-known class of realizations in the literature, Mostowski's β-models, is not closed under intersection.

[25] These are the so-called 'arithmetically definable' sets. This terminology (if taken literally) conflicts with the thesis of [5]. Briefly, the objection to the terminology is this: if existential number theoretic quantification is accepted as well defined at all, then not only each object of A_0 is well defined, but the whole class, i.e., A_0 can be enumerated by an intuitively-arithmetic definition.

[26] i.e., the conclusion of each of these rules has the form $\exists X A(X)$, possibly containing parameters, and some invariantly defined X_0 satisfies $A(X_0)$. Of course, there are logical *consequences* of these rules of existential form (with more complicated A) which do not have this property, see §11b(ii)2.

(i) Let A(n, X) be a formula in the language of analysis whose only set variable is X. Then

$$\{n: \forall X A(n, X) \text{ is true in } \mathscr{D}\} = \{n: \forall \mathbf{X} A(n, \mathbf{X})\},$$

where, as in §4, bold face capitals range over $\mathfrak{P}(N)$.

PROOF. By Kleene's basis theorem, for *any* $\mathscr{C} \supset \mathscr{D}_A$,

$$\{n: \forall X A(n, X) \text{ is true in } \mathscr{C}\} = \{n: \forall \mathbf{X} A(n, \mathbf{X})\}.$$

(ii) Suppose $A_1(n, X, Y)$ and $A_2(n, X, Y)$ are formulae whose only set variables are X and Y, and suppose also that

$$\forall n[\forall \mathbf{X} \exists \mathbf{Y} A_1(n, \mathbf{X}, \mathbf{Y}) \leftrightarrow \exists \mathbf{X} \forall \mathbf{Y} A_2(n, \mathbf{X}, \mathbf{Y})],$$

i.e., $\forall X \exists Y A_1$ defines a Δ_2^1-set in $\mathfrak{P}(N)$, i.e., when X and Y range over $\mathfrak{P}(N)$ and the l.c. variables over N. Then

$$\{n: \forall X \exists Y A_1(n, X, Y) \text{ is true in } \mathscr{D}\} = \{n: \forall \mathbf{X} \exists \mathbf{Y} A_1(n, \mathbf{X}, \mathbf{Y})\}$$

provided there is a Ramsey cardinal.

PROOF. For any $\mathscr{C} \supset \mathscr{D}_L$, by [97],

$$\{n: \forall X \exists Y A_1(n, X, Y) \text{ is true in } \mathscr{C}\} = \{n: \forall \mathbf{X} \exists \mathbf{Y} A_1(n, \mathbf{X}, \mathbf{Y})\}.$$

By [57], using Ramsey cardinals, $\mathscr{D}_A \supset \mathscr{D}_L$, and so, by assumption I, $\mathscr{D} \supset \mathscr{D}_L$. Substituting $\mathscr{D}$ for $\mathscr{C}$ and using II, we get the result claimed.

We can, if we wish, avoid Ramsey cardinals by using additional information contained in [97]; not all of $\mathscr{D}_L$ is needed, but only constructible sets of Δ_2^1-order.

(iii) Note that results (i) and (ii) are obtained by combining purely set theoretic results with the simple assumptions I and II; cf. Note Ib(i). Also note that, for the formulae considered in (i) and (ii), the *same* set is defined whether the (set) quantifiers range over $\mathscr{D}_A$, $\mathscr{D}$, or $\mathfrak{P}(N)$: it does not seem to be known whether all formulae of analysis have this property, even if large cardinals are assumed.

It seems worth remarking that the result $\mathscr{D} \supset \mathscr{D}_L$ in (ii) was not particularly plausible before Solovay's work in [57]; not even when one knew by [94] that $\mathscr{D}_L$ is countable without knowing *how* $\mathscr{D}_L$ can be enumerated (unless $\mathscr{D}$ includes definitions of sets of natural numbers by means of arbitrary ordinals when, by [76], $\mathscr{D} \supset \mathscr{D}_L$ holds under Assumption II).

(c) *Axioms of choice and definability.* Let us first recall some facts. As we have already seen,[26] avoiding the axiom of choice does not, by itself, ensure the realization of existential theorems by means of explicitly definable sets (see, however, the last problem of this Note). In the opposite direction: restriction to definable sets ensures the axiom of choice provided first, the definitions are explicitly well ordered, and second the relation between name (i.e., definition), and object named (i.e., set defined) is itself definable. For example, both of Gödel's notions of *constructible* and *ordinal-definable* sets [76] satisfy this proviso. Without the proviso the axiom of choice is not plausible in general when logical operations are interpreted set theoretically. Consider $\forall n \exists X A(n, X) \rightarrow \exists X \forall n A(n, X_n)$: for each n there may be a definable X satisfying A(n, X) but the *relation* between n and X need not be so definable. For example, the restriction to first order definable sets has this property [take A(n, X) to mean: X enumerates the arithmetic sets definable by means of n quantifiers].[27]

For the notion $\mathscr{D}$ in (b) above there is no obvious reason to assume that all (intelligible) definitions present themselves as arranged in a (definable) well order. In contrast the comprehension axiom is satisfied by $\mathscr{D}$ provided assumption II is accepted.

It seems fair to say that there was a genuinely interesting issue *behind* the early discussions of the axiom of choice, but the results above make it doubtful whether the question of logical dependence on this axiom really *pinpoints* the issue.

A much more promising idea was introduced by Gödel in [76]: without detour via the axiom of choice he considers *the role* of *notions of higher* (transfinite) *type* in questions of definability. Given A(X), are there sets satisfying A which are definable by use of higher type operations, but not without them? Some interesting results have been obtained for a slight (?) variant of this question involving *formal* proofs from given axioms rather than truth in the set theoretic hierarchy. Thus Gandy [75] shows that there are Π^1_2-formulae A which are provably satisfied by a set defined by use of countable ordinals, but not so by any set $\in \mathscr{D}_A$. Even more impressive are Solovay's results, still unpublished, which show that the *measure* of certain projective sets of real numbers is definable by use of symbols for Ramsey cardinals, but not by a definition $\in \mathscr{D}_A$ (even if the existence of such ordinals is assumed).

What light do these (and related) facts throw on mathematical practice? First, as far as proofs of elementary, say, arithmetic statements are concerned, we can justify the full use of the axiom of choice; by relativization to constructible sets the axiom can be eliminated; and what point is there in giving explicit definitions in the proof if they get lost in the statement of theorems? On the other hand the results cast doubt on a fairly popular 'aesthetic justification' of using Zorn's lemma instead of definitions by recursion on the countable ordinals, which was common 50 years ago: mathematicians do not like to 'mix' types; ordinals involve transfinite types, while Zorn's lemma is stated by use of only 2 or 3 types (above the individuals considered). Since most mathematicians do not even realize what is lost by their 'taste' (since they do not know the explicit definitions involved), they are in no position to judge the matter.

Warning. The last paragraph concerned the role of the axiom of choice for definability; as far as validity in $\mathfrak{P}(N)$ is concerned, its use is of course unobjectionable.

Let me conclude with two *problems.*

(i) What is the role of the axiom of choice in proofs of arithmetic statements by use of principles which are *not* satisfied by the constructible sets, for instance large cardinals?

[27] Clearly, if (philosophical) analysis of $\mathscr{D}$ were to convince us that (AC_{01}) is false when relativized to $\mathscr{D}$ we should have a painless independence proof of (AC_{01}) from (CA). Such an independence result would not be less convincing than the existing ones, but it would probably be less informative; for instance it might not show that (CA) $\cup \neg(\Pi^1_2\text{-}AC_{01})$ is a conservative extension of (CA) with respect to formulae $\forall f A$ where A is (essentially) Σ^1_2. In short, the situation would be quite similar to the cases treated in Note Ic(ii).

(ii) Recall §11b(ii)2: for the *particular* existential theorem $\exists X A(X)$ we have, by [57], an explicitly definable $X_0 \in \mathscr{D}_A$ which satisfies A. What can we say about the explicit realization of existential theorems proved without the axiom of choice if at the same time strong axioms of infinity are used?

§V. Technically useful systems: axiomatic theory of hyperarithmetic sets. The purpose of this note is to substantiate the points made in the informal discussion at the end of §4.

For background recall some standard exposition of the properties of hyperarithmetic sets. Inspection shows that the proofs fall 'naturally' into groups. For instance, if we consider the equation $\Delta^1_1 = \mathrm{HYP}$, where HYP is Kleene's original definition of the hyperarithmetic hierarchy, $\mathrm{HYP} \subset \Delta^1_1$ is more 'elementary' than $\Delta^1_1 \subset \mathrm{HYP}$. In this note we shall group the proofs according to the *complexity of the formulae that occur* in the *instances of* (CA), (AC_{01}), (DC_{11}) used.

Obviously such an axiomatic analysis is not the only way of looking at the matter! (For instance a simple minded approach would ask first which results about Δ^1_1 generalize to Δ^1_n for all n). It's a gamble, like any other axiomatic treatment, even in algebra or topology. But, it seems to me to have been rewarding already [74], and should be even more so when one develops generalized hyperarithmeticity, like generalized recursion theory, on structures other than arithmetic. For the present survey the corollaries to the analysis are particularly useful for the information which they give about the formal systems considered.

All systems below will be in the language of analysis. Besides the relations (between formal systems) listed in Note II, we shall use the notion of *uniform*-S-ω-*model* of S′, given by a definition which, in *each* model of S, is the identity on the domain of the l.c. variables, that is even the nonstandard integers of the model of S are left fixed[28]. All axioms considered below are tacitly assumed to be *added* to the system $\mathscr{E}$ of §8.

(a) *Basic systems and their properties.* Each of the axioms (Δ^1_1-CA), (Σ^1_1-AC_{01}), (Σ^1_1-DC_{11}) has the property that its minimum ω-model consists of just the hyperarithmetic sets ([28], [29]). What do they have in common? and what are the relations between them (either with respect to ordinary formal consequence or to ω-consequence)? We know generally that

$$(\Sigma^1_1\text{-}DC_{11}) \rightarrow (\Sigma^1_1\text{-}AC_{01}) \rightarrow (\Delta^1_1\text{-CA}).$$

(i) Friedman [74] has given a uniform (Δ^1_1-CA)-ω-model of (Σ^1_1-DC_{11}) with most satisfactory conservative extension properties; in particular the same theorems *about* hyperarithmetic sets (when defined in the original Kleene fashion), can be proved in all three systems.

(ii) Also he shows that (Σ^1_1-DC_{11}) cannot be derived from (Δ^1_1-CA), nor from (Σ^1_1-AC_{01}), not even in the sense of ω-consequence. His proof is necessarily different from (i):

[28] A somewhat weaker requirement on δ would be this: in each ω-model M of S, δ defines an ω-model of S′ leaving each integer of M fixed. The positive results quoted in this Note establish the stronger form, the negative results refute even the weaker form.

There can be no uniform (Δ^1_1-CA)-ω-model of (Δ^1_1-CA) $\wedge$ $\neg$(Σ^1_1-DC$_{11}$) because there is a minimum ω-model of (Δ^1_1-CA), namely HYP, and in HYP also (Σ^1_1-DC$_{11}$) is satisfied.[29]

It seems to be *open* whether there is a uniform (Δ^1_1-CA)-model of (Δ^1_1-CA) $\wedge$ $\neg$(Σ^1_1-DC$_{11}$). Also it is open whether (Σ^1_1-AC$_{01}$) can be derived from (Δ^1_1-CA) since the sketch in [71] is inconclusive.

(iii) The most important result of [74] is the determination of the exact proof theoretic strength of the three systems: (an elementary proof of) conservative extension results with respect to the usual formal system, in [5] or [52], for ramified analysis of level ϵ_0.[30]

This last result is not only technically interesting, because of its novel combination of proof theoretic results §8(3) and model theoretic constructions, but also foundationally. As stressed in [29] it was not intuitively plausible that the (Δ^1_1-CA) (*-axiom*) be valid when relativized to predicative sets, and, from the work of [5], it is not; however, we now know that (Δ^1_1-CA) has a predicative consistency proof.[31]

(b) *Axiomatic analysis.* As already mentioned the proofs of HYP $\subset \Delta^1_1$ and of $\Delta^1_1 \subset$ HYP strike one as being of different 'order'. Thus (Σ^0_∞-CA) is amply sufficient to derive the former. It still seems to be open whether the latter can be derived in (Σ^1_1-DC$_{11}$).

Here are some isolated result. (Here it seems best to use a different notation for the class $\mathscr{H}$ of hyperarithmetic functions and for their definitions, modulo formal equivalence, namely HYP.)

(i) The *usual* proof of $\Delta^1_1 \subset$ HYP uses the comparability of recursive well-orderings. This intermediate result can certainly not be proved from (Σ^1_1-DC$_{11}$) since it does not hold when relativized to $\mathscr{H}$, but the latter satisfies (Σ^1_1-DC$_{11}$).

To see this, recall that comparability implies, in conjunction with (Σ^0_1-CA), separability of any two disjoint Σ^1_1-sets by a Δ^1_1-set. If we relativize the conclusion to $\mathscr{H}$, we get separability of two disjoint Π^1_1-sets by a Δ^1_1-set, since any Π^1_1-set is Σ^1_1-over $\mathscr{H}$, and any set which is Δ^1_1-over-$\mathscr{H}$ is $\mathscr{H}$, which contradicts a well-known result of Kleene.

(ii) By [74], if for a recursive linear ordering (defined, say, by a formula without set quantifiers) the existence of a hierarchy can be *proved* from (Σ^1_1-DC$_{11}$), the ordering (defined by the formula in any ω-model) $< \epsilon_0$.

[29] This remark illustrates a very general principle which applies whenever the axioms considered have a minimum model in the class of models studied.

[30] This result has since also been proved by Howard as follows: by refining the argument of [58] he shows that the interpretation of Σ^1_1-DC$_{11}$ only requires bar recursion, §10(i), applied to G and H of the type of F itself. It then turns out that the ordinal 'of' this restricted schema is exactly the ordinal of formal ramified analysis of level ϵ_0. Full bar recursion of type 0 goes far beyond the limit Γ_0 of predicative analysis. (The present results correct the second assertion in [73]).

[31] *Correction.* At the end of [58], I described (Δ^1_1-CA) as *impredicative* and observed that Spector's method in [58] gave the first reduction of an impredicative comprehension principle to accepted intuitionistic methods. Friedman's result shows that (Δ^1_1-CA) is only *prima facie* impredicative. However, [58] *does* give a reduction of a genuinely impredicative principle (to accepted intuitionistic methods) because it applies to the schema: WF(R) $\to$ TI(R, A) discussed in §10.

At least it's better than for $(\Sigma^0_\infty\text{-DC}_{11})$! where such an ordering would be finite.

(iii) Again by [74], if for two first order formulae A(n, X) and B(n, X), $\forall n[\forall X A(n, X) \leftrightarrow \neg\forall X B(n, X)]$ can be proved from $(\Sigma^1_1\text{-DC}_{11})$, the set $\{n: \forall X A(n, X)\}$ belongs to ramified analysis of level $< \epsilon_0$.

(c) *The schema of well-foundedness in* §10: *model theoretic and proof theoretic analysis.* For a list of properties, see the beginning of §10. Here we shall establish that the system is 'weaker' than $(\Pi^1_1\text{-CA})$. As we know [16], it implies $(\Sigma^1_1\text{-DC}_{11})$, and it is not hard to show that the consistency of $(\Sigma^1_1\text{-DC}_{11})$ can be proved by means of the schema. So we have 'located' the schema strictly between $(\Sigma^1_1\text{-DC}_{11})$ and $(\Pi^1_1\text{-CA})$.[32]

(i) A $(\Sigma^1_1\text{-CA})$-ω-model for the schema $\text{WF}(R) \to \text{TI}(R, A)$ when R has no *set* parameters. Recall that R is arithmetic and A arbitrary, and that $(\Sigma^1_1\text{-CA})$ is equivalent to $(\Pi^1_1\text{-CA})$ of Note IIIc(i). Our ω-model is the class $\mathscr{C}$ of that Note, or, more precisely, its formal definition in $(\Pi^1_1\text{-CA})$. Let $f_{\mathscr{C}}$ be a function which enumerates $\mathscr{C}$. We now use two facts. First, $(\Sigma^1_1\text{-CA})$ implies Kleene's basis theorem, and hence the assertion: if there is a descending sequence in R, there is one which $\in \mathscr{C}$. Second, the relativization of A to $\mathscr{C}$ is equivalent to a formula $A_{\mathscr{C}}$ *arithmetic* in $f_{\mathscr{C}}$, and $(\Sigma^1_1\text{-CA})$ implies: $\text{WF}(R) \to \text{TI}(R, A_{\mathscr{C}})$. This completes the proof.—Note that our $\mathscr{C}$ does not satisfy the schema when R does have set parameters.

(ii) Is there a $(\Sigma^1_1\text{-CA})$-ω-model for the schema $\text{WF}(R) \to \text{TI}(R, A)$ when R *does* have set parameters? The *obvious* model, say $\mathscr{C}_P$, consists of the functions arithmetic in

$$O \cup O^O \cup O^{O^O} \cup \cdots.$$

But we certainly cannot prove in $(\Sigma^1_1\text{-CA})$ the existence of a function which enumerates $\mathscr{C}_P$ since $\mathscr{C}_P$ itself satisfies $(\Sigma^1_1\text{-CA})$! (It is clear that such a function can be established in $(\Delta^1_2\text{-CA})$ (even without parameters), as observed in [17, p. 327].) *Added in proof.* H. Friedman has answered this question positively (cf. footnote 34) which supersedes the result *stated* in (iii) below; the *argument* of (iii) establishes more: the schema with parameters is a conservative extension of the schema without parameters for arithmetic formulae.

(iii) A $(\Sigma^1_1\text{-CA})$-model for the schema $\text{WF}(R) \to \text{TI}(R, A)$ containing set parameters in R. By Note IIb(i) it is sufficient to prove the consistency of the schema in $(\Sigma^1_1\text{-CA})$. The only arguments I know are roundabout. First one applies the method of [58], or Howard's improvement [101], to reduce the system to quantifier-free bar recursion of type 0. Then one has a choice: either one defines a *model* for bar recursion or one proves *computability* of the bar recursion terms, both in $(\Sigma^1_1\text{-CA})$. In the former case one uses a variant of the continuous functionals of [24],[33] and proves the existence of bar recursion functionals by means of the schema $\text{WF}(R) \to \text{TI}(R, A)$ *without* parameters, for which, by (i), there is a

[32] As to the relevance of this schema for an axiomatic theory of hyperarithmetic sets, note that the schema implies comparability of recursive well orderings, mentioned in b(i) above.

[33] In the definition of [24], one takes *arbitrary* functions of type 1, and, say, recursive neighbourhood functions at higher type. (I had this model in mind when asserting the present result in the review of [21].)

$(\Sigma_1^1\text{-CA})$-ω-model. In the latter case, one analyzes the computations in terms of Howard's ordinal $\varphi_{\varepsilon_{\Omega+1}}(1)$ and proves its well ordering in $(\Sigma_1^1\text{-CA})$.

Though I expressed reservations about the foundational significance of the schema $WF(R) \to TI(R, A)$, at the end of §10, the system seems to be technically sufficiently useful to make a more direct proof of (iii) above worthwhile.[34]

§VI. Set theory without the power set axiom: its reduction to analysis. The principal purpose of this note is to summarize the precise results which constitute this 'reduction'. We shall formulate them by use of the notions of Note II. The reduction justifies the concentration on analysis in the present survey.

Let $(Z)^-$ and $(ZF)^-$ be, respectively, the first order versions of Zermelo's and Zermelo Fränkel's set theory without the power set axiom. Let a star denote the addition of the axiom (schema) of choice, (or of the countable axiom of choice: the results below are not affected).

(a) *Crude results* depending only on the familiar formalization of analysis in set theory, and on the description of hereditarily countable trees in analysis.

(i) There is a $(Z)^-$-ω-model of (CA), and a $(Z)_*^-$-ω-model of (AC_{01}). These models are provided by the familiar definition in set theory of the set of natural numbers, and hence of the collection of its subsets.

Since $(ZF)^- \supset (Z)^-$ it follows that the models defined are also $(ZF)^-$-ω-models.

(ii) Conversely there is a (CA)-ω-model of $(Z)^-$, and a (AC_{01})-ω-model of $(Z)_*^-$ and $(ZF)_*^-$.

The same *definition* will do for all three cases: one considers the countable trees coded by number theoretic functions and defines *set equality* $(a = b)$ by: there is an isomorphism between the trees a and b.

The most interesting property of the models so defined is this, in the notation of Note IIb(ii):

The [(CA), $(Z)^-$] *models commute*; *and so do the models for the pairs* [(AC_{01}), $(Z)_*^-$] *and* [(AC_{01}), $(ZF)_*^-$].

Consequently the same analytic theorems can be proved in $(Z)_*^-$ and $(ZF)_*^-$.

(iii) There is an (AC_{01})-ω-model of $(ZF)^-$, since, by (ii), there is an (AC_{01})-ω-model even of $(ZF)_*^-$. The need for (AC_{01}), and not merely (CA), arises as follows: the replacement axiom of $(ZF)^-$, namely

$$\forall a\{(\forall x \in a)\exists ! y A(x, y) \to \exists z \forall y[y \in z \leftrightarrow (\exists x \in a)A(x, y)]\},$$

translated into a model of isomorphism-*classes* of trees, requires us to *choose* a tree from the isomorphism class corresponding to y. The result b(iv) below, due to Gandy, shows that $(ZF)^-$ is *not* a conservative extension of (CA), and so the need for (AC_{01}) is not tied to the particular model considered.

[34] *Correction.* The model theoretic argument in footnote 25 of [31, p. 139] is not correct. It is true that the class of functions of hyper degree $< O'$ satisfies the schema *with* parameters, but it does not satisfy the pairing axiom! ([99], Theorem 18 provides α and β of hyper degree $< O'$ such that $\alpha \cup \beta$ has hyper degree $= O'$.) It is nice to think that the care taken in [17, p. 329] to list pairing axioms was not superfluous. *Added in proof.* Friedman's construction, mentioned in (ii) above, makes sense of the crude idea in [31].

(b) *Delicate results*, involving the theory of constructible sets and its development by use of generic sets.

(i) There is a (CA)-ω-model of $(ZF)^-_*$. The model consists of the (countable) trees defined by *constructible* sets of natural numbers. It is faithful, in the sense of Note IIb(ii), for all closed formulae which are essentially Σ^1_3 ([27, p. 386]).

Evidently an alternative method is to give a (CA)-ω-model of (AC_{01}), and then apply the 'crude' result in (ii) above.

(ii) By [75], $(ZF)^-$ is *not* a conservative extension of (CA); more specifically, there is an instance A_0 of $(\Pi^1_2\text{-}AC_{01})$ which is implied by $(ZF)^-$, but not by (CA).

The result is optimal in at least two respects. First, by Kondo's theorem, (CA) implies $(\Sigma^1_2\text{-}AC_{01})$ or, equivalently, $(\Pi^1_1\text{-}AC_{01})$, in fact even $(\Sigma^1_2\text{-}DC_{11})$. Second $(CA) \cup \{\neg A_0\}$ is a conservative extension for all (formulae which are the universal closure of) essentially Σ^1_2-formulae, while A_0 itself is essentially Π^1_3.[35]

(c) *Choice of subsystems of analysis.* Recall Note IVc. Just because we take subsystems seriously, we cannot be satisfied to let the formal results above 'speak for themselves': the history of mathematical logic suggests they are liable not to be listened to, or to be misheard. Since the subject is wide open, I confine myself to some elementary comments.

(i) What is the significance of (the set of consequences of) (CA)? In terms of the philosophically interesting structures considered in Note IV, it is either too big or too small: too big for IVa, since (CA) is not predicatively justifiable, too small for IVb (and of course for the principal model of §4) because, e.g., A_0 of b(ii) above, is satisfied in $\mathscr{D}$, and even in $\mathscr{D}_E$.

Perhaps a more significant class is $(CA) \cap \mathscr{A}$ where $\mathscr{A}$ is a *syntactically-restricted* class of formulae in the language of analysis. A nice paradigm is Friedman's discovery mentioned in Note Va(i), concerning formulae *about* hyperarithmetic sets, when (CA) is replaced by $(\Delta^1_1\text{-}CA)$. One would expect $(CA) \cap \mathscr{A}_2$ to have some interest where $\mathscr{A}_2$ consists of formulae all of whose quantifiers are relativized to Δ^1_2-definitions.

(ii) All the results in (a) and (b) above are for 'full' analysis or set theory (without the power set axiom). Let $(\Sigma_1\text{-}ZF)^-$ be obtained from $(ZF)^-$ if, in the replacement axiom in (a)(iii), *A* is restricted to be a Σ_1-formula in the sense of [41]. One would expect $(\Sigma_1\text{-}ZF)^-$ and also $(\Delta_1\text{-}Z)^-$ to be closely related to $(\Sigma^1_1\text{-}AC_{01})$ and $(\Delta^1_1\text{-}CA)$, which, if true, would round off nicely the results reported in Note V.[36]

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[35] A corresponding result for set theory was stated in [31, p. 110], but only for Σ^1_2 formulae, not for their universal closures. This result implies faithfulness for *essentially* Σ^1_2-formulae because, by Kondo's theorem, (CA) implies that any essentially Σ^1_2-formula is (equivalent to a) Σ^1_2 (formula). Note that we do not have 'complete' symmetry between the addition of the axiom of choice (or $V = L$) to (CA) and of its negation: by b(i) the former preserves all essentially Σ^1_3 statements, but the latter does not preserve all *essentially* Π^1_3 statements.

[36] But one does *not* expect $(\Sigma_n\text{-}ZF)^-$ to be closely connected with, say, $(\Sigma^1_n\text{-}AC_{01})$, no more than one expected the Δ^1_2-sets to be a 'minimum' model of $(\Delta^1_2\text{-}CA)$ merely because, by [28], the Δ^1_1-sets are the minimum ω-model of $(\Delta^1_1\text{-}CA)$! In fact, there is no minimum ω-model of $(\Delta^1_2\text{-}CA)$ at all, and the minimum β-model (footnote 23) of $(\Delta^1_2\text{-}CA)$ is strictly included in the class of Δ^1_2-sets.

§VII. Interpretations by means of functionals: conservative extension results. We consider here interpretations of (formal systems of) arithmetic and analysis, using either functionals of lowest type in the no-counter example-interpretation [81] or functionals of all finite types in Gödel's [12] extended in [24] and [58].

Viewed naively, *any* use of functionals in proof theoretic reductions of arithmetic or analysis appears *paradoxical* since one uses objects of type higher than those mentioned in the systems considered. The view is wrong because it overlooks the abstract operations implicit in the logical symbolism: the usual set theoretic interpretation of the classical logical laws (such as the law of the excluded middle applied to undecided propositions) presupposes evidently nonconstructive set theoretic operations, and Heyting's interpretation of the intuitionistic logical operations involves highly abstract operations on thoughts (proofs) especially in connection with the universal quantifier and implication. More concretely, the view overlooks the possibility of making *explicit* the information hidden in complex logical statements. On the other hand, the naive impression is right because the use of functional variables has to be supported by an *analysis*: it would be pointless to regard them as ranging over arbitrary functionals in the set theoretic hierarchy built up with the natural numbers as individuals.[37] Such an analysis could consist either in *describing* operations independently of the set theoretic hierarchy, cf. [12] (or [24] where functionals are treated as 'equivalence classes' of rules); or else in treating the use of functional variables only as a technical device which makes the work intelligible, and *eliminating* them. The latter alternative needs conservative extension results as in [59] or [98]. The relation between the alternatives is very similar to that between general philosophical analysis and conservative extension results discussed in Note Ic(ii) and footnote 27.

The simplest example to correct the naive impression is provided by the no-counterexample-interpretation. Suppose

$$A \text{ is } \exists x_1 \forall y_1 \cdots \exists x_n \forall y_n A_0(x_1, \cdots, x_n, y_1, \cdots, y_n).$$

Write

$$A_0(x, y) \quad \text{for} \quad A_0(x_1, \cdots, x_n, y_1, \cdots, y_n),$$

and

$$A_0(x, fx) \quad \text{for} \quad A_0[x_1, \cdots, x_n, f_1(x_1), \cdots, f_n(x_1, \cdots, x_n)].$$

Let F_i, $1 \le i \le n$, be variables for numerical valued functionals with arguments $f_1, \cdots, f_n$ and let $\tilde{F}_i$ be the value of $F_i(f_1, \cdots, f_n)$; write

$$A_0(F, f) \quad \text{for} \quad A_0[\tilde{F}_1, \cdots, \tilde{F}_n, f_1(\tilde{F}_1), \cdots, f_n(\tilde{F}_1, \cdots, \tilde{F}_n)].$$

[37] The error is completely parallel to that involved in the well-known objection to consistency proofs by ϵ_0-induction for ordinary, i.e., ω-induction, cf. [31, p. 165, 3.3322]. On the other hand the objection would be justified if one introduced ϵ_0 in terms of the set theoretic notion of ordinal! Hence the need for §6d. The following general point should, perhaps, have been mentioned in connection with the conflict between *size* and *complexity* in the case of ordinals in §6c(iv). From set theoretic foundations which are preoccupied with *existential assumptions*, one is accustomed to consider more inclusive concepts as more problematic. This certainly does not apply to constructive foundations; for instance the notion of natural number is evidently less problematic than, say, a definition of a subset of $\{0, 1\}$ by use of nonconstructive concepts!

We have two (classical) logical relations

$$A \leftrightarrow \neg \forall x_1 \exists y_1 \cdots \forall x_n \exists y_n \neg A_0(x, y)$$

and

$$\exists \mathbf{F} \forall f A_0(\mathbf{F}, f) \rightarrow \neg \exists f \forall x \neg A_0(x, fx);$$

and two semilogical equivalences (using the axiom of choice)

$$\forall x_1 \exists y_1 \cdots \forall x_n \exists y_n \neg A_0(x, y) \leftrightarrow \exists f \forall x \neg A_0(x, fx),$$

and

$$\exists \mathbf{F} \forall f A_0(\mathbf{F}, f) \rightarrow \neg \exists f \forall x \neg A_0(x, fx).$$

Putting them together we have

$$A \leftrightarrow \exists \mathbf{F} \forall f A_0(\mathbf{F}, f).$$

Inspection shows that if $\exists \mathbf{F} \forall f A_0(\mathbf{F}, f)$ is true at all, then there is an $\mathbf{F}_0$ which is recursive, and hence depends continuously on $f_1, \cdots, f_n$ (i.e., only on a finite number of values of the f), such that $\forall f A_0(\mathbf{F}_0, f)$. In contrast, if we took the usual Skolem form of A, i.e.,

$$\exists x_1 \exists g_2 \cdots \exists g_n \forall y_1 \cdots \forall y_n A_0[x_1, g_2(y_1), \cdots, g_n(y_1, \cdots, y_{n-1}), y_1, \cdots, y_n]$$

we could not choose g_i recursive. (Another way of looking at the matter is to observe that the equivalences above do not hold if relativized to recursive f and recursive **F**.)

Of course, recursiveness and continuity are only a first approximation to constructivity; see the *Remark* at the end of §11a. We have to make a fresh start.

(a) *General principle of interpretations*, in terms of an intuitive conception $\mathscr{P}$. We associate with each formula A of the formal system studied a relation $A^*(s, t)$ between (sequences of) functionals s and t, such that, nonconstructively, $A \leftrightarrow \exists s \forall t A^*(s, t)$, and also, from a formal derivation p of A we get an s_p such that, for variable t, $\mathscr{P} \vdash A^*(s_p, t)$.

In an obvious way, these relations ensure that the interpretation $A^*(s_p, t)$ makes the information contained in A more explicit, i.e., we lose nothing from the nonconstructive point of view, and, at the same time, we have a reduction to $\mathscr{P}$.

The execution of this programme depends on finding formal principles $\mathscr{F}$ which are valid both for the functional operations of $\mathscr{P}$ and for the nonconstructive ones and which suffice to derive the implications above.

(i) Having made a guess at A^*, which, in the case of the no-counterexample-interpretation above is $A_0(\mathbf{F}, f)$, we try to find, for each axiom A, a functional s_A such that

$$\mathscr{P} \vdash A^*(s_A, t) \quad \text{for variable t},$$

and for each rule of inference, deriving A from $\bar{A}$ and $\bar{\bar{A}}$ say, a functor Φ_A such that

$$\mathscr{P} \vdash A^*[\Phi_A(s_1, s_2), t] \quad \text{holds provided both } \mathscr{P} \vdash \bar{A}^*(s_1, t_1)$$

and $\mathscr{P} \vdash \bar{\bar{A}}^*(s_2, t_2)$ hold (for variables t_1 and t_2 of appropriate type).[38]

[38] To avoid the logical relation 'hidden' in this clause, one can make the dependence explicit by means of functors Ψ_1 and Ψ_2 such that

$$\mathscr{P} \vdash (\bar{A}^*[s_1, \Psi_1(s_1, s_2, t)] \wedge \bar{\bar{A}}^*[s_2, \Psi_2(s_1, s_2, t)]) \rightarrow A^*[\Phi(s_1, s_2), t]$$

Then, if $\xi_1, \xi_2, \cdots$ is an enumeration of the sequence of functionals generated by means of the Φ_A from the s_A above, we have the *pseudo-reflection-principle*, for all p and t,

$$\mathrm{Prov}(p, \ulcorner A \urcorner) \rightarrow A^*(\xi_{\nu(p)}, t)$$

for a suitable ν, as required.

(ii) To get the formal principles $\mathscr{F}$ one must describe explicitly the schemata for the functionals, and the derivation of $A^*[\Phi_A(s_1, s_2), t]$ from $\bar{A}^*(s_1, t)$ and $\bar{\bar{A}}^*(s_2, t)$; see footnote 38.

We get the *full* reflection principle by adjoining to $\mathscr{F}$ the principles needed to derive $\exists s \forall t A^*(s, t) \rightarrow A$, which in general will of course not be valid in $\mathscr{P}$. It follows that if the extended system, say $\mathscr{F}_1$, is a conservative extension of a system $\mathscr{S}$ (in the language of A) then the full reflection principle for the formal system studied is derivable in $\mathscr{S}$.

A detailed application of this idea, for the case of the no-counterexample-interpretation of arithmetic, is given in the proof of Theorem 12 of [36].

(iii) An interpretation of the kind described allows us to *separate* derivations of A into a 'mathematical' part followed by 'trivial' logical inferences. Specifically, given a formal derivation p_0, by use of the pseudo-reflection principle we obtain (in $\mathscr{F}$!) a $\xi_{\nu(p_0)}$, such that, for variable t, $A^*(\xi_{\nu(p_0)}, t)$ and then we apply the purely logical steps to get $\exists s \forall t A^*(s, t)$ and hence, trivially,[39] A.

This separation may be compared to that achieved in familiar cut-free systems for predicate logic, where the mathematical or combinatorial part consists in the construction of tautologies, *followed* by marginal[39] logical inferences. In the case of arithmetic the separation achieved by use of the no-counterexample-interpretation in (ii) above seems to me more complete than that of the 'cut free' system §6a, where, for $\alpha < \epsilon_0$, one applies the rule of α-induction also to quantified formulae: these inferences would have to be counted as 'mathematical'. I believe it would be worthwhile to formulate rigorously the distinction behind this impression.

(b) *Functionals of finite type.* Naively, one thinks of functionals of lowest type,

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for variables s_1, s_2, t. Since, by hypothesis, the relations $A^*, \bar{A}^*, \bar{\bar{A}}^*$ are decidable, the operations $\wedge$ and $\rightarrow$ are unproblematic truth functional operations.

The reader may wish to apply the interpretation to examples in the theory of (uniformly) continuous functions. If C(f) denotes

$$(\forall \epsilon > 0)(\exists \delta > 0)\forall x \forall x'[(|x - x'| < \delta \wedge 0 \leq x < x' \leq 1) \rightarrow |fx - fx'| < \epsilon]$$

we have $C(f) \rightarrow \exists y \forall x(0 \leq x \leq 1 \rightarrow fx \leq y)$. But y cannot be computed from f, i.e., from a rule for computing f arbitrarily closely. However, f *together* with a modulus $\delta(\epsilon)$ of continuity does allow the computation. Now the *interpretation* of the formula C(f) consists precisely of this extra information. In other words, a continuous function is regarded as a pair (f, δ) for the purposes of the theorem on the boundedness of continuous functions in closed intervals.

[39] 'Trivial' in the sense that for the natural rules needed in this inference, *derivability* is a decidable relation. Recall that Herbrand's or Gentzen's rules fall naturally into two parts: those used for constructing suitable tautologies, and those allowing quantificational inferences and contractions of disjunctions, inferring $A \vee B$ from $A \vee A \vee B$. Clearly, *given* the propositional formula P_1 and a (possibly) quantified formula P_2, one can decide whether P_2 can be derived from P_1 by means of the second set of rules. The step from $\exists s \forall t A^*(s, t)$ to A is similar to this second kind of inference.

as used in the no-counterexample-interpretation, as more 'elementary' than those of finite type. Now, on the one hand, by [24, p. 122, 5.32], there are in any case severe limitations to extending this interpretation beyond arithmetic or ramified analysis. But also, formal properties of functionals needed for the no-counterexample-interpretation according to the list in [59] (cf. the review of [60]), are *evidently* satisfied only for a somewhat abstract kind of functional, namely those introduced in terms of Brouwer's notion of ordinal.[40] In contrast, Gödel describes in [12] a more[40] elementary kind of functional of *all* finite types satisfying the properties needed for *his* interpretation of arithmetic. (This interpretation was mentioned in footnote 10. The reader is referred to [12] for the rules associating A^* with A, A^* being A_1 of footnote 10.)

The formal extension of Gödel's translation in [12] to the language of analysis is perfectly routine [24]. Using §9 it is easy to list the properties of functionals needed to interpret (DC_{00}) and (DC_{11}), namely bar recursion of all finite types, first introduced by Spector [58]. But nobody knows a constructive class of functionals which has these properties (and does not assume species of natural numbers)![40] At the present time, the only philosophically satisfactory use of this extension is its specialization to the subsystem considered in §10 which uses bar recursion of lowest type, that is Brouwer's notion of ordinal (as mentioned above, this is exactly the same notion which *evidently* possesses the properties needed for the no-counterexample-interpretation of *arithmetic*!) However, we have some useful conservative extension results.

(i) The *principal lemma* (cf. [24, p. 120]) concerns arithmetic or analytic formulae A built up using only $(\neg, \wedge, \forall)$:

For any class of functionals (of all finite types) *satisfying*, classically, *the quantifier free axiom of choice* (QF-AC)

$$\forall s\exists t Q(s, t) \rightarrow \exists T\forall s Q(s, Ts) \text{ for quantifier-free } Q$$

we have: $A \leftrightarrow \exists s\forall t A_1(s, t)$, where A_1 is the relation associated with A.

The corresponding result for the full axiom of choice or, equivalently, for purely universal Q, is immediate. The reduction to (QF–AC) is important, because (QF–AC) is satisfied by natural classes of continuous and even 'recursively continuous' functionals while (AC) itself is not.

Thus, from the point of view of recursiveness the present interpretation is more satisfactory than the no-counterexample-interpretation since, as observed above, the *steps* in the proof of $A \leftrightarrow \exists s\forall t A_0(s, t)$ do not relativize to the relevant classes of recursive functions.[41]

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[40] This matter is analyzed in detail in my Amsterdam lecture, cited in footnote 1. On this analysis, Gödel's notion described in [12] is essentially of the same strength as the predicative hierarchy Note Va. (Bar recursion of arbitrary finite type is certainly satisfied by a kind of functional defined by use of arbitrary species of natural numbers: but then one has a much easier reduction of classical analysis, as in §11a.)

[41] Let A' be the formula $\neg\forall x_1\exists y_1 \cdots \forall x_n\exists y_n \neg A_0(x, y)$ in the notation at the beginning of this note, and let A'' be

$$\neg\forall x_1 \neg \forall y_1 \cdots \neg \forall x_n \neg \forall y_n A_0(x, y).$$

Then the principal relations between [12] and the no-counterexample-interpretation are:

(ii) In the case of arithmetic, we get *conservative extension* results [86] very easily by observing that the effective operations of [24] satisfy both (QF–AC) and the axioms about functionals needed[42] for the interpretation [12]. Similarly, one sees that a formula A in the fragment $(\neg, \wedge, \forall)$ is derivable in classical arithmetic if and only if, for some s_A, $A_1(s_A, t)$ is derivable in [12].[42]

(iii) In the case of analysis, one gets corresponding conservative extension results, not however for the system (CA) but for (DC_{11}). (Evidently, by the conservative extension results of Note VI for Σ_3^1 formulae, we get partial conservative extension results of the system of bar recursion of finite type with respect to (CA). Here one uses the continuous (instead of effective) operations of [24].

(c) *Models and interpretations for formal independence proofs.* As far as, e.g., *constructive* independence proofs are concerned models are, by the nature of the case, useless; at best they serve as an auxiliary for a *relative* independence proof. (For an exception see [95] where very weak subsystems of arithmetic with quite constructive nonstandard models are treated.) Also, once we have an interpretation we can *manufacture* an independence problem which is instantaneously solved by means of the interpretation and, perhaps, not accessible to natural model theoretic methods, for instance the computability of the functionals used in the interpretation of a system $\mathscr{S}$ cannot be formally derived in $\mathscr{S}$.

Beside these obvious uses of interpretations there appear to be *potential* technical ones for formal independence proofs. We shall review three of them (which I mentioned in [22] or [83]; the reader may prefer to skip the discussion unless he actually plans to read those papers).

(i) To establish model theoretically the independence of a formula B from the set $\mathscr{A}$, we have to give a model for $\mathscr{A}$ which does not satisfy B. If we write $\mathscr{A}$ and B in Hilbert's ϵ-calculus, our model will satisfy *every* derivation from $\mathscr{A}$ (in the sense of assigning values to all the ϵ-terms appearing in the derivation). Interpretations are more *flexible* because they assign to every derivation d a model for its ϵ-terms, depending on the particular d. (For an elaboration, see [60, paragraph 5.1, pp. 184–185]).

But though this switch allows a more constructive (elementary) description of the models involved, it does not seem to make it *easier* to find them explicitly. Compare here the familiar covering theorem: Let a_n be a sequence of intervals on the real line, and $\Sigma|a_n| < 1$, $|a_n|$ denoting the length of a_n. True, for each N, there is a rational ξ_N in [0, 1] such that $\xi_N \notin \bigcup_{n \le N} a_n$, while there need be no rational ξ such that $\xi \notin \bigcup a_n$ (and no recursive real ξ even if the sequence a_n is a recursive sequence of

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A_1' and A_0 are *identical*; there are functionals Φ and Ψ of [12] such that $A_1''[s, \Psi(s, t)] \to A_1'[\Phi(s), t]$ (since $A'' \to A'$ holds intuitionistically). Since, for classical arithmetic, A, A′, A″ are all equivalent, we do not only get: $\mathrm{Prov}(p, \ulcorner A \urcorner) \to A_0(\xi_{\nu(p)}, t)$ but even $\mathrm{Prov}(p, \ulcorner A \urcorner) \to A_1''(\xi_{\nu''(p)}, t)$.

[42] There is an ambiguity, stressed in [98], in the formulation of the formal system T of [12]: do we have $s = t \vee \neg s = t$ for s and t which are *not* numerical valued? The extension is obviously conservative with respect to formulae in the fragment $(\neg, \wedge, \forall)$ since $\neg\neg(s = t \vee \neg s = t)$ is a theorem. But [98] also shows that the extension is conservative with respect to arbitrary formulae in *intuitionistic* arithmetic even when quantification rules are added to T.

rational intervals). But this does not mean it is hard to *define* a ξ in $[0, 1] - \bigcup a_n$. In fact, we get ξ automatically if we know $\Sigma|a_n| < 1$; and, in any case, we often know enough about a ξ to conclude $\xi \notin \bigcup a_n$, even if ξ is not (known to be) recursive.

Another similar switch is involved in footnote 22; instead of defining a model for ZF in which, say, the continuum hypothesis (CH) does not hold, one gives a method of defining models for the first n axioms ZF_n of ZF in which (CH) is false. This makes the independence proof of (CH) more elementary, not easier to find.

(ii) Models of quite weak systems of set theory, e.g., Gödel-Bernays without the axiom of infinity, are necessarily nonrecursive, and so are models useful for independence proofs in arithmetic, i.e., nonstandard models. In contrast, the functionals used in interpretations of much stronger systems are recursive (the functionals needed for the interpretation of a system $\mathscr{S}$ lie in a *sub-class* $R_{\mathscr{S}}$ of the recursive functions). Thus one cannot prove a (recursive) set X to be infinite unless there is a function $f \in R_{\mathscr{S}}$ such that f(n) exceeds the nth element of X. In other words, knowledge about the density of such *X could* lead to formal independence results.

But, at present, we know far too little about $R_{\mathscr{S}}$ (or about the density of such sets X as the set of primes n for which n + 2 is prime) to *apply* this observation. More generally, a nonrecursive model is often no harder to manipulate than a nonrecursive ξ in the case of the covering theorem above.

(iii) In the case of arithmetic, nonstandard models must go *beyond* the natural numbers, i.e., the intended model, while interpretations *cut down* the intended class of functionals.

But since there are other things, on earth, if not in heaven, than the intended model of arithmetic, there is no obvious practical advantage in (iii). The situation reminds one of examples (i) and (ii) in Note Va: in (ii) one cannot use the minimum (intended) ω-model of the system (Δ^1_1-CA) considered, while in (i) one does. But finding a suitable formal definition of this model for the solution of (i), is not much easier than solving (ii).[43]

[43] Friedman [74] presents his proof of (ii) as a combination of a (model theoretic) definition δ and an application of Gödel's second theorem as follows: he finds a proposition S such that (Σ^1_1-DC_{11}) $\cup$ {S} has a model, for instance the principal model, and shows that δ enumerates a (Σ^1_1-DC_{11}) $\cup$ {S}-model of (Σ^1_1-AC_{01}) $\cup$ {S}. By Gödel's second theorem, (Σ^1_1-DC_{11}) cannot be derivable from (Σ^1_1-AC_{01}) since otherwise (Σ^1_1-AC_{01}) $\cup$ {S} would prove its own consistency. So, in terms of models, there is *some* model M of (Σ^1_1-DC_{11}) such that the structure M′ defined by δ in M is not a model of (Σ^1_1-DC_{11}). But to get a 'truly' model theoretic result in the sense of [36], one would like to replace M′ by something more explicit. Suppose then that M_0 is *any* model of (Σ^1_1-DC_{11}) $\cup$ {S}, for instance the principal model, and define the sequence $M_1, M_2, \cdots$ as follows: If M_n is a model of (Σ^1_1-DC_{11}) $\cup$ {S}, let M_{n+1} be the structure defined when the variables in δ range over M_n; stop if M_n is not a model of (Σ^1_1-DC_{11}).

CONJECTURE. The sequence $M_1, M_2, \cdots$ terminates for *all* M_0. (Perhaps not even M_1 is a model of (Σ^1_1-DC_{11})!)

It might be remarked that the schema above may be suitable for a model theoretic proof of Gödel's second theorem itself. For by Note IIb, for any formula (axiom) A, we find effectively δ_A which defines the truth predicate for a model of A in first order arithmetic Z. Suppose now that Z is derivable from A, i.e., A 'contains' arithmetic, and suppose that $A \vdash$ Con A. Then for any model M of A, δ_A would define a model of A in M.

Summarizing the present situation we have three basic methods for independence proofs in arithmetic and weak systems of analysis: Gödel's diagonal method and its extensions, e.g., in [36]; interpretations and other proof theory leading to bounds on 'provable' ordinals as in §8; and model theoretic constructions. As pointed out in [36] (last paragraph of the introduction), it is unreasonable to be fanatic about the latter; but to counterbalance [36], let me conclude with some examples where these constructions are very much more attractive. In the review of [21] there is a model theoretic reduction to arithmetic of the subsystem of analysis where the comprehension axiom is replaced by the assertion: a finitary infinite tree has an infinite path; I much prefer this proof to my original one in [86]. Or again Friedman's analysis in [74] of (Δ_1^1-CA) using, from proof theory, only the *existence* of a model of (Δ_1^1-CA) in which ϵ_0-induction fails, is more sparkling than the alternative in footnote 30.

[22] appeared ten years ago. It will be interesting to see whether any of the properties (i)–(iii) of interpretations will find a use in the next ten years.

§VIII. Fragments of propositional and predicate calculus: a complement to §13. The sets of propositional operations considered here are not functionally complete; nor are all the 'usual' quantifiers such as $\forall$, $\exists$, or $\exists_\alpha$ of [80] and [100] included. However, the *meaning* of the logical operations is the usual set theoretic one. Fragments have been considered in the literature [77], with the requirement that *implication* be included. This is obviously not needed from the model theoretic point of view. It was connected with the habit of formalization by means of the rule of detachment (which is formulated by use of $\rightarrow$); it is not needed for Gentzen style formalizations which are in any case better adapted for proof theory.

Before reporting a few neat results in this subject, it is as well to say a word about its interest. First, granted that the choice of language was a principal and difficult problem in §13, it seems not unreasonable to experiment with 'sub' systems in simpler situations than the infinitary languages studied in §13: this prepares one for the technically more complicated subject. (From this point of view, our fragments are not fundamental, because, by hypothesis, they only *illustrate* points whose interest derives from the more complicated situation.) Second, there is a subject, as

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CONJECTURE. For *every* M_0, we can show by a purely model theoretic construction that the sequence $M_1, M_2, \cdots$ defined as above, terminates.

Added in proof. The first conjecture above is established by a slight modification of Friedman's proof of the following result. *Let* P *be an arithmetic relation* (between number theoretic functions) satisfying $\forall f \forall g \forall h([P(f, g) \wedge P(f, h)] \rightarrow g \equiv h)$. *Then there is no sequence* f_n *such that*, for each $n = 1, 2, \cdots, f_n$ *enumerates* (via an arithmetic pairing function) *an* ω*-model of* (Π_1^0-CA) *and* $P(f_n, f_{n+1})$ *holds*. Obviously the result holds therefore for an arbitrary extension of (Π_1^0-CA). For the case of *general models* I can prove the second conjecture for set theories A by use of the following special properties of the traditional definition δ_A: (i) δ_A takes the left most path in a certain primitive recursive finitary graph, (ii) the integers of M_{n+1} are *nonstandard* with respect to those of M_n, (iii) there is a specific (formula with) number n_A which, for all n, has different truth values in M_n and M_{n+1}. If a level of the graph, where n_A occurs, has width m_A our sequence $M_0, \cdots, M_m$ terminates for some $m \leq m_A$ since, by (ii), the 'left most' path in M_{n+1} is to the right of the 'left most' path in M_n.

yet wholly unexplored, of formulating the mathematical methods used in proof theory abstractly. One has the impression that cut elimination involves some quite general combinatorial, perhaps lattice theoretic principle; moreover such superficially different matters as the elimination of variables of type higher than τ from the definition of functionals of type τ (see Note VII) seem to involve 'similar' ideas. It is good to have *examples* on which one can test one's abstract formulation, and, I believe, fragments may serve a useful purpose, where one has a different language and not different rules (in contrast to the case of intuitionistic systems). In any case fragments provide attractive exercises; and when working them out the student automatically acquires the precise knowledge of the language of ordinary predicate calculus which is essential.

(a) *Propositional operations.* (i) It is obvious that every fragment has a complete set of rules with subformula property, i.e., cut free rules. A possible application of this result, to the second point above, is this: given an abstract formulation of cut elimination, we may ask whether it applies to the rules for fragments. (Note that the routine proof only yields completeness; not instructions for cut elimination.)

(ii) The interpolation lemma holds in an arbitrary fragment; see Ex.2 of [35], essentially due to F. Ville. In a rather trivial way, the definability theorem does not! Specifically, take a fragment in which $\top$ is not definable. The definability theorem requires, if (Ap, Ap$'$, p $\vdash$ p$'$), i.e., if *p* is *implicitly* defined by A, that p be *explicitly* defined; but for Ap $=_{\text{def}}$ p, this would be $\top$ itself. As it happens, this case presents the *only* exception since A($\top$) itself is an explicit definition of p.

The possible applications of this observation, i.e., the *sensitive* dependence of the interpolation and definability theorem on the exact choice of language, have already been mentioned in §13.

(iii) Reznikoff [93] shows, for instance, that, in the fragment consisting of $\rightarrow$ only, every set of formulae of cardinality $\leq \aleph_1$ has an independent axiomatization (in this fragment), but gives a set of cardinal $\aleph_2$ which does not. On the other hand, independent axiomatizability is not tied to a *functionally complete* set of operations, since every fragment *including* ($\wedge$, $\rightarrow$) permits independent axiomatizations. (As he remarks his result is not very strong because ($\wedge$, $\rightarrow$) is the only *proper* fragment of this kind.)

(b) *Predicate calculus.* The positive result about independent axiomatizability extends to the fragment ($\forall$, $\exists$, $\wedge$, $\rightarrow$), but the *full* extension of the other results in (a) is not known.

Incidentally, the *absence of prenex normal forms*, so familiar from the theory of infinitary languages (or intuitionistic logic), is illustrated by the fragment ($\forall$, $\exists$, $\leftrightarrow$).

Without wishing to exaggerate matters, I get the impression from the results above that fragments of classical logic are not without technical interest.

The subject of fragments is a quite *central* issue for intuitionistic logic, since even the usual propositional calculus is a fragment. It is therefore not proper to enter on this subject here.

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