3. *On a generalized logic calculus.*

By

Gaisi TAKEUTI.

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In this paper we shall give a generalization of the logic calculus, called LK by Gentzen in his dissertation (1), and prove some metatheorems in this generalized logic calculus, which will be denoted by GLC. The exact definition of GLC will be given in Chapter I, and the metatheorems will be proved in Chapter II. Most of these metatheorems are intended to formalize the common feeling of mathematicians such as: A consistent system remains consistent if the notions of sets, functions, etc. are added therein. Of course such "feeling" needs careful and strict logical analysis, and is to be proved under due formulation. Our GLC seems to be an adequate logical system to such a formulation.

As is well-known, Gentzen has proved the fundamental theorem, that any provable sequence in LK is provable without cut, by means of which he has succeeded in proving the consistency of the theory of numbers. The author is now unable to answer whether a corresponding fact is valid in GLC, or even in G^1LC (see Appendix). But we can easily see that, if it is valid in GLC (or even in G^1LC), the consistency of the theory of real numbers would immediately follow. Therefore the author would like to propose, though it would seem very bold, the assertion that any provable sequence in GLC (or in G^1LC) is provable without cut, as his fundamental conjecture.

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Contents.

Chapter I. Formalization of Generalized Logic Calculus.

§ 1. **Symbols.** We shall develop and generalize the logic system of G. Gentzen (1), with some modifications. For that purpose we shall first arrange the various symbols and then define the various notions, such as *formula*, *proof-figure* etc., recursively as the combinations of symbols.

1.1. Variables.

1.1.1. Free variables of type (0) (free variables without an argument-place) ; $a, b, c, \cdots\cdots$.

1.1.2. Bound variables of type (0) (bound variables without an argument-place) ; $x, y, z, \cdots\cdots$.

1.1.3. Special variables of type (0) (special variables without an argument-place) ; $1, \omega, \phi, \cdots\cdots$.

1.1.4. Free variables of type $(n_1, \cdots\cdots, n_i)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $\alpha(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i]$, $\beta(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i], \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the *k-th argument-place* of $\alpha(n_1, \cdots\cdots n_i)[*_1, \cdots\cdots, *_i]$.

1.1.5. Bound variables of type $(n_1, \cdots\cdots, n_i)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $\varphi(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i]$, $\psi(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i], \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the k-th argument-place of $\varphi(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i]$.

1.1.6. Special variables of type $(n_1, \cdots\cdots, n_i)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $\sigma(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i]$, $\tau(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i], \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the k-th argument-place of $\sigma(n_1, \cdots\cdots, n_i)[*_1, \cdots\cdots, *_i]$.

Special variable to type $(1, 1, \cdots\cdots, 1)$ is called a '*predicate*', and the predicates $= (1, 1)[*_1, *_2]$, $< (1, 1)[*_1, *_3]$, $\in (1, 1)[*_1, *_2]$ are denoted as usual also by $*_1 = *_2$, $*_1 < *_2$, $*_1 \in *_2$ respectively.

1.2. Functions.

1.2.1. Free functions of type $(n_1, \cdots\cdots, n_i)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $f(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$, $g(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i), \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the k-th argument-place of $f(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$.

1.2.2. Bound fnnctions of type $(n_1, \cdots\cdots, n_*)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $p(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$, $q(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i), \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the k-th argument-place of $p(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$.

1.2.3. Special functions of type $(n_1, \cdots\cdots, n_i)$ $(i, n_1, \cdots\cdots, n_i = 1, 2, 3, \cdots\cdots)$; $s(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$, $t(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i), \cdots\cdots$.

$*_k (1 \leqq k \leqq i)$ is called the k-th argument-place of $s(n_1, \cdots\cdots, n_i)(*_1, \cdots\cdots, *_i)$.

Special functions $+ (1, 1)(*_1, *_2)$, $\times (1, 1)(*_1, *_2)$, are also denoted by $*_1 + *_2$, $*_1 \times *_2$ respectively, as usual.

1.3. Logical symbols ; $\neg$, $\wedge$, $\vee$, $\forall$, E ; $\neg$, $\wedge$, $\vee$, $\forall$, E are read 'not', 'and', 'or', 'all', 'there is a' respectively.

§ 2. **Varieties and Formulas.** We shall consider, hereafter, a row of symbols, which is called a *figure*. First as the important notion of it we shall concepts of *varieties* and *formulas*.

We define the concepts of varieties and formulas as combination of symbols. They are defined recursively as follows :

2.1. Every free variable of type (0) is a variety of type (0).

2.2. Every special variable of type (0) is a variety of type (0).

2.3. Let $\alpha(n_1+1, \cdots\cdots, n_i+1)[*_1, \cdots\cdots, *_i]$ be a free variable of type $(n_1+1, \cdots\cdots, n_i+1)$.

If we put a variety of type (n_k) V^k into the k-th argument-place of $\alpha(n_i+1, \cdots\cdots, n_i+1)[*_1, \cdots\cdots, *_i]$ for all k, then the variable becomes a 'formula' $\alpha(n_i+1, \cdots\cdots, n_i+1)[V^1, \cdots\cdots, V^i]$.

2.4. Let $\sigma(n_i+1_i \cdots\cdots, n_i+1)[*_1, \cdots\cdots, *_i]$ be a special variable of type $(n_1+1, \cdots\cdots, n_i+1)$.

If we put a variety of type (n_k) V^k into the k-th argument-place of $\sigma(n_1+1, \cdots\cdots, n_i+1)[*_1, \cdots\cdots, *_i]$ for all k, then the variable becomes a 'formula' $\sigma(n_1+1, \cdots\cdots, n_i+1)[V^1, \cdots\cdots, V^i]$.

2.5. Let $f(n_1+1, \cdots\cdots, n_i+1)(*_1, \cdots\cdots, *_i)$ be a free or a special function of type $(n_1+1, \cdots\cdots, n_i+1)$.

If we put a variety of type (n_k) V^k into k-th argument-place of $f(n_1+1, \cdots\cdots, n_i+1)(*_1, \cdots\cdots, *_i)$ for all k, then the function becomes a variety of type (0).

2.6. Let $\alpha^k(n_k)[*_1]$ $(1 \leqq k \leqq i)$ be a free variable of type (n_k) assumed that $\alpha^j(n_j)[*_1]$ is not $\alpha^k(n_k)[*_1]$, when $j \neq k$ holds. Moreover, let $\varphi^k(n_k)[*_1]$ $(1 \leqq k \leqq i)$ be a bound variable of type (n_k) which is not contained in a formula $\mathfrak{A}$ assumed that $\varphi^i(n_j)[*_1]$ is not $\varphi^k(n_k)[*_1]$, when $j \neq k$ holds.

If we substitute $\varphi^k(n_k)[*_1]$ for $\alpha^k(n_k)[*_1]$ in a formula $\mathfrak{A}$ for all k $(1 \leqq k \leqq i)$ at all places where $\alpha^k(n_k)[*_1]$ are, and add $\{\varphi^1(n_1), \cdots\cdots, \varphi^i(n_i)\}$ in front of the so constructed formula, then we obtain a variety of type $(n_1+1, \cdots\cdots, n_i+1)$.

2.7. If $\mathfrak{A}$ and $\mathfrak{B}$ are formulas, then $\neg\mathfrak{A}$, $\mathfrak{A} \wedge \mathfrak{B}$, and $\mathfrak{A} \vee \mathfrak{B}$ are formulas.

2.8. Let $\varphi(n_1, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$ be a bound variable of type $(n_1, \cdots\cdots, n_i)$ which is not contained in a formula $\mathfrak{A}$.

If we substitute $\varphi(n_1, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$ for a free variable of type $(n_1, \cdots\cdots, n_i)$ $\alpha(n_1, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$ in a formula $\mathfrak{A}$ at all places where $\alpha(n_i, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$ are, and add $\mathrm{V}\varphi(n_1, \cdots\cdots, n_i)$ or $\mathrm{E}\varphi(n_1, \cdots\cdots, n_i)$ in front of the so constructed formula, then we obtain a formula.

Hereafter we shall think that a variable of type (0) is a special case of a variable of type $(n_1, \cdots\cdots, n_i)$.

2.9. Let $p(n_1, \cdots\cdots, n_i)(\ , \cdots\cdots, \ )$ be a bound function of type $(n_1, \cdots\cdots, n_i)$ which is not contained in a formula $\mathfrak{A}$.

If we substitute $p(n_1, \cdots\cdots, n_i)(\ , \cdots\cdots, \ )$ for a free function of type $(n_1, \cdots\cdots, n_i)$, $f(n_1, \cdots\cdots, n_i)(\ , \cdots\cdots, \ )$ in $\mathfrak{A}$ at all places where $f(n_1, \cdots\cdots, n_i)(\ , \cdots\cdots, \ )$ are, and add $\mathrm{V}p(n_1, \cdots\cdots, n_i)$ or $\mathrm{E}p(n_1, \cdots\cdots, n_i)$ in front of the so constructed formula, then we obtain a formula.

2.10. A variety of type (0) is often called a 'term'.

2.11. Type n may be written for type (n), in the sequel.

2.12. If no confusion is likely to occur, the notation may be abbreviated by the following conventions; we may write α, $\alpha[\ , \cdots\cdots, \ ]$, $\alpha(n_1, \cdots\cdots, n_i)$, φ, $\varphi[\ , \cdots\cdots, \ ]$, $\varphi(n_1, \cdots\cdots, n_i)$: $f, f(\ , \cdots\cdots, \ )$, $f(n_1, \cdots\cdots, n_i)$; $p, p(\ , \cdots\cdots, \ ), p(n_1, \cdots\cdots, n_i)$ respectively for $\alpha(n_1, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$; $\varphi(n_1, \cdots\cdots, n_i)[\ , \cdots\cdots, \ ]$; $f(n_1, \cdots\cdots, n_i)(\ , \cdots\cdots, \ )$ $p(n_1, \cdots\cdots n_i)(\ , \cdots\cdots, \ )$ etc.

2.13. The 'outermost symbol' of a formula is the logical symbol which is added at the end of the construction of the formula, if it exists.

2.14. The 'outermost variable' of a formula of the form $\alpha\,[V_1,\cdots\cdots,V_i]$ indicates the variable used at the final stage in the construction of this formula.

2.15. The 'outermost function' of a variety of the form $f\,(V_1,\cdots\cdots,V_i)$ indicates the function used at the final stage in the construction of this variety.

2.16. A formula is called normal, when it has neither $\forall$ which is contained in an argument-place of a variable or a function, nor E, and when its logical symbols coming later than $\forall$ in the construction of the formula are only $\forall$.

§ 3. **Several Notations.**

3.1. $\mathfrak{F}(\alpha\,(n_1,\cdots\cdots,n_i))$ denotes a formula which has a free variable of type $(n_1,\cdots\cdots,n_i)$ $\alpha\,(n_1,\cdots\cdots,n_i)\,[\ ,\cdots\cdots,\ ]$ in some indicated places. The indicated places need not be all the places where $\alpha(n_1,\cdots\cdots,n_i)\,[\ \cdots\cdots,\ ]$ are. Such a notation is used even if no place is indicated.

$\mathfrak{F}(\alpha^1(n_1,\cdots\cdots,n_i),\cdots\cdots,\alpha^k(m_1,\cdots\cdots,m_j))$ denotes a formula with free variables $\alpha^1(n_1,\cdots\cdots,n_i),\cdots\cdots,\alpha^k(m_1,\cdots\cdots,m_j)$ at some indicated places.

Similarly we use the following notations $\mathfrak{F}(f\,(n_1,\cdots\cdots,n_i))$, $\mathfrak{V}(f\,(n_1,\cdots\cdots,n_i))$ etc. as to the free function $f\,(n_1,\,\cdots\cdots.\,n_i)\,(\ ,\cdots\cdots,\ )$.

We can substitute some variables $\widetilde{\alpha}\,(h_1,\cdots\cdots,h_\nu),\cdots\cdots,\widetilde{\beta}\,(l_1,\,\cdots\cdots,l_\mu)$ for $\alpha\,(h_1,\cdots\cdots,h_\nu),\cdots\cdots,\beta\,(l_1,\cdots\cdots,l_\mu)$ at their indicated places in the formcla $\mathfrak{A}(\alpha\,(h_1,\cdots\cdots,h_\nu),\cdots\cdots,\beta(l_1,\cdots\cdots,l_\mu))$; the result is denoted by $\mathfrak{A}\,(\widetilde{\alpha}(h_1,\,\cdots\cdots,h_\nu),\cdots\cdots,\widetilde{\beta}\,(l_1,\cdots\cdots,l_\mu))$.

Analogous notations will also be used in making substitution of a variety or a function.

We shall often omit, if no confusion is alike, the type symbol ; thus, e.g. $\mathfrak{A}(\alpha\,(h_1\,\cdots\cdots,h_\nu),\cdots\cdots)$ may be abbreviated by $\mathfrak{A}(\alpha,\cdots\cdots)$ and $\forall\varphi\,(h_1,\cdots\cdots,h_\nu)\,\mathfrak{F}(\varphi\,(h_1,\cdots\cdots h_\nu))$ may be written by $\forall\varphi\mathfrak{F}(\varphi\,(h_1,\cdots\cdots h_\nu))$ or $\forall\varphi\mathfrak{F}(\varphi)$ etc..

3.2. Functional.

If $T(\alpha^1(n_1),\cdots\cdots,\alpha^i(n_i))$ is a term and $\varphi^1(n_1),\cdots\cdots,\varphi^i(n_i)$ are bound variables which are not contained in $T(\alpha^1(n_1),\cdots\cdots,\alpha^i(n_i))$, then the form $\{\varphi^1(n_1),\cdots\cdots,\varphi^i(n_i)\}\ T(\varphi^1(n_1),\cdots\cdots,\varphi^i(n_i))$ is called a '*functional*' of type $(n_1+1,\cdots\cdots,n_i+1)$. $\mathfrak{L}\,(\alpha\,(n_1,\cdots\cdots,n_i),\cdots\cdots,\beta\,(m_1,\cdots\cdots,m_j))$ denotes a functional which has free variables $\alpha\,(n_1,\cdots\cdots,n_i),\cdots\cdots,\beta(m_1,\cdots\cdots,m_j)$ at some indicated places.

The notation $\mathfrak{L}\,(\widetilde{\alpha}\,(n_1,\cdots\cdots,n_i),\cdots\cdots,\widetilde{\beta}(m_1,\cdots\cdots,m_j))$ and other notations $\mathfrak{L}(f\,(n_1,\cdots\cdots,n_i),\cdots\cdots,g(m_1,\cdots\cdots,m_j))$ etc. are used in the same way as in 3.1.

3.3. Let $A(\alpha)$ be a formula or a variety or a functional.

We define that $A(\alpha)$ is of a full indication, if and only if the α's which are contained in $A(\alpha)$ are indicated in $A(\alpha)$.

§ 4. **The concept 'homologous'.**

We define recursively as follows that a formula $\mathfrak{B}$ is homologous to a formula $\mathfrak{A}$ and that a variety V_2 is homologous to a variety V_1.

4.1. If $\mathfrak{A}$ contains neither a bound variable nor a bound function, then $\mathfrak{B}$ is $\mathfrak{A}$ itself.

Similarly, if V_1 contains neither a bound variable nor a bound function, then V_2 is V_1 itself.

4.2. If $\mathfrak{A}$ is of the form $\neg\mathfrak{A}_1$, $\mathfrak{A}_1\wedge\mathfrak{A}_2$ or $\mathfrak{A}_1\vee\mathfrak{A}_2$. then $\mathfrak{B}$ is accordingly of the form $\neg\mathfrak{B}_1$, $\mathfrak{B}_1\wedge\mathfrak{B}_2$ or $\mathfrak{B}_1\vee\mathfrak{B}_2$, where $\mathfrak{B}_i$ is homologous to $\mathfrak{A}_i$.

4.3. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, $E\varphi\mathfrak{F}(\varphi)$, $\forall q\mathfrak{G}(q)$, $Eq\mathfrak{G}(q)$, then $\mathfrak{B}$ is accordingly of the form $\forall\psi\widetilde{\mathfrak{F}}(\psi)$, $E\psi\widetilde{\mathfrak{F}}(\psi)$, $\forall p\widetilde{\mathfrak{G}}(p)$, or $Ep\widetilde{\mathfrak{G}}(p)$, where $\widetilde{\mathfrak{F}}(\alpha)$ is homologous to $\mathfrak{F}(\alpha)$ and $\widetilde{\mathfrak{G}}(f)$ is homologous to $\mathfrak{G}(f)$.

(α and f must not be contained in $\mathfrak{A}$ and ψ must not be contained in $\widetilde{\mathfrak{F}}(\alpha)$, and φ, ψ, α are variables of the same type, and q, p, f are functions of the same type.)

4.4. If $\mathfrak{A}$ is of the form $\alpha[V^1,\cdots\cdots, V^i]$ or $\sigma[V^1,\cdots\cdots, V^i]$, then $\mathfrak{B}$ is accordingly of the form $\alpha[\widetilde{V}^1,\cdots\cdots, \widetilde{V}^i]$ or $\sigma[\widetilde{V}^1,\cdots\cdots, \widetilde{V}^i]$, where each $\widetilde{V}^k$ is homologous to V^k.

4.5. If V_1 is of the form $f(V^1,\cdots\cdots, V^i)$ or $s(V^1,\cdots\cdots, V^i)$, then V_2 is of the form $f(\widetilde{V}^1,\cdots\cdots, \widetilde{V}^i)$ or $s(\widetilde{V}^1,\cdots\cdots, \widetilde{V}^i)$, where each $\widetilde{V}^k$ is homologous to V^k.

4.6. If V_1 is of the form $\{\varphi^1,\cdots\cdots, \varphi^i\}\mathfrak{F}(\varphi^1,\cdots\cdots, \varphi^i)$, then V_2 is of the form $\{\psi^1,\cdots\cdots, \psi^i\}\widetilde{\mathfrak{F}}(\psi^1,\cdots\cdots, \psi^i)$, where $\widetilde{\mathfrak{F}}(\alpha^1,\cdots\cdots, \alpha^i)$ is homologous to $\mathfrak{F}_1(\alpha^1,\cdots\cdots,\alpha^i)$. ($\alpha^1,\cdots\cdots, \alpha^i$ must not be contained in V_1, and $\psi^1,\cdots\cdots, \psi^i$ must not be contained in $\widetilde{\mathfrak{F}}(\alpha^1,\cdots\cdots, \alpha^i)$, and $\varphi^k, \psi^k, \alpha^k$ are variables of the same type for each k.)

4.7. Next we shall define that a functional $\mathfrak{T}$ is homologous to a functional $\{\varphi^1,\cdots\cdots, \varphi^i\}T(\varphi^1,\cdots\cdots, \varphi^i)$, if and only if $\mathfrak{T}$ is of the form $\{\psi^1\cdots\cdots, \psi^i\}\widetilde{T}(\psi^1,\cdots\cdots, \psi^i)$ and $\widetilde{T}(\alpha^1,\cdots\cdots, \alpha^i)$ is homologous to $T(\alpha^1,\cdots\cdots, \alpha^i)$.

4.8. Clearly we can see the following properties. A is homologous to A itself.

If A is homologous to B, then B is homologous to A.

If A is homologous to B and B is homologous to C, then A is homologous to C.

Let A be homologous to B. According as A is a formula or a variety of type $(n_1,\cdots\cdots, n_i)$ or a functional of type $(n_1,\cdots\cdots, n_i)$, B is a formula or a variety of type $(n_1,\cdots\cdots, n_i)$ or a functional of type $(n_1,\cdots\cdots, n_i)$ respectively.

4.9. According to 4.8 the concept 'homologous' is an equivalence relation. So the figures homologous to a formula or a variety or a functional constitute a 'homology class'.

We shall often confound from now on a figure with its homology class, if no confusion is to be feared.

4.10. Let $A(\alpha)$ be a formula or a variety or a functional and $A^1(\alpha)$ be a figure homologous to $A(\alpha)$. Where $A(\alpha)$ is of a full indication we define that $A^1(\alpha)$ has the same indication as $A(\alpha)$, if and only if $A^1(\alpha)$ is of a full indication.

In general cases, we define that $A^1(\alpha)$ has the same indication as $A(\alpha)$, if and only if $A^1(\beta)$ has the same indication as $A(\beta)$, where β is contained in neither $A(\alpha)$ nor $A^1(\alpha)$. Let $A(u_1,\cdots\cdots, u_n)$ be a formula or a variety or a functional and $A^1(u_1,\cdots\cdots, u_n)$ be a figure homologous to $A(u_1,\cdots\cdots u_n)$ and $u_i(i=1,2,\cdots\cdots, n)$ be a free variable or a free function. In the same way as above we define that $A^1(u_1,\cdots\cdots, u_n)$ has the same indication as $A(u_1,\cdots\cdots, u_n)$ or that $A^1(u_1,\cdots\cdots u_n)$ has the same indication as $A(u_1,\cdots\cdots, u_n)$.

4.11. We shall define a variety $\oplus_n(\alpha(n))$ of type n recursively as follows:

4.11.1. $\oplus_0(a)$ is a.

4.11.2. $\oplus_{n+1}(\alpha(n+1))$ is $\{\varphi(n)\}\alpha(n+1)[\oplus_n(\varphi(n))]$.

We prove by the induction on n that $\oplus_n(\alpha(n))$ is a variety of type n.

It is clear when n is zero according to 2.1.

So we assume that the proposition is proved when n is k. Now, we shall prove the proposition when n is $k+1$.

Then $\oplus_{k+1}(\alpha(k+1))$ is $\{\varphi(k)\}\alpha(k+1)[\oplus_k(\varphi(k)]$.

By the hypothesis of the induction $\oplus_k(\alpha(k))$ is a variety of type k. So $\alpha(k+1)[\oplus_k(\beta(k))]$ is a formula. Therefore $\oplus_{k+1}(\alpha^{(k+1)})$ is a variety of type $k+1$ according to 2.3 and 2.6.

From now on, $\oplus_n(\alpha(n))$ will denote the homology class of $\oplus_n(\alpha(n))$ and an arbitrary variety belonging to it. And $\oplus(\alpha(n))$ may be written for $\oplus_n(\alpha(n))$. and $\beta[\gamma_1, \cdots\cdots, \gamma_i]$ for $\beta[\oplus(\gamma_1), \cdots\cdots, \oplus(\gamma_i)]$.

§ 5. Substitution.

5.1. The 'height' of a figure, which is a variable, a function, a variety or a functional of type $(n_1, \cdots\cdots, n_i)$, is meant by the maximal number of $n_1, \cdots\cdots, n_i$.

5.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be a formula and a variety respectively and V be a variety of the same type as α.

Now we define by the induction on the height of α a complete substitution of V for α in $\mathfrak{A}$ or $\mathfrak{B}$.

The homology class of the so defined figure and the so defined homology class and an arbitrary figure belonging to it are denoted by $\mathfrak{A}\binom{V}{\alpha}$ or $\mathfrak{B}\binom{V}{\alpha}$ respectively.

Moreover we use the following notation: Let $A(\beta)$ be a formula or a variety of the full indication for β and β be not contained in V. We define that $B(\beta)$ is $A(\beta)\binom{V}{\alpha}$ if and only if $B(\beta)$ is the formula $A(\beta)\binom{V}{\alpha}$ of the full indication for β. In general we define that $B(\beta)$ is $A(\beta)\binom{V}{\alpha}$ if and only if $B(\gamma)$ is $A(\gamma)\binom{V}{\alpha}$, where γ is contained in neither $A(\beta)$ nor V and so $A(\gamma)$ is of the full indication for γ. We use often the notation $A\binom{V_1}{\alpha_1}\cdots\cdots\binom{V_n}{\alpha_n}$, which shows the figure $A\binom{V_1}{\alpha}\binom{V_2}{\alpha_2}\cdots\cdots\binom{V_n}{\alpha_n}$.

The height of a complete substitution of V for α is the height of α.

First, we define recursively as follows by the induction on the number of the stages to construct $\mathfrak{A}$ or $\mathfrak{B}$ the substitution in the case where the height of α is zero. In this case we suppose that α is a and V is a term t.

5.2.1. If $\mathfrak{A}$ has no a, then $\mathfrak{A}\binom{t}{a}$ is an arbitary formula homologous to $\mathfrak{A}$.

5.2.2. If $\mathfrak{B}$ has no a, then $\mathfrak{B}\binom{t}{a}$ is an arbitrary variety homologous to $\mathfrak{B}$.

5.2.3. $a\binom{t}{a}$ is an arbitrary term homologous to t.

5.2.4. If $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$, $\mathfrak{B}\wedge\mathfrak{C}$ or $\mathfrak{B}\vee\mathfrak{C}$, then $\mathfrak{A}\binom{t}{a}$ is $\neg\{\mathfrak{B}\binom{t}{a}\}$, $\{\mathfrak{B}\binom{t}{a}\}\wedge\{\mathfrak{C}\binom{t}{a}\}$ or $\{\mathfrak{B}\binom{t}{a}\}\vee\{\mathfrak{C}\binom{t}{a}\}$ respectively.

5.2.5. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, $\mathrm{E}\varphi\mathfrak{F}(\varphi)$, $\forall p\mathfrak{B}(p)$ or $\mathrm{E}p\mathfrak{B}(p)$, then $\mathfrak{A}\binom{t}{a}$ is $\forall\psi\mathfrak{G}(\psi)$, $\mathrm{E}\psi\mathfrak{G}(\psi)$, $\forall q\mathfrak{C}(q)$ or $\mathrm{E}q\mathfrak{C}(r)$ respectively, where $\mathfrak{G}(\alpha)$ is $\mathfrak{F}(\alpha)\binom{t}{a}$ and $\mathfrak{C}(f)$ is $\mathfrak{B}(f)\binom{t}{a}$ and ψ is not contained in $\mathfrak{G}(\alpha)$ and q is not contained in $\mathfrak{C}(f)$.

5.2.6. If $\mathfrak{A}$ is of the form $\alpha[\mathfrak{B}_1, \cdots\cdots, \mathfrak{B}_n]$, then $\mathfrak{A}\binom{t}{a}$ is $\alpha[\mathfrak{B}_1\binom{t}{a}, \cdots\cdots, \mathfrak{B}_n\binom{t}{a}]$.

5.2.7. If $\mathfrak{B}$ is of the form $f(\mathfrak{B}_1 \cdots\cdots, \mathfrak{B}_n)$, then $\mathfrak{B}\binom{t}{a}$ is $f(\mathfrak{B}_1\binom{t}{a}, \cdots\cdots, \mathfrak{B}_n\binom{t}{a})$.

5.2.8. If $\mathfrak{B}$ is of the form $\{\varphi_1, \cdots\cdots, \varphi_n\}\mathfrak{F}(\varphi_1, \cdots\cdots, \varphi_n)$, then $\mathfrak{B}\binom{t}{a}$ is $\{\psi_1, \cdots\cdots, \psi_n\}\mathfrak{G}(\psi_1, \cdots\cdots, \psi_n)$, where $\mathfrak{G}(\alpha_1, \cdots\cdots, \alpha_n)$ is $\mathfrak{F}(\alpha_1, \cdots\cdots, \alpha_n)\binom{t}{a}$ and $\psi_1, \cdots\cdots, \psi_n$ are different from each other and not contained in $\mathfrak{G}(\alpha_1, \cdots\cdots, \alpha_n)$.

We can see clearly the following properties.

5.2.9. If $\mathfrak{A}$ is a formula, then $\mathfrak{A}\binom{t}{a}$ is a formula.

5.2.10. If $\mathfrak{A}$ is a variety of type $(n_1,\cdots\cdots,n_i)$, then $\mathfrak{A}\binom{t}{a}$ is a variety of type $(n_1,\cdots\cdots,n_i)$.

5.2.11. If A^1 is homologous to A and t^1 is homologous to t, then $A^1\binom{t^1}{a}$ is homologous to $A\binom{t}{a}$.

5.2.12. $A\binom{t}{a}$ has neither a free variable nor a free function other than the variables and the functions contained in A or t.

5.2.13. $A\binom{t}{a}\binom{t'}{b}$ is $A\binom{t'}{b}\binom{t\binom{t'}{b}}{a}$, if t' has no a.

Let us suppose that a complete substitution of a variety V of type $(n_1,\cdots\cdots,n_i)$ for $\alpha(n_1,\cdots\cdots,n_i)$ is defined and satisfies the following properties, if the height of α is less than N.

5.2.14. If $\mathfrak{A}$ is a formula, then $\mathfrak{A}\binom{V}{\alpha}$ is a formula.

5.2.15. If $\mathfrak{A}$ is a variety of type $(m_1,\cdots\cdots,m_j)$, then $\mathfrak{A}\binom{V}{\alpha}$ is a variety of type $(m_1,\cdots\cdots,m_j)$.

5.2.16. If A^1 is homologous to A and V^1 is homologous to V, then $A^1\binom{V^1}{\alpha}$ is homologous to $A\binom{V}{\alpha}$.

5.2.17. $A\binom{V}{\alpha}$ has neither a free variable nor a free function other than the variables and the functions contained in A or V.

5.2.18. $A\binom{\oplus(\alpha)}{\alpha}$ is A itself.

5.2.19. $\oplus\alpha\binom{V}{\alpha}$ is V itself.

5.2.20. $A\binom{V}{\alpha}\binom{V'}{\beta}$ is $A\binom{V'}{\beta}\binom{V\binom{V'}{\beta}}{\alpha}$, if V' has no α.

Now we shall define a complete substitution of height N.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be a formula and a variety respectively and α be a free variable, whose height is N, and V be a variety of the same type as α.

The definition of $\mathfrak{A}\binom{V}{\alpha}$ and $\mathfrak{B}\binom{V}{\alpha}$ is performed by the induction on the number of stages to construct $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Without loss of generality we can assume that V is $\{\varphi_1,\cdots\cdots,\varphi_i\}\mathfrak{F}(\varphi_1,\cdots\cdots,\varphi_i)$ and φ_k is of type n_k and α is of type $(n_1+1,\cdots\cdots,n_i+1)$ and so N is the maximal number of $n_1+1,\cdots\cdots,n_i+1$.

5.2.21. If $\mathfrak{A}$ has no α, then $\mathfrak{A}\binom{V}{\alpha}$ is an arbitrary formula homologous to $\mathfrak{A}$.

5.2.22. If $\mathfrak{B}$ has no α, then $\mathfrak{B}\binom{V}{\alpha}$ is an arbitrary variety homologous to $\mathfrak{B}$.

5.2.23. If $\mathfrak{A}$ is or the form $\neg\mathfrak{B}$, $\mathfrak{B}\wedge\mathfrak{C}$ or $\mathfrak{B}\vee\mathfrak{C}$, then $\mathfrak{A}\binom{V}{\alpha}$ is

$$\neg\{\mathfrak{B}\tbinom{V}{\alpha}\},\ \{\mathfrak{B}\tbinom{V}{\alpha}\}\wedge\{\mathfrak{C}\tbinom{V}{\alpha}\}\ \text{or}\ \{\mathfrak{C}\tbinom{V}{\alpha}\vee\{\mathfrak{C}\tbinom{V}{\alpha}\}$$

respectively.

5.2.24. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, $\mathrm{E}\varphi\mathfrak{F}(\varphi)$, $\forall p\mathfrak{B}(p)$ or $\mathrm{E}p\mathfrak{B}(p)$, then $\mathfrak{A}\binom{V}{\alpha}$ is $\forall\psi\mathfrak{G}(\psi)$, $\mathrm{E}\psi\mathfrak{G}(\psi)$, $\forall q\mathfrak{C}(q)$ or $\mathrm{E}q\mathfrak{C}(q)$ respectively, where $\mathfrak{G}(\beta)$ is $\mathfrak{F}(\beta)\binom{V}{\alpha}$ and $\mathfrak{C}(f)$ is $\mathfrak{B}(f)\binom{V}{\alpha}$ and ψ is not contained in $\mathfrak{G}(\beta)$ and q is not contained in $\mathfrak{C}(f)$.

5.2.25. If $\mathfrak{A}$ is of the form $\beta[\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_n]$, where β is not α, then $\mathfrak{A}\binom{V}{\alpha}$ is $\beta[\mathfrak{B}_1\binom{V}{\alpha},\cdots\cdots,\mathfrak{B}_n\binom{V}{\alpha}]$.

5.2.26 If $\mathfrak{A}$ is of the form $\alpha[\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_i]$, then $\mathfrak{A}\binom{V}{\alpha}$ is

$$\mathfrak{F}(\beta_1,\cdots\cdots,\beta_i)\begin{pmatrix}\mathfrak{V}_1\binom{V}{\alpha}\\ \beta_1\end{pmatrix}\cdots\cdots\begin{pmatrix}\mathfrak{V}_i\binom{V}{\alpha}\\ \beta_i\end{pmatrix}.$$

Here the heights of $\beta_1,\cdots\cdots,\beta_i$ are less then N, and so $\begin{pmatrix}\mathfrak{V}_k\binom{V}{\alpha}\\ \beta_k\end{pmatrix}(1\leqq k\leqq i)$ is already defined.

5.2.27. If $\mathfrak{V}$ is of the form $f(\mathfrak{V}_1,\cdots\cdots,\mathfrak{V}_n)$, then $\mathfrak{V}\binom{V}{\alpha}$ is $f(\mathfrak{V}_1\binom{V}{\alpha},\cdots\cdots\mathfrak{V}_n\binom{V}{\alpha})$.

5.2.28. If $\mathfrak{V}$ is of the form $\{\psi_1,\cdots\cdots,\psi_m\}\mathfrak{G}(\psi_1,\cdots\cdots,\psi_m)$, then $\mathfrak{V}\binom{V}{\alpha}$ is $\{\xi_1,\cdots\cdots,\xi_m\}\mathfrak{H}(\xi_1,\cdots\cdots,\xi_m)$, where $\mathfrak{H}(\beta_1\cdots\cdots,\beta_m)$ is $\mathfrak{G}(\beta_1,\cdots\cdots,\beta_m)\binom{V}{\alpha}$ and $\xi_1,\cdots\cdots,\xi_m$ are different from each other and not contained in $\mathfrak{H}(\beta_1,\cdots\cdots,\beta_m)$.

Now we prove the following properties.

5.2.29. If $\mathfrak{A}$ is a formula, then $\mathfrak{A}\binom{V}{\alpha}$ is a formula.

5.2.30. If $\mathfrak{V}$ is a variety of type $(m_1\cdots\cdots,m_j)$, tnen $\mathfrak{V}\binom{V}{\alpha}$ is a variety of type $(m_1,\cdots\cdots,m_j)$.

5.2.31. If A^1 is homologous to A and V^1 is homologous to V, then $A^1\binom{V^1}{\alpha}$ is homologous to $A\binom{V}{\alpha}$.

5.2.32. $\mathfrak{A}\binom{V}{\alpha}$ has neither a free variable nor a free function other than the variables ank the functions contained in A or V.

5.2.33. $A\begin{pmatrix}\oplus(\alpha)\\ \alpha\end{pmatrix}$ is A itself.

5.2.34. $\oplus(\alpha)\binom{V}{\alpha}$ is V itself.

5. 2.35, $A\binom{V}{\alpha}\binom{V'}{\beta}$ is $A\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}$, where V' has no α and $\widetilde{V}$ is $V\binom{V'}{\beta}$.

As other cases are simple, we shall prove 5.2.33, 5.2.34 and 5.2.35. Proof of 5.2.33 and 5.2.34.

We prove at a time 5.2.33 and 5.2.34 by the induction of the number of stages to construct A or V respectively.

As other cases are easy, we treat the cases, where A is of the form $\alpha[V']$ and V is of the form $\{\psi\}\mathfrak{F}(\psi)$.

Since $\oplus(\alpha(N))$ is $\{\varphi\}\alpha[\oplus_{N-1}(\varphi)]$, $A\begin{pmatrix}\oplus(\alpha)\\ \alpha\end{pmatrix}$ is $\alpha\left[\oplus_{N-1}(\beta)\left(V'\begin{pmatrix}\oplus\alpha\\ \alpha\\ \beta\end{pmatrix}\right)\right]$.

Hence by the hypothesis of the induction $A\begin{pmatrix}\oplus(\alpha)\\ \alpha\end{pmatrix}$ is $\alpha[\oplus_{N-1}(\beta)\binom{V'}{\beta}]$.

Hence $A\begin{pmatrix}\oplus\alpha\\ \alpha\end{pmatrix}$ is $\alpha[V']$ and 5.2.33 is proved.

Moreover $\oplus(\alpha)\binom{V}{\alpha}$ is $\{\psi\}\mathfrak{G}(\psi)$, where $\mathfrak{G}(\beta)$ is $\mathfrak{F}(\gamma)\begin{pmatrix}\oplus(\beta)\\ \gamma\end{pmatrix}$.

Hence by the hypothesis of the induction $\oplus(\alpha)\binom{V}{\alpha}$ is V.

Proof of 5.2.35. We prove the proposition by the double induction of the sum of the height of α and the height of β and the number of the stages to construct A. As other cases are simple, we treat the case where A is of the form $\alpha[\mathfrak{V}_1,\cdots\cdots,\mathfrak{V}_i]$ or $\beta[\mathfrak{V}_1'\cdots\cdots,\mathfrak{V}'_j]$.

Without the loss of generality we assume that V is of the form $\{\varphi_1,\cdots\cdots,\varphi_i\}\mathfrak{F}(\varphi_1,\cdots\cdots,\varphi_i)$ and V' is of the form $\{\varphi_1,\cdots\cdots,\varphi_j\}\mathfrak{G}(\varphi_1,\cdots\cdots,\varphi_j)$.

First we treat the case when A is $\alpha[\mathfrak{V}_1,\cdots\cdots,\mathfrak{V}_i]$.

Then by the definition 5.2.26 $A\binom{V}{\alpha}$ is $(\mathfrak{F}(\gamma_1,\cdots\cdots,\gamma_i))\begin{pmatrix}\mathfrak{V}_1\binom{V}{\alpha}\\ \gamma_1\end{pmatrix}\cdots\cdots\begin{pmatrix}\mathfrak{V}_i\binom{V}{\alpha}\\ \gamma_i\end{pmatrix}$, where $\gamma,\cdots\cdots,\gamma_i$ are different from each other and not contained in any of $\mathfrak{V}_1,\cdots\cdots,\mathfrak{V}_i$, V and V'.

Hence by the definition,

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{V}{\alpha}\binom{\mathfrak{B}_1\binom{V}{\alpha}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V}{\alpha}}{\gamma_i}\binom{V'}{\beta}.$$

By the hypothesis of the induction,

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{V}{\alpha}\binom{V'}{\beta}\binom{\mathfrak{B}_1\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_i},$$

and moreover

$$(\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}\binom{\mathfrak{B}_1\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_i},$$

where $\widetilde{V}$ is $V\binom{V'}{\beta}$.

Hence by the definition,

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{\widetilde{V}}{\alpha}\binom{\mathfrak{B}_1\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_i}.$$

And by the definition 5.2.26

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{\mathfrak{B}_1\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V'}{\beta}}{\gamma_i}\binom{\widetilde{V}}{\alpha},$$

that is,

$$(\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{V'}{\beta}\binom{\mathfrak{B}_1\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i\binom{V'}{\beta}}{\gamma_i}\binom{\widetilde{V}}{\alpha}.$$

Again by the hypothesis of the induction

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\alpha[\gamma_1,\cdots\cdots,\gamma_i])\binom{\mathfrak{B}_1}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_i}{\gamma_i}\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}.$$

Therefore $$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } A\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}.$$

Next we treat the case where A is $\beta[\mathfrak{B}_1',\cdots\cdots,\mathfrak{B}_j']$. Then, by the definition $A\binom{V}{\alpha}$ is $\beta\left[\mathfrak{B}_1'\binom{V}{\alpha},\ \cdots\cdots,\ \mathfrak{B}_j'\binom{V}{\alpha}\right]$. And by the definition 5.2.26

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\beta[\gamma_1,\cdots\cdots,\gamma_j])\binom{V'}{\beta}\binom{\mathfrak{B}_1'\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_j'\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_j}.$$

that is,

$$(\beta[\gamma_1,\cdots\cdots,\gamma_j])\binom{V}{\alpha}\binom{V'}{\beta}\binom{\mathfrak{B}_1'\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_j'\binom{V}{\alpha}\binom{V'}{\beta}}{\gamma_j},$$

where $\gamma_1,\cdots\cdots,\gamma_j$ are different from each other and not contained in any of $\mathfrak{B}_1'\cdots\cdots,\mathfrak{B}_j'$, V and V'.

Hence by the hypothesis of the induction

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\beta[\gamma_1,\cdots\cdots,\gamma_j])\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}\binom{\mathfrak{B}_1'\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_j'\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}}{\gamma_j},$$

and moreover

$$(\beta[\gamma_1,\cdots\cdots,\gamma_j])\binom{V'}{\beta}\binom{\mathfrak{B}_1'\binom{V'}{\beta}}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_j'\binom{V'}{\beta}}{\gamma_j}\binom{\widetilde{V}}{\alpha}.$$

And finally by the definition 5.2.26

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } (\beta[\gamma_1\cdots\cdots,\gamma_j])\binom{\mathfrak{B}_1'}{\gamma_1}\cdots\cdots\binom{\mathfrak{B}_j'}{\gamma_j}\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}.$$

Therefore

$$A\binom{V}{\alpha}\binom{V'}{\beta} \text{ is } A\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}.$$

5.3. Let $\mathfrak{F}$ be a functional and V be a variety of the same type as α.

We can suppose that $\mathfrak{F}$ is of the form $\{\varphi_1,\cdots\cdots,\varphi_n\}T(\varphi_1,\cdots\cdots,\varphi_n)$ without the loss of generality.

Then any figure of the form $\{\psi_1,\cdots\cdots,\psi_n\}T'(\psi_1,\cdots\cdots,\psi_n)$, where $T'(\beta_1,\cdots\cdots,\beta_n)$ is $T(\beta_1,\cdots\cdots,\beta_n)\binom{V}{\alpha}$ and $\psi_1,\cdots\cdots,\psi_n$ are different from each other and not contained in $T'(\beta_1,\cdots\cdots,\beta_n)$, is called a functional completely substituted V for α.

The so defined figure determines only a homology class. Both the so determined homology class and a functional belonging to it are simply denoted by $\mathfrak{F}\binom{V}{\alpha}$.

We can see easily the following properties.

5.3.1. If $\mathfrak{F}$ is a functional of type $(n_1,\cdots\cdots,n_i)$, then $\mathfrak{F}\binom{V}{\alpha}$ is a functional of type $(n_1,\cdots\cdots,n_i)$.

5.3.2. If $\mathfrak{F}'$ is homologous to $\mathfrak{F}$ and V' is homologous to V, then $\mathfrak{F}'\binom{V'}{\alpha}$ is homologous to $\mathfrak{F}\binom{V}{\alpha}$.

5.3.3. $\mathfrak{F}\binom{V}{\alpha}$ has neither a free variable nor a free function other than the variables and the functions contained in $\mathfrak{F}$ or V.

5.3.4. $\mathfrak{F}\binom{V}{\alpha}\binom{V'}{\beta}$ is $\mathfrak{F}\binom{V'}{\beta}\binom{\widetilde{V}}{\alpha}$, where V' has no α and $\widetilde{V}$ is $V\binom{V'}{\beta}$.

5.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be a formula and a variety and F be a functional of the same type as f.

Now, we define recursively a homology class completely substituted F for f in $\mathfrak{A}$ and $\mathfrak{B}$ as follows and both the so constructed homology class and one figure belonging to it are simply denoted by $\mathfrak{A}\binom{F}{f}$ or $\mathfrak{B}\binom{F}{f}$ respectively.

Moreover we use the following notation: Let $A(\beta)$ be a formula or a variety of the full indication for β and β be not contained in F. We define that $B(\beta)$ is $A(\beta)\binom{F}{f}$ if and only if $B(\beta)$ is the formula $A(\beta)\binom{F}{f}$ of the full indication for β. In general we define that $B(\beta)$ is $A(\beta)\binom{F}{f}$ if and only if $B(\gamma)$ is $A(\gamma)\binom{F}{f}$, where γ is not contained in $A(\beta)$ and F and so $A(\gamma)$ is of the full indication for γ.

In the same way we define that $B(u_1,\cdots\cdots,u_i)$ is $A(u_1,\cdots\cdots,u_i)\binom{F}{f}$ where each $u_k(1\leq k\leq i)$ is a free variable or a free function.

The definition is performed by the induction on the number of the stages to construct $\mathfrak{A}$ or $\mathfrak{B}$ respectively. Without the loss of generality we assume that F is $\{\varphi_1,\cdots\cdots,\varphi_n\}T(\varphi_1,\cdots\cdots,\varphi_n)$.

5.4.1. If $\mathfrak{A}$ has no f, then $\mathfrak{A}\binom{F}{f}$ is an arbitrary formula homologous to $\mathfrak{A}$.

5.4.2. If $\mathfrak{B}$ has no indicated place, then $\mathfrak{B}\binom{F}{f}$ is an arbitrary variety homologous to $\mathfrak{B}$.

5.4.3. If $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$, $\mathfrak{B}\wedge\mathfrak{C}$ or $\mathfrak{B}\vee\mathfrak{C}$, then $\mathfrak{A}\binom{F}{f}$ is $\neg\{\mathfrak{B}\binom{F}{f}\}$, $\{\mathfrak{B}\binom{F}{f}\}\wedge\{\mathfrak{C}\binom{F}{f}\}$ or $\{\mathfrak{B}\binom{F}{f}\}\vee\{\mathfrak{C}\binom{F}{f}\}$ respectively.

5.4.4. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, $E\varphi\mathfrak{F}(\varphi)$, $\forall p\mathfrak{G}(p)$, or $Ep\mathfrak{G}(p)$, then $\mathfrak{A}\binom{F}{f}$ is $\forall\psi\mathfrak{F}'(\psi)$, $E\psi\mathfrak{F}'(\psi)$, $\forall q\mathfrak{G}'(q)$ or $Eq\mathfrak{G}'(q)$ respectively, where $\mathfrak{F}'(\alpha)$ is

$\mathfrak{F}(\alpha)\binom{F}{f}$ and ψ is not contained in $\mathfrak{F}'(\alpha)$ and $\mathfrak{G}'(g)$ is $\mathfrak{G}(g)\binom{F}{f}$ and q is not contained in $\mathfrak{G}'(g)$.

5.4.5. If $\mathfrak{A}$ is of the form $\alpha[\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_i]$, then $\mathfrak{A}\binom{F}{f}$ is $\alpha[\mathfrak{B}_1\binom{F}{f},\cdots\cdots,\mathfrak{B}_i\binom{F}{f}]$.

5.4.6. If $\mathfrak{B}$ is of the form $g(\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_i)$ and g is not f, then $\mathfrak{B}\binom{F}{f}$ is $g(\mathfrak{B}_1\binom{F}{f},\cdots\cdots,\mathfrak{B}_i\binom{F}{f}))$.

5.4.7. If $\mathfrak{B}$ is of the form $f(\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_n)$, then $\mathfrak{B}\binom{F}{f}$ is $T(\alpha_1,\cdots\cdots,\alpha_n)$ $\binom{\mathfrak{B}_1\binom{F}{f}}{\alpha_1}\cdots\cdots\binom{\mathfrak{B}_n\binom{F}{f}}{\alpha_n}$, where $\alpha_1,\cdots\cdots,\alpha_n$ are different from each other and not contained in F.

5.4.8. If $\mathfrak{B}$ is of the form $\{\psi_1,\cdots\cdots,\psi_m\}\mathfrak{F}(\psi_1,\cdots\cdots,\psi_n)$, then $\mathfrak{B}\binom{F}{f}$ is $\{\xi_1,\cdots\cdots,\xi_m\}\mathfrak{G}(\xi_1,\cdots\cdots,\xi_m)$, where $\mathfrak{G}(\alpha_1,\cdots\cdots,\alpha_m)$ is $\mathfrak{F}(\alpha_1,\cdots\cdots,\alpha_m)\binom{F}{f}$ and $\xi_1,\cdots\cdots,\xi_m$ are different from each other and not contained in $\mathfrak{G}(\alpha,\cdots\cdots,\alpha_m)$.

We can see easily the following properties.

5.4.9. If $\mathfrak{A}$ is a formula and F is a functional of the same type as f, then $\mathfrak{A}\binom{F}{f}$ is a formula.

5.4.10. If $\mathfrak{B}$ is a variety of type $(n_1,\cdots\cdots,n_i)$ and F is a functional of the same type as f, then $\mathfrak{B}\binom{F}{f}$ is a variety of type $n_1\cdots\cdots,n_i)$.

5.4.11. If A' is homologous to A and F' is homologous to F, then $A'\binom{F'}{f}$ is homologous to $A\binom{F}{f}$.

5.4.12. $A\binom{F}{f}$ has neither a free variable nor a free function other that the variables and the functions contained in A or F.

5.4.13. $A\binom{F}{f}\binom{F'}{g}$ is $A\binom{F'}{g}\binom{\widetilde{F}}{f}$, where F' has no f and $\widetilde{F}$ is $F\binom{F'}{g}$.

5.4.14. $A\binom{V}{\alpha}\binom{F'}{g}$ is $A\binom{F'}{g}\binom{\widetilde{V}}{\alpha}$, where F' has no α and $\widetilde{V}$ is $V\binom{F'}{g}$.

5.4.15. $A\binom{F}{f}\binom{V}{\alpha}$ is $A\binom{V}{\alpha}\binom{\widetilde{F}}{f}$, where V has no f and $\widetilde{F}$ is $F\binom{V}{\alpha}$.

5.5. Let $\mathfrak{J}$ be a functional and F be a functional of the same type as f. We suppose that $\mathfrak{J}$ is of the form $\{\varphi_1,\cdots\cdots,\varphi_n\}T(\varphi_1,\cdots\cdots,\varphi_n)$.

Then any figure of the form $\{\psi_1,\cdots\cdots,\psi_n\}T'(\psi_1,\cdots\cdots,\psi_n)$, where $T'(\beta_1,\cdots\cdots,\beta_n)$ is $T(\beta_1,\cdots\cdots,\beta_n)\binom{F}{f}$ and $\psi_1,\cdots\cdots,\psi_n$ are different from each other and not contained in $T'(\beta_1,\cdots\cdots,\beta_n)$, is called a functional completely substituted F for f in $\mathfrak{J}$.

The so defined figure determines only a homology class. Both the so determined homology class and a functional belonging to it are simply denoted by $\mathfrak{J}\binom{F}{f}$.

We can see easily the following properties.

5.5.1. If $\mathfrak{J}$ is a functional of type $(n_1,\cdots\cdots,n_i)$, then $\mathfrak{J}\binom{F}{f}$ is a functional of type $(n_1,\cdots\cdots,n_i)$.

5.5.2. If $\mathfrak{J}'$ is homologous to $\mathfrak{J}$ and F' is homologous to F, then $\mathfrak{J}'\binom{F'}{f}$ is homologous to $\mathfrak{J}\binom{F}{f}$.

5.5.3. $\mathfrak{J}\binom{F}{f}$ has neither a free variable nor a free function other than the variables and the functions contained in $\mathfrak{J}$ or F.

5.5.4. Let A be a formula or a variety or a functional and let $u_i\,(i=1,2)$ be a free variable or a free function and $U_i\,(i=1,2)$ be a variety of a functional of the same type as U_i respectively. Then we have $A\binom{U_1}{u_1}\binom{U_2}{u_2}$ is $A\binom{U_2}{u_2}\binom{\widetilde{U}_1}{u_1}$, where U_2 has no u_1 and $\widetilde{U}_1$ is $U_1\binom{U_2}{u_2}$.

5.6. Let $A(u)$ be a formula or a variety of a functional and u be a free variable or a free function and U be a variety or a functional of the same type as u respectively. If $A(u)$ is of a full indication for u, then we define the homology class substituted U for u at all the indicated places in $A(u)$ as the figure $A(u)\binom{U}{u}$ and in general, we define the homology class substituted U for u at all the indicated places in $A(u)$ as the fugure $A(v)\binom{U}{v}$, where v is not contained in $A(u)$ and so $A(v)$ is of a full indication.

And both the so constructed homology class and one figure belonging to it are simply denoted by $A(U)$.

5.6.1. Let $A(u_1, u_2)$ be a formula or a variety or a functional and let $u_i(i=1,2)$ be a free variable or a free function and $U_i(i=1,2)$ be a variety or a functional of the same type as u_i respectively.

If U_1 has no u_2 and U_2 has no u_1 and $A(u_1, u_2)$ is of a full indication for u_1 and u_2, then by 5.5.6 we have that $A(u_1, u_2)\binom{U_1}{u_1}\binom{U_2}{u_2}$ is $A(u_1, u_2)\binom{U_2}{u_2}\binom{U_1}{u_1}$. Therefore $A(U_1, U_2)$ is defined uniquely by this homology class.

5.6.2. Let $A(u)$ be a formula or a variety or a functional and let u and v be a free variable or a free function and $U(v)$ be a variety or a functional of the same type as u respectively, and V be a variety or a functional of the same type as v respectively.

If V has no u and $A(u)$ and $U(v)$ are of a full indication for u and v respectively, then according to 5.5.6 we have that $A(u)\binom{U(v)}{u}\binom{V}{v}$ is $A(u)\binom{U(V)}{u}$.

Therefore $A(U(V))$ is defined uniquely by this homology class.

5.6.3. In the same way a homology class $A(U(V), V)$ is obtained uniquely by all the orders of substitutions. Moreover we have uniquely $A(U_1(V_1), \cdots\cdots, U_n(V_n))$, $A(U_1(U_2(U_3)))$ etc.

§ 6. Proof-figure.

6.1. If $\mathfrak{A}, \cdots\cdots, \mathfrak{A}_\mu, \mathfrak{B}_1, \cdots\cdots, \mathfrak{B}_\nu$ are formulas; the figure $\mathfrak{A}_1, \cdots\cdots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_1, \cdots\cdots, \mathfrak{B}_\nu$ is called a sequence. We read it 'if $\mathfrak{A}_1, \cdots\cdots, \mathfrak{A}_{\mu-1}$ and $\mathfrak{A}_\mu$ then $\mathfrak{B}_1$ or $\mathfrak{B}_2$ or$\cdots\cdots$, or $\mathfrak{B}_\nu$'. $\mathfrak{A}_2, \cdots\cdots, \mathfrak{A}_\mu$ is called the left side and $\mathfrak{B}_1, \cdots\cdots, \mathfrak{B}_\nu$ the right side of this sequence. Either side or even both sides of a sequence need not contain a formula. They may be void. Thus '$\mathfrak{A}_1, \cdots\cdots, \mathfrak{A}_\mu \rightarrow$' '$\rightarrow \mathfrak{B}_1, \cdots\cdots, \mathfrak{B}_\nu$' and '$\rightarrow$' are sequences, read respectively '$\mathfrak{A}_1, \cdots\cdots$, and $\mathfrak{A}_\mu$ contains a contradiction', 'Either $\mathfrak{B}$, or$\cdots\cdots$or $\mathfrak{B}_\nu$ holds' and 'There is a contradiction'.

6.2. Inference-figure.

If $S_1, \cdots\cdots, S_\nu$ and S are sequences, a figure

$$\frac{S_1, \cdots\cdots\cdots\cdots, S_\nu}{S}$$

is called an inference figure. We read it '$S_1, \cdots\cdots, S_\nu$, therefore S', $S_1, \cdots\cdots, S_\nu$ are called the upper sequences, and S is called the lower sequence.

6.3. Proof-figure.

If a finite number of sequences is arranged in a figure, so that the following conditions may be fulfilled, we call it a 'proof-figure'.

1) All sequences but one exception, called the end-sequence, are upper sequences of some other sequences.

2) There is no circle: i.e. if we start from a sequence and go down to its lower sequences one after another, we never return the original sequences. In other words, the sequences form a 'tree'. Sequences, which are not lower sequences of other sequences in a proof-figure, are called the beginning sequences.

A series of sequences in a proof-figure with the following property, is called a 'string'. The series begins with a beginning sequences and ends with a end-sequence; every sequence of the series, except the last, is followed immediately by just one lower sequence of it. 'A sequence is above another sequence' means that there is a string which contains the former sequence in a former order than the latter.

6.4. Now G. Gentzen has developed his 'Logistischer-Kalkül' (LK) in his precited paper, and defined a proof-figure (Herleitung) as that belonging to his LK, when it has the following property.

6.4.1. The beginning sequences are always of the form $\mathfrak{D} \to \mathfrak{D}$ ($\mathfrak{D}$ is an arbitrary formula).

6.4.2. The inference-figure of the proof-figure is obtained by some substitutions from the following 'inference-schemata', whereby the Greek capital letters $\Gamma, \Delta, \cdots\cdots$ denote the (occasionally void) series of formulas in the left and the right sides of sequences.

I) Inference-schemata on the structure of the sequences:

'Weakening'

left: $\dfrac{\Gamma \to \Delta}{\mathfrak{D}, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta}{\Gamma \to \Delta, \mathfrak{D}}$

'Contraction'

left: $\dfrac{\mathfrak{D}, \mathfrak{D}, \Gamma \to \Delta}{\mathfrak{D}, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, \mathfrak{D}, \mathfrak{D}}{\Gamma \to \Delta, \mathfrak{D}}$

'Exchange'

left: $\dfrac{\Pi, \mathfrak{D}, \mathfrak{B}, \Gamma \to \Delta}{\Pi, \mathfrak{B}, \mathfrak{D}, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, \mathfrak{B}, \mathfrak{D}, \Lambda}{\Gamma \to \Delta. \mathfrak{D}, \mathfrak{B}, \Lambda}$

Formulas such as $\mathfrak{B}$, $\mathfrak{D}$ in a schema of these types will be called the chief formulas of the schema.

'Version'

$$\frac{\Gamma \to \Delta}{\widetilde{\Gamma} \to \widetilde{\Delta}}$$

The series $\widetilde{\Gamma}$ and $\widetilde{\Delta}$ in 'Version' are obtained from Γ and Δ respectively by the substitution of a homologous formula for each formula.

II) 'Cut'

$$\frac{\Gamma \to \Delta, \mathfrak{D} \quad \mathfrak{D}, \Pi \to \Lambda}{\Gamma, \Pi, \to \Delta, \Lambda}$$

In this inference-figure we call $\mathfrak{D}$ the 'cut formula'.

III) Inference-schemata on the logical symbols.

$\neg$: left: $\dfrac{\Gamma\to\Delta, \mathfrak{A}}{\neg\mathfrak{A}, \Gamma\to\Delta}$ right: $\dfrac{\mathfrak{A}, \Gamma\to\Delta}{\Gamma\to\Delta, \neg\mathfrak{A}}$

$\wedge$: left: $\dfrac{\mathfrak{A}, \Gamma\to\Delta}{\mathfrak{A}\wedge\mathfrak{B}, \Gamma\to\Delta}$ or $\dfrac{\mathfrak{B}, \Gamma\to\Delta}{\mathfrak{A}\wedge\mathfrak{B}, \Gamma\to\Delta}$ right: $\dfrac{\Gamma\to\Delta, \mathfrak{A} \quad \Gamma\to\Delta, \mathfrak{B}}{\Gamma\to\Delta, \mathfrak{A}\wedge\mathfrak{B}}$

$\vee$: left: $\dfrac{\mathfrak{A}, \Gamma\to\Delta \quad \mathfrak{B}, \Gamma\to\Delta}{\mathfrak{A}\vee\mathfrak{B}, \Gamma\to\Delta}$ right: $\dfrac{\Gamma\to\Delta, \mathfrak{A}}{\Gamma\to\Delta, \mathfrak{A}\vee\mathfrak{B}}$ or $\dfrac{\Gamma\to\Delta, \mathfrak{B}}{\Gamma\to\Delta, \mathfrak{A}\vee\mathfrak{B}}$

$\forall$: left: $\dfrac{\mathfrak{F}(t), \Gamma\to\Delta}{\forall x\mathfrak{F}(x), \Gamma\to\Delta}$
(t is an arbitrary term.)

right: $\dfrac{\Gamma\to\Delta, \mathfrak{F}(a)}{\Gamma\to\Delta, \forall x\mathfrak{F}(x)}$
(There is no a in the lower sequence.)

E : left: $\dfrac{\mathfrak{F}(a), \Gamma\to\Delta}{\mathrm{E}x\mathfrak{F}(x), \Gamma\to\Delta}$
(There is no a in the lower sequence.)

right: $\dfrac{\Gamma\to\Delta, \mathfrak{F}(t)}{\Gamma\to\Delta, \mathrm{E}x\mathfrak{F}(x)}$
(t is an arbitrary term.)

Now we shall extend Gentzen's LK to a 'generalized logic calculus' (GLC). We shall add the following ones to Gentzen's inference-schemata in the logical symbols, and define a proof-figure as that belonging to GLC, when it has the above property 1), 2), the latter having now a wider sense:

$\forall$ for variables of type $(n_1, \cdots\cdots, n_i)$

left : $\dfrac{\mathfrak{F}(V), \Gamma\to\Delta}{\forall\varphi\mathfrak{F}(\varphi(n_1, \cdots\cdots, n_i)), \Gamma\to\Delta}$
(V is an arbitrary variety of type $(n_1 \cdots\cdots, n_i)$.)

right : $\dfrac{\Gamma\to\Delta, \mathfrak{F}(\alpha(n_1, \cdots\cdots, n_i))}{\Gamma\to\Delta, \forall\varphi\mathfrak{F}(\varphi(n_1, \cdots\cdots, n_i))}$
(There is no $\alpha(n, \cdots\cdots, n)$ in the lower sequence.)

E for variables of type $(n_1, \cdots\cdots, n_i)$

left : $\dfrac{\mathfrak{F}(\alpha(n_1, \cdots\cdots, n_i)), \Gamma\to\Delta}{\mathrm{E}\varphi(\varphi(n_1, \cdots\cdots, n_i)), \Gamma\to\Delta}$
(There is no $\alpha(n_1, \cdots\cdots, n_i)$ in the lower sequence.)

right : $\dfrac{\Gamma\to\Delta, \mathfrak{F}(V)}{\Gamma\to\Delta, \mathrm{E}\varphi\mathfrak{F}(\varphi(n_1, \cdots\cdots, n_i))}$
(V is an arbitrary variety of type $(n_1, \cdots\cdots, n_i)$.)

$\forall$ for functions of type $(n_1, \cdots\cdots, n_i)$

left : $\dfrac{\mathfrak{F}(F), \Gamma\to\Delta}{\forall p\mathfrak{F}(p(n_1, \cdots\cdots, n_i)), \Gamma\to\Delta}$
(F is an arbitrary functional of type $(n_1, \cdots\cdots n_i)$.)

right : $\dfrac{\Gamma\to\Delta, \mathfrak{F}(f(n_1, \cdots\cdots, n_i))}{\Gamma\to\Delta, \forall p\mathfrak{F}(p(n_1, \cdots\cdots, n_i))}$
(There is no $f(n_1, \cdots\cdots, n_i)$.) in the lower sequence.)

E for functions of type $(n_1, \cdots\cdots, n_i)$

left : $\dfrac{\mathfrak{F}(f(n_1, \cdots\cdots n_i)), \Gamma\to\Delta}{\mathrm{E}p\mathfrak{F}(p(n_1, \cdots\cdots, n_i)), \Gamma\to\Delta}$
(There is no $f(n_1, \cdots\cdots, n_i)$ in the lower sequence.)

right : $\dfrac{\Gamma\to\Delta, \mathfrak{F}(F)}{\Gamma\to\Delta, \mathrm{E}p\mathfrak{F}(p(n_1, \cdots\cdots, n_1))}$
(F is an arbitrary functional of type $(n_1, \cdots\cdots, n_i)$.)

Following Gentzen, we call the formulas such as $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{F}(t)$, etc. in the inference-schemata III) 'subformula', namely, the one which appears in the upper sequences and not represented by Greek capital letters in the above table, and 'chief formulas' the corresponding one in the lower sequences such as $\mathfrak{A}\wedge\mathfrak{B}$, $\mathfrak{A}\vee\mathfrak{B}$, $\neg\mathfrak{A}$ etc.

6.4.3. A proof-figure of GLC, which has S as its end-sequence, is called a 'proof-figure to S'. A sequence S is called 'provable' or 'provable in GLC', when there exists such a proof-figure.

6.4.4. A logical symbol in a formula is called improper, when it is contained in a variety in the construction of the formula; it is called proper in all other cases.

6.5. In the same way as Gentzen (2) we have in GLC the sequence '$\rightarrow$' (whose left and right sides are both void) is not provable.

6.6. A logical system is called consistent when '$\rightarrow$' is not provable in the system. So 6.5. means that GLC is consistent.

6.7. We call a formula without free variables and free functions 'axiom'. If $A_1,\cdots\cdots,A_k$ are axioms, and $A_1,\cdots\cdots,A_k\rightarrow$ is not provable in this logical system, we say that these axioms are consistent in this logical system.

A system of axioms $A_1, A_2, A_3,\cdots\cdots$ is said to be consistent, when every subsystem of a finite number of axioms is always consistent.

6.8. Let S be a beginning sequence with a proper logical symbol of a proof-figure H.

If S is of the form $\forall x\mathfrak{F}(x)\rightarrow\forall x\mathfrak{F}(x)$ or of the form $\neg\mathfrak{A}\rightarrow\neg\mathfrak{A}$, then we may substitute respectively

$$\cfrac{\cfrac{\mathfrak{F}(a)\rightarrow\mathfrak{F}(a)}{\mathfrak{F}(a)\rightarrow\forall x\mathfrak{F}(x)}}{\forall x\mathfrak{F}(x)\rightarrow\forall x\mathfrak{F}(x)} \qquad \text{or} \qquad \cfrac{\cfrac{\cfrac{\mathfrak{A}\rightarrow\mathfrak{A}}{\rightarrow\mathfrak{A}\neg\mathfrak{A}}}{\rightarrow\neg\mathfrak{A},\mathfrak{A}}}{\neg\mathfrak{A}\rightarrow\neg\mathfrak{A}}$$

for S in H.

In this way we can decrease the number of proper logical symbols in every beginning sequences in H.

So we may assume that every beginning sequence in H has no proper logical symbol.

In a similar way we may assume moreover that the chief-formula of every weakening in H has no proper logical symbol.

6.9. Let V be a variety of the same type as α.

If the sequence $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m$ is provable, then the sequence $\widetilde{\mathfrak{A}}_1,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$ is provable, where $\widetilde{\mathfrak{A}}_i$ is $\mathfrak{A}_i\binom{V}{\alpha}$ for each $i(1\leqq i\leqq n)$ and $\widetilde{\mathfrak{B}}_j$ is $\mathfrak{B}_j\binom{V}{\alpha}$ for each $j(1\leqq j\leqq m)$ respectively.

Proof. We prove the proposition by the induction on the number N of the inference-figures in the proof-figures to $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m$.

If N is one, then $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m$ is of the from $\mathfrak{D}\rightarrow\mathfrak{D}$, and the proposition is clcar, because $\widetilde{\mathfrak{A}}_1,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$ is $\mathfrak{D}\binom{V}{\alpha}\rightarrow\mathfrak{D}\binom{V}{\alpha}$ in the case.

Next we shall prove the case where N is M, assuming that the proposition holds when N is less than M.

As other cases simple or similar, we treat only the case where the last inference-figure to $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m$ is the following

$$\frac{\mathfrak{F}(U),\mathfrak{A}_2,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m}{\forall\varphi\mathfrak{F}(\varphi),\mathfrak{A}_2,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m}$$

where $\forall\varphi\mathfrak{F}(\varphi)$ means $\mathfrak{A}_1$ in the question.

By the hypothesis of the induction the following sequence is provable $\mathfrak{F}(U)\binom{V}{\alpha},\widetilde{\mathfrak{A}}_2,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$.

According to 5.54 $\mathfrak{F}(U)\binom{V}{\alpha}$ is homologous to $\widetilde{\mathfrak{F}}(\widetilde{U})$, where $\widetilde{\mathfrak{F}}(\beta)$ is $\mathfrak{F}(\beta)\binom{V}{\alpha}$ and $\widetilde{U}$ is $U\binom{V}{\alpha}$.

Therefore the sequence $\widetilde{\mathfrak{F}}(\widetilde{U}),\widetilde{\mathfrak{A}}_2,\cdots\cdots\ \widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$ is provable, hence by the inference-figure

$$\cfrac{\cfrac{\widetilde{\mathfrak{F}}(\widetilde{U}),\widetilde{\mathfrak{A}}_2,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots\widetilde{\mathfrak{B}}_m}{\forall\varphi\widetilde{\mathfrak{F}}(\varphi),\widetilde{\mathfrak{A}}_2,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m}}{\widetilde{\mathfrak{A}}_1,\widetilde{\mathfrak{A}}_2,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m}\ \text{Version}$$

$\widetilde{\mathfrak{A}}_1,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$ is provable.

In the same way we have the following proposition.

6.10. Let F be a functional of the same type of f.

If the sequence $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n\rightarrow\mathfrak{B}_1,\cdots\cdots,\mathfrak{B}_m$ is provable, then the sequence $\widetilde{\mathfrak{A}}_1,\cdots\cdots,\widetilde{\mathfrak{A}}_n\rightarrow\widetilde{\mathfrak{B}}_1,\cdots\cdots,\widetilde{\mathfrak{B}}_m$ is provable, where $\widetilde{\mathfrak{A}}_i$ is $\widetilde{\mathfrak{A}}_i\binom{F}{f}$ for each $i(1\leqq i\leqq n)$ and $\widetilde{\mathfrak{B}}_j$ is $\mathfrak{B}_j\binom{F}{f}$ for each $j(1\leqq j\leqq m)$ respectively.

Chapter II. Metatheorems in Generalized Logic Calculus

§ 7. **Restriction.** As we have to use repeatedly the logical symbols, we shall write $\neg\mathfrak{A}\vee\mathfrak{B}$, $\mathfrak{A}\vee\neg\mathfrak{B}$, $\neg\mathfrak{A}\wedge\mathfrak{B}$ and $\mathfrak{A}\wedge\neg\mathfrak{B}$ for $(\neg\mathfrak{A})\vee\mathfrak{B}$, $\mathfrak{A}\vee(\neg\mathfrak{B})$, $(\neg\mathfrak{A})\wedge\mathfrak{B}$ and $\mathfrak{A}\wedge(\neg\mathfrak{B})$ respectively. Further, we shall write $\mathfrak{A}\vdash\mathfrak{B}$ and $\mathfrak{A}\mapsto\mathfrak{B}$ for $\neg\mathfrak{A}\vee\mathfrak{B}$ and $(\neg\mathfrak{A}\vee\mathfrak{B})\wedge(\neg\mathfrak{B}\vee\mathfrak{A})$ respectively. The metamathematical meaning of these notations will be clear. Those symbols $\vdash$, $\mapsto$ have a weaker adhering power than $\neg$, $\wedge$, $\vee$, e.g. $\mathfrak{A}\wedge\mathfrak{B}\vdash\mathfrak{C}$, $\mathfrak{A}\vee\mathfrak{B}\mapsto\mathfrak{C}$, $\neg\mathfrak{A}\mapsto\mathfrak{B}$ mean $(\mathfrak{A}\wedge\mathfrak{B})\vdash\mathfrak{C}$, $(\mathfrak{A}\vee\mathfrak{B})\mapsto\mathfrak{C}$, $(\neg\mathfrak{A})\mapsto\mathfrak{B}$ respectively.

7.1. We define the metamathematical concept of 'restriction' depending on a 'system of restriction'. A system of restriction is given when at most one formula of the form $\mathfrak{F}\langle n_1,\cdots\cdots,n_i\rangle(\alpha(n_1,\cdots\cdots,n_i))$ called a restriction formula for a variable of type $(n_1,\cdots\cdots,n_i)$, and at most one formula of the form $\mathfrak{G}\langle n_1,\cdots\cdots,n_i\rangle(f(n_1,\cdots\cdots,n_i))$ called a restricting formula for a function of type $(n_1,\cdots\cdots,n_i)$ are associated with each type $(n_1,\cdots\cdots,n_i)$, the restricting formula should not contain any free variable or any free function other than $\alpha(n_1,\cdots\cdots,n_i)$ or $f(n_1,\cdots\cdots,n_i)$ respectively, and $\alpha(n_1,\cdots\cdots,n_i)$ and $f(n_1,\cdots\cdots,n_i)$ are of a full indication.

This system of restriction is often denoted by the notation

$$\{\mathfrak{F}\langle n_1,\cdots\cdots,n_i\rangle(\alpha(n_1,\cdots\cdots,n_i));\ \mathfrak{G}\langle m_1,\cdots\cdots,m_j\rangle(f(m_1,\cdots\cdots,m_j))\}.$$

Let R be a system of restriction

$$\{\mathfrak{F}<n_1,\cdots\cdots,n_i>(\alpha(n_1,\cdots\cdots,n_i));\ \mathfrak{G}<m_1,\cdots\cdots,m_j>(f(m_1,\cdots\cdots,m_j))\}.$$

The restriction A^r of a formula A or a variety A depending on R is defined recursively as follows and we call r a restricting operator of R.

7.1.1. If A contains neither a bound variable nor a bound function then A is an arbitrary figure homologous to A. So we may consider that A^r is A.

7.1.2. $(\urcorner\mathfrak{A})^r$ is $\urcorner\mathfrak{A}^r$.

$$(\mathfrak{A}\wedge\mathfrak{B})^r \text{ is } \mathfrak{A}^r\wedge\mathfrak{B}^r.$$

$$(\mathfrak{A}\vee\mathfrak{B})^r \text{ is } \mathfrak{A}^r\vee\mathfrak{B}^r.$$

7.1.3. Let $A(\alpha)$ be a formula or a variety and β be not contained in $A(\alpha)$.

We define $A^r(\beta)$ as $(A(\beta))^r$, where $A^r(\beta)$ is of a full indication for β. So $A^r(\alpha)$, $A^r(\varphi)$ or $A^r(V)$ is obtained from $A^r(\beta)$ by substituting α, φ or V respectively for β at all the indicated places in $A^r(\beta)$.

In the same way we define $A^r(f)$, $A^r(p)$, $A(F)$ or $A^r(\alpha,\varphi,V,f,p,F)$ etc..

7.1.4. From now on, we define that $\mathfrak{A}\vdash\mathfrak{B}$ represents $\mathfrak{B}$ and $\mathfrak{A}\wedge\mathfrak{B}$ represents $\mathfrak{B}$, if $\mathfrak{A}$ is void.

$$(\forall\varphi(n_1,\cdots\cdots,n_i)\mathfrak{F}(\varphi))^r \text{ is } \forall\psi(\mathfrak{F}<n_1,\cdots\cdots,n_i>(\psi(n_1,\cdots\cdots,n_i))\vdash\mathfrak{F}^r(\psi)),$$

where ψ is contained in neither $\mathfrak{F}<n_1,\cdots\cdots,n_i>(\alpha)$ nor $\mathfrak{F}^r(\alpha)$.

$$(\mathrm{E}\varphi(n_1,\cdots\cdots,n_i)\mathfrak{F}(\varphi))^r \text{ is } \mathrm{E}\psi(\mathfrak{F}<n_1,\cdots\cdots,n_i>(\psi(n_1,\cdots\cdots,n_i))\wedge\mathfrak{F}^r(\psi)),$$

where ψ is contained in neither $\mathfrak{F}<n_1,\cdots\cdots,n_i>(\alpha)$ nor $\mathfrak{F}^r(\alpha)$.

$$(\forall p(n_1,\cdots\cdots,n_i)\mathfrak{A}(p))^r \text{ is } \forall p(\mathfrak{G}<n_1,\cdots\cdots,n_i>(q(n_1,\cdots\cdots,n_i))\vdash\mathfrak{A}^r(q)),$$

where q is contained in neither $\mathfrak{G}<n_1,\cdots\cdots,n_i>(f)$ nor $\mathfrak{A}^r(f)$.

$$(\mathrm{E}p(n_1,\cdots\cdots,n_i)n(p))^r \text{ is } \mathrm{E}q(\mathfrak{G}<n_1,\cdots\cdots,_i>(q(n_1,\cdots\cdots,n_i))\wedge\mathfrak{A}_i(q)),$$

where q is contained in neither $\mathfrak{G}<n_1,\cdots\cdots,n>(f)$ nor $\mathfrak{A}^r(f)$.

7.1.5. $(\{\varphi_1,\cdots\cdots,\varphi_n\}\mathfrak{F}(\varphi_1,\cdots\cdots,\varphi_n))^r$ is $\{\psi_1,\cdots\cdots,\psi_n\}\mathfrak{F}^r(\psi_1,\cdots\cdots,\psi_n)$, where $\psi_1,\cdots\cdots,\psi_n$ are different from each other and not contained in $\mathfrak{F}^r(\alpha_1,\cdots\cdots,\alpha_n)$.

7.1.6. $(\alpha[V_1,\cdots\cdots,V_n])^r$ is $\alpha[V_1^r,\cdots\cdots,V_n^r]$.

7.1.7. $(f(V_1,\cdots\cdots,V_n))^r$ is $f(V_1^r,\cdots\cdots,V_n^r)$.

7.2. Let F be a functional of the form $\{\varphi_1,\cdots\cdots,\varphi_n\}T(\varphi_1,\cdots\cdots,\varphi_n)$. We define the restriction F^r of F as $\{\psi_1,\cdots\cdots,\psi_n\}T^r(\psi_1,\cdots\cdots,\psi_n)$, where $\psi_1,\cdots\cdots,\psi_n$ are different from each other and not contained in $T^r(\alpha_1,\cdots\cdots,\alpha_n)$.

7.3. Next we shall prove the following theorems.

7.3.1. Let $A(V_1,\cdots\cdots,V_n)$ be a formula or a variety. $(A(V_1,\cdots\cdots,V_n))^r$ is homologous to $A^r(V_1^r,\cdots\cdots,V_1^r)$.

Proof. We prove the proposition by the double induction on the maximal number of the heights of $V_1,\cdots\cdots,V_n$ and the number of stages to construct $A(\alpha_1,\cdots\cdots,\alpha_n)$.

As other cases are simple, we treat only the case where $A(\alpha_1,\cdots\cdots,\alpha_n)$ is of the form $\alpha[\mathfrak{B}_1(\alpha_1,\cdots\cdots,\alpha_n),\cdots\cdots,\mathfrak{B}_m(\alpha_1,\cdots\cdots,\alpha_n)]$. Without the loss of generality we can assume that V_1 is $\{\varphi_1,\cdots\cdots,\varphi_m\}\mathfrak{A}(\varphi_1,\cdots\cdots,\varphi_m)$. First we consider

$A^r(V_1^r, \cdots\cdots, V_n^r)$. Clearly $A^r(\alpha_1, \cdots\cdots, \alpha_n)$ is $\alpha_1[\mathfrak{B}_1^r(\alpha_1, \cdots\cdots, \alpha_n), \cdots\cdots, \mathfrak{B}_m^r(\alpha_1, \cdots\cdots, \alpha_n)]$ and V_1^r is $\{\psi_1, \cdots\cdots, \psi_m\}\mathfrak{A}^r(\psi_1, \cdots\cdots, \psi_m)$. Therefore $A^r(V_1^r, \cdots\cdots, V_n^r)$ is $\mathfrak{A}^r(\mathfrak{B}_1^r(V_1^r, \cdots\cdots, V_n^r), \cdots\cdots, \mathfrak{B}_m^r(V_1^r, \cdots\cdots, V_n^r))$. Next we concider $(A(V_1, \cdots\cdots, V_n))^r$. Since the height of $\mathfrak{B}_i(V_1, \cdots\cdots V_n)$ $(i=1, \cdots\cdots, m)$ is less than the height of V_1 and so less than the maximal number of the heights of $V_1, \cdots\cdots, V_n$, by the hypothesis of the induction it follows that $(\mathfrak{A}(\mathfrak{B}_1(V_1, \cdots\cdots, V_n), \cdots\cdots, \mathfrak{B}_m(V_1, \cdots\cdots, V_n)))^r$ is $\mathfrak{A}^r((\mathfrak{B}_1(V_1, \cdots\cdots, V_n))^r, \cdots\cdots, (\mathfrak{B}_m(V_1, \cdots\cdots, V_n))^r)$ and so $(A(V_1, \cdots\cdots, V_n))^r$ is $\mathfrak{A}^r((\mathfrak{B}_1(V_1, \cdots\cdots, V_n))^r \cdots\cdots, (\mathfrak{B}_m(V_1, \cdots\cdots, V_n))^r)$.

Moreover by the hypothesis of the induction it follows that $(\mathfrak{B}_i(V_1, \cdots\cdots, V_n))^r$ is $\mathfrak{B}_i(V_1^r, \cdots\cdots, V_n^r))$for each $i(1 \leqq i \leqq m)$. Hence it follows that $(A(V_1, \cdots\cdots, V_n))^r$ is $\mathfrak{A}^r(\mathfrak{B}_1^r(V_1^r, \cdots\cdots, V_n^r), \cdots\cdots, \mathfrak{B}_m^r(V_1^r, \cdots\cdots, V_n^r))$ and so the proposition is proved.

According to 7.3.1. we have easily,

7.3.2. Let $F(V_1, \cdots\cdots, V_n)$ be a functional. $(F(V_1, \cdots\cdots, V_n))^r$ is homologous to $F^r(V_1^r, \cdots\cdots, V_n^r)$.

In the same way as in 7.3.1. we have easily,

7.3.3. Let $A(F_1, \cdots\cdots, F_n)$ be a formula or a variety. $(A(F_1, \cdots\cdots, F_n))^r$ is homologous to $A^r(F_1^r, \cdots\cdots, F_n^r)$.

According to 7.3.1, 7.3.2 and 7.3.3. we have easily,

7.3.4. Let $A(U_1, \cdots\cdots, U_n)$ be a formula or a variety of a functional and $U_i(i=1, \cdots\cdots, n)$ be a variety or a functional.

Then $(A(U_1, \cdots\cdots, U_n))^r$ is homologous to $A^r(U_1^r, \cdots\cdots, U_n^r)$.

Let Γ be a series of formulas $\mathfrak{A}_1, \cdots\cdots, \mathfrak{A}_n$. Then Γ^r represents $\mathfrak{A}_1^r, \cdots\cdots, \mathfrak{A}_n^r$.

7.5. Let R be a system of restriction

$$\{\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha(n_1, \cdots\cdots, n_i));\ \mathfrak{G}<m_1, \cdots\cdots, m_j>(f(m_1 \cdots\cdots, m_j))\}$$

The notation may be abbreviated by the following conventions; $cl(\beta(n_1, \cdots\cdots, n_i))$ may be written for $\mathfrak{F}<n_1, \cdots\cdots, n_i>(\beta(n_1, \cdots\cdots, n_i))$ and $cl(g(m_1, \cdots\cdots, m_j))$ for $\mathfrak{G}<m_1, \cdots\cdots, m_j>(g(m_1, \cdots\cdots, m_j))$. e.g. We can write $cl(\alpha(1)), cl(\beta(2)) \rightarrow \mathfrak{A}$ for $\mathfrak{F}<1>(\alpha(1)), \mathfrak{F}<2>(\beta(2)) \rightarrow \mathfrak{A}$.

7.6. The Theorem on Restriction

Let $A_1, \cdots\cdots, A_N$ be arbitrary axioms and let R be a system of restricting $\{\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha);\ \mathfrak{G}<m_1, \cdots\cdots, m_j>(f)\}$ and r be its restricting operator.

Now, we consider the following axioms;

$$A_1^r, \cdots\cdots, A_N^r$$

$\mathfrak{F}<n_1, \cdots\cdots, n_i>(\sigma(n_1, \cdots\cdots, n_i))$ for all the special variables which are contained in $A_1, \cdots\cdots, A_N$.

$\mathfrak{G}<m_1, \cdots\cdots, m_j>(s(m_1, \cdots\cdots, m_j))$ for all the special functions which are contained in $A_1, \cdots\cdots, A_N$.

$$\mathrm{E}\varphi\mathfrak{F}<n_1, \cdots\cdots, n_i>(\varphi(u_1 \cdots\cdots, n_i)) \text{ for all } (n_1, \cdots\cdots, n_i)$$

$$\mathrm{E}p\mathfrak{G}<m_1, \cdots\cdots, m_j>(p(m_1, \cdots\cdots, m_j)) \text{ for all } (m_1, \cdots\cdots, m_j)$$

and Γ_0 denoteds the series of these axioms.

If the following conditions are fulfilled, then the axioms $A_1, \cdots\cdots, A_N$ are consistent in GLC.

1) There exists a series of axioms $\widetilde{\Gamma}$ such that for an arbitrary axiom A of Γ_0 $\widetilde{\Gamma}\rightarrow A$ is provable in GLC and $\widetilde{\Gamma}$ is consistent in GLC.

2) If V is an arbitrary variety of type $(\overline{n}_1, \cdots\cdots, \overline{n}_k)$ and $\alpha(n_1, \cdots\cdots, n_i), \cdots \cdots, f(m_1, \cdots\cdots, m_j), \cdots\cdots$ are all the free or special variables and all the free or special functions which are contained in V, then $\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha), \cdots\cdots, \mathfrak{G}<m_1, \cdots\cdots, m_j>(f), \cdots\cdots, \widetilde{\Gamma}\rightarrow\mathfrak{F}<\overline{n}_1, \cdots\cdots, \overline{n}_k>(V^r)$ is provable in GLC.

3) If F is an arbitrary functional of type $(\overline{m}_1, \cdots\cdots, \overline{m}_k)$ and $\alpha(n_1, \cdots\cdots, n_i)$, $\cdots\cdots, f(m_1, \cdots\cdots, m_j), \cdots\cdots$ are all the free or special variables and all the free or special functions which are contained in F, then

$$\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha), \cdots\cdots, \mathfrak{G}<m_1, \cdots\cdots, m_j>(f), \cdots\cdots, \widetilde{\Gamma}\rightarrow\mathfrak{G}<\overline{m}_1, \cdots\cdots, \overline{m}_k>(F^r)$$

is provable in GLC.

Proof. If $A_1, \cdots\cdots, A_N$ were not consistent in GLC, then there should exist a proof-figure H to $A_1, \cdots\cdots, A_N\rightarrow$, which does contain neither a special variable nor a special function other than those in $A_1, \cdots\cdots, A_N$.

Let $\alpha^1, \cdots\cdots, \alpha^N, f^1, \cdots\cdots, f^M$ be all the free variables and functions in H respectively and $\Gamma\rightarrow\Delta$ an arbitrary sequence in H.

Now, we shall prove that the following sequence is provable in GLC.

7.6.1. $cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \widetilde{\Gamma}, \Gamma_0, \Gamma^r\rightarrow\Delta^r$.

If $\Gamma\rightarrow\Delta$ is of the form $\mathfrak{D}\rightarrow\mathfrak{D}$, then it is trivial. So we prove the proposition by the induction on the number of inference-figures to $\Gamma\rightarrow\Delta$.

As other cases are easy, we shall treat only the case where the last inference-figure $\forall$.

1) the case; $\forall$ left for variable: Then the inference-figure to $\Gamma\rightarrow\Delta$ is of the form

$$\frac{\mathfrak{F}(V), \Gamma'\rightarrow\Delta'}{\forall\varphi\mathfrak{F}(\varphi), \Gamma'\rightarrow\Delta'}$$

$\Gamma\rightarrow\Delta$ is $\forall\varphi\mathfrak{F}(\varphi),\ \Gamma'\rightarrow\Delta'$ here.

By the hypothesis of the induction the following sequence is provable

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \widetilde{\Gamma}, \Gamma_0,\ \mathfrak{F}^r(V^r),\ \Gamma'^r\rightarrow\Delta^r.$$

Moreover by the hypothesis of the theorem we have

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N).\ cl(f^1), \cdots\cdots, cl(f^M)\ \widetilde{\Gamma}, \Gamma_0\rightarrow cl(V^r).$$

Therefore

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \widetilde{\Gamma}, \Gamma_0,\ cl(V^r)\vdash\mathfrak{F}^r(V^r),\ \Gamma'^r\rightarrow\Delta^r.$$

is provable, and so is also

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f'), \cdots\cdots, cl(f^M).\ \widetilde{\Gamma}, \Gamma_0,\ \forall\varphi(cl(\varphi)\vdash\mathfrak{F}^r(\varphi)),\ \Gamma'^r\rightarrow\Delta^r,$$

that is

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \widetilde{\Gamma}, \Gamma_0,\ \Gamma^r\rightarrow\Delta^r.$$

2) the case; $\forall$ right for variables: Then the inference-figure to $\Gamma \to \Delta$ is of the form

$$\frac{\Gamma \to \Delta', \mathfrak{F}(\alpha)}{\Gamma \to \Delta', \forall\varphi\mathfrak{F}(\varphi)}$$

$\Gamma \to \Delta$ is $\Gamma \to \Delta'$, $\forall\varphi\mathfrak{F}(\varphi)$ here. By the hypothesis of the induction the following sequence is provable.

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \tilde{\Gamma}, \Gamma_0, \Gamma^r \to \Delta'^r, \mathfrak{F}^r(\alpha).$$

Without the loss of generality we can assume that α is α^1. Henceforth it follows succesively that the following sequence are provable

$$cl(\alpha^2), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \tilde{\Gamma}, \Gamma_0, \Gamma^r \to \Delta'^r,\ cl(\alpha^1) \vdash \mathfrak{F}^r(\alpha^1),$$

that is

$$cl(\alpha^2), \cdots\cdots, cl(\alpha^N),\ cl(f'), \cdots\cdots, cl(f^M),\ \tilde{\Gamma}, \Gamma_0, \Gamma^r \to \Delta'^r,\ \forall_\varphi(cl(\varphi) \vdash \mathfrak{F}^r(\varphi)).$$

Therefore, it follows,

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f'), \cdots\cdots, cl(f^M),\ \tilde{\Gamma}, \Gamma_0, {}^r\Gamma \to \Delta^r.$$

3) the case; $\forall$ left for functions: This case is treated in the same way as in 1).

4) the case; $\forall$ right for functions: This case is treated in the same way as in 3) and 2).

Now if we regard $\Gamma \to \Delta$ as $A_1, \cdots\cdots, A_N \to$ in 7.6.1, then we can see that the following sequence is provable

$$cl(\alpha^1), \cdots\cdots, cl(\alpha^N),\ cl(f^1), \cdots\cdots, cl(f^M),\ \tilde{\Gamma}, \Gamma_0,\ A_1, \cdots\cdots, A_N \to$$

Therefore

$$\mathrm{E}\varphi^1 cl(\varphi'), \cdots\cdots, \mathrm{E}\varphi^N cl(\varphi^N),\ \mathrm{E}p^1 cl(p^1), \cdots\cdots, \mathrm{E}p^M cl(p^M),\ \tilde{\Gamma}, \Gamma_0 \to$$

is provable.

Hence $\tilde{\Gamma}, \Gamma_0 \to$ is provable.

By the above and the hypothesis of the theorem it follows that $\tilde{\Gamma} \to$ must be provable, which is impossible. The proof is thus completed.

7.7. Let $e(*)$ be a special variable of type 1. We call restriction $e(*)$ the system of restriction, which contains only the formula $e(a)$.

The restriction A^r of A depending on this system of restriction is denoted by $A^{e(\ )}$.

7.8. Corollary

Let $e(\ )$ be a special variable of type 1. We consider the axioms;

$A_1^{e(\ )}, \cdots\cdots, A_N^{e(\ )}$

$e(s)$ for all the special variables s of type 0 in $A_1, \cdots\cdots, A^N$.

$\mathrm{E}x\, e(x)$

$\forall\varphi_1, \cdots\cdots, \forall\varphi_i(e(\varphi_1) \wedge \cdots\cdots \wedge e(\varphi_i) \vdash e(f(n_1+1, \cdots\cdots, n_i+1)(\varphi_1, \cdots\cdots, \varphi_i)))$ for all the special functions $f(n_1+1, \cdots\cdots, n_i+1)$ in $A_1, \cdots\cdots, A_N$, in which $e(\varphi_k)$ $(k=l, \cdots\cdots, i)$ is void if φ_k is not of type 0,

Γ_0 denoted these series of these axioms.

If Γ_0 is consistent in GLC, then $A_1, \cdots\cdots, A_N$ are consistent in GLC.

Proof. We regard Γ_0 here as $\widetilde{\Gamma}$ in the theorem 7.6.

Let t be an arbitrary term and $a_1, \cdots\cdots, a_m$ be all the free or special variables of type 0 which are contained in t.

Then we have only to prove that the following sequence is provable

$$e(a_1), \cdots\cdots, e(a_m),\ \Gamma_0 \to e(t^{e(\ )}).$$

If t is an free or special variable of type 0, then it is trivial.

So we shall prove the proposition by the induction on the number of the stages to constract t.

We assume here that t is of the form $f(V_1, \cdots\cdots, V_i)$.

Then we have clearly $e(V_1^{e(\ )}), \cdots\cdots, e(V_i^{e(\ )}),\ \Gamma_0 \to e(t^{e(\ )})$, where $V_k (k=1, \cdots\cdots, i)$ is void, if V_k is not a term. Moreover by the hypothesis of the induction, if $V_k (k=1, \cdots\cdots, i)$ is a term, the following sequence is provable

$$e(a_1), \cdots\cdots, e(a_m),\ \Gamma_0 \to e(V_k^{e(\ )})$$

Hence it follows that $e(a_1), \cdots\cdots, e(a_m),\ \Gamma_0 \to e(t^{e(\ )})$ is provable.

7.9. Let R be an arbitrary system of restriction and r be its restricting operator.

If A is an arbitrary normal formula (See 2.16), then the following sequence is provable in GLC

$$A \to A^r$$

Proof. We prove the proposition by the induction on the number of $\forall$'s in A.

If A has no $\forall$, then A^r is A and so the proposition is evident. Therefore we assume that A is of the form $\forall \varphi \mathfrak{F}(\varphi)$ and the following sequence is provable

$$\mathfrak{F}(\alpha) \to \mathfrak{F}^r(\alpha)$$

Then the following sequences are successively proved.

$$\mathfrak{F}(\alpha) \to cl(\alpha) \vdash \mathfrak{F}^r(\alpha), \quad \forall \varphi \mathfrak{F}(\varphi) \to cl(\alpha) \vdash \mathfrak{F}^r(\alpha).$$

And
$$A \to \forall \varphi (cl(\varphi) \vdash \mathfrak{F}^r(\varphi)).$$

And finally, $A \to A^r$. q. e. d.

7.10. Let $*_1 = *_2$ be a special variable of type (1, 1).

Then we define recursively as follows the special variables $*_1 \overset{n}{\equiv} *_2$ of type $(n+1, n+1)$ and the system of equality restriction

$$\{\mathfrak{F}\langle n_1+1, \cdots\cdots, n_i+1\rangle (\alpha(n_1+1, \cdots\cdots, n_i+1));$$
$$\mathfrak{G}\langle m_1+1, \cdots\cdots, m_j+1\rangle (f(m_1+1, \cdots\cdots, m_j+1))\}$$

depending on $*_1 \underset{0}{=} *_2$.

7.10.1. $a \equiv b$ means $a = b$.

7.10.2. $\mathfrak{F}\langle n_1+1, \cdots\cdots, n_i+1\rangle(\alpha)$ is an abbreviation of

$$\forall \varphi^1 \forall \psi^1 \cdots\cdots \forall \varphi^i \forall \psi^i \{\varphi^1 \overset{n_1}{\equiv} \psi^1 \wedge \cdots\cdots \wedge \varphi^i \overset{n_i}{\equiv} \psi^i \wedge \alpha[\varphi^1, \cdots\cdots, \varphi^i] \vdash \alpha[\psi^1, \cdots\cdots, \psi^i]\}$$

7.10.3. $\mathfrak{G}<m_1+1,\cdots\cdots, m_j+1>(f)$ is an abbreviation of

$$\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^j\forall\varphi^j\{\varphi^1\overset{m_1}{\equiv}\psi^1\wedge\cdots\cdots\wedge\varphi^j\overset{m_j}{\equiv}\psi^j\vdash f(\varphi^1,\cdots\cdots,\varphi^j)=f(\psi^1\cdots\cdots,\psi^j)\}$$

7.10.4. $\alpha(n+1)\overset{n+1}{\equiv}\beta(n+1)$ is an abbreviation of

$$\mathfrak{F}<n+1>(\alpha)\wedge\mathfrak{F}<n+1>(\beta)$$
$$\wedge\forall\varphi(n)\{\mathfrak{F}<n>(\varphi)\vdash(\alpha[\oplus_n(\varphi)]\dashv\vdash\beta[\oplus_n(\varphi)])\}.$$

If no confusion is alike, then $\alpha\overset{*}{\equiv}\beta$ may be written for $\alpha\overset{n}{\equiv}\beta$.

7.11. Clearly we have the following sequences,

7.11.1. $\mathfrak{F}<n+1>(\alpha)\rightarrow\alpha\overset{*}{\equiv}\alpha$

7.11.2. $\alpha\overset{*}{\equiv}\beta\rightarrow\beta\overset{*}{\equiv}\alpha$

7.11.3. $\alpha\overset{*}{\equiv}\beta,\ \beta\overset{*}{\equiv}\gamma\rightarrow\alpha\overset{*}{\equiv}\gamma$.

7.12. The Theorem on Equality

Let $*_1=*_2$ be a special variable of type $(1,1)$ and R be the system of equality restriction depending on $*_1=*_2$ and of the form $\{\mathfrak{F}<n_1+1,\cdots\cdots,n_i+1>(\alpha(n_1+1,\cdots\cdots,n_i+1));\ \mathfrak{G}<m_1+1,\cdots\cdots,m_j+1>(f(m_1+1,\cdots\cdots,m_j+1))\}$ and let $A_1,\cdots\cdots,A_N$ be normal axioms and contain three following axioms;

$$\forall x(x=x)$$

$$\forall x\forall y(x=y\vdash y=x)$$

$$\forall x\forall y\forall z(x=y\wedge y=z\vdash x=z)$$

If $A_1,\cdots\cdots,A_N$ are consistent and for each special variable $\sigma(n_1+1,\cdots\cdots,n_i+1)$ which is contained in $A_1,\cdots\cdots,A_N$ the sequence $A_1,\cdots\cdots,A_N\rightarrow\mathfrak{F}<n_1+1,\cdots\cdots,n_i+1>(\sigma)$ is provable and for each special function $s(m_1+1,\cdots\cdots,m_j+1)$ which is contained in $A_1,\cdots\cdots,A_N$ the sequence $A_1,\cdots\cdots,A_N\rightarrow\mathfrak{G}<m_1+1,\cdots\cdots,m_j+1>(s)$ is provable, then the following axioms which will be denoted by Π_0 are consistent;

$A_1,\cdots\cdots,A_N$

$\forall\varphi(n_1+1,\cdots\cdots,n_i+1)\mathfrak{F}<n_1+1,\cdots\cdots,n_i+1>(\varphi)$ for all the types $(n_1+1,\cdots\cdots,n_i+1)$

$\forall p(m_1+1,\cdots\cdots,m_j+1)\mathfrak{G}<m_1+1,\cdots\cdots,m_j+1>(p)$ for all the types $(m_1+1,\cdots\cdots,m_j+1)$

Proof. Regarding $A_1,\cdots\cdots A_N$ as $\widetilde{\Gamma}$ and Π_0 as $A_1,\cdots\cdots,A_N$ respectively in the theorem 7.6, we shall first 1), 2), 3) in the same theorem.

The condition 1) is evident, if we prove that the following sequences are provable for each type $(n_1+1,\cdots\cdots,n_i+1)$,

$$A_1,\cdots\cdots,A_N\rightarrow\mathrm{E}\varphi\mathfrak{F}<n_1+1,\cdots\cdots,n_i+1>(\varphi)$$
$$A_1,\cdots\cdots,A_N\rightarrow\mathrm{E}\varphi\mathfrak{G}<n_1+1,\cdots\cdots,n_i+1>(p)$$

But we have easily

$$A_1,\cdots\cdots,A_N\rightarrow\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i\{\varphi^1\overset{*}{\equiv}\psi^1\wedge\cdots\cdots\wedge\varphi^i\overset{*}{\equiv}\psi^i\wedge a=a\vdash a=a\},$$

$$A_1,\cdots\cdots, A_N \to \forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i\{\varphi^1 \overset{*}{\equiv} \psi^1 \wedge \cdots\cdots \wedge \varphi^i \overset{*}{\equiv} \psi^i \vdash a=a\}.$$

Therefore we have

$$A_1,\cdots\cdots, A_N \to \mathrm{E}\varphi\mathfrak{F}<n_1+1,\cdots\cdots, n_i+1>(\varphi)$$
$$A_1,\cdots\cdots, A_N \to \mathrm{E}p\mathfrak{G}<n_1+1,\cdots\cdots, n_i+1>(p)$$

Now we shall prove at a time the conditions 2) and 3).

Let $\{\varphi^1,\cdots\cdots,\varphi^n\}\mathfrak{A}(\varphi^1,\cdots\cdots,\varphi^n)$ be an arbitrary variety and $\{\psi^1,\cdots\cdots,\psi^m\}$ $T(\psi^1,\cdots\cdots,\psi^m)$ be an arbitrary functional and let $\alpha^1,\cdots\cdots,f^1,\cdots\cdots$ be all the free or special variables and all the free or special functions which are contained in $\{\varphi^1,\cdots\cdots,\varphi^n\}\mathfrak{A}(\varphi^1,\cdots\cdots,\varphi^n)$ and let $\beta^1,\cdots\cdots, g^1,\cdots\cdots$ be all the free or special variables and all the free or special functions respectively which are contained in $\{\psi,^1\cdots\cdots,\psi^m\}T(\psi^1,\cdots\cdots,\psi^m)$.

We have only to prove the following sequences ;

$$cl(\alpha^1),\cdots\cdots, cl(f^1),\cdots\cdots, A_1,\cdots\cdots, A_N$$
$$\gamma^1 \overset{*}{\equiv} \delta^1,\cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma^1,\cdots\cdots,\gamma^n) \to \mathfrak{A}^r(\delta^1,\cdots\cdots,\delta^n)$$
$$cl(\beta^1),\cdots\cdots, cl(g^1),\cdots\cdots, A_1,\cdots\cdots, A_N$$
$$\gamma^1 \overset{*}{\equiv} \delta^1,\cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m \to T^r(\gamma^1,\cdots\cdots,\gamma^m) = T^r(\delta^1,\cdots\cdots,\delta^m).$$

Instead of proving the proposition, we prove the following generalized proposition, that is ;

Let $\mathfrak{A}(\gamma^1,\cdots\cdots,\gamma^n)$ be an arbitrary formula and $V(\gamma^1,\cdots\cdots,\gamma^m)$ be an arbitrary variety of type k, and let $\alpha^1,\cdots\cdots,f^1,\cdots\cdots$ be all the free or special variables and all the free or special functions which are contained in $\mathfrak{A}(\varphi^1,\cdots\cdots,\varphi^n)$ and let $\beta^1,\cdots\cdots, g^1,\cdots\cdots$ be all the free or special variable and all the free or special functions respectively which contained in $T(\psi^1,\cdots\cdots,\psi^m)$.

Then the following sequences are provable;

$$cl(\alpha^1),\cdots\cdots, cl(f^1),\cdots\cdots, A_1,\cdots\cdots, A_N$$
$$\gamma^1 \overset{*}{\equiv} \delta^1,\cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma^1,\cdots\cdots,\gamma^n) \to \mathfrak{A}^r(\delta^1,\cdots\cdots,\delta^n)$$
$$cl(\beta^1),\cdots\cdots, cl(g^1),\cdots\cdots, A_1,\cdots\cdots, A_N$$
$$\gamma^1 \overset{*}{\equiv} \delta^1,\cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m \to V^r(\gamma^1,\cdots\cdots,\gamma^m) \overset{*}{\equiv} V^r(\delta^1,\cdots\cdots,\delta^m).$$

We prove this proposition by the induction on the number l of the stages to construct $\mathfrak{A}(\gamma^1,\cdots\cdots,\gamma^n)$ or $V(\gamma^1,\cdots\cdots,\gamma^m)$.

If l is zero, then the proposition is evident. When l is less that L, suppose the proposition be proved. And we shall treat the case where l is L.

Then the proposition is devided into the following cases;

1) the case, where the outermost symbol of $\mathfrak{A}(\gamma^1,\cdots\cdots,\gamma^m)$ is $\forall$, E, $\wedge$, $\vee$, or $\neg$.

As other cases are similar, we treat only the on $\forall$. We assume that $\mathfrak{A}(\gamma^1,\cdots\cdots\gamma^n)$ is of the form $\forall\xi\mathfrak{B}(\gamma^1,\cdots\cdots,\gamma^n,\xi)$. Then $\mathfrak{A}^r(\gamma^1,\cdots\cdots,\gamma^n)$ is $\forall\xi(cl(\xi) \vdash \mathfrak{B}^r(\gamma^1,\cdots\cdots,\varphi^n,\xi))$.

By the hypothesis of the induction we have

$$cl(\alpha^1),\cdots\cdots, cl(f^1),\cdots\cdots, A_1,\cdots\cdots, A_N,\ cl(\xi),$$

$$\gamma' \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{B}^r(\gamma^1, \cdots\cdots, \gamma^n, \xi) \rightarrow \mathfrak{B}^r(\gamma^1, \cdots\cdots, \gamma^n, \xi)$$

Therefore we have successively the following sequences,

$$\begin{array}{l} cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N,\ cl(\xi) \\ \gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ cl(\xi) \vdash \mathfrak{B}^r(\gamma^1, \cdots\cdots, \gamma^n, \xi) \rightarrow \mathfrak{B}^r(\delta^1, \cdots\cdots, \delta^n, \xi) \\ cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N,\ cl(\xi) \\ \gamma^1 \overset{*}{\equiv} \delta^1 \cdots\cdots. \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma^1, \cdots\cdots, \gamma^n) \rightarrow \mathfrak{B}^r(\delta^1, \cdots\cdots, \delta^n, \xi) \\ cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\ \gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma^1, \cdots\cdots, \gamma^n) \rightarrow cl(\zeta) \vdash \mathfrak{B}^r(\delta^1, \cdots\cdots, \delta^n, \xi). \end{array}$$

And finally, $cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N$

$$\gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma', \cdots\cdots, \gamma^n) \rightarrow \mathfrak{A}^r(\delta^1, \cdots\cdots, \delta^n).$$

2) the case, where the outer most variable of $\mathfrak{A}(\gamma,^1 \cdots\cdots, \gamma^n)$ is α. We assume that α is α^1 and $\mathfrak{A}(\gamma^1, \cdots\cdots, \gamma^n)$ is of the form

$$\alpha^1[V_1(\gamma^1, \cdots\cdots, \gamma^n), \cdots\cdots, V_i(\gamma^1, \cdots\cdots, \gamma^n)].$$

Then $\mathfrak{A}^r(\gamma^1, \cdots\cdots, \gamma^n)$ is $\alpha^1[V_1^r(\gamma^1, \cdots\cdots, \gamma^n), \cdots\cdots, V_i^r(\gamma^1, \cdots\cdots, \gamma^n)]$.

By the hypothesis of the induction we have for each $j\,(l \leqq j \leqq i)$

$$\begin{array}{l} cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\ \gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n \rightarrow V_j^r(\gamma^1, \cdots\cdots, \gamma^n) \overset{*}{\equiv} V_j^r(\delta^1, \cdots\cdots, \delta^n). \end{array}$$

On the other hand we have easily

$$\begin{array}{l} cl(\alpha^1),\ V_1^r(\gamma^1, \cdots\cdots, \gamma^n) \overset{*}{\equiv} V_1^r(\delta^1, \cdots\cdots, \delta^n), \cdots\cdots, \\ \qquad V_i^r(\gamma^1, \cdots\cdots, \gamma^n) \overset{*}{\equiv} V_i^r(\delta^1, \cdots\cdots, \delta^n),\ \mathfrak{A}(\gamma^1, \cdots\cdots, \gamma^n) \rightarrow \mathfrak{A}^r(\delta^1, \cdots\cdots, \delta^n). \end{array}$$

Hence we have

$$\begin{array}{l} cl(\alpha^1), \cdots\cdots, cl(f^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\ \gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^n,\ \mathfrak{A}^r(\gamma^1, \cdots\cdots, \gamma^n) \rightarrow \mathfrak{A}^r(\delta^1, \cdots\cdots, \delta^n). \end{array}$$

3) the case, where the outermost function of $V(\gamma^1, \cdots\cdots, \gamma^m)$ is g. We assume that g is g' and $V(\gamma^1, \cdots\cdots, \gamma^m)$ is of the form $g^1(V_1(\gamma^1, \cdots\cdots, \gamma^m), \cdots\cdots, V_i(\gamma', \cdots\cdots, \gamma^m))$.

Then $V^r(\gamma^1, \cdots\cdots, \gamma^m)$ is $g^1(V^r(\gamma^1, \cdots\cdots, \gamma^m), \cdots\cdots, V_i^r(\gamma^1, \cdots\cdots, \gamma^m))$.

By the hypothesis of the induction we have for each $j\,(l \leqq j \leqq i)$

$$\begin{array}{l} cl(\beta^1), \cdots\cdots, cl(g^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\ \gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m \rightarrow V_j^r(\gamma^1, \cdots\cdots, \gamma^m) \equiv V_j^r(\delta^1, \cdots\cdots, \delta^m). \end{array}$$

On the other hand we have easily

$$\begin{array}{l} cl(g'),\ V_1^r(\gamma^1, \cdots\cdots, \gamma^m) \overset{*}{\equiv} V_1^r(\delta', \cdots\cdots, \delta^m), \cdots\cdots, \\ \qquad V_i^r(\gamma^1, \cdots\cdots, \gamma^m) \overset{*}{\equiv} V_i^r(\delta^1, \cdots\cdots, \delta^m) \rightarrow V^r(\gamma^1, \cdots\cdots, \gamma^m) = V^r(\delta^1, \cdots\cdots, \delta^m). \end{array}$$

Hence we have

$$cl(\beta^1), \cdots\cdots, cl(g^1), \cdots\cdots, A_1, \cdots\cdots. A_N$$

$$\gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^n \overset{*}{\equiv} \delta^m \rightarrow \mathfrak{A}^r(\gamma^1, \cdots\cdots, \gamma^m)\, \mathfrak{A}^r(\delta^1, \cdots\cdots, \delta^m).$$

Hence we have

4) the case, where $V(\gamma^1, \cdots\cdots, \gamma^m)$ is of the form $\{\varphi\}\mathfrak{F}(\varphi, \gamma^1, \cdots\cdots, \gamma^m)$.
Then $V^r(\gamma^1, \cdots\cdots, \gamma^m)$ is $\{\varphi\}\mathfrak{F}^r(\varphi, \gamma^1, \cdots\cdots, \gamma^m)$.

We have only to prove that the following sequence is provable.

$$\begin{aligned}
&\gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m \\
&\quad \rightarrow \forall\varphi\forall\psi\{\varphi \overset{*}{\equiv} \psi \wedge \mathfrak{F}^r(\varphi, \gamma^1, \cdots\cdots, \gamma^m) \vdash \mathfrak{F}^r(\psi, \gamma^1, \cdots\cdots, \gamma^m)\} \\
&\quad \wedge \forall\varphi\forall\psi\{\varphi \overset{*}{\equiv} \psi \wedge \mathfrak{F}^r(\varphi, \delta^1, \cdots\cdots, \delta^m) \vdash \mathfrak{F}^r(\psi, \delta^1, \cdots\cdots, \delta^m)\} \\
&\quad \wedge \forall\varphi\{cl(\varphi) \vdash (\mathfrak{F}^r(\oplus_n(\varphi), \gamma^1, \cdots\cdots, \gamma^m) \vdash\dashv \mathfrak{F}^r(\oplus_n(\varphi), \delta^1, \cdots\cdots, \delta^m))\}.
\end{aligned}$$

On the other hand the following sequence is provable by tpe hypothesis of the induction.

$$\begin{aligned}
&cl(\beta^1), \cdots\cdots, cl(g^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\
&\gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m,\ \gamma \overset{*}{\equiv} \delta,\ \mathfrak{F}^r(\gamma, \gamma^1, \cdots\cdots, \gamma^m) \rightarrow \mathfrak{F}^r(\delta, \gamma^1, \cdots\cdots, \gamma^m)
\end{aligned}$$

Therefore we have only to prove

$$\begin{aligned}
&cl(\beta^1), \cdots\cdots, cl(g^1), \cdots\cdots, A_1, \cdots\cdots, A_N \\
&\gamma' \overset{*}{\equiv} \delta', \cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m,\ cl(\alpha) \\
&\quad \rightarrow \mathfrak{F}^r(\oplus_n(\alpha), \gamma', \cdots\cdots, \gamma^m) \vdash\dashv \mathfrak{F}^r(\oplus_n(\alpha), \delta^1, \cdots\cdots\ \delta^m).
\end{aligned}$$

According to 7.11 we have only to prove

$$\begin{aligned}
&cl(\beta'), \cdots\cdots, cl(g'), \cdots\cdots, A_1, \cdots\cdots, A_N, \\
&\gamma^1 \overset{*}{\equiv} \delta^1, \cdots\cdots, \gamma^m \overset{*}{\equiv} \delta^m,\ cl(\alpha) \rightarrow \mathfrak{F}^r(\alpha, \gamma^1, \cdots\cdots, \gamma^m) \vdash\dashv \mathfrak{F}^r(\alpha, \gamma^1, \cdots\cdots, \delta^m).
\end{aligned}$$

And this is clear by the hypothesis of the induction. q.e.d.

7.13. Let $\alpha(n)$ and $\beta(n)$ be free variables of type n.

Then we define $\alpha \equiv \beta$ $(n=0.\ 1, 2, \cdots\cdots)$ as an abbreviation of the following formula or an arbitrary formula homologous to it.

7.13.1. $\alpha = \beta$, when n is zero.

7.13.2. $\forall\varphi(m)\{\alpha[\oplus_m(\varphi)] \vdash\dashv \beta[\oplus_m(\varphi)]\}$, when n is $k+1$ $(k=0, 1, 2, \cdots\cdots)$.

Clearly the following sequence are provable.

7.13.3. $\rightarrow \alpha \equiv \alpha$

7.13.4. $\alpha \equiv \beta \rightarrow \beta \equiv \alpha$

7.13.5. $\alpha \equiv \beta,\ \beta \equiv \gamma \rightarrow \alpha \equiv \gamma$

7.4. We define the equality-axioms which will be denoted by Γ_e as the series of the following axioms;

$$\begin{aligned}
&\forall\varphi(n_1+1, \cdots\cdots, n_i+1)\forall\varphi^1\forall\psi^1 \cdots\cdots \forall\varphi^i\forall\psi^i\{\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^i \\
&\qquad \equiv \psi^i \vdash \varphi[\varphi^1, \cdots\cdots, \varphi^i] \vdash\dashv \varphi[\psi^1, \cdots\cdots, \psi^i]
\end{aligned}$$

for all the types $(n_1+1. \cdots\cdots, n_i+1)$

$$\begin{aligned}
&\forall p(m_1+1, \cdots\cdots, m_j+1)\forall\varphi^1\forall\psi^1 \cdots\cdots \forall\varphi^j\forall\psi^j\{\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^j \\
&\qquad \equiv \psi^j \vdash p(\varphi^1, \cdots\cdots, \varphi^j) = p(\psi^1, \cdots\cdots, \psi^j)\}
\end{aligned}$$

for all the types $(m_1+1, \cdots\cdots, m^j+1)$.

7.15. The Theorem of Equality in the Ordinary Sense

Under the same hypothesis of 7.12 the following axioms are consistent;

$$\Gamma^e,\ A_1, \cdots\cdots, A_N$$

Proof. Clearly we have

$$\forall\varphi\mathfrak{F}<n+1>(\varphi(n+1)),\quad \forall\psi\mathfrak{F}<n>(\psi(n))\to\alpha\overset{n+1}{\equiv}\beta \vdash\dashv \alpha\equiv\beta.$$

Therefore this theorem holds according to 7.12.

From now on, we call 7.14 the Axioms on Equality.

The following corollaries are evident by the proof of 7.13 and 7.15.

7.16. Corollary.

Let $*_1=*_2$ be a special variable of type (1, 1) and R be the system of equality restriction depending on $*_1=*_2$ and of the from $\{\mathfrak{F}<n_1+1. \cdots\cdots, n_i+1>(\alpha(n_1+1, \cdots\cdots, n_i+1))$; $\mathfrak{G}<m_1+1, \cdots\cdots, m_j+1>(f(m_1+1, \cdots\cdots, m_j+1))\}$ and let $A_1, \cdots\cdots, A_N$ be consistent axioms and have neither $\forall$ of the form $\forall\varphi$, $\forall p$, nor E of the form Eφ, Ep other than of the form $\forall x$, Ey and contain the following axioms;

$$\forall x(x=x)$$
$$\forall x\forall y(x=y \vdash y=x)$$
$$\forall x\forall y\forall z(x=y \wedge y=z \vdash x=z)$$

If the sequence $A_1, \cdots\cdots, A_N\to\mathfrak{F}<n_1+1, \cdots\cdots, n_i+1>(\sigma)$ is provable for each special variable $\sigma(n_1+1, \cdots\cdots, n_i+1)$ which is contained in $A_1, \cdots\cdots, A_N$ and the Sequence $A_1, \cdots\cdots, A_N\to\mathfrak{G}<m_1+1, \cdots\cdots, m_j+1>(s)$ is provable for each special function $s(m_1+1, \cdots\cdots, m_j+1)$ contained in $A_1, \cdots\cdots, A_N$, then the following the axioms are consistent;

$$A_1,\ \cdots\cdots,\ A_N,\ \Gamma_e.$$

7.17. Corollary.

Let $*_1=*_2$ be a special variable of type (1, 1) and R be the system of equality restriction depending on $*_1=*_2$ and of the form $\{\mathfrak{F}<n_1+1, \cdots\cdots, n_i+1>(\alpha(n_1+1, \cdots\cdots, n_i+1))$; $\mathfrak{G}<m_1+1, \cdots\cdots, m_j+1>(f(m_1+1, \cdots\cdots, m_j+1))\}$ and let $A_1, \cdots\cdots, A_N$ be consistent axioms and have neither a form $\forall\varphi$, $\forall p$ nor a form Eφ, Ep other than the form $\forall x$, Ey, $\forall\varphi(1, \cdots\cdots, 1)E\psi(1, \cdots\cdots, 1)$ and contain following axioms

$$\forall x\forall y\forall\varphi(1)\{x=y \vdash\dashv (\varphi[x] \vdash\dashv \varphi[y])\}.$$

If the sequence $A_1, \cdots\cdots, A_N\to\mathfrak{F}<n_1+1, \cdots\cdots, n_i+1)(\sigma)$ is provable for each special variable $\sigma(n_1+1, \cdots\cdots, n_i+1)$ contained in $A_1, \cdots\cdots A_N$ and the sequence $A_1,\cdots\cdots, A_N\to\mathfrak{G}<m_1+1, \cdots\cdots, m_j+1>(s)$ is provable for each special function $s(m_1+1, \cdots\cdots, m_j+1)$ contained $A_1, \cdots\cdots A_N$, then the following axioms are onsistent $A_1, \cdots\cdots A_N, \Gamma_e$.

7.18. Γ_a is denoted by the series of the following axioms;

$$\forall x\ (x=x)$$
$$\forall x\forall y\ (x=y \vdash y=x)$$
$$\forall x\forall y\forall z\ (x=y \wedge y=z \vdash x=z)$$
$$\forall x \neg (x'=1)$$
$$\forall x\forall y\ (x=y \vdash\dashv x'=y')$$

7.19. If Γ_a is consistent, then Γ_a and Γ_e are consistent.

Proof. It is clear according to 7.15 and 7.16.

7.20. The system of axioms Γ is equivalent to the system of axioms Π, if and only if $\Pi \rightarrow A$ is provable for an arbitrary formula A of Γ and $\Gamma \rightarrow B$ is provable for an arbitrary formula B of Π.

7.21. Γ_e is equivalent to the following system of axioms which will be denoted by $\tilde{\Gamma}_e$;

$$\forall\varphi\forall x\forall y\,(x=y \vdash (\varphi[x] \dashv\vdash \varphi[y]))$$

$$\forall p(n+1)\,\forall\varphi'(n)\,\forall\psi'(n)\,\{\varphi' \equiv \psi' \vdash (\varphi[\varphi'] \dashv\vdash \varphi[\psi'])\} \qquad \text{for all } n.$$

$$\forall p(m+1)\,\forall\varphi(m)\,\forall\psi(m)\,\{\varphi \equiv \psi \vdash p(\varphi) = p(\psi)\} \qquad \text{for all } m.$$

Proof. We have only to prove that $\tilde{\Gamma}_e \rightarrow A$ is provable for an arbitrary axiom A of Γ_e.

As other cases are similar, we treat only the following special case;

$$\tilde{\Gamma}_e \rightarrow \forall\varphi(n+1, m+1)\,\forall\varphi'(n)\,\forall\psi'(n)\,\forall\varphi^2(m)\,\forall\psi^2(m)$$
$$\{\varphi' \equiv \psi' \wedge \varphi^2 \equiv \psi^2 \vdash (\varphi[\varphi', \varphi^2] \dashv\vdash \varphi[\varphi', \psi^2])\}$$

Therefore we have only to prove

$$\tilde{\Gamma}_e,\ \alpha' \equiv \beta',\ \alpha^2 \equiv \beta^2 \rightarrow \gamma[\alpha', \alpha^2] \dashv\vdash \gamma[\beta', \beta^2].$$

Now we have easily the following sequences

$$\tilde{\Gamma}_e,\ \alpha' \equiv \beta' \rightarrow \gamma[\alpha', \alpha^2] \dashv\vdash \gamma[\beta', \beta^2].$$

and

$$\tilde{\Gamma}_e,\ \alpha^2 \equiv \beta^2 \rightarrow \gamma[\beta', \alpha^2] \dashv\vdash \gamma[\beta', \beta^2],$$

Hence we have easily

$$\tilde{\Gamma}_e,\ \alpha' \equiv \beta',\ \alpha^2 \equiv \beta^2 \rightarrow \gamma[\alpha', \alpha^2] \dashv\vdash \gamma[\beta', \beta^2].$$

7.22. If Γ_a is consistent, then the following axioms are consistent.

$$\Gamma_a,\ \tilde{\Gamma}_e,\ \forall\varphi\forall x\{\varphi[1] \wedge \forall y\,(\varphi[y] \vdash \varphi[y']) \vdash \varphi[x]\}.$$

Proof. $e(a)$ denotes $\forall\varphi\{\varphi[1] \wedge \forall y\,(\varphi[y] \vdash \varphi[y']) \vdash \varphi[a]\}$.

Then by the corollary 7.8 we have only to prove that the following axioms are consistent;

$$\Gamma_a^{e(\ )},\quad \tilde{\Gamma}_e^{e(\ )}$$

$$e(1)$$

$$\mathrm{E}x\ e(x)$$

$$\forall x\,(e(x) \vdash e(x'))$$

$$\forall\varphi\forall x\{e(x) \wedge \varphi[1] \wedge \forall y\,(\varphi[y] \vdash \varphi[y']) \vdash \varphi[x]\}.$$

And each axiom A of this form is provable under Γ_a and $\tilde{\Gamma}_e$, that is $\Gamma_a, \tilde{\Gamma}_e \rightarrow A$ is provable.

7.23. By the way we shall arrange the systems of Equality Axioms.

Γ_e.

$$\forall\varphi\forall\varphi^1\forall\psi^1 \cdots\cdots \forall\varphi^i\forall\psi^i\{\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^i \equiv \psi^i$$
$$\vdash (\varphi[\varphi^1, \cdots\cdots, \varphi^i] \dashv\vdash \varphi[\psi^1, \cdots\cdots, \psi^i]\}$$

$$\forall p\forall\varphi^1\forall\psi^1 \cdots\cdots \forall\varphi^j\forall\psi^j\{\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^j \equiv \psi^j$$
$$\vdash p(\varphi^1, \cdots\cdots, \varphi^j) = p(\psi^1, \cdots\cdots, \psi^j)\}$$

Γ_e'.

$$\forall\varphi\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i\{\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^i\equiv\psi^i \vdash (\varphi[\varphi^1,\cdots\cdots,\varphi^i]\dashv\vdash\varphi[\psi^1,\cdots\cdots,\psi^i])\}$$

$$\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^j\forall\psi^j\{\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^j\equiv\psi^j \vdash s(\varphi^1,\cdots\cdots,\varphi^j)=s(\psi^1,\cdots\cdots,\psi^j)\}$$

for each special function s.

$\tilde{\Gamma}_e$.

$$\forall\varphi\forall x\forall y(x=y\vdash(\varphi[x]\dashv\vdash\varphi[y]))$$

$$\forall\varphi\forall\varphi^1\forall\psi^1\{\varphi^1\equiv\psi^1\vdash(\varphi']\dashv\vdash[\psi']))\}$$

$$\forall p\forall\varphi\forall\psi\{\varphi\equiv\psi\vdash p(\varphi)=p(\psi)\}$$

$\tilde{\Gamma}_e'$.

$$\forall\varphi\forall x\forall y(x=y\vdash(\varphi[x]\dashv\vdash\varphi[y]))$$

$$\forall\varphi\forall\varphi'\forall\psi'\{\varphi'\equiv\psi'\vdash(\varphi[\varphi']\dashv\vdash\varphi[\psi'])\}$$

$$\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^j\forall\psi^j\{\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^j\equiv\psi^j\vdash s(\varphi^1,\cdots\cdots,\varphi^j)=s(\psi^1,\cdots\cdots,\psi^j)\}$$

for each special function s.

7.24. a strict restriction.

Let R be a system of restriction

$$\{\mathfrak{F}<n_1,\cdots\cdots,n_i>(\alpha(n_1,\cdots\cdots,n_i));\ \mathfrak{G}<m_1,\cdots\cdots,m_j>(f(m_1\ \cdots\cdots,m_j))\}.$$

A strict restriction $A^{\tilde{r}}$ of a formula A or a variety A depending on R is defined recursively as follows and we call $\tilde{r}$ a strictly restricting operator of R.

7.24.1. If A contains neither a bound variable nor a bound function, then $A^{\tilde{r}}$ is an arbitrary figure homologous to A. So we may consider that $A^{\tilde{r}}$ is A.

7.24.2. $(\neg\mathfrak{A})^{\tilde{r}}$ is $\neg\mathfrak{A}^{\tilde{r}}$.

$(\mathfrak{A}\wedge\mathfrak{B})^{\tilde{r}}$ is $\mathfrak{A}^{\tilde{r}}\wedge\mathfrak{B}^{\tilde{r}}$.

$(\mathfrak{A}\vee\mathfrak{B})^{\tilde{r}}$ is $\mathfrak{A}^{\tilde{r}}\vee\mathfrak{B}^{\tilde{r}}$.

7.24.3. Let $A(\alpha)$ be a formula or a variety and β be not contained in $A(\alpha)$. We define $A^{\tilde{r}}(\beta)$ as $(A(\beta))^{\tilde{r}}$ of a full indication for β. So $A^{\tilde{r}}(\alpha)$, $A^{\tilde{r}}(\varphi)$ or $A^{\tilde{r}}(V)$ is of obtained from $A^{\tilde{r}}(\beta)$ be substituted α, φ or V respectively for β at all the indicated places in $A^{\tilde{r}}(\beta)$. In the same way we define $A^{\tilde{r}}(f)$, $A^{\tilde{r}}(p)$, $A^{\tilde{r}}(F)$ or $A^{\tilde{r}}(\alpha,\varphi,V,f,p,F)$ etc.

7.24.4.

$(\forall\varphi\mathfrak{F}(\varphi))^{\tilde{r}}$ is $\forall\psi(cl(\psi)\vdash\mathfrak{F}^{\tilde{r}}(\psi))$, where ψ is contained in neither $cl(\alpha)$ nor $\mathfrak{F}^{\tilde{r}}(\alpha)$.

$(E\varphi\mathfrak{F}(\varphi))^{\tilde{r}}$ is $E\psi(cl(\psi)\wedge\mathfrak{F}^{\tilde{r}}(\psi))$, where ψ is contained in neither $cl(\alpha)$ nor $\mathfrak{F}^{\tilde{r}}(\alpha)$.

$(\forall p\mathfrak{A}(p))^{\tilde{r}}$ is $\forall q(cl(q)\vdash\mathfrak{A}^{\tilde{r}}(q))$, where q is contained in neither $cl(f)$ nor $\mathfrak{A}^{\tilde{r}}(f)$.

7.24.5. $(\{\varphi_1, \cdots\cdots, \varphi_n\}\mathfrak{F}(\varphi_1, \cdots\cdots \varphi_n))^{\tilde{r}}$ is $\{\psi_1, \cdots\cdots, \psi_n\}cl(\psi_1) \wedge \cdots\cdots \wedge cl(\psi_n) \wedge \mathfrak{F}^{\tilde{r}}(\psi_1, \cdots\cdots, \psi_n)$, where $\psi_1, \cdots\cdots, \psi_n$ are different from each other and not contained in $\mathfrak{F}^{\tilde{r}}(\alpha_1, \cdots\cdots, \alpha_n)$.

7.24.6. $(\alpha[V_1, \cdots\cdots, V_n])^{\tilde{r}}$ is $\alpha(V_1^{\tilde{r}}, \cdots\cdots, V_n^{\tilde{r}})$.

7.24.7. $(f(V_1, \cdots\cdots, V_n))^{\tilde{r}}$ is $f(V_1^{\tilde{r}}, \cdots\cdots, V_n^{\tilde{r}})$.

7.25. Let F be a functional of the form $\{\varphi_1, \cdots\cdots, \varphi_n\}T(\varphi_1, \cdots\cdots, \varphi_n)$. We define the strict restriction $F^{\tilde{r}}$ of F as $\{\psi_1, \cdots\cdots, \psi_n\}T^{\tilde{r}}(\psi_1, \cdots\cdots, \psi_n)$, where $\psi_1, \cdots\cdots, \psi_n$ are different from each other and not contained in $T^{\tilde{r}}(\alpha_1, \cdots\cdots, \alpha_n)$.

7.26. Let R be a system of restriction $\{\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha);\ \mathfrak{G}<m_1, \cdots\cdots, m_j>(f)\}$ and $\tilde{r}$ be its strictly restricting opeartor. Moreover let Γ_0 be a series of axioms containing the equality axioms and fulfil the following conditions;

1) If V is an arbitrary variety and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all the free or special variables and functions respectively which are contained in V, then $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow cl(V^{\tilde{r}})$ is provable in GLC.

2) If F is anarbitrary functional and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all the free or special variables and functions respectively which are contained in F, then $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow cl(F^{\tilde{r}})$ is prevable in GLC.

Then the following sequences are provable in GLC.

7.26.1. $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{F}(V_1, \cdots\cdots, V_i))^{\tilde{r}} \vdash\!\dashv \mathfrak{F}^{\tilde{r}}(V_1^{\tilde{r}}, \cdots\cdots, V_i^{\tilde{r}})$

$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{V}(V_1, \cdots\cdots, V_i))^{\tilde{r}} \equiv \mathfrak{V}^{\tilde{r}}(V_1^{\tilde{r}}, \cdots\cdots, V_i^{\tilde{r}})$

Here $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots$ are all the free or special variables and functions which are contained in $\mathfrak{F}(V_1, \cdots\cdots, V_i)$ and $\mathfrak{V}(V_1, \cdots\cdots, V_i)$.

7.26.2. $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{V}(F))^{\tilde{r}} \vdash\!\dashv \mathfrak{F}^{\tilde{r}}(F^{\tilde{r}})$

$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{V}(F))^{\tilde{r}} \equiv \mathfrak{V}^{\tilde{r}}(F^{\tilde{r}})$.

Here $cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots$ are all the free or special variables and functions which are contained in $\mathfrak{F}(F)$ and $\mathfrak{V}(F)$.

Proof. First we prove 7.26.1 by the double induction of the maximal number of heights of $V_1, \cdots\cdots, V_i$ and the number of stages to construct $\mathfrak{F}(\alpha_1, \cdots\cdots, \alpha_i)$ or $\mathfrak{V}(\alpha_1, \cdots\cdots, \alpha_i)$. As other sases are simple, we treat only the essential case, where $\mathfrak{F}(\alpha_1, \cdots\cdots, \alpha_i)$ is $\alpha_1[\mathfrak{V}_1(\alpha_1, \cdots\cdots, \alpha_i), \cdots\cdots, \mathfrak{V}_j(\alpha_1, \cdots\cdots, \alpha_i)]$ and $\mathfrak{V}(\alpha_1, \cdots\cdots, \alpha_i)$ is $\{\psi\}\mathfrak{A}(\alpha_1, \cdots\cdots, \alpha_i)$.

Then by the hypothesis of the induction we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{V}_1(V_1, \cdots\cdots, V_i))^{\tilde{r}} \equiv \mathfrak{V}_1^{\tilde{r}}(V_1^{\tilde{r}}, \cdots\cdots, V_i^{\tilde{r}})$$

$$\vdots$$

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow (\mathfrak{V}_j(V_1, \cdots\cdots, V_i))^{\tilde{r}} \equiv \mathfrak{V}_j^{\tilde{r}}(V_1^{\tilde{r}}, \cdots\cdots, V_i^{\tilde{r}}).$$

Without the loss of generality we can assume that V_1 is of the from

$$\{\varphi_1, \cdots\cdots, \varphi_j\}\mathfrak{G}(\varphi_1, \cdots\cdots, \varphi_j).$$

Then we see that $V_1^{\tilde{r}}$ is

$$\{\varphi_1, \cdots\cdots \varphi_j\}(cl(\varphi_1) \wedge \cdots\cdots \wedge cl(\varphi_j) \wedge \mathfrak{G}^{\tilde{r}}(\varphi_1, \cdots\cdots, \varphi_j)).$$

Therefore by the hypothesis of the theorem we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{\mathfrak{F}^r}(\widetilde{V_1^r}, \cdots\cdots, \widetilde{V_i^r})$$
$$\vdash \widetilde{\mathfrak{G}^r}(\widetilde{(\mathfrak{B}_1(V_1, \cdots\cdots, V_i))^r}, \cdots\cdots, \widetilde{(\mathfrak{B}_j(V_1, \cdots\cdots, V_i))^r})$$

As the maximal number of heights of $\mathfrak{B}_1(V_1, \cdots\cdots, V_i), \cdots\cdots, \mathfrak{B}_j(V_1, \cdots\cdots, V_i)$ is less than the maximal number of heights of $V_1, \cdots\cdots, V_i$, by the hypothesis of the induction we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{\mathfrak{G}^r}(\widetilde{(\mathfrak{B}_1(V_1, \cdots\cdots, V_i))^r},$$
$$\cdots\cdots, \widetilde{(\mathfrak{B}_j(V_1, \cdots\cdots, V_i))^r}) \vdash \widetilde{(\mathfrak{G}(\mathfrak{B}_1(V_1, \cdots\cdots, V_i),}$$
$$\widetilde{\cdots\cdots, \mathfrak{B}_j(V_1, \cdots\cdots, V_i)))^r}$$

Therefore we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{\mathfrak{F}^r}(V_1^r, \cdots\cdots, \widetilde{V_i^r}) \vdash \widetilde{(\mathfrak{F}(V_1, \cdots\cdots, V_i))^r}$$

Now we consider the case where $\mathfrak{B}(\alpha_1, \cdots\cdots, \alpha_i)$ is $\{\psi\}\mathfrak{A}(\alpha_1, \cdots\cdots, \alpha_i, \psi)$. $\widetilde{\mathfrak{B}^r}(\alpha_1, \cdots\cdots, \alpha_i)$ being $\{\psi\}(cl(\psi) \wedge \widetilde{\mathfrak{A}^r}(\alpha_1, \cdots\cdots, \alpha_i, \psi))$, $\widetilde{\mathfrak{B}^r}(\widetilde{V_1^r}, \cdots\cdots, \widetilde{V_i^r})$ is

$$\{\psi\}(cl(\psi) \wedge \widetilde{\mathfrak{A}^r}(\widetilde{V_1^r}, \cdots\cdots, \widetilde{V_i^r}, \psi))$$

And by the hypothesis of the induction we see

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \{\psi\}(cl(\psi) \wedge \widetilde{(\mathfrak{A}(V_1, \cdots\cdots, V_i, \psi))^r})$$
$$\equiv \{\psi\}(cl(\psi) \wedge \mathfrak{A}^r(\widetilde{V_1^r}, \cdots\cdots, \widetilde{V_i^r}, \psi))$$

From the above holds the proposition.

Now we prove 7.26.2 by using of 7.26.1 and by the induction on the number of the stages to construct $\mathfrak{F}(f)$ and $\mathfrak{B}(f)$. As other case are simple, we treat only the essential case, where $\mathfrak{B}(f)$ is $f(V_1(f), \cdots\cdots, V_i(f))$. Without the loss of generality we can assume that F is of the form $\{\varphi_1, \cdots\cdots, \varphi_i\}$ $T(\varphi_1, \cdots\cdots, \varphi_i)$.

Then $\mathfrak{B}(F)$ is $T(V_1(F), \cdots\cdots, V_i(F))$ and $\widetilde{\mathfrak{B}^r}(\widetilde{F^r})$ is $\widetilde{T^r}(\widetilde{V_1^r}(\widetilde{F^r})$ $\cdots\cdots, \widetilde{V_i^r}(\widetilde{F^r}))$.
By the hypothesis of the induction we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{V_1^r}(\widetilde{F^r}) \equiv \widetilde{(V_1(F))^r}$$
$$\vdots$$
$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{V_i^r}(\widetilde{F^r}] \equiv \widetilde{(V_i(F))^r}.$$

By the above and 7.26.1 we have

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \Gamma_0 \rightarrow \widetilde{\mathfrak{B}^r}(\widetilde{F^r}) \equiv \widetilde{(T(V_1(F), \cdots\cdots, V_i(F)))^r}.$$

And the proposition is proved.

7.27. (The Theorem on Strict Restriction)

Let $A, \cdots\cdots, A_N$ be arbitrary axioms and let R be a system of restriction $\{\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha);\ \mathfrak{G}<m_1, \cdots\cdots, m_j>(f)\}$ and $\tilde{r}$ be its strictly restricting operator.

Now we consider the following axioms;

$A_1, \cdots\cdots, A_N$

$cl(\sigma(n_1, \cdots\cdots, n_i))$ for all the special variables σ which are contained in $A_1, \cdots\cdots, A_N$

$cl(s)$ for all the special functions s which are contained in $A_1, \cdots\cdots,$ A_N

$$\mathrm{E}\varphi cl(\varphi(n_1, \cdots\cdots, n_i)) \text{ for all } (n_1, \cdots\cdots, n_i).$$
$$\mathrm{E}p cl(p(m, \cdots\cdots, m)) \text{ for all } (m_1, \cdots\cdots, m_j).$$

and Γ_0 denotes the series of these axioms.

If the following conditions are fulfilled, then the axioms $A_1, \cdots\cdots, A_N$ are consistent in GLC.

1) There exists a series of axioms $\widetilde{\Gamma}$ containing the equality axioms such that for an arbitary axiom A of $\widetilde{\Gamma}_0$ $\widetilde{\Gamma}\to A$ is provable in GLC and $\widetilde{\Gamma}$ is consistent in GLC.

2) If V is an arbitary variety and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all the free or opecial variables and all the free or opecial functions respectively which are contained in V, then

$$cl(\alpha), \cdots\cdots, cl(f), \cdots\cdots, \widetilde{\Gamma}\to cl(V^{\widetilde{r}})$$

is provable in GLC.

3) If F is an arbitrary functional and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all the freeor special variables and functions respectively which are contained in F, then

$$cl(\alpha), \cdots\cdots, cl(f), \widetilde{\Gamma}\to cl(F^{\widetilde{r}})$$

is provable in GLC.

Proof. By the proposition 7.26 we can perform the proof of this theorem in the same way as in the proof of the theorem on restriction.

§ 8. Type-elevation.

8.1. GLC without bound function

A proof-figure of GLC is called a proof-figure of GLC without bound function, if and only if it contains no bound function. A sequence S is called provable in GLC without bound function, if and only if there exists a proof-figure of GLC without bound function, whose end-sequence is S.

8.2. Word

If $A(\alpha_1, \cdots\cdots, \alpha_n, f_1, \cdots\cdots, f_m)$ is a formula or a variety, then the figure of the form $A(\varphi_1, \cdots\cdots, \varphi_n, p_1, \cdots\cdots, p_m)$ is called a 'word', where $\varphi_1, \cdots\cdots, \varphi_n, p_1, \cdots\cdots, p_m$ are different from each other and not contained in $A(\alpha, \cdots\cdots, \alpha_n, f_1, \cdots\cdots, f_m)$.

In particular, any formula and any variety are words.

8.3. The total number of variables, functions and logical symbols which is contained in a word A is called a word-number of A.

8.4. Let A be a word. We define recursively as follows the type-elevated figure A^ρ of A and we call ρ a type-elevator.

8.4.1. a^ρ is $\{x_1\}\bar{a}(1)[x_1]$.

x^ρ is $\{x_1\}\bar{x}(1)[x_1]$.

8.4.2. If A is of the from $\neg B, B\wedge C$ or $B\vee C$, then A^ρ is $\neg B^\rho$, $B^\rho\wedge C^\rho$ or $B^\rho\vee C^\rho$ respedtively.

8.4.3. If A is of the form $\forall\varphi(n_1, \cdots\cdots, n_i)B$, $\mathrm{E}\varphi(n_1, \cdots\cdots, n_i)B$, $\forall p(m_1, \cdots\cdots, m_j)C$ or $\mathrm{E}p(m_1, \cdots\cdots, m_j)C$, then A^ρ is $\forall\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)B^\rho$, $\mathrm{E}\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)B^\rho$, $\forall\bar{p}(m_1+1, \cdots\cdots, m_j+1, 1)C^\rho$ or $\mathrm{E}\bar{p}(m_1+1, \cdots\cdots, m_j+1, 1)C^\rho$ respectively.

8.4.4. If A is of the form $\alpha(n_1, \cdots\cdots, n_i)[A_1, \cdots\cdots, A_i]$ or $\varphi(n_1, \cdots\cdots, n_i)[A_1, \cdots\cdots, A_i]$, then A^ρ is $\bar{\alpha}(n_1+1, \cdots\cdots, n_i+1)[A_1^\rho, \cdots\cdots, A_i^\rho]$ or $\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)[A_1^\rho, \cdots\cdots, A_i^\rho]$ respectively.

8.4.5. If A is of the form $f(n_1, \cdots\cdots, n_i)(A_1, \cdots\cdots, A_i)$ or $p(n_1, \cdots\cdots, n_i)(A_1, \cdots\cdots, A_i)$, then A^ρ is $\{x_m\}\bar{f}(n_1+1, \cdots\cdots, n_i+1, 1)[A_1^\rho, \cdots\cdots, A_i^\rho, x_m]$ or $\{x_m\}\bar{p}(n_1+1, \cdots\cdots, n_i+1, 1)[A_1^\rho, \cdots\cdots, A_i^\rho, x_m]$ respectively, where m is a word-number of A.

8.4.6. If A is of the form $\{\varphi_1(n_1), \cdots\cdots, \varphi_i(n_i)\}B$, then A^ρ is $\{\bar{\varphi}_1(n_1+1), \cdots\cdots, \varphi_i(n_i+1)\}B^\rho$.

8.5. From now on in this section we suppose the following properties.

8.5.1. If α is a free variable or type $(n_1, \cdots\cdots, n_i)$, then $\bar{\alpha}(n_1+1, \cdots\cdots, n_i+1)$ is a free variable of type $(n_1+1, \cdots\cdots, n_i+1)$.

8.5.2. If φ is a bound variable of type $(n_1, \cdots\cdots, n_i)$, then $\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)$ is a bound variable of type $(n_1+1, \cdots\cdots, n_i+1)$.

8.5.3. If f is a free function of type $(n_1, \cdots\cdots, n_i)$, then $\bar{f}(n_1+1, \cdots\cdots, n_i+1, 1)$ is a free variable of type $(n_1+1, \cdots\cdots, n_i+1, 1)$.

8.5.4. If p is a bound function of type $(n_1, \cdots\cdots, n_i)$, then $\bar{p}(n_1+1, \cdots\cdots, n_i+1, 1)$ is a bound variable of type $(n_1+1, \cdots\cdots, n_i+1, 1)$.

8.6. We can see easily the following properties.

8.6.1. If A is a word and its word-number is m and all the variables and all the functions which are contained in A are $\alpha, \cdots\cdots, \varphi, \cdots\cdots, f, \cdots\cdots, p, \cdots\cdots$, then A^ρ has neither a function nor a variable other than $x_1, \cdots\cdots, x_m, \bar{\alpha}, \cdots\cdots, \varphi, \cdots\cdots, \bar{f}, \cdots\cdots, \bar{p}, \cdots\cdots$.

8.6.2. If A is a word and A^ρ coutains the form $\forall\bar{\varphi}$ or $\mathrm{E}\bar{\varphi}$ or $\forall\bar{p}$ or $\mathrm{E}\bar{p}$ or $\{\cdots\cdots, \bar{\varphi}_i, \cdots\cdots\}$, then A contains the form $\forall\varphi$ or $\mathrm{E}\varphi$ or $\forall p$ or $\mathrm{E}p$ or $\{\cdots\cdots, \varphi_i, \cdots\cdots\}$ respectively, where φ_i is not of the form x_j and so is not type 0.

8.7. Let $A(\alpha(n_1, \cdots\cdots, n_i))$ be a word of a full indication for α. Then we define $A^\rho(\bar{\alpha}(n_1+1, n_i+1))$ as $(A(\alpha(n_1, \cdots\cdots, n_i)))^\rho$ of a full indication for $\bar{\alpha}(n_1+1, \cdots\cdots, n_i+1)$. In general we define $A^\rho(\bar{\beta}(n_1+1, \cdots\cdots, n_i+1))$ as a figure which is obtained from $A^\rho(\bar{\gamma}(n_1+1, \cdots\cdots, n_i+1))$ by substituting $\bar{\beta}$ for $\bar{\gamma}$ at all the indicated places in $A^\rho(\bar{\gamma}(n_1+1, \cdots\cdots, n_i+1))$, where γ is not contained in $A(\beta)$ and so $A(\gamma)$ is of a full indication for γ.

In the same way we define $A^\rho(\bar{\alpha}, \cdots\cdots, \bar{\beta}, \bar{f}, \cdots\cdots, \bar{g})$ etc.

Now the following propositions are evident by this definition.

8.7.1. $(\forall\varphi\mathfrak{F}(\varphi))^\rho$ is $\forall\bar{\varphi}\mathfrak{F}^\rho(\bar{\varphi})$ and $(\mathrm{E}\varphi\mathfrak{F}(\varphi))^\rho$ is $\mathrm{E}\bar{\varphi}\mathfrak{F}^\rho(\bar{\varphi})$.

8.7.2. $(\forall p\mathfrak{G}(p))^\rho$ is $\forall\bar{p}\mathfrak{G}^\rho(\bar{p})$ and $(\mathrm{E}\bar{p}\mathfrak{G}(p))^\rho$ is $\mathrm{E}p\mathfrak{G}^\rho(\bar{p})$.

8.7.3. $(\{\varphi_1, \cdots\cdots, \varphi_i\}\mathfrak{F}(\varphi_1, \cdots\cdots, \varphi_i))^\rho$ is $\{\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i\}\mathfrak{F}^\rho(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i)$.

8.8. If A is a formula or a variety of type $(n_1, \cdots\cdots, n_i)$, then A^ρ is a formula or a variety of type $(n_i+1, \cdots\cdots, n_i+1)$ respectively.

Proof. We prove this proposition by the induction on the word-number of A. As the other cases are simple from 8.6 and 8.7, we treat only the case where A is of the form $f(n_1, \cdots\cdots, n_i)(A_1, \cdots\cdots, A_i)$. By the hypothesis of the induction we have that A_k^ρ is a variety of type n_k for all $k(1 \leqq k \leqq i)$.

Therefore $\bar{f}(n_1+1, \cdots\cdots, n_i+1, 1)[A_1^\rho, \cdots\cdots, A_i^\rho, a]$ is a formula, where α is an arbitrary free variable of type 0. From 8.6 it follows that $A_1^\rho, \cdots\cdots, A_i^\rho$ have no x_m, where m is a word-number of A. Hence A^ρ is a variety of type 1.

We can see easily the following proposition.

8.9. Let A be a formula or a variety and let A' be a figure homologous to A. Then A'^ρ is homologous to A^ρ.

8.10. Now let F be a functional of the form $\{\varphi_1, \cdots\cdots, \varphi_i\} T(\varphi_1, \cdots\cdots, \varphi_i)$. Since $T(\varphi_1, \cdots\cdots, \varphi_i)$ is a word, $(T(\varphi_1, \cdots\cdots, \varphi_i))^\rho$ can be defined and is of the form $\{x_j\} T'(\bar{\varphi}_1, \cdots\cdots. \bar{\varphi}_i, x_j)$, where $T'(\bar{\alpha}_1, \cdots\cdots, \bar{\alpha}_i, a)$ is a formula and $\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i$ and x_j are not contained in $T'(\bar{\alpha}_1, \cdots\cdots, \bar{\alpha}_i, a)$. We define the type-elevated figure F^ρ of F as the figure of the form $\{\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i, x_j\} T'(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i, x_j)$.

8.10.1. If F is a functional of type $(n_i, \cdots\cdots, n_i)$, then F^ρ is a variety of type $(n_i+1, \cdots\cdots, n_i+1, 1)$.

8.10.2. If F' is homologous to F, then F'^ρ is homologous to F^ρ.

Moreover we have,

8.11. $(\oplus_n(\alpha(n)))^\rho$ is $\oplus_{n+1}(\bar{\alpha}(n+1))$.

Proof. We prove the proposition by the induction on n.

$$(\oplus_0(a))^\rho \text{ is } \{x_1\}\bar{a}(1)[x_1] \text{ and so } \oplus_1(\bar{a}(1)).$$

Now we assume that the proposition holds when $n=m$. $(\oplus_{m+1}(\alpha(m+1)))^\rho$ is $\{\varphi(m+1)\}\bar{\alpha}(m+2)[(\oplus_m(\varphi(m))^\rho]$ and so it is $\{\bar{\varphi}(m+1)\}\bar{\alpha}(m+2)[\oplus_{m+1}(\bar{\varphi}(m+1))]$ by the hypothesis of the induction and it is $\oplus_{m+2}(\bar{\alpha}(m+2))$.

Now we shall prove the following proposition.

8.12. Let $A(\alpha)$ be a formula or a variety and V be a variety of the same type as α. Then $(A(V))^\rho$ is homologous to $A^\rho(V^\rho)$. Moreover let $A(f)$ be a formula or a variety and F be a functional of the same type as f. Then $(A(F))^\rho$ is homologous to $A^\rho(F^\rho)$.

Proof. We prove the proposition by the double induction on the height M of α or f and the number N of stages to construct $A(\alpha)$ or $A(f)$ respectively.

If N is one, then the theorem is evident.

We assume that the theorem is proved, when M is less than M' and when M is M' and N is less than N'.

Now we prove the theorem when M is M' and N is N'.

As other cases are simple and similar, we treat only the case, where $A(f)$ is $f(A_1(f), \cdots\cdots, A_n(f))$ and F is

$$\{\varphi_1, \cdots\cdots, \varphi_n\} g(B_1(\varphi_1, \cdots\cdots, \varphi_n), \cdots\cdots, B_m(\varphi_1, \cdots\cdots, \varphi_n)).$$

Then by the definition $A^\rho(\bar{f})$ is $\{x_i\}\bar{f}(A_1^\rho(\bar{f}), \cdots\cdots. A_n^\rho(\bar{f}), x_i)$ and F^ρ is $\{\bar{\varphi}_1 \cdots\cdots, \bar{\varphi}_n, x_j\}\bar{g}(B_1^\rho(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_n), \cdots\cdots, B_m^\rho(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_n), x_j)$.

Therefore $A^\rho(F^\rho)$ is

$$\{x\}\bar{g}(B_1^\rho(A_1^\rho(F^\rho), \cdots\cdots, A_n^\rho(F^\rho)), \cdots\cdots, B_m^\rho(A_1^\rho(F^\rho), \cdots\cdots, A_n^\rho(F^\rho)), x),$$

where x is not contained in

$$\bar{g}(B_1^\rho(A_1^\rho(F^\rho),\cdots\cdots,A_n^\rho(F^\rho)),\cdots\cdots,B_m^\rho(A_1^\rho(F^\rho),\cdots\cdots,A_n^\rho(F^\rho)),a).$$

Hence by the hypothesis of the induction $A^\rho(F^\rho)$ is homologous to

$$\{x\}\bar{g}(B_1^\rho((A_1(F))^\rho,\cdots\cdots,(A_n(F))^\rho,\cdots\cdots,B_m^\rho((A_1(F))^\rho,\cdots\cdots,(A_n(F))^\rho),x)$$

Moreover by the hypothesis of the induction $A^\rho(F^\rho)$ is homologous to

$$\{x\}\bar{g}((B_1(A_1(F),\cdots\cdots,A_n(F)))^\rho,\cdots\cdots,(B_m(A_1(F),A_n(F)))^\rho,x).$$

Hence by the definition $A^\rho(F^\rho)$ is homologous to

$$(g(B_1(A_1(F),\cdots\cdots,A_n(F)),\cdots\cdots,B_m(A_1(F),\cdots\cdots,A_n(F))))^\rho.$$

Therefore $A^\rho(F^\rho)$ is homologous to $(A(F))^\rho$ and the theorem is proved.

8.13. Let Γ be a series of formulas $\mathfrak{A}_1,\cdots\cdots,\mathfrak{A}_n$. We define Γ^ρ as $\mathfrak{A}_1^\rho,\cdots\cdots,\mathfrak{A}_n^\rho$.

8.14. The Theorem of Type-elevation

Let $\Gamma\to\Delta$ be a provable sequence in GLC. Then $\Gamma^\rho\to\Delta^\rho$ is provable in GLC without function, moreover there exists a proof-figure, which has no function and whose end-sequence is $\Gamma^\rho\to\Delta^\rho$.

Proof. We prove the proposition by the induction on the number N of the inference-figures which are contained in a proof-figure to $\Gamma\to\Delta$.

If $\Gamma\to\Delta$ is of the form $\mathfrak{D}\to\mathfrak{D}$, then $\Gamma^\rho\to\Delta^\rho$ is $\mathfrak{D}^\rho\to\mathfrak{D}^\rho$ and so the theorem is evident.

Assuming that the theorem is proved, when N is less than N', we shall prove the theorem when N is N'. As other cases are simple and similar, we treat only the case, where the last inference-figure to $\Gamma\to\Delta$ is the following form:

$$\frac{\mathfrak{F}(F),\Gamma'\to\Delta}{\forall p\mathfrak{F}(p),\Gamma'\to\Delta}$$

Here $\Gamma\to\Delta$ is $\forall p\mathfrak{F}(p),\Gamma'\to\Delta$.

By the hypothesis of the induction $(\mathfrak{F}(F))^\rho$, $\Gamma'^\rho\to\Delta^\rho$ is provable in GLC without function. And so $\mathfrak{F}^\rho(F^\rho)$, $\Gamma'^\rho\to\Delta^\rho$ is provable in GLC without a bound function.

Hence $\forall p\mathfrak{F}^\rho(p)$, $\Gamma'^\rho\to\Delta^\rho$ and so $\Gamma^\rho\to\Delta^\rho$ are provable in GLC without a bound function and the theorem is proved.

8.15. Let a be an arbitrary free or special variable of type 0. We use an abbreviated notation $\widetilde{a}$ for the homology class of the variety $\{x\}(x=a)$.

If y is an arbitrary bound variable of type 0, then $\widetilde{y}$ is the figure which is obtained from $\widetilde{a}$ by substituting y for a, where we assume without the loss of generality that $\widetilde{a}$ has no y.

Now we can see easily the following proposition.

8.15.1. $\Gamma_e'\to\widetilde{a}\equiv\widetilde{b}\mapsto a=b$.

8.16. Now we define the homology class $\widetilde{\alpha}$ of the variety of $(n_1+1,\cdots\cdots,n_i+1)$ for an arbitrary free or special variable $\alpha(n_1,\cdots\cdots,n_i)$ by the induction on the height of α.

$$\widetilde{\alpha}\text{ is }\{\varphi_1(n_1),\cdots\cdots,\varphi_i(n_i)\}\mathrm{E}\psi_1(n_1-1)\cdots\cdots\mathrm{E}\psi_i(n_i-1)$$
$$(\varphi_1\equiv\widetilde{\psi}_1\wedge\cdots\cdots\wedge\varphi_i\equiv\widetilde{\psi}_i\wedge\alpha[\psi_1,\cdots\cdots,\psi_i])$$

If V_k is a variety of type n_k for all $k(1 \leqq k \leqq i)$, then $\widetilde{\alpha}[V_1, \cdots\cdots, V_i]$ represents $\beta[V_1, \cdots\cdots, V_i](\frac{\widetilde{\alpha}}{\beta})$, where $V_1, \cdots\cdots, V_i$ have no β.

Let $\varphi(n_1, \cdots\cdots, n_i)$ be an arbitrary bound variable, then $\widetilde{\varphi}$ and $\varphi[V_1, \cdots\cdots, V_i]$ are obtained from $\widetilde{\alpha}$ and $\widetilde{\alpha}[V_1, \cdots\cdots, V_i]$ respectively by substituting φ for α, where $V_1, \cdots\cdots, V_i$ have no α.

Now we prove the following proposition by the induction on the height of α.

8.16.1. $\Gamma_e' \to \widetilde{\alpha} \equiv \widetilde{\beta} \dashv\vdash \alpha \equiv \beta$.

Proof. $\widetilde{\alpha} \equiv \widetilde{\beta}$ is $\forall\psi(\widetilde{\alpha}[\psi] \dashv\vdash \widetilde{\beta}[\psi]$, that is,

$$\forall\psi(\mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \alpha[\varphi]) \dashv\vdash \mathrm{E}\varphi(\varphi \equiv \widetilde{\varphi} \wedge \beta[\varphi])).$$

Therefore we have only to prove

$$\Gamma_e', \forall\psi\{\mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \alpha[\varphi]) \vdash \mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \beta[\varphi])\}, \quad \alpha[\gamma] \to \beta[\gamma].$$

Now we have easily

$$\Gamma_e', \forall\psi\{\mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \alpha[\varphi]) \vdash \mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \beta[\varphi])\}, \quad \alpha[\gamma] \to \mathrm{E}\varphi(\widetilde{\gamma} \equiv \widetilde{\varphi} \wedge \beta[\varphi])$$

By the hypothesis of the induction and from this we have

$$\Gamma_e', \forall\psi\{\mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \alpha[\varphi]) \vdash \mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \beta[\varphi])\}, \quad \alpha[\gamma] \to \beta[\gamma]$$

Hence the proposition is proved.

8.17.1. $\Gamma_e' \to \forall\varphi_1 \cdots\cdots \forall\varphi_i\{\widetilde{\alpha}[\widetilde{\varphi}_1, \cdots\cdots, \widetilde{\varphi}_i] \dashv\vdash \alpha[\varphi_1, \cdots\cdots, \varphi_i[\}$

8.17.2. $\Gamma_e' \to \mathrm{E}\psi\forall\varphi_1 \cdots\cdots \forall\varphi_i\{\alpha[\widetilde{\varphi}_1, \cdots\cdots, \widetilde{\varphi}_i] \vdash \widetilde{\psi}[\widetilde{\varphi}_1, \cdots\cdots, \widetilde{\varphi}_i]\}$

Proof. $\widetilde{\alpha}[\widetilde{\gamma}_1, \cdots\cdots \widetilde{\gamma}_i]$ is $\mathrm{E}\varphi_1 \cdots\cdots \varphi_i(\widetilde{\gamma}_1 \equiv \widetilde{\varphi}_i \wedge \cdots\cdots \wedge \widetilde{\gamma}_i \equiv \widetilde{\varphi}_i \wedge \alpha[\varphi_1, \cdots\cdots, \varphi_i])$. And so from 8.16.1 we have 8.17.1. And 8.17.2 is obtained from 8.16.1.

18.8. Let $f(n_1, \cdots\cdots, n_i)$ be an arbitrary free or special function of type $(n_1, \cdots\cdots, n_i)$. We use an abbreviated notation $\widetilde{f}$ for the homology class of the variety of type $(n_1+1, \cdots\cdots, n_i+1, 1)$

$$\{\varphi_1, \cdots\cdots, \varphi_i, x\}[\mathrm{E}\psi_1 \cdots\cdots \mathrm{E}\psi_i(\varphi_1 \equiv \widetilde{\psi}_1 \wedge \cdots\cdots \wedge \varphi_i \equiv \widetilde{\psi}_i \wedge x = f(\psi_1, \cdots\cdots, \psi_i))$$
$$\vee\{(\forall\psi_1(\varphi_1 \equiv \urcorner\widetilde{\psi}_1) \vee \cdots\cdots \vee \forall\psi_i\urcorner(\varphi_i \equiv \widetilde{\psi}_i)) \vdash x = s_0\}],$$

where s_0 is a fixed special variable of type 0.

If V_k is a variety of type n_k for all $k(1 \leqq k \leqq i)$ and t is a term, then $\widetilde{f}[V_1 \cdots\cdots, V_i, t]$ represents $\beta[V_1, \cdots\cdots, V_i\, t](\frac{\widetilde{f}}{\beta})$, where $V_1, \cdots\cdots, V_i$ and t have no β. Let $p(n_1, \cdots\cdots, n_i)$ be an arbitrary bound variable, then $\widetilde{p}$ and $\widetilde{p}[V_1, \cdots\cdots, V_i, t]$ are obtained from $\widetilde{f}$ and $\widetilde{f}[V_1, \cdots\cdots, V_i, t]$ respectively by substituting p for f, where $V_1, \cdots\cdots, V_i$ and t have no α.

Now we prove the following proposition.

8.18.1. $\Gamma_e' \to \widetilde{f}[\widetilde{\alpha}_1, \cdots\cdots, \widetilde{\alpha}_i, a] \dashv\vdash a = f(\alpha_1, \cdots\cdots, \alpha_i)$

Proof. $\widetilde{f}[\widetilde{\alpha}, \cdots\cdots, \widetilde{\alpha}_i, a]$ is

$$\mathrm{E}\psi_1 \cdots\cdots \mathrm{E}\psi_i(\widetilde{\alpha}_1 \equiv \widetilde{\psi}_1 \equiv \wedge \cdots\cdots \wedge \widetilde{\alpha}_i \equiv \widetilde{\psi}_i \wedge a = f(\alpha_1, \cdots\cdots, \alpha_i))$$
$$\vee\{(\forall\psi_1\urcorner(\widetilde{\alpha}_1 \equiv \widetilde{\psi}_1) \vee \cdots\cdots \vee \forall\psi_i\urcorner(\widetilde{\alpha}_i \equiv \widetilde{\psi}_i)) \vdash a = s_0\}$$

Therefore the proposition is clear from 8.16.1.

8.19. Class-elevation

Let $\mathfrak{A}$ and $\mathfrak{B}$ be a formula and a variety of monotype, which means of the type $n(n=0, 1, 2, \cdots\cdots)$. We define recursively as follows the class-elevated homology class $\mathfrak{A}^\sigma$ and $\mathfrak{B}^\sigma$. And σ is called a class-elevator.

If $\mathfrak{F}(\alpha)$ and $\mathfrak{B}(\alpha)$ are of a full indication for α, then $\mathfrak{F}^\sigma(\bar{\alpha})$ and $\mathfrak{B}^\sigma(\bar{\alpha})$ represent $(\mathfrak{F}(\bar{\alpha}))^\sigma$ and $(\mathfrak{B}(\bar{\alpha}))^\sigma$ of a full indication for $\bar{\alpha}$ respectively. Moreover if $\mathfrak{F}(f)$ and $\mathfrak{B}(f)$ are of a full indication for f, then $\mathfrak{F}^\sigma(\bar{f})$ and $\mathfrak{B}^\sigma(\bar{f})$ represent $(\mathfrak{F}(\bar{f}))^\sigma$ and $(\mathfrak{B}(\bar{f}))^\sigma$ of a full indication for f respectively. Here $\bar{\alpha}$ and $\bar{f}$ have the same meaning as those in the case of the type-elevation.

8.19.1. a^σ is $\{x\}(\bar{a}(1)[x])$.

8.19.2. If $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$, $\mathfrak{B}\wedge\mathfrak{C}$ or $\mathfrak{B}\vee\mathfrak{C}$, then $\mathfrak{A}^\sigma$ is $\neg\mathfrak{B}^\sigma$, $\mathfrak{B}^\sigma\wedge\mathfrak{C}^\sigma$ or $\mathfrak{B}^\sigma\vee\mathfrak{C}^\sigma$ respectively.

8.19.3. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, or $E\varphi\mathfrak{F}(\varphi)$ then $\mathfrak{A}^\sigma$ is $\forall\psi\mathfrak{F}^\sigma(\widetilde{\psi})$ or $E\psi\mathfrak{F}^\sigma(\widetilde{\psi})$ respectively, where $\mathfrak{F}^\sigma(\widetilde{\alpha})$ denotes $\mathfrak{F}^\sigma(\widetilde{\alpha})\left(\frac{\tilde{\alpha}}{\alpha}\right)$ and ψ is not contained in $\mathfrak{F}^\sigma(\widetilde{\alpha})$.

8.19.4. If $\mathfrak{A}$ is of the form $\forall p\mathfrak{F}(p)$ or $Ep\mathfrak{F}(p)$, then $\mathfrak{A}^\sigma$ is $\forall q\mathfrak{F}^\sigma(\widetilde{q})$ or $Eq\,\mathfrak{F}^\sigma(\widetilde{q})$ respectively, where q is not contained in $\mathfrak{F}^\sigma(\widetilde{f})$.

8.19.5. If $\mathfrak{A}$ is of the form $\alpha[V_1, \cdots\cdots, V_i]$, then $\mathfrak{A}^\sigma$ is $\bar{\alpha}[V_1^\sigma, \cdots\cdots, V_i^\sigma]$.

8.19.6. If V is of the from $f(V_1, \cdots\cdots, V_i)$, then V^σ is $\{x\}\bar{f}[V_1, \cdots\cdots, V_i, x]$.

8.19.7. If V is of the form $\{\varphi\}\mathfrak{F}(\varphi)$, then V^σ is $\{\psi\}E\varphi(\psi\equiv\widetilde{\varphi}\wedge\mathfrak{F}^\sigma(\widetilde{\varphi}))$.

Then the following proposition in clear.

8.19.8. If $\mathfrak{A}$ is a formula, then $\mathfrak{A}^\sigma$ is a formula.

8.19.9. If $\mathfrak{B}$ is a variety of type n, then $\mathfrak{B}^\sigma$ is a variety of type $n+1$.

8.20 Operation $\rho\eta$ and $\sigma\eta$

Let A be a formula or a variety of monotype, and all the free or special variables and functions which are contained in A be $\alpha, \cdots\cdots, f, \cdots\cdots$ We define $A^{\rho\eta}$ and $A^{\sigma\eta}$ as $(A^\rho)\left(\frac{\widetilde{\alpha}}{\alpha}\right)\cdots\cdots\left(\frac{\widetilde{f}}{f}\right)\cdots\cdots$ and $(A^\sigma)\left(\frac{\widetilde{\alpha}}{\alpha}\right)\cdots\cdots\left(\frac{\widetilde{f}}{f}\right)\cdots\cdots$ respectively.

Clearly we can see the following propositions.

8.20.1. If A is a formula, then $A^{\rho\eta}$ and $A^{\sigma\eta}$ are formulas. And if A is a variety of type m, then $A^{\rho\eta}$ and $A^{\sigma\eta}$ are varieties of type $m+1$.

8.20.2. Each free or special variable and each free special function which are contained in $A^{\rho\eta}$ or $A^{\sigma\eta}$ are contained in A.

8.21. The following sequence is provable in GLC without function

$$\Gamma_e',\ \forall\varphi_1(m_1+1)\cdots\cdots\forall\varphi_i(m_i+1)E\psi_1(m_1)\cdots\cdots E\psi_i(m_i)$$
$$(\alpha[\varphi_1, \cdots\cdots, \varphi_i] \vdash \varphi_1\equiv\widetilde{\psi}\wedge\cdots\cdots\wedge\varphi_i\equiv\widetilde{\psi}_i)\rightarrow E\beta\,(\alpha\equiv\widetilde{\beta})$$

Proof. $\Gamma_e',\ \forall\psi_1\cdots\cdots\forall\psi_i\{\widetilde{\alpha}[\psi_1, \cdots\cdots, \widetilde{\psi}_i\} \dashv\vdash \widetilde{\beta}[\widetilde{\psi}_1, \cdots\cdots, \widetilde{\psi}_i]\}$

$$\forall\varphi_1\cdots\cdots\forall\varphi_i E\psi_1\cdots\cdots E\psi_i(\alpha[\varphi_1, \cdots\cdots, \varphi_i]\vdash \varphi_i\equiv\widetilde{\psi}_1\wedge\cdots\cdots\wedge\varphi_i\equiv\widetilde{\psi}_i)$$
$$\forall\varphi_1\cdots\cdots\forall\varphi_i E\psi_1\cdots\cdots E\psi_i\{\widetilde{\beta}[\varphi_1, \cdots\cdots, \varphi_i]\vdash \varphi_1\equiv\widetilde{\psi}_1\wedge\cdots\cdots\wedge\varphi_i\equiv\widetilde{\psi}_i\}$$
$$\rightarrow\alpha\equiv\widetilde{\beta}$$

Therefore from 8.17 the proposition is clear. And from 8.21 and the definition of $\widetilde{\beta}$, we have easily.

8.22. The following sequence is provable in GLC without bound function

$$\Gamma_e'\rightarrow\forall\varphi_1\cdots\cdots\forall\varphi_i E\psi_1\cdots\cdots E\psi_i(\alpha[\varphi_1, \cdots\cdots\varphi_i]\vdash$$
$$\varphi_1\equiv\widetilde{\psi}_1\wedge\cdots\cdots\wedge\widetilde{\varphi}_i\equiv\psi_i)\dashv\vdash E\beta\,(\alpha\equiv\widetilde{\beta}).$$

8.23. The following sequences are provable in GLC without function

$$\Gamma_e' \quad \overline{a}\simeq\overline{b}\rightarrow\overline{a}\equiv\overline{b}$$
$$\Gamma_e'\rightarrow\mathfrak{F}^{\rho\eta}(\overline{\alpha}, \overline{\beta})\dashv\vdash\overline{\alpha}\equiv\overline{\beta}.$$

where $\mathfrak{F}(\alpha, \beta$ is $\alpha\equiv\beta)$ of a full indication for α and β.

Proof. Since $\overline{a\simeq b}$ is $\mathrm{E}x\mathrm{E}y(\overline{a\equiv\tilde{x}}\wedge\overline{b\equiv\tilde{y}}\wedge x=y)$, and $\mathfrak{F}^{\rho\eta}(\overline{\alpha}, \overline{\beta})$ is $\forall\overline{\varphi}\{\overline{\alpha[\varphi]}$ $\dashv\vdash\overline{\beta}[\overline{\varphi}]$, the proposition is clear.

8.24. Let $\mathfrak{A}$, V and T be an arbitrary formula, an arbitrary variety of the form $\{\varphi\}\mathfrak{F}(\varphi)$ and a term respectively. And if $\mathfrak{A}$ and V have no bound function then the following sequence is provable in GLC without a bound function.

$$\Gamma_e'\rightarrow\mathfrak{A}^{\sigma\eta}\dashv\vdash\mathfrak{A}$$
$$\Gamma_e'\rightarrow V^{\sigma\eta}\equiv\{\psi\}\mathrm{E}\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}(\varphi))$$
$$\Gamma_e'\rightarrow T^{\sigma\eta}\equiv\{x\}(x=T).$$

Proof. We proposition by the induction on the number of stages to construct $\mathfrak{A}$ or V.

First $\Gamma_e'\rightarrow a^{\sigma\eta}\equiv\{x\}(x=a)$ and $\Gamma_e'\rightarrow a^{\sigma\eta}\equiv\{x\}(x=a)$ are evident. We treat only several essential cases. We consider the case on $\mathfrak{A}$. If $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$, then $\mathfrak{A}^{\sigma\eta}$ is $\forall\varphi\mathfrak{F}^{\sigma\eta}(\tilde{\varphi})$. By the hypothesis of the induction we have $\Gamma_e\rightarrow\mathfrak{F}^{\sigma\eta}(\tilde{\alpha})\dashv\vdash\mathfrak{F}(\alpha)$. Therefore the proposition is clear.

If $\mathfrak{A}$ is of the form $\alpha[V_1, \cdots\cdots, V_i]$, then $\mathfrak{A}^{\sigma\eta}$ is $\tilde{\alpha}[V_1^{\sigma\eta}, \cdots\cdots, V_i^{\sigma\eta}]$. Without the loss of generality we can assume α is of type $(n+1, 1)$ and $\mathfrak{A}$ is $\alpha[\{\varphi\}\mathfrak{F}(\varphi), T]$.

Then by the hypothesis of the induction we have

$$\Gamma_e'\rightarrow\mathfrak{A}^{\sigma\eta}\dashv\vdash\tilde{\alpha}[\{\varphi\}\mathrm{E}\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}(\varphi)), \{x\}(x=T)].$$

From 8.17.1. we have

$$\Gamma_e'\rightarrow\tilde{\alpha}[\{\varphi\}E\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}(\varphi)), \{x\}(x=T)]\dashv\vdash\alpha[\{\varphi\}\mathfrak{F}(\varphi), T].$$

Hence $\Gamma_e\rightarrow\mathfrak{A}^{\sigma\eta}\dashv\vdash\mathfrak{A}$ is provable.

Now we consider the cases of V and T.

If V is of the from $\{\varphi\}\mathfrak{F}(\varphi)$, then $V^{\sigma\eta}$ is $\{\psi\}\mathrm{E}\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}^{\sigma\eta}(\tilde{\varphi}))$. By the hypothesis of the induction we have $\Gamma_e'\rightarrow\mathfrak{F}^{\sigma\eta}(\alpha)\dashv\vdash\mathfrak{F}(\alpha)$.

Therefore the proposition is clear.

If T is of the from $f(V_1,\cdots\cdots, V_i)$, then $T^{\sigma\eta}$ is

$$\{x\}[\mathrm{E}\varphi_1\cdots\cdots\mathrm{E}\varphi_i(V_1^{\sigma\eta}\equiv\tilde{\varphi}_1\wedge\cdots\cdots\wedge V_i^{\sigma\eta}\equiv\tilde{\varphi}_i\wedge x=f(\varphi_1,\cdots\cdots, \varphi_i)).$$
$$\vee\{(\forall\varphi_1 7(V_1^{\sigma\eta}\equiv\tilde{\varphi}_1)\vee\cdots\cdots\vee\forall\varphi_i 7(V_i^{\sigma\eta}\equiv\tilde{\varphi}_i))\vdash x=s_0\}]$$

Without the loss of generality we can assume that f is of the type $(n+1,1)$ and T is of the form $f(\{\varphi\}\mathfrak{F}(\varphi), T')$.

Then $T^{\sigma\eta}$ is

$$\{x\}[\mathrm{E}\psi\mathrm{E}y\{(\{\varphi\}\mathfrak{F}(\varphi))^{\sigma\eta}\equiv\tilde{\psi}\wedge T'^{\sigma\eta}\equiv\tilde{y}\wedge x=f(\psi, y)\}$$
$$\vee\{\forall\psi 7((\{\varphi\}\mathfrak{F}(\varphi))^{\sigma\eta}\equiv\tilde{\psi})\vee\forall y 7(T'^{\sigma\eta}\equiv\tilde{y}))\vdash x=s_0\}].$$

By the hypothesis of the induction we have

$$\Gamma_e' \rightarrow (\{\varphi\}\mathfrak{F}(\varphi))^{\sigma\eta} \equiv \{\psi\}\mathrm{E}\varphi(\psi \equiv \widetilde{\varphi} \wedge \mathfrak{F}(\varphi))$$

and
$$\Gamma_e' \rightarrow T'^{\sigma\eta} \equiv \{z\}(z = T').$$

Therefore we have from 8.15.1

$$\Gamma_e' \rightarrow \{(\{\varphi\}\mathfrak{F}(\varphi))^{\sigma\eta} \equiv \widetilde{\alpha}\} \mathrel{\vdash\!\dashv} (\{\varphi\}\mathfrak{F}(\varphi) \equiv \alpha)$$

and
$$\Gamma_e' \rightarrow (T'^{\sigma\eta} \equiv \widetilde{a} \mathrel{\vdash\!\dashv} T' = a,$$

and
$$\Gamma_e' \rightarrow T^{\sigma\eta} \equiv \{x\}(x = f(\{\varphi\}\mathfrak{F}(\varphi), T')).$$

Therefore the proposition is proved. From 8.24 we have

8.25. Under the same hypothesis as that in 8.24 the following sequences are provable in GLC without a bound function

$$\Gamma_e' \rightarrow \mathrm{E}\varphi(V^{\sigma\eta} \equiv \widetilde{\varphi}) \qquad \Gamma_e' \rightarrow \mathrm{E}x(T^{\sigma\eta} \equiv \widetilde{x}).$$

8.26. Let $e(*)$ be a special variable of type 1. Then we define recursively as follows the restriction system $R\ \{\mathfrak{F}<n_1, \cdots\cdots, n_i>(\alpha)\}$ which is generated by $e(\ )$.

8.26.1. $\mathfrak{F}<0>(a)$ is $e(a)$.

8.26.2. $\mathfrak{F}<n_1+1, \cdots\cdots, n_i+1>(\alpha)$ is

$$\forall\varphi_1 \cdots\cdots \forall\varphi_i(\alpha[\varphi_1, \cdots\cdots, \varphi_i] \vdash \mathfrak{F}<n_1>(\varphi_1) \wedge \cdots\cdots \wedge \mathfrak{F}<n_i>(\varphi_i)).$$

8.27. $$\Gamma_e' \rightarrow \mathfrak{F}^{\rho\tau}<n_1+1, \cdots\cdots, n_i+1>(\bar{\alpha}) \mathrel{\vdash\!\dashv} \mathrm{E}\xi(\bar{\alpha} \equiv \widetilde{\xi})$$

where the operation τ means the operation $\begin{pmatrix} \{\varphi\}\,\mathrm{E}x(\varphi \equiv \widetilde{x}) \\ \overline{e(\ )} \end{pmatrix}$.

Proof. We prove the proposition by the induction of the height H of α.

If theh eight of α is zero, then $\mathfrak{F}^{\rho\tau}<0>(a)$ is $\bar{e}(\bar{a})^{\tau}$ and so $\mathrm{E}x(\bar{a} \equiv \widetilde{x})$, whence the proposition is clear.

Now we assume that the proposition holds, if the height of α is less than H. By the definition $\mathfrak{F}^{\rho\tau}<n_1+1, \cdots\cdots, n_i+1>(\bar{\alpha})$ is

$$\forall\bar{\varphi}_1 \cdots\cdots \forall\bar{\varphi}_i(\bar{\alpha}[\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_j] \vdash \mathfrak{F}^{\rho\tau}<n_1>(\bar{\varphi}_1) \wedge \cdots\cdots \wedge \mathfrak{F}^{\rho\tau}<n_i \wedge (\bar{\varphi}_i))$$

By the hypothesis of induction we have

$$\Gamma_e' \rightarrow \mathfrak{F}^{\rho\tau}<n_1+1, \cdots\cdots, n_i+1>(\bar{\alpha}) \mathrel{\vdash\!\dashv} \forall\bar{\varphi}_1 \cdots\cdots \forall\bar{\varphi}_i(\bar{\alpha}[\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i]$$
$$\vdash \mathrm{E}\psi_1\bar{\varphi}_1 \equiv \widetilde{\psi}_1) \wedge \cdots\cdots \wedge \mathrm{E}\psi_i(\bar{\varphi}_i \equiv \widetilde{\psi}_i))$$

Therefore by virtue of 8.22 we have

$$\Gamma_e' \rightarrow \mathfrak{F}^{\rho\tau}<n_1+1, \cdots\cdots, n_i+1>(\bar{\alpha}) \mathrel{\vdash\!\dashv} \mathrm{E}\xi(\bar{\alpha} \equiv \widetilde{\xi}) \quad \text{q.e.d.}$$

8.28. Let $\mathfrak{A}$ be a formula and V be a variety without a bound functian and $e(*)$ be a special variable of type 1 which is not contained in $\mathfrak{A}$ and V and let R be the restriction system which is generated by $e(\ )$ and $\widetilde{r}$ be its strictly restricting operator. Then the following sequence are provable in GLC without a bound function.

$$\Gamma_e' \rightarrow \mathfrak{A}^{\widetilde{r}\rho\eta\tau} \mathrel{\vdash\!\dashv} \mathfrak{A}^{\sigma\eta} \qquad \Gamma_e' \rightarrow V^{\widetilde{r}\rho\eta\tau} \equiv V^{\sigma\eta}.$$

Proof. We prove the proposition by the induction on the number N of the stages to construct $\mathfrak{A}$ or V respectively.

If N is one, then the proposition is evident.

Therefore we assume that the proposition holds, when the height of α is less than N.

We treat only several essential cases.

1) $\mathfrak{A}$ is of the form $\forall\varphi\mathfrak{F}(\varphi)$.

Then $\mathfrak{A}^{r\rho\eta_\tau}$ is $\qquad (\forall\varphi(\mathfrak{F}<n_1>(\varphi)\vdash\tilde{\mathfrak{F}}^{r}(\varphi)))^{\rho\eta_\tau}$

and so $\qquad \forall\bar{\varphi}(\mathfrak{F}^{\rho\tau}<n_1>(\bar{\varphi})\vdash\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\bar{\varphi}))$.

Therefore from 8.27 we have $\Gamma_e'\rightarrow\tilde{\mathfrak{A}}^{r\rho\eta_\tau}\dashv\vdash\forall\bar{\varphi}(\mathrm{E}\xi(\bar{\varphi}\equiv\tilde{\xi})\vdash\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\tilde{\xi}))$, so $\Gamma_e'\rightarrow\tilde{\mathfrak{A}}^{r\rho\eta_\tau}\dashv\vdash\forall\xi\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\tilde{\xi})$.

By the hypothesis of the induction we have $\Gamma_e'\rightarrow\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\tilde{\alpha})\dashv\vdash(\mathfrak{F}(\alpha))^{\sigma\eta}$ so $\Gamma_e'\rightarrow\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\tilde{\alpha})\dashv\vdash\mathfrak{F}^{\sigma\eta}(\tilde{\alpha})$.

Hence we have $\qquad \Gamma_e'\rightarrow\tilde{\mathfrak{A}}^{r\rho\eta_\tau}\dashv\vdash\forall\xi\mathfrak{F}^{\sigma\eta}(\tilde{\xi})$

and so $\qquad \Gamma_e'\rightarrow\tilde{\mathfrak{A}}^{r\rho\eta_\tau}\dashv\vdash\mathfrak{A}^{\sigma\tau}$.

2) V is of the form $\{\varphi_1,\cdots\cdots,\varphi_i\}\mathfrak{F}(\varphi_1,\cdots\cdots,\varphi_i)$.

Then $\tilde{V}^{r\rho\eta_\tau}$ is

$$\{\bar{\varphi}_1,\cdots\cdots,\bar{\varphi}_i\}(\mathfrak{F}^{\rho\tau}<n_1>(\bar{\varphi}_1)\wedge\cdots\cdots\wedge\mathfrak{F}^{\rho\tau}<n_i>(\bar{\varphi}_i)\wedge\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\bar{\varphi}_1\cdots\cdots,\bar{\varphi}_i)).$$

Therefore from 8.27 we have

$$\Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv\{\bar{\varphi}_1,\cdots\cdots,\bar{\varphi}_i\}(\mathrm{E}\xi_1(\bar{\varphi}_1\equiv\tilde{\xi}_1)\wedge\cdots\cdots\wedge\mathrm{E}\xi_i(\bar{\varphi}_i\equiv\tilde{\xi}_i)\wedge\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\bar{\varphi}_1,\cdots\cdots,\bar{\varphi}_i))$$

and

$$\Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv\{\bar{\varphi}_1,\cdots\cdots,\bar{\varphi}_i\}\mathrm{E}\xi_1\cdots\cdots\mathrm{E}\xi_i(\bar{\varphi}_1\equiv\tilde{\xi}_1\wedge\cdots\cdots\wedge\bar{\varphi}_i\equiv\tilde{\xi}_i\wedge\tilde{\mathfrak{F}}^{r\rho\eta_\tau}(\tilde{\xi}_1,\cdots\cdots,\tilde{\xi}_i)).$$

By the hypothesis of the induction we have

$$\Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv\{\bar{\varphi}_1,\cdots\cdots,\bar{\varphi}_i\}\mathrm{E}\xi_1\cdots\cdots\mathrm{E}\xi_i(\bar{\varphi}_1\equiv\tilde{\xi}_1\wedge\cdots\cdots\wedge\bar{\varphi}_i\equiv\tilde{\xi}_i\wedge\mathfrak{F}^{\sigma\eta}(\tilde{\xi}_1,\cdots\cdots,\tilde{\xi}_i))$$

Hence we have $\qquad \Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv V^{\sigma\eta}$

3) V is of the form $f(V_1,\cdots\cdots,V_i)$.

Then $\tilde{V}^{r\rho\eta_\tau}$ is $(\bar{f}(\tilde{V}_1^{r\rho},\cdots\cdots,\tilde{V}_i^{r\rho}))^{\eta_\tau}$ and so it is

$$\{x\}[\mathrm{E}\psi_1\cdots\cdots\mathrm{E}\psi_i(\tilde{V}_1^{r\rho\eta_\tau}\equiv\tilde{\psi}_1\wedge\cdots\cdots\wedge\tilde{V}_i^{r\rho\eta_\tau}\equiv\tilde{\psi}_i\wedge x=f(\psi_1,\cdots\cdots,\psi_i)))$$
$$\vee\{\forall\psi_1\neg(\tilde{V}_1^{r\rho\eta_\tau}\equiv\tilde{\psi}_1)\vee\cdots\cdots\vee\forall\psi_i\neg(\tilde{V}_i^{r\rho\eta_\tau}\equiv\tilde{\psi}_i))\vdash x=s_0\}].$$

By the hypothesis of the induction we have

$$\Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv\{x\}[\mathrm{E}\psi_1\cdots\cdots\mathrm{E}\psi_i(V_1^{\sigma\eta}\equiv\tilde{\psi}_1\wedge\cdots\cdots\wedge V_i^{\sigma\eta}\equiv\tilde{\psi}_i\wedge x=f(\psi_1,\cdots\cdots,\psi_i))$$
$$\vee\{(\forall\psi_1\neg(V_1^{\sigma\eta}\equiv\tilde{\psi}_1)\vee\cdots\cdots\vee\forall\psi_i\neg(V_1^{\sigma\eta}\equiv\tilde{\psi}_i))\vdash x=s_0\}]$$

and so $\qquad \Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv V^{\sigma\eta}$.

4) $\mathfrak{A}$ is of the form $\alpha[V_1,\cdots\cdots,V_i]$.

Then the proposition is clear by the hypothesis of the induction.

8.29. Under the same hypothesis as that of 8.28 the following sequences are provable in GLC without bound function

$$\Gamma_e'\rightarrow\tilde{\mathfrak{A}}^{r\rho\eta_\tau}\dashv\vdash\mathfrak{A}.$$

$$\Gamma_e'\rightarrow\tilde{V}^{r\rho\eta_\tau}\equiv\{\psi\}\mathrm{E}\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}(\varphi))$$

if V is of the form $\{\varphi\}\mathfrak{F}(\varphi)$.

$$\Gamma_e'\rightarrow\tilde{T}^{r\rho\eta_\tau}\equiv\{x\}(x=T),$$

if T is a term.

8.30. The following sequence is probable in GLC without a bonud function

$\Gamma_e' \to \mathfrak{A}^{\rho\eta}$. Here $\mathfrak{A}$ is an arbitrary axiom of Γ_e'.

Proof.

1) The case where $\mathfrak{A}$ is $\forall\varphi\forall x\forall y\{x=y \vdash \varphi[x] \dashv\vdash \varphi[y])\}$.

Then $\mathfrak{A}^{\rho\eta}$ is $\quad \forall\bar{\varphi}\forall\bar{x}\forall\bar{y}\{\bar{x}\simeq\bar{y} \vdash (\bar{\varphi}[\bar{x}] \dashv\vdash \bar{\varphi}[\bar{y}])\}$.

Therefore the proposition is clear from 8.23.

2) The case where $\mathfrak{A}$ is $\forall\xi\forall\varphi\forall\psi\{\varphi\equiv\psi \vdash (\xi[\varphi] \dashv\vdash \xi[\varphi])\}$.

Then $\mathfrak{A}^{\rho\eta}$ is $\forall\bar{\xi}\forall\bar{\varphi}\forall\bar{\psi}\{\mathfrak{F}^{\rho\eta}(\bar{\varphi},\bar{\psi}) \vdash (\bar{\xi}[\bar{\varphi}] \vdash \bar{\xi}[\bar{\psi}])$, where $\mathfrak{F}^{\rho\eta}(\bar{\varphi},\bar{\psi})$ is the same as that of 8.23.

3) The case where $\mathfrak{A}$ is

$\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^j\forall\psi^j\{\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^j\equiv\psi^j \vdash s(\varphi,^1\cdots\cdots,\varphi^j)=s(\psi^1,\cdots\cdots,\psi^j)\}$.

Then $\mathfrak{A}^{\rho\eta}$ is

$$\begin{aligned}&\forall\bar{\varphi}^1\forall\bar{\psi}^1\cdots\cdots\forall\bar{\varphi}^j\forall\bar{\psi}^j\{\mathfrak{F}^{\rho\eta}(\bar{\varphi}^1,\bar{\psi}^1)\wedge\cdots\cdots\wedge\mathfrak{F}^{\rho\eta}(\bar{\varphi}^j,\bar{\psi}^j)\\ &\vdash (\{x\}[\mathrm{E}\varphi^1\cdots\cdots\mathrm{E}\varphi^j(\bar{\varphi}^1\equiv\tilde{\varphi}^1\wedge\cdots\cdots\wedge\bar{\varphi}^j\equiv\tilde{\varphi}^j\wedge x=s(\varphi^1,\cdots\cdots,\varphi^j))\\ &\vee\{(\forall\bar{\varphi}^1\neg(\bar{\varphi}^1\equiv\tilde{\varphi}^1)\vee\cdots\cdots\vee\forall\varphi^j\neg(\bar{\varphi}^j\equiv\tilde{\varphi}^j))\vdash x=s_0\}]\\ &\simeq\{x\}[\mathrm{E}\psi^1\cdots\cdots\mathrm{E}\psi^j(\bar{\psi}^1\equiv\tilde{\psi}^1\wedge\cdots\cdots\wedge\bar{\psi}^j\equiv\tilde{\psi}^j\wedge x=s(\psi^1,\cdots\cdots,\psi^j))\\ &\vee\{(\forall\psi^1\neg(\bar{\psi}^1\equiv\tilde{\psi}^1)\vee\cdots\cdots\vee\forall\psi^j\neg(\bar{\psi}^j\equiv\tilde{\psi}^j))\vdash x=s_0\}])\},\end{aligned}$$

where $\mathfrak{F}^{\rho\eta}(\bar{\varphi},\bar{\psi})$ is the same as that of 8.23.

Therefore we have only to prove the following sequences.

$$\Gamma_e',\ \bar{\alpha}^1\equiv\bar{\beta}^1,\cdots\cdots,\bar{\alpha}^j\equiv\bar{\beta}^j$$

$$\mathrm{E}\varphi^1(\bar{\alpha}^1\equiv\tilde{\varphi}^1),\cdots\cdots,\mathrm{E}\varphi^j(\bar{\alpha}^j\equiv\tilde{\varphi}^j)\to s(\alpha^1,\cdots\cdots,\alpha^j)=s(\beta^1,\cdots\cdots,\beta^j)$$

and

$$\Gamma_e',\ \bar{\alpha}^1=\bar{\beta}^1,\cdots\cdots,\bar{\alpha}^j\equiv\bar{\beta}^j$$

$$\forall\varphi^1\neg(\bar{\alpha}^1\equiv\tilde{\varphi}^1)\vee\cdots\cdots\vee\forall\varphi^j\neg(\bar{\alpha}^j\equiv\tilde{\varphi}^j)\to s_0=s_0.$$

Therefore the proposition is clear.

8.31. Let $e(\ )$ be a special variable of type 1. Then we define recursively as follows the restricting system $\tilde{R}$ $\{\mathfrak{F}<n_1,\cdots\cdots,n_i>(\alpha)\ ;\ \mathfrak{G}<m_1,\cdots\cdots,m^j>(f)\}$ which is strongly generated by $e(\ )$.

8.31.1. $\mathfrak{F}<0>(a)$ is $e(a)$.

8.31.2. $\mathfrak{F}<n_1+_1,\cdots\cdots,n_i+1>(\alpha)$ is

$$\forall\varphi_1\cdots\cdots\forall\varphi_i(\alpha[\varphi_1,\cdots\cdots,\varphi_i]\vdash\mathfrak{F}<n_1>(\varphi_1)\wedge\cdots\cdots\wedge F<n_i>(\varphi_i)).$$

8.31.3. $\mathfrak{G}<m_1+1,\cdots\cdots,m_j+1>(f)$ is

$$\forall\varphi_1\cdots\cdots\forall\varphi_i(\mathfrak{F}<m_1>(\varphi_1)\wedge\cdots\cdots\wedge\mathfrak{F}<m_i>(\varphi_i)\vdash e(f(\varphi_1,\cdots\cdots,\varphi_i))).$$

We can see easily the following proposition.

8.32. If A is an axiom without a bound function, then $A^{\tilde{r}}$ is $A^{\tilde{\tilde{r}}}$, where $\tilde{r}$ is a strictly restricting operator which is generated by $e(\ )$ and $\tilde{\tilde{r}}$ is a strictly restricting operator which is strongly generated by $e(\ )$.

8.33. If $\tilde{\tilde{r}}$ is the strictly restricting operator which is strongly generated by $e(\ )$. Then the following sequences are provable.

8.33.1. $\mathrm{E}xe(x) \to \mathrm{E}\varphi\, cl\,(\varphi(n_1 \cdots\cdots, n_i))$.

8.332. $\mathrm{E}xe(x) \to \mathrm{E}p\, cl\,(p(m_1, \cdots\cdots, m_j))$.

Proof. We see easily 8.33.1 by proving the following sequence

$$\mathrm{E}xe(x) \to cl(\{\varphi_1, \cdots\cdots, \varphi_i\}(cl(\varphi_1) \wedge \cdots\cdots \wedge cl\,(\varphi_i)))\,.$$

Moreover we can see easily 8.33.2 by the following provable sequence

$$e(a) \to \forall\varphi_1 \cdots\cdots \forall\varphi_i\,(cl(\varphi_1) \cdots\cdots cl(\varphi_i) \vdash e(\{\varphi_1, \cdots\cdots, \varphi_i\}a)),$$

8.34. If $\tilde{\tilde{r}}$ is the strictly restricting operator which is strongly generated by $e(\ )$. Then the following propositions hold.

8.34.1. If V is an arbitrary variety and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all fhe free or special variables and all the free or special functions which are contained in V, then $cl(\alpha), \cdots\cdots cl\ (f), \cdots\cdots \to cl(V^{\tilde{\tilde{r}}})$ is provable.

8.34.2. If F is an arbitrary functional and $\alpha, \cdots\cdots, f, \cdots\cdots$ are all the free or special variables and functions which are contained in F, then $cl(\alpha), \cdots\cdots, cl\,(f), \cdots\cdots \to cl(F^{\tilde{\tilde{r}}})$ is provable.

Proof. First we shall prove 8.34.1.

If V is of the form $\{\varphi_1, \cdots\cdots, \varphi_i\}\mathfrak{F}(\varphi_1, \cdots\cdots, \varphi_i)$, then $V^{\tilde{\tilde{r}}}$ is

$$\{\varphi_1, \cdots\cdots, \varphi_i\}(cl(\varphi_1) \wedge \cdots\cdots \wedge cl(\varphi_i) \wedge \mathfrak{F}^{\tilde{\tilde{r}}}(\varphi_1, \cdots\cdots, \varphi_i))\,.$$

Therefore the sequence $\to cl(V^{\tilde{\tilde{r}}})$ is provable. Now we shall prove 8.34.2. Without the loss generality we can assume that F is of the form

$$\{\varphi_1, \cdots\cdots, \varphi_i\}f(V_1(\varphi_1, \cdots\cdots, \varphi_i), \cdots\cdots, V_j(\varphi_1, \cdots\cdots, \varphi_i))\,.$$

We have only to prove

$$cl(f) \to e(f\,((V_1(\alpha_1, \cdots\cdots, \alpha_i))^{\tilde{\tilde{r}}}, \cdots\cdots, (V_j(\alpha_1, \cdots\cdots, \alpha_i))^{\tilde{\tilde{r}}})),$$

which will be obtained immediately by proving the following provable sequences

$$\to cl((V_1(\alpha_1, \cdots\cdots, \alpha_i))^{\tilde{\tilde{r}}})$$
$$\vdots$$
$$\to cl((V_j(\alpha_1, \cdots\cdots, \alpha_i))^{\tilde{\tilde{r}}})\,.$$

8.35. Let $\tilde{\tilde{r}}$ be the strictly restricting operator which is strongly generated by $e(\ )$ and let the following axioms be consistent

$$A_1^{\tilde{\tilde{r}}}, \cdots\cdots, A_N^{\tilde{\tilde{r}}}$$

$cl(\sigma)$ for each special variable σ which is contained in $A_1, \cdots\cdots, A_N$

$cl(s)$ for each special function s which is contained in $A_1, \cdots\cdots, A_N$

$$\mathrm{E}xe(x)\,.$$

Then the axioms $A_1, \cdots\cdots, A_N$ are consistent.

Proof. By virtue of 7.28, 8.33 and 8.34 the proof is evident. From 8.32 and 8.35 we have easily.

8.36 Let r be the strictly restricting operator generated by $e(\ )$ and $A_1, \cdots\cdots, A_N$ be axioms without a bound function and, moreover, let the following axioms be consistent

$$A_1^{\tilde{r}}, \cdots\cdots, A^{\tilde{r}}_N$$

$cl(\sigma)$ for each special variable σ which is contained in $A_1, \cdots\cdots, A_N$

$$\forall\varphi_1 \cdots\cdots \forall\varphi_i\{cl(\varphi_1) \wedge \cdots\cdots \wedge cl(\varphi_i) \vdash e(s(\varphi_1, \cdots\cdots, \varphi_i))\}$$

for each special function s which is contained in $A_1, \cdots\cdots A_N$

$$\mathrm{E}xe(x)$$

Then the axioms $A_1, \cdots\cdots, A_N'$ are consistent.

§ 9. The Concepts of 'Set' and 'Function'.

9.1. The Theorem on Set of a Primary Form.

Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and $e(\ )$ be a special variable of type 1 which is not contained in $A_1, A_2, \cdots\cdots$ and R be the restriction system generated by $e(\ )$ and $\tilde{r}$ be its strictly restriction operator.

Moreover let M be a special function of type 2 which is not contained in $A_1, A_2, \cdots\cdots$ and $*\in*$ be a special variable of type $(1, 1)$ which is not contained in $A_1, A_2, \cdots\cdots$.

If $A_1, A_2, \cdots\cdots$ and equality axioms Γ_e' are consistent in GLC without bound function, then the following axioms are consistent in GLC.

$$A_1^{\tilde{r}}, A_2^{\tilde{r}}, \cdots\cdots, \Gamma_e', e(s_0)$$
$$\forall x \forall y \{x \in y \vdash e(x)\}$$
$$\forall x \{x \in x \dashv\vdash e(x)\}$$
$$\forall x \forall y \{x \in y \wedge e(y) \vdash x = y\}$$
$$\forall x \{Ey(y \in x) \wedge \forall y \forall z (y \in x \wedge z \in x \vdash y = z) \vdash e(x)$$
$$\forall \varphi \forall x \forall y \{\forall z (z \in x \dashv\vdash z \in y) \vdash (\varphi[x] \dashv\vdash \varphi[y]\}$$
$$\forall \varphi \forall x \{x \in M\{z\}\varphi[z] \dashv\vdash \varphi[z] \wedge e(z)\}$$
$$\forall \varphi_1 \cdots\cdots \forall \varphi_i e(s(\varphi_1, \cdots\cdots, \varphi_i))$$

for each special function s which is contained in $A_1, A_2, \cdots\cdots$.

$cl(\sigma)$ for each special variable σ which is contained in $A_1, A_2, \cdots\cdots$.

Proof. We write all these axioms for $\mathfrak{M}$. We have only to prove that $\mathfrak{M}^\rho$ is consistent in GLC without function.

We consider that all the special variables and functions which are contained in $A_1, A_2, \cdots\cdots$ are $\sigma, \cdots\cdots, s, \cdots\cdots$.

Then we define an operator $\tilde{\eta}$ by

$$\begin{pmatrix}\tilde{\sigma}\\ \sigma\end{pmatrix}\cdots\cdots\begin{pmatrix}\tilde{s}\\ s\end{pmatrix}\cdots\cdots\begin{pmatrix}\hat{e}(\ )\\ e(\ )\end{pmatrix}\begin{pmatrix}*\hat{\in}*\\ *\in*\end{pmatrix}\begin{pmatrix}\hat{M}\\ M\end{pmatrix}$$

Here $\hat{e}(\ )$, $\hat{\in}$, $\hat{M}$ are defined as follows.

$$\{\varphi\}\hat{e}(\varphi) \text{ is } \{\varphi\}\mathrm{E}x(\varphi \equiv \tilde{x})$$
$$\{\varphi, \psi\}\varphi \hat{\in} \psi \text{ is}$$
$$\{\varphi, \psi\}(\hat{e}(\varphi) \wedge \mathrm{E}x(\varphi[x] \wedge \psi]x]))$$
$$\{\varphi(2), x\}\hat{M}[\varphi(2), x] \text{ is } \{\varphi(2), x\}\mathrm{E}\psi\{\varphi[\psi] \wedge \psi[x] \wedge \hat{e}(\psi)\}.$$

Then we prove $\mathfrak{M}^{\rho\tilde{\eta}}$ is consistent. To prove this, it is sufficient that the following sequence $A_1, A_2, \cdots\cdots, \Gamma_e' \rightarrow \mathfrak{A}^{\rho\tilde{\eta}}$, where $\mathfrak{A}$ is an axiom of $\mathfrak{M}$.

1) the case where $\mathfrak{A}$ is one of $A_1^{\tilde{r}}, A_2^{\tilde{r}}, \cdots\cdots$.

Then $\mathfrak{A}^{\rho\tilde{\eta}}$ is $A^{\tilde{r}\rho\eta_r}$ and so the proposition is clear.

2) the case where $\mathfrak{A}$ is $e(s_0)$ or one of $cl(\sigma)$.

Then $\mathfrak{A}^{\rho\eta}$ is $cl^{\rho\tau}(\tilde{\sigma})$ or $\hat{e}(s_0)$ and so the proposition is clear from 8.27.

3) the case where $\mathfrak{A}$ is $\forall\varphi_1\cdots\cdots\forall\varphi_i e(s(\varphi_1,\cdots\cdots,\varphi_i))$.

$\mathfrak{A}^{\rho\tilde{\tau}}$ is $\forall\bar{\varphi}_1\cdots\cdots\forall\bar{\varphi}_i\hat{e}(\{x\}\tilde{s}(\bar{\varphi}_1,\cdots\cdots,\bar{\varphi},x))$ and so the proposition is clear from the definition of $\tilde{s}$.

4) the case where $\mathfrak{A}$ is $\forall x\forall y\{x\in y\vdash e(x)\}$.

$\mathfrak{A}^{\rho\eta}$ is $\forall\bar{x}\forall\bar{y}\{\bar{x}\mathrel{\hat{\in}}\bar{y}\vdash\hat{e}(\bar{x})\}$ and so the proposition is clear from the definition of $\hat{e}(*)$ and $*\mathrel{\hat{\in}}*$.

5) the case where $\mathfrak{A}$ is $\forall x\{x\in x\dashv\vdash e(x)\}$.

$\mathfrak{A}^{\rho\tilde{\eta}}$ is $\forall\bar{x}\{\bar{x}\mathrel{\hat{\in}}\bar{x}\dashv\vdash\hat{e}(\bar{x})\}$ and so is

$$\forall\bar{x}\{\mathrm{E}y(\bar{x}\equiv\tilde{y})\wedge\mathrm{E}y(\bar{x}[y]\wedge\bar{x}[y]\dashv\vdash\mathrm{E}y(\bar{x}\equiv\tilde{y})).$$

Hence the proposition is clear.

6) the case where $\mathfrak{A}$ is

$$\forall x\forall y\{x\in y\wedge e(y)\}\vdash x=y\}.$$

$\mathfrak{A}^{\rho\eta}$ is

$$\forall\bar{x}\forall\bar{y}\{\bar{x}\mathrel{\hat{\in}}\bar{y}\wedge\hat{e}(\bar{y})\vdash\bar{x}\simeq\bar{y}\}.$$

Therefore we have only to prove

$$\Gamma_e',\qquad \mathrm{E}x(\alpha\equiv\tilde{x}),\ \mathrm{E}x(\beta\equiv\tilde{x}),$$

$$\mathrm{E}x(\alpha[x]\wedge\beta[x])\to\mathrm{E}x\mathrm{E}y(\alpha\equiv\tilde{x}\wedge\beta\equiv\tilde{y}\wedge x=y).$$

We can see easily $\quad \Gamma_e',\ \alpha\equiv\tilde{c},\ \alpha[d]\wedge\beta[d]\to c=d$

and $\quad \Gamma_e',\ \beta\equiv\tilde{b},\ \alpha[d]\wedge\beta[d]\to b=d.$

Therefore we have easily

$$\Gamma_e',\ \mathrm{E}x(\alpha\equiv\tilde{x}),\ \mathrm{E}x(\beta\equiv\tilde{x}),$$

$$\mathrm{E}x(\alpha[x]\wedge\beta[x])\to\mathrm{E}x\mathrm{E}y(\alpha\equiv\tilde{x}\wedge\beta\equiv\tilde{y}\wedge x=y).$$

7) the case where $\mathfrak{A}$ is $\forall\varphi\forall x\{x\in M(z)\varphi(z)\dashv\vdash\varphi[x]\wedge e(x)\}$.

$\mathfrak{A}^{\rho\tilde{\eta}}$ is $\forall\bar{\varphi}\forall\bar{x}\{\hat{e}(\bar{x})\wedge\mathrm{E}y(\bar{x}[y]\wedge\mathrm{E}\psi(\varphi[\psi]\wedge\psi[v]\wedge\hat{e}(\psi)))\dashv\vdash\bar{\varphi}[\bar{x}]\wedge\hat{e}(\bar{x})\}$.

First we shall prove $\Gamma_e',\ \hat{e}(\bar{a}),\bar{a}[b],\tilde{\alpha}[\beta],\ \beta[b],\ \hat{e}(\beta)\to\tilde{\alpha}[\bar{a}]$.

We can see easily $\Gamma_e',\ \hat{e}(\bar{a}),\ \hat{e}(\beta),\ \bar{a}[b],\ \beta[b]\to\bar{a}\equiv\beta$.

Moreover we see $\Gamma_e',\ \bar{a}\equiv\beta,\ \tilde{\alpha}[\beta]\to\tilde{\alpha}[a]$, whence the proof is accomplished.

Next we shall prove

$$\Gamma_e',\ \hat{e}(\bar{a}),\ \tilde{\alpha}[\bar{a}]\to\mathrm{E}y(\bar{a}[y]\wedge\mathrm{E}\psi(\bar{\alpha}[\psi]\wedge\psi[y]\wedge\hat{e}(\psi))).$$

We can see easily $\quad \Gamma_e',\ \bar{a}\equiv\tilde{b}\to\bar{a}[b]$

and $\quad \Gamma_e',\ \bar{a}\equiv\tilde{b},\ \tilde{\alpha}[\bar{a}]\to\tilde{\alpha}[\bar{a}]\wedge\bar{a}[b]\wedge\hat{e}(\bar{a})$

whence the proof is accomplished.

8) the case where $\mathfrak{A}$ is

$$\forall x\{\mathrm{E}y(y\in x)\wedge\forall y\forall z(y\in x\wedge z\in x\vdash y\in z)\vdash e(x))\}.$$

$$\mathfrak{A}^{\rho\tilde{\eta}} \text{ is } \forall\bar{x}\{\mathrm{E}\bar{y}(\bar{y}\hat{\in}\bar{x}) \wedge \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{x}\wedge\bar{z}\hat{\in}\bar{x} \vdash \bar{y}\simeq\bar{z}) \vdash \hat{e}(\bar{x}))\}.$$

We have only to prove

$$\Gamma_e{}',\ \bar{b}\hat{\in}\bar{a},\ \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{a}\wedge\bar{z}\hat{\in}\bar{a} \vdash \bar{y}\simeq\bar{z}) \to \hat{e}(\bar{a}).$$

We can see easily

$$\Gamma_e{}',\ \mathrm{E}x(\bar{b}\equiv\tilde{x}),\ \mathrm{E}x(\bar{b}[x]\wedge\bar{a}[x]) \to \mathrm{E}x\bar{a}[x],$$

and $$\Gamma_e{}',\ \bar{a}[c] \to \tilde{c}\hat{\in}\bar{a}.$$

Hence $$\Gamma_e{}',\ \bar{a}[c],\ \bar{a}[d],\ \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{a}\wedge\bar{z}\hat{\in}\bar{a} \vdash \bar{y}=\bar{z}) \to \tilde{c}\simeq\tilde{d}$$

and $$\Gamma_e{}',\ \bar{a}[c],\ \bar{a}[d],\ \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{a}\wedge\bar{z}\hat{\in}\bar{a} \vdash \bar{y}=\bar{z}) \to c=d$$

and $$\Gamma_e{}',\ \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{a}\wedge\bar{z}\hat{\in}\bar{a} \vdash \bar{y}\simeq\bar{z}) \to \forall\bar{y}\forall\bar{z}(\bar{a}[y]\wedge\bar{a}[z] \vdash y=z).$$

Therefore we have

$$\Gamma_e{}',\ \bar{b}\hat{\in}\bar{a},\ \forall\bar{y}\forall\bar{z}(\bar{y}\hat{\in}\bar{a}\wedge\bar{z}\hat{\in}\bar{a} \vdash \bar{y}\simeq\bar{z}) \to \hat{e}(\bar{a}).$$

9) the case where $\mathfrak{A}$ is

$$\forall\varphi\forall x\forall y\{\forall z(z\in x \dashv\vdash z\in y) \vdash (\varphi[x] \dashv\vdash \varphi[y])\}.$$

Then $\mathfrak{A}^{\rho\tilde{\eta}}$ is

$$\forall\bar{\varphi}\forall\bar{x}\forall\bar{y}\{\forall\bar{z}(\bar{z}\hat{\in}\bar{x} \dashv\vdash \bar{z}\hat{\in}\bar{y}) \vdash (\bar{\varphi}[x] \dashv\vdash \bar{\varphi}[y])\}.$$

Therefore we have only to prove

$$A_1, A_2, \cdots\cdots, \forall\bar{z}(\bar{z}\hat{\in}\alpha \dashv\vdash \bar{z}\hat{\in}\beta) \to \gamma[\alpha] \dashv\vdash \gamma[\beta].$$

Now $\forall\bar{z}(\bar{z}\hat{\in}\alpha \dashv\vdash \bar{z}\hat{\in}\beta)$ is equivalent to

$$\forall\bar{z}(\bar{z}\in\alpha \vdash \bar{z}\hat{\in}\beta) \wedge \forall\bar{z}(\bar{z}\hat{\in}\beta \vdash z\hat{\in}\alpha)$$

and $\forall\bar{z}(\bar{z}\hat{\in}\alpha \vdash z\hat{\in}\beta)$ is $\forall\bar{z}(\hat{e}(\bar{z}) \wedge \mathrm{E}x(\bar{z}[x]\wedge\alpha[x]) \vdash \bar{z}\hat{\in}\beta)$.

Since the following sequence is provable

$$A_1, A_2, \cdots\cdots \to \forall\bar{z}(\hat{e}(\bar{z}\wedge\mathrm{E}x(\bar{z})\,[x]\wedge\alpha[x]) \vdash \bar{z}\hat{\in}\beta) \dashv\vdash \forall z(\alpha[z] \vdash \tilde{z}\hat{\in}\beta).$$

We have only to prove

$$A_1, A_2, \cdots\cdots, \forall z(\alpha[z] \dashv\vdash \beta[z]) \to \gamma[\alpha] \dashv\vdash \gamma[\beta].$$

Hence the proposition is clear.

10) the case where $\mathfrak{A}$ is one of $\Gamma_e{}'$.

Then $\mathfrak{A}^{\rho\tilde{\eta}}$ is $\mathfrak{A}^{\rho\eta}$ and so the proposition is clear from 8.30.

9.2. Let R be a restricting system which is generated by $e(\ \ )$ and $\tilde{r}$ be its a strictly resticting operator, then we have

$$\Gamma_e{}'^{\tilde{r}},\ cl(\alpha) \to (\oplus(\alpha))^{\tilde{r}} \equiv \oplus(\alpha).$$

Proof. We prove the proposition by the induction on the height of α.

If the height of α is zero, then the proposition is clear. In the general case we have only to prove

$$\Gamma_e{}',\ cl(\alpha) \to \forall\varphi\{cl(\varphi) \wedge \alpha[(\oplus(\varphi))^{\tilde{r}}] \dashv\vdash \alpha[\oplus(\varphi)]\}.$$

By the hypothesis of the induction we have only to prove

$$\Gamma_e{}',\ cl(\alpha) \to \forall\varphi\{cl(\varphi) \wedge \alpha[\varphi] \dashv\vdash \alpha[\varphi]\}$$

and the proposition is evident.

9.3. Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and contain Γ_e' and be consistent in GLC without a bound function and the following sequence be provable in GLC without a bound function

$$A_1, A_2, \cdots\cdots \to \forall\varphi^1 \cdots\cdots \forall\varphi^i \mathrm{E}x\mathfrak{F}(x, \varphi^1, \cdots\cdots, \varphi^i)$$

$$A_1, A_2, \cdots\cdots \to \forall\varphi^1 \cdots\cdots \forall\varphi^i \forall x \forall y\{\mathfrak{F}(x, \varphi^1, \cdots\cdots, \varphi^i) \wedge \mathfrak{F}(y, \varphi^1, \cdots\cdots, \varphi^i) \vdash x=y\},$$

where $\forall\varphi^1 \cdots\cdots \forall\varphi^i \mathrm{E}x\mathfrak{F}(x, \varphi^1, \cdots\cdots, \varphi^i)$ is an axiom and has no bound function.

Then the following axioms are consistent in GLC

$$A_1, A_2, \cdots\cdots$$

$$\forall\varphi^1 \cdots\cdots \forall\varphi^i \mathfrak{F}(s_0(\varphi^1, \cdots\cdots, \varphi^i), \varphi^1, \cdots\cdots, \varphi^i)$$

$$\forall\varphi^1\forall\psi^1 \cdots\cdots \forall\varphi^i\forall\psi^i(\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^i \equiv \psi^i \vdash s_0(\varphi^1, \cdots\cdots, \varphi^i) = s_0(\psi^1, \cdots\cdots, \psi^i))$$

where s_0 is a special function which is not contained in $A_1, A_2, \cdots\cdots$.

Proof. By the hypothesis of the theorem

$$A_1, A_2, \cdots\cdots, \forall\varphi^1 \cdots\cdots \forall\varphi^i \mathrm{E}x\mathfrak{F}(x, \varphi^1, \cdots\cdots, \varphi^i)),$$

$\forall\varphi^1 \cdots\cdots \forall\varphi^i \forall x \forall y\{\mathfrak{F}(x, \varphi^1, \cdots\cdots, \varphi^i) \wedge \mathfrak{F}(y, \varphi^1, \cdots\cdots, \varphi^i) \vdash x=y\}$ are consistent in GLC without bound function.

Let us take $e(\ )$, $* \in *$ and M which are described in the theorem on the the set of a primary form.

Then by virtue of the theorem on the set of a primary form the following axioms are consistent in GLC

$$A_1^{\tilde{r}}, A_2^{\tilde{r}}, \cdots\cdots$$

$$\forall\varphi^1 \cdots\cdots \forall\varphi^i \mathrm{E}x(cl(\varphi^1) \wedge \cdots\cdots \wedge cl(\varphi^i) \vdash e(x) \wedge \tilde{\mathfrak{F}}^r(x, \varphi^1, \cdots\cdots, \varphi^i))$$

$$\forall\varphi^1 \cdots\cdots \forall\varphi^i \forall x \forall y(cl(\varphi^1) \wedge \cdots\cdots \wedge cl(\varphi^i) \wedge e(x) \wedge e(y)$$
$$\wedge \tilde{\mathfrak{F}}^r(x, \varphi^1, \cdots\cdots, \varphi^i) \tilde{\mathfrak{F}}^r(y, \varphi^1, \cdots\cdots, \varphi^i) \vdash x=y)$$

$$\forall x \forall y(x \in y \vdash e(x)\}$$

$$\forall x\{\mathrm{E}y(y \in x) \wedge \forall y \forall z(y \in x \wedge z \in x \vdash y \in z) \vdash e(x)\}$$

$$\forall x\{x \in x \dashv\vdash e(x)\}$$

$$\forall x \forall y\{x \in y \wedge e(v) \vdash x=y\}$$

$$\forall\varphi \forall x(x \in M(z)\varphi[z] \dashv\vdash \varphi[z] \wedge e(z)\}$$

$$\forall\varphi^1 \cdots\cdots \forall\varphi^i e(s(\varphi^1 \cdots\cdots, \varphi^i))$$

$$cl(\sigma).$$

Now we set these axioms $\mathfrak{M}$ and $s_0(\alpha^1, \cdots\cdots, \alpha^i)$ $M(z)\tilde{\mathfrak{F}}^r(z, \alpha', \cdots\cdots, \alpha^i)$. Then $\mathfrak{M} \to a \in M(z)\tilde{\mathfrak{F}}^r(z, \alpha^1, \alpha^i) \dashv\vdash e(a) \wedge \tilde{\mathfrak{F}}^r(z, \alpha^1, \cdots\cdots, \alpha^i)$. We can see easily $\mathfrak{M}$, $cl(\alpha^1), \cdots\cdots, cl(\alpha^i) \to \mathrm{E}x(x \in M(z)\tilde{\mathfrak{F}}^r(z, \alpha^1, \cdots\cdots, \alpha^i))$ and $\mathfrak{M}$, $cl(\alpha^1) \cdots\cdots, cl(\alpha^i), e(a), e(b)$

$$a \in M(z)\tilde{\mathfrak{F}}^r(z, \alpha^1, \cdots\cdots, \alpha^i), \quad b \in M(z)\tilde{\mathfrak{F}}^r(z, \alpha^1, \cdots\cdots, \alpha^i) \to a=b.$$

Therefore we have

$$\mathfrak{M},\ cl(\alpha^1), \cdots\cdots, cl(\alpha^i) \to e(M(z)\tilde{\mathfrak{F}}^r(z, \alpha^1, \cdots\cdots, \alpha^i)),$$

that is,

$$\mathfrak{M},\ cl(\alpha^1), \cdots\cdots, cl(\alpha^i) \to e(s_0(\alpha^1, \cdots\cdots, \alpha^i)).$$

Therefore

$$\mathfrak{M},\ cl(\alpha^1),\ \cdots\cdots,\ cl(\alpha^i)\rightarrow s_0(\alpha^1,\ \cdots\cdots,\ \alpha^i)\in M(z)\widetilde{\mathfrak{F}^r}(z,\alpha^1,\ \cdots\cdots,\ \alpha^i).$$

Hence

$$\mathfrak{M},\ cl(\alpha^1),\ \cdots\cdots,\ cl(\alpha^i)\rightarrow\widetilde{\mathfrak{F}^r}(s_0(\alpha^1,\ \cdots\cdots,\ \alpha^i)),\ (\alpha^1,\ \cdots\cdots,\ \alpha^i).$$

Hence

$$\mathfrak{M}\rightarrow\forall\varphi^1\cdots\cdots\forall\varphi^i(cl(\varphi^1)\wedge\cdots\cdots\wedge cl(\varphi^i)\vdash\widetilde{\mathfrak{F}^r}(s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i),\ \varphi^1,\ \cdots\cdots,\ \varphi^i)\}$$

Now we shall prove

$$\begin{aligned}&\mathfrak{M},\ cl(\alpha^1),\ \cdots\cdots,\ cl(\alpha^i),\ cl(\beta^1),\ \cdots\cdots,\ cl(\beta^i),\\ &\quad\forall\xi^1(cl(\xi^1)\vdash(\alpha^1[\xi^1]\dashv\vdash\beta^1[\xi^1])),\ \cdots\cdots,\ \forall\xi^i(cl(\xi^i)\\ &\qquad\vdash(\alpha^i[\xi^i]\dashv\vdash\beta^i[\xi^i])))\rightarrow s_0(\alpha^1,\ \cdots\cdots,\ \alpha^i)=s_0(\beta^1,\ \cdots\cdots,\ \beta^i)\end{aligned}$$

We see clearly $\mathfrak{M},\ cl(\alpha^1),\ \alpha^1[\gamma']\rightarrow cl(\gamma^1)$.

Therefore we have only to prove

$$\begin{aligned}&\mathfrak{M},\ cl(\alpha^1),\ \cdots\cdots,\ cl(\alpha^i),\ cl(\beta^1),\ \cdots\cdots,\ cl(\beta^i),\\ &\qquad\alpha^1\equiv\beta^1,\ \cdots\cdots,\ \alpha^i\equiv\beta^i\rightarrow s_0(\alpha^1,\ \cdots\cdots,\ \alpha^i)=s_0(\beta^1,\ \cdots\cdots,\ \beta^i).\end{aligned}$$

And this is clear from

$$\begin{aligned}&\mathfrak{M},\ cl(\alpha^1)\ \cdots\cdots,\ cl(\alpha^i),\ \cdots\cdots,\ cl(\beta^1),\ \cdots\cdots,\ cl(\beta^i),\\ &\qquad\alpha^1\equiv\beta^1,\ \cdots\cdots,\ \alpha^i\equiv\beta^i\rightarrow s_0(\alpha^1,\ \cdots\cdots,\ \alpha^i)\in s_0(\beta^1,\ \cdots\cdots,\ \beta^i).\end{aligned}$$

Therefore we have the following sequence from 9.2

$$\begin{aligned}&\mathfrak{M}\rightarrow\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i\{cl(\varphi^1)\wedge\cdots\cdots\wedge cl(\varphi^i)\wedge cl(\psi^1)\wedge\cdots\cdots\wedge cl(\psi^i)\\ &\qquad\wedge\forall\xi_1(cl(\xi^1)\vdash(\varphi^1[\widetilde{(\oplus(\xi^1))^r}]\dashv\vdash\psi^1[\widetilde{(\oplus(\xi^1))^r}]))\wedge\cdots\cdots\\ &\qquad\wedge\forall\xi^i(cl(\xi^i)\vdash(\varphi^i[\widetilde{(\oplus(\xi^i))^r}]\dashv\vdash\psi^i[\widetilde{(\oplus(\xi^i))^r}]))\\ &\quad\vdash s_0(\widetilde{((\oplus(\varphi)^1)^r},\ \cdots\cdots,\ \widetilde{(\oplus(\varphi^i))^r})=s_0(\widetilde{((\oplus(\psi)^1)^r},\ \cdots\cdots,\ \widetilde{(\oplus(\psi^i))^r})\}.\end{aligned}$$

By the above considerations we see the following axioms are consistent in GLC

$$\begin{aligned}&\mathfrak{M},\ \forall\varphi^1\cdots\cdots\forall\varphi^i(cl(\varphi^1)\wedge\cdots\cdots\wedge cl(\varphi^i)\vdash e(s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i)))\\ &\quad\widetilde{(\forall\varphi\cdots\cdots\forall\varphi^i\mathfrak{F}(s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i),\ \varphi^1,\ \cdots\cdots,\ \varphi^i))^r}\\ &\quad(\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i(\varphi^1\equiv\psi'\wedge\cdots\cdots\wedge\varphi^i\equiv\psi^i\vdash s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i)\\ &\qquad\widetilde{=s_0(\psi^1,\ \cdots\cdots,\ \psi^i)))^r},\end{aligned}$$

if we consider $s_0(*,\ \cdots\cdots,\ *)$ as a special function and don't consider as a functional.

Therefore by virtue of 8.36 the following axioms are consistent.

$$\begin{aligned}&A_1,\ A_2,\ \cdots\cdots\\ &\forall\varphi^1\cdots\cdots\forall\varphi^i\mathfrak{F}(s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i),\ \varphi^1,\ \cdots\cdots,\ \varphi^i)\\ &\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i(\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^i\equiv\psi^i\vdash s_0(\varphi^1,\ \cdots\cdots,\ \varphi^i)\\ &\qquad=s_0(\psi^1,\ \cdots\cdots,\ \psi^i)).\end{aligned}$$

As an easy special case of 9.3 we have following theorem.

9.4. Let $A_1,\ A_2,\ \cdots\cdots$ be axioms without bound function and contain Γ_e' and be consistent in GLC without bound function.

Then these axioms are consistent in GLC.

9.5. Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and let $A_1, A_2, \cdots\cdots$ and Γ_e' be consistent in GLC.

Then $A_1, A_2, \cdots\cdots$ and Γ_e are consistent in GLC.

Proof. Let $\tilde{\tilde{r}}$ be a strictly restricting operator which is strongly generated by $e(\ )$, which is a special variable of type 1 which is not contained in $A_1, A_2, \cdots\cdots$. From 8.35 and 8.13 we have only to prove that the following axioms are consistent in GLC.

$$A_1^{\tilde{\tilde{r}}\rho},\ A_2^{\tilde{\tilde{r}}\rho},\ A_3^{\tilde{\tilde{r}}\rho},\ \cdots\cdots$$

$(cl(\sigma))^\rho$ for each special variable σ which is contained in $A_1, A_2, \cdots\cdots$

$(cl(s))^\rho$ for each special function s which is contained in $A_1, A_2, \cdots\cdots$

$(\mathrm{E}xe(x))^\rho$

$\Gamma_e^{\tilde{\tilde{r}}\rho}$.

Let $\tilde{\eta}$ be the operator $\binom{\tilde{\sigma}}{\sigma}\cdots\cdots\binom{\tilde{s}}{s}\cdots\cdots\binom{\hat{e}(\ )}{e(\ )}$, where $\sigma, \cdots\cdots, s, \cdots\cdots$ are all the sepecial variables and all the special functions which are contained in $A_1, A_2, \cdots\cdots$ and $\{\varphi\}\hat{e}\{\varphi\}$ means $\{\varphi\}\mathrm{E}x(\varphi\equiv\tilde{x})$. In the same way as in the proof of the theorem on the set of a primary form, we see easily

$$A_1, A_2, A_3, \cdots\cdots, \Gamma_e'\to A_i^{\tilde{\tilde{r}}\rho\tilde{\eta}}$$

$$A_1, A_2, A_3, \cdots\cdots, \Gamma_e'\to(cl(\sigma))^{\rho\tilde{\eta}}\wedge(\mathrm{E}xe(x))^{\rho\tilde{\eta}}$$

$$A_1, A_2, A_3, \cdots\cdots, \Gamma_e'\to(\Gamma_e')^{\tilde{\tilde{r}}\rho\tilde{\eta}}.$$

Therefore we have only to prove

9.5.1. $\Gamma_e'\to(cl(s))^{\rho\tilde{\eta}}$

9.5.2. $\Gamma_e'\to(\forall p\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i(\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^i\equiv\psi^i \dashv\vdash p(\varphi^1, \cdots\cdots, \varphi^i)$
$=p(\psi^1, \cdots\cdots, \psi^i)))^{\tilde{\tilde{r}}\rho\tilde{\eta}}$.

First we shall prove 9.5.1.

$$(cl(s)^{\rho\tilde{\eta}} \text{ is } (\forall\varphi_1\cdots\cdots\forall\varphi_i(cl(\varphi_1)\wedge\cdots\cdots\wedge cl(\varphi_i)\vdash e(\varphi_1, \cdots\cdots, \varphi_i))))^{\rho\tilde{\eta}}.$$

Therefore we have only to prove

$$\Gamma_e'\to\forall\varphi_1\cdots\cdots\forall\varphi_i(\hat{e}(\{x\}(\tilde{s}(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i, x)))).$$

Hence from 8.17.1 the proposition is clear.

Now we shall prove 9.5.2. We have only to prove

$$\Gamma_e'\to\forall\bar{p}\forall\varphi^1\forall\psi^1\cdots\cdots\forall\varphi^i\forall\psi^i(cl^{\rho\tilde{\eta}}(\bar{p})\wedge\varphi^1\equiv\psi^1\wedge\cdots\cdots\wedge\varphi^i\equiv\psi^i$$

$$\vdash\{x\}\bar{p}[\tilde{\varphi}^1, \cdots\cdots, \tilde{\varphi}^i, x]\simeq\{x\}\bar{p}[\tilde{\varphi}^1, \cdots\cdots, \tilde{\varphi}^i x]).$$

Now $cl^{\rho\tilde{\eta}}(f)$ is

$$\forall\bar{\varphi}_1\cdots\cdots\forall\bar{\varphi}_i(cl^{\rho\tilde{\eta}}(\bar{\varphi}_1)\wedge\cdots\cdots\wedge cl^{\rho\tilde{\eta}}(\bar{\varphi}_i)\vdash\hat{e}(\{x\}\bar{f}[\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i, x])).$$

Therefore

$$\Gamma_e'\to cl^{\rho\tilde{\eta}}(\bar{f})\dashv\vdash\forall\varphi_1\cdots\cdots\forall\varphi_i(\hat{e}\{x\}\bar{f}[\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i, x])).$$

And then we have only to prove

$$\Gamma_e',\ \forall\varphi_1\cdots\cdots\forall\varphi_i \mathrm{E}y(\{x\}\bar{f}[\tilde{\varphi}_1,\cdots\cdots,\tilde{\varphi}_i,x]\equiv\tilde{y}),\ \alpha^1\equiv\beta^1,\cdots\cdots,\alpha^i\equiv\beta^i$$
$$\rightarrow\{x\}\bar{f}[\tilde{\alpha}^1,\cdots\cdots,\tilde{\alpha}^i,x]\simeq\{x\}\bar{f}[\tilde{\beta}^1,\cdots\cdots,\tilde{\beta}_i,x].$$

Therefore we have only to prove

$$\Gamma_e',\ \{x\}\bar{f}[\tilde{\alpha}_1,\cdots\cdots,\tilde{\alpha}_i,x]\equiv\tilde{a},\ \{x\}\bar{f}[\tilde{\beta}_1,\cdots\cdots,\tilde{\beta}_i]\equiv\tilde{b},\ \alpha^1\equiv\beta^1,\cdots\cdots,\alpha^i\equiv\beta^i$$
$$\rightarrow\{x\}\bar{f}[\tilde{\alpha}^1,\cdots\cdots,\tilde{\alpha}^i,x]\simeq\{x\}\bar{f}[\tilde{\beta}^1,\cdots\cdots,\tilde{\beta}^i,x].$$

Since $\alpha\simeq\beta$ means $\mathrm{E}x\mathrm{E}y(\alpha\equiv\tilde{x}\wedge\tilde{\beta}\equiv y\wedge x=y)$ we have only to prove

$$\Gamma_e',\ \{x\}\bar{f}[\tilde{\alpha}_1,\cdots\cdots,\tilde{\alpha}_i,x]\equiv\tilde{a},\ \{x\}\bar{f}[\tilde{\beta},\ \cdots\cdots,\tilde{\beta}_i]\equiv\tilde{b},$$
$$\alpha^1\equiv\beta^1,\cdots\cdots,\alpha^i\equiv\beta^i\rightarrow a=b$$

that is,

$$\Gamma_e',\ \bar{f}[\tilde{\alpha}_1,\cdots\cdots,\tilde{\alpha}_i\,a],\ \bar{f}[\tilde{\beta}_1,\cdots\cdots,\tilde{\beta},b],$$
$$\forall x(\bar{f}[\tilde{\alpha}_1,\cdots\cdots,\tilde{\alpha}_i,x]\vdash x=a),\ \forall x(\bar{f}[\tilde{\beta}_1,\cdots\cdots,\tilde{\beta}_i,x]\vdash x=b),$$
$$\alpha^1\equiv\beta^1,\cdots\cdots,\alpha^i\equiv\beta^i\rightarrow a=b,$$

which can be easily proved.

9.6. The Theorem on Function

Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and let $A_1, A_2, \cdots\cdots$ and Γ_e' be consistent in GLC without bound function and, moreover, let the following sequence be provable in GLC without bound function

$$A_1, A_2, \cdots\cdots, \Gamma_e'\rightarrow\forall\varphi^1\cdots\cdots\forall\varphi^i\mathrm{E}x\mathfrak{F}(x,\varphi^1,\cdots\cdots,\varphi^i)$$

and

$$A_1, A_2, \cdots\cdots, \Gamma_e'\rightarrow\forall\varphi^1\cdots\cdots\forall\varphi^i\forall x\forall y\{\mathfrak{F}(x,\varphi^1,\cdots\cdots,\varphi^i)$$
$$\wedge\mathfrak{F}(y,\varphi',\cdots\cdots,\varphi^i)\vdash x=y\},$$

where $\forall\varphi^1\cdots\cdots\forall\varphi^i\mathrm{E}x\mathfrak{F}(x,\varphi^1,\cdots\cdots,\varphi^i)$ is an axiom and has no bound function. Then the following axioms are consistent in GLC

$$A_1, A_2, \cdots\cdots$$
$$\forall\varphi^1\cdots\cdots\forall\varphi^i\mathfrak{F}(s_0(\varphi^1,\cdots\cdots,\varphi^i),\varphi^1,\cdots\cdots,\varphi^i)$$
$$\Gamma_e,$$

where s_0 is a special function which not contained in $A_1, A_2, \cdots\cdots$.

Proof. By virture of 9.3 and 9.5 the proposition is clear. As a special case of 9.6, we have

9.7. Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and let $A_1, A_2, \cdots\cdots$ and Γ_e' be consistent in GLC without bound function.

Then the following axioms are consistent in GLC

$$A_1, A_2, \cdots\cdots, \Gamma_e.$$

9.8. Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and $e(\ )$ be a special variable of type 1 which is not contained in $A_1, A_2, \cdots\cdots$ and R be the restricting system which generated by $e(\ )$ and $\tilde{r}$ be its strictly restricting operator.

Moreover let M be a special function of type 2 which is not contained in $A_1, A_2, \cdots\cdots$ and $*\in*$ be a special variable of type $(1, 1)$ which is not contained in $A_1, A_2, \cdots\cdots$ and let $A_1, A_2, \cdots\cdots$ which are be not contain $\underset{\circ}{=}$ and Γ_e is an equality axioms, whose equality notation is $\underset{\circ}{=}$ and not $=$.

If $A_1, A_2, \cdots\cdots$ and Γ_e' are consistent in GLC without bound function, then the following axioms are consistent in GLC.

$$A_1^{\tilde{r}}, A_2^{\tilde{r}}, \cdots\cdots$$
$$\forall x \forall y \{x \in y \vdash (x)\}$$
$$\forall x \{x \in x \dashv\vdash e(x)\}$$
$$\forall x \forall y \{x \in y \wedge e(y) \vdash x \underset{\circ}{=} y\}$$
$$\forall \varphi \forall x \{x \in M(z)\varphi \dashv\vdash [z] \wedge e(z)\}$$
$$\forall \varphi_1 \cdots\cdots \forall \varphi_i e(s(\varphi_1, \cdots\cdots, \varphi_i))$$
$$e(s_0)$$
$$cl(\sigma)$$
$$\forall x \{\mathrm{E}y(y \in x) \wedge \forall y \forall z (y \in x \wedge z \in x \vdash y \underset{\circ}{=} z) \vdash e(x)\}$$
$$\forall x \forall y \{\forall z (z \in x \dashv\vdash z \in y) \vdash x = y\}$$
$$\Gamma_e'.$$

Proof. By virtue of the theorem on the Set of a Primary Form the proposition is clear, setting $\{x, y\}(x=y)$ for $\{x, y\}\forall z(z \in x \dashv\vdash z \in y)$.

9.9. The Theorem on Set

Let $A_1, A_2, \cdots\cdots$ be axioms without a bound function and $e(\ \ )$ be a special variable of type 1 which is not contained in $A_1, A_2, \cdots\cdots$ and $\tilde{r}$ be its strictly restricting operator. Moreover, let M be a special function of type 2 which is not contained in $A_1, A_2, \cdots\cdots$ and $* \in *$ be a special variable of type (1, 1) which is not contained in $A_1, A_2, \cdots\cdots$. If $A_1, A_2, \cdots\cdots$ and Γ_e' are consistent in GLC witnout bound function, then the following axioms are consistent in GLC without bound function, then the following axioms are consistent in GLC.

$$A_1^{\tilde{r}\varepsilon}, \ A_2^{\tilde{r}\varepsilon}, \ \cdots\cdots$$
$$\forall x \forall y \{x \in y \vdash (x)\}$$
$$\forall x (x \in x \dashv\vdash e(x)\}$$
$$\forall x \forall y \{x \in y \wedge e(y) \vdash x = y\}$$
$$\forall \varphi \forall x \{x \in M(z)\varphi[z] \dashv\vdash \varphi[x] \wedge e(x)\}$$
$$\forall \varphi_1 \cdots\cdots \forall \varphi_i e(s(\varphi_1, \cdots\cdots, \varphi_i))$$
$$e(s_0)$$
$$cl(\sigma)$$
$$\forall x \{\mathrm{E}y(y \in x) \wedge \forall y \forall z (y \in x \wedge z \in x \vdash y = z) \vdash e(x)\}$$
$$\forall x \forall y \{\forall z (z \in x \dashv\vdash z \in y) \vdash x = y)\}$$
$$\Gamma_e,$$

where ε means $\begin{pmatrix} \{x, y\}(e(x) \wedge e(y) \wedge x = y) \\ \{x, y\}(x, y) \end{pmatrix}$.

Proof. From 9.5 and 8.8 the proposition is clear.

From now on in this section we consider another type-elevation, another class-elevation and another theorem on set, which are called the secondary type-elevation, the secondary class-elevation and the theorem on set of the secondary form. The detail proofs and definitions are omitted because they are similar as before.

9.10. Let A be a word. We define recursively as follows the secondary type-elevated figure $A^{\rho'}$ of A and we call ρ' the secondary type-elevator.

9.10.1. $a^{\rho'}$ is $\{x_1\}\forall\varphi(\varphi[x_1] \dashv\vdash \varphi[a])$.

$x^{\rho'}$ is $\{x_1\}\forall\varphi(\varphi[x_1] \dashv\vdash \varphi[x])$.

9.10.2. If A is of the form $\neg B, B\wedge C$ or $B\vee C$, then $A^{\rho'}$ is $\neg B^{\rho'}, B^{\rho'}\wedge C^{\rho'}$, $B^{\rho'}\vee C^{\rho'}$ respectively.

9.10.3. If A is of the form $\forall\varphi(n_1, \cdots\cdots, n_i)B$, $\mathrm{E}\varphi(n_1, \cdots\cdots, n_i)B$, $\forall p(m_1, \cdots \cdots, m_j)C$ or $\mathrm{E}p(m_1, \cdots\cdots, m_j)C$, then $A^{\rho'}$ is $\forall\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)B^{\rho'}$, $\mathrm{E}\bar{\varphi}(n_1+1, \cdots\cdots, n_i+1)B^{\rho'}$, $\forall\bar{\bar{p}}(m_1+1, \cdots\cdots, m_j+1)C^{\rho'}$ or $\mathrm{E}\bar{\bar{p}}(m_1+1, \cdots\cdots, m_j+1)C^{\rho'}$ respectstively.

Hereafter $\bar{\bar{p}}$ means a bound function and $\bar{\bar{f}}$ means a free function.

9.10.4. If A is of the form $\alpha[A_1, \cdots\cdots, A_i]$ or $\varphi[A_1, \cdots\cdots, A_i]$, then $A^{\rho'}$ is $\bar{\alpha}[A_1^{\rho'}, \cdots\cdots, A_i^{\rho'}]$ or $\bar{\varphi}[A_1^{\rho'}, \cdots\cdots, A_i^{\rho'}]$ respectively.

9.10.5. If A is of the form $f(A_i, \cdots\cdots, A_i)$ or $p(A_1, \cdots\cdots, A_i)$, then $A^{\rho'}$ is $\{x_m\}\forall\varphi(\varphi[x_m] \dashv\vdash \varphi[\bar{\bar{f}}(A_1^{\rho'}, \cdots\cdots, A_i^{\rho'})])$ or $\{x_m\}\forall\varphi(\varphi[x_m] \dashv\vdash \varphi[\bar{\bar{p}}(A_1^{\rho'}, \cdots\cdots, A_i^{\rho'})])$.

9.10.6. If A is of the form $\{\varphi_1, \cdots\cdots, \varphi_i\}B$, then $A^{\rho'}$ is $\{\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i\}B^{\rho'}$.

9.11. Let F be a functional of the form $\{\varphi_1, \cdots\cdots, \varphi_i\}T(\varphi_1, \cdots\cdots, \varphi_i)$. Since $T(\varphi_1, \cdots\cdots, \varphi_i)$ is a word, $(T(\varphi_1, \cdots\cdots, \varphi_i))^{\rho'}$ is well defined and is of the form $\{x_j\}\forall\varphi(\varphi[x_j] \dashv\vdash \varphi[\bar{\bar{T'}}(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i)])$. We define the secondary type-elevated figure $F^{\rho'}$ of F as the figure of the form $\{\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i\}T'(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_i)$.

9.12 Let $A(\alpha)$ be a formula or a variety and V be a variety of the same type as α. Then $(A(V))^{\rho'}$ is homologous to $A^{\rho'}(V^{\rho'})$. Moreover let $A(f)$ be a formula or a variety and F be a functional of the same type as f.

Then $(A(F))^{\rho'}$ is homologous to $A^{\rho'}(F^{\rho'})$.

Proof. We prove the proposition by the double induction as in the past. We treat only the essential new case, that is, we assume that $A(f)$ is $f(A_1(f), \cdots\cdots, A_n(f))$ and F is $\{\varphi_1, \cdots\cdots, \varphi_n\}g(B_1(\varphi_1, \cdots\cdots, \varphi_n), \cdots\cdots, B(\varphi_1, \cdots\cdots, \varphi_n))$. Then by the definition, $A^{\rho'}(\bar{\bar{f}})$ is $\{x_i\}\forall\varphi(\varphi[x_i] \dashv\vdash \varphi[(\bar{\bar{f}}(A_1^{\rho'}(\bar{\bar{f}}), \cdots\cdots, A_n^{\rho'}(\bar{\bar{f}}))])$ and $F^{\rho'}$ is

$$\{\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_n\}\bar{\bar{g}}(B_1^{\rho'}(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_n), \cdots\cdots, B_m^{\rho'}(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_n)).$$

Therefore $A^{\rho'}(F^{\rho'})$ is

$$\{x\}\forall\varphi(\varphi[x] \dashv\vdash \varphi[\bar{\bar{g}}(B_1^{\rho'}(A_1^{\rho'}(F^{\rho'}), \cdots\cdots, A_n^{\rho'}(F^{\rho'})), \cdots\cdots, B_m^{\rho'}(A_1^{\rho'}(F^{\rho'}), \cdots\cdots, A_n^{\rho'}(F^{\rho'})))]).$$

Hence by the hypothesis of the induction $A^{\rho'}(F^{\rho'})$ is homologous to

$$\{x\}\forall\varphi(\varphi[x] \vdash \varphi[\overline{\overline{g}}((B_1(A_1(F), \cdots\cdots, A_n(F)))^{\rho'}, \cdots\cdots, (B_m(A_1(F), \cdots\cdots, A_n(F)))^{\rho'})]).$$

Therefore by the definition $A^{\rho'}(F^{\rho'})$ is homologous to $(g(B_1(A_1(F), \cdots\cdots, A_n(F)), B_m(A_1(F), \cdots\cdots, A_n(F)))^{\rho'}$ and so the proposition is proved.

9.13. The Theorem on the Secondary Type-Elevation.

Let $\Gamma\rightarrow\Delta$ be a provable sequence in GLC. Then $\Gamma^{\rho'}\rightarrow\Delta^{\rho'}$ is provable in GLC.

Proof. We shall have the proof in the same way as in the past.

9.14. Let $f(n_1, \cdots\cdots, n_i)$ be an arbitrary function. We use an abbreviated notation $\tilde{\tilde{f}}$ for the homology class of the functional of type $(n_1+1, \cdots\cdots, n_i+1)$ $\{\varphi_1, \cdots\cdots, \varphi_i\}f(\{\psi_1\}\varphi_1[\tilde{\psi}_1], \cdots\cdots, \{\psi_i\}\varphi_i[\tilde{\psi}_i])$.

Now we prove the following proposition.

9.15. $\Gamma_e \rightarrow \tilde{\tilde{f}}(\tilde{\alpha}_1, \cdots\cdots, \tilde{\alpha}_i) = f(\alpha_1, \cdots\cdots, \alpha_i)$.

Proof. $\tilde{\tilde{f}}(\tilde{\alpha}_1, \cdots\cdots, \tilde{\alpha}_i)$ is $f(\{\psi_1\}\tilde{\alpha}[\tilde{\psi}_i], \cdots\cdots, \{\psi_i\}\tilde{\alpha}_i[\tilde{\psi}_i])$ and $\tilde{\alpha}[\tilde{\beta}]$ is $\mathrm{E}\psi(\tilde{\beta} \equiv\tilde{\psi}\wedge\alpha[\psi])$. Therefore from 8.15.1 $\Gamma_e\rightarrow\tilde{\alpha}[\tilde{\beta}] \vdash \alpha[\beta]$. Hence the proposition is clear.

9.16. $\Gamma_e\rightarrow\mathrm{E}p\forall\varphi_1\cdots\cdots\forall\varphi_i\{f(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i) = \tilde{\tilde{p}}(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i)\}$.

Proof. $\Gamma_e\rightarrow\forall\varphi_i\cdots\cdots\forall\varphi_i\{f(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i) = f(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i)\}$

Hence we have

$$\Gamma_e\rightarrow\mathrm{E}p\forall\varphi_1\cdots\cdots\forall\varphi_i\{f(\tilde{\varphi}_1, \cdots\cdots, \tilde{\varphi}_i) = p(\varphi_1, \cdots\cdots, \varphi_i)\}.$$

Therefore from 9.15 the proposition is proved.

9.17. Secondary Class-Elevation.

Let $\mathfrak{A}$ and V be a formula and a variety of monotype respectively. We define recursively as follows the secondary class-elevated homology class $\mathfrak{A}^{\sigma'}$ and $V^{\sigma'}$. And σ' is called the secondary class-elevator.

9.17.1. $a^{\sigma'}$ is $\{x\}\forall\varphi(\varphi[x] \vdash \varphi[a])$

9.17.2. If $\mathfrak{A}$ is of the form $\neg\mathfrak{B}, \mathfrak{B}\wedge\mathfrak{C}$ or $\mathfrak{B}\vee\mathfrak{C}$, then $\mathfrak{A}^{\sigma'}$ is $\neg\mathfrak{B}^{\sigma'}$, $\mathfrak{B}^{\sigma'}\wedge\mathfrak{C}^{\sigma'}$ or $\mathfrak{B}^{\sigma'}\vee\mathfrak{C}^{\sigma'}$ respectively.

9.17.3. If $\mathfrak{A}$ is of the from $\forall\varphi\mathfrak{F}(\varphi)$ or $\mathrm{E}\varphi\mathfrak{F}(\varphi)$, then $\mathfrak{A}^{\sigma'}$ is $\forall\psi\mathfrak{F}^{\sigma}(\tilde{\psi})$ or $\mathrm{E}\psi\mathfrak{F}^{\sigma'}(\psi)$ respectively.

9.17.4. If $\mathfrak{A}$ is of the form $\forall p(\mathfrak{F}(p))$ or $\mathrm{E}p\mathfrak{F}(p)$, then $\mathfrak{A}^{\sigma'}$ is $\forall q\mathfrak{F}^{\sigma'}(\tilde{\tilde{q}})$ or $\mathrm{E}q\mathfrak{F}^{\sigma'}(\tilde{\tilde{q}})$ respectively.

9.17.5. If $\mathfrak{A}$ is of the form $\alpha[V_1, \cdots\cdots, V_i]$, then $\mathfrak{A}^{\sigma'}$ is $\alpha[V_1^{\sigma'}, \cdots\cdots, V_i^{\sigma'}]$.

9.17.6. If V is of the form $f(V_1, \cdots\cdots, V_i)$, then $V^{\sigma'}$ is

$$\{x\}\forall\varphi(\varphi[x] \vdash \varphi[\overline{\overline{f}}(V_1^{\sigma'}, \cdots\cdots, V_i^{\sigma'})]).$$

9.17.7. If V is of the form $\{\varphi\}\mathfrak{F}(\varphi)$, then $V^{\sigma'}$ is $\{\psi\}\mathrm{E}\varphi(\psi\equiv\tilde{\varphi}\wedge\mathfrak{F}^{\sigma'}(\tilde{\varphi}))$.

9.18 Operation $\rho'\eta'$ and $\sigma'\eta'$.

Let A be a formula or a variety of monotype, and all the free or special variables and functions which are contained in A be, $\alpha\cdots\cdots, f, \cdots\cdots$. Then we defin e$A^{\rho'\eta'}$ and $A^{\sigma'\eta'}$ as $(A^{\rho'})\binom{\tilde{\alpha}}{\alpha}\cdots\cdots\binom{\tilde{\tilde{f}}}{\overline{\overline{f}}}\cdots\cdots$ and $(A^{\sigma'})\binom{\tilde{\alpha}}{\alpha}\cdots\cdots\binom{f''}{\overline{\overline{f}}}\cdots\cdots$ respectively.

9.19. Let $\mathfrak{A}$, V and T be an arbitrary formula, an arbitrary variety of the form $\{\varphi\}\mathfrak{F}(\varphi)$ and a term.

Then the following sequences are provable in GLC.

9.19.1. $\Gamma_e \to \mathfrak{A}^{\sigma'\eta'} \vdash\dashv \mathfrak{A}$.

9.19.2. $\Gamma_e \to V^{\sigma'\eta'} \equiv \{\psi\}\mathrm{E}\varphi(\psi \equiv \tilde{\varphi} \wedge \mathfrak{F}(\varphi))$.

9.19.3. $\Gamma_e \to T^{\sigma'\eta'} \equiv \{x\}(x = T)$.

Proof. We prove the proposition by the induction as in the past.

We treat only the essential new cases.

1) the case where $\mathfrak{A}$ is of the form $\forall p \mathfrak{F}(p)$.

Then $\mathfrak{A}^{\sigma'\eta'}$ is $\forall p \mathfrak{F}^{\sigma'\eta'}(\tilde{\tilde{p}})$. By the hypothesis of the induction we have $\Gamma_e \to \mathfrak{F}^{\sigma'\eta'}(\tilde{\tilde{f}}) \vdash\dashv \mathfrak{F}(f)$. Therefore the proposition is clear.

2) the case where T is of the form $f(V_1, \cdots\cdots, V_i)$.

We assume, further, that V_j is of the form $\{\varphi_j\}\mathfrak{F}_j(\varphi_j)$ for each $j\,(1 \leqq j \leqq i)$. Then $T^{\sigma'\eta'}$ is $\{x\}\forall\varphi(\varphi[x] \vdash\dashv \varphi[\tilde{\tilde{f}}(\{\varphi_1\}\mathrm{E}\psi_1$

$$(\tilde{\varphi}_1 \equiv \tilde{\psi}_1 \wedge \mathfrak{F}^{\sigma'\eta'}(\tilde{\psi}_1)), \cdots\cdots, \{\varphi_i\}\mathrm{E}\psi_i(\varphi_i \equiv \psi_i \wedge \mathfrak{F}^{\sigma'\eta'}(\tilde{\psi}_i)))).$$

Therefore we can see easily $\Gamma_e \to T^{\sigma'\eta'} \equiv \{x\}(x = f(V_1, \cdots\cdots, V_i))$, and the proposition is proved.

9.20. Under the same hypothesis of 9.19 the following sequences are provable in GLC

$$\Gamma_e \to \mathrm{E}\varphi(V^{\sigma'\eta'} \equiv \tilde{\varphi})$$

$$\Gamma_e \to \mathrm{E}x(T^{\sigma'\eta'} \equiv \tilde{x}).$$

Proof. The proposition is clear from 9.19.

9.21. Let $e(\ )$ be a special variable of type 1. We define recursively as follows the secondary restricting system $R\{\mathfrak{F}\langle n_1, \cdots\cdots, n_i\rangle(\alpha); \mathfrak{G}\langle m_1, \cdots\cdots, m_j\rangle(f)\}$ which is generated by $e(\ )$.

9.21.1. $\mathfrak{F}\langle 0\rangle(a)$ is $e(a)$.

9.21.2. $\mathfrak{F}\langle n_1+1, \cdots\cdots, n_i+1\rangle(\alpha)$ is

$$\forall\varphi_1, \cdots\cdots, \forall\varphi_i(\alpha[\varphi_1, \cdots\cdots, \varphi_i] \vdash \mathfrak{F}\langle n_1\rangle(\varphi_1) \wedge \cdots\cdots \wedge \mathfrak{F}\langle n_i\rangle(\varphi_i)).$$

9.21.3. $\mathfrak{G}\langle m_1+1, \cdots\cdots, m_j+1\rangle(f)$ is

$$\forall\varphi_1 \cdots\cdots \forall\varphi_j(f(\varphi_1, \cdots\cdots, \varphi_j) = f(\{\psi_1\}(cl(\psi_1) \wedge \varphi_1[\psi_1]), \cdots\cdots, \{\psi_j\}(cl(\psi_j) \wedge \varphi_j[\psi_j]))).$$

9.22. We have the following sequences.

9.22.1. $\Gamma_e \to \mathfrak{F}^{\rho'\tau}\langle n_1+1, \cdots\cdots, n_i+1\rangle(\bar{\alpha}) \vdash\dashv \mathrm{E}\xi(\bar{\bar{\alpha}} \equiv \tilde{\xi})$.

9.22.2. $\Gamma_e \to \mathfrak{G}^{\rho'\tau}\langle m_1+1, \cdots\cdots, m_j+1\rangle(\bar{\bar{f}})$

$$\vdash\dashv \mathrm{E}p\forall\varphi_1 \cdots\cdots \forall\varphi_i(\bar{\bar{f}}(\varphi_1, \cdots\cdots, \varphi_i) = \tilde{\tilde{p}}(\varphi_1, \cdots\cdots, \varphi_i)),$$

where τ means $\begin{pmatrix} \bar{\sigma}_0 \\ \tilde{=} \end{pmatrix}\begin{pmatrix} \bar{e}(\ ) \\ \hat{e}(\ ) \end{pmatrix}$ and σ_0 means $=$.

Proof. The proof of 9.21.1 goes in the same way as in the proof as that of 8.27. Therefore we prove 9.22.2.

$cl^{\rho'\tau}(\bar{\bar{f}})$ is

$$\forall\bar{\varphi}_1\cdots\cdots\forall\bar{\varphi}_j(\{x\}\forall\varphi(\varphi[x] \vdash \varphi[\bar{\bar{f}}(\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_j)]) \\ \simeq\{x\}\forall\varphi(\varphi[x] \vdash \varphi[\bar{\bar{f}}((\{\psi_1\}(cl(\psi_1) \wedge\varphi_1[\psi_1]))^{\rho'\tau}, \\ \cdots\cdots, (\{\psi_j\}(cl(\psi_j) \wedge\varphi_j[\psi_j]))^{\rho'\tau})]).$$

Therefore we see

$$\Gamma_e\to cl^{\rho'\tau}(\bar{\bar{f}}) \vdash \forall\bar{\varphi}_1\cdots\cdots\forall\bar{\varphi}_j(\bar{\bar{f}}[\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_j] \\ =\bar{\bar{f}}((\{\psi_1\}(cl(\psi_1) \wedge\varphi_1[\psi_1]))^{\rho'\tau}, \cdots\cdots, (\{\psi_j\}(cl(\psi_j) \wedge\varphi_j[\psi_j]))^{\rho'\tau})).$$

Moreover

$$(\{\psi\}(cl(\psi) \wedge\varphi[\psi]))^{\rho'\tau} \text{ is } \{\bar{\psi}\}(cl^{\rho'\tau}(\bar{\psi}) \wedge\bar{\varphi}[\bar{\psi}]).$$

From 9.22.1 we see

$$\Gamma_e\to (\{\psi\}(cl(\psi) \wedge\alpha[\psi]))^{\rho'\tau}\equiv\{\bar{\psi}\}\mathrm{E}\psi(\bar{\psi}\equiv\tilde{\psi}\wedge\bar{\alpha}[\bar{\psi}]).$$

Therefore we see

$$\Gamma_e\to cl^{\rho'\tau}(\bar{\bar{f}}) \vdash \forall\bar{\varphi}_1\cdots\cdots\forall\bar{\varphi}_j(\bar{\bar{f}}[\bar{\varphi}_1, \cdots\cdots, \bar{\varphi}_j] \\ =\bar{\bar{f}}(\{\bar{\psi}_1\}\mathrm{E}\psi_1(\bar{\psi}_1\equiv\tilde{\psi}_1\wedge\bar{\varphi}_1[\tilde{\psi}_1]),\cdots\cdots,\{\bar{\psi}_j\}\psi_j(\bar{\psi}_j\equiv\tilde{\psi}_j\mathrm{E}\wedge\bar{\varphi}_j[\tilde{\psi}_j]))).$$

$$\Gamma_e,\ cl^{\rho'\tau}(\bar{\bar{f}}),\ \beta_1\equiv\{\psi_1\}\alpha_1[\tilde{\psi}_1], \cdots\cdots, \beta_j\equiv\{\psi_j\}\alpha_j[\tilde{\psi}_j] \\ \to\bar{\bar{f}}(\alpha, \cdots\cdots, \alpha_j) =\bar{\bar{f}}(\tilde{\beta}_1, \cdots\cdots, \tilde{\beta}_j).$$

And we see

$$\Gamma_e,\ cl^{\rho'\tau}(\bar{\bar{f}}),\ g(\beta_1, \cdots\cdots, \beta_j) =\bar{\bar{f}}(\tilde{\beta}, \cdots\cdots, \tilde{\beta}_j),\ \beta_1\equiv\{\psi_1\}\alpha_1[\tilde{\psi}_1], \\ \cdots\cdots, \beta_j\equiv\{\psi_j\}\alpha_j[\tilde{\psi}_j]\to\bar{\bar{f}}(\alpha_1, \cdots\cdots, \alpha_j) =\tilde{\tilde{g}}(\alpha_1, \cdots\cdots, \alpha_j).$$

Therefore we see

$$\Gamma_e,\ cl^{\sigma'\tau}(\bar{\bar{f}}),\ \beta_1\equiv\{\psi_1\}\alpha_1[\tilde{\psi}_1], \cdots\cdots, \beta_j\equiv\{\psi_j\}\alpha_j[\tilde{\psi}_j] \\ \to\bar{\bar{f}}(\alpha_1, \cdots\cdots, \alpha_j) =\tilde{\tilde{g}}(\alpha_1, \cdots\cdots, \alpha_j).$$

And we have

$$\Gamma_e,\ cl^{\rho'\tau}(\bar{\bar{f}})\to Ep\forall\varphi_1\cdots\cdots\forall\varphi_j(\bar{\bar{f}}(\varphi_1, \cdots\cdots, \varphi_j) =\tilde{\tilde{p}}(\varphi_1, \cdots\cdots, \varphi_j)).$$

And so we have only to prove

$$\Gamma_e,\ \mathrm{E}p\forall\varphi_1\cdots\cdots\forall\varphi_j(\bar{\bar{f}}(\varphi_1, \cdots\cdots, \varphi_j) =\tilde{\tilde{p}}(\varphi_1, \cdots\cdots, \varphi_j))\to cl^{\sigma'\tau}(\bar{\bar{f}}).$$

To prove this we have only to prove

$$\Gamma_e\to\tilde{\tilde{g}}(\alpha_1, \cdots\cdots, \alpha_j) =\tilde{\tilde{g}}(\{\psi_1\}\mathrm{E}\varphi_1(\psi_1\equiv\tilde{\varphi}_1\wedge\alpha_1[\tilde{\varphi}_1]), \cdots\cdots, \\ \{\psi_j\}\mathrm{E}\varphi_j(\psi_j\equiv\tilde{\varphi}_j\wedge\alpha_j[\tilde{\varphi}_j])).$$

that is,

$$\Gamma_e,\ \beta_1\equiv\{\psi_1\}\alpha_1[\tilde{\psi}_1], \cdots\cdots, \beta_j\equiv\{\psi_j\}\alpha_j[\tilde{\psi}_j] \\ \to\tilde{\tilde{g}}(\alpha_1, \cdots\cdots, \alpha_j) =\tilde{\tilde{g}}(\tilde{\beta}_1, \cdots\cdots, \tilde{\beta}_j).$$

Therefore we have only to prove

$$\Gamma_e,\ \beta_1\equiv\{\psi_1\}\alpha_1[\tilde{\psi}_1], \cdots\cdots, \beta_j\equiv\{\psi_j\}\alpha_j[\tilde{\psi}_j] \\ \to\tilde{\tilde{g}}(\alpha_1, \cdots\cdots, \alpha_j) =g(\beta_1, \cdots\cdots, \beta_j)$$

which is clear by the definition of $\tilde{\tilde{g}}$.

9.23 Let $\mathfrak{A}$ and V be a formula and a variety respectively and $e(\ )$ be a special variable of type 1 which is not contained in $\mathfrak{A}$ and V and let $\boldsymbol{R}$ be the secondary restricting system which is generated by $e(\ )$ and $\tilde{r}$ be its strictly restricting operator.

Then the following sequences are provable in GLC.

9.23.1. $\Gamma_e \to \mathfrak{A}^{\tilde{r}\rho'\tau'} \dashv\vdash \mathfrak{A}^{\sigma'\eta'}$.

9.23.2. $\Gamma_e \to V^{\tilde{r}\rho'\tau'} \equiv V^{\sigma'\eta'}$.

where τ' means $\binom{\tilde{\sigma}}{\sigma} \cdots\cdots \binom{\overset{|}{f}}{\overline{f}} \cdots\cdots \binom{\hat{e}(\ )}{e(\ )}$ provided that $\sigma, \cdots\cdots, f, \cdots\cdots$ are all the free or special variables and functions except $e(\ )$.

Proof. The proof goes in the same way as in the proof of 8.28.

9.24. Under the hypothesis as that of 9.23 the following sequences are provable in GLC.

9.24.1. $\Gamma_e \to \mathfrak{A}^{\tilde{r}\rho'\tau'} \dashv\vdash \mathfrak{A}$.

9.24.2. $\Gamma_e \to V^{\tilde{r}\rho'\tau'} \equiv \{\psi\}\mathrm{E}\varphi(\psi \equiv \tilde{\varphi} \wedge \mathfrak{F}(\varphi))$, if V is of the from $\{\varphi\}\mathfrak{F}(\varphi)$.

9.24.3. $\Gamma_e \to V^{\tilde{r}\rho'\tau'} \equiv \{x\}(x = V)$, if V is a term.

9.25. The following sequence is provable in GLC $\Gamma_e \to \mathfrak{A}^{\rho'\eta'}$. Here $\mathfrak{A}$ is an arbitrary axiom of Γ_e.

Proof. We have only to treat the case where $\mathfrak{A}$ is

$$\forall p \forall \varphi^1 \forall \psi^1 \cdots\cdots \forall \varphi^j \forall \psi^j \{\varphi^1 \equiv \psi^1 \wedge \cdots\cdots \wedge \varphi^j \equiv \psi^j \vdash p(\varphi^1, \cdots\cdots, \varphi^j) \\ = p(\psi^1, \cdots\cdots, \psi^j).$$

Then $\mathfrak{A}^{\rho'\eta'}$ is

$$\forall \overline{\overline{p}} \forall \overline{\varphi}^1 \forall \overline{\psi}^1 \cdots\cdots \forall \overline{\varphi}^j \forall \overline{\psi}^j \{\mathfrak{F}^{\rho\eta}(\overline{\varphi}^1, \overline{\psi}^1) \wedge \cdots\cdots \wedge \mathfrak{F}^{\rho\eta}(\overline{\varphi}^j, \overline{\psi}^j) \\ \vdash \{x\}\forall \varphi(\varphi[x] \dashv\vdash \varphi[\overline{\overline{p}}(\overline{\varphi}_1, \cdots\cdots, \overline{\varphi}^j)]) \simeq \{x\}\forall \varphi(\varphi[x] \\ \dashv\vdash \varphi[\overline{\overline{p}}(\overline{\psi}^1, \cdots\cdots, \overline{\psi}^j)])\},$$

where $\mathfrak{F}^{\rho\eta}(\overline{\varphi}, \overline{\psi})$ is the same as that of 8.23.

Therefore we see easily

$$\Gamma_e \to \mathfrak{A}^{\rho'\eta'} \dashv\vdash \forall \overline{\overline{p}} \forall \overline{\varphi}^1 \forall \overline{\psi}^1 \cdots\cdots \forall \overline{\varphi}^j \forall \overline{\psi}^1 \{\overline{\varphi}^1 \equiv \overline{\psi}^1 \wedge \cdots\cdots \wedge \overline{\varphi}^j \equiv \overline{\psi}^j \\ \vdash \overline{\overline{p}}(\overline{\varphi}', \cdots\cdots, \overline{\varphi}^j) = \overline{\overline{p}}(\overline{\psi}', \cdots\cdots, \overline{\psi}^j)\}.$$

Hence we have $\Gamma_e \to \mathfrak{A}^{\rho'\eta'}$.

9.26. $A_1, A_2, \cdots\cdots$ be axioms without the special variable $e(\ )$ of type 1 and the special variable $* \in *$ of type $(1, 1)$ and let R be the secondary restricting system which is generated by $e(\ )$ and $\tilde{r}$ be its strictly restricting operator.

If $A_1, A_2, \cdots\cdots$ and Γ_e are consistent in GLC, then the following axioms are consistent in GLC.

$$A_1^{\tilde{r}}, A_2^{\tilde{r}}, \cdots\cdots$$

$$\Gamma_e$$

$$\forall x \forall y (x \in y \vdash e(x)\}$$

$$\forall x \{x \in x \dashv\vdash e(x)\}$$

$$\forall x \forall y \{x \in y \wedge e(y) \vdash x = y\}$$

$$\forall x \{\mathrm{E}y(y \in x) \wedge \forall y \forall z (y \in x \wedge z \in x \vdash y = z) \vdash e(x)\}$$

$$\forall \varphi \forall x \forall y \{\forall z (z \in x \dashv\vdash z \in y) \vdash (\varphi[x] \dashv\vdash \varphi[y])\}$$

$$\forall \varphi \mathrm{E}x \forall y \{y \in x \dashv\vdash \varphi[y] \wedge e(y)\}$$

$$e(s_0)$$

$\forall\varphi_1\cdots\cdots\forall\varphi_i e(s(\varphi_1, \cdots\cdots, \varphi_i))$ for each special function s which is contained in $A_1, A_2, \cdots\cdots$

$cl(\sigma)$ for each special variable σ which is contained in $A_1, A_2, \cdots\cdots$

$cl(s)$ for each special function s which is contained in $A_1, A_2, \cdots\cdots$.

Proof. We write all these axiom as $\mathfrak{M}$. We have only to prove that $\mathfrak{M}^{\rho'}$ is consistent in GLC.

We consider that all the special variables and functions which are contained in $A_1, A_2, \cdots\cdots$ are $\sigma, \cdots\cdots, s, \cdots\cdots$.

Then we define operator $\tilde{\eta}$ as

$$\binom{\tilde{\sigma}}{\bar{\sigma}}\cdots\cdots\binom{\tilde{\tilde{s}}}{\bar{s}}\cdots\cdots\binom{\hat{e}(\ )}{e(\ )}\binom{*\hat{\in}*}{*\in*}.$$

Then we shall prove $\mathfrak{M}^{\rho\tilde{\eta}}$ is consistent.

To prove this it is sufficient that the following sequence is pravable $A_1, A_2, \cdots\cdots, \Gamma_e \to \mathfrak{A}^{\rho'\tilde{\eta}}$, where $\mathfrak{A}$ is an axiom of $\mathfrak{M}$.

The proof goes in the same way as in 9.1.

Therefore we treat only the essential new cases.

1) the case where $\mathfrak{A}$ is

$$\forall\varphi \mathrm{E}x\forall y\{y \in x \mathrel{\mathrm{H}} \varphi[y] \wedge e(y)\}.$$

Then $\mathfrak{A}^{\rho'\tilde{\eta}}$ is

$$\forall\bar{\varphi}\mathrm{E}\bar{x}\forall\bar{y}\{\hat{e}(\bar{y}) \wedge \mathrm{E}z(\bar{y}[z]) \wedge \bar{x}[z] \mathrel{\mathrm{H}} \bar{\varphi}[\bar{y}] \wedge \hat{e}(y)\}.$$

Since clearly we have

$$\to \mathfrak{A}^{\rho'\tilde{\eta}} \mathrel{\mathrm{H}} \forall\bar{\varphi}\mathrm{E}\bar{x}\forall\bar{y}\{\hat{e}(\bar{y}) \vdash (\mathrm{E}z(\bar{y}[z] \wedge \bar{x}[z]) \mathrel{\mathrm{H}} \bar{\varphi}[\bar{y}])\},$$

we have only to prove

$$A_1, A_2, \cdots\cdots, \Gamma_e \to \forall\varphi\mathrm{E}\psi\forall y\{\mathrm{E}z(z=y \wedge \psi[z]) \mathrel{\mathrm{H}} \varphi[\tilde{y}]\},$$

that is

$$A_1, A_2, \cdots\cdots, \Gamma_e \to \forall\varphi\mathrm{E}\psi\forall y\{\psi[y] \mathrel{\mathrm{H}} \varphi[\tilde{y}]\}$$

which is evident.

2) the case where $\mathfrak{A}$ is $cl(s)$.

Then $\mathfrak{A}^{\rho\tilde{\eta}}$ is $cl^{\rho'\tilde{\eta}}(\tilde{\tilde{s}})$ and the proposition is clear from 9.22.

9.27. The Theorem on Set of the Secondary Form.

Let $A_1, A_2, \cdots\cdots$ be axioms without the special variable $e(\ )$ of type 1 and the secondary restricting system which is generated by $e(\ )$ and $\tilde{r}$ be its strictly restricting operator.

If $A_1, A_2, \cdots\cdots$ and Γ_e are consistent in GLC, then the following axioms are consistent in GLC.

$A_1^{\tilde{r}\varepsilon}, A_2^{\tilde{r}\varepsilon}, \cdots\cdots$

Γ_e

$\forall x\forall y\{x \in y \vdash e(x)\}$

$\forall x\{x \in x \mathrel{\mathrm{H}} e(x)\}$

$\forall x\forall y\{x \in y \wedge e(y) \vdash x=y\}$

$$\forall x\{\mathrm{E}y(y \in x)) \wedge \forall y \forall z(y \in x \wedge z \in x \vdash y \in z) \vdash e(x)\}$$
$$\forall x \forall y\{\forall z(z \in x \dashv\vdash z \in y) \vdash x=y\}$$
$$\forall \varphi E x \forall y\{y \in x \dashv\vdash \varphi[y] \wedge e(y)\}$$
$$e(s_0)$$
$$\forall \varphi_1 \cdots\cdots \forall \varphi_i e(s(\varphi_1, \cdots\cdots, \varphi_i))$$
$$cl(\sigma)$$
$$cl(s).$$

where $\mathcal{E}$ means the same as that in 9.9.

Proof. By virtue of 9.26 the proof goes in the same way as in the proof of 9.9.

9.28. If the following axioms Γ_a are consistent, then the natural number theory is consistent.

$$\forall x(x=x)$$
$$\forall x \forall y(x=y \vdash y=x)$$
$$\forall x \forall y \forall z(x=y \wedge y=z \vdash x=z)$$
$$\forall x \neg(x'=1)$$
$$\forall x \forall y(x=y \dashv\vdash x'=y')$$

Proof. If Γ_a is consistent, then from 7.24 Γ_a, Γ_e and

$$\forall \varphi \forall x\{\varphi[1] \wedge \forall y(\varphi[y] \vdash \varphi[y']) \vdash \varphi[x]\}$$

are consistent.

We define $a<b$ as

$$\mathrm{E}\varphi \mathrm{E}x\{\forall s \forall t \forall u(\varphi[s, u] \wedge \varphi[t, u] \vdash s=t) \wedge \forall u \mathrm{E}v(\varphi[v, u])$$
$$\wedge \varphi[a, l] \wedge \forall u \forall v(\varphi[u, v] \varphi[u', v'] \wedge \varphi[b, x] \wedge \neg(x=1)\}.$$

Then we have the following axioms,

$$\forall x \forall y(x<y \vee x=y \vee y<x)$$
$$\forall x \forall y \neg(x<y \wedge x=y)$$
$$\forall x \forall y \neg(x<y \wedge y<x)$$
$$\forall x \forall y \forall z(x<y \wedge y<z \vdash x<z)$$
$$\forall x(x<x')$$
$$\forall x(1<x \vee 1=x)$$
$$\forall x \forall y(x<y \vdash x'<y \vee x'=y).$$

Moreover by virtue of the theorem on function we have the following axiom.

$$\forall \varphi(\mathrm{E}x\varphi[x] \vdash \varphi[\mathrm{Min}(\{x\}\varphi[x])])$$
$$\forall \varphi \forall x(\varphi[x] \vdash x \geqq \mathrm{Min}(\{x\}\varphi[x]))$$

where $a \geqq b$ means $b<a \vee b=a$.

And the concepts in the theory of natural numbers are represented by using of Min and $\forall \varphi$ etc.

9.29. If the following axioms Γ_r are consistent, then the real number theory is consistent.

$\forall x(x=x)$

$\forall x\forall y(x=y \vdash y=x)$

$\forall x\forall y\forall z(x=y \wedge y=z \vdash x=z)$

$\forall x\forall y\forall z(x=v \vdash x+z=y+z)$

$\forall x(0+x=x)$

$\forall x\forall y(x+y=y+x)$

$\forall x\forall y\forall z\{(x+y)+z=x+(y+z)\}$

$\forall x\forall y(x=y \vdash -x=-y)$

$\forall x(-x+x=0)$

$\forall x\forall y\forall z(x=y \vdash x\cdot z=y\cdot z)$

$\forall x(1\cdot x=x)$

$\forall x\forall y(x\cdot y=y\cdot x)$

$\forall x\forall y\forall z\{x\cdot(y\cdot z)=(x\cdot y)\cdot z\}$

$\forall x(x=0\vee x^{-1}\cdot x=1)$

$\forall x\forall y\forall z\{(x+y)\cdot z=x\cdot z+y\cdot z\}$

$\forall x\forall y\forall z\{(x=y\wedge y<z) \vdash x<z\}$

$\forall x\forall y\forall z\{(x=y\wedge z<y) \vdash z<x\}$

$0<1$

$\forall x\forall y(x=y\vee x<y\vee y<x)$

$\forall x\forall y(\neg(x<y)\vee\neg(x=y))$

$\forall x\forall y(\neg(x<y)\vee\neg(y<x))$

$\forall x\forall y\forall z\{(x<y\wedge y<z) \vdash x<z\}$

$\forall x\forall y\forall z(x<y \vdash x+z<y+z)\}$

$\forall x\forall y\forall z(0<x\wedge y<z \vdash x\cdot y<x\cdot z)$

Proof. If Γ_r are consistent, then in the same way as in 7.24 the following axioms $\tilde{\Gamma}_r$ are consistent.

Γ_r

Γ_e

$g(0)$

$g(1)$

$\forall x\forall y\{g(x)\wedge x=y \vdash g(y)\}$

$\forall x\forall y\{g(x)\wedge g(y) \vdash g(x+y)\}$

$\forall x\forall y(g(x)\wedge g(y) \vdash g(x\cdot y))$

$\forall x(g) \vdash x=0\vee 1<x\vee 1=x)$

$\forall x\mathrm{E}y\mathrm{E}z(0<x \vdash x=y^{-1}\cdot z\wedge g(y)\wedge g(a))$

$\forall\varphi\forall x\{\varphi[0]\wedge\forall y(\varphi[y] \vdash \varphi[y+1])\wedge g(x) \vdash \varphi[x]\}$

Therefore the proposition is evident by applying the theorem on the set for $\tilde{\Gamma}_r$.

Appendix.

A.1. We call a formula, an axiom or proof-figure that of G^iLC ($i=0, 1, 2, \cdots\cdots$), if and only if the heights of the variables which are contained in it are not greater than i and the heights of fnnctions which are contained in it are not greater than $i+1$.

Therefore G^0LC is *LK*.

Let $A_1, \cdots\cdots, A_N$ be axioms in G^iLC and consistent in G^iLC, the following question is hard but essential.

'Are $A_1, \cdots\cdots, A_N$ consistent in GLC?'

The special case where $i=0$ can be obtained from the fundamental conjecture of GLC.

A.2. If the fundamental conjecture of GLC holds, then we cannot improve the theorem on the set of the secondary form by substituting an axiom

$$\forall\varphi\forall y\{y \in M(z)\varphi[z] \mapsto \varphi[y] \wedge e(y)\} \text{ for } \forall\varphi Ex\forall y\{y \in x \mapsto \varphi[y] \wedge e(y)\}.$$

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