March 8 Notes for "On the Formally Proven Inconsistency of Zermello Frankel Set Theory and Related Systems". On 1 March 2008 while at child's fair in the Albany government plaza, I (Dan Willard) began to work out (with some careless errors) the proof of ZFC's omega inconsistency. I wrote a first darft of what I had in mind on march 2 and the morning or march 3, but did not finally decide what was proven until the evening of March 3. After notarizing the March 4 result, I further strengthened on the evening of March 4 to verify that ZF is also unable to prove its own inconsistency and both ZF and ZFC are inconsistent (in the context of different levels of the replacement schema present). The latter document (from which a proof can be quite easily extrapolated) was notarized on March 7), and on March 8 I notatarized the current more polished title — whose title is planned to be similar to the title of Godel's 1931 paper.

Given an axiomization of set theory (including ZF), we will use the following notation:

1. INT is the set of nonegative integers.
2. P_1 is the power set of INT.
3. P_2 is the power set of P_1
4. Given any logical language L, the symbol ENUM_L will denote a an injective function from INT into INT that enumerates all the Godel numbers of the sentences of L. More precisely, for an integer $i \in \text{INT}$, the symbol $\text{ENUM}_L(i)$ will denote the godel number of the i-th sentence in L.
5. Let α denote the godel number some r.e. axiom system, and e denote an element of P_1. The symbol $\text{Decipher}_{L,\alpha}(i,e)$ will denote a Boolean value that equals TRUE when either $\text{ENUM}_L(i)$ is an axiom of α or $i \in e$. More formally in a context where AxiomSet(α) represents the set of godel numbers for α's axioms, $\text{Decipher}_{L,\alpha}(i,e)$ is formally encoded by the formula:

$$\text{Decipher}_{L,\alpha}(i,e) \;=\; [\;\text{ENUM}_L(i) \in \text{AxiomSet}(\alpha) \quad \vee \quad i \in e\;] \tag{4}$$

6. The symbol $\lceil \Psi \rceil$ will have its usual meaning of denoting Ψ's godel number.
7. The symbol $\text{System}_{L,\alpha}(e)$ will denote the axiom system that is natually associated with Equation (4)'s Decipher function. It will thus contain the axiom sentence Ψ iff there exists an integer i such that:

$$\text{ENUM}_L(i) \;=\; \lceil \Psi \rceil \quad \wedge \quad \text{Decipher}_{L,\alpha}(i,e) \;=\; \text{TRUE} \tag{5}$$

8. The symbol $\text{ConsistentSyst}_{L,\alpha}(e)$ will denote a Boolean value that equals TRUE if and only if the formalism $\text{System}_{L,\alpha}(e)$ is consistent.
9. The symbol $\text{CompleteSyst}_{L,\alpha}(e)$ will denote a Boolean value that equals TRUE if and only if the formalism $\text{System}_{L,\alpha}(e)$ is complete (i.e. for each sentence Ψ, it must be true that either Ψ or $\neg\Psi$ is a formal axiomatic sentence belonging to $\text{System}_{L,\alpha}(e)$.
10. The symbol SupportSet(L,α) will will denote the set of all e satisfying both $\text{ConsistentSyst}_{L,\alpha}(e)$ and $\text{CompleteSyst}_{L,\alpha}(e)$.
11. ZF will be abbreviations for the godel number for the Zermello Franklin axiom system (without the Axiom of Choice). Its formal structure can be found in for example [?, ?]
12. "Choose" denotes a function whose domain is P_2 and which maps each $x \in P_2$ onto some $e \in P_1$ such that $e \in x$. The Axiom of Choice implies that the function "Choose" exists and thus our nomenclature is well defined.
13. ZFC will be abbreviations for the godel number for the Zermello Franklin axiom system with the Choice axiom added. Without loss in generality, we may assume that it contains a special function symbol added to our language for denoting the above "Choose" function. (We do not actually need the "Choose" function to have an especially named function symbol in our language, but it makes the notation in our discussion much more convenient.)
14. Combining the notation from the last four items, the symbol Support-ZFC will denote the special degenerate version of Item 10's SupportSet(L,α) where α now represents the ZFC axiom system and L is ZFC's language. (Thus, it intuitively reperesents the set of all $e \in P_1$ whose associated axiomatic sentences are consistent with ZFC.)
15. Likewize, Support-ZF will denote the special degenerate version of Item 10's SupportSet(L,α) where α represents the ZF axiom system and L represents its language.

Our immediate goal in the current setion of this paper is to present a diagonalization argument that will show that ZFC will be able to fomally prove the theorem that "Support-ZFC represent the empty element belonging to P_2 ". Since Zermello Franklin Set Theory can prove Godel's Completeness Theorem, this will imply that ZFC can prove a theorem declaring its own inconsistency. The analgous result where the ZF axiom system replaces ZFC is sligtly more complicatd (and it will therefore be postponed until Section ???)

The presence of the function "Choose" within the system ZFC is not crucial for the theorems proven formally in this section. However, it does simplify our notation, and we will therefore often employ it. (The intutive reason the "Choose" function symbol is not formally needed is because the Axiom of Choice implies the existence of a technically unnamed semantic object possessing all its features.)

To start our construction, we will use the what Mendelson [?] calls the Fixed Point Theorem. This theorem was first explicitly introduced into the logic literature by Carnap [?] — although both Carnap and Mendelson describe it as being implicit in Godel's historic 1931 paper [?]. Its formal statement is given below:

Theorem 1 (Godel-Carnap Fixed Point Theorem) *Suppose α is an axiom system that makes representable the recursive functions. Then for any wff $\psi(x)$ which is free in only the single variable x, it is possible to consruct a sentence ϕ such that α can prove the validity of the statement:*

$$\phi \;\Leftrightarrow\; \psi(\lceil \phi \rceil) \tag{6}$$

In order to review how Godel and Carnap would have us construct ϕ from $\psi(x)$, we shall use the following notation:

Subst(g, h) will denote Gödel's classic substitution formula — which yields TRUE when g is an encoding of a formula and h is an encoding of a sentence that replaces all occurrence of free variables in g with a constant representing g's Godel number.

Also, let $\Upsilon(y)$ denote the following formula:

$$\forall x \quad \text{Subst}(y,x) \;\Rightarrow\; \psi(x) \tag{7}$$

Then ϕ has been defined by the Godel-Carnap construction to be the sentence $\Upsilon(\lceil \phi \rceil)$. Mendelson's textbook [?] provides one example of a very nicely formulated proof showing that this particular defintion for the sentence ϕ has the property that the axiom system α can verify Equation (6)'s statement.

Our objective in the current paper is to employ the Fixed Point Theorem to prove the inconsistency of the axiom systems ZF and ZFC. We will do so by using the Fixed Point Theorem to construct two sentences, called Paradox-ZFC and Paradox-ZF, that enable us to formalize *non-constructive* proofs that these two respective systems are inconsistent. The final result of this paper (Theorem ???) will actually consist of a constructive proof of the inconsistency of these two systems. However, a very surprising aspect of this discourse in this paper is that it will *formally need* the non-constructive proofs, centering around Paradox-ZFC and Paradox-ZF, as vital intermediate steps to formalize the contstructives contradcition proofs, that will appear in the last stage of this paper.

Definition of Paradox-ZFC. Similar to the Liar's Paradox and its variation that had appeared in Godel's seminal 1931 paper [?], the sentence Paradox-ZFC will be a self-referencing mathematical precept that is built with the help of Theorem 1's Fixed Point Principle. Its analog in Godel's centenial paper consisted of a relatively simple application of the notion of self-reference that consisted of the following sentence:

- There is no proof of this sentence from the axiom system of Peano Arithmetic

The reason that the mathematics literature has awaited essentially 100 years for a proof that ZF and ZFC are inconsistent is that these proofs will require a much more complicated form of Fixed Point sentence than the example Godel used in •. Thus, the formal definition of construct Paradox-ZFC appears below, and we shall posptone defining a yet more elaborate version of our evolving paradigm, called Paradox-ZF, until the next section of this paper.

•• If Support-ZFC is a nonempty set then the application of the function "Choose" to the domain element "Support-ZFC" produces an element $e \in$ Support-ZFC such that the negation of this sentence is true under e's assignment of truth values to the sentences of the language L.

Sworn to me on the 8th day of march 2008.

[signature]

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