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Referee's report on

**On the Tender Line Separating Generalizations and Boundary-Case Exceptions for the Second Incompleteness Theorem under Semantic Tableaux Deduction**

by Dan E. Willard

This paper is a summary and explanation, for a broad audience, of part of the author's earlier work on what he calls boundary-case exceptions to Gödel's second incompleteness theorem. That is, it concerns deductive systems that, while intuitively including plenty of arithmetic, nevertheless prove (statements that intuitively express) their own consistency. It turns out that (at least) three aspects of the deductive system are important here: (1) Are the fundamental arithmetical operations (successor, addition, multiplication) given as functions and thus necessarily total, or are they given as relations that might denote only partial functions? (2) Is logic formalized as a Hilbert-style system, as semantic tableaux, etc.? (3) Is the law of the excluded middle given as a single axiom scheme, or must each instance be proved when it is needed? The main point of this paper is that seemingly small changes in these aspects can make the difference between incompleteness and an exception.

The paper contains none of the difficult proofs of the relevant theorems; instead it refers to the author's earlier publications. It contains careful explanations of the ideas involved, both in the results and in their proofs. Although I am not a proof-theorist, I found the explanations easy to follow and useful for my understanding of the issues. In addition, the paper contains interesting historical information, in particular about Gödel's and Hilbert's attitudes toward exceptions to the second incompleteness theorem, exceptions that might revive some version of Hilbert's program.

In my opinion, this paper will be interesting and useful for a wide range of readers, and I recommend that it be accepted for publication, subject to some minor corrections and suggestions.

I'll send, with this report, a copy of the paper in which I've marked, with notes in the right margin, various local corrections and suggestions. Let me list here some general comments.

The author uses some rather unusual terminology. Even in the paper's title, the use of "tender" strikes me as unidiomatic; I would have expected "fine" (since "fine line" is a standard expression) or perhaps "delicate" or "subtle". Similarly, in the "Keywords and Phrases" at the bottom of page 1, I'd expect the standard phrase "Hilbert's second problem", not "Hilbert's



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# On the Tender Line Separating Generalizations and Boundary-Case Exceptions for the Second Incompleteness Theorem under Semantic Tableaux Deduction

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**Abstract**

Our previous research has studied the semantic tableau deductive methodology, of Fitting and Smullyan, and observed that it permits boundary-case exceptions to the Second Incompleteness Theorem, when multiplication is viewed as a 3-way relation (rather than as a total function). It is known that tableau methodologies do prove a schema of theorems, verifying all instances of the Law of the Excluded Middle. But yet we show that if *one promotes* this schema of theorems into formalized logical axioms, then the meaning of the pronoun "I", in our self-referencing engine, does change. Our partial evasions of the Second Incompleteness Theorem, thus, then come to a complete halt.

Delete comma before "verifying"

"But yet" - delete one

**CITATION INFORMATION** : A shorter, less-polished conference-style "Extended Abstract" version of this article [57] was published in Volume 11972 of Springer's LNCS Series. (The LFCS conference met on January 4-7, 2020 in Deerfield, Florida.)

**KEYWORDS and PHRASES:** Semantic Tableau deduction, Hilbert's Second Open Question, "Hidden Implication" of Gödel's Second Incompleteness Theorem, Hilbert's Consistency Program and its "Future Extensions"'.

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# 1 Introduction

hidden significance and subtle

This article is intended to explore the *"hidden significance"* ~~and unexplored~~ implications of Gödel's Second Incompleteness Theorem and its ~~various~~ generalizations. ~~In particular,~~ the existence of a deep chasm separating the goals of Hilbert's consistency program from the implications of the Second Incompleteness Theorem was evident, immediately, after Gödel published [20]'s seminal announcement. We exhibited in [46, 47, 48, 50, 51, 52, 53, 54] a large number of articles about generalizations and boundary case exceptions to the Second Incompleteness Theorem, starting with our 1993 article [46]. These papers, which included six papers published in the JSL and APAL, showed every extension $\alpha$ of Peano Arithmetic can be mapped onto an axiom system $\alpha^*$ that can recognize its own consistency and prove analogs of all $\alpha$'s $\Pi_1$ theorems (in a slightly different language, called $L^*$). FIRST PARAGRAPH FIXED

Look at text

Editors should decide whethe mentions of JSL and APAL ar appropriate. They sound to m unnecessary and awkward cla of respectability.

The term "Self-Justifying" arithmetic was employed in our articles. [47, 50, 51, 52, 54]. These papers were able to verify their own consistency by containing a built-in self-referencing axiom that declared *"I am consistent"* (as will be explained later). In particular, our axiom systems $\alpha^*$ used the Fixed-Point Theorem to assure $\alpha^*$'s self-referencing analogs of the pronoun "I" would enable it to refer to itself in the context of its *"I am consistent"* axiomatic declaration.

The period after "articles" sl be a comma (or just deletec

"These papers" or "The syste studied in these papers"?

It turns out that such a self-referencing mechanism will produce unacceptable Gödel-style diagonalizing contradictions, when either $\alpha^*$ or its particular deployed definition of consistency are too strong. This is because our methodologies *only* become contradiction-free *when* $\alpha^*$ ~~uses~~ rests on sufficiently weak underlying structures.

These weak structures obviously have significant disadvantages. Their virtue is that their formalisms $\alpha^*$ can be arranged to prove more $\Pi_1$ like theorems than Peano Arithmetic, while offering *some type* of *partial* knowledge about their own consistency. We will call such formalisms **"Declarative Exceptions"** to the Second Incompleteness Theorem.

An alternative type of exception to the Second Incompleteness Theorem, which we shall call an **"Infinite-Ranged Exception"**, was recently developed by Sergei Artemov [4] (It is related to the works of Beklemishev [6] and Artemov-Beklemishev [5].) Artemov observed Peano Arithmetic can verify its own consistency, from a special infinite-ranging perspective. This means PA will generate an infinite set of

Add a period after "[4]".

theorems $T_1$, $T_2$, $T_3$ ... where each $T_i$ shows some subset $S_i$ of PA is unable to prove $0 = 1$ and where PA equals the formal union of these special selected $S_i$ satisfying the inclusion property of $S_1 \subset S_2 \subset S_3 \subset \ldots$ .

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This perspective, which is certainly very useful, is also not a panacea. Thus, the abstract in [4] cautiously used the adjective of "somewhat" to describe how it sought to partially achieve the goals sought by Hilbert's Consistency Program (with an infinite collection of theorems $T_1$, $T_2$, $T_3$ ... replacing Hilbert's intended goal of finding one unifying formal consistency theorem).

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Our "Declarative"' exceptions to the Second Incompleteness Theorem and Artemov's "Infinite Ranging" exceptions are two quite different rigorous results, which are nicely compatible with each other. This is because each acknowledged that the Second Incompleteness Theorem is a strong result, that *will admit no full-scale exceptions*. Also, these results are of interest because Gödel openly conjectured that Hilbert's Consistency Program would ultimately, reach *some levels of partial success* (see next section). We will explain, herein, how Gödel's conjecture can be *partially justified*, due to an unusual consequence of the Law of the Excluded Middle.

More specifically, we shall focus on the semantic tableau deductive mechanisms of Fitting and Smullyan [15, 40] and their special properties from the perspective of our JSL-2005 article [50]. Each instance of the Law of the Excluded Middle has been treated by most tableau mechanisms as a provable theorem, rather than as a built-in logical axiom. This may, at first, appear to be an insignificant distraction because most deductive methodologies do not have their consistency reversed when a theorem is promoted into becoming a logical axiom.

Our self-justifying axiom systems are *different*, however, because their built-in self-referencing *"I am consistent"* axioms have their meanings change, fundamentally, when their self-referencing concept of "I" involves promoting a schema of theorems verifying the Law of Excluded Middle *into formal explicitly declared logical axioms*.

This effect is counterintuitive because similar distinctions exist almost nowhere else in Logic. Thus some confusion, that has surrounded our prior work, can be clarified when one realizes that *an interaction* between the self-referencing concept of "I" with the Law of Excluded Middle causes the Second Incompleteness Theorem to become activated *precisely when* the Law of Excluded Middle *is promoted* into becoming a schema of logical axioms.

The intuitive reason for this unusual effect is that the transforming of derived theorems *into* logical axioms can shorten proofs under the Fitting-Smullyan semantic tableau technology. In the particular context where §3's formalism uses self-referencing *"I am consistent"* axioms and views multiplication as a 3-way relationship, these conditions will be sufficient for enacting the full power of the Second Incompleteness Theorem.

The next chapter will explain how these issues are related to questions raised by Gödel and Hilbert about feasible boundary-case exceptions to the Second Incompleteness Effect.

# 2 Revisiting Some Intuitions of Gödel and Hilbert

Interestingly, neither Gödel (unequivocally) nor Hilbert (after learning about Gödel's work) would dismiss the possibility of a compromise solution, whereby a *fragment* of the goals of Hilbert's Consistency Program would remain intact. Thus, Hilbert never withdrew [26]'s statement $*$ for justifying his program:

> $*$ *" Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?"*

Gödel was, also, cautious (especially during the early 1930's) not to speculate whether all facets of Hilbert's Consistency program would come to a termination. He thus inserted the following cautious caveat into his famous 1931 paper [20]:

> $**$ *"It must be expressly noted that Theorem XI"* (e.g. the Second Incompleteness Theorem) *"represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ... "*

Change "e.g."(which means "for exam to "i.e." (which means "that is".

Several biographies of Gödel [11, 22, 58] have noted that Gödel's intention (prior to 1930) was to establish Hilbert's proposed objectives, before he formalized his famous result that led in an opposite direction. Moreover, Yourgrau's biography [58] of Gödel records how von Neumann found it necessary during the early 1930's to *"argue against Gödel himself"* about the definitive termination of Hilbert's consistency program, which *"for several years"* after [20]'s publication, Gödel *"was cautious not to prejudge"*.

It is known that Gödel hinted the Second Incompleteness Theorem was more significant in a 1933 Vienna lecture [21]. Yet, Gödel (who published only about 85 pages during his career) was frequently ambivalent about this point. Thus, a YouTube talk by Gerald Sacks [39] recalled Gödel telling Sacks some type of revival of Hilbert's Consistency Program was likely (see footnote [1] for more details). Moreover~~, Anil~~ had Nerode has told us [32] he recalled Stanley Tennenbaum ~~having~~ similar conversations with Gödel, where Gödel ~~again stated~~ his suspicion that Hilbert's Consistency Program would be partially revived. Many scholars have been caught by surprise by Gödel's private hesitation ~~about the broader implications of the Se~~cond Incompleteness Effect. ~~This is because Gödel only published roughly 85 pages during his career, and he never publicly expanded upon [20]'s statement~~ ** .

The "only about 85 pages" fac already stated 8 lines earlier. [ one of the two occurrences.

that he privately

The research that followed Gödel's seminal 1931 discovery has technically focused on studying mostly generalizations of the Second Incompleteness Theorem (instead of also examining its boundary-case exceptions). Many of these generalizations of the Second Incompleteness Theorem [2, 3, 7, 8, 9, 10, 13, 16, 23, 24, 25, 29, 33, 34, 35, 36, 37, 41, 42, 43, 44, 45, 47, 48, 49, 51] are quite subtle.

The author of this paper is especially impressed by a generalization of the Second Incompleteness Effect, arrived at by the combined work of Pudlák and Solovay together with added research by Nelson and Wilkie-Paris [31, 36, 42, 45]. These results, which have been further amplified in [10, 16, 23, 43, 47], show the Second Incompleteness Theorem does not require the presence of the Principle of Induction to apply to most formalisms that use a Hilbert-Frege style of deduction.

The next chapter's Remark 3.5 will helpfully summarize such generalizations of the Second Incompleteness Effect.

[1] Some quotes from Sacks's YouTube talk [39] are that Gödel *"did not think"* the objectives of Hilbert's Consistency Program *"were erased"* by the Incompleteness Theorem, and Gödel believed (according to Sacks) it left Hilbert's program *"very much alive and even more interesting than it initially was"*.

when he saw the Second Incompleteness Theorem as a very powerful result that also owned limitations

## 3 Main Notation and Background Literature

Let us call an ordered pair $(\alpha, D)$ a **Generalized Arithmetic Configuration** (abbreviated as a **"GenAC"** ) when its first and second components are defined as follows:

1. The **Axiom Basis** " $\alpha$ " for a GenAC is defined as its set of proper axioms.

2. The second component " $D$ " of a GenAC, called its **Deductive Apparatus**, is defined as the union of its logical axioms " $L_D$" with its rules for obtaining inferences.

**Example 3.1** This notation allows us to separate the logical axioms $L_D$ , associated with $(\alpha, D)$ , from its "basis axioms", denoted as " $\alpha$ ". It also allows us to compare different deductive apparatuses from the literature. Thus, the $D_E$ apparatus, from Enderton's textbook [12], uses only modus ponens as a rule of inference, but it deploys a complicated 4-part schema of logical axioms. This differs from the $D_M$ and $D_H$ apparatuses in the Mendelson [30] and Hájek-Pudlák [25] textbooks. (They used a more reduced set of logical axioms but employed "generalization" as a second rule of inference.) In contrast, the $D_F$ apparatus, from Fitting's and Smullyan's textbooks [15, 40], uses *no logical axioms,* ~~but~~ employs a broader "tableau style" rule of inference. ~~AN IMPORTANT POINT is that~~ while proofs have different lengths under different apparatuses, all the common apparatuses produce the same set of final theorems from an initial common "axiom basis" ~~of~~ $\alpha$ (as footnote [2] explains).

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Delete "of" before \alpha, beca \alpha is the axiom basis.

does

**Definition 3.2** Let $\alpha$ again denote an axiom basis, $D$ designate a deduction apparatus, and $(\alpha, D)$ denote their GenAC. Henceforth, the configuration $(\alpha, D)$ will be called **Self-Justifying** when

i. one of $(\alpha, D)$'s theorems (or possibly one of $\alpha$'s axioms) states that the deduction method $D$, applied to the basis system $\alpha$, produces a consistent set of theorems, and

ii. the GenAC formalism $(\alpha, D)$ is actually, in fact, consistent.

[2] This is because all the common apparatuses satisfy the requirements of Gödel's Completeness Theorem.

~~In essence~~!

Thus

**Example 3.3** Using Definition 3.2's notation, our prior research [46, 47, 50, 51, 54] constructed GenAC pairs $(\alpha, D)$ that were "Self Justifying". We also proved that the Incompleteness Theorem implies specific limits beyond which self-justifying formalisms simply cannot transgress. For any $(\alpha, D)$, all our articles observed it was easy to construct a system $\alpha^D \supseteq \alpha$ that satisfies the Part-i condition (in an isolated context *where the Part-ii condition is not also satisfied*). In essence, $\alpha^D$ could consist of all of $\alpha$'s axioms plus the added "**SelfRef**$(\alpha, D)$" sentence, defined below:

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> $\oplus$ There is no proof (using $D$'s deduction method) of $0 = 1$ from the *union* of the axiom system $\alpha$ with *this* sentence "SelfRef$(\alpha, D)$" (looking at itself).

Kleene [28] was the first to notice how to encode analogs of SelfRef$(\alpha, D)$'s above statement, which we often call an "I AM CONSISTENT" axiom. Each of Kleene, Rogers and Jeroslow [28, 38, 27] emphasized $\alpha^D$ may be inconsistent (e.g. violate Part-ii of self-justification's definition *despite* the assertion in SelfRef$(\alpha, D)$'s particular statement). This is because if the pair $(\alpha, D)$ is too strong then a quite conventional Gödel-style diagonalization argument can be applied to the axiom basis of $\alpha^D = \alpha +$ SelfRef$(\alpha, D)$, where the added presence of the statement SelfRef$(\alpha, D)$ will cause this extended version of $\alpha$, ironically, to become automatically inconsistent. Thus, an encoding for "SelfRef$(\alpha, D)$" is relatively easy, via an application of the Fixed Point Theorem, but this sentence is *potentially devastating.*

**Definition 3.4** Let $Add(x, y, z)$ and $Mult(x, y, z)$ denote two 3-way predicates specifying $x + y = z$ and $x * y = z$. (Obviously, arithmetic's classic associative, commutative, identity and distributive axioms will have $\Pi_1$ encodings when they are expressed using these two predicates.) We will say that a formalized axiom basis system of $\alpha$ recognizes successor, addition and multiplication as **Total Functions** iff it can prove all of (1) - (3) as theorems:

$$\forall x \; \exists z \quad Add(x, 1, z) \tag{1}$$
$$\forall x \; \forall y \; \exists z \quad Add(x, y, z) \tag{2}$$
$$\forall x \; \forall y \; \exists z \quad Mult(x, y, z) \tag{3}$$

We will call the GenAC system $(\alpha, D)$ a **Type-M** formalism iff it proves (1) - (3) as theorems, **Type-A** if it proves only (1) and (2), and it will be called **Type-S** if it proves only (1) as a theorem. Also, $(\alpha, D)$ will be called **Type-NS** iff it can prove none of (1) - (3).

**Remark 3.5** The separation of GenAC systems into the categories of Type-NS, Type-S, Type-A and Type-M systems helps summarize the prior literature about generalizations and boundary-case exceptions for the Second Incompleteness Theorem. This is because:

**i.** The combined research of Pudlák, Solovay, Nelson and Wilkie-Paris [31, 36, 42, 45], as formalized by Theorem ++, implies that no natural Type−S system $(\alpha, D)$ can recognize its own consistency when $D$ represents one of Example 3.1's three Hilbert-Frege deductive methods of $D_E$, $D_H$ and $D_M$. It thus establishes the following result:

> ++ *(Solovay's modification [42] of Pudlák [36]'s formalism using some of Nelson and Wilkie-Paris [31, 45]'s methods)* : Let $(\alpha, D)$ denote a Type-S GenAC system which assures the successor operation will provably satisfy both $x' \neq 0$ and $x' = y' \Leftrightarrow x = y$. Then $(\alpha, D)$ cannot verify its own consistency whenever simultaneously $D$ is some type of a Hilbert-Frege deductive apparatus and $\alpha$ treats addition and multiplication as 3-way relations, satisfying their usual associative, commutative, distributive and identity axioms.

Essentially, Solovay [42] privately communicated to us in 1994 an analog of theorem ++. Many authors have noted Solovay has been reluctant to publish his nice privately communicated results on many occasions [10, 25, 31, 34, 36, 45]. Thus, approximate analogs of ++ were explored subsequently by Buss-Ignjatović, Hájek and Švejdar in [10, 23, 43], as well as in Appendix A of our paper [47] and in [49]. Also, Pudlák's initial 1985 article [36] captured the majority of ++'s essence, chronologically before Solovay's observations. Also, Friedman did some ~~closely~~ related work in [16].

**ii.** Part of what makes ++ interesting is that [47, 50, 51] presented two types of self-justifying GenAC systems, whose natural hybrid is precluded by ++. Specifically, these results involve using Example 3.3's self-referencing *"I am* *consistent"* axiom (from statement $\oplus$ ). Thus, they establish that some (not all) Type-NS systems [47, 51] can verify their own consistency under a Hilbert-Frege style deductive apparatus [3], and some (not all) Type-A systems [46, 47, 50, 52] can, likewise, corroborate their consistency under a more restrictive semantic tableau apparatus. Also, we observed in [48, 53] how one could refine ++ with Adamowicz-Zbierski's methods [2] to show most Type-M systems cannot recognize their semantic tableau consistency.

**Remark 3.6** . Several of our papers, starting with our 1993 article [46], have used Example 3.3's *"I am consistent"* axiomatic declaration $\oplus$ for evading the Second Incompleteness Effect. Other possible types of evasions rest on the cut-free methods of Gentzen and Kreisel-Takeuti [19, 29], an interpretational approach (such as what Adamowicz, Bigorajska, Friedman, Nelson, Pudlák and Visser had applied in [1, 17, 31, 36, 44]), or Artemov's Infinite-Range perspective [4] (where an infinite schema of theorems replaces one single unified consistency theorem). We encourage the reader to examine all these articles, each of which has their own separate merits. "their own" —> "its own"
Our focus, in this paper, will be primarily on the next section's Theorems 4.4 and 4.5. They show that some partial *(and not full)* evasions of the Second Incompleteness Effect can arise under a semantic tableau deductive apparatus.

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## 4 Main Theorems and Related Notation

A function $F$ is called Non-Growth when $F(a_1, \ldots, a_j) \leq Maximum(a_1, \ldots, a_j)$ holds. Six examples of non-growth functions are:

1. *Integer Subtraction* (where $x - y$ is defined to equal zero when $x \leq y$ ),
2. *Integer Division* (where $x \div y$ equals $x$ when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise),
3. $Maximum(x, y)$,
4. $Log_{\spadesuit}(x)$ which is an abbreviation for $\lceil \mathrm{Log}_2( x + 1 ) \rceil$ under the conventional notation. (The footnote [4] explains the significance of this concept.)

[3] The Example 3.1 had provided three examples of Hilbert-Frege style deduction operators, called $D_E$, $D_H$ and $D_M$ . It explained how these deductive operators differ from a tableau-style deductive apparatus by containing a modus ponens rule.

[4] The Hájek-Pudlák textbook [25] uses the notation " $| x |$ " to designate what we shall call " $Log_{\spadesuit}(x)$ " Thus for $x \geq 1$, $Log_{\spadesuit}(x)$ denotes the number of symbols that will encode the number $x$, when it is written in a binary format.

5. $Root(x, y) = \lceil x^{1/y} \rceil$ , and also

6. $Count(x, j)$ which designates the number of physical "1" bits that are stored among $x$'s rightmost $j$ bits.

Our papers used the term **Grounding Function** to refer to these six non-growth operations. Also, the term **U-Grounding Function** referred to a function that corresponds to either one of these six grounding primitives or the *growth-oriented* functional operations of Addition and $Double(x) = x + x$.

Our language $L^*$, defined in [50], was built out of the eight U-Grounding function operations plus the primitives of "0", "1", "=" and "$\leq$". This language differs from a conventional arithmetic by *excluding* a formal multiplication function symbol. (Instead, it treats multiplication as a 3-way relation, via the obvious employment of its Division primitive.) This notation leads to a surprisingly strong and tempting evasion of the Second Incompleteness Effect.

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**Definition 4.1** In a context where $t$ is any term in our language $L^*$, the special quantifiers used in the wffs $\forall\, v \leq t \;\; \Psi(v)$ and $\exists\, v \leq t \;\; \Psi(v)$ will be called **bounded quantifiers.** Also, any formula in our language $L^*$, all of whose quantifiers are so bounded, will be called a $\Delta_0^*$ formula. The $\Pi_n^*$ and $\Sigma_n^*$ formulae are, thus, defined by the usual rules, EXCEPT they DO NOT contain multiplication function symbols. These rules are that:

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1. Every $\Delta_0^*$ formula will also be a " $\Pi_0^*$ " and " $\Sigma_0^*$ " formula.

2. A wff will be called $\Pi_n^*$ when it is encoded as $\forall v_1 \ldots \forall v_k \; \Phi$ with $\Phi$ being $\Sigma_{n-1}^*$.

3. A wff will be called $\Sigma_n^*$ when it is encoded as $\exists v_1 .. \exists v_k \; \Phi$, with $\Phi$ being $\Pi_{n-1}^*$.

precede

**Remark 4.2** . A sentence $\Psi$ will be called **Rank-1*** when it can be encoded as either a $\Pi_1^*$ or $\Sigma_1^*$ sentence. Our definitions for $\Pi_1^*$ or $\Sigma_1^*$ formulae ~~will~~ do differ from Arithmetic's conventional counterparts by excluding multiplication function symbols. (This issue will turn out to be central to our evasions of the Second Incompleteness Effect.)

format

What exactly is allowed in "enco a wff? Prenex operations? Migh alter proof complexity in weak sy

will differ —> differ [because t definitions have already been g

There will be three variants of formal deductive apparatus methods, which we will compare. The first is *semantic tableau.* It will receive an abbreviated name of "Tab" and correspond to Fitting's textbook formalism [15]. (Its definition can also be found in the attached Appendix.) Thus, a Tab-proof for a theorem $\Psi$, from an axiom basis $\alpha$, is a tree-structure that begins with the sentence $\neg\ \Psi$ stored inside the tree's root and whose every root–to–leaf path establishes a contradiction by containing some pair of contradictory nodes that will "close" its path. The rules for generating internal nodes, along each root–to–leaf path, are that each node must be *either* a proper axiom of $\alpha$ *or* a deduction from an ancestor node via one of the Appendix's stated "elimination" rules for the $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, and $\exists$ symbols.

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should be hyphens, but the au
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only the internal nodes of thes
but ~~also~~ the leaves.

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Our second explored deductive apparatus is called *Extended Tableau*, and shall be abbreviated as "**Xtab**". Its definition is identical to Tab-deduction, except that for any sentence $\phi$ in our language $L^*$, the sentence $\phi \vee \neg\phi$ is allowed as an internal node in an Xtab proof tree. (In other words, *Xtab*-deduction differs from *Tab*-deduction by allowing all instances of the Law of Excluded Middle to appear as permitted logical axioms. In contrast, *Tab*-deduction will view these instances only as derived theorems.)

Our third deductive apparatus was called Tab-1 in [50]. It is, essentially, a compromise between Tab and Xtab, where a "Tab-1" proof for $\Psi$ from an axiom basis $\alpha$ corresponds to a set of ordered pairs $(p_1, \phi_1)$, $(p_2, \phi_2)$, ...., $(p_k, \phi_k)$ where

Statements 1 and 2 imply that \Psi mu
be a Rank-1* sentence. No more
complex sentences can be proved in
this system. Is that intentional?

1. $\phi_k \ = \ \Psi$

2. Each $p_j$ is a Tab-proof of what we have called ~~a~~ To Rank-1* sentence $\phi_j$ from the union of $\alpha$ with the preceding Rank-1* sentences of $\phi_1,\ \phi_2,\ ....,\ \phi_{j-1}$ .

The Rank-1* constraint ( defined by Remark 4.2 and utilized by the above Item 2) is significant. This is because Tab-1 deduction is less efficient than Xtab when *the former requires* $\phi_j$ be a Rank-1* sentence. (In contrast, Xtab *does not impose* a similar Rank-1* requirement upon $\phi$ when its Law of the Excluded Middle allows $\phi \vee \neg\phi$ to appear *anywhere* as a permissible logical axiom, *for fully arbitrary* $\phi$. ) Thus, Xtab is more desirable than Tab-1 *when it can actually be feasibly (?) employed.*

Let us say an axiom system $\alpha$ owns a **Level-1** appreciation of its own self-consistency (under a deductive apparatus $D$) iff it can verify that $D$ produces no two simultaneous proofs for a $\Pi_1^*$ sentence and its negation. Within this context, where $\beta$ denotes any basis axiom system using $L^*$'s U-Grounding language, $\mathrm{IS}_D(\beta)$

was defined in [50] to be an axiomatic formalism capable of recognizing all of $\beta$'s $\Pi_1^*$ theorems and corroborating its own Level-1 consistency under $D$'s deductive apparatus. It consists of the following four groups of axioms:

**Group-Zero:** Two of the Group-zero axioms will define the constant-symbols, $\bar{c}_0$ and $\bar{c}_1$, designating the integers of 0 and 1. The Group-zero axioms will also define the growth functions of Addition and $Double(x) = x + x$. (They will enable our formalism to define any integer $n \geq 2$ using fewer than $3 \cdot \lceil \text{Log } n \rceil$ logic symbols.)

**Group-1:** This axiom group will consist of a finite set of $\Pi_1^*$ sentences, denoted as $F$, which can prove any $\Delta_0^*$ sentence that holds true under the standard model of the natural numbers. (Any finite set of $\Pi_1^*$ sentences $F$, with this property, may be used to define Group-1, as [50] had noted.)

**Group-2:** Let $\ulcorner\Phi\urcorner$ denote $\Phi$'s Gödel Number, and $\text{HilbPrf}_\beta(\ulcorner\Phi\urcorner, p)$ denote a $\Delta_0^*$ formula indicating that $p$ is a Hilbert-Frege styled proof of theorem $\Phi$ from axiom system $\beta$. For each $\Pi_1^*$ sentence $\Phi$, the Group-2 schema will contain the below axiom (4). (Thus $\text{IS}_D(\beta)$ can trivially prove all $\beta$'s $\Pi_1^*$ theorems.)

$$\forall\, p \quad \{ \ \text{HilbPrf}_\beta(\ulcorner\Phi\urcorner, p) \ \Rightarrow \ \Phi \ \} \tag{4}$$

**Group-3:** The final part of $\text{IS}_D(\beta)$ will be a self-referencing $\Pi_1^*$ axiom, that indicates $\text{IS}_D(\beta)$ is "Level-1 consistent" under $D$'s deductive apparatus. It thus amounts to the following declaration:

> \# *No two proofs exist for a $\Pi_1^*$ sentence and its negation, when $D$'s deductive apparatus is applied to an axiom system, consisting of the union of Groups 0, 1 and 2 with* this sentence *(looking at itself).*

One encoding for # as a self-referencing $\Pi_1^*$ axiom, had appeared in [50]. Thus, Line (5) is a $\Pi_1^*$ representation for # when:

a. $\text{Prf}_{\,\text{IS}_D(\beta)}(a, b)$ is a $\Delta_0^*$ formula indicating that $b$ is a proof of a theorem $a$ from the axiom basis $\text{IS}_D(\beta)$ under $D$'s deductive apparatus, and

b. $\text{Pair}(x, y)$ is a $\Delta_0^*$ formula indicating that $x$ is a $\Pi_1^*$ sentence and $y$ represents $x$'s negation.

and Sections 5 and 6 will explain the core intuitions ~~to~~ that enable us to ~~justify Theorem 4.5~~ conclude that Remark 3.15's can be expanded to justify Theorem 4.5's

$$\forall x\, \forall y\, \forall p\, \forall q \quad \neg\, [\ \mathrm{Pair}(x,y)\ \wedge\ \mathrm{Prf}_{\ \mathrm{IS}_{D(\beta)}}(x,p)\ \wedge\ \mathrm{Prf}_{\ \mathrm{IS}_{D(\beta)}}(y,q)\ ]$$

For the sake of brevity, we will not provide exact details about how Line (5) c
encoded under the Fixed Point Theorem. Adequate details are provided in [47

**Definition 4.3** *Let "D" denote any one of the Tab, Xtab or Tab-1 ded apparatus. Then we will say that the resulting mapping of $IS_D(\ \bullet\ )$ is* Consistency Preserving *iff $IS_D(\beta)$ is automatically consistent whenever all the axioms of $\beta$ hold true under the standard model of the natural numbers.*

The preceding definition raises questions about whether the mappings of $\mathrm{IS}_{Tab}(\ \bullet\ )$, $\mathrm{IS}_{Tab-1}(\ \bullet\ )$, and $\mathrm{IS}_{Xtab}(\ \bullet\ )$ are consistency preserving. It turns out that Theorem 4.4 will show the first two of these mappings are consistency preserving, while Theorem 4.5 explores how the Law of the Excluded Middle conflicts with $\mathrm{IS}_{Xtab}(\ \bullet\ )$'s Group-3 axiom.

**Theorem 4.4** *The $IS_{Tab-1}(\ \bullet\ )$ and $IS_{Tab}(\ \bullet\ )$ mappings are consistency preserving. (I.e. the axiom systems $IS_{Tab-1}(\beta)$ and $IS_{Tab}(\beta)$ are automatically consistent whenever all $\beta$'s axioms hold true under the standard model of the Natural Numbers.)*

**Theorem 4.5** *In contrast, $IS_{Xtab}(\ \bullet\ )$ fails to be a consistency-preserving mapping. (More specifically, $IS_{Xtab}(\ \beta\ )$ is automatically inconsistent whenever $\beta$ proves some conventional $\Pi_1^*$ theorems stating that addition and multiplication satisfy ~~their~~ usual associative, commutative, distributive and identity properties.)*

This is

The proofs of Theorems 4.4 and 4.5 would be quite lengthy, if they were derived from first principles. Fortunately, it is unnecessary for us to do so here, because we already gave a detailed justification of Theorem 4.4's result for $\mathrm{IS}_{Tab-1}(\ \bullet\ )$ in [50], and one ~~can incrementally modify the Remark 3.5's~~ special Invariant of ++ to justify Theorem 4.5. Thus, it will be possible for the next two sections of this paper to adequately summarize the intuition behind Theorems 4.4 and 4.5, without delving into the full formal details.

Some people use "incrementall modify" to mean "modify only si Others use it to mean "modify i sequence of steps". Clarify whi mean.

(Our formal discussion in Sections 5 and 6 will certainly be abbreviated, but they will be sufficient to explain the core intuitions)

Part of the reason Theorems 4.4 and 4.5 are of interest is because of their surprising contrast. Thus, some historians have wondered whether Hilbert and Gödel were entirely incorrect when their statements * and ** suggested some form of the

Consistency Program would likely be viable. Moreover Gerald Sacks's YouTube talk [39], as well as some added comments by Anil Nerode [32], have reinforced this point. This is because Gödel repeated analogs of **'s statement on several occasions, during the later part of his career. Thus, the contrast between Theorems 4.4 and 4.5 provides possible evidence that a *fractional portion* of what Hilbert and Gödel had advocated, might become feasible.

This paper will not have the page space to go into the full details, but the next several sections will summarize the gist behind the proofs for Theorems 4.4 and 4.5.

# 5 Intuition Behind Theorem 4.4

Let us recall the acronym "Tab" stands for semantic tableau deduction. This was defined by Fitting [14, 15] to be a tree-like proof of a theorem $\Psi$ from an axiom basis $\alpha$, whose root consists of the temporary negated assumption of $\neg\,\Psi$ and whose every root–to–leaf path establishes a contradiction by containing some pair of contradictory nodes that "close" its path. Each internal node along these paths must *either* be a proper axiom of $\alpha$ *or be* a deduction from an ancestor node via one of the "elimination" rules associated with the logic symbols of $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, or $\exists$ (that are illustrated in the Appendix.)

New yellow

**Example 5.1** Let $\text{IS}^{M}_{Tab}(\ \bullet\ )$ denote a mapping transformation identical to Theorem 4.4's formalism of $\text{IS}_{Tab}(\ \bullet\ )$, *except that* $\text{IS}^{M}_{Tab}$ shall contain a further multiplication function operation and, accordingly, have its Group-3 "I am consistent" axiom statements *updated* to recognize multiplication as a total function. It turns out this change will cause $\text{IS}^{M}_{Tab}(\ \bullet\ )$ to stop satisfying the consistency-preservation property, which Theorem 4.4 attributed to $\text{IS}_{Tab}(\ \bullet\ )$.

reword

The intuition behind this change can be roughly summarized if we let $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ denote the sequences defined by:

$$x_0 \;=\; 2 \;=\; y_0 \tag{6}$$

$$x_i \;=\; x_{i-1} + x_{i-1} \tag{7}$$

$$y_i \;=\; y_{i-1} * y_{i-1} \tag{8}$$

For $i > 0$, let $\phi_i$ and $\psi_i$ denote the sentences in (7) and (8) respectively. Also, let $\phi_0$ and $\psi_0$ denote (6)'s sentence. Then $\phi_0, \phi_1, ... \phi_n$ imply $x_n = 2^{n+1}$, and $\psi_0, \psi_1, ... \psi_n$ imply $y_n = 2^{2^n}$. Thus, the latter sequence shall grow at an exponentially faster rate than the former. It turns out that this change in growth speed causes the $\text{IS}^M_{Tab}( \bullet )$, and $\text{IS}_{Tab}( \bullet )$ to have quite opposite self-justification properties.

In particular, let the quantities $\text{Log}(y_n) = 2^n$ and $\text{Log}(x_n) = n+1$ represent the lengths for the binary codings for $y_n$ and $x_n$. Thus, $y_n$'s coding will have a length $2^n$, which is *much larger* than the $n+1$ steps of $\psi_0, \psi_1, ... \psi_n$ (used to define $y_n$'s existence). In contrast, $x_n$'s binary encoding will have a sharply smaller length of size $n+1$. These observations are significant because every proof establishing a variant of the Second Incompleteness Effect involves a Gödel number $z$ encoding a capacity to self-reference its own definition.

What happened to the \sp subscript on Log earlier?

The faster growing series $y_0, y_1, , ... y_n$ should, intuitively, have this self-referencing capacity because $y_n$'s binary encoding has a $2^{n+1}$ length that greatly exceeds the size of the $O(n)$ steps used to define its value. Leaving aside many of [48, 53]'s further details, this fast growth explains roughly why a Type-M logic, such as $\text{IS}^M_{Tab}$, satisfies the semantic tableau version of the Second Incompleteness Theorem, unlike $\text{IS}_{Tab}$.

Our paradigm also explains why $\text{IS}_{Tab}$'s Type-A formalism produces boundary-case exceptions for the semantic tableau version of the Second Incompleteness Theorem. This is because [50] showed that it was unable to construct numbers $z$ that can self-reference their own definitions (when only the *more slowly growing* addition primitive is available). In particular assuming only two bits are needed to encode each sentence in the sequence $\phi_0, \phi_1, ... \phi_n$, the length $n+1$ for $x_n$'s binary encoding is insufficient for encoding this sequence.

Leaving aside many of [50]'s details, this short length for $x_n$ explains the central intuition behind [50]'s evasion of the Second Incompleteness Theorem under $\text{IS}_{Tab}$. It arises essentially because of the *sharp* difference between the growth rates of the two sequences of $x_1, x_2, x_3 ...$ and $y_1, y_2, y_3 ...$.

There is obviously insufficient space for this extended abstract to provide more details, here. A fully detailed proof of Theorem 4.4 is available in [50]. It establishes

It's no longer an extended abs so those space limitations don apply. Are there new limitation apply here?

(see [5] ) that Peano Arithmetic can prove $\beta$'s consistency implies *both* the consiste
and also the self-justifying property of $\text{IS}_{Tab-1}(\beta)$.

Our more modest goal, within the present abbreviated paper, has been to *me*
summarize the intuition behind Theorem 4.4's surprising evasion of the Second
completeness Effect. It arises, intuitively, because of the striking difference in
growth rates between the two series of $x_1, x_2, x_3 ...$ and $y_1, y_2, y_3 ...$ .

# 6 Summary of Theorem 4.5's Proof

A formal proof of Theorem 4.5 is complex, but it can be nicely summarized. This is because this proposition's proof is similar to the formal justification for Remark 3.5's Invariant of ++ . (The latter's insight has come from the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris [36, 42, 31, 45]. It was, also, subsequently verified by several other authors [10, 16, 23, 43, 47] in slightly different forms.)

The crucial aspect of the Hilbert-Frege deductive methodology is that its modus ponens rule assures thar a proof of a theorem $\psi$ from an axiom system $\alpha$ has a length no more than proportional to the sum of the proof-lengths used to derive $\phi$ and $\phi \rightarrow \psi$ . This **"Linear-Sum Effect"** *does not* apply also to *Tab*-deduction (because the latter lacks a modus ponens rule).

thar –> that

done

The *Xtab* deductive methodology is , however, quite different from the *Tab* form of deduction, in that *only Xtab* supports an analog of the prior paragraph's "Linear-Sum Effect". This is because any node of an Xtab proof-tree is allowed to store any sentence of the form $\phi \vee \neg \phi$ (as a consequence of its allowed use of the Law of Excluded Middle). This added feature will allow an *Xtab* proof for $\psi$ to have a length proportional to the sum of the proof lengths for $\phi$ and $\phi \rightarrow \psi$ . In particular, such an Xtab proof for $\psi$ will consist of the following four steps:

1. The root of an Xtab proof for $\psi$ consists of the usual temporary negated hy-

[5] The *exact* meaning of this implication is subtle. This is because Peano Arithmetic (PA) **CANNOT KNOW** whether $\beta$ is consistent when $\beta = PA$. Thus, *unlike* the quite different formalism of $\text{IS}_{Tab-1}(PA)$ , the system of PA shall linger in a state of self-doubt, about whether both PA and $\text{IS}_{Tab-1}(PA)$ are consistent. *The* **main point** *is, however, that we humans* **believe** *PA is consistent, and we can use this fact to* **confirm** *that $IS_{Tab-1}(PA)$ is* **BOTH** *consistent and able to verify its self-consistency via its* **"I am consistent"** *axiom.*

pothesis of $\neg\psi$ (which the remainder of the proof tree will show is impossible to hold).

2. The child of this root node consists of an allowed invocation of the Law of the Excluded Middle of the *particular* form $\phi \vee \neg\phi$ .

3. The relevant Xtab proof tree will next employ the Appendix's branching rule for allowing the two sibling nodes of $\phi$ and $\neg\phi$ to descend from Item 2's node.

4. Finally, our Xtab proof will insert below (3)'s left sibling node of $\phi$ a subtree that is no longer than a proof for $\phi \rightarrow \psi$ , and likewise insert a proof for $\phi$ below (3)'s right sibling of $\neg \phi$ .

The point is that the very last step of the above 4-part proof has a length no greater than the sum of the two proof lengths for $\phi$ and $\phi \rightarrow \psi$. (This is analogous to the proof expansions resulting from a conventional modus ponens operation.) Its first three steps will have *entirely inconsequential* effects that increase the overall proof length by no more than a *tiny* amount, that is proportional to the trivial sum of the lengths for the two individual sentences of " $\phi$ " and " $\psi$ ".

Hence, the preceeding "Linear-Sum Effect" allows us to construct an analog of Remark 3.5 's earlier Theorem $++$ for Xtab deduction. It is formalized by the statement $\odot$ below:

$\odot$ Any axiom system $\mathcal{A}$ is *automatically inconsistent* whenever it satisfies the following three conditions:

I. $\mathcal{A}$ can verify Successor is a total function (as Line (1) formalized).

II. $\mathcal{A}$ can prove addition and multiplication (viewed as 3-way relations) satisfy their usual associative, commutative, distributive and identity-operator properties.

III. $\mathcal{A}$ proves an added theorem (which turns out to be false) affirming its own consistency when the Xtab deductive apparatus is used.

It is not possible to provide a short proof for statement $\odot$ because it will rest upon the very detailed "Definable Cut" machinery from pages 172-174 of the Hájek-Pudlák textbook [25]. The intuition behind $\odot$ is, however, quite simple.

It is that statement $\odot$ causes ++'s mechanism to generalize from Hilbert-Frege deduction to Xtab (because both satisfy the Linear-Sum Effect).

The nice aspect of $\odot$ is that its machinery establishes Theorem 4.5. This is because if $\beta$ satisfies Theorem 4.5's hypothesis then $IS_{Xtab}(\beta)$ will satisfy[6] the conditions I-III that cause $IS_{Xtab}(\beta)$ to become inconsistent.

# 7 More Elaborate Forms of Theorems 4.4 and 4.5

the precise variant of

Our results in Theorems 4.4 and 4.5 demonstrate self-justifying methodologies apply to "Tab"', *but not* also "Xtab" deduction. (This is because Xtab treats the the Law of Excluded Middle as a formal schema of logical axioms, and the latter activates the power of the Second Incompleteness Effect.)

Our goal in this section will be to view this machinery in more meticulous detail. Thus, we will explore *at what exact juncture* the boundary is crossed between generalizations of the Second Incompleteness Theorem and its permissible exceptions.

**Definition 7.1** Let $L^{*}$ again denote the base arithmetic language (that was defined in §3 ), and $Z$ denote an arbitrary set of sentences appearing in the language $L^{*}$ (such as its set of $\Pi_2^{*}$ sentences). Let us recall that the Appendix defined a semantic tableau proof of a theorem $\Psi$ from $\alpha$'s axiom system. Then a **Z-Enriched** modification for a semantic tableau proof of a theorem $\Psi$, from $\alpha$'s set of proper axioms, will be defined as the particular refinement of the Appendix's proof-tree formalism that allows Line (9) as an added permissible logical axiom, for any $\Upsilon \in Z$.

$$\Upsilon \ \vee \ \neg \, \Upsilon \tag{9}$$

The terminology of a formula t free in a variable is very unusu normally says a variable is fre formula

**Definition 7.2** It is also of interest to consider a slight modification of the preceding nomenclature, where $Z$ is a set of formulae that are allowed to be free in the single variable of $x$ (instead of representing a sentence that contains no free variables). In this case, $\Upsilon(x)$ will designate a formula, within the subset of $Z$, and Line (10) will replace Line (9) as the added permissible logical axiom that can be allowed to appear inside a **"Z-Base Variable Enriched"** proof.

a sentence that contains – sentences that contain

$$\forall \, x \quad \Upsilon(x) \ \vee \ \neg \, \Upsilon(x) \tag{10}$$

[6]Actually, $IS_{Xtab}(\beta)$ will satisfy a requirement stronger than Item I because it recognizes addition as a total function.

A fully detailed justification will not be provided here, but it turns out our results from [47] can be expanded to show that their evasions of the semantic tableau version of the Second Incompleteness Theorem can be extended to both the cases of Z-Enriched and "Z-Base Variable Enriched" mechanisms, when Z represents the $\Delta_0^*$ class of formulae. We can also extend our results from [49] to show that the comparable evasions of the semantic tableau version of the Second Incompleteness Effect will fail at and above the $\Pi_2^*$ level.

Why quotation marks around "Z-Base Variable Enriched" but not around "Z-Enriched"?

We conjecture the preceeding $\Delta_0^*$ evasions of the Second Incompleteness Theorem will continue at the $\Pi_1^*$ level, but this fact has not yet been formally proven.

A fascinating aspect about this subject is that semantic tableau deduction satisfies its particular variant of Gödel's Completeness Theorem [15, 40]. Thus, the set of theorems proven by an axiom system $\alpha$ , via a conventional (unenriched) version of semantic tableau deduction, *is identical to* those theorems proven by a Z-enriched deductive mechanism. Yet despite this invariance, the proof-lengths change, *quite sharply,* under the Z-enriched formalisms of Defintions 7.1 and 7.2. This extreme change in proof-length causes the deployment of an *"I am consistent"* axiom to become *fully infeasible* when $\Upsilon$ in Line (9) is allowed to represent any arbitrary $\Pi_2^*$ sentence (see footnote [7] ).

Clarify (here or earlier) that "infeasible" refers to inconsis not to complexity.

# 8 Further Generalizations

For the sake of simplicity, the previous sections had focused on the semantic tableau deductive apparatus. However, it is known [15] that resolution shares numerous characteristics with tableau. Therefore, it turns out that Theorems 4.4 and 4.5 do generalize when resolution replaces semantic tableau.

In particular, let us say a theorem $\mathcal{T}$ has a *Res*−proof from $\alpha$ 's set of proper axioms when there is a resolution-based proof [15] of $T$ from $\alpha$ . Also, the term *Xres*−proof of $T$ refers to the obvious extension of a *Res*−proof that allows all instances of the Law of Excluded Middle (from the base language of $L^*$ ) to appear as formalized logical axioms.

[7] The point is that the sharp compression in proof lengths produces Gödel-like Diagonalization compressions, similar to those particular Second Incompleteness Effects applicable to $\Pi_2^*$ sentences, that are examined in [49].

ts

WOK

paradigms that pertain

It turns out $Xres$ differs from $Res$ in the same manner $Xtab$ differed from $Tab$. Thus, the obvious generalizations of Theorems 4.4 and 4.5 hold for $Res$ and $Xres$. In particular, $\mathrm{IS}_{Res}(\ \bullet\ )$ is a consistency preserving transformation, but $\mathrm{IS}_{Xres}(\ \bullet\ )$ again is not.

Some logicians may, also, wish to examine special speculations in [55]'s arXiv article. It contemplated an alternative approach, where self-justifying arithmetics employ an unconventional "indeterminate" functional object, called the $\Theta$ primitive, to formalize the traditional properties of an endless sequence of integers.

If a conjecture stated in [55] is correct (as we are almost certain it is), then such a self-justifying machine will be plausible for constructing the entire set of natural numbers, *without encountering* the usual incompleteness difficulties that the Theorem ++ (of Pudlák and Solovay) associated with Type-S formalisms (that recognize merely Successor as a total function). Interestingly, the $\theta$ function primitive of [55] should allow a substantial Type-NS arithmetic to exist that can *simultaneously* recognize its own Hilbert-Frege consistency and possess a formalized ability to constructively enumerate the full infinite collection of integers $0,\ 1,\ 2,\ 3,\ \ldots$.

This \theta was a capital \The in the previous paragraph.

# 9 Concluding Remarks

Our main results in this article are surprising because it is quite unusual for an initially consistent formalism $\alpha$ *to become inconsistent* when its initial schema of theorems (establishing the widespread validity of the Law of the Excluded Middle) is transformed into being a schema of logical axioms.

This unusual effect arose because the meaning of a Group-3 *"I am consistent"* axiom changes, *quite substantially,* when theorems are transformed into logical axioms (as illustrated by footnote [8] ). Thus, unacceptable diagonalizing contradictions can occur when an *"I am consistent"* axiom is able to reference itself *in the context of a SUFFICIENTLY POWERFUL mathematical machine.*

The contrast between Theorems 4.4 and 4.5 (where only the former eschews diagonalization effects) helpfully explains how Hilbert and Gödel appreciated the Second Incompleteness Effect, *while they were simultaneously cautious about it.* Moreover,

I conjecture that you meant "avoids" or "evades", not "esch

[8] The point is that proofs are compressed when theorems are transformed into logical axioms, and such compressions can produce diagonalizing contradictions under some Type-A logics using *"I am consistent"* axioms.

Don't Forget Acknowledgment FIV,

Gödel's particular remark ** should not be ignored when comments from Gerald Sacks and Stanley Tennenbaum [32, 39] recalled how Gödel reiterated the gist of his 1931-published remark, many years after its printing. Indeed, it is noteworthy Harvey Friedman recorded a YouTube lecture [18], stating he was also tentatively open to the possibility that the Second Incompleteness Theorems might allow partial exceptions.

Thus, while there is no doubt that the Second Incompleteness Theorem will be remembered for its seminal impact, its part-way exceptions are also significant. This is because futuristic high-tech computers will better understand their self-capacities, if they own some *partial* awareness about their own consistency.

earlier,

There is no page space to delve into all details here. However, the distinction between the initial "IS(A)" system, from our ~~1993 and 2001~~ papers [46, 47], with the more sophisticated $\mathrm{IS}_{Tab-1}(\beta)$ formalism of our ~~year 2005~~ article [50] should, also, be briefly mentioned. Our older "IS(A)" formalism was actually simpler, but it was substantially weaker because it only recognized the non-existence of a proof of $0 = 1$ from itself. In contrast, $\mathrm{IS}_{Tab-1}(\beta)$'s Group-3 axiom can corroborate that *no two simultaneous proofs* exist for a Rank-1* sentence and its negation. This is an important distinction, because the First Incompleteness Theorem indicates no decision procedure exists for separating all true from false Rank-1* sentences. (See [51, 52, 54, 55] for other particular refinements for our "IS(A)" formalism.)

I suggest deleting "year-" so that the 2005 reference will look like the 1993 and 2001 one line earlier.

subsequent

In summary, the main purpose of this article has been to explore the contrast between the opposing Theorems 4.4 and 4.5. The latter theorem, thus, provides *another helpful reminder* about the millennial importance of Gödel's seminal Second Incompleteness Theorem. Yet at the same time, Theorem 4.4 illustrates how some *partial exceptions* to Gödel's result do arise, as Hilbert and Gödel predicted in their statements * and **.

In essence, the 2-way contrast between Theorems 4.4 and 4.5 may be as significant as their individual actual results. This is because the Second Incompleteness Theorem is fundamental to Logic. Many historians have, thus, been perplexed by the *partial* reluctance that Hilbert and Gödel had expressed about it in * and **. A partial reason for this reluctance is, perhaps, related to the contrast between these two opposing theorems.

**ACKNOWLEDGMENTS:** I thank Seth Chaiken and James P. Torre, IV for several quite helpful comments about how to improve the presentation, and I thank the referee for his many helpful comments

# Appendix providing a formal definition for a Semantic Tableau proof:

Our definition of a semantic tableau proof is similar to analogs from the textbooks by Fitting and Smullyan [15, 40]. A tableau proof of a theorem $\Psi$ from a set of proper axioms (denoted as $\alpha$ ) is therefore a tree structure, whose root contains the temporary contradictory assumption of $\neg\Psi$ and whose every descending root-to-leaf branch affirms a contradiction by containing both some sentence $\phi$ and its negation $\neg\phi$. Each internal node in this tree will be either a proper axiom of $\alpha$ or a deduction from a higher ancestor in this tree via one of six elimination rules for the logical connective symbols of $\wedge$ , $\vee$ , $\rightarrow$ , $\neg$ , $\forall$ and $\exists$ . (These rules use a notation where "$\mathbf{A} \implies \mathbf{B}$" is an abbreviation for a sentence $\mathbf{B}$ being an allowed deduction from its ancestor of $\mathbf{A}$ .) Delete "of" after ancestor".

1. $\Upsilon \wedge \Gamma \implies \Upsilon$ and $\Upsilon \wedge \Gamma \implies \Gamma$ .
2. $\neg\neg\Upsilon \implies \Upsilon$. Other rules for the "$\neg$" symbol are: $\neg(\Upsilon \vee \Gamma) \implies \neg\Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \rightarrow \Gamma) \implies \Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \wedge \Gamma) \implies \neg\Upsilon \vee \neg\Gamma$, $\neg\exists v\, \Upsilon(v) \implies \forall v \neg\, \Upsilon(v)$ and $\neg\forall v\, \Upsilon(v) \implies \exists v\, \neg\Upsilon(v)$
3. A pair of sibling nodes $\Upsilon$ and $\Gamma$ is allowed when their ancestor is $\Upsilon \vee \Gamma$.
4. A pair of sibling nodes $\neg\Upsilon$ and $\Gamma$ is allowed when their ancestor is $\Upsilon \rightarrow \Gamma$.
5. $\forall v\, \Upsilon(v) \implies \Upsilon(t)$ where $t$ may denote any term.
6. $\exists v\, \Upsilon(v) \implies \Upsilon(p)$ where $p$ is a newly introduced parameter symbol.

One minor difference in notation is we treat "$\forall\, v \leq s \quad \Phi(v)$" as an abbreviation for $\forall v \ \{ \ v \leq s \ \rightarrow \ \Phi(v) \ \}$ and "$\exists\, v \leq s \quad \Phi(v)$" as an abbreviation for $\exists v \ \{ \ v \leq s \ \wedge \ \Phi(v) \ \}$ . Therefore, Rules 5 and 6 imply the following hybrid rules for processing bounded universal and bounded existential quantifiers:

**a.** $\forall v \leq s\, \Upsilon(v) \implies t \leq s \rightarrow \Upsilon(t)$ where $t$ may be any arithmetic term.

**b.** $\exists v \leq s \ \Upsilon(v) \implies p \leq s \ \wedge \ \Upsilon(p)$ where $p$ is a new parameter symbol.

**Added Comment:** The preceding paragraph has formalized what §4 called the "Tab" version of a semantic tableau proof. Its "Xtab" variant is identical except that any node may optionally store a sentence of the form $\mho \ \vee \ \neg\mho$ (for arbitrary $\mho$ ), as a manifestation of its allowed use of the Law of the Excluded Middle.

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Capitalize "Sätze" and "Syste
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Be consistent about (non)capitalization in titles of (English) articles.

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