From Dynamic to Static Semantics, Quantitatively

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Abstract

We exhibit a new relationship between dynamic and static semantics. We define the categorical outlay needed to define Interaction Graphs models, a generalisation of Girard's Geometry of Interaction models, which strongly relate to game semantics. We then show how this category is mapped to weighted relational models of linear logic. This brings into vision a new bridge between the dynamic and static approaches, and provides formal grounds for considering interaction graphs models as quantitative versions of GoI and game semantics models. We finally proceed to show how the interaction graphs models relate to a very general notion of quantitative coherence spaces.

1. Introduction

This paper is about denotational semantics of (a fragment of) linear logic. The focus is on (denotational) static and dynamic semantics, i.e. respectively *semantics of proofs* and *semantics of proofs and their cut-elimination*. Denotational semantics were introduced by Scott [16] as mathematical models for programming languages. Through the proofs-as-programs correspondence, it is equivalent to study semantics of programs and semantics of proofs, and the current paper will focus on a fragment of linear logic.

Denotational semantics are data-driven semantics, i.e. their focus is on data and data types. A program then corresponds to a function that maps some input data to some output data. This view of *programs as functions* is however arguably too coarse, as it models only the *what* (a program computes) and not the *how* (the program computes). Consider a program P and an input data A, and the result B that P outputs when given A as input. Denotational semantics will represent both $P(A)$ – P given input A – and B – the result of P's computation given input A – by the same object, i.e. $\|P(A)\| = \|B\|$. Dynamic semantics try to provide mathematical models that capture also *how* programs compute. Formally a dynamic semantic will be such that $\|P(A)\| \neq \|B\|$, but will provide an interpretation to the computation as well, i.e. there exists an operation $\rightarrow$ in the model such that $\|P(A)\| \rightarrow \|B\|$.

This paper presents a functor mapping the interpretations of Multiplicative Linear Logic (MLL) proofs in dynamic semantics to the interpretations of MLL proofs in denotational semantics. Intuitively, one can understand this functor as taking a dynamic semantics and collapsing the interpretation of execution to an equality. More precisely, we show how this collapse can be performed in a way that preserves *quantitative information*. Formally, the functor is defined as the antisymmetric tensor algebra construction.

Since the paper is concerned with semantics of a fragment of linear logic, we will now speak of proofs rather than programs and of cut-elimination rather than program execution. However, the reader should make a note to herself that, through the Curry-Howard correspondence, proof-theoretic and computer science terminologies are interchangeable.

1.1 Static semantics.

Static semantics are denotational models that *do not account for* the dynamics, i.e. the interpretation of a proof in a static model should be an invariant of the cut-elimination. A well-studied static semantics for linear logic is the so-called *relational model* where formulas A,B are interpreted as sets $\|A\|, \|B\|$ and proofs of the linear implication $A \multimap B$ are interpreted as binary relations on $\|A\| \times \|B\|$. Another example of static semantics considered in this paper is Girard's coherence spaces, which can be defined categorically as a tight double-glueing construction [12] applied to the relational model.

Quantitative denotational semantics are those which capture quantifiable properties of the programs it represents, such as time, space, and resource consumption. That is, quantitative semantics are denotational models of computation that mirror more information than the so-called *qualitative* models, e.g. the relational model, coherence spaces. Origins of this approach can be traced back to Girard's work on functor models for lambda-calculus [7] – that inspired linear logic [6] – which exhibited for the first time a decomposition of the semantic interpretation of lambda-terms as Taylor series.

Quantitative semantics has also provided static semantics for various algebraic extensions of lambda calculus such as probabilistic lambda calculi [4]. By considering a refinement of the relational model, Laird *et al.* [14] provided a uniform account of several static models accounting for quantitative notions. Finally, some quantitative generalisations of coherence space semantics have been considered, namely probabilistic [4, 8] and quantum coherence spaces [8]. However, no uniform account of those has yet appeared in the literature.

1.2 Dynamic Semantics.

We regroup under this terminology those semantics that do not solely account for proofs but also for their *dynamics*, e.g. game semantics and various *geometry of interaction* (GoI) constructions. Formally, the interpretation of a proof of $A \multimap B$ cut with a proof of A will differ from the interpretation of the cut-free proof of B obtained by cut-elimination. Although different models and techniques regrouped under this name may differ on certain aspects, their interpretation of programs/proofs is quite similar [1].

On the technical level, we will work in this paper with the author's *interaction graphs* construction. These models include all of Girard's geometry of interaction (GoI) models [20], and provide a far-reaching generalisation of those. Indeed, interaction graphs should be understood as a quantitative variant of geometry of interaction. Moreover, since GoI models are strongly related to game semantics (even more so when we restrict to MLL as it done in this paper), interaction graphs can be thought of as a quantitative generalisation of those as well. Interaction graphs thus provide a natural framework to deal with quantitative dynamic interpretations with great generality.

One should note that GoI models differ from those of game semantics by how the formulas are interpreted. Game semantics start with interpretations of formulas (games) and then define the interpretations of proofs. On the contrary, and though proof interpretations are the same, GoI constructions define the interpretation of formulas from the "untyped" interpretation of proofs. Categorically, the latter construction is realised by a (tight) double-glueing construction [12] defined from a notion of orthogonality between morphisms.

1.3 Contributions.

The first contribution of this paper is the definition of the categorical foundations of the interaction graphs constructions through the introduction of *triskells*. This categorical apparatus is of interest in itself as it generalises both the bicategory of spans and the categories of matrices and arrays over a semiring. Furthermore, unlike in the work of Laird *et al.*, our definition is performed internally to the underlying category and does not refer to external objects. This opens the way to defining even wider generalisations based on the replacement of the underlying category of sets with e.g. any topos.

Secondly, we shed light onto a new relationship between dynamic and denotational semantics. This relationship is not only shown to hold for *qualitative semantics*, but also for *quantitative semantics*. This provides formal grounds to the claim that the author's interaction graphs construction should be understood as a quantitative version of dynamic semantics – GoI and game semantics. Though not the first work bridging dynamic and static semantics [3, 11], the technique used here is novel and is the first to be applied to quantitative models.

Finally, we describe how this functor not only preserves the interpretation of proofs but also maps *orthogonalities*. I.e. we show how this functor lifts to a map from the double-glueing constructions considered in interaction graphs and the double-glueing construction of coherence spaces. In particular, the specific model detailed in the first paper by the author [17] is shown to map to "strict" probabilistic coherence spaces. This correspondence, although confined to specific cases when considering categories of relations, can be lifted to a much more general correspondence when considering the generalisation of weighted relations provided by triskells. We end the paper by sketching a very general construction of *quantitative coherence spaces* suggested by the previous sections.

1.4 Organisation of the paper.

The first section introduces the notion of *triskell* in a category. Triskells are generalisations of spans that will account for the quantitative aspects available in the interaction graphs constructions. The category of triskells will then (Section section 3) be shown to have the structure of a *traced monoidal category*, i.e. it has the structure to define a GoI interpretation of proofs. We also show how triskells can be mapped to weighted relations in the sense of Laird *et al.* [14]. These two results show that the category of triskells can accomodate both quantitative GoI and quantitative static semantics. The monoidal products interpreting the linear logic tensor in these two kinds of models are not the same however. We thus introduce (section 4) an endofunctor which maps one monoidal product to the other. This functor thus maps the interpretation of a proof of MLL in (quantitative) dynamic models to the interpretation of the same proof in (quantitative) denotational models. Finally, we study in section 5 the double-glueing construction of GoI and show that it relates to the coherence space double-glueing through the functor just exhibited. This result is however quite specific as it is based upon the notion of determinant and trace of matrices. We thus end this paper by defining a notion of determinant and trace of triskells, allowing for the definition of both quantitative GoI models and quantitative coherence space semantics in a very general setting

2. Triskells

The idea of triskells comes from the author's work on Interaction Graphs [17, 20]. Triskells are a generalisation of spans [2]. While the latter can be used to represent directed graphs, triskells possess the extra structure needed to interpret weighted edges.

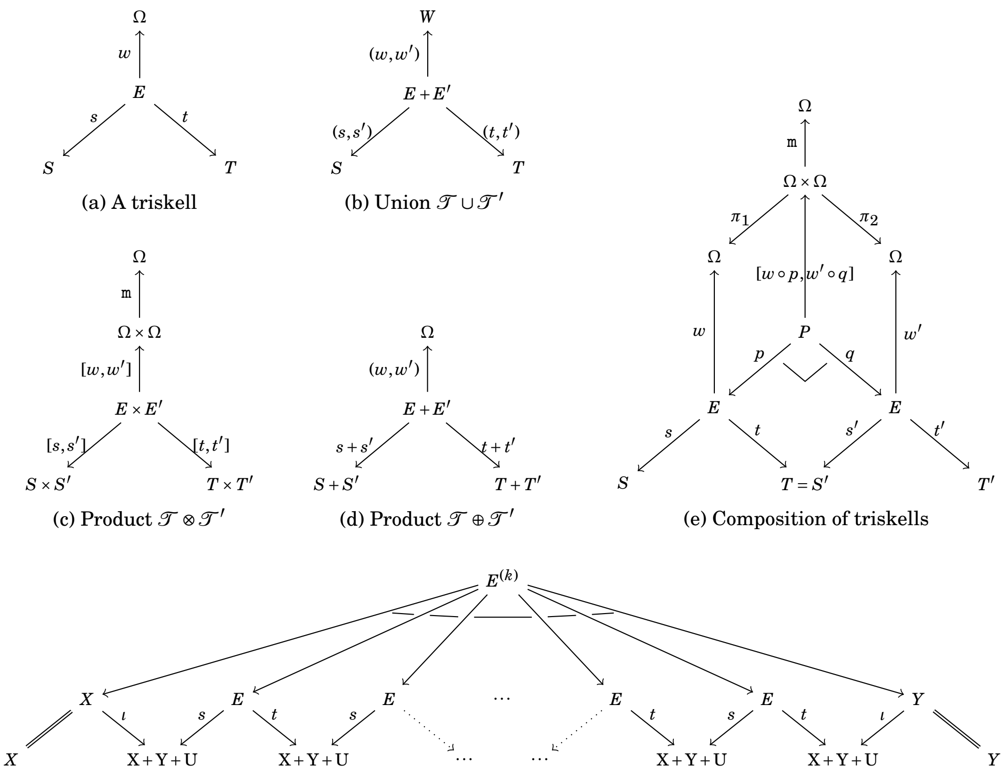
Definition 1. A *triskell* $\mathscr{T}$ in a category $\mathbb{C}$ is defined as three morphisms sharing the same source: it is a tuple $(\mathrm{E},\mathrm{S},\mathrm{T},\Omega,s,t,w)$ with $\mathrm{E},\mathrm{S},\mathrm{T},\Omega \in \mathrm{Obj}_{\mathbb{C}}$, $s \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{E},\mathrm{S})$, $t \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{E},\mathrm{T})$, $w \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{E},\Omega)$. We call S (resp. T, resp. Ω) and s (resp. t, resp. w) the *source object* (resp. *target object*, resp. *weight object*) and *source map* (resp. *target map*, resp. *weight map*). We denote by $\mathbf{Tk}^{\Omega}(\mathrm{S},\mathrm{T})$ the set of all triskells with source object S, target object T and weight object Ω.

For composition to be defined, the set of weights must be fixed once and for all and endowed with an associative binary product, i.e. Ω should have the structure of a monoid.

Definition 2. Let $(\mathbb{C},\otimes,1)$ be a monoidal category with α,ρ,λ the associativity, right and left unit maps of the monoidal product. A monoid object $(\mathrm{M},\mathtt{m},\mathtt{u})$ in $\mathbb{C}$ is an object M together with a multiplication $\mathtt{m} \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{M}\otimes\mathrm{M},\mathrm{M})$ and a unit $\mathtt{u} \in \mathrm{Hom}_{\mathbb{C}}(1,\mathrm{M})$, which satisfy associativity, i.e. $\mathtt{m}\circ(\mathrm{Id}_{\mathrm{M}}\otimes\mathtt{m}) = \mathtt{m}\circ(\mathtt{m}\otimes\mathrm{Id}_{\mathrm{M}})\circ\alpha$, and identity, i.e. $\mathtt{m}\circ(\mathrm{Id}_{\mathrm{M}}\otimes\mathtt{u}) = \rho_{\mathrm{M}}$ and $\mathtt{m}\circ(\mathtt{u}\otimes\mathrm{Id}_{\mathrm{M}}) = \lambda_{\mathrm{M}}$.

For our purposes, we will suppose that $\mathbb{C}$ is a cartesian category, i.e. the monoidal product is the cartesian product and the unit is a terminal object. We will write $\times$ the product, and $[\cdot,\cdot]$ the associated operation of *pairing* of morphisms. This hypothesis is enough to ensure the existence, for each object E of a unit map $\mathtt{u}_{\mathrm{E}} \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{E},\Omega)$, i.e. a map such that $\mathtt{m}\circ[f,\mathtt{u}_{\mathrm{E}}] = f = \mathtt{m}\circ[\mathtt{u}_{\mathrm{E}},f]$ for all $f \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{E},\Omega)$. From now on, all monoid objects considered will be monoid objects for the cartesian monoidal structure over $\mathbb{C}$.

We now fix a cartesian category $\mathbb{C}$ with pullbacks, i.e. $\mathbb{C}$ is finitely complete. We also pick a monoid object Ω, and two triskells $\mathscr{T},\mathscr{T}'$ with weight object Ω and such that the source object of $\mathscr{T}'$ coincides with the target object of $\mathscr{T}$. We can define (Figure 1e) their *composition* $\mathscr{T}'\circ\mathscr{T}$ as the tuple $(\mathrm{P},\mathrm{S},\mathrm{T},\mathrm{W},q\circ s,p\circ t,\mathtt{m}\circ[w\circ p,w'\circ q])$, where (P,p,q) is a designated pullback of t and s'. As when working with spans, associativity is only satisfied up to isomorphism and the natural structure we obtain is not that of a category but rather a bicategory. However, identifying isomorphic objects leads to a category.



(a) A triskell (b) Union $\mathscr{T} \cup \mathscr{T}'$ (c) Product $\mathscr{T} \otimes \mathscr{T}'$ (d) Product $\mathscr{T} \oplus \mathscr{T}'$ (e) Composition of triskells (f) Underlying span of the triskell of length k paths from X to Y

Figure 1: Operations on triskells

Definition 3. Let $\mathbb{C}$ be a complete category and Ω a monoid in $\mathbb{C}$. The bicategory $\mathbb{T}\mathrm{k}^{\Omega}(\mathbb{C})$ is:

$$\mathrm{Obj}_{\mathbb{T}\mathrm{k}^{\Omega}(\mathbb{C})} = \mathrm{Obj}_{\mathbb{C}},$$
$$\mathrm{Hom}_{\mathbb{T}\mathrm{k}^{\Omega}(\mathbb{C})}(\mathrm{A},\mathrm{B}) = \mathbf{T}\mathrm{k}^{\Omega}(\mathrm{A},\mathrm{B}).$$

The category $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{C})$ is then defined as the quotient of $\mathbb{T}\mathrm{k}^{\Omega}(\mathbb{C})$ w.r.t. 2-isormorphisms.

Notations 4. When they exist, coproducts will be denoted by $+$, while the *copairing* of morphisms will be denoted by $(\cdot,\cdot)$. Given $f \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{X},\mathrm{Y})$ and $g \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{X}',\mathrm{Y}')$, we also define the morphism $f+g \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{X}+\mathrm{X}',\mathrm{Y}+\mathrm{Y}')$ as the copairing $(\iota_1\circ f,\iota_2\circ g)$ where $\iota_1 \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Y},\mathrm{Y}+\mathrm{Y}')$ and $\iota_2 \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Y}',\mathrm{Y}+\mathrm{Y}')$ are the natural inclusions.

The product of the underlying category $\mathbb{C}$ naturally induces a monoidal product on triskells written $\otimes$ (Figure 1c). If $\mathbb{C}$ has coproducts, then $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{C})$ has two additional operations: the sum (Figure 1d) another monoidal product written $\oplus$, and the union, written $\cup$ (Figure 1b), which is defined between two triskells with same source and target objects.

Notations 5. We will denote by $1 = (\{\star\},\{\star\},\{\star\},\Omega,!_{\{\star\}},!_{\{\star\}},\mathrm{u}_{\{\star\}})$ – where $!_{\{\star\}}$ is the unique morphism $\{\star\} \rightarrow \{\star\}$ – the unit of the monoidal product $\otimes$. Whenever the underlying category has coproducts, we will denote by $\emptyset = (\emptyset,\emptyset,\emptyset,\Omega,\emptyset_{\emptyset},\emptyset_{\emptyset},\emptyset_{\Omega})$ – where $\emptyset_A$ denotes the unique map $\emptyset \rightarrow A$ – the unit of the monoidal product $\oplus$.

The category of triskells is of interest since it is the ambient category in which the author's Interaction Graphs construction takes place [17] (taking $\mathbb{C}$ as the category of countable sets and restricting to finite objects). Although later works [20, 22] make use of a larger category (in particular, one considers a sort of abelian enrichment to deal with additives [20]), we restrict our discussion to the category of triskells in this paper.

Example 6. If $\mathbb{C}$ is the category of countable sets $\mathbb{S}\mathrm{et}_{\aleph_1}$, then a triskell over Ω whose source and target objects are finite sets is exactly a Ω-weighted directed graph[1] as defined in earlier work by the author [17]. Composition corresponds to the composition of graphs by considering length 2 paths, while the trace defined below coincides with the *execution* [17].

The previous remarks explain the following terminology. When considering a triskell $(\mathrm{E},\mathrm{S},\mathrm{T},\Omega,s,t,w)$ over the category of sets, we will refer to the set E as the *set of edges*, and we will refer to any element of E as an *edge*.

3. Triskells over $\mathbb{S}\mathrm{et}$ and weighted relations

We now fix once and for all the underlying $\mathbb{C}$ to be a category of sets, i.e. $\mathbb{C}$ is the category $\mathbb{S}\mathrm{et}_{\kappa}$ of all sets of cardinality less than κ where κ is any fixed cardinal. We suppose moreover that $\mathbb{S}\mathrm{et}_{\kappa}$ has countable coproducts, i.e. $\kappa \geqslant \aleph_1$.

3.1 Traced monoidal structure

We now explain how one can define a categorical trace [13] on $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{C})$ by constructing the *triskell of paths*. We start from a triskell $\mathscr{T}$ whose source and target objects are of the form X+U

[1] Notice that in the presence of corpoducts, the notion of span captures exactly that of *directed graphs*, which are usually represented by a pair of morphisms s,t from a set of edges E to a set of vertices V.

and Y + U respectively. One can define from it a triskell with same source and target X + Y + U which we denote $\mathcal{T}'$. The composition of k copies of this triskell $\mathcal{T}'$ with the same source and target objects X + Y + U defines the "triskell of paths of length k". Pre- and post-composition with "projections" onto Y and X respectively gives rise to the triskell of paths of length k from X to Y (Figure 1f). Finally, countable coproducts allows to build the triskell $\mathrm{Tr}_{\mathrm{U}}(\mathcal{T})$ of all (finite) paths from X to Y in $\mathcal{T}$, through the countable version of the union of triskells (Figure 1b).

Theorem 7. *For any monoid Ω in $\mathbb{S}\mathrm{et}_\kappa$, $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et}_\kappa)$ is traced monoidal.*

Proof. Let $\mathcal{T}$ be a triskell with source object $X+U$ and target object $Y+U$. Then $\mathcal{T}$ is the union of the four triskells $\mathcal{T}_{X,Y}$, $\mathcal{T}_{X,U}$, $\mathcal{T}_{U,Y}$ and $\mathcal{T}_{U,U}$ defined by pre- and post-compositions with natural inclusions. It is then not difficult to show that $\mathrm{Tr}_U(\mathcal{T})$ can be written as the union of $\mathcal{T}_{X,Y}$ and the triskells $\mathcal{T}_{X,U} \circ \mathcal{T}^k_{U,U} \circ \mathcal{T}_{U,Y}$, i.e. $\mathrm{Tr}_U(\mathcal{T})$ is computed through the standard trace formula [10, Proposition 8]. The proof that it satisfies the trace axioms [13] is then standard. □

Remark 8. The above theorem should hold in a more general setting. First, the construction of the trace can be performed as long as the underlying category $\mathbb{C}$ is finitely complete and has countable coproducts. Then, the proof that $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{C})$ is indeed traced monoidal relies mainly on two additional properties of the category of sets, namely that coproducts are disjoint and stable under pullbacks. In particular, one should be able to replace $\mathbb{S}\mathrm{et}_\kappa$ by any topos.

3.2 Simple triskells and weighted relations

When Ω is a complete semiring, one can define a functor from the category of triskells over Ω to the category of weighted relations over Ω through the following *contraction of triskells*.

Given a triskell $T = (\mathrm{E},\mathrm{S},\mathrm{T},\Omega,s,t,w)$, the map $[s,t]$ factors uniquely as $\mathrm{E} \xrightarrow{e} \bar{\mathrm{E}} \xrightarrow{m} \mathrm{S}\times\mathrm{T}$, where m is a monomorphism (and e is an epimorphism). We can additionally define a map $\bar{w} : \bar{\mathrm{E}} \to \mathrm{W}$ as follows: $\bar{w}(a) = \sum_{b\in \mathrm{E}, e(b)=a} w(b)$. We can therefore defined a triskell $\curlywedge(\mathcal{T})$ – the contraction of $\mathcal{T}$ – as $(\bar{\mathrm{E}},\mathrm{S},\mathrm{T},\mathrm{W},\pi_1\circ m,\pi_2\circ m,\bar{w})$. This triskell has a particular property: the pairing of the source and target maps is a monomorphism; such triskells will be called *simple*.

Remark 9. This mimics a natural contracting operation [17] that maps a (weighted) graph $G = (V,E,s,t,w)$ to a simple (weighted) graph $\curlywedge(G)$ with the same set of vertices V.

This operation on triskells actually defines a functor from the category of triskells to the category of weighted relations defined by Laird *et al.* [14] – or equivalently the category $\mathbb{M}\mathrm{at}_\Omega$ of matrices with coefficients in a (complete) semiring Ω.

Definition 10. Let Ω be a complete semiring. We define the category $\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et})$ as follows. Objects are sets and morphisms from A to B are matrices in $\Omega^{\mathrm{A}\times\mathrm{B}}$. Composition, written $\bullet$, is defined as the matrix product. The identity morphism is the identity matrix.

Now, a matrix M in $\Omega^{\mathrm{A}\times\mathrm{B}}$ can be written as a triskell $\mathcal{T}_M$: writing $w : (a,b) \mapsto m_{a,b}$, we define this triskell by $\mathcal{T}_M = (\mathrm{A}\times\mathrm{B},\mathrm{A},\mathrm{B},\Omega,\pi_1,\pi_2,w)$. This correspondence is not one-to-one however, as zero coefficients can be represented both by edges with weight zero (as in the triskell defined above) or by a lack of edge. We therefore define also the minimal triskell associated to M as $(\mathrm{E}_M,\mathrm{A},\mathrm{B},\Omega,\pi_1,\pi_2,\bar{w})$ where $\mathrm{E}_M = \{(a,b)\in \mathrm{A}\times\mathrm{B} \mid m_{a,b} \neq 0\}$ and $\bar{w}$ is the restriction of w to E_M. Conversely, every simple triskell $T = (\mathrm{E},\mathrm{A},\mathrm{B},\Omega,s,t,w)$ defines a matrix $M_{\mathcal{T}}$ in $\Omega^{\mathrm{A}\times\mathrm{B}}$. Since $[s,t]$ is a monomorphism, we can consider E a subset of $\mathrm{S}\times\mathrm{T}$. Then the coefficient $m_{a,b}$ is equal to $w((a,b))$ if $(a,b)\in\mathrm{E}$ and null otherwise.

Remark 11. To obtain an exact correspondence, one should consider relations weighted in a complete semiring Ω (with a zero) and simple triskells over $\Omega^* - \Omega$ bereft of its zero element.

It should be noted that the composition of relations is *not* the composition of (simple) triskells: the composition $\mathcal{T}\circ\mathcal{T}'$ of two simple triskells $\mathcal{T},\mathcal{T}'$ need not be simple in general[2]. However, a simple computation shows that $\curlywedge(\mathcal{T}_M\circ\mathcal{T}_N) = \mathcal{T}_{N\bullet M}$ for all matrices M,N in $\Omega^{\mathrm{A}\times\mathrm{B}}$.

Remark 12. Two monoidal products are considered by Laird *et al.* on the category $\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et})$, namely the *sum* $\oplus$ and the *tensor* $\otimes$. It is not difficult to check that, through the identification of relations with simple triskells, those correspond to the monoidal products $\oplus$ and $\otimes$ on triskells. Furthermore, the operation of union of triskells corresponds to matrix addition.

Definition 13. Let Ω be a complete semiring. We define the contraction functor $\mathfrak{C}$ from $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{C})$ to $\mathbb{R}\mathrm{el}_\Omega(\mathbb{C})$ as follows. It acts as the identity on objects, and maps a triskell $\mathcal{T}$ to the matrix $M_{\curlywedge(\mathcal{T})}$ of the contracted triskell $\curlywedge(\mathcal{T})$.

Theorem 14. *The functor $\mathfrak{C}$ is a strict monoidal functor for the two monoidal structures, i.e. $\mathfrak{C}(\mathcal{T}\otimes\mathcal{T}') = \mathfrak{C}(\mathcal{T})\otimes\mathfrak{C}(\mathcal{T}')$ and $\mathfrak{C}(\mathcal{T}\oplus\mathcal{T}') = \mathfrak{C}(\mathcal{T})\oplus\mathfrak{C}(\mathcal{T}')$.*

This suggests the category $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})$ as a natural generalisation of the category of weighted relations. We will try to follow this intuition in the next sections.

We have now shown that triskells can be used to define both quantitative denotational and dynamic semantics for MLL. However, these interpretations are based upon the interpretation of the tensor product as two distinct monoidal products. In the next section, we explain how one can define a strict monoidal functor on the category of triskells which maps the monoidal product $\oplus$ onto the monoidal product $\otimes$.

4. Fock Functor

We now define a functor from $\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et})$ to itself which we call the *Fock functor*, by analogy with the *antisymmetric Fock functor*. We recall that this functor is induced by the construction of the *Grassman algebra* or *exterior algebra* [15, XIX, Âğ1, page 733].

4.1 The relational Fock functor

For the definition to make sense in our setting, we will need now to suppose that the monoid Ω has a structure of a continuous commutative ring, i.e. we need additive inverses in Ω. It is important to note that such algebraic structures do exist (for instance, taking the ring of formal power series over a group), though they are not as usual as the well-known semiring $\mathbf{R}_{\geqslant 0}\cup\{\infty\}$. To consider simpler examples, one can restrict to the subcategory of relations (resp. triskells) between *finite* sets. The continuity requirement is then useless and one could consider simple rings, such as the real numbers.

Notations 15. Consider a triskell $\mathcal{T}$ in $\mathbf{Tk}^{\Omega}(\mathrm{A},\mathrm{B})$, and choose $\bar{a} = \{a_1,\dots,a_k\}\in \mathrm{P}_{\mathrm{fin}}(\mathrm{A})$ and $\bar{b} = \{b_1,\dots,b_k\}\in\mathrm{P}_{\mathrm{fin}}(\mathrm{B})$. We define the set $M_{\mathcal{T}}[\bar{a},\bar{b}]$ as the set of all pairs $(\sigma,\vec{e})$ where σ is a permutation and $\vec{e}$ is a sequence of edges such that $s(e_i) = a_i$

[2] This is already true for spans. To see this, it is enough to consider any triskells whose underlying spans are $(\{(1,2),(1,3)\},\{1\},\{2,3\},\pi_1,\pi_2)$ and $(\{(2,4),(3,4)\},\{2,3\},\{4\},\pi_1,\pi_2)$, with π_i the natural projections.

and $t(e_i) = b_{\sigma(i)}$. Then for all such pairs $(\sigma, \vec{e})$ we define the weight $\omega_{\mathscr{T}}[\bar{a}, \bar{b}](\vec{e}) = \prod_{e \in \vec{e}} w(e)$. We then define $\omega_{\mathscr{T}}[\bar{a}, \bar{b}] = \sum_{(\sigma, \vec{e}) \in M_{\mathscr{T}}[\bar{a}, \bar{b}]} \epsilon(\sigma) \omega_{\mathscr{T}}[\bar{a}, \bar{b}](\vec{e})$ where $\epsilon(\sigma)$ is the signature of the permutation σ.

Remark 16. Notice that $\omega_{\mathscr{T}}[\bar{a}, \bar{b}]$ is the Leibniz formula for the determinant of the submatrix of M defined as $M[\bar{a}, \bar{b}] = (m_{a,b})_{a \in \bar{a}, b \in \bar{b}}$.

Notations 17. For any simple triskell $\mathscr{T}$ with source object A and target object B, we denote by $a \sim_\omega b$ the existence of $e \in \mathrm{E}$ with $s(e) = a$, $t(e) = b$ and $w(e) = \omega$. Since $\mathscr{T}$ is supposed simple, at most one such $e \in \mathrm{E}$ exists given a pair $(a, b) \in \mathrm{A} \times \mathrm{B}$.

Definition 18. We define the *Fock* functor $\mathfrak{F}$ from the category $\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et})$ to itself as follows (we consider morphisms defined as triskells). On objects $\mathfrak{F}$ acts as the finite powerset functor: $\mathrm{A} \mapsto \mathrm{P_{fin}}(\mathrm{A})$. On morphisms, we have $\bar{a} = \{a_1, \dots, a_k\} \sim_w \{b_1, \dots, b_{k'}\} = \bar{b}$ in $\mathfrak{F}(T)$ if and only if $k = k'$, $S_{\mathscr{T}}[\bar{a}, \bar{b}] \neq \emptyset$ and $w = w_{\mathscr{T}}[\bar{a}, \bar{b}]$.

Example 19. Let us consider the relation R corresponding to the matrix $\begin{pmatrix} a & b \end{pmatrix}$ between $\{1,2\}$ and $\{3\}$. Then $\mathfrak{F}(R)$ is a relation between $\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $\{\emptyset, \{3\}\}$ defined as follows. First, $\emptyset$ is in relation with $\emptyset$ with weight 1. Then $\{1\}$ and $\{2\}$ are both in relation with $\{3\}$ with respective weights a and b. Figure 2 illustrates two more involved examples.

Theorem 20. *The functor $\mathfrak{F}$ is strict monoidal from $(\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et}), \oplus, \emptyset)$ to $(\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et}), \otimes, 1)$.*

Proof (sketch). We begin by showing functoriality. Consider relations R, R', and compute both $\mathfrak{F}R \circ R'$ and $\mathfrak{F}R \circ \mathfrak{F}R'$. By definition, we have $\bar{a} \sim_w \bar{b}$ in $\mathfrak{F}R \circ R'$ if and only if $w = \sum_{\sigma \in S[\bar{a}, \bar{b}]} \epsilon(\sigma) w(\sigma)$ in $R \circ R'$. We rewrite this as $w = \sum_{\sigma \in S[\bar{a}, \bar{b}]} \epsilon(\sigma) \sum_i w_i^1 w_i^2$. On the other hand, $\bar{a} \sim_w \bar{b}$ in $\mathfrak{F}R \circ \mathfrak{F}R'$ if and only $w = \sum_{\bar{c}} w[\bar{a}, \bar{c}] w[\bar{c}, \bar{b}]$, which can be rewritten

$$w = \sum_{\bar{c}} \left(\sum_{\sigma \in S[\bar{a}, \bar{c}]} \epsilon(\sigma) \sum_i w_i^1 \right) \left(\sum_{\sigma \in S[\bar{c}, \bar{b}]} \epsilon(\sigma) \sum_i w_i^2 \right)$$

We can, up to distributivity, rewrite the first sum as

$$\sum_{\bar{c}} \left(\sum_{\sigma \in S[\bar{a}, \bar{c}]} \epsilon(\sigma) \sum_i w_i^1 \right) \left(\sum_{\sigma \in S[\bar{c}, \bar{b}]} \epsilon(\sigma) \sum_i w_i^2 \right) + Res$$

where Res is a residue (i.e. the paths that share a middle vertex). We then only need to show that this residue is null, but this is a consequence of symmetry (i.e. each edge has an inverse edge which cancels it).

On objects, monoidality is clear from the well-known identity $\mathrm{P_{fin}}(A + B) = \mathrm{P_{fin}}(A) \times \mathrm{P_{fin}}(B)$.

On morphisms, monoidality is obtained by a simple computation showing that $\bar{a} \sim \bar{b}$ is in $\mathfrak{F}(R \oplus R')$ if and only if it is in $\mathfrak{F}(R) \otimes \mathfrak{F}(R')$. This can be done directly, or through the proof of monoidality of Theorem 23 (below) by noticing that for fixed source and target $\bar{a}$ and $\bar{b}$, any edge in $\mathfrak{F}(R \oplus R')$ is the contraction of the edges in $\mathfrak{F}_\uparrow(R \oplus R')$ and any edge in $\mathfrak{F}(R) \otimes \mathfrak{F}(R')$ is the contraction of the edges in $\mathfrak{F}_\uparrow(R) \otimes \mathfrak{F}_\uparrow(R')$. Since there is a bijective correspondence between these sets of edges (preserving the weights), this allows to conclude. $\square$

This shows that the image of $(\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{S}\mathrm{et}), \oplus, 0)$ through the composition $\mathfrak{F} \circ \mathfrak{C}$ is the category $(\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et}), \otimes, 1)$. I.e. the Interaction Graphs' interpretation $\|\pi\|_{\mathrm{IG}}$ of a *cut-free* MLL proof π is mapped to its weigthed relational model's interpretation $\|\pi\|_{\mathrm{WR}}$. This result, however, can be improved by lifting the Fock functor to the category of triskells.

4.2 The Triskells Fock Functor

We will now show how to lift the fock functor from simple triskells to general triskells, i.e. we will show there exists a functor $\mathfrak{F}_\uparrow$ such that the following diagram commutes. We will define $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{C})^0$ later on; let us point out for now that it is a quotient of $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{C})$ w.r.t. a simple congruence.

$$\begin{tikzcd}
(\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{C}), \uplus) \arrow[r, dotted, "\mathfrak{F}_\uparrow"] \arrow[d, "\mathfrak{C}"'] & (\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{C}), \times)^0 \arrow[d, "\mathfrak{C}"] \\
(\mathbb{R}\mathrm{el}_\Omega(\mathbb{C}), \uplus) \arrow[r, "\mathfrak{F}"] & (\mathbb{R}\mathrm{el}_\Omega(\mathbb{C}), \times)
\end{tikzcd}$$

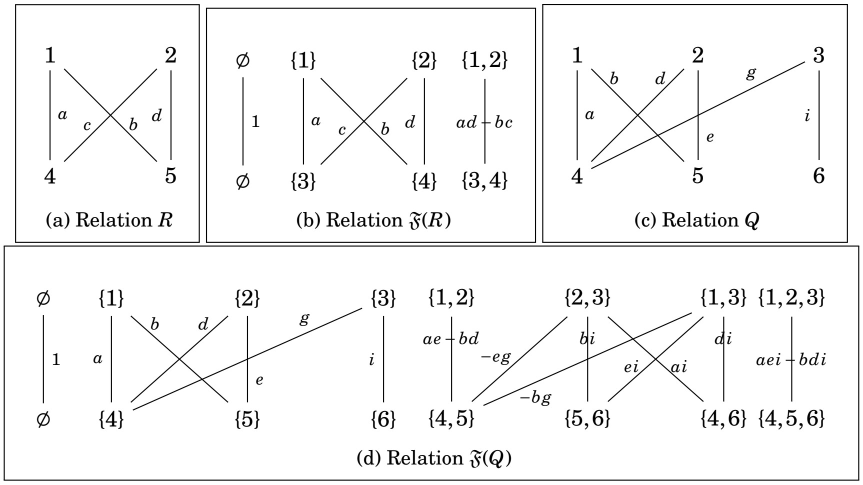
Although it is necessary that Ω have the structure of a complete ring (or simply a ring if we restrict to the subcategories where objects are finite sets) for the above diagram to commute, we will in fact not need much more than a monoid structure on Ω for defining $\mathfrak{F}_\uparrow$. This is yet another argument in favour of considering triskells instead of relations. This lifted functor is defined in a way which is similar to the definition $\mathfrak{F}$, but simpler as one does not perform any summations. I.e. every element of $M_{\mathscr{T}}[\bar{a}, \bar{b}]$ will give rise to a new edge. The only thing we will need are signs, i.e. ways to interpret $\epsilon(\sigma)$ for a permutation σ. This is done by picking a monoid Ω that can be written as the product of a monoid with the monoid $\{-1, 1\}$ (with multiplication). In the following, we will call such structure a *signed monoid*.

Remark 21. We chose here to fix Ω as a signed monoid, but we could have as well defined $\mathfrak{F}_\uparrow$ as a functor from $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{S}\mathrm{et})$ to $\mathbb{T}\mathrm{k}^{\{-1,1\} \times \Omega}_\simeq(\mathbb{S}\mathrm{et})$ for any monoid Ω.

Definition 22. We define $\mathfrak{F}_\uparrow$ as follows. On objects, $\mathfrak{F}_\uparrow$ acts as the finite powerset functor. On morphisms, it maps the triskell T to the triskell $\mathfrak{F}_\uparrow(T)$ defined as follows: for each $(\sigma, \vec{e})$ in $M_{\mathscr{T}}[\bar{a}, \bar{b}]$, there is exactly one edge of source $\bar{a}$, target $\bar{b}$ and weight $\epsilon(\sigma) \omega_\sigma[\bar{a}, \bar{b}](\vec{e})$.

It turns out that $\mathfrak{F}_\uparrow$ is *not* a functor from $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{C})$ to itself. To understand this, and understand why the same situations are not a problem with relations, let us consider the triskells (actually these are relations) $\mathscr{T}$ and $\mathscr{T}'$ shown in Figure 3a and Figure 3b. Then the fact that $\mathfrak{F}_\uparrow$ is not a functor can be seen by computing $\mathfrak{F}_\uparrow(\mathscr{T}' \circ \mathscr{T})$ (Figure 3c) and $\mathfrak{F}_\uparrow(\mathscr{T}') \circ \mathfrak{F}_\uparrow(\mathscr{T})$ (Figure 3d). As the reader can see for herself, the contraction operation will cancel the edges from $\{1,2\}$ to $\{4,5\}$ in the triskell $\mathfrak{F}_\uparrow(\mathscr{T}' \circ \mathscr{T})$. This is why $\mathfrak{F}$ is indeed a functor while $\mathfrak{F}_\uparrow$ is not.

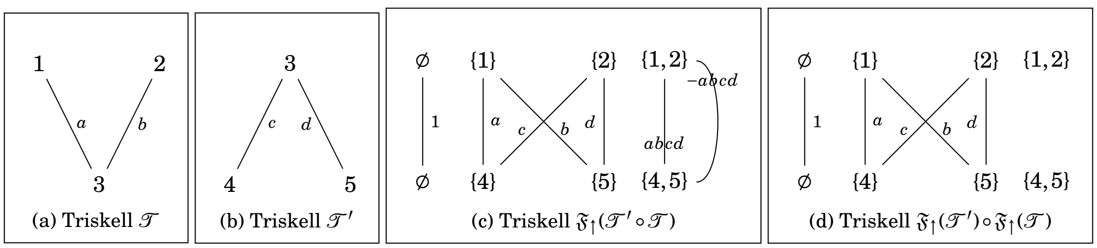
However, one can show that $\mathfrak{F}_\uparrow$ is a functor *up to some equivalence* on morphisms. For this, we define a congruence on the set of triskells over $\{1, -1\} \times \Omega$ as follows: two triskells $\mathscr{T}$ and $\mathscr{T}'$ are equivalent if and only if there exists two "zero triskells" $\mathscr{Z}$ and $\mathscr{Z}'$ and a triskell $\mathscr{A}$ such that $\mathscr{T} = \mathscr{A} \cup \mathscr{Z}$ and $\mathscr{T}' = \mathscr{A} \cup \mathscr{Z}'$. Here a *zero triskell* is a triskell whose set of edges E can be decomposed as $E^+ + E^-$ so that there is a bijection $\theta : E^+ \to E^-$ satisfying $s(e) = s(\theta(e))$, $t(e) = t(\theta(e))$ and $w(e) = -w(\theta(e))$. Then $\mathfrak{F}_\uparrow$ can be shown to be a functor from $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{S}\mathrm{et})$ to the quotient $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{S}\mathrm{et})^0$ of $\mathbb{T}\mathrm{k}^\Omega_\simeq(\mathbb{S}\mathrm{et})$ by this congruence, and this congruence is the smallest that makes $\mathfrak{F}_\uparrow$ a functor. Notice moreover that this congruence is contained into the equivalence defined from the contracting functor $\mathfrak{C}$, i.e. the equivalence defined by $\mathscr{T} \sim \mathscr{T}'$ if and only if $\mathfrak{C}(\mathscr{T}) = \mathfrak{C}(\mathscr{T}')$. This implies that the functor $\mathfrak{C}$ is well-defined on the quotient



(a) Relation R (b) Relation $\mathfrak{F}(R)$ (c) Relation Q

(d) Relation $\mathfrak{F}(Q)$

Figure 2: The functor $\mathfrak{F}$ on two examples



(a) Triskell $\mathscr{T}$ (b) Triskell $\mathscr{T}'$ (c) Triskell $\mathfrak{F}_\uparrow(\mathscr{T}' \circ \mathscr{T})$ (d) Triskell $\mathfrak{F}_\uparrow(\mathscr{T}') \circ \mathfrak{F}_\uparrow(\mathscr{T})$

Figure 3: $\mathfrak{F}_\uparrow$ is not a functor

category $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})^0$, as two equivalent triskells in $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})$ have the same image through $\mathfrak{C}$. Lastly, one easily checks that this quotient is compatible with the monoidal structure, i.e. $(\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})^0, \otimes, 1)$ is a monoidal category.

Theorem 23. *The functor $\mathfrak{F}_\uparrow$ is a strict monoidal functor from $(\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et}), \oplus, \emptyset)$ to $(\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})^0, \otimes, 1)$, and $\mathfrak{F} \circ \mathfrak{C} = \mathfrak{C} \circ \mathfrak{F}_\uparrow$.*

Proof (sketch). We just show the monoidality on morphisms, as the rest is quite clear[3].

On morphisms, we consider an edge e of source and target $\bar{a}$ and $\bar{b}$ respectively. Then there is such an edge e is in $\mathfrak{F}_\uparrow(R \oplus R')$ if and only if there exists a permutation σ such that $a_i = b_{\sigma(i)}$. It is here clear that σ is the union of two permutations τ and ρ such that for all i in the domain of τ (resp. of ρ) a_i is an element of the source of R (resp. of R') and $b_{\sigma(i)}$ is an element of the target of R (resp. of R'). Thus we can rewrite $\bar{a}$ as $(\bar{m}_1, \bar{m}_2)$ and $\bar{b}$ as $(\bar{n}_1, \bar{n}_2)$. Thus, there is such an edge e in $\mathfrak{F}_\uparrow(R \oplus R')$ if and only if there exists two edges e_1 and e_2 in $\mathfrak{F}_\uparrow(R)$ and $\mathfrak{F}_\uparrow(R')$ respectively such that $s(e_i) = \bar{m}_i$, $t(e_i) = \bar{n}_i$, i.e. if and only if there is an edge in $\mathfrak{F}_\uparrow(R) \times \mathfrak{F}_\uparrow(R')$ with source and targets $\bar{a}$ and $\bar{b}$ respectively. □

A consequence of Theorem 20 is the fact that a *cut-free* MLL proof's interpretation in the dynamic models is mapped, through $\mathfrak{F} \circ \mathfrak{C}$, to the interpretation of the same proof in the relational model, modulo the fact that we are working with a continuous semiring (or a ring when considering only finite sets as objects). More importantly, this last theorem 23 extends this result when one considers triskells instead relations. I.e. a cut-free MLL proof's interpretation in the interaction graphs model for a monoid Ω is mapped through $\mathfrak{F}_\uparrow$ to the interpretation of the same proof in the triskells denotational model for the monoid $\{-1, 1\} \times \Omega$ (modulo the simple quotient considered above). We will now show how this result actually extends to the double-glueing constructions, and how this extension is not restricted to cut-free proofs as the above functor lifts to a closed functor between the induced *-autonomous categories.

[3] To show functoriality, we notice that the residue which appears in the proof of Theorem 20 is cancelled by the congruence, i.e. the triskell consisting in exactly all edges contributing to the residue is a "zero triskell".

5. Orthogonalities

5.1 Some preliminary remarks

In this section, we start discussing the image of the double-glueing construction considered in Interaction Graphs. First, we limit our discussion to the subcategory of $\mathbb{R}\mathrm{el}_\Omega(\mathbb{S}\mathrm{et}_{\aleph_1})$ where all objects are finite sets. A further restriction will be that Ω is chosen as a subring of the complex numbers $\mathbf{C}$. These constraints are quite strong but they were introduced by the author [17] to obtain a combinatorial version of Girard's hyperfinite geometry of interaction [9].

In this particular case, we can define in the category of relations both the (normalised) trace – i.e. the mean of diagonal coefficients – and the determinant of matrices. We now recall the following well-known theorem from linear algebra.

Theorem 24. *Let A be a matrix with complex coefficients, then $\mathrm{tr}(\mathfrak{F}(A)) = \det(1 + A)$.*

We now fix the subring $[0,1]$ of the complex numbers as our weight object Ω, and we restrict to matrices with operator norm at most 1. This second restriction implies that the operations interpreting multiplicative linear logic will not create weights greater than 1 [17], i.e. the fact that $[0,1]$ is not a subring (i.e. not closed under addition) will not induce incoherences in the model.[4] These restrictions still provide an adequate framework for defining the combinatorial variant of Girard's hyperfinite model [9, 17] and probabilistic coherent spaces.

Girard's hyperfinite GoI uses the Fuglede-Kadison determinant $\det^{\mathrm{FK}}$ [5], a (positive-)real-valued determinant which corresponds, in finite dimensions, to the absolute value of the usual determinant (to the power $1/k$). This determinant is used to define the orthogonality considered in Girard's GoI models, i.e. two matrices A,B are (det-)orthogonal, noted $\perp_{\mathrm{goi}}$, when $\det^{\mathrm{FK}}(1-AB)\neq 0,1$, i.e. when $|\det(1-AB)|\neq 0,1$. The trace, on the other hand, can be used to define (a variant of) the orthogonality considered in probabilistic/quantum coherence spaces, i.e. two matrices A,B are (trace-)orthogonal, denoted $\perp_{\mathrm{coh}}$, when $\mathrm{tr}(AB)\in]0,1[$.[5]

The theorem above shows that $\det^{\mathrm{FK}}(1-AB) = |\mathrm{tr}(\mathfrak{F}(AB))|$, i.e. when we restrict to norm 1 matrices with coefficients in $[0,1]$:

$$\begin{aligned} A\perp_{\mathrm{goi}} B &\iff \det{}^{\mathrm{FK}}(1-AB)\in]0,1[\\ &\iff |\mathrm{tr}(\mathfrak{F}(A)\mathfrak{F}(B))|\in]0,1[\\ &\iff \mathfrak{F}(A)\perp_{\mathrm{coh}}\mathfrak{F}(B) \end{aligned}$$

This implies that the interaction graph model detailed in the author's first work [17] is mapped to the model of *strict* coherent spaces through the functor $\mathfrak{F}$. This result, while interesting on its own, can be generalised if one works with triskells instead of relations. The reason triskells provide a better framework is that we will be able to get rid of the restrictions we imposed in this section. Indeed, we asked for objects to be restricted to finite sets: the reason for this is that dealing with infinite sets implies dealing with infinite sums. Thus infinite sets force us out of the usual domain of definition of matrix determinants. The second restriction, i.e. that the weight object is a subring of the complex numbers, is also linked to the definition of the determinant. Indeed, the definition of a determinant for matrices over arbitrary semirings is quite involved and has no clear solution [23]. Lastly, the condition on the norm comes from the very definition of Girard's model: execution cannot be defined in general when the norm is greater than 1 [19].

We will now show how one can define a much more general and satisfying correspondence when dealing with triskells. Such a generalisation will provide many more models since there are fewer restrictions on the weight object Ω. This is done by introducing notions of determinant and trace of a triskell that generalise the usual definition for complex matrices. We ask but one thing about these extensions, namely that they map the determinant to the trace through $\mathfrak{F}_{\uparrow}$. That is, we ensure that a generalisation of Theorem 24 holds.

[4] The norm restriction allows to bypass the ring structure as it insures that the operations interpreting linear logic connectives will not create triskells (or graphs) with weights outside $[0,1]$. This is clear for the disjoint union used to interpret the tensor product, it is proved for the trace in earlier works [17].

[5] The original definition allows for the values 0 and 1 which are here forbidden. This slight change however do not impact the result, i.e. we still define from this orthogonality a model of multiplicative linear logic.

5.2 Determinant and trace for triskells

The remark above motivates a definition of determinant and trace in the categories of triskells. This can be done in a coherent way, namely such that the equivalent of Theorem 24 holds. Note also that the trace satisfies the property that $\mathrm{tr}_m(AB)=\mathrm{tr}_m(BA)$, and will be used to define a notion of quantitative coherent spaces in $\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et})$.

Let $(\mathrm{K},+)$ be a complete monoid – i.e. a monoid with a notion of infinite sums –, let Ω be a signed monoid, and $m:\Omega\rightarrow\mathrm{K}$ be any map, i.e. not necessarily a monoid (or equivalently group) homomorphism. We define the m-trace and the m-determinant of any triskell with same source and target objects the finite set A as

$$\begin{aligned} \mathrm{tr}_m(\mathscr{T}) &= \sum_{e\in E, s(e)=t(e)} m(w(e)) \\ \det{}_m(\mathscr{T}) &= \sum_{\sigma\in\mathfrak{S}_A}\ \sum_{(e_i)\in\prod_{i\in A}E[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i)) \end{aligned}$$

In the last expression, $\mathfrak{S}_A$ should be understood as the set of permutation over A, and for all $x,y\in A$ we denote by $E[x,y]$ the subset of edges e such that $s(e)=x$ and $t(e)=y$.

Remark 25. If one keeps to the subcategory where objects are finite sets, then completeness of K is not necessary and any monoid would do. As a particular case, the identity map on complex numbers allows one to recover the usual determinant and trace of matrices.

Although the trace defined as it is above can be used to deal with matrices over infinite sets, the determinant considered there cannot, as one needs to define infinite products for the expression to make sense. Given a monoid Ω, we define the extended monoid Ω^0 by adding an absorbing element 0, i.e. $0.a=a.0=0$ for all $a\in\Omega^0$. One can then define, in this extended monoid, a satisfying notion of infinite products, namely $\prod_{i\geqslant 0}a_i$ is defined equal to $\prod_{i=0}^{k}a_i$ if for all $j>k$ we have $a_j=1$ and equal to 0 otherwise. By implicitely considering elements of Ω as their image in Ω^0 and extending m to Ω^0 by defining $m(0)$ as being equal to the unit of the sum in K, the expression above for determinant makes sense for arbitrary triskells. With this definition of determinant and trace, we obtain the following theorem.

Theorem 26. *For any triskell $\mathscr{A}$ with same source and target sets,* $\det_m(1+\mathscr{A})=\mathrm{tr}_m(\mathfrak{F}_{\uparrow}(\mathscr{A}))$.

Proof. The proof for the usual matrix identity is shown by the following computation:

$$\begin{aligned} \det(1+A) &= \sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)\prod_{i=1}^{n}(a_{\sigma(i),i}+\delta_{\sigma(i),i}) \\ &= \sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)\prod_{i\in\mathrm{Fix}(\sigma)}(a_{i,i}+1)\prod_{i\not\in\mathrm{Fix}(\sigma)}a_{\sigma(i),i} \\ &= \sum_{\sigma\in\mathfrak{S}_n}\ \sum_{\mathrm{Fix}(\sigma)\subset K\subset\{1,\dots,n\}}\epsilon(\sigma)\prod_{i\in K}a_{i,i}\prod_{i\not\in\mathrm{Fix}(\sigma)}a_{\sigma(i),i} \\ &= \sum_{K\subset\{1,\dots,n\}}\ \sum_{\sigma\in\mathfrak{S}_K}\epsilon(\sigma)\prod_{i\in K-\mathrm{Fix}(\sigma)}a_{i,i}\prod_{i\not\in\mathrm{Fix}(\sigma)}a_{\sigma(i),i} \\ &= \sum_{K\subset\{1,\dots,n\}}\ \sum_{\sigma\in\mathfrak{S}_K}\epsilon(\sigma)\prod_{i\in K}a_{\sigma(i),i} \\ &= \sum_{K\subset\{1,\dots,n\}}\det(A_K) \end{aligned}$$

The adaptation to the triskells setting is the following computation. We fix a triskell A of source and target objects X. We recall that $\mathfrak{S}_X$ denotes the set of permutations of X. We denote by

Fix(σ) the subset of the support consisting of elements x such that $\sigma(x) = x$ and by supp(σ) the set of $x \in X$ such that $\sigma(x) \neq x$. For any two elements $x, y \in X$, we denote by $A[x,y]$ the set of edges e in A such that $s(e) = x$ and $t(e) = y$. We then follow the computation shown in Figure 4. When considering only finite sets as objects, this last line of the computation in Figure 4 can be rewritten as $\sum_{K \in \mathrm{P}_{\mathrm{fin}}(X)} \det_m(A_K)$.
Now, the trace of $\mathfrak{F}(A)$ is equal to

$$\mathrm{tr}_m(\mathfrak{F}(A)) = \sum_{\bar{a} \in \mathrm{P}_{\mathrm{fin}}(A)} \sum_{\sigma \in S_{\mathcal{T}}[\bar{a},\bar{a}]} m(\epsilon(\sigma)\omega_\sigma[\bar{a},\bar{a}](\sigma))$$

which becomes $\sum_{K \in \mathrm{P}_{\mathrm{fin}}(X)} \det_m(A_K)$ (Figure 5). □

6. Quantitative coherence spaces

6.1 The models

We will describe here a notion of *quantitative coherence spaces*. Although coherence spaces and their probabilistic and quantum variations have been defined as a double-glueing construction on the category of (weighted) relations, we here describe them as a double-glueing construction applied to the category of triskells. This change in the underlying category allows for a wider array of definable models, as the category of weighted relations uses a complete semiring to represent quantitative features, while triskells need only the simpler structure of a monoid. We stress that this double-glueing on triskells gives rise to a double-glueing construction on weighted relations through the contraction functor $\mathfrak{C}$.

Definition 27. Given a complete monoid $(\mathrm{K}, +)$, a map $m : \Omega \to \mathrm{K}$ and a subset $\perp\!\!\!\perp$ of K, we define a binary relation – the orthogonality – on triskells by $\mathcal{T} \perp \mathcal{T}'$ if and only if $\mathrm{tr}_m(\mathcal{T} \circ \mathcal{T}') \in \perp\!\!\!\perp$. We define the orthogonal of a set of triskells A as the set $A^{\perp} = \{\mathcal{T} \mid \forall \mathcal{A} \in A, \mathcal{T} \perp \mathcal{A}\}$.

Definition 28. A quantitative coherence space (QCS) A is a pair $(X^A, \mathbf{A})$ of a set X^A and a bi-orthogonally closed set $\mathbf{A}$ of triskells over X^A (i.e. of source and target X^A), i.e. $\mathbf{A} = (\mathbf{A}^{\perp})^{\perp}$.

Definition 29. Let $A = (X^A, \mathbf{A})$ and $B = (X^B, \mathbf{B})$ be QCS. Their tensor product is defined as

$$A \otimes B = (X^A \times X^B, \{\mathcal{T} \times \mathcal{T}', \mathcal{T} \in \mathbf{A}, \mathcal{T}' \in \mathbf{B}\}^{\perp\perp})$$

As it is usual with this way of representing coherence spaces, no nice representation of the dual connective $\parr$ can be obtained. The linear implication, however, is nicely characterised.

Definition 30. Let $\mathcal{F}$ be a triskell over $X \times Y$, and $\mathcal{A}$ be a triskell over X. We define the application $[\mathcal{F}]\mathcal{A}$ as the triskell $(E, Y, Y, \Omega, s, t, \omega)$, where:

$$E = \{(f, a) \mid x, x' \in X, y, y' \in Y, f \in E^F[(x,y),(x',y')], a \in E^A[x,x']\}$$

$$s(f,a) = \pi_Y(s(f)) \qquad t(f,a) = \pi_Y(t(f)) \qquad \omega(f,a) = \omega(f)\omega(a)$$

Definition 31. If $A = (X^A, \mathbf{A})$ and $B = (X^B, \mathbf{B})$ are QCS, the linear arrow is defined as follows:

$$A \multimap B = (X^A \times X^B, \{\mathcal{F} \in \mathbf{Tk}^{\Omega}(\mathrm{A} \times \mathrm{B}, \mathrm{A} \times \mathrm{B}) \mid \forall \mathcal{A} \in \mathbf{A}, [\mathcal{F}]\mathcal{A} \in \mathbf{B}\})$$

Theorem 32. *For any m and $\perp\!\!\!\perp$, this defines a model of* MLL.

Proof. We show that the category and constructions we defined have the structure of a $*$-autonomous category. One easily checks that the tensor is commutative and associative, and has as unit the QCS $1 = (\{*\}, \perp\!\!\!\perp^{\perp})$. The proof that $A \multimap B = (A \otimes B^{\perp})^{\perp}$ is a direct consequence of the property that $\mathrm{tr}_m([\mathcal{F}]\mathcal{A} \circ \mathcal{B}) = \mathrm{tr}_m(\mathcal{F} \circ (\mathcal{A} \times \mathcal{B}))$. Lastly, the fact that $1^{\perp}$ is dualizing is immediate: the natural morphism from a QCS $A = (X, \mathbf{A})$ to $(A \multimap \perp) \multimap \perp$ is the following triskell which is clearly an isomorphism: $(X \times \{*\}, X, X \times \{*\}^2, \pi_X \circ \mathrm{Id}_{X \times \{*\}}, [\mathrm{Id}_{X \times \{*\}}, \mathrm{Id}_*], \mathrm{u}_{\times\{*\}})$ □

6.2 Relations to previous models

These models generalise previously introduced coherence spaces models. To see this, we will need some additional definitions. Say a triskell $\mathcal{T} = (\mathrm{E}, \mathrm{S}, \mathrm{T}, \Omega, s, t, w)$ is *diagonal* when $\mathrm{S} = \mathrm{T}$ and for all $e \in \mathrm{E}$, $s(e) = t(e)$. Restricting morphisms to diagonal (resp. hermitian) triskells defines a subcategory of the category of triskells. Restricting the definitions of QCS just considered to this subcategory leads to the usual notions of coherence spaces and probabilistic coherence spaces (in the style of Girard [8]). Restricting in the same manner to the straightforward notion of *hermitian* triskells[6] leads to Girard's quantum coherence spaces model.

Proposition 33. *The restriction of* QCS *to* simple diagonal triskells *(and finite objects) yields usual coherence spaces – when $\Omega = \{1\}$ – and probabilistic coherence spaces – when $\Omega = \mathbf{R}_{\geqslant 0}$ – (for* MLL). Restriction to *simple hermitian triskells* (and finite objects) leads to quantum coherence spaces (for MLL).

Proof. The proof is straightforward and more or less the same in all three cases. First, notice that the restriction to simple diagonal (resp. simple hermitian) triskells over finite sets boils allows us to consider our objects as diagonal (resp. hermitian) matrices (over finite-dimensional Hilbert spaces) with weights in $\{0,1\}$ or $\mathbf{R}_{\geqslant 0}$ (resp. weights in $\mathbf{C}$). It is a simple check that the tensor product defined on triskells corresponds to the usual tensor product of matrices. Moreover, the choice of $m(x) = x$ and $\perp\!\!\!\perp = \mathbf{R}_{\geqslant 0}$ makes the orthogonality definable directly in terms of matrices as $A \perp B$ if and only if $0 \leqslant \mathrm{tr}(AB) \leqslant 1$. The correspondence between the quantitative coherence space models and Girard's models are then either direct (in the quantum case) – the definitions are exactly the same – or obtained by identifying $\mathbf{R}_{\geqslant 0}$-valued (resp. $\{0,1\}$-valued) diagonal matrices over a space of dimension k with functions from $\{1, 2, \dots, k\}$ to $\mathbf{R}_{\geqslant 0}$ (resp. with subsets of $\{1, 2, \dots, k\}$). □

Now, probabilistic coherence spaces were generalised to deal with infinite objects by Danos and Ehrhard; this allowed them to define exponentials, something unavailable in Girard's construction. It turns out that their generalisation is also a special case of our construction. To formalise this, we consider the subcategory of diagonal triskells and the QCS one can define in this category. To simplify things, first notice that it is not necessary to consider $\Omega = \mathbf{R}_{\geqslant 0}$: since we are allowing multiple edges with same source and target, all $\mathbf{R}_{\geqslant 0}$-weighted (simple) triskells can be obtained from $[0,1]$-weighted triskells through the collapse functor. Secondly, Danos and Ehrhard require two additional conditions that can be translated in our setting by considering the following definition. We will use here *(weighted) partial identities* to express it, i.e. if $B \subset X^A$, the partial identity on B weighted by $\lambda \in \Omega$ is the triskell $(B, X^A, X^A, \Omega, \iota, \iota, \lambda \mathrm{u_B})$ where ι denotes the inclusion of B in X^A and $\lambda \mathrm{u_B}$ is the constant function equal to λ (definable through the "unit" map $\mathrm{u_B}$).

Definition 34. We say that a QCS A is *everywhere bounded below* if for every $x \in X^A$ there is a $\lambda > 0$ such that the partial identity of x weighted by λ lies in $\mathbf{A}$. If both A and $A^{\perp}$ are everywhere bounded below, we say that A is *bounded*.

[6] A triskell $\mathcal{T} = (\mathrm{E}, \mathrm{S}, \mathrm{T}, \Omega, s, t, w)$ is hermitian if $\Omega = \mathbf{C}$, $\mathrm{E} = \mathrm{F} \uplus \mathrm{G}$, and $\exists \phi : \mathrm{F} \to \mathrm{G}$ bijective s.t. $w(\phi(e)) = \bar{w}(e)$.

$$
\begin{aligned}
\det_m(1+A) &= \sum_{\sigma\in\mathfrak{G}_X} \sum_{(e_1,\dots,e_n)\in\prod_{i\in X}(1+A)[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i))\\
&= \sum_{\sigma\in\mathfrak{G}_X} \sum_{(e_1,\dots,e_n)\in\prod_{i\in \mathrm{Fix}(\sigma)}(1+A)[\sigma(i),i]\prod_{i\in\mathrm{supp}(\sigma)}A[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i))\\
&= \sum_{\sigma\in\mathfrak{G}_X} \sum_{\mathrm{supp}(\sigma)\subset K\subset X} \sum_{(e_1,\dots,e_{\mathrm{Card}(K)})\in\prod_{i\in K}A[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i))\\
&= \sum_{K\subset X} \sum_{\sigma\in\mathfrak{G}_K} \sum_{(e_1,\dots,e_{\mathrm{Card}(K)})\in\prod_{i\in K}A[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i))\\
&= \sum_{K\subset X} \det_m(A_K)
\end{aligned}
$$

Figure 4: Proof of Theorem 26, first computation

$$
\begin{aligned}
\mathrm{tr}_m(\mathfrak{F}(A)) &= \sum_{K\in \mathrm{P}_{\mathrm{fin}}(X)} \sum_{\sigma\in S_{\mathscr{T}}[K,K]} m(\epsilon(\sigma)\omega_\sigma[K,K](\sigma))\\
&= \sum_{K\in \mathrm{P}_{\mathrm{fin}}(X)} \sum_{\sigma\in S_{\mathscr{T}}[K,K]} m(\epsilon(\sigma)\prod_i \omega_i)\\
&= \sum_{K\in \mathrm{P}_{\mathrm{fin}}(X)} \sum_{\sigma\in\mathfrak{G}_K} \sum_{(e_1,\dots,e_{\mathrm{Card}(K)})\in\prod_{i\in K}A[\sigma(i),i]} m(\epsilon(\sigma)\prod w(e_i))\\
&= \sum_{K\in \mathrm{P}_{\mathrm{fin}}(X)} \det_m(A_K)
\end{aligned}
$$

Figure 5: Proof of Theorem 26, second computation

Remark 35. This restriction to specific bi-orthogonally closed sets is reminiscent of a similar restriction in the IG models [20]. It is not clear if the two restrictions are related, but they are strikingly similar at a first glance. A more detailed study of these aspects will have to be relegated to later work, as it is intimately related to the interpretation of additives.

The constructions for multiplicative connectives in Danos and Ehrhard paper are then the same as the one considered here. The essential point is that the tensor product of bounded QCS is again a bounded coherence space. This leads to the following proposition.

Proposition 36. *The restriction to* bounded QCS *in the subcategory of* diagonal triskells *(when $\Omega = [0,1]$) yields Danos and Ehrhard probabilistic coherence spaces model (for* MLL*).*

Proof. The proof follows the proof above. Through the contraction functor $\mathfrak{C}$, a $[0,1]$-weighted diagonal triskell on a set X^A yields a $\mathbf{R}_{\geqslant 0}\cup\{\infty\}$-weighted diagonal matrix, which can be identified with a function from X^A to $\mathbf{R}_{\geqslant 0}\cup\{\infty\}$. The use of $m(x) = x$ from $\mathbf{R}_{\geqslant 0}$ to $\mathbf{R}_{\geqslant 0}\cup\{\infty\}$ and $\perp\!\!\!\perp = [0,1]$ translates as before the orthogonality as a trace computation on the product. This boils down to the following orthogonality between functions from X^A to $\mathbf{R}_{\geqslant 0}$: $f \perp g$ if and only if $0 \leqslant \sum_{x\in X^A} f(x)g(x) \leqslant 1$. To this point, we are in the same case as in Proposition 33, except that we allow for infinite coefficients: this is not a problem when we consider bounded QCS since a function taking an infinite value on a point – say x – cannot belong to a bounded QCS as it cannot be orthogonal to (weighted) partial identities on x. Moreover, the fact that a bounded QCS (resp. the orthogonal of a bounded QCS) is everywhere bounded below corresponds exactly to Danos and Ehrhard second condition (resp. third condition) in the definition of probabilistic coherence spaces: to understand that our definition is just a reformulation of theirs, one just need to notice that Danos and Ehrhard second and third conditions are dual of each other w.r.t. the orthogonality expressed above. ☐

We have shown how QCS can be related to previous coherence space semantics. We will now explain how it relates to the author's Interaction Graphs (IG) models. We first prove a number of properties for the introduced notions of determinant and trace, which will then lead us to Theorem 43 stating a formal link between the two families of models.

Proposition 37. *If $(\mathrm{K},\cdot,+)$ is a semiring in which infinite sums are invariant w.r.t. every permutation of the index set (not only finite ones) and m is a monoid morphism from Ω to $(\mathrm{K},\cdot)$, i.e. $m(a.b) = m(a).m(b)$, then tr_m is linear, i.e. $\mathrm{tr}(T \uplus T') = \mathrm{tr}(T) + \mathrm{tr}(T')$ and $\mathrm{tr}(a.T) = m(a)\mathrm{tr}(T)$.*

Proof. The proof is a simple computation. If we denote by E and E' the respective sets of edges of the triskells $\mathscr{T}$ and $\mathscr{T}'$, then using the first hypothesis (invariance w.r.t. all permutations) we obtain:

$$
\begin{aligned}
\mathrm{tr}_m(\mathscr{T}\cup\mathscr{T}') &= \sum_{e\in E+E', s(e)=t(e)} m(w(e))\\
&= \sum_{e\in E, s(e)=t(e)} m(w(e)) + \sum_{e'\in E', s(e')=t(e')} m(w(e'))\\
&= \mathrm{tr}_m(\mathscr{T}) + \mathrm{tr}_m(\mathscr{T}')
\end{aligned}
$$

The second part is also direct. If $\mathscr{T} = (E,S,T,\Omega,s,t,w)$ and $a.\mathscr{T}$ denotes the triskell $(E,S,T,\Omega,s,t,x\mapsto a.w(x))$, then using the second hypothesis (that m is a monoid morphism):

$$\begin{aligned}\mathrm{tr}_m(a.\mathscr{T}) &= \sum_{e\in E,s(e)=t(e)} m(a.w(e)) \\ &= \sum_{e\in E,s(e)=t(e)} m(a).m(w(e)) \\ &= m(a).\sum_{e\in E,s(e)=t(e)} m(w(e)) \\ &= m(a).\mathrm{tr}_m(\mathscr{T})\end{aligned}$$

□

We can now relate these notions of trace and determinant to the measurement introduced in earlier work on interaction graphs [20], and denoted by $[\![\cdot,\cdot]\!]_m$. We consider here triskells over finite sets, and the real numbers as the semiring K. Notice the latter is not complete, but completeness is needed only for dealing with triskells over infinite sets.

Proposition 38. *Suppose* $\mathrm{K} = \mathbf{R}$ *and that* m *is a monoid morphism such that* $m(-1) = -1$. *Suppose moreover that there is a family* $a_k \in \Omega, k \in \mathbf{N}$ *with* $m(a_k) = 1/k$ *and that* A, B *are triskells over finite sets. Writing* $\hat{m}(x) = -\log(1-m(x))$, *we have*[7]*:*

$$-\log(\det_m(1-AB)) = \mathrm{tr}_m(\sum a_k(AB)^k) = [\![A,B]\!]_{\hat{m}}$$

Proof. Recall that we consider triskells over finite sets. Since m is a signed monoid morphism from Ω into $\mathbf{R}$ considered with the multiplication, $m(-1) = -1$. Then $\det_m(A)$ relates to the usual determinant of matrices as follows: $\det_m(A) = \det(m(A))$. Moreover, it should be clear that $m(1-A) = 1-m(A)$ since $1-A$ stands for the sum of the identity triskell and the triskell A in which the weight map has been composed with the map $x \mapsto -x$. Hence $\det_m(1-A) = \det(1-m(A))$. Since the latter expression is the usual determinant of a real (hence complex) matrix, one can use the well-known equality $\log(\det(A)) = \mathrm{tr}(\log(A))$ to obtain that $-\log(\det_m(1-A)) = \mathrm{tr}(-\log(1-m(A)))$. By definition $-\log(1-m(A)) = \sum_k \frac{m(A)^k}{k}$. Thus, by the linearity of the trace and the fact that $m(A)^k = m(A^k)$ (since m is a monoid morphism), $-\log(\det_m(1-A)) = \sum_k \frac{1}{k}\mathrm{tr}(m(A^k))$. Now, it is quite clear that $\mathrm{tr}_m(A) = \mathrm{tr}(m(A))$ in the particular case we consider. Thus $-\log(\det_m(1-A)) = \sum_k \frac{1}{k}\mathrm{tr}_m(A^k)$, and we conclude by the linearity of the m-trace, which states that $\sum_k \frac{1}{k}\mathrm{tr}_m(A^k) = \mathrm{tr}_m(\sum_k a_k A^k)$.

We first give some definitions. Following previous work [17, 20], we call a *circuit* π the equivalence class of cycles – paths with their source equal to their target – up to cyclic permutation. I.e. a circuit π thus represents the geometric notion of "cycle", in which the starting point is meaningless: whereas the syntactic notion of cycle in a graph will consider the two sequences a,b and b,a as two different cycles (when a,b is indeed a cycle), they correspond to a unique circuit. A 1-circuit π is then a circuit which is not a proper power (for the concatenation operation) of a smaller circuit, i.e. there is no circuit ρ and integer $k > 1$ such that $\pi = \rho^k$. We recall that $[\![A,B]\!]_m$ is defined as $\sum_{\pi \text{ 1-circuit}} m(\omega(\pi))$, where $\omega(\pi)$ is the weight of π. We then have the computation shown in Figure 6, which can be found (except for the first equality) in the author's earlier work [17].

□

In other words, the orthogonality defined in interaction graphs by $A \perp B$ iff $[\![A,B]\!]_{\hat{m}} \in \perp\!\!\!\perp$ and the orthogonality of QCS

$$\begin{aligned}\mathrm{tr}_m(\sum_k a_k A^k) &= \sum_k \frac{1}{k}\mathrm{tr}_m(A^k) \\ &= \sum_k \frac{1}{k}\sum_{\pi \text{ cycle},\mathrm{lg}(\pi)=k} \frac{m(w(\pi))}{\mathrm{pow}(\pi)} \\ &= \sum_k \sum_{\pi \text{ circuit},\mathrm{lg}(\pi)=k} \frac{m(w(\pi))}{\mathrm{pow}(\pi)} \\ &= \sum_{\pi \text{ circuit}} \frac{m(w(\pi))}{\mathrm{pow}(\pi)} \\ &= \sum_{\pi \text{ 1-circuit}} \sum_k \frac{m(w(\pi^k))}{k} \\ &= \sum_{\pi \text{ 1-circuit}} \sum_k \frac{m(w(\pi)^k)}{k} \\ &= \sum_{\pi \text{ 1-circuit}} \sum_k \frac{m(w(\pi))^k}{k} \\ &= \sum_{\pi \text{ 1-circuit}} -\log(1-m(w(\pi))) \\ &= [\![A]\!]_{-\log(1-m)}\end{aligned}$$

Figure 6: Computation for the proof of Proposition 38

are mapped one to the other when we restrict to the hypotheses of the theorem. This will yield to a theorem formally relating IG models to QCS models. Before stating and proving this result, we need to discuss *observational equivalence*. We mimic here the definition of the observational equivalence considered in IG models [20]. As it is the case in this original setting, quotienting w.r.t. this equivalence preserves the $*$-autonomous structure of the initial category.

Definition 39. Let $A = (X^A, \mathbf{A})$ be a QCS, and $\mathscr{A}, \mathscr{B}$ be triskells in $\mathbf{A}$. We say $\mathscr{A}, \mathscr{B}$ are *observationally equivalent*, noted $\mathscr{A} \sim_A \mathscr{B}$, when for all triskell $\mathscr{T} \in \mathbf{A}^{\perp}$, $\mathrm{tr}_m(\mathscr{A}\mathscr{T}) = \mathrm{tr}_m(\mathscr{B}\mathscr{T})$.

The proof of the following lemma is then similar to the proof of the equivalent propositions for Interaction Graphs [20, Proposition 86, Corollaries 87 and 88].

Lemma 40. *The observational equivalence is a congruence on the category of* QCS.

Definition 41 (The quotiented QCS models)**.** The quotiented QCS model is then defined as follows: objects are QCS while morphisms are equivalence classes of triskells w.r.t. observational equivalence, i.e. the morphisms from A to B is the set of equivalence classes w.r.t. $\sim_{A\multimap B}$ in the QCS $A \multimap B$.

Proposition 42. *The quotiented* QCS *model is a* $*$*-autonomous category.*

Proof. The identity $\mathrm{tr}_m([\mathscr{F}]\mathscr{A} \circ \mathscr{B}) = \mathrm{tr}_m(\mathscr{F} \circ (\mathscr{A} \times \mathscr{B}))$ is used to show that the observational equivalence is a "monoidal" congruence, i.e. if $\mathscr{A} \sim_A \mathscr{A}'$ then $\mathscr{A} \times \mathscr{B} \sim_{A\otimes B} \mathscr{A}' \times \mathscr{B}$. The quotiented QCS category thus inherits the monoidal structure of the initial QCS category. To finalise the proof, we follow the proof of the similar proposition for IG models [20, Proposition 91]. We can use the same arguments to verify that the congruence is compatible with the isomorphism between $\mathrm{Hom}_{\mathbb{C}}(\mathrm{A}, \mathrm{B} \multimap \mathrm{C})$ and $\mathrm{Hom}_{\mathbb{C}}(\mathrm{A} \otimes \mathrm{B}, \mathrm{C})$, and that $\perp$ is dualising. □

We can now state the main theorem of this section. Although the trace on triskells does not map to the composition through

[7] We identify in this statement triskells and weighted directed graphs (the measurement being defined [20] on this type of graphs).

the (lifted) Fock functor, this mismatch is rubbed out by the observational equivalence. Therefore, the lifted Fock functor lifts to a closed functor from IG models to quotiented QCS models.

Theorem 43. *The lifted Fock functor maps the* IG *model (for* MLL) to the quotiented QCS models (for MLL) defined on its image.

Proof. We recall that IG models interpret proofs (and therefore defines morphisms) as pairs (a,A) where a is a real number and A is a triskell (well, a directed weighted graph in the original papers [17, 20]). To take the additional real number, we define the map from IG to QCS as follows.

$$\Psi : (a,A) \mapsto \exp(a)\mathfrak{F}_{\uparrow}(A)$$

The following computation shows that $[\Psi(f,F)]\Psi(a,A)$ is observationally equivalent to $\mathfrak{F}_{\uparrow}\mathrm{Tr}_{X^A}(\Psi(f,F)\circ\Psi(a,A))$.

$$\begin{aligned}
&\mathrm{tr}_m([\Psi(f,F)]\Psi(a,A)\circ\Psi(b,B)) \\
&\quad= \mathrm{tr}_m(\Psi(f,F)\circ\Psi(a,A)\times\Psi(b,B)) \\
&\quad= -\log(\det_m(1-\Psi(f,F)\circ(\Psi(a,A)\oplus\Psi(b,B))) \\
&\quad= -\log(\det_m(1-\mathrm{Tr}_{X^A}(\Psi(f,F)\circ\Psi(a,A))\circ\Psi(b,B))) \\
&\quad= \mathrm{tr}_m(\mathfrak{F}_{\uparrow}\mathrm{Tr}_{X^A}(\Psi(f,F)\circ\Psi(a,A))\circ\mathfrak{F}_{\uparrow}\Psi(b,B))
\end{aligned}$$

As a consequence, the map Ψ is a functor from the IG model to the QCS model defined on the image of Ψ. Moreover, this functor is strong monoidal as a consequence of $\mathfrak{F}_{\uparrow}$ being strong monoidal (Theorem 23). Thus, it is a closed functor since both IG and QCS are *-autonomous categories, hence monoidal closed categories. □

6.3 Beyond multiplicatives

It is natural to wonder about the generalisation of QCS to larger fragments of linear logic. First, the model extends quite straightforwardly to additives.

Definition 44. Given QCS $A = (X^A, \mathbf{A})$ and $B = (X^B, \mathbf{B})$, we define

$$A \,\&\, B = (X^A \uplus X^B, \{\mathscr{A} + \mathscr{B} \mid \mathscr{A} \in \mathbf{A}, \mathscr{B} \in \mathbf{B}\}^{\perp\perp})$$

As usual, the dual $(A^{\perp} \,\&\, B^{\perp})^{\perp}$ is denoted $A \oplus B$. With this definition, Proposition 33 and Proposition 36 are easily extended to the MALL fragment. However, the most interesting question concerns the extension to exponential connectives. It turns out that one can define exponentials using the *symmetric Fock functor*.

Definition 45. We define $\mathfrak{S}$ as follows. On objects, $\mathfrak{S}$ acts as the finite multisets $\mathrm{M_{fin}}(\cdot)$ functor. On morphisms, the functor maps the triskell T to the triskell $\mathfrak{S}(T)$: for each $(\sigma, \vec{e})$ in $M_{\mathscr{T}}[\bar{a},\bar{b}]$, there is exactly one edge in $\mathfrak{S}(T)$ of source $\bar{a}$, target $\bar{b}$ and weight $\omega_\sigma[\bar{a},\bar{b}](\vec{e})$.

Proposition 46. *The functor* $\mathfrak{S}$ *is strict monoidal from* $(\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et}),\oplus,\emptyset)$ *to* $(\mathbb{T}\mathrm{k}^{\Omega}_{\simeq}(\mathbb{S}\mathrm{et}),\otimes,1)$.

Proof. As for the antisymmetric case, we first show it is functorial. This is easier in this case. There exists in $\mathfrak{S}(\mathscr{T}\circ\mathscr{T}')$ an edge e of source $\bar{a} = (a_i)_{i=1}^{k}$ and target $\bar{b} = (b_i)_{i=1}^{k'}$ if and only if $k = k'$ and there exists a permutation σ of $\{1,2,\dots,k\}$ and edges $(e_i)_{i=1}^{k}$ in $\mathscr{T}\circ\mathscr{T}'$ with $s(e_i) = a_i$ and $t(e_i) = b_i$. The edges e_i are obtained through composition and are therefore pairs (e_i^1, e_i^2) of an edge in $\mathscr{T}$ and an edge in $\mathscr{T}'$ such that $s(e_i^2) = t(e_i^1)$. This is equivalent to the fact that there exists edges $\bar{e}^1$ and $\bar{e}^2$ in $\mathfrak{S}\mathscr{T}$ and $\mathfrak{S}\mathscr{T}'$ respectively with $s(\bar{e}^1) = \bar{a}$, $t(\bar{e}^1) = s(\bar{e}^2)$ and $t(\bar{e}^2) = \bar{b}$, i.e. there is an edge of source $\bar{a}$ and target $\bar{b}$ in $\mathfrak{S}\mathscr{T}\circ\mathfrak{S}\mathscr{T}'$. The facts that the correspondence we just described is bijective and that it preserves the weights are straightforward, i.e. $\mathfrak{S}$ is indeed a functor.

Now, monoidality on objects is the well-known fact that $\mathrm{M_{fin}}(A+B) = \mathrm{M_{fin}}(A)\times\mathrm{M_{fin}}(B)$. To show monoidality on morphisms, we pick an edge $e = (\sigma;\bar{e})$ in $\mathfrak{S}(\mathscr{T}+\mathscr{T}')$ where $\mathscr{T} = (E,A,B,\Omega,s,t,w)$ and $\mathscr{T}' = (E',A',B',\Omega,s',t',w')$. We write $\bar{a}$ and $\bar{b}$ its respective source and target. Then $\bar{a}$ (resp. $\bar{b}$) can be written as a pair $(\bar{a}_1,\bar{a}_2)$ (resp. $(\bar{b}_1,\bar{b}_2)$) of a multiset of elements in A (resp. in B) and a multiset of elements in A' (resp. B'). Now, since there are no edges going from A to B' nor from A' to B, we can write σ as $\sigma_1+\sigma_2$ and $\bar{e}$ as a pair $(\bar{e}_1,\bar{e}_2)$ and see that, up to a reindexing, $(\sigma_1;\bar{e}_1)$ (resp. $(\sigma_2;\bar{e}_2)$) is an edge in $\mathfrak{S}(\mathscr{T})$ (resp. $\mathfrak{S}(\mathscr{T}')$) from $\bar{a}_1$ (resp. $\bar{a}_2$) to $\bar{b}_1$ (resp. $\bar{b}_2$): In other words, the edge $(\sigma;\bar{e})$ we picked corresponds to an edge in $\mathfrak{S}(\mathscr{T})\times\mathfrak{S}(\mathscr{T}')$. It is not difficult to see that this correspondence is bijective. □

Definition 47. Given a QCS $A = (X^A, \mathbf{A})$ we define the following QCS.

$$!A = (\mathrm{M_{fin}}(X^A), \{\mathfrak{S}(\mathscr{A}) \mid \mathscr{A} \in \mathbf{A}\}^{\perp\perp})$$

Theorem 48. *For any* m *and* $\perp\!\!\!\perp$*, this defines a model of* LL.

Proof. This is quite straightforward. It is easy to check that the additives & and $\oplus$ defined as above are respectively a product and a coproduct. Then, since we already showed that $\mathfrak{S}$ is strict monoidal, we have that $!(A \,\&\, B) = !A \otimes !B$. We then show that a co-monad structure on $!A$ can be defined by the following maps:

$$\eta_{!A} = (A, A, \mathrm{M_{fin}}(A), \Omega, \mathrm{Id}, a \mapsto \{a\}, 1)$$

$$\mu_{!A} = (\mathrm{M_{fin}}(\mathrm{M_{fin}}(A)), \mathrm{M_{fin}}(A), \Omega, \mathrm{Id}, [\bar{a}_1,\bar{a}_2,\dots,\bar{a}_k] \mapsto \sum_i \bar{a}_i, 1)$$

□

Let us notice that this definition of exponentials is similar to that of exponentials in weighted relations models of Laird *et al.* [14]. More formally, given a triskell $\mathscr{T}$ contracting to the weighted relation $R_{\mathscr{T}}$, the triskell obtained by applying the symmetric Fock functor $\mathfrak{S}\mathscr{T}$ is mapped, through the contraction functor, to the relation denoted by $!R_{\mathscr{T}}$ in Laird *et al.*

We finally explain how this differs from the exponentials considered by Danos and Ehrhard [4]. First, we remark that if $\mathscr{T}$ is a diagonal triskell, then $\mathfrak{S}\mathscr{T}$ will be diagonal as well. Then starting from a simple diagonal triskell $\mathscr{T}$ corresponding to the relation R over a set X with weights in $[0,1]$, we can compute the triskell $\mathfrak{C}\mathfrak{S}\mathscr{T} = (w_m)_{m\in\mathrm{M_{fin}}(\mathrm{X})}$ and compare it to Danos and Ehrhard exponential $!R = (R^!_m)_{m\in\mathrm{M_{fin}}(\mathrm{X})}$. For any multiset m, we can then show that $w_m = [m].R^!_m$, where $[m]$ denotes the multinomial coefficient $[m] = (\sum_{x\in\mathrm{X}} m(x))!/\prod_{x\in\mathrm{X}} m(i)!$ (identifying $[m]$ as a function $\mathrm{X}\to\mathbf{N}$). Although not equal, the two interpretations of exponentials are therefore related. Of course, the exact definition of Danos and Ehrhard can be considered in our setting by defining a variant of the exponential in which weights are scaled by a $\frac{1}{[m]}$ factor. This is however a less general construction as it can only be performed when the latter scalars can be defined.

7. Perspectives

The small mismatch between the traced monoidal composition and the relational composition which is rubbed out by the observational equivalence seems to be worth investigating.

Indeed, the author believes that small changes in the definitions of the categories could lead to a more precise correspondence where the observational quotient is not needed. This is based on the following observation: the mismatch arises from the fact that the antisymmetric Fock builds the triskell of simple paths, i.e. paths that do not go through the same edge twice, while the trace on triskells is defined from the set of all paths[8]. Considering the *simple paths* execution [18], another family of IG models, one would expect such a result.

It appears that the category of triskells provides an interesting generalisation to the category of weighted relations when it comes to defining quantitative denotational models. In this paper, we have presented the structure only for multiplicative linear logic, and it thus makes sense to study extensions to interpret proofs of larger fragments, e.g. additives and exponential connectives of linear logic, fixed points. As such extensions were considered by the author for the interaction graphs models, a related question is what these extensions are mapped to through the "functor" $\mathfrak{F}_{\uparrow}$. Furthermore, these extensions can be studied not only at the level of proof interpretations, but also with respect to the double-glueing constructions. Indeed, quantitative coherence spaces models were defined on the category of triskells for larger fragments in the last section. It should be noted, however, that to define satisfying exponential connectives, one has to consider generalisations of graphs [21, 22].

Lastly, the notion of triskell seems of interest in itself, especially considering the fact that its definition is internal to the underlying category $\mathbb{C}$. It then appears that both dynamic and denotational models can be defined in categories of triskells over any topos instead of a category of sets.

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[8] Thus, the trace as defined in the present paper would correspond to the *symmetric* Fock functor, which unfortunately does not provide the nice results about orthogonality we obtain when working with the antisymmetric Fock functor.