

**BLASTING Through The Information
Theoretic Barrier With FUSION TREES**

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BLASTING Through The Information Theoretic Barrier With FUSION TREES

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1. Introduction

This paper introduces a new sublogarithmic data structure for searching, the Fusion Tree. These trees lead to improved worst-case algorithms for sorting and searching, surpassing the limitations of the information theoretic lower bound.

The information theoretic bound asserts that sorting N numbers requires $N \log N$ comparisons in the worst case. The degree of universality of this bound has not been fully resolved. Paul and Simon [4] have established that any unit cost random access machine algorithm whose operations include addition, subtraction, multiplication, and comparison with 0 (but not division or bit-wise Boolean operations) requires $\Omega(N \log N)$ time to sort N numbers in the worst case. On the other hand, with the inclusion of division and Boolean operations, they obtain a linear time algorithm. This linear time algorithm, however, suffers from an abuse of the unit cost assumption. In particular, the algorithm generates operands during the course of its computation whose individual lengths are N^2 times the length of the largest of the N numbers to be sorted.

What seems needed is a computational model that avoids the potential abuses of the unit cost random access machine, but allows for unit cost operations among operands of "reasonable" size, i.e. operands of size commensurate with the sizes of the numbers to be sorted. Accordingly, we consider a reformulation of the sorting problem wherein our machine has a b -bit word size, and

each of the N input numbers is assumed to be a non-negative integer less than 2^b (and hence each fits into one word). Moreover, it is desirable that a sorting algorithm use only $O(N)$ words of memory.

We assume throughout this paper that the number of data items that are present never exceeds 2^b , the universe size. (To cope with a larger number of items, we could proceed by storing bucket pointers in our data structure, with equal items placed in common buckets.)

Our "finite universe" reformulation of the sorting problem suggests the use of finite universe priority queues and similar types of data structures. For example, there is the priority queue of Van Emde Boas et al. [5] which accommodates priority queue operations in time $\log \log U$ per operation, where $U = 2^b$ is the universe size. This structure requires U words of memory, however. An alternative is Willard's Y-fast tries [6] in conjunction with the dynamic perfect hashing methods of Dietzfelbinger et al. [3], which brings the space down to linear in N , and accommodates priority queue operations in amortized randomized time $\log \log U$ per operation. Besides the fact that either too much space or randomization is required, these approaches improve upon conventional sorting only for sufficiently large values of N . (i.e. when $\log N \gg \log \log U$).

Another finite universe data structure for implementing priority queue operations is described in Ajtai, Fredman, and Komlos [1]. This data structure is defined only within the cell probe model of computation and cannot be implemented with normal machine instructions. However, it is tantalizing in that it accommodates priority queue operations in constant time per operation for N sufficiently small compared with U . Ajtai et al.

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[1] state as an open problem whether some variant of their data structure can be implemented in a more realistic machine model. This paper contains a partial solution to this problem; one however which is good enough to be embedded in a new type of search tree data structure: the "Fusion Tree".

Fusion Trees accommodate dynamic search operations in amortized time $O(\log N/\log\log N)$ per operation on a random access machine with b -bit word size, whose instruction set includes addition, subtraction, multiplication, comparison, and bitwise AND operations. With the use of Fusion trees, we can construct a linear space $O(N\log N/\log\log N)$ worst-case time sorting algorithm. We emphasize that the implicit constant inside the O -notation is independent of b . Moreover, if we allow randomization and integer division, we show that dynamic searching and sorting can be accomplished in times, respectively $O(\sqrt{\log N})$ and $O(N\sqrt{\log N})$.

2. Notation and Other Preliminaries

We let $\text{bin}(a_1, a_2, \dots)$ denote the sum $2^{a_1} + 2^{a_2} + \dots$ for non-negative integers a_i .

We assume that the multiplication instruction produces a two word product, one consisting of the most significant b bits, the other containing the least significant b bits. This fact enables us to effect a right shift; upon multiplying x by $\text{bin}(b-r)$, we obtain the result of right shifting x by r bits (in the significant portion of the product). This procedure depends, however, on having available the quantity $\text{bin}(b-r)$, either as a program constant or as the result of a previous computation.

We occasionally require operands that slightly exceed the size of a single machine word. In such circumstances, we use double precision methods.

We observe that the bitwise Boolean operations, OR and XOR (exclusive OR), can be implemented on our machine by making use of the constant $2^b - 1 = \text{bin}(0, 1, \dots, b-1)$, subtraction (so that we can complement a word), along with the AND operation.

Given a set of numbers S and a number x , we denote by $\text{rank}_S(x)$ the value $|\{t \mid t \in S, t \leq x\}|$.

3. Overview

The Fusion tree can be roughly regarded as an implementation of a B-tree [2] such that (a) the approximate degree B of an internal node is an increasing, unbounded function of N , and (b) when performing a key search, the correct child of each node along the search path can be determined in constant time, independently of B . This suffices to achieve $O(\log N)$ search time. It will be the case, however, that updating an internal node (inserting or deleting a child, splitting the node etc.) will require time B^4 . To keep the amortized cost of update operations under control, we modify our structure so that it can be viewed as a B-tree whose leaves are identified as the roots of weight balanced binary search trees of size approximately B^4 . For example, when the size of one of these binary search trees gets too large, it is split into two trees with its associated root (B-tree leaf) likewise splitting. This reduces the amortized number of updates to the internal portion of the B-tree by a factor of B^4 , while increasing the search times only by an additive amount of $\log B$. (We require that every key in the set represented by the Fusion tree reside in these binary trees; when a key gets deleted it is unnecessary to remove its presence from the B-tree nodes, where it serves only as a search discriminator.) Since we will be able to maintain $\log B = \Theta(\log\log N)$ using routine methods, we achieve a search tree whose operations can be performed in $O(\log N/\log\log N)$ amortized time.

Our algorithms require a fixed number of constants whose values depend on the word size b . We assume that these constants are given as part of the algorithms, and we are not charged for their computations, which could depend on the word size, b . This introduces a small amount of non-uniformity (in terms of b) into our algorithms.

The core of our data structure lies in our ability to represent a B-tree node so that (a) we can determine the correct child of a node in constant time when conducting a search, and (b) we can update a node in time B^4 . This aspect of our data structure is motivated by and extends the work of Ajtai et al. [1]. We turn our attention to these considerations.

Consider a B-tree node consisting of k keys ($B/2 \leq k \leq B$) given by $S = \{u_1, \dots, u_k\}$. In order to determine the correct child of the node when conducting a search for u , it suffices to compute $\text{rank}_S(u)$. This is to be accomplished in con-

start time. Central to the representation of our binary code is the notion of compressed key representation, which we proceed to describe.

4. Compressed Key Representation

Given a set $S = \{u_1, \dots, u_k\}$ consisting of b -bit numbers, we proceed to construct a set $B(S)$ of distinguishing bit positions and a set $K(S) = \{v_1, \dots, v_g\}$ of r -bit numbers where $r = |B(S)| \leq b-1$.

Consecutively number the b bit positions so that bit position 1 corresponds to the most significant bit. Our definition of $B(S)$ proceeds recursively. If $|S| = 1$, then $B(S)$ is the empty set. Otherwise, let p be the most significant bit position that distinguishes between two of the numbers in S . Let $S_0 = \{v_1, \dots, v_g\}$ consist of those u_i 's in S for which u_i has a zero in bit position p , and let $S_1 = \{w_1, \dots, w_h\}$ consist of those u_i 's for which u_i has a one in bit position p . Note that $g, h \geq 1$ and $g+h = k$. Then $B(S) = \{p\} \cup B(S_0) \cup B(S_1)$.

Now define u_i to be the result of deleting from u_i all bits except for those occupying positions contained in $B(S)$. More formally, let $c_1 < c_2 < \dots < c_r$ denote the elements of $B(S)$ in sorted order. Then the j^{th} bit of u_i is the c_j -th bit of u_i . We refer to u_i as the compressed key representative of u_i , and we define $K(S) = \{u_1, \dots, u_g\}$. Observe that this compression operation is order preserving among the elements of S .

Given an arbitrary b -bit number u not necessarily in S , we define $\hat{u}(S)$ likewise to be the result of extracting just those bits of u occupying positions in $B(S)$. The following Lemma presents an important property of the compressed key representation.

Lemma 1. Let \hat{u}_i be an arbitrary compressed key in $K(S)$, and suppose that for an arbitrary b -bit number u , the most significant bit position in which $\hat{u}(S)$ and \hat{u}_i differ is position m . (If $\hat{u}(S) = \hat{u}_i$, then we define $m = r + 1$.) Let $c_1 < \dots < c_r$ be the elements of $B(S)$ in sorted order and let $c_0 = 0, c_{r+1} = b+1$. Assume that the most significant bit position p in which (the full keys) u and u_i differ satisfies $p < c_m$. Then $\text{rank}_S(u)$ is uniquely determined by the interval $(c_{j-1}, c_j]$ containing p , together with the relative order between u and u_i .

Proof. We proceed by induction on $k = |S|$. If $k = 1$, then the result is trivial. If $p \in (c_0, c_1]$, then the assumption $p < c_m$ together with the fact

that u and u_i agree in bit positions c_1, c_2, \dots, c_m imply $p < c_1$. By definition of c_1 , all elements in S agree through bit position c_1-1 , and therefore, $\text{rank}_S(u) = 0$ or k , depending on whether $u < u_i$ or $u > u_i$. Now suppose that $p > c_1$. Then u and u_i agree at least through bit position c_1 . Assume without loss of generality that u and u_i have a 0 in bit position c_1 . Then (using the notation preceding the statement of the Lemma), $u_i \in S_0 = \{v_1, \dots, v_g\}$, and $\text{rank}_S(u) \leq g$ since $u \leq w_1, \dots, w_h$ (as each of the w_j 's have a one in bit position c_1). It follows that $\text{rank}_S(u) = \text{rank}_{S_0}(u)$, and we proceed (below) to reduce to the case $S = S_0$ and apply the induction hypothesis. We observe that $B(S_0) \subseteq B(S)$, and therefore, the sorted sequence, $c'_1 < c'_2 < \dots$, which comprises the elements of $B(S_0)$, is a subsequence of $c_1 < c_2 < \dots$. The most significant (smallest) bit position c'_q in which u and u_i differ (from among just the positions c'_1, c'_2, \dots) clearly satisfies $c'_q \geq c_m$. Therefore, $p < c'_q$. Applying the induction hypothesis for S_0 and observing that the intervals $(c'_{j-1}, c'_j]$ refine the intervals $(c_{j-1}, c_j]$, the Lemma follows.

The above Lemma enables us to reduce the computation of $\text{rank}_S(u)$ to computing $\text{rank}_{K(S)}(\hat{u}(S))$ as we now sketch. By computing $\text{rank}_{K(S)}(\hat{u}(S))$, we determine the predecessor \hat{u}_j and the successor \hat{u}_{j+1} of $\hat{u}(S)$ in $K(S)$ (one of the two quantities \hat{u}_j or \hat{u}_{j+1} may not exist). Next, we compare u with u_j and u_{j+1} ; we are done if $u_j \leq u \leq u_{j+1}$. Otherwise choose \hat{u}_l ($l = j$ or $j+1$) to be the value among \hat{u}_j and \hat{u}_{j+1} having the longest prefix of significant bits in common with $\hat{u}(S)$. Since $u < u_j$ or $u > u_{j+1}$ (by assumption), it must be the case that (using the notation of Lemma 1) $p < c_m$, and therefore, Lemma 1 can be applied to determine the rank of u once we know the interval $(c_j, c_{j+1}]$ containing p .

A convenient alternative description for the set $B(S)$ that will be used below is given as follows. Assume that the u_i 's are sorted, so that $u_1 < u_2 < \dots$. Let d_i be the most significant bit position in which u_i and u_{i+1} differ. Then $B(S) = \{d_1, d_2, \dots, d_{k-1}\}$.

For the sake of notational convenience, throughout the remainder of this paper we number bit positions from right-to-left, with position 0 corresponding to the least significant bit. Thus, the quantity $\text{bin}(a_1, a_2, \dots)$ contains 1's precisely in bit positions a_1, a_2, \dots .

A difficulty with using compressed keys as well as with computing the transformation that

maps u into $\hat{u}(S)$ (in constant time) is that there is no obvious way to relocate the appropriate extracted bits so that they form a consecutive block of bits. The desired properties of the representation, however, are preserved by any similar transformation that relocates the extracted bits into (possibly nonconsecutive) positions without changing their relative order. We assume that the unused positions are filled with 0's. Because we ultimately need to concatenate the compressed representations of the keys in S into a single b -bit word, it is desirable that the extracted bits be relocated into a relatively narrow, if not consecutive, field of bits. This can be accomplished by an appropriate multiplication along with bitwise AND operations (to clear out the unwanted bits) as we proceed to describe.

Let $c_1 < \dots < c_r$ be the bit positions (elements of $B(S)$) that need to be relocated, and let u be the quantity from which we wish to extract these bits. Let $C = \text{bin}(c_1, c_2, \dots, c_r)$. Assuming that the quantity C is available to us, we compute $v = u \text{ AND } C$ to zero out the irrelevant portion of u . Next, we multiply v by a suitable quantity M to relocate the appropriate bits of v into a narrow field, and once again use AND to zero out irrelevant bits. Our task is to prove the existence (and ultimately construct within our update algorithm) this quantity M . Writing $M = \text{bin}(m_1, m_2, \dots, m_r)$, if we ignore for the moment the interfering effects of cross terms and carry's that occur with multiplication, we can regard the effect of multiplying v and M as the relocation of the bits in positions c_1, c_2, \dots, c_r to the respective locations c_1+m_1, \dots, c_r+m_r . Our choosing to ignore the other cross terms will be justified if we can also arrange for the r^2 sums, c_i+m_j , $1 \leq i, j \leq r$, to be distinct (which, in particular, implies that no carry's can take place). Our goal, therefore, is to construct a set of integers, m_1, \dots, m_r such that (a) $m_1+c_1 < m_2+c_2 < \dots < m_r+c_r$, (b) the m_i+c_i are distinct, and (c), the sums in (a) fall into a "small" interval. Lemma 2 (below) provides the desired result. We further require that $m_1+c_1 = b$, so that our newly compressed key is right justified in the significant portion of the product. This requirement is easily satisfied by translating the m_i 's of Lemma 2 by a suitable constant. We redefine the notation $\hat{u}(S)$ (and likewise the definition of the compressed keys $\hat{u}_1, \dots, \hat{u}_k$), to reflect the computationally accessible variant offered by this approach, noting again that the desirable proper-

ties are preserved.

Lemma 2. Given a sequence of r integers, $c_1 < c_2 < \dots < c_r$, there exists a sequence of r integers m_1, m_2, \dots, m_r such that

- (a) the r^2 sums c_i+m_j , $1 \leq i, j \leq r$ are distinct,
- (b) $m_1+c_1 < m_2+c_2 < \dots < m_r+c_r$,
- (c) $(m_r+c_r) - (m_1+c_1) \leq r^4$.

Proof. First, we prove that there exist integers m'_1, \dots, m'_r such that $m'_i+c_g \not\equiv m'_j+c_h \pmod{r^3}$ whenever $i \neq j$. Assuming for the moment that such integers can be found, it follows that the r^2 (non-reduced) sums m'_i+c_j are all distinct, and moreover, by adding suitable multiples of r^3 to the m'_i to obtain the final values m_i , we can further satisfy the requirement $m_i+c_i < m_{i+1}+c_{i+1} \leq m_i+c_i+r^3$, so that (a), (b) and (c) are satisfied.

To prove the existence of m'_1, \dots, m'_r , suppose that $t < r$ and that we have succeeded in choosing m'_1, \dots, m'_t satisfying $m'_i+c_g \not\equiv m'_j+c_h \pmod{r^3}$ when $i \neq j$. We show how to construct a suitable m'_{t+1} . Observing that $m'_{t+1}+c_i \equiv m'_s+c_j$ implies $m'_{t+1} \equiv m'_s+c_j-c_i$, we can choose m'_{t+1} to be the least residue not represented among the fewer than r^3 residues of the form $m'_s+c_j-c_i$, $1 \leq s \leq t$, $1 \leq i, j \leq r$. In this way, we successively compute the quantities m'_1, \dots, m'_r .

5. Fusion

We proceed to describe the representation of a B-tree node containing the keys $u_1 < \dots < u_k$, $k \leq B$. Let $S = \{u_1, \dots, u_k\}$ and let $B(S) = \{c_1, c_2, \dots, c_r\}$. Our B-tree node contains the quantity $C = \text{bin}(c_1, c_2, \dots)$ as well as the quantity $M = \text{bin}(m_1, m_2, \dots)$ which provides the multiplier used in the computation of $\hat{u}(S)$. The final AND operation performed in the computation of $\hat{u}(S)$ requires the quantity, $\text{bin}(c_1+m_1, c_2+m_2, \dots)$, which is likewise to be made available. Also present is an array containing the u_i 's in sorted order and an array containing the u_i 's in sorted order.

We will describe below a constant time method for computing $\text{rank}_{K(S)}(\hat{u}(S))$ and $\text{rank}_{B(S)}(m)$, where m is an integer in $[0, b]$. We will also describe a constant time method for computing $\text{msb}(u, v)$, which is defined to give the most significant bit position in which the two b -bit quantities u and v differ. For the present, we assume these capabilities.

As outlined in the previous Section, after computing $j = \text{rank}_{K(S)}(\hat{u}(S))$, we compare u with u_j and u_{j+1} . Assuming that u does not fall in the interval $[u_j, u_{j+1}]$, we determine which of u_j and u_{j+1} has the longest prefix of significant bits in common with $\hat{u}(S)$ by comparing $\hat{u}(S) \text{ XOR } u_j$ with $\hat{u}(S) \text{ XOR } u_{j+1}$ (the smaller value corresponds to the longer prefix). Assuming that u_h , $h = j$ or $j+1$, has the longer prefix in common with $\hat{u}(S)$, we compute $m = \text{msb}(u, u_h)$ and then identify the interval $[c_i, c_{i+1})$ containing m by computing $i = \text{rank}_{B(S)}(m)$. We finally obtain $\text{rank}_S(u)$ by executing a table look-up indexed on h , i , and whether (a) $u < u_h$ or (b) $u > u_h$. The space requirements of our B-tree node is dominated by this table, which requires $O(B^2)$ space. Since the number of B-tree nodes is $O(N/B^4)$, the total amount of space required to represent the Fusion tree is $O(N)$.

The constant time computation of $\text{rank}_{K(S)}(\hat{u}(S))$ proceeds as follows. Contained within our B-tree node is a quantity K_S into which the compressed keys from $K(S)$ are "fused" together - hence the name of our structure. More specifically, the bits of K_S are partitioned into $E = \lfloor b^{1/6} \rfloor$ equal size fields. (If b is not divisible by E , then the rightmost $E \lfloor b/E \rfloor$ bits are partitioned into equal size fields.) Each compressed key will occupy a separate field. As a consequence, the maximum allowable degree B of a node is constrained to satisfy $B \leq E$. Since, as indicated earlier, we can assume that $N \leq 2^8$, we have sufficiently many fields to maintain $\log B = \Omega(\log \log N)$, which achieves the stated speed-up over conventional sorting. Our limitation on the size of E comes from the fact that our compressed keys require up to k^4 bits per key. The compressed keys are right justified in their respective fields, and the leading two bits in each such field are 0's. The fields which do not contain compressed keys are filled with all 1's except for a leading 0. The structure of K_S is shown below.

$K_S = \underbrace{0 1 \cdots 1}_{\text{Field } E} \cdots \underbrace{0 0 \cdots \hat{u}_k}_{\text{Field } k} \cdots \underbrace{0 0 \cdots}_{\text{Field } 1}$

Now by executing a suitable multiplication and bitwise OR operation involving $\hat{u}(S)$, we obtain the number Y shown below. (The fields of Y correspond to those of K_S .)

$Y = \underbrace{1 0 \cdots 0}_{\text{Field } E} \underbrace{\hat{u}(S)}_{\cdots} \cdots \underbrace{1 0 \cdots 0}_{\text{Field } 1} \underbrace{\hat{u}(S)}$

Next, we subtract K_S from Y and zero out all but the lead bits of the fields (performing a bitwise AND). The result contains a 1 only in those positions that are leading bits of fields for which K_S contained a \hat{u}_i with $\hat{u}_i \leq \hat{u}(S)$. A suitable multiplication will now sum these bits, so that a portion of the resulting product will contain the desired rank, which is then extracted by an AND operation.

We use the same method to compute $\text{rank}_{B(S)}(m)$ in constant time; instead of using the quantity K_S this computation utilizes the quantity B_S in which the elements of $B(S)$ are fused together.

Our next task is to describe the constant time computation of $\text{msb}(u, v)$, the most significant bit position in which the b -bit quantities u and v differ. For the additional purpose of performing update operations on B-tree nodes as described later, we wish to augment this computation so that as a by-product of computing $m = \text{msb}(u, v)$, we also obtain the quantities, $\text{bin}(m)$ and $\text{bin}(b-m)$. Our computation trivially reduces to determining the position of the leftmost 1 in $u \text{ XOR } v$, which we can write as $\lfloor \lg(u \text{ XOR } v) \rfloor$ (\lg denotes the binary logarithm). We will need the following lemma.

Lemma 3. We say that a number x is d -sparse provided that the positions of all of its 1 bits belong to a set of the form, $Y = \{a+di \mid 0 \leq i < d\}$, which consists of d consecutive terms of an arithmetic progression with common difference d . (Not all of these positions have to be occupied by 1's, however.) If x is d -sparse, then there exist constants y_1 and y_2 such that for $z = (y_1 x) \text{ AND } y_2$, the i^{th} bit of the significant part of z equals the bit in the position $a+di$ of x , $0 \leq i < d$. In other words, z is the result of "perfectly compressing" x .

Proof: Let $a_i = a+di$ and let $t_i = b - a + (1-d)i$, $0 \leq i < d$. Then $a_i + t_i = b + i$ and all sums $a_i + t_j$, $0 \leq i, j < d$ are distinct. We choose $y_1 = \text{bin}(t_1, t_2, \dots)$, and the Lemma follows easily.

The computation for $\lfloor \lg x \rfloor$ is divided into two phases. We assume that $x \neq 0$. Let $s = \lceil \sqrt{b} + 1 \rceil$. Consider a partitioning of the b bits of our word x into right justified contiguous blocks blocks of s

bits. (The left-most block has possibly fewer than s bits.) The number of blocks is given by $\lceil b/s \rceil$. The first phase of the computation determines the left-most block in which x contains a 1, extracts, and then right justifies this block. The second phase of the computation finds the left-most 1 in this extracted block.

Define C_1 to have 1's precisely in the left-most bit position of each block: i.e. $C_1 = \text{bin}(s-1, 2s-1, \dots)$. Define C_2 to be the bitwise complement of C_1 . We first compute the function, $\text{lead}(x)$, which enjoys the property that for each of its blocks, the left-most bit within the block equals 1 if and only if the corresponding block in x contains a 1. (All other bits in $\text{lead}(x)$ equal 0.) $\text{lead}(x)$ is given by $(C_1 - [(C_1 - x \text{ AND } C_2) \text{ AND } C_1]) \text{ OR } (x \text{ AND } C_1)$. Because $\text{lead}(x)$ is s -sparse, we can apply Lemma 3 to obtain (in constant time) a compression of the left-most bits from each of the blocks of $\text{lead}(x)$. Let $\text{compress}(x)$ denote the result of this compression. Now let $P = \{\text{bin}(0), \dots, \text{bin}(\lceil b/s \rceil - 1)\}$ denote the first $\lceil b/s \rceil$ non-negative powers of 2. We set $b_1 = \text{rank}_P(\text{compress}(x))$: This rank computation proceeds in the same manner in which we computed $\text{rank}_{K_s}(s)$, except that we use $\lceil b/s \rceil$ blocks each consisting of s bits in which the relevant quantities are right justified. In place of the quantity K_s , a constant consisting of the powers of 2 fused together in right-to-left ascending order is used. (It is also necessary to use double precision arithmetic since the quantities involved slightly fill up more than single machine words.) The quantity b_1 identifies the block number (counting from the right) of the left-most block of x containing a 1.

In addition to obtaining the block number of the left-most block containing a 1, we can also obtain the quantity $\text{bin}(r)$, where r is the position of the left-most 1 of $\text{lead}(x)$. Reviewing the computation for $\text{rank}_P(\text{compress}(x))$, after performing the subtraction and zeroing of all but the lead bits from each of the blocks of s bits (the moment just before we multiply), we obtain a quantity which we denote as L . Among the lead bits of the blocks of L , the rightmost b_1 of them will be 1's, and the others will be 0's. Because the 1's of L appear periodically with period s , we can extract the left-most such 1 (the other 1's being replaced by 0's) by subtracting from L the result of right shifting L by s bits. The position of this extracted 1, however, exactly coincides with the position of the left-most 1 of $\text{lead}(x)$ since the blocks of $\text{lead}(x)$

and L have a common size of s bits. In a similar manner, we can also obtain $\text{bin}(b-r)$ by redoing our rank computation, but using instead the powers of two fused together in left-to-right ascending order. Now if $b_1 = 1$ then we proceed directly to the next phase (only the rightmost block contains a 1). Otherwise, we multiply x by $\text{bin}(b-r+s-1) = \text{bin}(b-r) \cdot \text{bin}(s-1)$ to extract (and right justify in the significant portion of the product) the left-most block of x containing a 1. This completes the description of the first phase of the computation for $\lfloor \lg x \rfloor$.

The second phase of the computation computes the position j of the left-most 1 in our extracted block of x consisting of s bits. As before, we perform a rank computation of these s bits relative to the set consisting of the first s powers of two. To obtain $\text{bin}(j)$, we proceed as in the first phase, and then employ Lemma 3 to perfectly compress the leading bits of the blocks. We obtain $\text{bin}(s-j)$ by repeating this computation, but reversing the order in which the powers of two are concatenated. The results, b_1 , $\text{bin}(r)$, $\text{bin}(b-r)$, j , $\text{bin}(j)$, and $\text{bin}(s-j)$, of these two phases can now be combined to yield the result $m = \lfloor \lg x \rfloor$ of the $\text{msb}(u,v)$ computation, augmented to include $\text{bin}(m)$ and $\text{bin}(b-m)$.

To complete our discussion of the Fusion tree, we need to indicate how a B-tree node can be built in time B^4 . The various update operations on B-tree nodes can be regarded as special instances of the node building operation. We assume that we are given the set of discriminator keys, $S = \{u_1, u_2, \dots, u_k\}$. First, we sort the keys obtaining (say) $u_1 < \dots < u_k$. Next, we compute the set $B(S)$ which is given by $\{\text{msb}(u_i, u_{i+1}) \mid 1 \leq i < k\}$. The augmented computation of the $c_i = \text{msb}(u_i, u_{i+1})$ also yields the $\text{bin}(c_i)$ values. These values are sorted to obtain distinct copies and then summed to obtain the constant $C = \text{bin}(c_1, \dots, c_r)$. The proof of Lemma 2 provides an algorithm to compute the m_i positions which define the multiplier M . However, we actually need the values $\text{bin}(m_i)$. Since the m_i 's of Lemma 2 satisfy $b \leq c_i + m_i \leq b + r^4$, by starting with the $\text{bin}(b-c_i)$ values (provided by the augmented msb computations), we can obtain each $\text{bin}(m_i)$ value by performing $O(\log r)$ multiplications (in effect constructing addition chains). All of the required computations for the construction of a B-tree node, including the construction of the look-up table stored in such a node, involve

straightforward algorithms that can be performed in time polynomial in k . (k^4 time is easily seen to be sufficient.)

The inclusion of a fixed number of constants in our algorithms (whose values depend only on b) is necessitated by considerations typified by the following. The quantity K_S (in which the compressed keys are fused together) contains $b^{1/6}$ fields, all but the right-most k of which have the form (*) 011...1. Let w denote the constant obtained by concatenating together fields having the form (*), and let $w' = \text{bin}(f)$, where f is the common size of these fields. Using w and w' we can compute K_S in $O(k)$ time. (This assumes we have already computed the compressed keys which make up K_S .) Similarly, given $\hat{u}(S)$, the constant time computation of the quantity Y used in the $\text{rank}_{K(S)}(\hat{u}(S))$ computation utilizes certain constants.

6. Extensions

We remark first that by suitably refining our Fusion Tree structure, we obtain worst case as opposed to amortized time bounds.

If we are willing to either (a) relax the linear space restriction, or (b) preserve the linear space restriction but use randomization and integer division, then we can achieve $\sqrt{\log N}$ amortized time bounds for dynamic search operations. This is accomplished by using the data structure of van Emde Boas et al.[5] in the case of (a), (or, in the case of (b), using dynamic Y-fast tries [3,6]) when N exceeds

$$(*) 2^{(\log b)^2/36}.$$

For such N , these data structures perform search operations in time $O(\log \log U) = O(\log b) = O(\sqrt{\log N})$. For smaller N , we use the Fusion Tree, but modified to maintain $B = \Theta(2^{\sqrt{\log N}})$. With this choice for B , the amortized complexities of our operations are given by $O(\log B + \log N / \log B) = O(\sqrt{\log N})$. (We also need to check that the constraint $B \leq E$ is satisfied when N does not exceed (*).)

Thus, with the use of randomization and integer division, sorting can be accomplished in time $O(N\sqrt{\log N})$ and linear space.

7. Applications

Sorting and searching frequently occur as algorithmic components. Moreover, the commonly occurring priority queue data structure can

be implemented as a special case of searching. The field of computational geometry offers some immediate applications of the Fusion Tree or simple modifications of the Fusion Tree. For instance, we have improved algorithms for planar convex hull construction, priority search trees and rectangle intersections, fractional cascading, and certain orthogonal range query problems.

Another application leads to an improved version of Willard's Q-fast tries[7]. As described above, the Y-fast trie in conjunction with dynamic perfect hashing [3,6] provides a dynamic linear space data structure for searching with randomized amortized $O(\log \log U)$ time operations. This data structure provides the best known performance for large subset sizes. The Q-fast trie, on the other hand, provides the best known performance among linear space deterministic data structures, again for large subset sizes. Balanced search trees constitute a component part of the Q-fast trie. We can improve the performance of Q-fast tries by using Fusion Trees to implement these search tree components, and changing the parameters of the trie so that its height is given by $\sqrt{\log U / \log \log U}$, and its node degree is given by $2^{\sqrt{\log U \cdot \log \log U}}$. The resulting data structure provides $\sqrt{\log U / \log \log U}$ operation times, improving upon the original Q-fast trie by a $\sqrt{\log \log U}$ factor.

8. Open Questions

Two immediate open questions are: *How fast can we sort?* and *How fast can we search?* Some interesting variations of these questions are as follows. First, we ask if there exist possible exotic machine instructions that can lead to yet faster sorting and searching algorithms. Put another way, *Is RISC Risky?* We note that the instructions used in our algorithms belong to the class NC¹. This can be considered a reasonable restriction to impose on possible exotic instructions, although the question remains interesting even without this restriction. Several people have pointed out to the authors that while comparison is an AC⁰ operation, our algorithms use multiplication, which does not belong to AC⁰. Thus, it would be interesting to know if there exist fast sorting and searching algorithms that can be implemented with AC⁰ instructions. Put another way, *Can Fusion Trees be implemented with brisk RISC?*

9. Concluding Remark

The decision tree model of computation enjoys the properties of being reasonably general, tractable, and often times quite challenging. The beguiling successes of this appealing model of computation have lead to many claims that various $N \log N$ algorithms are optimal; conclusions that should not be taken too seriously.

10. References

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