On the Computational Meaning of Axioms

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Abstract

This paper investigates an anti-realist theory of meaning suitable for both logical and proper axioms. Unlike other anti-realist accounts such as Dummett–Prawitz verificationism, the standard framework of classical logic is not called into question. This account also admits semantic features beyond the inferential ones: computational aspects play an essential role in the determination of meaning. To deal with these computational aspects, a relaxation of syntax is necessary. This leads to a general kind of proof theory, where the objects of study are not typed objects like deductions, but rather untyped ones, in which formulas are replaced by geometrical configurations.

1 Introduction

1.1 Between models and proofs: The standard conception of axioms

In the standard conception of axioms, the notion of *structure* has a conceptual and ontological priority. Starting from a certain « body of facts [*Tatsachenmaterial*] » (see Hilbert 1905, translated in Hallett 1995, p. 136) composed by propositions, theorems, conjectures, and proof methods belonging to different mathematical systems, it is possible to single out some *invariants* that allow to identify the common features of these systems. In this process of *abstraction*, a general and univocal form is pointed out. This form, that « might be called a relational structure » (Bernays 1967, p. 497) is fixed at the linguistic level by the axioms. When formalized in a set-theoretical way, this notion of structure becomes ontologically concrete and can play the role of an *interpretation structure* – i.e. a model – both for the axioms and for the sentences derivable from them. In this sense, we can say that the grounding idea of the axiomatic method is to capture a class of models sharing some relevant properties that distinguish them from other classes of models. An immediate consequence is that the proper axioms of a certain theory $\mathscr{T}$ are considered meaningful because they are *true* exactly in those classes of models that they identify. It seems then that the notion of axiom fits well with a truth-conditional, or model-based, theory of meaning (see Naibo 2013, ch. 3).[1] For instance, Hintikka's remark that the genuine relation between axioms and theorems is the model-theoretical relation of logical consequence, rather than the syntactical relation of derivability goes in this direction (Hintikka 2011, pp. 73-75). Derivations are then subordinated to semantical aspects, in the sense that their role is reduced to that of guaranteeing truth transmission (see Dummett 1973a, p. 434). This idea finds a further confirmation in the difficulty of constructing an inferentialist theory of meaning – for example, in the style of Dummett-Prawitz verificationism – when it has to deal with axioms. In particular, in the presence of proper axioms, the fundamental notion of *canonical proof* is lost. Consider, for example, the derivation in natural deduction of the sentence $\forall x(x = 0 \vee \exists y(x = s(y)))$ from Peano's axioms (regardless of whether we are using classical or intuitionistic inference rules). The derivation terminates with an application of the $\rightarrow$ elimination rule having as a major premiss an instance of the axiom scheme of induction. This is not a *canonical proof* in the sense of the (inferential) verificationism of Dummett-Prawitz, because it does not terminate with the introduction rule of the principal connective of the sentence under analysis – in this case the $\forall$ introduction rule.[2] In general, this means that in the presence of proper axioms it is not possible to reduce to a common form – or to identify a common feature of – all possible proofs of the sentences that have the same principal connective. The immediate consequence is that the notion of proof cannot be used to explain the meaning of sentences: in absence of a common form to which they can be reduced, different proofs of the same sentence would turn out to confer different meanings to it, so we could not refer to it as the *same sentence*.[3] This would be particularly problematic for mathematics, where a theorem is supposed to always possess the same meaning, even if proved in different ways.

However, an inferentialist theory of meaning resting on the notion of proof seems to be particularly attractive in the case of mathematical theories, since

[1]As Dummett (1976) remarks, a truth-conditional theory of meaning is itself presented in an axiomatic way. In particular, the axioms fix the reference of the primitive terms of the language.

[2]Notice that here we prefer to exclude from the set of canonical proofs those that are *trivially canonical*, i.e. those terminating with a sequence of c-elim/c-intro rules, with c as the principal connective of the conclusion. Another well known example of axiomatic theories that prevent from the possibility of obtaining canonical proofs, namely of canonical proofs of disjunctive or existential sentences, is represented by those theories the axioms of which contain strictly positive occurrences of disjunctive and existential sentences (see Troelstra & Schwichtenberg 2000, pp. 6, 106-107).

[3]This position is usually identified with a Wittgensteinian position (see Wittgenstein 1956, Part II, §31, Part V, §7), nevertheless we think that this is a mistake. The point is that Wittgenstein is not speaking of the meaning of a sentence considering it as something abstract, invariant and objective – as the propositional content of that sentence could be – but he is speaking instead of how each single agent gets to a specific understanding of that sentence, depending on the particular place she assigns to it inside her own web of beliefs.

in mathematical practice proofs are usually reckoned as a privileged way to access to mathematical objects (especially when proofs are considered as constructions) and to the properties of these objects (especially when proofs are considered as demonstrations).[4] This position has been in fact endorsed also by some champions of the axiomatic method, like the Bourbaki group, who opened its seminal book on set theory with these words:

> Ever since the time of the Greeks, mathematics has involved proof; and it is even doubted by some whether proof, in the precise and rigorous sense which the Greeks gave to this word, is to be found outside mathematics. (Bourbaki 1968, p. 7)

Axioms represent then the meeting point of different moments of the development of mathematical theories, or of the mathematical enterprise in general. On the one hand, the process of abstraction leading to the definition of an axiomatic system is connected to a *synthetic* moment: the axioms are required to capture all the relevant information belonging to a certain domain of discourse, in the sense that they should compactify and synthesize everything we know about a certain domain.[5] On the other hand, the *analytical* moment of the mathematical enterprise is represented by proofs: the information present in the axioms should be extracted and deployed just by the use of pure logical derivations (see Pasch 1925, pp. 194-195; Hempel 1945, p. 7).

Axioms have thus a double role: they are the points of entrance of the semantics into the syntax (i.e. they single out a class of models) and the starting points of derivations. This distinctive feature of axioms of connecting semantics and syntax takes its formal characterization in the *soundness and completeness theorem* (for Hilbert-style deductive systems):

$$ax_{\mathscr{T}} \vDash A \text{ if and only if } ax_{\mathscr{T}} \vdash A$$

where $ax_{\mathscr{T}}$ is the set of axioms of a certain theory $\mathscr{T}$, and A is a sentence of the language of $\mathscr{T}$.

In this sense, even if from the axiomatic point of view the most fundamental relation between axioms and theorems are still that of logical consequence, which is in fact a relation that allows to express results holding between the sentences of the theory. However, if we do not simply want to present already established results (if we do not simply want to present these relationships between the sentences of the theory), but we also want also to explain how they have been obtained, or even to discover new ones, it seems

[4]The characterization of proofs either as constructions – or better, as methods of construction – or as demonstrations can be originally found in Proclus (1970, p. 157ff). A more contemporary discussion about these distinctions can be found in Sundholm (1993; 1998).

[5]This aspect is strictly connected to the ideal of deductive completeness (cf. Awodey and Reck 2002, p. 4).

that the notion of proof naturally plays a fundamental role. In particular, when proofs are considered, it becomes clear why in logic a central place is reserved to the soundness and completeness theorem: it allows to link the truth of sentences – supposed to be guaranteed by some kind of (abstract) structures – with the way in which it can be achieved by human agents, that is via proofs. In other words, when the soundness and completeness theorem is analyzed with respect to proofs, and not only with respect to provability,[6] it seems to open the way to an epistemic interpretation of semantical concepts otherwise transcending a human-based dimension. But then, why not to let proofs play a genuine and autonomous semantic role?[7] Do we really need (abstract) structures in order to define semantical concepts – like those of truth and meaning (of mathematical sentences) – or is possible to recover them in some more ontological parsimonious way?

1.2 Our proposal: A proof-based account of axiomatic theories

What we try to investigate here is a way to take into account a standard presentation of mathematical theories based on axioms in which proofs (and operations on them) are the only genuine semantical objects, so that there is no need to postulate other notions – like that of (set-theoretic) structure – which are highly problematic to define, since they seem to invoke a reference to some kind of abstract objects, which differently from proofs are not immediately accessible to human agents.[8] In this sense, inferentialist theories of meaning are closely related to anti-realist positions according to which semantical concepts must not to transcend our epistemic capacities. Now, if in logic and mathematics we should not abandon this epistemic-based perspective, then we have not to abandon the semantical key concept of canonical proof, since a canonical proof is a finite object the nature of which does not go beyond our epistemic capacities. In order not to abandon this key concept, it seems that the only possible solution is to give up the notion of axiom in favor of other alternative notions, namely that of *non-logical rule of inference* (Negri & von Plato 1998) or that of *rewrite rule* (Dowek et al. 2003). However, from a philosophical point of view, this way of proceeding eventually leads to a substantially revisionist position. Indeed, embracing an inferentialist and anti-realist theory of meaning not only leads to revisionism about logical

[6]It is worth noting that the fact of not limiting to the simple analysis of the provability level, but to investigate theorems also from the point of view of the the structural analysis of proofs is one of the leitmotifs of Kreisel's work (see for example Kreisel 1987, p. 399).

[7]This seems to be indeed the idea expressed by Bourbaki (1994, p. 17) when they say: « "Mathematical truth" resides thus uniquely in logical deduction starting from premises arbitrarily set by axioms.»

[8]This immediacy corresponds to the idea that proofs are *epistemic transparent* objects (Usberti 1997, p. 535): it is not possible that something is a proof without the possibility (for a human agent) to recognize it as such (cf. Kreisel 1962, p. 202).

constants (by abandoning classical operators for intuitionistic ones; see e.g. Dummett 1973; Dummett 1991, pp. 291-300), but it also leads to revisionism about the commonly accepted view of mathematical theories, namely by abandoning the standard conception of a theory as a set of axioms and replacing it with the conception of a theory as a set of postulates (i.e. hypothetical actions or *Erzeugungsprinzipien*; see e.g. von Plato 2007, p. 199) or as an algorithm (i.e. a set of computational instructions, see Dowek 2010). These solutions are analyzed in details in Naibo (2013, Part III).

What we propose in this paper is a way to save a theory of meaning essentially based on the notion of inference and proof, but without necessarily abandoning the notion of axiom. In order to do that, we will adopt what can be called an *interactional* point of view. In contrast to standard Dummettian inferentialism, we will extend our set of key semantic concepts by accepting not only objects that exclusively contain correct instantiations of axioms and rules – as in the case of canonical proofs – but also objects that contain incorrect ones. For this reason, such objects will be called *paraproofs*. Differently from proofs, they are essentially *untyped* objects. This means that types – i.e. propositions or sentences – are no longer conceived as the primitive entities on which the inference rules act, but become the outcome of the interaction between paraproofs. A quite natural setting to model this notion of interaction is the *computational* one and especially the so-called Curry-Howard correspondence will be our starting point (see Sørensen & Urzyczyn 2006 for a comprehensive presentation). From such of a perspective, a formal correlation between proofs and programs is established; in particular, proofs can be seen as the surface linguistic "description" of the inner intensional behavior of programs. Our idea is then to show how it is possible to construct semantical aspects starting from the behavior of programs. More precisely, our aim is to show that knowing the meaning of an axiom does not consist in knowing which objects and structures make it true, but in knowing what is the computational behavior of the program associated to it, once the axiom in question has been put in interaction with other programs.

In a nutshell, we put forward an inferentialist approach, alternative to the usual Dummett-Prawitz verificationism. Our account is compatible with the viewpoint of classical logicians, in particular those who wish to remain (ontologically) parsimonious in the definition of semantical notions like meaning and truth.

2 Axioms and computation

From a historical point of view, the proofs-as-programs correspondence was first established between (deductive systems for) constructive logics and abstract functional programming languages, and particularly between minimal or intuitionistic natural deduction and λ-calculus. In this setting, the execution of a program – i.e. a λ-term – having specification A corresponds to the normalization of a natural deduction proof having A as conclusion. More precisely, each (local) elimination of a *detour* taking the form of a c-introductio/c-elimation sequence – where c is a logical connective – corresponds to a step of program execution. For example, the elimination of a $\rightarrow$ *detour* corresponds to the execution of one step of β-reduction, that is, one step of the computation of the value taken by the function/program $\lambda x.t$ once the latter is applied to the input u:

$$\dfrac{\dfrac{\begin{matrix}[x:A]^{(m.)}\\ \vdots \\ t:B\end{matrix}}{\lambda x.t:A\rightarrow B}\rightarrow\text{intro}\qquad \begin{matrix}\vdots\\ u:A\end{matrix}}{(\lambda x.t)u:B}\rightarrow\text{elim }(m.)\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ u:A\\ \vdots\\ t[u/x]:B\end{matrix}$$

Computation seems then to be necessarily tied to non-atomic – i.e. complex – types, namely the maximal formulas of the *detours*.[9] In this respect, it is worth noting that there are two types of formulas that can never appear as maximal formulas: proper axioms and assumptions.[10] The reason is trivial. Proper axioms and assumptions are always the starting points of derivations, therefore they cannot be preceded by an introduction rule and thus no detour can be created.

The analysis just sketched can be made more precise by appealing to two properties resulting from a generalization of Prawitz's translation of natural deduction into sequent calculus (Prawitz 1965, pp. 90-91; von Plato 2003, § 5) . Such generalization allow to work not only with normal proofs, but also with non-normal ones (for details see Pereira 1982, Part C).[11] We will work with this translation because it is faithful from the computational point of view: *only* detours are translated into instances of the cut rule. In this way, cut-formulas coincide with maximal formulas and thus cuts are always non-

[9]For the notion of maximal formula see Dummett (1977, p. 152). For the notation of λ-calculus see Krivine (1993).

[10]Notice that the difference between proper axioms and assumptions is that the former can never be discharged, while the latter are in principle always dischargeable (even if *de facto* they are not). At the level of proof-objects – i.e. at the level of the objects used for codifying derivational steps, as λ-terms (see Sundholm 1998, pp. 196-197) – the difference is that proper axioms correspond to proof-term constants, while assumptions to proof-term variables. Roughly speaking, proper axioms are sentences which are to be considered as already having been proved, and therefore which can always be justified (see Heyting 1962, p. 239). Assumptions, on the other hand, are placeholders: they wait to be justified by a proof that we neither possess nor know to be constructible.

[11]The original Prawitz' translation works for systems of minimal, intuitionistic and classical logic. It is worth noting that Prawitz treats sequents as composed by sets of formulas. However, his translation can be adapted to the case of sequents considered as composed by multisets. In this case, the translation is directed either to the sequent calculus system **G1[mic]** or **G2[mic]** (see Troelstra & Schwichtenberg 2000, pp. 61-66 for a presentation of these systems).

atomic.[12] Furthermore, Prawitz' translation operates by transforming proper axioms – or their instances, in the case of axiom schemes – into part of the derivation, i.e. by moving proper axioms – or their instances – from the top position in a natural deduction derivation to the lefthand side of sequents. For example, consider the theory of equality presented by the two axioms

$$\textbf{(Ref)} \quad \forall x(x = x)$$

$$\textbf{(Euc)} \quad \forall x \forall y \forall z(x = y \wedge x = z \to y = z)$$

The non-normal proof in natural deduction shown in Figure 1(a) is translated as the sequent calculus proof shown in Figure 1(b).

The immediate consequence of the translation is that two types of formulas are excluded from the set of cut-formulas:

1. The formulas that are proper axioms.

2. The principal formulas of logical axioms (Negri & von Plato 2001, p. 16), also called *identity axioms* (Girard et al. 1989, p. 30).

And since a fundamental property of any "good" sequent calculus system is the possibility of "atomizing" the principal formulas of identity axioms (Wansing 2000, pp. 10-11),[13] we can replace 2. by

2*. All the atomic formulas appearing in the logical (i.e. identity) axioms.

This means that from the computational point of view, proper axioms and (atomic) identity axioms are identified: neither of them plays any role in the execution of a program.[14] They have no genuine computational content, as they are just the external borders of proofs. In other words, the *dynamics* of

[12] In Gentzen's translation (Gentzen 1934-35, § 4), differently from the translation chosen here, normal proofs are also translated into proofs with non-atomic cuts, because elimination rules are translated by appealing to cuts.

[13] By the translation provided above we can assign a well defined computational content to cut elimination, i.e., β-reduction. Analogously, the property of identity axiom atomization can be assigned a computational operation, i.e., η-expansion. This operation guarantees the possibility of working in an extensional setting even in the case of programs, which are by definition intensional objects (see Hindley & Seldin 2008, pp. 76-77). For further details see Naibo & Petrolo (2014).

[14] The λ-term associated to the previous natural deduction proof is $(\mathtt{t})\langle(\lambda x(\mathtt{t})\langle x,\mathtt{r}\rangle)\pi_1(z),\pi_2(z)\rangle$, where $\mathtt{r}$ and $\mathtt{t}$ are two proof-constants associated with the reflexivity and Euclidean axioms respectively, and $\pi_1(z)$ and $\pi_2(z)$ are the first and second projection of z. Reducing the *β-redex* contained in this λ-term – which corresponds to the detour of the proof that this λ-term codifies – we get $(\mathtt{t})\langle(\mathtt{t})\langle\pi_1(z),\mathtt{r}\rangle,\pi_2(z)\rangle$. It is not difficult to see that the constants $\mathtt{t}$ and $\mathtt{r}$, as well as the variable z, are not involved in the process of reduction. This means that proof-constants have no interaction with the other proof-constructors and that we cannot assign to them any computational content. Proof-objects in this case have only the role of codifying the structure of the proofs to which they are associated with, but they cannot be interpreted as programs.

$$
\begin{prooftree}
\AxiomC{$\mathbf{Euc}$}
\noLine
\UnaryInfC{$\forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z)$}
\RightLabel{\scriptsize $\forall$ elim}
\UnaryInfC{$b = a \wedge b = c \rightarrow a = c$}
\AxiomC{$\mathbf{Euc}$}
\noLine
\UnaryInfC{$\forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z)$}
\RightLabel{\scriptsize $\forall$ elim}
\UnaryInfC{$a = b \wedge a = a \rightarrow b = a$}
\AxiomC{$[a = b]^{(1.)}$}
\AxiomC{$\mathbf{Ref}$}
\noLine
\UnaryInfC{$\forall x(x = x)$}
\RightLabel{\scriptsize $\forall$ elim}
\UnaryInfC{$a = a$}
\RightLabel{\scriptsize $\wedge$ intro}
\BinaryInfC{$a = b \wedge a = a$}
\RightLabel{\scriptsize $\rightarrow$ elim}
\BinaryInfC{$b = a$}
\RightLabel{\scriptsize $\rightarrow$ intro (1.)}
\UnaryInfC{$a = b \rightarrow b = a$}
\AxiomC{$a = b \wedge b = c$}
\RightLabel{\scriptsize $\wedge$ elim$_1$}
\UnaryInfC{$a = b$}
\RightLabel{\scriptsize $\rightarrow$ elim}
\BinaryInfC{$b = a$}
\AxiomC{$a = b \wedge b = c$}
\RightLabel{\scriptsize $\wedge$ elim$_2$}
\UnaryInfC{$b = c$}
\RightLabel{\scriptsize $\wedge$ intro}
\BinaryInfC{$b = a \wedge b = c$}
\RightLabel{\scriptsize $\rightarrow$ elim}
\BinaryInfC{$a = c$}
\end{prooftree}
$$

(a) Natural Deduction Proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$a = b \vdash a = b$}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$a = a \vdash a = a$}
\RightLabel{\scriptsize $\forall_L$}
\UnaryInfC{$\forall x(x = x) \vdash a = a$}
\RightLabel{\scriptsize $\wedge_R$}
\BinaryInfC{$a = b, \forall x(x = x) \vdash a = b \wedge a = a$}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$b = a \vdash b = a$}
\RightLabel{\scriptsize $\rightarrow_L$}
\BinaryInfC{$a = b, \forall x(x = x), a = b \wedge a = a \rightarrow b = a \vdash b = a$}
\RightLabel{\scriptsize $\forall_L$}
\UnaryInfC{$a = b, \forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z) \vdash b = a$}
\RightLabel{\scriptsize $\forall_L$}
\UnaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z) \vdash a = b \rightarrow b = a$}
\AxiomC{}
\UnaryInfC{$a = b \vdash a = b$}
\RightLabel{\scriptsize $\wedge_{L_1}$}
\UnaryInfC{$a = b \wedge b = c \vdash a = b$}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$b = a \vdash b = a$}
\RightLabel{\scriptsize $\rightarrow_L$}
\BinaryInfC{$a = b \wedge b = c, a = b \rightarrow b = a \vdash b = a$}
\RightLabel{\scriptsize Cut}
\BinaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z), a = b \wedge b = c \vdash b = a$}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$b = c \vdash b = c$}
\RightLabel{\scriptsize $\wedge_{L_2}$}
\UnaryInfC{$a = b \wedge b = c \vdash b = c$}
\RightLabel{\scriptsize $\wedge_R$}
\BinaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z), a = b \wedge b = c \vdash b = a \wedge b = c$}
\AxiomC{}
\RightLabel{\scriptsize ax}
\UnaryInfC{$a = c \vdash a = c$}
\RightLabel{\scriptsize $\rightarrow_L$}
\BinaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z), a = b \wedge b = c, b = a \wedge b = c \rightarrow a = c \vdash a = c$}
\RightLabel{\scriptsize $\forall_L$}
\UnaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z), a = b \wedge b = c, \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z) \vdash a = c$}
\RightLabel{\scriptsize Ctr}
\UnaryInfC{$\forall x(x = x), \forall x \forall y \forall z(x = y \wedge x = z \rightarrow y = z), a = b \wedge b = c \vdash a = c$}
\end{prooftree}
$$

(b) Sequent Calculus Proof

Figure 1: Example of translation from natural deduction to sequent calculus

proofs, when conceived of as cut elimination, do not tell us much about the role played by both proper and identity axioms in formal proofs. But what if we look at other dynamical aspects of proofs such as proof-search procedures (i.e. bottom-up proof reconstructions)? Do these procedures give us more information about the proof-theoretical role of axioms?

First, it should be noticed that by shifting the attention from cut-elimination procedures – seen as a program executions – to proof-search procedures also entails a shift of logical framework, from intuitionistic logic – or more generally constructive logics – to classical logic. The reason is that classical systems are much more suitable to proof-search than intuitionistic ones, because of the invertibility of all their logical rules (Troelstra & Schwichtenberg 2000, p. 79). Let us then concentrate on classical sequent calculus and consider a provable sequent of the form $\Gamma, A \vdash \Delta$, where A is a proper axiom or an instance of a proper axiom scheme. If we have an algorithmic procedure allowing to reconstruct the proof of the sequent, for example by working with a system of classical logic like **G3c**, the best result we can get is to decompose A into atomic sentences belonging to some initial identity axioms.[15] Again, we should conclude that proper axioms have no particular role in proofs: they are not different from other context formulas used in purely logical proofs, and everything can eventually be reduced to logical combinations of identity axioms. In order to prevent the transformation of proofs containing proper axioms into purely logical proofs, we have to block the logical decomposition of the axiom A. A tentative strategy would be to apply a proof-search procedure on $\Gamma, A \vdash \Delta$ within a system without left-rules. This strategy is equivalent to a proof-search procedure in a right-handed system with an additional initial sequent $\vdash A$. However, in such a system, every proof using $\vdash A$ uses cuts that cannot be eliminated (Girard 1987a, p. 125, Troelstra & Schwichtenberg 2000, p. 127). In general, cuts are an obstacle to the root-first reconstruction of proof; in order to determine whether a sequent $\Gamma \vdash \Delta$ is derivable by using a cut rule, we should check the derivability of the two sequents $\Gamma \vdash C$ and $C, \Gamma \vdash \Delta$ for any arbitrary formula C, which produces the immediate consequence of removing any bound on the proof-search. Therefore, a proof-search procedure that allows the recognition of all the theorems of the theory $\mathscr{T}$ containing the axiom A does not always terminate.

As a solution to this problem, we could still operate a proof-search on a given theorem B belonging to a certain theory $\mathscr{T}$ without requiring that the derivation closes – that is, without necessarily using the axioms of $\mathscr{T}$ as the initial sequent of the form $\vdash A$. More precisely, suppose that B is a formula such that there are no positive occurrences of existential formulas and no

[15]This point becomes even clearer when applied to proof-objects . While in natural deduction the proof-objects associated with proper axioms are constants, in sequent calculus they are complex λ-terms not containing any constant. This is because in sequent calculus proper axioms are constructed in the context of derivations, i.e. in the antecedent.

negative occurrences of universal formulas.[16] When we work in **G3c**, a sequent $\vdash B$ having an empty antecedent can always be univocally decomposed in a set of *basic sequents* of the form

$$\bot,\ldots,\bot,P_1,\ldots,P_m \vdash Q_1,\ldots,Q_n,\bot,\ldots,\bot \tag{1}$$

where P_i and Q_j are atoms (Negri & von Plato 2001, p. 50).

If B is a non-logical theorem, or even an axiom, then its decomposition leads to a (possibly infinite) set of « basic mathematical sequents » (Gentzen 1938, p. 257) containing at least one sequent of the form

$$P_1,\ldots,P_m \vdash Q_1,\ldots,Q_n,\bot,\ldots,\bot \tag{2}$$

where $P_i \not\equiv Q_j$ for every i and j (Negri & von Plato 2001, p. 51).

It is worth noticing that atomic identity axioms are just a particular case of (1), namely when there are no $\bot$ and $P_i \equiv Q_j$ for some i and j. This remark suggests the possibility of identifying a unique way to deal with both proper axioms and identity axioms. In the next section we propose a solution along these lines[17] by introducing a *generalized* axiom rule inspired to Girard (2001).

3 From proofs to models

In this section the attention will be focused on classical logic. This choice is not simply due to practical – if not even opportunistic – reasons related to the efficacy of classical systems over intuitionistic ones with respect to proof-search problems. There is in fact a deeper, conceptual reason that has its roots in the discussion carried out in § 1. As we mentioned there, our aim is to do justice to a non-revisionist stance with respect to the architecture of mathematical theories, where one of the characteristic features of the standard view is precisely that the underlying logic of mathematical theories is classical logic, and not intuitionistic logic.

3.1 Schütte's completeness proof revisited

We will now introduce a system that allows us to study both the syntactical and the semantical role played by identity and proper axioms from the unifying perspective of Schütte's completeness proof (see Schütte 1956). This system should be considered as a general framework for carrying out an abstract

[16] For the standard definition of positive and negative occurrences of a formula see Troelstra & Schwichtenberg (2000, p. 6).

[17] In the next section we will restrict to one-sided sequent systems. This choice is only dictated by a wish to ease the proof analysis. Notice that the result we presented above could be adapted to one-sided systems (i.e. without any formulas on the left of sequents) by replacing the two-sided notion of *basic sequent* with the corresponding notion of one-sided basic sequent, that is $\vdash P_1,\ldots,P_n,\bot,\ldots,\bot$ where the P_i are now either atoms or negations of atoms.

and formal study of the axioms, rather than a genuine deductive system. The reason for this, as it will be explained in more detail below, is that in order to have sufficient expressive power for speaking of every possible axioms we are obliged to flirt with inconsistency (cf. Propositions 4 and 16). Despite this feature of the system, its proper logical part can be singled out through the definition of some kind of *correctness criterion*. The idea is that this system represents a compact way to simultaneously deal with a family of deductive systems: by imposing some specific restrictions on the use of the generalized axiom rule it is possible to single out one specific deductive system at a time, and to characterize it as a logical or non-logical system. In other words, what we propose here is a system for studying proofs from an *abstract* point of view, where the abstraction concerns the form taken by the initial sequents.[18]

We start by presenting the propositional case, namely the system $\mathrm{pLK}^{\maltese}_{R}$. We will then move to the more interesting case of first order logic. Looking at the propositional part of the system will be sufficient to grasp its main features and understand how it works. Readers who are not interested in a finer analysis of the system are recommended to skip § 3.2.

Let $\mathscr{A}$ be a set of atomic formulas.

Definition 1 (Formulas and Sequents)**.** The set of formulas is inductively defined by the following grammar

$$F := P, Q \mid \neg P \mid F \vee F \mid F \wedge F \qquad (P, Q \in \mathscr{A})$$

A sequent $\Gamma \vdash \Delta$ is an ordered pair of multisets Γ, Δ of formulas.

Definition 2. The system $\mathrm{pLK}^{\maltese}_{R}$ is defined by the rules of figure 2.

$$\frac{}{\vdash \Gamma}\ \maltese^{At} \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge$$

Figure 2: $\mathrm{pLK}^{\maltese}_{R}$ rules

The only proviso on the application of $\maltese^{At}$, the generalized axiom rule, is that the formulas appearing in the sequent $\vdash \Gamma$ (if any at all) have to be atomic formulas or negations thereof.

Proposition 3 (Invertibility)**.** *The rules $\vee$ and $\wedge$ are invertible.*

[18]It must be remarked that at present there is no such discipline as *abstract proof theory* as a branch of mathematical logic, in the same sense as which there is, instead, an *abstract model theory*. Our proposal can be considered as a contribution to the attempt of defining such a discipline, which is different in nature from other attempts such as those undertaken in the categorial analysis of proofs (cf. Hyland 2002, §1).

Proof. See Appendix A. $\square$

Proposition 4. *Every sequent can be derived in* $\mathrm{pLK}_R^{\maltese}$.

Proof. By induction on the number of connectives in the sequent. $\square$

Definition 5. Given a proof π in $\mathrm{pLK}_R^{\maltese}$, $\mathfrak{L}(\pi)$ is the multiset of sequents introduced by the $\maltese$ rules in π. $\mathfrak{L}(\pi)$ is called the *set of leaves* of π.

Lemma 6. *Let* π *and* π' *be derivations of* $\vdash \Gamma$ *in* $\mathrm{pLK}_R^{\maltese}$, *then* $\mathfrak{L}(\pi) = \mathfrak{L}(\pi')$.

Proof. See Appendix A. $\square$

Remark 7. By Proposition 4 and the previous lemma we can now use the notation $\mathfrak{L}(\vdash \Gamma)$, for any sequent $\vdash \Gamma$.

Definition 8. A sequent $\vdash \Gamma$ is *correct* when it is atomic and there exists an atom P, such that P and $\neg P$ are both in $\vdash \Gamma$. By extension, a $\maltese^{At}$ rule is correct when the sequent it introduces is correct.

An *incorrect* sequent is an atomic sequent that is not correct.

Definition 9. The system pLK_R is obtained by replacing the $\maltese^{At}$ rule with a rule that introduces only correct sequents. This rule will be called logical axiom rule and noted in the following manner

$$\frac{}{\vdash \Gamma, P, \neg P}\ \mathrm{ax}$$

Proposition 10. *A sequent* $\vdash \Gamma$ *is derivable in* pLK_R *if and only if* $\mathfrak{L}(\vdash \Gamma)$ *contains only correct sequents.*

Proof. See Appendix A. $\square$

Definition 11. Let $\delta : \mathscr{A} \to \{0,1\}$ be a valuation, and $\bar{\delta}$ its extension to formulas of pLK_R.

- $\delta \vDash \Gamma$ if and only if there exists at least one $A \in \Gamma$ such that $\bar{\delta}(A) = 1$
- $\vDash \Gamma$ if and only if for all valuation δ, $\delta \vDash \Gamma$.

Lemma 12. *For any valuation* δ, $\delta \vDash \Gamma$ *if and only if for all* $\vdash \Delta$ *in* $\mathfrak{L}(\vdash \Gamma)$, $\delta \vDash \Delta$.

Proof. See Appendix A. $\square$

Proposition 13. $\vDash \Gamma$ *if and only if all sequents in* $\mathfrak{L}(\vdash \Gamma)$ *are correct.*

Proof. See Appendix A. $\square$

3.2 Embedding semantics in the syntax

We adapt the previous proof to the first-order case. We must take particular care of the $\maltese$ rule and of the definition of correctness. Notice that this case is more liberal than the propositional case: no conditions are imposed on the application of the $\maltese$ rule, which can be used at every point of the derivation.

Definition 14. Let $\mathrm{LK}_R^{\maltese}$ be the system defined by the rules of figure 3.

$$\frac{}{\vdash \Gamma}\ \maltese$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge$$

$$\frac{\vdash \Gamma, A[y/x]}{\vdash \Gamma, \forall x A(x)}\ \forall \quad (y \text{ fresh}) \qquad\qquad \frac{\vdash \Gamma, A(t), \exists x A(x)}{\vdash \Gamma, \exists x A(x)}\ \exists$$

Figure 3: The rules of $\mathrm{LK}_R^{\maltese}$ sequent calculus

Definition 15. A derivation of $\mathrm{LK}_R^{\maltese}$ is a finite tree obtained from the rules of figure 3 where all leaves are closed by using a $\maltese$ rule.

Proposition 16. *Every sequent $\vdash \Gamma$ is derivable in $\mathrm{LK}_R^{\maltese}$.*

Proof. Take $\vdash \Gamma$ and close the derivation by an instance of $\maltese$ rule. □

Definition 17. The $\maltese$ rule that introduces the sequent $\vdash \Gamma$ is correct if there exists a formula A such that both A and $\neg A$ are in Γ.

A derivation is correct if all its $\maltese$ rules are correct.

Definition 18. A $\maltese$ rule introducing a sequent $\vdash \Gamma$ is *admissible* if either it is correct or there exists a correct derivation of $\vdash \Gamma$ in $\mathrm{LK}_R^{\maltese}$.

Lemma 19. *If there exists a derivation π of $\vdash \Gamma$ in $\mathrm{LK}_R^{\maltese}$ such that $\mathfrak{L}(\pi)$ contains only admissible $\maltese$ rules, then π can be extended to a derivation π' of $\vdash \Gamma$ such that $\mathfrak{L}(\pi')$ contains only correct $\maltese$ rules.*

Proof. See Appendix B. □

Definition 20. The system LK_R is obtained by replacing the $\maltese$ rule with a rule that introduces only correct sequents. This rule will be called logical axiom rule and noted in the following manner

$$\frac{}{\vdash \Gamma, A, \neg A}\ \mathrm{ax}$$

Theorem 21. *The sequent $\vdash \Gamma$ is derivable in LK_R if and only if there exists a derivation π of $\vdash \Gamma$ in $\mathrm{LK}_R^{\maltese}$ and $\mathfrak{L}(\pi)$ contains only admissible $\maltese$ rule.*

Proof. See Appendix B. □

Definition 22. A ✠ rule is *simple* if the sequent $\vdash \Gamma$ it introduces contains only atoms, negations of atoms, and existential formulas.

A *simple derivation* is a derivation in which all ✠ rules are simple.

Lemma 23. *Let π be a derivation in $\mathrm{LK}_R^{✠}$. There exists a simple extension of π.*

Proof. See appendix B. □

Definition 24. Let $B = \exists x A$ be an existential formula. We define the *instances* of B to be the formulas $A[t/x]$ where t is a term.

More generally, if A is a formula and C a subformula of A, the set of *instances* of C is the set of formulas $C[t_1/x_1,\ldots,t_n/x_n]$ where $t_1,\ldots,t_n$ are terms and $x_1,\ldots,x_n$ are the bound variables of C.

Lemma 25. *Let π be the following derivation:*

$$\frac{}{\vdash \Gamma}\ ✠$$

If π is non-admissible, then we can find a sequence of extensions of π containing all the instances of the subformulas of Γ.

Proof. See Appendix B. □

Theorem 26. *Let π be a derivation in $\mathrm{LK}_R^{✠}$ of a sequent $\vdash \Gamma$ containing a non-admissible ✠ rule. Then there exists a model $\mathscr{M}$ such that $\mathscr{M} \not\models \Gamma$.*

Proof. See Appendix B. □

Theorem 27. *If π is a derivation of $\vdash \Gamma$ in $\mathrm{LK}_R^{✠}$ containing only admissible ✠ rules, then for all model $\mathscr{M}$, $\mathscr{M} \models \Gamma$.*

Corollary 28. *Let π and π' be derivations in $\mathrm{LK}_R^{✠}$ of the same sequent $\vdash \Gamma$. Then π contains only admissible ✠ rules if and only if π' contains only admissible ✠ rules.*

3.3 Axioms and models

Differently from what happens with the standard Schütte's completeness proof for classical logic, in the revisited proof we proposed – where $\vdash \Gamma$ is derived with non-admissible instances of ✠ rule – we cannot always conclude that $\bigvee \Gamma$ is an antilogy (i.e. a sentence which is false in every possible model).[19] In fact, $\bigvee \Gamma$ could be valid in some particular theories, namely theories whose models make true every non-admissible instance of ✠ rule used in the derivation of $\bigvee \Gamma$. From the derivation of $\bigvee \Gamma$ we can thus know more about the axioms and theorems of $\mathscr{T}$. In particular, the incorrect instances of ✠ rule correspond to a set of sequents $\mathscr{S}_1,\ldots,\mathscr{S}_n$ such that: either

[19] In other words, an antilogy is the negation of a tautology.

i. each $\mathscr{S}_i$ is provable in every axiomatic system containing Γ, or

ii. $\vdash \Gamma$ is provable in every axiomatic system containing $\mathscr{S}_1, \ldots, \mathscr{S}_n$.

This means that the proof-search in the system $\mathrm{LK}^{\maltese}_{R}$ does not always constitute a method for invalidating non-logical sentences. It can also be used to make explicit the set of conditions under which a non-tautology – i.e. a sentence which is a theorem of a particular theory $\mathscr{T}$[20] – is valid. The role played by the incorrect instances of $\maltese$ rule is to identify the particular class of models validating the non-tautology into question. Differently from the case of tautologies or antilogies, where either the whole class or the empty class of models is considered, in the case of non- tautologies the instances of the $\maltese$ rule single out a very specific and non-trivial class of models. It seems then that the proof-search dynamics leads to corroborate the standard view presented in §1 according to which the role of proper axioms is to identify classes of relational structures. However, the conceptual order is inverted here: structures are not primitive entities that are later syntactically fixed by the axioms, but they become generated from the syntactical features of the proof-search dynamics.

Nevertheless, these structures are not yet homogeneous with syntactical entities, since they are still set-theoretical entities. In what follows, we will present a general framework that allows to treat proofs and models – or better, countermodels – from a homogeneous point of view. In order to do that, we need to relax the usual notion of syntax. In order to make this idea clear, we will start with the problem of distinguishing between antilogies and non-tautologies.

4 Distinction between non-tautologies and antilogies

The framework of $\mathrm{LK}^{\maltese}_{R}$ presented above does not yet allow to distinguish between sequents that are non-tautologies, and antilogies. Let us consider the two following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \neg A, B}\maltese}{\vdash \neg A \vee B}\vee \qquad \dfrac{}{\vdash A}\maltese}{\vdash (\neg A \vee B) \wedge A}\wedge \qquad \dfrac{\dfrac{}{\vdash \neg B}\maltese \qquad \dfrac{}{\vdash \neg C}\maltese}{\vdash \neg B \wedge \neg C}\wedge}{\vdash ((\neg A \vee B) \wedge A) \wedge (\neg B \wedge \neg C)}\wedge$$

[20]In the literature, this kind of formulas are usually called *neutral formulas*. However, we prefer to avoid this terminology; even if these formulas are neutral form a logical point of view – they are neither tautology nor antilogy –, they are not neutral from the point of view of a particular theory T. Since our aim is to provide a framework that is applicable to specific mathematical theories, using this terminology could be misleading.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\maltese$}
\UnaryInfC{$\vdash A, \neg B$}
\RightLabel{$\vee$}
\UnaryInfC{$\vdash A \vee \neg B$}
\AxiomC{}
\RightLabel{$\maltese$}
\UnaryInfC{$\vdash A$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash (A \vee \neg B) \wedge A$}
\AxiomC{}
\RightLabel{$\maltese$}
\UnaryInfC{$\vdash \neg B$}
\AxiomC{}
\RightLabel{$\maltese$}
\UnaryInfC{$\vdash \neg C$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash \neg B \wedge \neg C$}
\RightLabel{$\wedge$}
\BinaryInfC{$\vdash ((A \vee \neg B) \wedge A) \wedge (\neg B \wedge \neg C)$}
\end{prooftree}
$$

The simple observation that the two proofs appeal to non correct instances of the $\maltese$ rule does not tell us anything about the fact that the concluding sequent is an antilogy or a non-tautology. The only way to distinguish between antilogies and non-tautologies is to check if there exists a valuation rendering all $\maltese$ rules true. For instance, such a valuation exists in the case of the second proof presented above (making A true, and B and C false), while it does not exist for the first one. We can thus conclude that the latter is a non-tautology, and the former is an antilogy. The problem with this way of distinguishing between non-tautologies and antilogies is that it appeals to the inspection of all possible valuations of all the $\maltese$ rules in the proof. Hence, this method is not exclusively based on proofs' inspection; moreover, it is not really effective (especially when dealing with first order logic). Ideally, we want to be able to recognize a non-tautology or an antilogy by means of a simple mechanical inspection of the proof.

A possible way to decide if a sentence is a non-tautology or an antilogy is to analyze not only the proof of this sentence, but also the proof of its negation. First, it is worth recalling that we are working in a framework where everything is derivable: no particular constraints were imposed on the application of the $\maltese^{At}$ rule. For example, the $\maltese^{At}$ rule can be applied also when $\Gamma = \emptyset$, and thus the empty sequent "$\vdash$" can be derived in $\mathrm{pLK}_R^{\maltese}$. In the object language, the empty sequent represents the idea that an unspecified absurdity[21] – namely, the empty succedent – is derivable from any kind of hypothesis – namely, the empty antecedent (see Paoli 2002, p. 32). Deriving the empty sequent corresponds therefore to deriving an absurdity as a theorem, and thus to showing that $\mathrm{pLK}_R^{\maltese}$ is inconsistent. In order to prevent the system from being inconsistent, an *ad hoc* solution is to impose a constraint on the application of the $\maltese^{At}$ rule, namely that $\Gamma \neq \emptyset$. In this way, even if any formula can still become a theorem, the system remains consistent. We would be then in a situation complementary to the one advocated by paraconsistent logics: $\mathrm{pLK}_R^{\maltese}$ would be a trivial but consistent system. In fact, it would be possible to obtain an empty sequent only by appealing to the cut rule, which would allow us to derive the empty sequent from the derivable sequents $\vdash A$ and $\vdash \neg A$, for a certain A. If it was the case, we would be able to characterize absurdity negatively, as something for which we do not possess a canonical derivation – that is a derivation terminating with the rule corresponding to the principal connective of the conclusion-formula. This is exactly the expla-

[21]Absurdity is seen here in a Brouwerian perspective, that is, as something that interrupts a derivation (Brouwer 1908, p. 109). In absence of any formula, no rule corresponding to a logical connective can be applied, and thus the derivation cannot be further carried on.

nation of absurdity given by verificationist accounts (see Sundholm 1983, p. 485; Martin-Löf 1996, p. 51).

Such an explanation, however, does not fit with the notion of proof as characterized by the system $\mathrm{pLK}_R^{\maltese}$ (and more generally, $\mathrm{LK}_R^{\maltese}$). Firstly, using a multi-succedent calculus makes it difficult to identify the conclusion-formula of a derivation. Secondly, and more importantly, it is not always clear *what kind* of absurdity is obtained by cutting a proof of $\vdash A$ with one of $\vdash \neg A$. More specifically, if A is a tautology, then $\neg A$ is an antilogy, and therefore the empty sequent is also an antilogy. But if both A and $\neg A$ are non-tautologies, then we are not entitled to conclude that the empty sequent is an antilogy. We would then be in a situation where we have different proofs of the empty sequent but we cannot establish whether they are a proof of the same theorem. Despite such difficulties, we could still recognize different proofs of the same theorem if we had a cut elimination procedure that allows us to show that all these proofs of the empty sequent are reducible to the same cut-free proof. Now, cut admissibility for the system LK_R is a corollary of Schütte's completeness proof but this result is not effective with respect to proofs transformations.[22] The result simply states that if we have a proof with cuts then there exists a proof of the same sequent without cuts, but it does not provide an algorithm for transforming the proof with cuts into the cut free one, nor it tells us anything about the form of the cut free proof. As a consequence, it cannot then be used to establish the identity of proof results. If we want to define a cut elimination algorithm for $\mathrm{LK}_R^{\maltese}$ we will have to appeal to the admissibility of the structural rules of weakening and contraction. Here, the problem is that the cut elimination procedure defined in this way is not local, and it does not give any information about the evolution of the set of instances of the $\maltese$ rule during the process of cut reduction. This last point is particularly crucial because the system $\mathrm{LK}_R^{\maltese}$ is intended to be a tool for the study of what set of instances of the $\maltese$ rule are used in order to derive a particular sequent. If the set of instances changes during the process of cut elimination, then our analysis is likely to fail.

5 Liberalizing syntax

In this section we define a framework that – like $\mathrm{LK}_R^{\maltese}$ – allows us to define logically incorrect proofs, but that at the same time also allows algorithmic

[22]It would be incorrect to claim that Schütte's demonstration of cut admissibility through completeness is *tout court* non-effective. In a purely logical setting, given a valid sequent $\vdash \Gamma$ – i.e. $\vDash \Gamma$ – it is possible to effectively enumerate all the proofs of the theorems of LK_R until a proof of Γ is found. Since LK_R does not contain the cut rule, this is a cut-free proof. However, as Kreisel (1958, p. 167) remarks, « [it] is not an algorithm at all in the sense of a working mathematician, because it depends on 'trying out all proofs of the subject', i.e. it is a systematic method of trial and error. » In other words, the algorithm in question in not an *efficient* one.

cut elimination procedures that do not alter the set of proper axioms, i.e. the set of instances of the ✠ rule. Our general aim is to produce a framework which is capable of both (i) generating a semantics from syntactical procedures, and (ii) assigning an interesting computational interpretation to these procedures, an interpretation that is not limited to the mere availability of a proof search algorithm. In such a framework, the semantical role of axioms should be explained without appealing to the set-theoretic notion of a model, which is based on a primitive and epistemic-transcending notion of truth. In order to do that, we will define the notion of truth over that of proof.

This approach differs from the standard inferentialist one in that proofs are analyzed with respect to their computational content, while standard inferentialism only focuses on the order of applications of the rules. Within this perspective, proofs are not regarded as singular objects of study but they are always considered in connection with a given environment: if proofs correspond to programs, then their computational behavior can be detected only inside a context of evaluation, namely a context composed by other proofs/programs. Since proofs are studied in their interaction with other proofs by means of the Cut rule, the perspective adopted here can be characterised as a *global* perspective on proofs. On the contrary, a *local* perspective on proofs consists in studying how the structure of a given (single) proof can be rearranged by means of proofs transformations (e.g. rules commutations).[23]

In order to define a framework for a global account of proof, we need to liberalize our syntax. This is due to the fact that proofs are usually presented as trees, and this presentation forces us to interpret them as ordered sequences of inference rules. In contrast to this approach, we propose to look at proofs from a geometrical point of view, where the order of application of the rules is irrelevant. On this view, the emphasis is put on the "spatial" configuration of the premisses and conclusions of the rules, and on the transformations that can be operated on these configurations while keeping them invariant. Hence, the most appropriate objects for codifying such perspective are no longer the syntactical objects inductively generated by a grammar, but should be some kind of mathematical objects which do not necessarily represent ordered or inductive structures. Furthermore, the operations definable over these objects will have a computational content, so that the desired proofs-as-programs paradigm is respected.

Let us now present in some more detail the perspective just sketched.

[23]The local/global distinction is inspired by the terminology adopted in computer science to specify how the behavior of programs is studied. The application of this distinction has been first introduced by Paoli (2002) in order to analyze the inferential properties of proofs, and successively used by Poggiolesi (2011) and Hjortland (2012). More precisely, they claim that the meaning of logical constants depends both on the shape of the premisses and conclusions of the rules governing the inferential behavior of each specific connective, and on the way these rules interact with the others, particularly during the process of cut elimination.

5.1 Proof nets and axioms

The most suitable framework for liberalizing syntax is, in our opinion, *linear logic* (Girard 1987b). First of all, it should be noticed that adopting such a point of view does not put into question the classical point of view that we defend in this paper. The reason is that linear logic is nothing but a way to analyze classical logic at the microscope, namely by controlling the use of structural rules of weakening and contraction. This control is obtained from the decomposition of standard implication $A \to B$ into two distinct operations: a linear implication $\multimap$, and an exponential modality ! allowing the repeated use of the argument of type A. This refinement let emerge from the set of rules for classical logic two sets of rules: the set of rules with shared derivational contexts – also known as *additive rules* – and the set of rules with independent derivational contexts – also known as *multiplicative rules* (see Di Cosmo & Miller 2001, §2.1).[24] For the purposes of the paper we can limit our analysis to the multiplicative fragment of linear logic, **MLL**, which is composed of: a closure operator $\perp$ corresponding to an involutive negation (more on this will be said later), multiplicative conjunction $\otimes$, and its De Morgan's dual, i.e. multiplicative disjunction $⅋$. Linear implication, instead, is definable in the same way as in classical logic, i.e. $A \multimap B \equiv A^{\perp} ⅋ B$. The rules corresponding to these connectives are the following:

$$\frac{}{\vdash P, P^{\perp}}\,\text{ax}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad\qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}\,⅋$$

and the cut rule is:

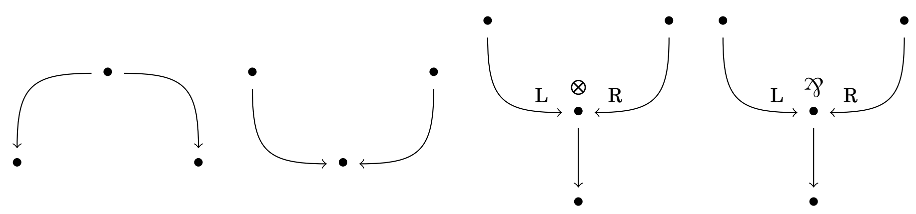
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,\text{Cut}$$

What makes linear logic particularly interesting for our discussion is the way in which proofs can be represented, or better, interpreted. The idea is to consider semantical entities which allow to deal with proof systems analogous to multi-conclusion natural deduction. In order to do that, we have to abandon the syntactic and linguistic analysis of proofs and replace it with a purely geometrical analysis. In this context, formulas are no longer interpretable as linguistic acts, but objets organized according to certain spatial relations over

[24]This is correct only in the case of binary rules, while it is less clear in the case of unary rules. When there is only one premiss there is only one context of derivation, and thus the problem of sharing it or splitting it does not arise. On the other hand, the presence of all the immediate subformulas of the conclusion of a unary rule in the premiss signals that in order to reconstruct the proof we need to follow all these formulas, since they are not derivable form the very same context. And this means that we are in presence of a multiplicative rule. Hence, the distinction we traced is not ambiguous as it could have been thought at first sight.

which invariant transformations can be executed. In other words, a formula is identified by the position it occupies, and not by its syntactical form.

The notion of *proof nets* introduced by Girard (1987b) rests on this very idea. More precisely, a proof is represented by a graph[25] constructed from basic elements representing the axioms, the connectives, and the cut rule (see Figure 4). A graph obtained in this way is called a *proof structure*[26]. Every sequent calculus proof can be represented as a proof structure, even though this correspondence is not injective. The non injectivity of this representation is the main motivation for the definition of proof structures which are meant to represent the quotient of sequent calculus proofs up to uninformative commutations of rules.



(a) Axiom Brick (b) Cut Brick (c) Tensor Brick (d) Parr Brick

Figure 4: Basic bricks for proof structures

The main interest of proof nets lies in the fact that the syntax of proof structures is extremely tolerant, and it allows to construct graphs that do not come from a sequent calculus proof, such as the proof structures in Figure 5. Those proof structures arising as representations of sequence calculus proofs are called sequentializable. Of course, this would not be helpful at all if we were not able to distinguish sequentializable proof structures: for this purpose one defines the notion of *proof net* as a proof structure that satisfies a given geometrical or topological property – called a correctness criterion, and then shows that proof nets are exactly the sequentializable proof structures.

[25]Here we consider *directed simple graphs*, i.e. directed graphs with, for all vertices a, b, at most one edge of source a and target b.

[26]The terminological choice adopted here is intended to convey the idea that a proof refers to a structure not only as a syntactical entity, but also as a semantical one, as we have seen in §3.3 and will clarify later, especially in §5.2.

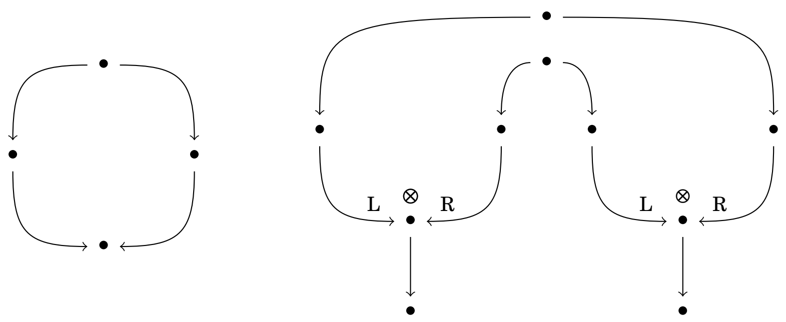


Figure 5: Proof Structures that are not Proof Nets

5.1.1 Correctness criterions

Many correctness criterions are available (see Seiller 2012c, §2.3 for a survey), though they all share the same global idea (Seiller 2012c, pp. 24-25). Given a proof structure $\mathscr{R}$:

1. we define a family S of objects (call them *tests*);
2. we show that $\mathscr{R}$ is sequentializable if and only if all elements in S satisfy a given property.

The similarity runs deeper if we are a little more precise: in each case, the elements of S can be defined by a proof structure without its axioms, (i.e. a proof structure $\mathscr{R}$ where the axiom vertices and their ingoing/outgoing edges have been erased). The second part of the criterion then describes how the axioms interact with this axiom-less part of the proof structure, which will be denoted by $\mathscr{R}_t$. The only difference between two proof nets corresponding to the same formula A consists in their axioms, the graph $\mathscr{R}_t$ being defined uniquely from A. A set of axioms can thus be considered as an *untyped proof* – noted with $\mathscr{R}_a$ – and $\mathscr{R}_t$ as a type. The correctness criterion is then simply a typing criterion: if a set of axioms (an untyped proof) together with a type $\mathscr{R}_t$ yields a proof net, then this proof can be typed by the formula defining $\mathscr{R}_t$.

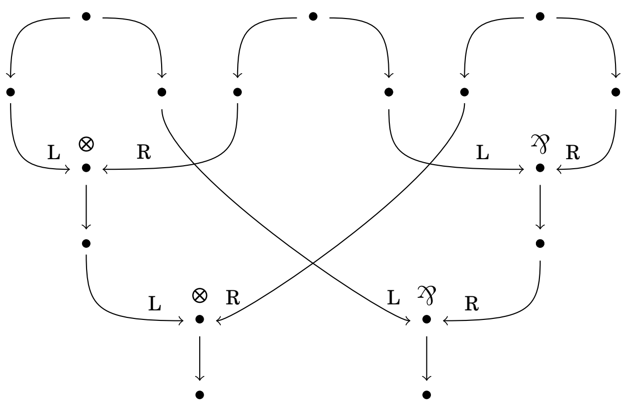
From now on, we will consider the correctness criterion based on the use of permutations (Girard 2011; Seiller 2102c):[27]

- from $\mathscr{R}_a$ one defines a permutation σ_a;
- from $\mathscr{R}_t$, the set of tests S is defined as a set of permutations;

[27] Roughly speaking, the idea is to take a graph, drop all the formulas labelling its nodes, and label again only the nodes of the axiom bricks with natural numbers (counting from left to right). The different paths that can be defined through the graph – or through its subgraphs $\mathscr{R}_a$ and $\mathscr{R}_t$ – induce a set of permutations on the given (finite) set of natural numbers.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash P,P^{\perp}}\,\mathrm{ax} \qquad \dfrac{}{\vdash Q,Q^{\perp}}\,\mathrm{ax}}{\vdash P\otimes Q,P^{\perp},Q^{\perp}}\,\otimes \qquad \dfrac{}{\vdash R,R^{\perp}}\,\mathrm{ax}}{\vdash (P\otimes Q)\otimes R,P^{\perp},Q^{\perp},R^{\perp}}\,\otimes}{\vdash (P\otimes Q)\otimes R,P^{\perp},Q^{\perp}\parr R^{\perp}}\,\parr}{\vdash (P\otimes Q)\otimes R,P^{\perp}\parr(Q^{\perp}\parr R^{\perp})}\,\parr$$

(a) Sequent Calculus Proof



(b) Proof Net

Figure 6: Proof of Associativity of the Tensor.
Note that atoms have disappeared when representing the proof as a proof net. This illustrates the useful and deep peculiarity of proof nets: the same proof net represent the proof of associativity for the tensor between atoms P,Q,R as shown above, as well as the proofs of associativity for the tensor of any triple A,B,C of formulas. In a way, a proof net represents a *scheme of proof* more than a single proof.

- $\mathscr{R}$ is a proof net if and only if for all $\tau \in S$, $\sigma_a \tau$ is a cyclic permutation.

We will say that two permutations σ, τ are *orthogonal* when their composition $\sigma\tau$ is cyclic.

Interestingly, the tests associated to $\mathscr{R}_t$ can be understood as counter-models. Indeed, if an untyped proof $\mathscr{R}_a$ cannot be typed by $\mathscr{R}_t$, it means that $\mathscr{R}_a$ is not a proof of the formula corresponding to $\mathscr{R}_t$. But the fact that $\mathscr{R}_a$ cannot be typed by $\mathscr{R}_t$ amounts to the existence of a test in S such that the product of σ_a (the permutation associated to $\mathscr{R}_a$) and τ is not cyclic. Showing that an untyped proof $\mathscr{R}_a$ is not a proof of a formula A then boils down to finding a test of A that is not passed by $\mathscr{R}_a$, in the same way in which a derivation in $\mathrm{LK}_R^{\maltese}$ can be shown incorrect by finding a counter-model that falsifies the set of axioms.

5.1.2 Cut elimination

A cut elimination procedure can be defined; such procedure is compatible with the interpretation of proof nets $\mathscr{R}$ as a pair consisting in an untyped proof $\mathscr{R}_a$ together with a type $\mathscr{R}_t$. This cut elimination procedure is strongly normalizing and we can therefore choose particular strategies of reduction. Let $(\mathscr{R}_a, \mathscr{R}_t)$ and $(\mathscr{P}_a, \mathscr{P}_t)$ be two proof nets linked by a cut rule. We now consider the reduction strategies that first eliminate all cuts between $\mathscr{R}_t$ and $\mathscr{P}_t$, and then eliminate cuts between $\mathscr{R}_a$ and $\mathscr{P}_a$. This decomposition leads to the following interpretation:

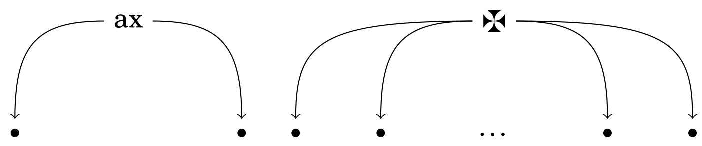
- the cut elimination between types ensures that the specifications are compatible: if a cut cannot be eliminated then the strategy stops, which indicates that the two untyped proofs were not typed properly;

- the cut elimination between types, when successful, has no real computational meaning: it only defines a type $\mathscr{Q}_t$ and describes how the untyped proofs $\mathscr{R}_a$ and $\mathscr{P}_a$ are *plugged* together;

- the cut elimination between untyped proofs bears the computational content, and yields an untyped proof $\mathscr{Q}_a$ such that $(\mathscr{Q}_a, \mathscr{Q}_t)$ is a proof net.

5.1.3 Generalized axioms

As it is the case for the framework described in Section 3, it is possible to extend the proof structure syntax by considering generalized axioms.



(a) Identity Axiom (b) Generalized Axiom

Figure 7: Generalizing Axioms in Proof Structures

In this setting, the generalized axioms represent a cyclic permutation and allow the derivation of any formula, in the same way as the as the $\maltese$ rule allowed the derivation of any formula in the system pLK$^{\maltese}_{R}$. But the change of paradigm, from sequent calculus to proof nets, reinforced the role of these generalized axioms. Once again, we can write such a generalized proof net as a pair $(\mathscr{R}_a, \mathscr{R}_t)$ composed by a type $\mathscr{R}_t$ and a *paraproof* $\mathscr{R}_a$. Paraproofs do not necessarily contain correct instantiations of axioms and rules. In this setting, one can prove that there is a correspondance between paraproofs of a formula A and the tests of its dual $A^{\perp}$. Let $\mathfrak{P}(A)$ denote the set of paraproofs $\mathscr{R}_a$ such that the pair $(\mathscr{R}_a, \mathscr{R}_t)$ is a proof net, where $\mathscr{R}_t$ is the type corresponding to a formula A. Then $\mathfrak{P}(A)$ corresponds to the orthogonal closure of the set of tests defined by the type $\mathscr{P}_t$ corresponding to the formula $A^{\perp}$.

The circle is now complete:

- an untyped paraproof can be given a type A if and only if it is orthogonal to the tests for A;
- an untyped paraproof is a test for the type A if and only if it is orthogonal to the proofs of A, i.e. proofs are tests for tests.

These remarks on proof structures support the idea that generalized axioms are a way of adding counter-models to the syntax. As we will see in the next section, this idea can be even used to redefine a logic where the objects are generalized untyped proofs, and the formulas are defined interactively.

5.2 Untyped proof theory

The ideas expounded in § 5.1.3 lead to the definition of the first version of a *geometry of interaction*, where basic objects are permutations (Girard 1988). This construction was then generalized to include of more expressive fragments of linear logic. In this section, we will describe this type of constructions in a very general way. We will focus in particular on *Ludics* (Girard 2001), which inspired the ideas developed in §6. Such a framework will be called *untyped proof theory*.

Definition 29. An untyped proof theory is given by:

- A set of *untyped paraproofs* $\mathscr{U}$;

- A notion of execution $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$, denoted here by $a, b \mapsto a :: b$;
- A notion of termination given by a set of untyped proofs $\Omega \subset \mathcal{U}$.

We can then construct everything from the notion of execution. First, the notion of *orthogonality* is defined: two paraproofs a, b are orthogonal – denoted $a \perp b$ – if and only if their execution $a :: b$ is in Ω. It is worth noting that no particular constraints are imposed on Ω; this means that the notion of termination is not absolute, but relative to what we consider a terminating configuration for the untyped proofs under analysis. Hence, untyped proof theory represents an extremely flexible computational framework. From a more mathematical point of view, the notion of orthogonality intuitively corresponds to the phenomenon that occurs between generalized axioms in proof structures: if $a \perp b$, then a (resp. b) must be a paraproof of a formula A (resp. $A^{\perp}$), or equivalently a test for $A^{\perp}$ (resp. for A). Following up on this parallel, *types* can be defined as the sets of paraproofs that are equal to their bi-orthogonal. Equivalently, a type is defined as a set of paraproofs A such that there exists another set of paraproofs B with $A = B^{\perp} = \{a \mid \forall b \in B, a \perp b\}$, i.e. a type is a set of paraproofs that pass a given set of tests B.

We can now define logical constants as constructions on paraproofs; these constructions in turn induce constructions on types. Specific constructions would then allow to recover fragments or even full linear logic. Below are three examples of such ideas.[28]

Example 30. The first construction based on permutations mentioned at the beginning of this section can be enriched in order to obtain a construction for **MLL** with units (Seiller 2012b), for **MALL** with additive units (Seiller 2012a), for Elementary Linear Logic (Seiller 2013, 2014) – a subsystem of linear logic that characterizes the set of functions computable in elementary time. In this setting, the set of paraproofs is a set of pairs of a graph and a real number, the notion of execution is based on the graph of alternating paths between two graphs, and the notion of termination is given by the set of pairs $(a, \emptyset)$, where $a \neq 0$ is a real number and $\emptyset$ denotes the empty graph on an empty set of vertices.

Example 31. In Ludics, the set of untyped paraproofs is the set of *designs-desseins*, the execution is the cut elimination procedure over these objects, and the notion of termination is the set of dessein containing a single design: the daimon.

Example 32. In the latest version of geometry of interaction, the set of untyped paraproofs is defined as the set of *projects*, while the execution is defined as the solution to the *feedback equation* and the termination is defined as the set of conducts of empty carrier with a non-null wager.

[28]A full analysis of logical constants from the untyped perspective is part of ongoing researches. See Naibo et al. (2011).

It is the possible to characterize in theses frameworks which paraproofs correspond to proofs in the same way as it is possible to identify correct derivations among the derivations of $\mathrm{LK}^{\maltese}_{R}$. These objects, which are called *successful* – or *winning*, to emphasize their relation to winning strategies in game semantics – can be tested against unsuccessful ones. The latter correspond to counter-models. We are thus in a framework where the syntactical and semantical (in the classical sense) aspects of logic are both represented in a homogeneous way. More specifically, distinguishing antilogies from non-tautologies does not involve anymore the verification of each counter-model of some formula A, but only requires deciding whether the set of tests of A, i.e. $A^{\perp}$, contains a successful paraproof.

6 Philosophical considerations

Hitherto we presented some ideas for developing a computational account of proofs which is general enough to allow the justification of logical statements, as well as of proper axioms. However, our general aim is to show that this computation setting can constitute an appropriate framework for the development of a uniform and epistemic-based understanding of logic and mathematics. In order to do this, we will adopt a linguistic point of view principally based on the analysis of the meaning of logical and mathematical sentences. Our meaning-theoretical analysis will be focused on the untyped framework presented in § 5, and based on Girard's geometry of interaction.[29]

6.1 Normative *vs.* descriptive theories of meaning

As we said in §1, in this paper we made an attempt of reconciling a standard notion of axiom with a certain kind of inferentialism based on the computational interpretation of proofs, or better, of proof structures. However, we have not yet clarified the philosophical extent of this kind of inferentialism. In particular, we have not yet made explicit which kind of theory of meaning can be induced by this computational perspective. Our claim is that such a theory of meaning is rather different from the one associated to standard inferentialism; the main difference being that the latter is *normative* while the former is *descriptive*. Let us try to clarify this crucial point.

Standard inferentialism is usually identified with Dummett-Prawitz verificationism (cf. Tennant 2012, §2). On to this approach, the rules governing our linguistic practice – i.e. the rules we are supposed to master in order to have successful linguistic exchanges – have to be governed by a principle of harmony that prevents the generation of new informations in a non-conservative way. This principle can be formally captured by the *inversion*

[29] A similar analysis is proposed by Bonnay (2007) on the basis of Krivine's classical realizability (see Krivine 2003).

principle formulated by Prawitz (1965, p. 33; 1973, pp. 232-233): anything that follows from the assertion of a certain (complex) sentence A, cannot exceed what directly follows from the grounds for asserting it, i.e. from the premisses of the introduction rule for A. This principle plays the role of a *norm* because it transcends the situation that it regulates: by imposing it at the beginning of the construction of a language, it guarantees in principle a "perfect" communication, avoiding misunderstandings as well as other linguistically pernicious situations (cf. Dummett 1973a, p. 454, for the well known example of 'Boche').[30] It would not be an exaggeration to say that from the verificationist point of view what counts is the way in which the speakers structure the contents of what that they want to communicate (i.e. the messages). If these contents are structured using expressions which respect the inversion principle, then this would already be sufficient in order to guarantee communication to work correctly. From such a perspective the responsibility for a good communication completely rests on the person who sent the message and not on the one who receive it.

Both the computational perspective we adopted in this paper and standard inferentialism take proofs to be the meaning-conferring objects. However, there is an essential difference between the two perspectives. As already seen in § 5.1.3, the computational perspective asks programs (i.e. paraproofs) to be tested in order to evaluate their behavior, and testing requires a context of evaluation (i.e. a set of paraproofs). Under the proof-as-programs correspondence, proofs are still necessary to determine the meaning of a sentence, but they are no longer sufficient. More specifically, knowing the order in which the rules have been applied in a proof is not sufficient: we also need to know what play the role of context of evaluation. In the perspective presented here, this is achieved by using the notion of paraproof. As we have seen, paraproofs do not necessarily represent correct – i.e. logically valid – (linguistic) arguments: the correctness of a paraproof depends on the interactional properties it displays in the presence of other paraproofs. From the linguistic point of view, if a proof corresponds to a correct justification for the *assertion* of a sentence (i.e. a correct justification for judging that sentence as true; cf. Martin-Löf 1987; Sundholm 1997), a paraproof corresponds to an *argument* supporting the *utterance* (see Lecomte & Quatrini 2011a) of a certain sentence in a particular context of discourse, regardless of whether the argument is (logically) correct and the sentence is true. In the same vein,

[30]This normative conception of language shares some similarities with the *Orthosprache-project* promoted by the so-called *Erlangen School* (or *Erlangen Constructivism*; Kamlah & Lorenzen 1972, Lorenzen & Schwemmer 1973). On this view, « language is not just a fact that we discover, but a human cultural accomplishment whose construction reason can and should control » (Rahman & Clerbout 2013, p. 4). However, the *Orthosprache-project* differs from Dummett-Prawitz verificationism in that it endorses a pluralistic – and not a monistic – conception of logic which allows to justify, for example, both intuitionistic and classical logic (see Rahman & Clerbout 2013, p. 9, 67, note 28; Sørensen & Urzyczyn 2006, §§4.5,4.6, 6.5,7.5).

the process of interaction between two paraproofs can be seen as a dispute between two speakers who use arguments to convince each other to accept their own position. In this sense, truth ceases to be an absolute notion and becomes an interactional and "social" one: a sentence can be judged as true when the speaker *always* possesses a convincing argument, i.e. when the speaker possesses a winning strategy (cf. p. 26, *supra*).

A further fundamental feature of this framework is that the meaning of sentences is not fixed by a set of rules obeying pre-established principles, but it is determined within the linguistic activity itself: knowing the meaning of a sentence corresponds to knowing which counter-arguments can be used in a dispute in order to terminate it. Since these counter-arguments strictly depend on the specific context and situation considered – namely, the arguments used by the other speaker – this means that they cannot be determined in advance of a dispute nor from "outside" the linguistic exchange itself. In this sense, the untyped approach induces a sort of game-theoretic semantics.

An important difference with standard game-theoretic semantics[31] is that in this framework knowledge of the meaning of a sentence does not correspond to knowledge of how to win a dispute. In order to determine the meaning of a sentence it is only requested that the dispute terminate; whether it does so with a gain or with a loss is irrelevant. As we mentioned before, having a winning strategy corresponds to knowing that the sentence in question is true. This implies that the meaning of a sentence neither coincides with a truth-definition nor depends on a primitive, unanalyzed notion of truth. It is for this very reason that, in Dummettian terms, the computational and untyped approach to semantics can be characterized as an anti-realist position: « [. . .] the notion of *truth*, considered as a feature, which each mathematical statement either determinately possesses or determinately lacks, [. . .] cannot be the central notion for a theory of the meanings of mathematical statements », on the contrary, « [. . .] it is in the mastery of [a] practice that our grasp of the meaning of the statements must consists » (Dummett 1973b, p. 225). Furthermore, characterizing the truth of a sentence as the possession of a winning strategy also allows to do justice to Dummett's *manifestability* requirement (Dummett 1973b, pp. 93-95; Dummett 1976, pp. 79-82 ; Dummett 1977, pp. 193-195): the fact that a sentence is true makes a noticeable difference at the level of our linguistic practice, since it implies that there is someone who would always gain a dispute if they were to argue for it.[32]

To sum up, the computation-based theory of meaning introduced here is still within the "meaning as use" paradigm. A distinguishing feature of this

[31] Classic texts in game-theoretic semantics are Hintikka (1983) for a model-based account of games, and Lorenzen & Lorenz (1978) for a syntactical and operative – or dialogical – account. For detailed surveys of recent developments of dialogical approaches to game-theoretic semantics see Rahman & Keiff (2004) and Keiff (2009). For textbook presentations see Redmond & Fontaine (2011) and Rückert (2011).

[32] A similar idea is presented in Marion (2012).

theory with respect to other "meaning as use" theories is that the set of licensed uses of a sentence is not defined by an absolute and external norm, but by the dispositions of the other speakers to respond to those uses. In this sense the notion of the correct use of a sentence emerges from the linguistic practice itself. More generally, standard inferentialist theories of meaning hold that the aim of a theory of meaning is to fix *in abstracto* the rules that a language must respect in order to work properly, i.e. in order to do what we expect it to do – allow communication between speakers. The perspective adopted here departs from standard inferentialism on this respect. In analogy with the position endorsed by Wittgenstein in the *Philosophical Investigations*, we hold that the aim of a theory of meaning is not to determine the « essence of a language » (Wittgenstein 1953, §97), that is, the set of characteristic features that a (abstract and idealized) language should possess. The mere fact that there is an established linguistic activity[33] and that it allows successful communication should lead us to think that linguistic activity should be considered as something that already works properly, not as something that should be rectified (Wittgenstein 1953, §98). From this perspective, linguistic ambiguities, usually considered to be sources of possible misunderstandings, are considered to be proper parts of the linguistic activity and as such, they are not rejected as incorrect.[34] This is a natural consequence of the absence of any *a priori* principles distinguishing between correct and incorrect (instances of) sentences. In other words, the idea is that the rules governing our linguistic activity are *immanent* to it. This view has a twofold consequence. On the one hand, we become aware of the way in which meaning is assigned to sentences by describing the linguistic activity. On the other hand, the knowledge of the meaning of a sentence is manifested in the capacity of taking part in a linguistic exchange where this sentence is used. On this view there is no need to make the rules governing the linguistic exchange explicit – if this where so, we would be forced to adopt an externalist approach. This feature calls attention to a first difference with standard Dummett-Prawitz verificationism. Other relevant differences will be analyzed in the following sections.

6.2 Feasibility and interaction

The computational perspective described above is compatible with an inferentialist point of view as long as the application of an inference rule within a paraproof corresponds to the successful performance of a linguistic act within a linguistic exchange. The peculiarity of our perspective consists in the fact that the choice of which rule to apply is constrained by the type of linguistic acts previously performed both by the speaker and by her opponent. In other

[33]This fact can be established "empirically".

[34]Indeed, in Ludics it is possible to represent fallacies in a formal and precise way as it has been shown by Lecomte & Quatrini (2011b).

words, the speaker chooses what rules to apply strictly on the basis of the specific linguistic situation she is confronted with. This has two main consequences. First, inference rules act on linguistic objects that – as utterances – are situation-dependent; they do not act on more abstract or "absolute" linguistic entities such as assertions (see Lecomte & Quatrini 2011a).[35] Second, the fact that linguistic acts are situation-dependent means that the inference rules used to perform them have to take into consideration the particular type of resources available in each situation.[36]

In this respect, the theory of meaning which emerges from the computational untyped setting presented above is still characterized by anti-realist features, as it is articulated on the basis of the linguistic competences possessed and manifested by the speaker, rather than on a primitive and unanalyzed notion of truth. Nonetheless, it retains certain differences with respect to standard forms of anti-realism, such as Dummettian verificationism. More specifically, in our computational untyped setting mastering the meaning of a sentence does not consists in knowing what can be done with that sentence *in principle*, but it consists in knowing what can be *practically* done with it in some particular situations. In this sense, accepting a computational untyped approach leads to accepting some sort of *radical anti-realist* position: to know the meaning of a sentence is to know how it can be *feasibly* used during a concrete linguistic exchange. The computational approach does not consider idealized scenarios, but it focuses on concrete dialogical situations: its aim is not to determine the principles necessary for the construction of a language, but to *represent* (the abstract structure of) a language. For this reason, practical constraints are not obtained by imposing on verificationist's principles a constraint on proof-size bounds in advance,[37] for example by imposing a polynomial growth of proof-length by changing the usual connectives with linear ones during normalization or cut elimination (see Dubucs 2002; Dubucs & Marion 2003):[38] such a change of connectives could only be justified

[35]Furthermore, paraproofs are not necessarily correct proofs and thus their definition does not involve the notion of truth. On the contrary, in the Bolzanian tradition an assertion takes the form of the judgment 'A is true' (where A is a sentence or a proposition). Thus, both in the case in which assertion is taken to be a primitive notion – e.g. by realist positions – and in the case in which is taken to be non primitive – by anti-realist positions – assertion is defined is defined with respect to the very notion of truth.

[36]These considerations become evident when we consider that the untyped setting introduced here validates the logical rules of linear logic; it it indeed widely acknowledged that linear logic is the clearest example of a resources-oriented logic (Di Cosmo & Miller 2010).

[37]These kinds of bounds are essentially dictated by two reasons: 1) guaranteeing that the process of *verification* that something is a proof can be practically done by human beings; 2) guaranteeing the semantic key objects, i.e. canonical proofs, to be objects that can be practically *constructed* by human beings. By respecting these two conditions it should be assured that, in the verificationist account, both truth and meaning never make appeal to entities transcending concrete human capacities, as it could be the existence of proofs the size of which goes beyond physical limits.

[38]The standard justification for the choice of polynomial bounds can be found in Wang (1981,

ad hoc. On the contrary, it is the very nature of the interactional approach that ensures the existence of bounds which guarantee that the knowledge of the meaning of sentences is based on linguistic skills manifestable by the speakers: the presence of two speakers, instead of only one, guarantees that the actions of each speaker are always constrained by the actions of the other one, and *vice versa*.

6.3 Holism and molecularism

Our last observations will concern a typically Dummettian theme, namely the debate between molecularist accounts of meaning to a holistic ones.

Generally speaking, it is usually argued that the adoption of an axiomatic approach comes with the acceptance of some kind of holism (see Troelstra & van Dalen 1988, pp. 851-852). This is because presenting a theory in an axiomatic way has two main consequences with respect to our understanding of the set of sentences constituting the theory itself. Firstly, the syntactical behavior of the expressions composing the language is fixed holistically: an expression is defined on the basis of the relations it entertains with the other expressions of the language, and there is no bound fixed in advance on the number of expressions that can be mutually related by the axioms. Secondly, the inferential behavior of an axiom can be fully determined only when it is used in conjunction with other formulas in order to extract some relevant information from it, and also in this case no bound can be imposed on the size of the set of these formulas in advance.[39] This picture seems to clash with the molecularist approach defended by Dummett. On such view, mastering a limited and well-determined fragment of a language is sufficient to understand the meaning of a given sentence – that is, linguistic competence does

§6.5) and it has been well summarized by Marion (2009), p. 424:

> It is generally agreed that polynomial-time computability captures the capacities of digital computing machines, as opposed to their *idealized* counterparts, the Turing machines. Digital computing machines do *not* have access to unlimited resources, and this seems to be the key point for a radical anti-realist program. It is only asked here from the radical anti-realist that she grants that digital computing machines are an unproblematic extension of human cognitive capacities, so that, with polynomial-time computability, one remains within the sphere of what is humanly feasible.

In fact, it seems to us that there is a further, and usually neglected, argument supporting this choice. Schematically, it can be presented in the following way: i) from the verificationist point of view meaning is based on proofs, and the only logical rules allowed for constructing these proofs are the intuitionsitic ones; ii) via the Curry-Howard correspondence each proof of intuitionistic logic can be associated to a computable function, and *vice versa*; iii) a fundamental property of a theory of meaning is compositionality; iv) by restricting to polynomial computable functions, compositionality between functions (i.e. proofs) is preserved: the composition of two polynomial computable functions is still a polynomial computable function.

[39]The other way round, this situation corresponds to the idea that to understand the meaning of an axiom it is necessary to understand the *totality* of the consequences that can be drawn from it (see Dummett 1991, p. 228).

not require an ability to master the whole totality of the expressions of a language (Dummett 1976, p. 79). However, this aspect appears to be in conflict with the axiom-based perspective adopted in this paper. Thus, the question arises of whether our approach is *essentially* holistic or whether it could be compatible with a kind of molecularism akin to the Dummettian perspective. More specifically, while the untyped computational perspective allows to define types – and thus also formulas and sentences – as sets of paraproofs (cf. §5.2 *supra*), it does not provide a standard inductive definition of them. In particular, there is no such notion as that of atomic type.

Prima facie, this seems to contrast with Dummett molecularism in so far as this requires the possibility of ranking sentences in a hierarchy of increasing complexity.[40] In absence of a way of fixing such a hierarchy of types-formulas, we might be unable to impose a bound on the complexity of the set of formulas that have to know in order to understand another given formula. Now, as seen above, in a computational framework knowing the meaning of a sentence amounts to being able to participate in a linguistic exchange once the sentence in question occurs. In absence of a way of fixing in advance a hierarchy of types-formulas according to their complexity, it may seem that there is no way of fixing in advance the kind of formulas that will be involved in the exchange. It would then follow that the understanding of the sentence in question may only be explained by appealing to the understanding of all the expressions of the language, which would make our perspective incompatible with Dummett's analysis. In the remainder of this section, we will argue that our perspective is indeed compatible with Dummettian verificationism.

Dummettian verificationism holds that understanding the meaning of a complex sentence A is reducible to understanding the meaning of its principal connective; in order to do so, we only need to consider the set of sentences in which this connective appears as the principal connective. This means that only a limited fragment of the language has to be analyzed in order to understand the meaning of a certain expression. This analysis can be carried out by focusing on the properties of the inference rules involved in the (direct) justification of A. This amounts to the ability to recognize what counts as a canonical proof of A and whether the inference rules used in the (putative) proof are correct, i.e. valid. The correctness of the rules is usually ensured by the inversion principle we mentioned in §6.1: what can be drawn from the elimination rules of a certain connective must already be drawn from the premisses of its corresponding introduction rules. As Sundholm (2004, p. 454) remarked, the peculiarity of this principle is that it « [...] leads straightforwardly to a resurrection of the old idea that the validity of an inference resides in the analytic containment of the conclusion in the premisses ». It

[40]In particular, see Dummett (1991, p. 223): « Compositionality demands that the relation of dependence imposes upon the sentences of the language a hierarchical structure deviating only slightly from being a partial order. »

is this very possibility of reducing proofs to "analytic proofs" that plays a crucial role in the molecularist approach: from the syntactical form of a given complex sentence A it is possible to extract a relevant information which allows to impose a bound on the set of sentences necessary to understand A.[41] In particular, if the inversion principle is respected, it could be possible to prove the so-called *subformula property*, which guarantees that if a complex sentence A is provable, then there exists a proof the rules of which are applied only to subformulas of the conclusion.[42]

Despite the analogies between the Dummettian and the computational approaches, we can see more precisely that there are some crucial differences. In analogy with the Dummettian perspective, the computational approach reduces the understanding of the meaning of a sentence A to the understanding of the principal connective of A. This may lead to thinking that it is sufficient to consider only a fragment of the language in order to understand A, namely the set of sentences in which the principal connective of A is principal. However, unlike the verificationist approach, the computational perspective presented here does not assume that the introduction rules and the harmony requirement do, by themselves, confer meaning on a certain connective. This is because knowing the meaning of A – or better, the meaning of its principal connective – does not require knowing how to directly justify A, but instead requires knowing how to construct an argument against A. What counts is to have a strategy to refute A, or equivalently, a strategy in support of $\neg A$. Thus, the meaning-conferring objects are represented by argumentative strategies as a whole, and not by single inference rules. The only property that these strategies are required to have is to terminate when an argument in support of A is introduced. More specifically, there are no a priori contraints on the order of the inference rules in the argument for $\neg A$, and it is not required that all the rules applied in the argument are correct (i.e. valid). In summary, no analyticity constraints are imposed on these arguments. What really matters is the strategy that must be followed in order to refute A, while the formulas used in applying this strategy take a back seat. The upshot is that there are no *a priori* limitations on the formulas in-

[41]It is worth noting that Girard's latest work, known under the name of *transcendental syntax*, aims at studying these issues from a formal point of view. In particular, it tries to prove how in some specific cases - namely when purely logical formulas are considered - the sets of tests can be shown to be *finite* (see Girard 2013, in part. § 3.2). The framework developed by Girard perfectly fits into our definition of untyped proof theory, as it is a particular case of Seiller's interaction graphs construction (Example 30).

[42]In fact, sometimes it could already be sufficient to prove a weaker property, like the subterm property. There are some theories – like the theory of equality, of groupoids and of lattices – for which the fact that a proof of A can make appeal to no other terms than those appearing in A is already sufficient to impose a bound on the set of formulas that should be known in order to know the meaning of A. The reason is that, from a technical point of view, for these theories the subterm property works like the subformula property: it allows to define proof-search methods by limiting the proof-search space (cf. Negri & von Plato 2011, §4).

volved in the arguments used for refuting A, and this leads towards a holistic account of the computational approach presented in this article.

Lastly, we would like to consider an objection to this interpretation. It may be thought that the claim that our computational approach and Dummettian verificationism are compatible is in conflict with what we said in §6.2 about feasibility properties, and therefore with some kind of internal limitations that seem intrinsic to the computational approach. We argue, however, that the contradiction is only apparent. In §6.2, we saw that the knowledge of the meaning of a sentence by a certain speaker is manifested by participating in a linguistic exchange with another speaker, where this exchange involves only a bounded amount of resources. What is relevant here, instead, is the impossibility of establishing in advance which fragment of the language the speaker has to master in order to perform linguistic exchanges with other speakers. This impossibility is not a particularly surprising feature for a "computational theory of meaning".

As discussed above, the computational perspective comes with a certain understanding of what a *descriptive* theory of meaning is. From this perspective, what counts are the competences of those speakers who effectively participate in linguistic exchanges, but there is no requirement to fix those competences in advance. In particular, there is no need to explain what features a language must possess in order to be learnable. On the contrary, by focussing on the molecularity property we can see that the problem of *learning* a language plays a special role in the verificationist theory of meaning (see Dummett 1993, p. ix). Human agents can process only a limited amount of information at a time, but can nevertheless learn languages. Learning the meaning of an expression therefore requires mastery of only a finite fragment of the language in question (see Dummett 1973a, p. 515). Otherwise the task would go beyond human capacities, which are *ex hypothesi* finite ones.

7 Conclusion

The aim of this paper is to give a new account of the meaning of mathematical axioms that does not appeal either to a primitive notion of truth or to other realist assumptions. Our approach can therefore, broadly speaking, be characterized as an anti-realist one.

A major difficulty arises when one tries to interpret the notion of axiom through a Dummettian anti-realist semantics. The addition of axioms to standard proof systems entails the loss of (the notion of) canonical proofs, which is in fact the cornerstone of a verificationist theory of meaning. The existing solutions to this problem require a deep change in the epistemological status of axioms: axioms are turned into specific kind of rules. As a result, these solutions lead to revisionist positions with respect to the architecture of mathematical theories. Our computational approach overcomes these difficulties, at least in part, by enriching the set of primitive semantic concepts in the underlying theory of meaning. Moreover, our approach is compatible with an inferentialist, anti-realist understanding of meaning.

Our strategy in this paper was twofold. First, we explored the computational aspects of a proof, considered as an "isolated" object, *via* proof-search algorithmic techniques. A careful analysis of the occurrences of the $\maltese$ rule allowed us to show a precise correspondence between (logically incorrect) generalized axiom rules and counter-models using a homogenous proof-theoretical setting. Secondly, we explored the computational aspects of the interaction between proofs, considered as objects interacting through the Cut rule. This approach allows one to forget formulas, by focusing only on the geometry of rules and their interactions. In this setting, generalized axioms provide a characterization of the crucial notion of paraproof. Both of these computational viewpoints reinforce the idea that generalized axioms are a way of working with (counter-)models inside the syntax. Axioms can thus be seen as fundamental entities at the crossroad between the syntax and the semantics of proof systems.

In the final part of the paper we made explicit some philosophical assumptions allowing us to integrate the analysis of untyped proofs with an inferentialist theory of meaning. We carried out such an analysis by pointing out some crucial differences between the inferentialist account based on Dummettian verificationism and the one based on the interactional approach presented in §5. Despite the fact that neither account considers the notion of truth as primitive but as epistemically dependent, a major divide exists between them. It amounts to the difference between a normative (and "solipsistic") theory of meaning and a descriptive (and "social") one. In the end, we showed in which sense the shift from the former to the latter leads us to embrace an even more radical form of anti-realism. Finally, we concluded our work with a short discussion around holism and molecularity. In order to fully comprehend the theory of meaning standing behind the computational and interactional approach presented in this paper, it is necessary to establish whether this theory of meaning is more harmonious with a holistic or a molecularistic approach. The answer to this question is not yet established and seems to us a valuable direction for future research.

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Appendices

A Properties of system $\mathrm{pLK}^{\maltese}_{R}$

A.1 Proof of Proposition 3

By induction on the number of connectives in the sequent. Since the proofs for $\vee$ and for $\wedge$ are similar, we only show the invertibility of the $\vee$. The base case for the induction is a sequent containing only one connective, i.e. a $\maltese$ rule followed by a $\vee$-rule. In this case the $\maltese^{At}$ rule itself gives a derivation of the premiss of the $\vee$-rule. Let us now assume that this is true for any sequent containing at most n connectives, and let us take a derivable sequent $\vdash \Gamma, A \vee B$, containing $n+1$ connectives, and let π be one of its derivations. If π ends with a $\vee$-rule, the subderivation obtained dropping the last rule gives a derivation of the premiss. If π does not end with a $\vee$ rule, then it must end with a $\wedge$ rule:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \vdash \Delta, A \vee B, C\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \vdash \Delta, A \vee B, D\end{array}}{\vdash \Delta, A \vee B, C \wedge D}\,\wedge$$

Applying the induction hypothesis on the premisses we get two derivations π_1' and π_2' of $\vdash \Delta, A, B, C$ and $\vdash \Delta, A, B, D$ respectively; thus the desired derivation is:

$$\dfrac{\begin{array}{c}\pi_1'\\ \vdots\\ \vdash \Delta, A, B, C\end{array} \qquad \begin{array}{c}\pi_2'\\ \vdots\\ \vdash \Delta, A, B, D\end{array}}{\vdash \Delta, A, B, C \wedge D}\,\wedge$$

A.2 Proof of Lemma 6

By induction on the number of connectives in $\vdash \Gamma$. The base case is obvious. Assume that the lemma is true when the number of connectives in Γ is at most n. We show that the derivations of $\vdash \Gamma, F, G$ terminating with a rule where F is principal have the same set of leaves as the derivations terminating with a rule where G is principal. We only show how the proof is done in the case $F = A \wedge B$ and $G = C \wedge D$, which is the more complicated case. By the invertibility of the $\wedge$-rule we have the two derivations:

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash \Gamma, A, C\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \vdash \Gamma, A, D\end{matrix}}{\vdash \Gamma, A, C \wedge D}\wedge \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\\ \vdash \Gamma, B, C\end{matrix} \qquad \begin{matrix}\pi_4\\ \vdots\\ \vdash \Gamma, B, D\end{matrix}}{\vdash \Gamma, B, C \wedge D}\wedge}{\vdash \Gamma, A \wedge B, C \wedge D}\wedge
$$

$$
\dfrac{\dfrac{\begin{matrix}\rho_1\\ \vdots\\ \vdash \Gamma, A, C\end{matrix} \qquad \begin{matrix}\rho_2\\ \vdots\\ \vdash \Gamma, B, C\end{matrix}}{\vdash \Gamma, A \wedge B, C}\wedge \qquad \dfrac{\begin{matrix}\rho_3\\ \vdots\\ \vdash \Gamma, A, D\end{matrix} \qquad \begin{matrix}\rho_4\\ \vdots\\ \vdash \Gamma, B, D\end{matrix}}{\vdash \Gamma, A \wedge B, D}\wedge}{\vdash \Gamma, A \wedge B, C \wedge D}\wedge
$$

Using the induction hypothesis, we have that $\mathfrak{L}(\pi_k) = \mathfrak{L}(\rho_k)$ for all k in $\{1,2,3,4\}$. Using these equalities and the induction hypothesis once again, we obtain that:

- any derivation π of $\vdash \Gamma, A, C \wedge D$ satisfies $\mathfrak{L}(\pi) = \mathfrak{L}(\pi_1) + \mathfrak{L}(\pi_2)$;
- any derivation π of $\vdash \Gamma, B, C \wedge D$ satisfies $\mathfrak{L}(\pi) = \mathfrak{L}(\pi_3) + \mathfrak{L}(\pi_4)$;
- any derivation π of $\vdash \Gamma, A \wedge B, C$ satisfies $\mathfrak{L}(\pi) = \mathfrak{L}(\pi_1) + \mathfrak{L}(\pi_3)$;
- any derivation π of $\vdash \Gamma, A \wedge B, D$ satisfies $\mathfrak{L}(\pi) = \mathfrak{L}(\pi_2) + \mathfrak{L}(\pi_4)$;

We have therefore shown that if π is any derivation of $\vdash \Gamma, A \wedge B, C \wedge D$ terminating with a $\wedge$-rule on $A \wedge B$ and, if ρ is any derivation of the same sequent terminating with a $\wedge$-rule on $C \wedge D$, they have the same set of leaves, namely:

$$
\mathfrak{L}(\pi) = \sum_{i=1,\dots,4} \mathfrak{L}(\pi_i) = \mathfrak{L}(\rho)
$$

The other cases are similar.

A.3 Proof of Proposition 10

Suppose π is a derivation of $\vdash \Gamma$ in pLK_R, then by replacing every axiom rule by a $\maltese$ rule we obtain a derivation π' of $\vdash \Gamma$ in $\mathrm{pLK}_R^{\maltese}$. Every $\maltese^{At}$ rule in π' is correct, since the sequent was introduced by an axiom rule in pLK_R.

Conversely, if π' is a derivation of $\vdash \Gamma$ in $\mathrm{pLK}_R^{\maltese}$ such that $\mathfrak{L}(\pi')$ contains only correct sequent, then each sequent in $\mathfrak{L}(\pi')$ can be derived from an axiom rule in pLK_R. Therefore we obtain a derivation π of $\vdash \Gamma$ in pLK_R by replacing every $\maltese^{At}$ rule in π' by an axiom rule.

A.4 Proof of Lemma 12

Notice that, by definition of satisfiability of a sequent (and associativity of $\vee$), $\delta \vDash \Delta, A, B$ if and only if $\delta \vDash \Delta, A \vee B$.

In the case of the $\wedge$-rule, assume first that $\delta \vDash \Delta, A$ and $\delta \vDash \Delta, B$. Either there is satisfiable formula in Δ, either both A and B are satisfiable, therefore $\delta \vDash \Delta, A \wedge B$. Conversely, assume that $\delta \nvDash \Delta, A$, then all formulas in Δ are unsatisfiable and A is not satisfiable, therefore $A \wedge B$ is not satisfiable. We conclude that $\delta \nvDash \Delta, A \wedge B$. The lemma is then proved by simple induction.

A.5 Proof of Proposition 13

Suppose $\mathfrak{L}(\vdash \Gamma)$ contains only correct sequents, then for any valuation δ and sequents $\vdash \Delta$ in $\mathfrak{L}(\vdash \Gamma)$, $\delta \vDash \Delta$ from the definition of correct sequent. Then, by Lemma 12, $\delta \vDash \Gamma$.

Conversely, let us assume that $\mathfrak{L}(\vdash \Gamma)$ contains at least an incorrect sequent $\vdash P_1, \ldots, P_n, \neg Q_1, \ldots \neg Q_p$, such that for all integer $i \in [1, n]$ and $j \in [1, p]$, $P_i \not\equiv Q_j$. We can now take a valuation δ satisfying $\delta(P_i) = 0$ and $\delta(Q_j) = 1$. Then $\delta \nvDash P_1, \ldots, P_n, \neg Q_1, \ldots \neg Q_p$ and by Lemma 12 this means that $\delta \nvDash \Gamma$.

B Properties of system $\mathrm{LK}_R^{✠}$

B.1 Proof of Lemma 19

Let π be such a derivation of the sequent $\vdash \Gamma$. Let $\mathfrak{L}(\pi)$ be the set of sequents introduced by ✠ rules in $\vdash \Gamma$, $\mathfrak{L}_c(\pi)$ the subset of $\mathfrak{L}(\pi)$ containing the sequents introduced by correct ✠ rules, and $\mathfrak{L}_a(\pi) = \mathfrak{L}(\pi) - \mathfrak{L}_c(\pi)$. By assumption, the sequents in $\mathfrak{L}_a(\pi)$ are introduced by admissible ✠ rules that are not correct. Hence there exists correct derivations π_i of $\vdash \Gamma_i$. Then, replacing the ✠ rules introducing the sequents $\vdash \Gamma_i$ in π by the derivations π_i, we obtain a derivation π' of $\vdash \Gamma$ extending π and containing only correct ✠ rules.

B.2 Proof of Theorem 21

Suppose we have a derivation of π in LK_R of a sequent $\vdash \Gamma$. Then, replacing every axiom rule by a ✠ yields a derivation π' of $\vdash \Gamma$ in $\mathrm{LK}_R^{✠}$. Moreover, the ✠ rules are all correct (since they were axiom rules in LK_R), hence admissible.

Conversely, suppose we have a derivation π' of a sequent $\vdash \Gamma$ in $\mathrm{LK}_R^{✠}$ that contains only admissible rules. Then, by lemma 19 we can find a derivation π'' extending π' such that π'' contains only correct ✠ rules. Then, we can replace these ✠ rules by axiom rules to get a derivation π of $\vdash \Gamma$ in LK_R .

B.3 Proof of Lemma 23

Suppose π contains at least one ✠ rule introducing a sequent $\vdash \Gamma$ containing a formula B that is not an atom, a negation of an atom or an existential formula. Then the principal connective in B is either a $\wedge$, a $\vee$ or a $\forall$. Replacing the ✠ rule introducing $\vdash \Gamma$ by the rule introducing the principal connective and closing the derivation we obtain by ✠ rules then gives us a new derivation π_1 of π. After a finite number of iterations of this process, we obtain the wanted extension.

B.4 Proof of Lemma 25

Suppose now that the sequent $\vdash \Gamma$, $\Gamma = A_1, \dots, A_m$, introduced by the non-admissible ✠ rule contains at least one quantifier. We fix an enumeration of the terms $t_1, \dots, t_n, \dots$ of the language and we will define an iterative process indexed by pairs (s, k_s) where s is a finite sequence of integers (the first step will be indexed by the null sequence of length m which will be written $(0)_m$) of length p_s and k_s is an integer in $[1, \dots, p_s]$. The process we describe consists in extending the derivation by applying $\exists$ rules in a way that insures us that for all existential formula $\exists x A(x)$ and term t_i, there exists a step where the $\exists$ rule is used on the formula $A[t_i/x]$. To insure all terms appear at some point in the process we will use the enumeration but we need to keep track of the last term used for each existential formula. Moreover, applying a $\exists$ rule on a formula containing two existential connectives will produce new existential formulas on which we must apply the same procedure. The sequence will therefore keep track, for each existential formula, of the last term we used. Its length may vary, but due to our choice of existential rule it can only expand. The integer, on the other hand, will keep track of the last existential formula we decomposed, so that we can ensure that all formulas are taken into account.

First, let us write $A_1, \dots, A_{p_{(0)_m}}$ the formulas in Γ that contain quantifiers. By lemma 23 we can suppose, without loss of generality, that the ✠ rules in π are simple. We will denote π by $\pi_{(0)_m}$, i.e. π will be the initial step of the process. The integer $k_{(0)_m}$ is defined to be 1, so we consider the derivation π_{A_1,t_1} obtained from π by replacing the ✠ rule introducing $\Gamma = \Delta, A_1, \dots, A_p$ by the derivation consisting of a ✠ rule introducing $\Delta, A'_1[t_1/x], A_2, \dots, A_{p_{(0)_m}}$ followed by an existential rule introducing A_1. It follows from Lemma 23 that this derivation can be extended to a simple derivation $\bar{\pi}_{A_1,t_1}$. Then, by the non-admissibility of the ✠ rule, this derivation contains at least one non-admissible ✠ rule introducing a sequent $\vdash \Gamma'$. Amongst the formulas of Γ are the all the formulas A_i for $1 \leqslant i \leqslant p$, but Γ' may contain more existential formulas. We thus denote by $A_1, \dots, A_p$ the existential formulas of Γ'. We write $(0)^+_m = (1, 0, \dots, 0)$ the sequence of length p: we thus obtained an extension $\pi_{(0)^+_m} = \bar{\pi}_{A_1,t_1}$ containing a non-admissible ✠ rule introducing a sequent $\Gamma_{(0)^+_m} = \bar{\Gamma}$. Defining $k_{(0)^+_m} = 2$, we arrived at the next step, indexed by $((0)^+_m, k_{(0)^+_m})$ and we can then iterate the process.

More generally, suppose we are at step (s, k_s) with $s = (s(0), \dots, s(p))$: we have a simple derivation π_s with a non-admissible ✠ rule introducing a sequent $\Gamma_s = \Delta_s, A_1, \dots, A_{p_s}$ (Δ_s contains only atoms and negations of atoms). We obtain a derivation $\pi_{A_{k_s}, t_{s(k_s)+1}}$ by replacing the ✠ rule introducing Γ_s with the derivation:

$$\dfrac{\overline{\vdash \Delta_s, A_1, \ldots, A_{p_s}, A'_{k_s}[t_{s(k_s)+1}]}\ \maltese}{\vdash \Gamma_s}\ \exists$$

This derivation $\pi_{A_{k_s}, t_{s(k_s)+1}}$ can then be extended by Lemma 23 to a simple derivation $\bar{\pi}_{A_{k_s}, t_{s(k_s)+1}}$ which contains a non-admissible $\maltese$ rule. The sequent Γ' introduced by this rule contains all the formulas $A_1, \ldots, A_{p_s}$ and may contain additional existential formulas $A_{p_s+1}, \ldots, A_n$. Let s^+ to be the sequence of length n defined by $(s(0), \ldots, s(k_s-1), s(k_s)+1, s(k_s+1), \ldots, s(p_s), 0, \ldots, 0)$, and:

$$k_{s^+} = \begin{cases} k_s + 1 & \text{if } k_s + 1 \leqslant n \\ 1 & \text{otherwise} \end{cases}$$

Let us write $n = p_{s^+}$. We thus obtained the next step in the process, indexed by (s^+, k_s^+): a simple derivation $\pi_{s^+} = \bar{\pi}_{A_{k_s}, t_{s(k_s)+1}}$ with a non-admissible $\maltese$ rule introducing a sequent $\Gamma_{s^+} = \Gamma' = \Delta_{s^+}, A_1, \ldots, A_{p_{s^+}}$.

We claim that for all pairs (i, j) of natural numbers (different from 0), there is a step s in the process such that $s(i) = j$. We will write len(s) the length of a sequence s. Notice the formulas $A_{p_s+1}, \ldots, A_{p_{s^+}}$ are instantiations of an existential formulas A_i for $1 \leqslant i \leqslant p_s$ and therefore the number of existential connectives in a B_j is strictly less than the number of existential connectives in the corresponding A_i. We will show that the value of k_s returns to 1 in a finite number of steps using this remark. We will denote the number of existential connectives in a formula A by $\natural(A)$. Suppose that we are at a given step s such that $k_s = 1$ and write $o_s = (\max_{k_s < i \leqslant p_s} \natural(A_i), p_s - k_s)$. This pair somehow measures the number of steps one has to make before k_s returns to 1. Then, after $p_s - k_s$ steps in the process – let us write the resulting step as s^1, we have $k_{s^1} = p_s$ and $p_{s^1} - p_s$ new formulas, each one such that $\natural(A) < \max_{k_s < i \leqslant p_s} \natural(A_i)$. Therefore, the pair $o_{s^1} = (\max_{k_{s^1} < i \leqslant p_{s^1}} \natural(A_i), p_{s^1} - p_s)$. Since $\max_{k_{s^1} < i \leqslant p_{s^1}} \natural(A_i) < \max_{k_s < i \leqslant p_s} \natural(A_i)$, we have $o_{s^1} < o_s$ in the lexicographical order and this is enough to show the claim.

B.5 Proof of Theorem 26

If the sequent $\vdash \Gamma$ introduced by the non-admissible $\maltese$ rule does not contain any quantifiers, then the proof reduces to the proof of Proposition 13. Indeed, the derivation of $\vdash \Gamma$ in $pLK_R^{\maltese}$ is incorrect (if it were correct, it would contradict the assumption since any correct derivation in $pLK_R^{\maltese}$ is a correct derivation in $\mathrm{LK}_R^{\maltese}$), hence we can find a model $\mathscr{M}$ such that $\mathscr{M} \not\models \Gamma$.

If $\vdash \Gamma$ contains existential formulas, we use Lemma 25 to obtain a sequence $(\pi_i)_{i \in \mathbf{N}}$ of extensions. From this sequence of extensions, one can obtain a sequence of sequents $\vdash \Gamma_i$ where for each i, there exists N such that $\vdash \Gamma_{i+1}$ is the premise of a rule whose conclusion is $\vdash \Gamma_i$ in all derivations π_j

with $j \geqslant N$. Moreover, this sequence can be chosen so as to contain all instances of the subformulas of Γ. We now define a model whose base set is the set of terms. The interpretations of function symbols and constants are straightforward. The only thing left to define is the interpretation of predicates: if P is a n-ary predicate symbol, then $(t_1, \ldots, t_n)$ is in the interpretation of P if and only if $\forall i \geqslant 0$, $Pt_1 \ldots t_n \notin \Gamma_i$.

We can now check that $\mathscr{M} \not\models \Gamma$. We chose A a formula in Γ and prove by induction on the size of the formula A that $\mathscr{M} \not\models A$:

- if A is an atomic formula, then $\mathscr{M} \not\models A$ by definition of the model;
- if $A = B \wedge C$, then there exists a sequent $\vdash \Gamma_i$ such that either $B \in \Gamma_i$ or $C \in \Gamma_i$. We suppose $B \in \Gamma_i$ without loss of generality. Then, by the induction hypothesis, we have $\mathscr{M} \not\models B$, hence $\mathscr{M} \not\models A$;
- if $A = B \vee C$, then there exists a sequent $\vdash \Gamma_i$ containing both B and C. By induction, these two formulas are not satisfied in the model $\mathscr{M}$, hence $\mathscr{M} \not\models A$;
- if $A = \forall x B(x)$, then there is a sequent $\vdash \Gamma_i$ containing $B[y/x]$. By induction, $\mathscr{M} \not\models B[y/x]$, hence $\mathscr{M} \not\models A$;
- if $A = \exists x B(x)$, then for every term t there exists a sequent $\vdash \Gamma_i$ such that $B[t/x] \in \Gamma_i$. By the induction hypothesis, $\mathscr{M} \not\models B[t/x]$. This being true for all term t, we conclude that $\mathscr{M} \not\models A$.

This concludes the proof: since all formula $A \in \Gamma$ is such that $\mathscr{M} \not\models A$, we have that $\mathscr{M} \not\models \Gamma$.

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