1 Example 7: Instationary Stokes' equations

In this example we consider the instationary incompressible Stokes equation. As in example 3, we use the symmetric stress tensor which has a little consequence when using the do-nothing outflow condition. In strong formulation we have

$$\begin{aligned} \partial_t v - \nabla \cdot (\nabla v + \nabla v^T) + \nabla p &= f \\ div\, v &= 0 \end{aligned}$$

on time interval $I = [0, T]$ and the domain $\Omega = [-6, 6] \times [0, 2]$. We choose for simplicity $f = 0$.

Time discretization is based on the implicit Euler scheme (BE) which is strongly A-stable but only from first order and very dissipative. The BE-scheme is well suited for stationary numerical examples.

We formulate the time stepping scheme as *One-step-θ scheme*. The time interval is given by $I = [0, T]$. Let v^n, p^n and the time step $k = t^{n+1} - t^n$ be given. Find $v = v^{n+1}, p^{n+1}$ such that:

$$\begin{aligned} v - k\theta(\nabla \cdot (\nabla v + \nabla v^T) + \nabla p) &= k\theta f^{n+1} + k(1-\theta) f^{n+1} \\ &\quad + v^n + k(1-\theta)(\nabla \cdot (\nabla v^n + \nabla (v^n)^T) + \nabla p^n) \\ div\, v &= 0 \end{aligned}$$

In the case of the BE-scheme, $\theta = 1$, and the equation is reduced to

$$\begin{aligned} v - k\theta(\nabla \cdot (\nabla v + \nabla v^T) + \nabla p) &= k\theta f^{n+1} + v^n \\ div\, v &= 0 \end{aligned}$$

Note, that one should prefer a complete implicit treatment of the pressure p. Instead of using $\theta p^{n+1} + (1-\theta)p^n$, the pressure appears only with θp^{n+1}.

After discretization in time, the space is treated, as ususally, with a Galerkin finite element scheme, here based on the Taylor-Hood element Q_2^c/Q_1^c.

Complete variational formulation reads:

Find $v := v^{n+1} \in V$ and $p := p^{n+1} \in L$:

$$\begin{aligned} (v, \phi^v) + k\theta(\nabla v + \nabla v^T, \nabla \phi^v) + k\theta(\nabla p, \phi^v) &= k\theta(f^{n+1}, \phi^v) + (v^n, \phi^v) \\ (div\, v, \phi^p) &= 0 \end{aligned}$$

for all suitable test functions $\phi^v, \phi^p \in V \times L$.