Singular Value Decomposition (SVD)

Supplementary "Recommendation Systems" Lecture

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Prerequisites

- Matrix Transpose
- Matrix Multiplication
- Identity Matrix
- Orthogonal Matrix
- Orthonormal Matrix
- Diagonal Matrix
- Determinant of a Matrix
- Eigen Values
- Eigen Vectors
- Gram-Schmidt orthonormalization process

What is SVD?

Given a rectangular matrix A, the linear algebra theorem SVD specifies that

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T$$

In other words, A can be broken down into the product of three matrices - an orthogonal matrix U, a diagonal matrix S, and the transpose of an orthogonal matrix V

Additional properties that hold good are :

$U^TU = I$, $V^TV = I$; the columns of U are orthonormal eigen vectors of AA^T , the columns of V are orthonormal eigen vectors of A^TA, and S is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order.

Start with the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

In order to find U, we have to start with AA^T. The transpose of A is

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

so

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

Next, we have to find the eigenvalues and corresponding eigenvectors of AA^T. We know that eigenvectors are defined by the equation $A\vec{v} = \lambda\vec{v}$, and applying this to AA^T gives us

$$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We rewrite this as the set of equations

$$11x_1 + x_2 = \lambda x_1$$

$$x_1 + 11x_2 = \lambda x_2$$

and rearrange to get

$$(11 - \lambda)x_1 + x_2 = 0$$
$$x_1 + (11 - \lambda)x_2 = 0$$

Solve for λ by setting the determinant of the coefficient matrix to zero,

$$\begin{vmatrix} (11-\lambda) & 1 \\ 1 & (11-\lambda) \end{vmatrix} = 0$$

which works out as

$$(11-\lambda)(11-\lambda) - 1 \cdot 1 = 0$$
$$(\lambda - 10)(\lambda - 12) = 0$$
$$\lambda = 10, \lambda = 12$$

our eigenvectors For $\lambda = 10$ we get

$$(11-10)x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$x_1 = -x_2$$

which is true for lots of values, so we'll pick $x_1 = 1$ and $x_2 = -1$ since those are small and easier to work with. Thus, we have the eigenvector $[1, -1]$ corresponding to the eigenvalue $\lambda = 10$. For $\lambda = 12$ we have

$$(11 - 12)x_1 + x_2 = 0$$

$$x_1 = x_2$$

and for the same reason as before we'll take $x_1 = 1$ and $x_2 = 1$

These eigenvectors become column vectors in a matrix ordered by the size

the eigenvector for $\lambda = 12$ is column one
the eigenvector for $\lambda = 10$ is column two

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Finally, we have to convert this matrix into an orthogonal matrix which we do by applying the Gram-Schmidt orthonormalization process to the column vectors. Begin by normalizing $\vec{v_1}$.

$$\vec{u_1} = \frac{\vec{v_1}}{|\vec{v_1}|} = \frac{[1,1]}{\sqrt{1^2+1^2}} = \frac{[1,1]}{\sqrt{2}} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$$

Compute

$$\vec{w_2} = \vec{v_2} - \vec{u_1} \cdot \vec{v_2} * \vec{u_1} =$$

$$[1,-1] - [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \cdot [1,-1] * [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] =$$

$$[1,-1] - 0 * [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] = [1,-1] - [0,0] = [1,-1]$$

and normalize

$$\vec{u_2} = \frac{\vec{w_2}}{|\vec{w_2}|} = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$$

to give

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

The calculation of V is similar

V is based on A^TA,

Find the eigenvalues of A^TA

find corresponding eigenvectors

use the Gram-Schmidt orthonormalization process to convert that to an orthonormal matrix.

All this to give us

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

when we really want its transpose

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

For S we take the square roots of the non-zero eigenvalues and populate the diagonal with them, putting the largest in s_{11}, the next largest in s_{22} and so on until the smallest value

$$S = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Now we have all the pieces of the puzzle

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\sqrt{12}}{\sqrt{2}} & \frac{\sqrt{10}}{\sqrt{2}} & 0 \\ \frac{\sqrt{12}}{\sqrt{2}} & \frac{-\sqrt{10}}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$