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Faculty of Mathematics and Physics

BACHELOR THESIS

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Operads and Field Theory

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When I started to learn the theory of operads, I knew virtually nothing. Through the last year, I dabbled in various mathematical disciplines. For what I learnt I mainly thank Braňo Jurčo and Martin Doubek, for helping me to orient in a flood of new concepts, explaining the big picture as well as omnipresent $(-1)^{\varepsilon}$ issues.

I also hold a deep gratitude to my family and friends, whose support, mental and material, got me through.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Operády a teorie pole

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Abstrakt: Operády a ich varianty, modulárne a cyklické operády, prirodzene popisujú skladanie objektov rôznych typov. Práca poskytuje prístupný úvod do teórie operád, formalizmu používaného v [1] a modernej aplikácie modulárnych operád vo fyzike [2]. S pomocou príkladov uvedieme Batalin-Vilkovisky formalizmus ako nástroj na kohomologickú integráciu dráhového integrálu v kvantovej teórii poľa. Master rovnica, podmienka na akciu, plynie z tohoto formalizmu. Riešenia master rovnice ale taktiež popisujú algebry nad Feynmanovou transformáciou modulárnej operády. Preskúmame master rovnicu takto definovanú na modulárnej operáde a zhrnieme aplikáciu tejto teórie do uzavretej strunovej teórie poľa.

Klíčová slova: operády, algebry nad operádami, master rovnice, teorie pole

Title: Operads and field theory

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Abstract: Operads and their variants, modular and cyclic operads, naturally describe compositions of objects of various types. We provide an accessible introduction to the theory of operads, the formalism for modular operads from [1] and modern application of modular operads to physics, due to Barannikov [2]. Through examples, we introduce Batalin-Vilkovisky formalism as a tool for cohomological integration of path integral in quantum field theories. A master equation, consistency condition for action, follows from this formalism. Solutions to master equation also describe algebras over Feynman transform of a modular operad. We explore the master equation defined in terms of modular operad and review an application to closed string field theory.

Keywords: operads, algebras over operads, master equation, field theory

Contents

Introduction

Operads have appeared in work of many distinguished mathematical physicists (Maxim Kontsevich, Jim Stasheff). Coming originally from homotopy theory, one of the connections of operads to physics is recent Barannikov's work [2]. Here, he relates *algebras over Feynman transform of a modular operad* to solutions of quantum master equation, physical construct from quantum field theory. This thesis is a record of author's effort to understand this connection.

The work is written by a physicist and even though its content is mostly mathematical, it is not always fully rigorous. We try to make the most murky points at least believable.

Bulk of the work is inspired by or rephrased from [1], an application of Barannikov's theory to string field theory.

First chapter of the thesis starts with motivation for introduction of operads. It is meant to be as comprehensible as possible, casting away all the additional concepts until later. Afterwards, we introduce the necessary machinery, adapted from [1].

Chapter two serves as a motivation for quantum master equation and BV algebras. This is the most physical part, and also the most "handwavy" one [1].

Third chapter briefly summarizes the reformulation of Barannikov's work by [1]. Here we see how modular operads give rise to BV algebra structures defined in chapter two.

The fourth, last chapter develops an example of constructions of chapter three, using simple vectors spaces and operads. The most technical calculations are done here, elaborating on some remarks in [1].

Most of this thesis is taken from other sources. We don't know of any explicit appearance of axioms of set-indexed endomorphism operad in literature.

[1] This property is borrowed from the path integral.

Notation

Notation

$\triangle$ and $\diamond$	denote the end of definition and example, respectively. We have chosen to mark this because bulk of this work is in definitions and examples.
$V^\#$	is a linear dual of V. All vector spaces are assumed to be finite-dimensional.
$\mathbb{S}_n$	is the permutation group of n elements.
1	is used to denote identity morphisms. The *number* 1 is rarely used, so there is no risk of confusion. Specifically, identities on set M are denoted 1_M.
$\Bbbk$	is a field of characteristic 0, we always have real or complex numbers on mind.
$\sqcup$	is a disjoint union of two sets. In the expression $A \sqcup B$, A and B are automatically assumed to be disjoint.
$[n]$	is a set $\{1, \dots, n\}$.

Abbreviations

RHS and LHS	are short for *right hand side* and *left hand side.*
BV	stands for Batalin-Vilkovisky.
BRST	stands for Becchi, Rouet, Stora and Tyutin.
QFT	stands for *quantum field theory.*

1. Operads

We start with an accessible introduction to the theory of operads. First few sections (1.2 to 1.7) cover the standard definition and canonical examples, taking a straightforward route of generalizing the natural structure of sets of functions with their compositions. The rest of this chapter then builds the formalism of set-indexed dg operads, needed for our application of operad theory.

There are several introductory texts demanding varying levels of mathematical maturity. First sections of Markl's [3] and Andor's [4] are good supplementary reading. Chapter 1 of part II of [5] provides a very general approach.

1.1 History

Operads were defined by J. Peter May in 1972 in his work [6][1], with many similar concepts appearing before (see [7] or part I of [5] for prehistory). Operads turned out to be useful as a machinery for cataloguing different structures with compositions and became used in many areas of mathematics – algebra, topology and physics from the name of [5], category theory, deformation theory and others.

In the first half of nineties, so-called *renaissance* of operads refers to a surge of activity reviving the research. Markl [8] reviews some of the themes of the renaissance.

Operads and mathematical physics interacted through various algebraic and topological constructions; BV algebras, moduli spaces of Riemann surfaces, with most of the applications concerning quantization of field theories. Markl [9] described Zwiebach's [10] results for action of closed string field theory using operads.

There are various generalizations of operads. Getzler and Kapranov introduced cyclic operads in [11] and modular operads in [12]. Recently, Barannikov [2] has shown a connection between modular operads and BV quantization.

1.2 Preliminaries

By $\mathbb{S}_n$, we mean a symmetric group of degree n; that is, a group of permutations of n-element set $\{1, \dots, n\}$ with group operation being composition. General permutation is denoted using Cauchy's two-line notation

$$\sigma = \left(\begin{smallmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{smallmatrix} \right) .$$

We will use symbols σ and τ for elements of $\mathbb{S}_n$.

Groups are usually important because they represent transformations. Accordingly, there is a notion of a *group action* on a set M.

Definition 1.1. For a group G and a set M, a *left action* is a function

$$G \times M \to M ,$$

denoted for $g \in G$, $m \in M$ by juxtaposition $gm \in M$, satisfying for all $m \in M$

$$\begin{aligned} g_2(g_1 m) &= (g_2 \cdot g_1) m \, , \ \forall g_1, g_2 \in G \, , \\ em &= m \, , \end{aligned}$$

[1]The name came from *monad*, suggested to MacLane by May as well. May says about the term *operad* that he coined it himself "spending a week thinking about nothing else" [7].

with $\cdot$ denoting the group operation and e the group unit.

Similarly, there's a *right action*, defined as function $M \times G \to M$, denoted by mg, satisfying for all $m \in M$

$$(mg_1)g_2 = m(g_1 \cdot g_2)\,,\ \forall g_1, g_2 \in G\,,$$

and

$$me = m\,.$$

$\triangle$

Notice that for fixed $g \in G$, the action is reduced to a function $g_M : M \to M$.

We will need an action of group $\mathbb{S}_n$ on a set of n-tuples. The most intuitive one is "permuting the elements" of the tuple by the permutation. That is, for a tuple $(a_1, \ldots, a_n)$, a_i goes from ith to $\sigma(i)$th position.

We should check if this is an action. Identity permutation doesn't exchange any elements and application of σ and then τ takes a_i to $\sigma(i)$th and then to $\tau(\sigma(i))$th position. We know that $\tau(\sigma(i)) = [\tau \cdot \sigma](i)$, and we see that action defined this way is *left action*.

The only slightly surprising part of this exercise is the explicit form of the action on the tuple. The ith element of $\sigma(a_1, \ldots, a_n)$, has to come from position j satisfying $\sigma(j) = i$, or $j = \sigma^{-1}(i)$

$$\sigma(a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})\,.$$

A minor generalization of $\mathbb{S}$ action on a set is a $\mathbb{S}$-module:

Definition 1.2. *Left/right* $\mathbb{S}$*-module* $\mathcal{P}$ is a sequence of sets

$$\mathcal{P} = \{\mathcal{P}(n) \mid \forall n \geq 1\}\,, \tag{1.1}$$

with left/right action of $\mathbb{S}_n$ on each $\mathcal{P}(n)$. $\triangle$

1.3 Endomorphism operad

Before introducing operads in their generality, let's look at a very important example, called *endomorphism operad.* For a fixed set M, there is a sequence $\{\mathcal{E}nd_M(n);\ n \geq 1\}$ of sets of functions defined by

$$\mathcal{E}nd_M(n) := \{p \mid M^{\times n} \to M\}\,, \tag{1.2}$$

that is, nth part of $\mathcal{E}nd_M$ is a set of all functions taking n objects from M and returning one. This sequence has a natural structure of a $\mathbb{S}$-module, for each function permuting its arguments

$$(p\sigma)(m_1, \ldots, m_n) := p(m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})\,. \tag{1.3}$$

This is a right action on $\mathcal{E}nd_M(n)$, as can be checked by applying two permutations. The reason for choosing right action is that $(p\sigma)(m_1, \ldots, m_n)$ can be interpreted as $p[\sigma(m_1, \ldots, m_n)]$, exactly the left action on an n-tuple we described in previous section. [2]

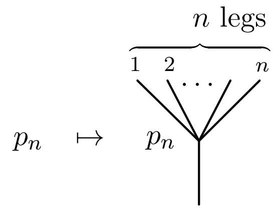
[2] The transition from right to left action agrees with notation

$$(p[\tau \cdot \sigma])(\ldots) = ((p\tau)\sigma)(\ldots) = (p\tau)[\sigma(\ldots)] = p[\tau[\sigma(\ldots)]]\,.$$

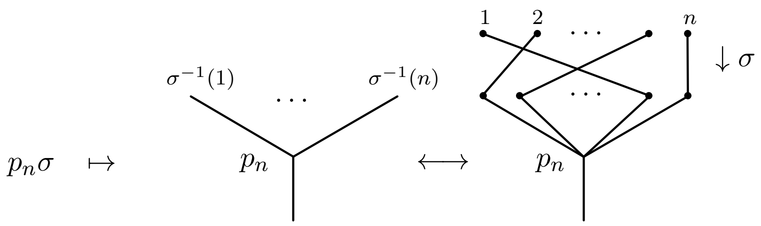
There is another natural operation on functions in $\mathcal{E}nd_M$ and that is composition, denoted $\circ$. For functions $p_k \in \mathcal{E}nd_M(k)$ and $q_l \in \mathcal{E}nd_M(l)$, there are k different ways of plugging q_l into p_k. The position of q_l is specified by index i, so we have operations

$$\circ_i : \mathcal{E}nd_M(k) \times \mathcal{E}nd_M(l) \to \mathcal{E}nd_M(k+l-1)$$
$$(p_k \circ_i q_l)(m_1, \ldots, m_{k+l-1}) := p_k(m_1, \ldots, m_{i-1}, q_l(m_i, \ldots, m_{i+l-1}), m_{i+l}, \ldots, m_{k+l-1}) .$$

These operations satisfy some non-trivial properties - the composition is associative and we should be able to exchange the order of $\mathbb{S}$ action and composition. This is where pictorial notation for these functions comes handy. Let us think of each function



(a) A function with n inputs.



(b) $\mathbb{S}$ action on a function.

Figure 1.1: Trees representing functions.

$p_n \in \mathcal{E}nd_M(n)$ as a one-vertex tree[3] with n leaves labelled with numbers 1 to n, as in figure 1.1a. The leaves correspond to function arguments and labels specify the object that goes into the function input. The $\mathbb{S}$ action works by reordering the labels by *inverse* permutation, or equivalently, by putting the permutation above the leaves. This correspondence is shown in picture 1.1b. Tracing the number i using the permutation leads to leaf $\sigma(i)$, or jth leaf is connected with number $\sigma^{-1}(j)$.

The $p_n \circ_i q_m$ works by grafting the root of tree q_m to the ith leave of p_n, as in figure 1.2

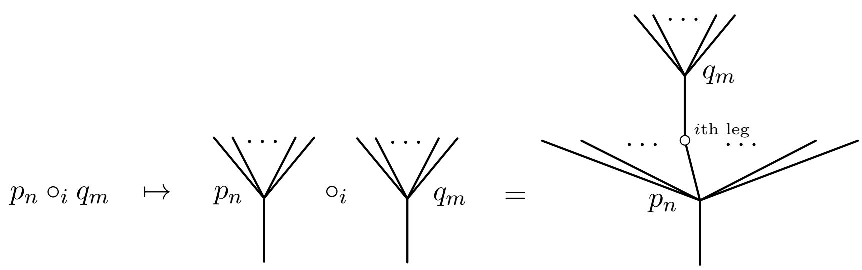


Figure 1.2: $\circ_i$ acting on trees.

[3] Trees are drawn with leaves up. This results in more natural $\mathbb{S}$ action. In the other case, the permutation would have to be flipped upside down.

We can omit the labels if we implicitly assume that the top level of the diagram (above the permutations and grafting) is labelled from left to right by numbers 1 to n (fig. 1.1b).

In $\mathcal{E}nd_M(1)$, there is a distinguished function $1_{\mathcal{E}nd_M}$, an identity on M, satisfying

$$p \circ_i 1_{\mathcal{E}nd_M} = p = 1_{\mathcal{E}nd_M} \circ_1 p$$

In the graphical language, this property says that $1_{\mathcal{E}nd_M}$ can be thought of as a straight line segment (figure 1.3).

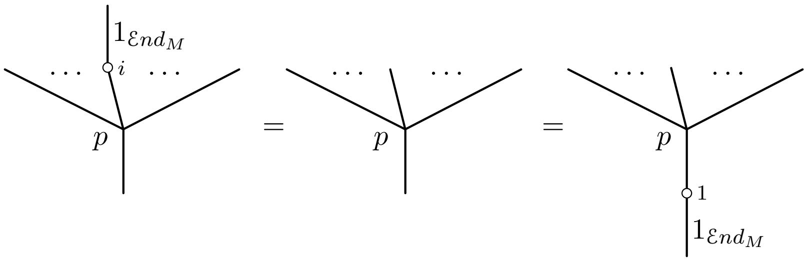
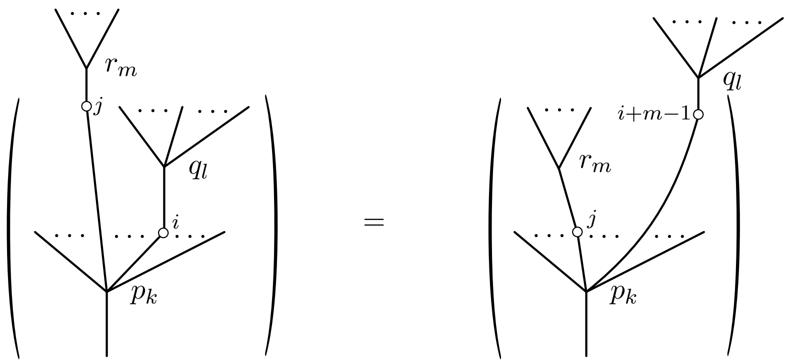


Figure 1.3: Unit axiom $p \circ_i 1_{\mathcal{E}nd_M} = p = 1_{\mathcal{E}nd_M} \circ_1 p$.

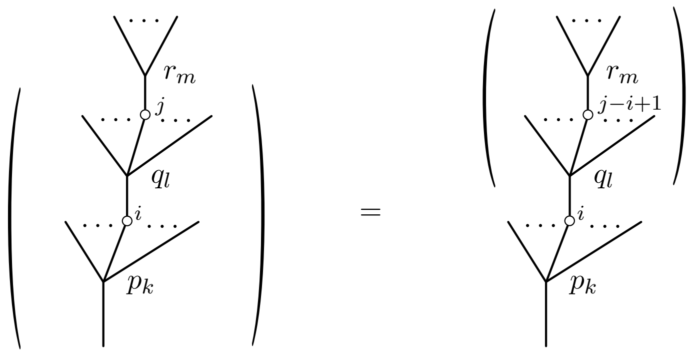
Associativity for $\circ_i$ is somewhat more complicated. There are three different ways of composing three different functions, for three placements of the innermost one. After composition, leaves are relabelled, so indices in $\circ_i$ have to change. These three cases are shown in figure 1.4, with corresponding formulas read from the figures.

The fact that $\mathbb{S}_n$ has a right action on p_n gives figure 1.5a.

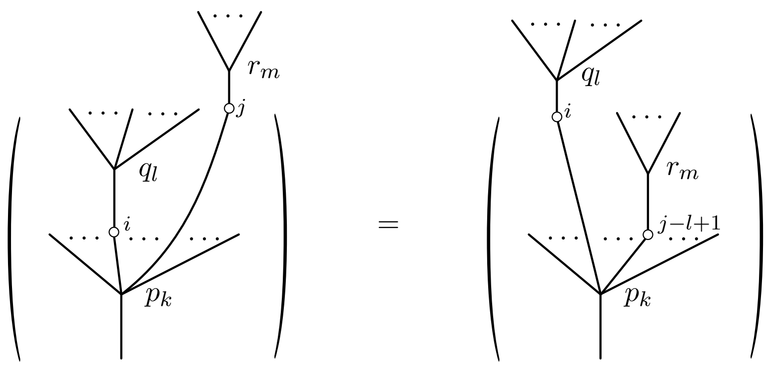
At last, it is possible to rewrite $(p_k\sigma) \circ_i (q_l\tau)$ using composition $p_k \circ_j q_l$ (for some j) followed by appropriate permutation. Figure 1.5b shows that $j = \sigma(i)$ and that resulting permutation, called $\sigma \circ_i \tau$, is created by pasting permutation τ into ith input of σ. This property of $\circ_i$ is called *equivariance*.



(a) The case of $j \in \{1 \ \ldots \ i-1\}$: $(p_k \circ_i q_l) \circ_j r_m = (p_k \circ_j r_m) \circ_{i+m-1} q_l$.

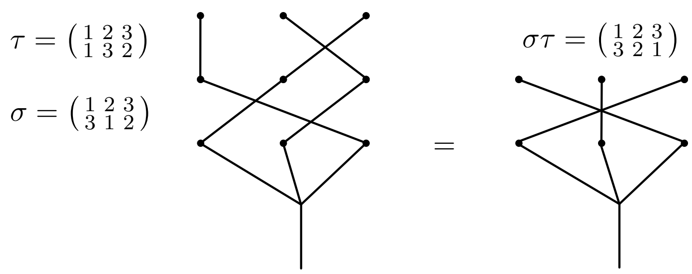


(b) The case of $j \in \{i \ \ldots \ i+l-1\}$: $(p_k \circ_i q_l) \circ_j r_m = p_k \circ_i (q_l \circ_{j-i+1} r_m)$.

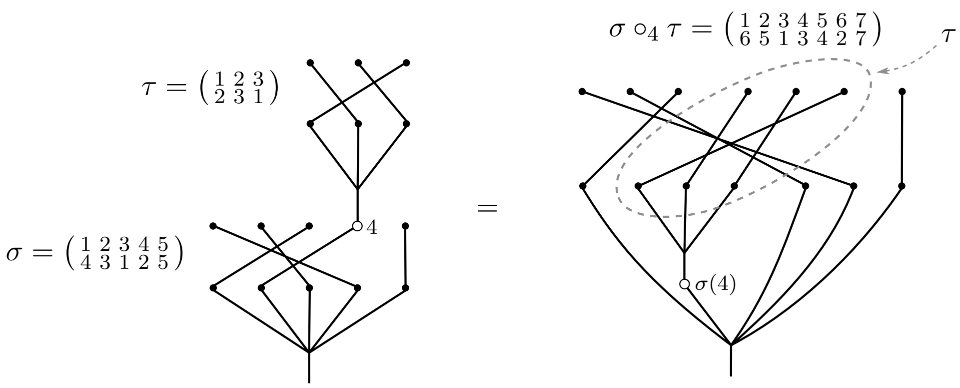


(c) The case of $j \in \{i+l \ \ldots \ k+l-1\}$: $(p_k \circ_i q_l) \circ_j r_m = (p_k \circ_{j-l+1} r_m) \circ_i q_l$.

Figure 1.4: Associativity axioms.



(a) $\mathbb{S}$ action – $(p\sigma)\tau = p[\sigma\tau]$.



(b) Equivariance – $(p_k\sigma) \circ_i (q_l\tau) = (p_l \circ_{\sigma(i)} q_l)[\sigma \circ_i \tau]$.

Figure 1.5: Relabelling of the legs.

1.4 Definition of an operad

In the last section, we have seen that collection of functions on set M, which we called *endomorphism operad*, carries two operations ($\mathbb{S}$ action and $\circ_i$) with certain properties. This notion is indeed a starting point for general definition of operad – abstracting away the structure of "functions over set", we are left with the operations and their properties.

Definition 1.3. *Operad* $\mathcal{P}$ is a right $\mathbb{S}$-module $\mathcal{P}$ with a unit

$$1_{\mathcal{P}} \in \mathcal{P}(1)$$

and binary operations[4]

$$\circ_i : P(n) \times P(m) \to P(n+m-1)\,,$$

defined $\forall\, n, m \geq 1$, $i \in \{1 \dots n\}$ satisfying following axioms:

1. unit law: For all $p_n \in \mathcal{P}(n)$ and for all $i \in \{1 \dots n\}$

$$p_n \circ_i 1_{\mathcal{P}} = p_n$$

and

$$1_{\mathcal{P}} \circ_1 p_n = p_n\,.$$

[4]One could be explicit and specify the n, m in the $\circ_i$, too, for example by writing $\circ_i^{(n,m)}$. This is, however, very cluttering and almost always superfluous.

2. associativity: For all $p_k \in \mathcal{P}(k)$, $q_l \in \mathcal{P}(l)$ and $r_m \in \mathcal{P}(m)$ and $\forall i \in \{1 \dots k\}$

$$
\begin{aligned}
(p_k \circ_i q_l) \circ_j r_m &= (p_k \circ_j r_m) \circ_{i+m-1} q_l \quad \text{if} \quad j \in \{1 \ \dots \ i-1\} \\
(p_k \circ_i q_l) \circ_j r_m &= p_k \circ_i (q_l \circ_{j-i+1} r_m) \quad \text{if} \quad j \in \{i \ \dots \ i+l-1\} \\
(p_k \circ_i q_l) \circ_j r_m &= (p_k \circ_{j-l+1} r_m) \circ_i q_l \quad \text{if} \quad j \in \{i+l \ \dots \ k+l-1\} \, .
\end{aligned}
$$

3. equivariance: of $\circ_i$ For all $p_l \in \mathcal{P}(k)$, $q_l \in \mathcal{P}(l)$ and for permutations $\sigma \in \mathbb{S}_k$, $\tau \in \mathbb{S}_l$

$$
(p_k \sigma) \circ_i (q_l \tau) = (p_l \circ_{\sigma(i)} q_l)[\sigma \circ_i \tau] \, ,
$$

where by $\sigma \circ_i \tau$ we mean a permutation in $\mathbb{S}_{k+l-1}$ created by inserting the permutation τ in the ith position of σ. Writing the $[\sigma \circ_i \tau](j)$ out explicitly, this means

$$
[\sigma \circ_i \tau](j) = \begin{cases}
\sigma(j), & \text{if } j \leq i-1 \ \text{ and } \sigma(j) < \sigma(i) \\
\sigma(j) + l - 1, & \text{if } j \leq i-1 \ \text{ and } \sigma(j) > \sigma(i) \\
\sigma(i) + \tau(j-i+1) - 1, & \text{if } j \in \{i \ \dots \ i+l-1\} \\
\sigma(j-l+1), & \text{if } j \geq i+l \ \text{ and } \sigma(j-l+1) < \sigma(i) \\
\sigma(j-l+1) + l - 1, & \text{if } j \geq i+l \ \text{ and } \sigma(j-l+1) > \sigma(i) \, .
\end{cases}
$$

Example for $\sigma \in \mathbb{S}_5$, $\tau \in \mathbb{S}_3$ and $\sigma \circ_4 \tau \in \mathbb{S}_7$ is shown in figure 1.6

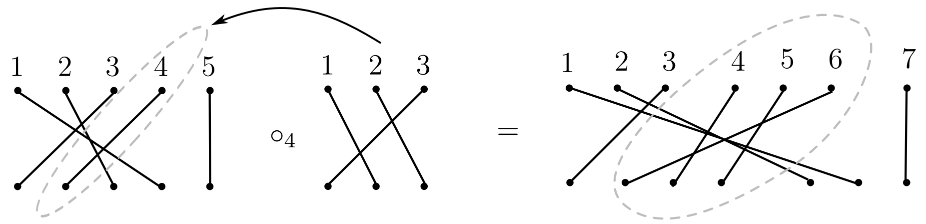


Figure 1.6: $\circ_i$ for permutations.

△

Only few comments are due. We used the example from endomorphism operad to demonstrate the form of $\sigma \circ_i \tau$ permutation. The prescription for this permutation is so complicated because both permutations are being relabelled.

We will call objects in $\mathcal{P}(n)$ the elements of operad $\mathcal{P}$ with arity n.

Operad defined this way is operad over category **Set**. This means that $\mathcal{P}(n)$ is a set for each n and $\circ_i$ takes a Cartesian product of two sets as an input. We will see operads over different categories in section 1.7.

1.5 Examples

We have already introduced an endomorphism operad, let's review its definition for future reference.

Definition 1.4. For a set M, the *endomorphism operad* $\mathcal{E}nd_M$ is defined as

$$
\mathcal{E}nd_M(n) := \{f : M^{\times n} \to M\} \, ,
$$

The $\mathbb{S}$ action on a function permutes the arguments

$$(f\sigma)(m_1, \ldots, m_n) := f(m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}) \,.$$

The $f_k \circ_i g_l$ inserts g_l to ith argument of f_k

$$(f_k \circ_i g_l)(m_1, \ldots, m_{k+l-1}) := f_k(m_1, \ldots, m_{i-1}, g_l(m_i, \ldots, m_{i+l-1}), m_{i+l}, \ldots, m_{k+l-1}) \,.$$

The operadic unit is a identity map on M. △

Example 1.5. The simplest operad consists of just 1 element, which has to be the operadic unit:

$$\mathcal{P}(1) := \{1_{\mathcal{P}}\} \,,$$
$$\mathcal{P}(n) := \emptyset \,, \quad n > 1 \,.$$

The symmetric group action is trivial (multiplying by permutation doesn't change a operad element) and the $\circ_i$ is defined as

$$1_{\mathcal{P}} \circ_1 1_{\mathcal{P}} := 1_{\mathcal{P}} \,.$$

◇

Example 1.6. If we want to have a non-trivial $\circ_i$, we have to populate every $\mathcal{P}(n)$ for $n \geq 1$. Let's take

$$\mathcal{P}(n) := \{n\} \,, \quad n \geq 1 \,, \tag{1.4}$$

where choosing the elements of operad to be integers is purely conventional. The symmetric group action is, again, trivial, but we have to define $\circ_i$ as

$$m \circ_i n := m + n - 1 \in \mathcal{P}(m + n - 1) \,.$$

This operad is called $\mathcal{C}om$[5]. Graphically, the elements of this operad are unlabelled trees (since the $\mathbb{S}$ action doesn't change the element). Grafting two m and n-leaved trees results in a tree with $m + n - 1$ leaves. ◇

Example 1.7. Bringing in the $\mathbb{S}$ action, we can define operad $\mathcal{A}ss$[6]

$$\mathcal{A}ss(n) := \mathbb{S}_n \,, \quad n \geq 1 \,, \tag{1.5}$$

Because we need to tell apart the elements of operad $\mathcal{A}ss$ and the symmetric group acting on it, we will indicate the operad elements using index $\mathcal{A}$, as in $\tau_{\mathcal{A}}$ or $\tau_{\mathcal{A},n} \in \mathcal{A}ss(n)$, if we want to specify the arity of the element.

The sigma action multiplies the operad element $\tau_{\mathcal{A}}$ by the symmetric group element σ from the right

$$\tau_{\mathcal{A}}\sigma := (\tau\sigma)_{\mathcal{A}} \,.$$

For identity permutations of $e_{\mathcal{A},n} \in \mathcal{A}ss(n)$, the $\circ_i$ is defined as for $\mathcal{C}om$ operad

$$e_{\mathcal{A},m} \circ_i e_{\mathcal{A},n} := e_{\mathcal{A},m+n-1}$$

and then it's extended using the equivariance axiom, writing the operad elements as $\sigma_{\mathcal{A},n} = e_{\mathcal{A},n}\sigma$

$$\sigma_{\mathcal{A},m} \circ_i \tau_{\mathcal{A},n} = (e_{\mathcal{A},m}\sigma) \circ_i (e_{\mathcal{A},n}\tau) = e_{\mathcal{A},m+n-1}[\sigma \circ_i \tau] = [\sigma \circ_i \tau]_{\mathcal{A},m+n-1} \,.$$

Graphically, these are exactly our labelled trees. The difference from the general graphical notation is that composition of identity permutations using $\circ_i$ (as operad elements) $e_{\mathcal{A},m}$ and $e_{\mathcal{A},n}$ gives another identity permutation in arity $m + n - 1$, regardless of the i. ◇

[5] *commutative* operad
[6] *associative* operad

1.6 Algebra over an Operad

We have seen that operads arise naturally from a notion of functions on a set. First, we need a notion of operad morphism.

Definition 1.8. Given two operads, $\mathcal{P}$ and $\mathcal{Q}$, with composition $\circ_i$ and $\tilde{\circ}_i$ respectively, f is an *operad* morphism $\mathcal{P} \to \mathcal{Q}$, if f[7] is a collection of functions between corresponding arities of operad element

$$f : \mathcal{P}(n) \to \mathcal{Q}(n),\ \forall n\,,$$

such that for all $p_m \in \mathcal{P}(m), p'_n \in \mathcal{P}(n)$, f commutes with $\circ_i$

$$f(p_m \circ_i p'_n) = f(p_m)\tilde{\circ}_i f(p'_n)\,,$$

is *equivariant* with respect to the $\mathbb{S}$ action

$$f(p_m\sigma) = f(p_m)\sigma\,,$$

for all $\sigma \in \mathbb{S}_n$, and maps operadic units

$$f(1_{\mathcal{P}}) = 1_{\mathcal{Q}}\,.$$

△

Operad morphisms make operads into a category **Operad**. Algebras over an operad $\mathcal{P}$ (also called $\mathcal{P}$-algebras) are morphisms with endomorphism operad as a codomain.

Definition 1.9. *Algebra over an operad* $\mathcal{P}$ is a operad morphism from $\mathcal{P}$ to endomorphism operad

$$\mathcal{P} \to \mathcal{E}nd_M\,,$$

for some set M △

With this notion, we can start interpreting general operads in terms of functions. Let's fix a set M and its endomorphism operad $\mathcal{E}nd_M$. We will be looking on the most general form of an algebra over an operad for examples from section 1.5. We will denote this operadic morphism by f.

Operad from example 1.5 necessarily has its only element, operadic unit, mapped into the unit function $1_{\mathcal{E}nd_M}$ on M.

Other two examples require a closer look.

Example 1.10. The operadic unit of $\mathcal{C}om$ from example 1.6 is again mapped to the identity on M. We can view any other element of $\mathcal{C}om$ as being generated by 2, the element of $\mathcal{C}om(2)$. Let's denote

$$\mu_{\mathcal{C}} := f(2)\,,$$

the operation

$$\mu_{\mathcal{C}} : M \times M \to M\,.$$

Trivial $\mathbb{S}$ action tells us that

$$\mu_{\mathcal{C}}(m_1, m_2) = \mu_{\mathcal{C}}(m_2, m_1)\,.$$

Every other operation $\mu_{\mathcal{C}}^{(n)} := f(n),\ n \geq 3$ is equal to any possible way of composing the $\mu_{\mathcal{C}}$ with itself $n-1$ times. For example,

$$\mu_{\mathcal{C}}^{(3)} = \mu_{\mathcal{C}} \circ (\mu_{\mathcal{C}} \times 1_{\mathcal{E}nd_M}) = \mu_{\mathcal{C}} \circ (1_{\mathcal{E}nd_M} \times \mu_{\mathcal{C}})\,,$$

[7] We abuse the notation, similarly as for $\circ_i$, to denote every f by the same letter, as opposed to e.g. f_n.

or, on elements

$$\mu_{\mathcal{C}}^{(3)}(m_1, m_2, m_3) = \mu_{\mathcal{C}}(\mu_{\mathcal{C}}(m_1, m_2), m_3) = \mu_{\mathcal{C}}(m_1, \mu_{\mathcal{C}}(m_2, m_3)) . \tag{1.6}$$

For $\mu_{\mathcal{C}}^{(n)}$, $\mathbb{S}$ action implies that it's totally symmetric in it's arguments.

Together, we see that algebra over $\mathcal{C}om$ consists of a associative commutative "product" $\mu_{\mathcal{C}}$, commutativity coming from trivial $\mathbb{S}$ action and associativity from the $\circ_i$ definition – the $\mu_{\mathcal{C}}$ product of n elements of M is independent of bracketing. ◇

Example 1.11. Again, the unit of $\mathcal{A}ss$ from example 1.7 is mapped to $1_{\mathcal{E}nd_M}$.

We will denote the product by $\mu_{\mathcal{A}} := f(e_{\mathcal{A},2})$. There is no symmetry, but for $\mu_{\mathcal{A}}^{(n)} := f(e_{\mathcal{A},n})$, $n \geq 3$ the associativity still holds, as e.g. in equation 1.6

$$\mu_{\mathcal{A}}^{(3)}(m_1, m_2, m_3) = \mu_{\mathcal{A}}(\mu_{\mathcal{A}}(m_1, m_2), m_3) = \mu_{\mathcal{A}}(m_1, \mu_{\mathcal{A}}(m_2, m_3)) .$$

The definition of $\circ_i$ for $\mathcal{A}ss$,

$$(\mu_{\mathcal{A}}^{(m)}\sigma) \circ_i (\mu_{\mathcal{A}}^{(n)}\tau) = \mu_{\mathcal{A}}^{(m+n-1)}[\sigma \circ_i \tau] ,$$

ensures that composition of products behaves exactly the way we expect it to, permuting arguments of the second product by τ *in the ith argument of* σ. We see that algebra over $\mathcal{A}ss$ describes an associative multiplication on M. ◇

1.7 Operads over different categories

So far, we dealt with an operad defined over the category **Set**; this means that the underlying $\mathbb{S}$ module is a collection of *sets* and $\circ_i$ is and morphism of sets, which means any ordinary function. Operads were, however, originally defined for topological spaces ([6]) and most of this thesis will use differential graded vector spaces.

The most general setting operads are usually defined in is that of *symmetric monoidal category*. Informally, this is a category with a product of objects satisfying certain properties. In the category of sets, this product is the Cartesian product, in vector spaces it's the tensor product.

Since we will not need such generality, we refer the reader to chapter 1 of part II of [5]. Note that in ibid., operads with $\circ_i$ product are called *pseudo-operads*, to distinguish them from operads defined in May's fashion (see section 1.11).

1.8 General indices on operad legs

We have seen a very direct way of defining an operad, generalizing the structure of an endomorphism operad, in which the "legs" of operad elements were naturally indexed by numbers. This, however, leads to complications in the definition. Because the legs are being relabelled, $\circ_i$ operation on permutations has to be introduced and multiple versions of the associativity axiom are needed[8].

So, we are lead to index the elements of an operad by non-conflicting indices, and a result of $\circ_a$ operation on two operad elements with index sets $C_1 \sqcup \{a\}$[9] and C_2 would be an element indexed by $C_1 \sqcup C_2$, the disjoint union of the two sets, without the element a.

[8][5] puts it nicely in the remark 5.10, (when talking about cyclic operads) "The operations ${}_i\circ_j$ have to satisfy certain obvious axioms whose written form is very complicated due to the linear structure of human language."

[9]$\sqcup$ is used to denote a disjoint union. Anywhere it appears, two sets it is combining are assumed to be disjoint.

One way of looking on this construction is a categorical generalization of the notion of an $\mathbb{S}$-module. Let us denote by $\mathbf{\Sigma}$ a category with objects being sets $[n] := \{1 \ldots n\}$ for each $n \geq 1$ and morphisms only the bijections. Then any functor from $\mathbf{\Sigma}$ to (arbitrary) category $\mathbf{A}$ consists of:

$A(n)$: collection of objects of $\mathbf{A}$, one for each $n \geq 1$

and for any morphisms $\sigma, \tau \in Hom_{\mathbf{\Sigma}}([n],[n])$, we have morphisms $\sigma_{\mathbf{A}}, \tau_{\mathbf{A}} \in Hom_{\mathbf{A}}(A(n), A(n))$ satisfying

$$\sigma_{\mathbf{A}} \circ \tau_{\mathbf{A}} = (\sigma \circ \tau)_{\mathbf{A}} \,.$$

This is exactly the definition of an left $\mathbb{S}$-module. To realize a right $\mathbb{S}$-module, a functor from $\mathbf{\Sigma}^{\mathrm{op}}$, the opposite category is needed. We can now replace $\mathbf{\Sigma}$ by category $\mathbf{Set_f}$, category of nonempty finite sets and their bijections.

Definition 1.12. Left $\mathbf{Set_f}$-module $\mathcal{P}$ in category $\mathbf{A}$ is a functor from $\mathbf{Set_f}$ to $\mathbf{A}$. Right $\mathbf{Set_f}$-module $\mathcal{P}$ in category $\mathbf{A}$ is a functor from $\mathbf{Set_f}^{\mathrm{op}}$ to $\mathbf{A}$.

△

Although we have used the right action so far, we will now *switch the convention*, following [1], to the left $\mathbf{Set_f}$-modules. Also from now on, we are working in a symmetric monoidal category of dg vector spaces. The basic definitions and properties summarized in appendix A.

To make the module $\mathcal{P}$ more explicit, note that it's equivalently described as a collection of objects $\mathcal{P}(C) \in \mathbf{dgVect}$ for $C \in \mathbf{Set_f}$ and (degree 0) morphisms $\mathcal{P}(\sigma) : \mathcal{P}(C) \to \mathcal{P}(D)$ for any $\sigma : C \to D$ a bijection, satisfying

$\mathcal{P}(1_C) = 1_{\mathcal{P}(C)}$, $\forall C \in \mathbf{Set_f}$, and

$\mathcal{P}(\sigma \circ \tau) = \mathcal{P}(\sigma) \circ \mathcal{P}(\tau)$, $\forall\ C \xrightarrow{\tau} D \xrightarrow{\sigma} E$.

We can now define a set-indexed operad in a category of dg vector spaces (since we won't need the general case of operad in symmetric monoidal category).

Definition 1.13. *Set-indexed operad* $\mathcal{P}$ *in* $\mathbf{dgVec}$ is a left $\mathbf{Set_f}$ module over a category $\mathbf{dgVec}$, with degree 0 homomorphisms

$$\circ_a : \mathcal{P}(C \sqcup \{a\}) \otimes P(D) \to P(C \sqcup D)\,,$$

satisfying

1. equivariance:

$$\mathcal{P}(\tau)(f) \circ_{\tau(a)} \mathcal{P}(\sigma)(g) = \mathcal{P}(\tau\big|_C \sqcup \sigma)(f \circ_a g)\,,$$

for $C \sqcup \{a\} \xrightarrow{\tau} C' \sqcup \{\tau(a)\}$ and $D \xrightarrow{\sigma} D'$ and f, g elements of dg vector spaces $\mathcal{P}(C \sqcup \{a\})$ and $\mathcal{P}(D)$. Disjoint union of two bijections $\tau\big|_C \sqcup \sigma$ is defined as

$$\left(\tau\big|_C \sqcup \sigma\right)(x) = \begin{cases} \tau(x) & : x \in C \\ \sigma(x) & : x \in D \end{cases}$$

This can be written without specifying the vectors as

$$\circ_{\tau(a)} \left(\mathcal{P}(\tau) \otimes \mathcal{P}(\sigma)\right) = \mathcal{P}(\tau\big|_C \sqcup \sigma) \circ_a \,.$$

We will say that this has to hold *if it makes sense*, meaning that σ and τ should be defined on the appropriate index sets and there have to be corresponding $\circ_x$ operations available for the two vectors f, g we are writing the equation for[10].

2. associativity: There are two versions,

$$(f \circ_a g) \circ_b h = \begin{cases} (-1)^{|g||h|}(f \circ_b h) \circ_a g & : \text{for } b \text{ in index set of } f \\ f \circ_a (g \circ_b h) & : \text{for } b \text{ in index set of } g\,. \end{cases}$$

Again, without specifying the vectors here,

$$\circ_b(\circ_a \otimes 1) = \begin{cases} \circ_a(\circ_b \otimes 1)(1 \otimes s) \quad \text{or} \\ \circ_a(1 \otimes \circ_b)\,, \end{cases}$$

whichever (if any) of the two expressions makes sense. The s is the transposition operator defined in equation A.6

3. unit element: Each dg vector space with one-element index set $\mathcal{P}(\{a\})$ has an element $e_a \in \mathcal{P}(\{a\})$ such that

$$e_a \circ_a f = f$$

for every vector f, and

$$f \circ_b e_a = f$$

for any $f \in \mathcal{P}(C \sqcup \{b\})$. The element also has to be unique, meaning that

$$\mathcal{P}(\tau)(e_a) = e_{\tau(a)}\,.$$

△

1.9 Set-indexed endomorphism operad

Labelling with sets does not come free, however. The important notion of an endomorphism operad works with linear morphisms, and their inputs are naturally indexed by numbers. To overcome this, construction called *unordered tensor product*[11] is used (see also [1] or [12]).

Definition 1.14. For C, finite set and V, vector space, define

$$\bigotimes_C V = \bigoplus_{\psi:C \xrightarrow{\sim} [|C|]} V^{\otimes |C|} \Big/ \sim\,, \tag{1.7}$$

[10] We are talking about equality of morphisms $\mathcal{P}(C \sqcup \{a\}) \otimes \mathcal{P}(D) \to \mathcal{P}(\tau(C) \sqcup \sigma(D))$, f and g are arguments of these morphisms.

[11] In mathematics and physics, functions are usually thought of as having a fixed order of unnamed arguments. Interestingly, some programming languages support named arguments, so that the name of the argument matters, not the order. This means that

```
f( arg1=val1, arg2=val2) == f( arg2=val2, arg1=val1) .
```

There is, a choice of order in this equality (the order of the arguments in the line), corresponding exactly to the isomorphisms ψ.

with ψ going through all the bijections. Note that there are $|C|!$ terms in the direct sum, indexed by the bijections. The equivalence relation is defined, using the $\mathbb{S}$ action, as

$$[v_1 \otimes \cdots \otimes v_{|C|}]^{\psi} = \pm[v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(|C|)}]^{\tau\circ\psi}\,, \tag{1.8}$$

for any permutation $\tau \in \mathbb{S}_{|C|}$. The upper index denotes the summand and the sign comes from permuting the graded elements. Every equivalence class has exactly $|C|!$ basis vectors, so the dimension of $\bigotimes_C V$ is the same as $V^{\otimes|C|}$. △

We will follow [1] and write

$$\imath_{\psi} : V^{\otimes|C|} \to \bigotimes_{C} V \tag{1.9}$$

for the inclusion into the ψ-th summand followed by projection into the equivalence class. Note that $\imath_\psi$ is a isomorphism, we will use it as a proxy for working with unordered tensor product. A useful formula is

$$\imath_{\psi} = \imath_{\tau\psi} \circ \tau\,, \tag{1.10}$$

where the τ on the RHS is the $\mathbb{S}$ action

This is still a dg vector space (see also [1], section III). Degree of its vectors is the same as the degree of elements of the equivalence class. Differential is transferred along any isomorphism $\imath_\psi$ as

$$D_C \circ \imath_{\psi} = \imath_{\psi} \circ d_{V^{\otimes|C|}}\,. \tag{1.11}$$

The identity $D_C^2 = 0$ follows as

$$D_C^2 \circ \imath_{\psi} = D_C \circ \imath_{\psi} \circ d_{V^{\otimes|C|}} = \imath_{\psi} \circ d^2_{V^{\otimes|C|}} = 0\,.$$

We can also check that the differential is well defined, i.e.

$$\imath_{\psi} \circ d_{V^{\otimes|C|}} \circ \imath_{\psi}^{-1} = \imath_{\phi} \circ d_{V^{\otimes|C|}} \circ \imath_{\phi}^{-1}$$

for any isomorphisms $\psi, \phi : C \xrightarrow{\sim} [|C|]$. We rewrite this into

$$d_{V^{\otimes|C|}} \circ \imath_{\psi}^{-1} \circ \imath_{\phi} = \imath_{\psi}^{-1} \circ \imath_{\phi} \circ d_{V^{\otimes|C|}}$$

and note that $\imath_{\psi}^{-1} \circ \imath_{\phi} = \imath_{\psi}^{-1} \circ \imath_{\psi\phi^{-1}\phi} \circ (\psi \circ \phi^{-1}) = (\psi \circ \phi^{-1})$. This says that differential has to commute with the $\mathbb{S}$ action, which is easily seen as a generalization of the case of $V \otimes V$:

$$s \circ (d \otimes 1 + 1 \otimes d) = (1 \otimes d) \circ s + (d \otimes 1) \circ s = (d \otimes 1 + 1 \otimes d) \circ s\,.$$

Example 1.15. Let's explicitly write out the unordered tensor product for a very simple scenario: V is two dimensional, with degree 0 basis vectors v and w; let $C = \{a, b\}$.

Then

$$\bigoplus_{\psi : \xrightarrow{\sim} [|C|]} V^{\otimes|C|} = (V \otimes V)^{ab} \oplus (V \otimes V)^{ba}\,,$$

where we denote the bijection $a \mapsto 1,\ b \mapsto 2$ by symbol ab and $a \mapsto 2,\ b \mapsto 1$ by symbol ba. The basis of $V \otimes V$ consists of elements $v \otimes v$, $v \otimes w$, $w \otimes v$ and $w \otimes w$. Only non-trivial equivalences come from permutation $\tau = \left(\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right)$:

$$\begin{aligned}
(v \otimes v)^{ab} &\sim (v \otimes v)^{ba} =: v^a v^b \,, \\
(v \otimes w)^{ab} &\sim (w \otimes v)^{ba} =: v^a w^b \,, \\
(w \otimes v)^{ab} &\sim (v \otimes w)^{ba} =: w^a v^b \,, \\
(w \otimes w)^{ab} &\sim (w \otimes w)^{ba} =: w^a w^b \,.
\end{aligned}$$

The first equivalence class can be represented by $a \to v$, $b \to v$, second class by $a \to v$, $b \to w$ and so on. The unordered tensor product thus has a basis consisting of pairings of elements of C and basis vectors of V.

To further develop the example, let's look on a linear function F from $\bigotimes_C V$ to e.g. V. Such function can be represented by a linear function $T : V^{\otimes 2} \to V$, defining $F = T(\text{vector } a, \text{vector } b)$. Composing F with $\imath_\psi$, we get $F\imath_\psi : V^{\otimes 2} \to V$. For $\psi = ab$, this means

$$\begin{aligned}
F\imath_{ab}(v \otimes v) &= F(v^a v^b) = T(v, v) \,, \\
F\imath_{ab}(v \otimes w) &= F(v^a w^b) = T(v, w) \,, \\
&\dots
\end{aligned}$$

and for $\psi = ba$

$$\begin{aligned}
F\imath_{ba}(v \otimes v) &= F(v^a v^b) = T(v, v) \,, \\
F\imath_{ba}(v \otimes w) &= F(w^a v^b) = T(w, v) \,. \\
&\dots
\end{aligned}$$

This means that, for example, $F\imath_{ab}(v \otimes v) = F\imath_{ba}(v \otimes v)$ and $F\imath_{ab}(v \otimes w) = F\imath_{ba}(w \otimes v)$. In the graded case, these would have signs $F\imath_{ab}(v \otimes w) = (-1)^{|v||w|} F\imath_{ba}(w \otimes v)$. $\diamond$

Now we can define the endomorphism operad.

Definition 1.16. *Endomorphism operad for a vector space V* is defined as

$$\mathcal{E}nd_V(C) = \mathrm{Hom}(\bigotimes_C V, V) \,. \tag{1.12}$$

$f \circ_a g$ is defined, as usually, by plugging the function $g \in \mathcal{P}(D)$ into the argument a of $f \in \mathcal{P}(C \sqcup \{a\})$. Writing this using isomorphism $\imath_\psi$

$$(f \circ_a g) \circ \imath_\psi := (f \circ \imath_{\psi_C})(g \circ \imath_{\psi_D} \otimes 1^{\otimes |C|})$$

for any[12] $\psi : C \sqcup D \to [|C| + |D|]$ satisfying $\psi(D) = [|D|]$, i.e. putting the elements of D before elements of C, and

$\psi_C : C \sqcup \{a\} \to [|C| + 1]$ is defined as $\psi_C(a) := 1$ and $\psi_C(c) := \psi(c) - |D| + 1$ for $c \in C$,

$\psi_D : D \to [|D|]$ is defined as $\psi_D(d) := \psi(d)$ for $d \in D$.

The $\mathbf{Set_f}$-module structure on $\mathcal{E}nd_V$ is defined for morphism $\sigma : C \to D$ as

$$\mathcal{E}nd_V(\sigma)(f) \circ \imath_\psi := f \circ \imath_{\psi\circ\sigma} = f \circ \imath_\psi \circ (\psi \circ \sigma^{-1} \circ \psi^{-1}) \,, \tag{1.13}$$

for any $\psi : D \to [|D|]$. The last equality follows from equation 1.10, the term $(\psi \circ \sigma^{-1} \circ \psi^{-1})$ is a permutation and represents the $\mathbb{S}_{|C|}$ action on tensor product. $\triangle$

[12] The correctness of this definition should also be proven. The proof is similar to the comments we made after definition 1.14; the different isomorphisms ψ, ϕ result in a permutation of vectors $\imath_\phi^{-1}\imath_\psi$, acting separately on numbers $1 \dots |C|$ and $|C| + 1 \dots |C| + |D|$

1.10 Cyclic and modular operads

With no indices colliding, operads can be straightforwardly generalized to cyclic operads and to modular operads. Cyclic operads are, intuitively, operads with their root leg made equal to other legs – graphically, we would represent them as trees without special root leg. The composition operation now has two indices, denoted ${}_a\circ_b$, specifying two legs that are being connected together.

The modular operad adds a possibility to connect legs a and b of one operad element together, with operation called ξ_{ab}. After connecting the legs, we are left with a loop. Information about loops is carried in the *genus* G of the operad element. Operad elements are thus numbered by pairs (C, G), with C the finite index set and G non-negative number. To prevent degenerate cases, *stability condition* is enforced on these pairs ([1], III.A):

$$2(G-1)+|C|>0\,. \tag{1.14}$$

This forbids the cases $(\{a\},0)$, $(\{a,b\},0)$, $(\emptyset,0)$ and $(\emptyset,1)$. Note that empty sets are allowed, representing loops.

Because the elements of operad are now indexed by pairs (C, G), the $\mathbf{Set_f}$-module has to be generalized to account for the genus.

Definition 1.17. Category **Cor** is a category with object pairs (C, G), C finite set and G non-negative integer, satisfying stability condition 1.14. Morphisms in this category are defined only between objects of the same genus, and are only bijections of sets. This category is called *category of stable corollas.* △

Definition 1.18. *Stable modular* $\mathbf{Set_f}$*-module* $\mathcal{P}$ *in a category* $\mathbf{A}$ is a functor from **Cor** to $\mathbf{A}$. △

Again, modular $\mathbf{Set_f}$-module is equivalently described as collection of objects $\mathcal{P}(C, G)$ and morphisms $\mathcal{P}(\sigma) : \mathcal{P}(C, G) \to \mathcal{P}(D, G)$ for each morphism $\sigma \in \mathrm{Hom}_{\mathbf{Cor}}((C, G),(D, G))$, satisfying the functor properties:

$\mathcal{P}(1_{(C,G)}) = 1_{\mathcal{P}(C,G)}$ for all (C, G) objects of **Cor**, and

$\mathcal{P}(\sigma \circ \tau) = \mathcal{P}(\sigma) \circ \mathcal{P}(\tau),\ \forall\ (C, G) \xrightarrow{\tau} (D, G) \xrightarrow{\sigma} (E, G)$.

Now, there are some properties that are natural to ask from the ${}_a\circ_b$ and ξ_{ab}, e.g. two non-conflicting ξ_{ab} and ξ_{cd} operations should commute. These conditions are captured in the following definition. Again, we are only interested in modular operads in **dgVec**.

Definition 1.19. *Modular operad* is a stable modular $\mathbf{Set_f}$-module $\mathcal{P}$ in **dgVec**, together with degree 0 morphisms

$$\begin{aligned} {}_a\circ_b &: \mathcal{P}(C \sqcup \{a\}, G_1) \otimes \mathcal{P}(D \sqcup \{b\}, G_2) \to P(C \sqcup D, G_1 + G_2)\,, & (1.15)\\ \xi_{ab} &: \mathcal{P}(C \sqcup \{a, b\}, G) \to P(C, G + 1)\,. & (1.16) \end{aligned}$$

satisfying following axioms:

1. **symmetry of ${}_a\circ_b$:** ${}_a\circ_b(x \otimes y) = (-1)^{|x||y|}{}_b\circ_a(y \otimes x)$, or ${}_a\circ_b = {}_b\circ_a s$

2. **equivariance of ${}_a\circ_b$:** $({}_{\tau(a)}\circ_{\sigma(b)})\mathcal{P}(\tau) \otimes \mathcal{P}(\sigma) = \mathcal{P}(\tau\big|_C \sqcup \sigma\big|_D)({}_a\circ_b)$, for arguments of ${}_a\circ_b$ as in definition 1.15 and for

$$\begin{aligned} &(C \sqcup \{a\}, G_1) \xrightarrow{\tau} (C' \sqcup \{\tau(a)\}, G_1)\,,\\ &(D \sqcup \{b\}, G_2) \xrightarrow{\sigma} (D' \sqcup \{\sigma(b)\}, G_2)\,. \end{aligned}$$

3. equivariance of ξ_{ab}: $\xi_{\tau(a)\tau(b)}\, \mathcal{P}(\tau) = \mathcal{P}(\tau\big|_C)\, \xi_{ab}$ for ξ_{ab} as in definition 1.16 and for

$$(C \sqcup \{a, b\}) \xrightarrow{\tau} (C' \sqcup \{\tau(a), \tau(b)\})\,.$$

4. two non-colliding ξ_{ab} operations: $\xi_{ab}\, \xi_{cd} = \xi_{cd}\, \xi_{ab}$, acting on vector with (a, b, c, d) in its index set.

5. ξ_{ab} after ${}_a\circ_b$: $\xi_{ab}\, {}_c\circ_d = \xi_{cd}\, {}_a\circ_b$, for the two arguments of $\circ$ having (a, c) and (b, d) in their index sets, respectively.

6. non-colliding ξ_{ab} and $\circ ab$: $\xi_{ab}\, {}_c\circ_d = {}_c\circ_d\, (\xi_{ab} \otimes 1)$ for the two arguments having (a, b, c) and (d) in their index sets, respectively.

7. associativity for ${}_a\circ_b$: ${}_a\circ_b(1 \otimes {}_c\circ_d) = {}_c\circ_d({}_a\circ_b \otimes 1)$, for (a), (b, c) and (d) in the three index sets of elements this equation acts on – i.e. this equation is an equation for morphisms[13]

$$\mathcal{P}(C \sqcup \{a\}, G_1) \otimes \mathcal{P}(D \sqcup \{b, c\}, G_2) \otimes \mathcal{P}(E \sqcup \{d\}, G_3) \to \mathcal{P}(C \sqcup D \sqcup E, G_1 + G_2 + G_3)\,.$$

$\triangle$

1.10.1 Cyclic operads

The cyclic operad is now just a special case of modular operad, without G-grading and ξ_{ab} operations. The underlying module now looks like the $\mathbf{Set_f}$-module for ordinary operads.

However, because cyclic (and modular) operads are a generalization of operads, they can be (and usually are) defined as operads with extended $\mathbb{S}_n$ action and $\circ_i$ operations. Going back to the $\mathbb{S}$-modules for a moment, cyclic $\mathbb{S}$-module $\mathcal{C}(n)$ is a collection of objects $\mathcal{C}(n)$ (sets, dg vector spaces) with action of $\mathbb{S}_n^+$, the group of permutations of $\{0, 1, \ldots, n\}$. Cyclic $\mathbb{S}$-module is associated with a (ordinary) $\mathbb{S}$-module by considering only the permutations $\tau \in \mathbb{S}_n^+$ such that $\tau(0) = 0$. Cyclic operad is then defined as a cyclic $\mathbb{S}$-module, such that the associated $\mathbb{S}$-module is an operad, satisfying generalized equivariance with respect to the bigger $\mathbb{S}_n^+$ action. For full treatment, we refer to [3], section 6 (especially proposition 42) and also [5], II.5.1.

Going the other way, we can define cyclic operad by forgetting structure of modular operad

Definition 1.20. Cyclic operad $\mathcal{C}$ is a modular $\mathbf{Set_f}$-module with operation ${}_a\circ_b$ as in definition of modular operad, satisfying axioms 1, 2 and 7. $\triangle$

For cyclic operads, one doesn't track the genus of elements, so the genus 0 part of modular operad gives rise to cyclic operad

$$\mathcal{C}(C) := \mathcal{C}(C, g = 0)\,.$$

Endomorphism cyclic operad has to be constructed with product $B : V \otimes V \to V$, satisfying certain properties. With this product, maps $V^{\otimes(n-1)} \to V$ and $V^{\otimes n} \to \Bbbk$ can be identified – linear maps taking n elements into a number can be then represented as unrooted trees (or corollas). ${}_a\circ_b$ operations are then defined using contraction with B, see [5], example 5.12.

[13] Note that there is only one associativity axiom for ${}_a\circ_b$, in contrast to two axioms $\circ_a$ for operad. For modular operads, the second variant follows from this variant and the symmetry of ${}_a\circ_b$.

1.11 Concluding remarks

Operads, as originally defined by May, used different kind of composition, usually denoted

$$\gamma : P_k \times P_{i_1} \times \cdots \times P_{i_k} \to P(i_1 + \cdots + i_k)\,,$$

plugging k operad elements into $p_k \in \mathcal{P}(k)$. In the presence of unit in the operad, one can recover the $\circ_i$ (initially defined by Markl in [8], therefore Markl's operads) by setting

$$p_k \circ_i q_l = \gamma(1_{\mathcal{P}}, \ldots, q_l, \ldots, 1_{\mathcal{P}})\,,$$

where q_l is inserted in ith argument. Going the other way around is always possible by repeated application of $\circ_i$. Formally, Markl states this in categorical language in [3], Proposition 13.

1.11.1 Non-unital operads

In the definition 1.3, one can omit the unit existence and unit axiom to get non-unital operads. This has a consequence relating previous discussion about γ and $\circ_i$ – without unit, one can only construct γ from $\circ_i$, not the other way around. Again, Markl states this in categorical language in [3], Proposition 18.

For cyclic operads, the unital and non-unital variants can be defined, but the unit element of modular operad would violate the stability condition.

1.11.2 Arity zero

Other variation on the definition we gave is allowing operad elements of arity 0, i.e. also have $\mathcal{P}(0)$ defined. This space is partly detached, since it can only be the right argument of $\circ_i$ – nothing can be plugged into functions with 0 inputs. The element $m_0 \in \mathcal{P}(0)$ behaves like the argument for the function, since map $q \mapsto q \circ_i m_0$ just lowers the arity of q by one. Markl defines such operads in [3], but e.g. [5] starts with $\mathbb{S}$-modules only with $n \geq 1$.

2. BV-BRST formalism

The path integral formulation of quantum mechanics was developed by Dirac and then extended and interpreted by Feynman. Roughly, in quantum (field) theory, every path (or history) of the system is assigned an amplitude

$$e^{\frac{i}{\hbar}S[\text{path}]},$$

where S is an classical action. The transition amplitude from state A to state B is an integral over *all* possible paths starting in A and ending in B. For measuring an observable $\mathcal{O}$ at a spacetime point $\mathbf{x}$ during this process, we have to integrate an expression

$$\mathcal{O}[\mathbf{x}]e^{\frac{i}{\hbar}S[\text{path}]}.$$

In the case of QFT, path is a time evolution of the field. Path integral treatment can be found in most textbooks on QFT, we refer to e.g. [13].

The path integral is famously ill-defined, since the space of paths is infinite-dimensional. Usually, it's "calculated" using analogies from finite dimensional case.

One such analogy is a *cohomological integration* (see [14]), called *Batalin-Vilkovisky formalism* when applied to quantum field theories. Informally, integration on a manifold can be restated as a cohomological problem in its de Rham cohomology.

Many quantum field theories are *gauge invariant* – different field configurations correspond to the same physical situation. Gauge transformations are just transformations between these equivalent fields. In the path integral approach, this poses a problem, because integrating over this classes results in infinities. Sometimes, this can be dealt with by fixing a gauge, but the resulting quantum theory isn't manifestly gauge invariant.

The BRST formalism cures the problems in some cases, e.g. if the Lie algebra of gauge symmetry is well behaved (it closes off-shell and its structure constants don't depend on fields). In more complicated cases (e.g. string field theory), BV-BRST formalism is needed.

The author does not claim a full understanding of these methods for quantizing gauge theories. This chapter is meant to physically motivate quantum master equation, which will turn out to be important in the context of modular operads.

Physical development of BRST and BV formalism can be found in Weinberg's [15]. The BV formalism is clearly explained in Gwilliam's [16], which we will follow. BV and BRST formalism is also discussed in nice exposition [17] by Fiorenza. Chapters 8 and following of [18] also discuss BRST from a geometrical standpoint, together with introduction to homological algebra. Chapters 18 and 19 of ibid. discuss the BV formalism.

2.1 BV algebras

A nice algebraical motivation for BV algebras is presented in notes [19], which we shortly rephrase. In appendix A, we have described a structure of a chain complex of vector spaces. If such chain complex has a multiplication, we talk about differential graded algebra, if the differential satisfies a (graded) Leibniz rule $(fg)' = f'g \pm fg'$.

What additional structure can be added? [19] argues that adding another differential isn't of a big interest, since the two differential wouldn't interact in an interesting way. Therefore, we look at differentials that don't satisfy the Leibniz rule – second derivative in the algebra of $C^\infty(\mathbb{R})$ is an example of such operator. Now, with two real functions

f and g, second derivative of their product can't be written using second derivatives only:

$$(fg)'' = f''g + 2f'g' + fg''\,.$$

However, for product of three functions:

$$(fgh)'' = (f'gh + fg'h + fgh')' = f''gh + fg''h + fgh'' + 2f'g'h + 2f'gh' + 2fg'h'$$

and using $2f'g' = (fg)'' - f''g - fg''$, we get so-called *seven-term identity*

$$\begin{aligned}(fgh)'' =& f''gh + fg''h + fgh'' \\ &+ [(fg)'' - f''g - fg'']h \\ &+ [(fh)'' - f''h - fh'']g \\ &+ [(gh)'' - g''h - gh'']f \\ =&(fg)''h + (fh)''g + (gh)''f - f''gh - fg''h - fgh''.\end{aligned} \tag{2.1}$$

Note that first derivative also satisfies the seven-term identity.

We will be working with dg vector spaces and our "derivative" will have degree $+1$, so signs will show up. Calling the second derivative operator Δ, we can state the definition of BV algebra

Definition 2.1. A *BV algebra* is a dg vector space (V, d) and following operators:

1. **the multiplication μ:** an associative graded-commutative degree 0 multiplication $\mu : V \otimes V \to V$ (also denoted as $\mu(f, g) \equiv fg$) compatible with the differential d. The associativity means

$$(fg)h = f(gh) \qquad \text{or} \qquad \mu(\mu \otimes 1) = \mu(1 \otimes \mu)$$

graded commutativity means, using just s for the symmetry operator s_{VV}

$$fg = (-1)^{|f||g|}gh \qquad \text{or} \qquad \mu = \mu s_{VV}$$

and the compatibility with the differential is the graded Leibniz rule

$$d(fg) = d(f)g + (-1)^{|f|}fd(g) \qquad \text{or} \qquad d\mu = \mu(d \otimes 1) + \mu(1 \otimes d)\,.$$

The structure so far is called a *commutative differential graded algebra*, or *cdga* for short.

2. **the differential Δ:** degree 1 square zero ($\Delta\Delta = 0$) operator $\Delta : V \to V$, compatible with the differential:

$$\Delta d + d\Delta = 0\,,$$

satisfying the graded version of 7-term identity

$$\begin{aligned}\Delta(fgh) =&\Delta(fg)h + (-1)^{|g||h|}\Delta(fh)g + (-1)^{|f|(|g|+|h|)}\Delta(gh)f \\ &- \Delta(f)gh - (-1)^{|f|}f\Delta(g)h - (-1)^{|f|+|g|}fg\Delta(h)\end{aligned} \tag{2.2}$$

or

$$\begin{aligned}\Delta(\mu(\mu \otimes 1)) =& [\mu((\Delta\mu) \otimes 1)]\,[1 + 1 \otimes s + (1 \otimes s)(s \otimes 1)] \\ &- \mu(\mu \otimes 1)\,[\Delta \otimes 1 \otimes 1 + 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta]\end{aligned}$$

$\triangle$

Another way to look at this structure is to look at a failure of Δ in being a derivation of μ. In the ungraded example with ordinary differentiation, this means

$$\text{the failure of }'' \text{ being a derivative is} \quad (fg)'' - f''g - fg'' = 2f'g'.$$

In the graded case, this failure is usually defined with sign $(-1)^{|f|}$ and is denoted by

$$\{f, g\} = (-1)^{|f|} \left[\Delta(fg) - \Delta(f)g - (-1)^{|f|} f\Delta(g)\right] . \tag{2.3}$$

The degree of the bracket is the same as the degree of Δ, $+1$. To make the Koszul convention work, we assign this degree to the comma in the middle of the bracket, so that f passing it picks up exactly the $(-1)^{|f|}$.

Now, we can look how these operators interact together:

1. **the symmetry of bracket:** Applying the graded commutativity to the definition of bracket, we get

$$\{f, g\} = (-1)^{|f||g|+|f|+|g|}\{g, f\} = (-1)^{1+(|f|+1)(|g|+1)}\{g, f\} . \tag{2.4}$$

2. **the bracket and the multiplication:** In the definition of $\{fg, h\}$, we employ the seven term identity and commute the terms to get

$$\{fg, h\} = f\{g, h\} + (-1)^{|g||h|+|g|}\{f, h\}g , \tag{2.5}$$

meaning that the bracket is a derivative of the product. This is called graded Poisson identity.

3. **d and the bracket:** Using $d\Delta = -\Delta d$ and Leibniz rule on the definition of the bracket, we get

$$d\{f, g\} = \{df, g\} + (-1)^{|f|-1}\{f, dg\} . \tag{2.6}$$

4. **Δ and the bracket:** Expanding $\Delta\{f, g\}$ and using $\Delta^2 = 0$, we get

$$\Delta\{f, g\} = \{\Delta f, g\} + (-1)^{|f|+1}\{f, \Delta g\} . \tag{2.7}$$

5. **graded Jacobi identity:** We expand $\Delta\{fg, h\}$ in two ways, using 2.5 or 2.7. Afterwards, we use 2.3 to expand Δ acting on product. With the help of 2.5 and 2.7, all the terms containing Δ cancel out and we are left with

$$\{\{f, g\}, h\} = \{f, \{g, h\}\} - (-1)^{(|f|+1)(|g|+1)}\{g, \{f, h\}\} \tag{2.8}$$

The fact that the bracket is a derivation of the product makes it into a variation on Poisson algebra, with modified degree of the bracket. This kind of algebra is usually called Gerstenhaber algebra

Definition 2.2. *Gerstenhaber algebra* is an commutative differential graded algebra V with a degree 1 bracket

$$\{\, ,\, \} : V \otimes V \to V \tag{2.9}$$

satisfying graded skew-symmetry (equation 2.4) the graded Poisson identity for the bracket (equation 2.5) and the graded Jacobi identity (equation 2.8). $\triangle$

This leads us naturally to equivalent definition of BV algebra. We have constructed the bracket solely from multiplication and Δ, but we can exchange the (somewhat unintuitive) seven-term identity for the bracket and some of its properties.

Definition 2.3. *BV algebra* is Gerstenhaber algebra V with square-zero degree 1 map $\Delta : V \otimes V$, such that

1. The Δ is compatible with d, i.e. $d\Delta + \Delta d = 0$.

2. Δ and d are derivatives of the bracket, i.e. equations 2.7 and 2.6 hold.

3. The bracket measures the failure of Δ being a derivation, as in equation 2.3

△

The seven-term identity now follows from the Poisson and Jacobi identities.

Example 2.4. We will now present a simple, abstract example of a BV algebra. Consider a chain complex

$$V = \mathbb{C}[x_1, \ldots, x_N, \xi_1, \ldots, \xi_N, \hbar]\,, \tag{2.10}$$

the linear combinations of monomials in variables x_i, ξ_i and $\hbar$ with graded-commutative product. The degrees of elements are defined as $|x_i| = |\hbar| = 0$ and $|\xi_i| = -1$. The derivative $\frac{\partial}{\partial \xi_i}$ is an object of degree 1, removing ξ_i after commuting with other variables left to it. Because of this definition, the partial derivatives also graded-commute with each another.

The differential of this chain complex can be taken to be 0 for now.

The operator Δ we will use is defined as

$$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial \xi_i}\,. \tag{2.11}$$

This removes exactly one degree -1 element from each monomial, thus is a degree 1 operator. The square zero property (nilpotency) follows from the usual "symmetric × antisymmetric" argument. The seven term identity is just a graded version of the calculation in the beginning of this chapter. ◇

2.1.1 Note on signs and degrees

We have chosen Δ to be degree 1. The other common choice is -1, motivated by the form of Δ is some cases – it contains one partial derivative erasing a graded element. Taking this element to be degree 1, the Δ decreases the degree exactly by one. Going between these two conventions amounts to reversing the degree for all vectors. There is a standard notation for this operation in graded vector spaces, denoted as $V_{-i} = V^i$. The BV algebra used in our reference for the next chapter, [1], has Δ and $\{\ ,\ \}$ of degree 1, but the axioms have different signs. This is because loc.cit. defines bracket, following [20], *without* the sign[1] $(-1)^{|f|}$ in the equation 2.3. If we define

$$\{f, g\}' := (-1)^{|f|}\{f, g\}\,, \tag{2.12}$$

the symmetry and Jacobi identity of the bracket take simpler forms

$$\{f, g\}' = (-1)^{|f||g|}\{g, f\}' \tag{2.13}$$

$$(-1)^{|g|+1}\{\{f, g\}', h\}' = (-1)^{|f|+|g|}\{f, \{g, h\}'\}' + (-1)^{|f||g|}\{g, \{f, h\}'\}'\,. \tag{2.14}$$

There are many forms of Jacobi identity, we will need an equivalent variant

$$\{\{f, g\}', h\}' + (-1)^{|h|(|f|+|g|)}\{\{h, f\}', g\}' + (-1)^{|f|(|g|+|h|)}\{\{g, h\}', f\}' = 0\,. \tag{2.15}$$

[1] We had chosen the convention with the sign because it seems standard in literature ([17], [21]). It is probably introduced to make the bracket a Gerstenhaber bracket, with convention defined already by Gerstenhaber [22]. See also remarks in section 1.1 of [20].

2.2 Integration in BV-BRST formalism

At first, we will focus on the BV part[2], postponing the BRST part, introduction of ghosts to deals with gauge invariance.

2.2.1 Cohomological integration

The connection of cohomology and integration comes from de Rham complex and Stokes theorem (see e.g. [23]). Let's look at a top form $\eta \in \Omega^n(M)$ for M compact n-dimensional manifold without boundary. Stokes theorem says that integral over the whole manifold M of an exact top form η is 0, since for $\eta = \mathrm{d}\omega$

$$\int_M \eta = \int_M \mathrm{d}\omega = \int_{\partial M} \omega = \int_\emptyset \omega = 0\,. \tag{2.16}$$

The same holds for compactly supported forms or forms decaying into 0 fast enough, with e.g coefficients being Schwartz functions (the M does not have to be compact in this case). For compactly supported functions, we just choose the integration domain larger than the support. For "Schwartz forms", the volume of the boundary grows polynomially with size, which is suppressed by the Schwartz function.

Because of this, the integral $\int_M : \Omega^n(M) \to \mathbb{R}$ is still well defined on the cohomology classes in $H^n_{\mathrm{dR}}(M)$, the cohomology in the highest degree. Indeed, elements in one cohomology class differ by exact forms, whose integral is 0.

For $\mathbb{R}^n$, the compact cohomology in the highest degree is $\mathbb{R}$ (we refer to [24]), so that the integration is actually isomorphism. If we want to calculate normalised integral of a function f with weight μ for functions and integral such that the $\int_M$ is isomorphism, we just find the cohomology classes of $[f\mu]$ and $[\mu]$ and then divide them

$$\frac{\int_M f\mu}{\int_M \mu} = \frac{[f\mu]}{[\mu]}\,. \tag{2.17}$$

Here, the μ is a volume form, corresponding to an integration weight in a usual Lebesgue integral after the choice of coordinates.

Example 2.5. Let's apply this formalism to one-dimensional manifold $\mathbb{R}$ and Schwartz-valued forms. Recalling the definition of Schwartz space

$$\mathcal{S} = \{f \in C^\infty_b(\mathbb{R}) |\, x^n f^{(m)} \in C^\infty_b(\mathbb{R}), \forall n, m\}\,, \tag{2.18}$$

we want to start by describing a cohomology in degree 1. As in the regular de Rham cohomology for $\mathbb{R}$, we can find primitive function $F(x) = \int_{-\infty}^x f(t)\mathrm{d}t$ for any top form $\omega = f(x)\mathrm{d}x$. However, F does not have to be Schwartz – its value tends to $\int_\mathbb{R} \omega$ in $+\infty$.

If we denote $A := \int_\mathbb{R} \omega = \int_\mathbb{R} f(x)\mathrm{d}x$, we can modify ω to be exact by substracting A, for example

$$f(x) \to g(x) = f(x) - \sqrt{\frac{a}{2\pi}} A e^{-\frac{a}{2}x^2}\,.$$

Primitive function $G(x) := \int_{-\infty}^x g(t)\mathrm{d}t$ is now Schwartz [3] and its derivative is $g(x)$,

[2] This distinction is somewhat non-standard, since BV and BRST formalisms are usually introduced together. Following remark 2.1.4 in [16], by BV part we mean just the cohomological integration; the BRST part is needed to deal with gauge theories.

[3] Boundedness of derivatives of $G(x)$ is trivial. $x^n G(x)$ is bounded if the limit for $x \to \pm\infty$ is finite. This follows from l'Hôpital's rule

$$\lim_{x\to\pm\infty} x^n G(x) = \lim_{x\to\pm\infty} \frac{G(x)}{x^{-n}} = \lim_{x\to\pm\infty} \frac{g(x)}{-nx^{-n-1}} = 0\,,$$

since g is Schwartz and we constructed G to vanish in $\pm\infty$, thanks to subtraction of the integrals of g and the Gaussian.

which is therefore exact. Rewriting this, we have

$$[f(x)\mathrm{d}x] = A\left[\sqrt{\frac{a}{2\pi}}e^{-\frac{a}{2}x^2}\mathrm{d}x\right] = \int_{\mathbb{R}} f(x)\mathrm{d}x \left[\sqrt{\frac{a}{2\pi}}e^{-\frac{a}{2}x^2}\mathrm{d}x\right], \tag{2.19}$$

which is exactly statement that the integration is isomorphism between Schwartz-valued top forms and $\mathbb{R}$ – functions with same integral lie in the same cohomology class. Any function with unit integral can be used instead of $\sqrt{\frac{a}{2\pi}}e^{-\frac{a}{2}x^2}$.

We can now use this formalism to find expectation value of x^n with Gaussian weight

$$\langle x^n\rangle := \frac{\int_{\mathbb{R}} x^n e^{-\frac{a}{2}x^2}\mathrm{d}x}{\int_{\mathbb{R}} e^{-\frac{a}{2}x^2}}. \tag{2.20}$$

We know that form

$$\mathrm{d}\left(x^n e^{-\frac{a}{2}x^2}\right)$$

is exact, so projecting into cohomology, we have

$$[0] = \left[nx^{n-1}e^{-\frac{a}{2}x^2}\mathrm{d}x - ax^{n+1}e^{-\frac{a}{2}x^2}\mathrm{d}x\right],$$

or

$$n\left[x^{n-1}e^{-\frac{a}{2}x^2}\mathrm{d}x\right] = a\left[x^{n+1}e^{-\frac{a}{2}x^2}\mathrm{d}x\right].$$

Thanks to equation 2.19, we can rewrite this as

$$n\left(\int_{\mathbb{R}} x^{n-1}e^{-\frac{a}{2}x^2}\mathrm{d}x\right)\left[\sqrt{\frac{a}{2\pi}}e^{-\frac{a}{2}x^2}\mathrm{d}x\right] = a\left(\int_{\mathbb{R}} x^{n+1}e^{-\frac{a}{2}x^2}\mathrm{d}x\right)\left[\sqrt{\frac{a}{2\pi}}e^{-\frac{a}{2}x^2}\mathrm{d}x\right],$$

or, since this is equality between non-zero vectors

$$\langle x^{n+1}\rangle = \frac{n}{a}\langle x^{n-1}\rangle.$$

◇

Solving the series is simple, using $\langle x^0\rangle = 1$ and $\langle x^1\rangle = 0$, we have

$$\langle x^{2n}\rangle = \frac{(2n-1)!!}{a^n},$$

where $(2n-1)!! = (2n-1)(2n-3)\dots(1)$.

Note that this calculation is equivalent to integration of $\int_{\mathbb{R}} x^n e^{-\frac{a}{2}x^2}\mathrm{d}x$ by parts.

2.2.2 Multivector fields

Let's again choose a manifold M. BV formalism consists of one more step, passing from de Rham complex to antisymmetric multivector fields. This connection of BV formalism to de Rham complex was pointed out by Witten [25].

Antisymmetric multivector fields $V^k = \Gamma(\Lambda^{-k}T_M X)$ are, like de Rham complex, graded vector space. We will use notation ξ_i for basis fields $\frac{\partial}{\partial x_i}$. We will take ξ_i to have degree -1, since they are duals to degree 1 forms.

We need to choose a nowhere vanishing top form μ on M. With this top form, we can contract multivector fields, making them into forms. Contraction i_μ of a multivector field of degree k with a top form μ will result in a form of degree $(n-k)$. Because we choose μ to be nowhere vanishing, this contraction is an isomorphism $V^k \to \Omega^{n-k}$ [4] and we can transfer exterior derivative d to a linear operator on the graded space V_k.

Let's transfer a basis vector $f = f(\mathbf{x})\xi_1 \wedge \cdots \wedge \xi_k \in V^k$. Contracting such vector with $\mu(\mathbf{x})\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ amounts for finding a new form ω such that

$$\omega(\xi_{k+1}, \dots, \xi_n) = f(\mathbf{x})\mu(\xi_1, \dots, \xi_n) = (-1)^{\frac{n(n-1)}{2}} f(\mathbf{x})\mu(\mathbf{x}) ,$$

where we put the vector f into the first k arguments of μ. The sign factor comes from commuting k vectors through $k-1, k-2, \dots, 1, 0$ 1-forms when evaluating [5] $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n(\xi_1, \dots, \xi_n)$. Such form ω is easy to find, it's

$$\omega = (-1)^{\frac{n(n-1)}{2} - \frac{(n-k)(n-k-1)}{2}} \mu(\mathbf{x}) f(\mathbf{x}) \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n .$$

Using d on ω, we get

$$\mathrm{d}\omega = (-1)^{\frac{n(n-1)}{2} - \frac{(n-k)(n-k-1)}{2}} \sum_{i=1}^{k} \frac{\partial(\mu(\mathbf{x})f(\mathbf{x}))}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n .$$

Let's denote a vector corresponding to this form $g := i_\mu^{-1}(\mathrm{d}\omega)$. g clearly has a form

$$g = \sum_{i=0}^{k} (-1)^{\epsilon_i} g_i(x) \xi_1 \wedge \cdots \wedge \hat{\xi_i} \wedge \cdots \wedge \xi_k ,$$

where the hat ˆ denotes missing element, $g_i(x)$ and ϵ_i are unknown. Computing $i_\mu(g)$, we get

$$\sum_{i=0}^{k} (-1)^{\epsilon_i + \frac{n(n-1)}{2} - \frac{(n-k+1)(n-k)}{2} + k - i} g_i(x)\mu(x) \mathrm{d}x_i \wedge \mathrm{d}x_{k+1} \wedge \cdots \wedge \mathrm{d}x_n ,$$

where the sign $k - i$ comes from commuting ξ_i from position left to ξ_{k+1} to its original place. Comparing this to $\mathrm{d}\omega$, we have

$$g_i(x) = \frac{1}{\mu(x)} \frac{\partial(\mu(\mathbf{x})f(\mathbf{x}))}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} + \frac{\partial \ln \mu(\mathbf{x})}{\partial x_i} f(x)$$

and

$$(-1)^{\epsilon_i} = (-1)^{n-i} .$$

This can be written as

$$\Delta f = i_\mu^{-1} d i_\mu f = (-1)^{n-1} \sum_{i=1}^{k} \left(\frac{\partial \ln \mu}{\partial x_i} + \frac{\partial}{\partial x_i}\right) \frac{\partial f}{\partial \xi_i} \tag{2.21}$$

and we can immediately extend the sum to n. The derivative $\frac{\partial}{\partial \xi_i}$ has degree 1, commuting it to ξ_i gives precisely the sign factor $(-1)^{i-1}$. The proof of correctness of this

[4] Since it's a map between spaces of dimension $\binom{n}{k}$, we only need the kernel to be zero. The antisymmetry of indices of multivector $\frac{1}{k!} a_{i_1 \dots i_k} \xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$ ensures that the indices of the contracted form won't cancel out. These coefficients are all multiplied by $\mu(x_1, \dots, x_n) \equiv \mu(\mathbf{x})$, the coefficient in front of μ in some coordinates. With nonzero μ, the resulting form is zero only when all coefficients of the multivector are zero everywhere.

[5] The evaluation is defined as

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n(\xi_1, \dots, \xi_n) = \sum_\sigma \sigma(\mathrm{d}x_1 \otimes \cdots \otimes \mathrm{d}x_n)(\xi_1 \otimes \cdots \otimes \xi_n) ,$$

hence the Koszul sign. σ acts on tensor product as defined in appendix A, see also chapter 4 for more examples.

formula for general vector $f(\mathbf{x})\xi_{i_1}\wedge\cdots\wedge\xi_{i_k}$ is a good exercise, following from cancellation of the permutation factors.

Furthermore, we also drop the unimportant factor[6] $(-1)^{n-1}$, so that Δ has form

$$\Delta = \sum_{i=1}^{n} \frac{\partial \ln \mu}{\partial x_i}\frac{\partial}{\partial \xi_i} + \frac{\partial^2}{\partial x_i \partial \xi_i}\,. \tag{2.22}$$

With constant μ, we get the algebra from example 2.4 The corresponding bracket, defined by equation 2.3, is

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} + (-1)^{|f|}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i}\,. \tag{2.23}$$

Few remarks, putting this into context, are due. This bracket, when defined on vector fields, is called Schouten-Nijenhuis bracket. The operator Δ associated with μ is called divergence (Marle [26] reviews multivector fields, [27], section 7 transfers the exterior derivative). Getzler's [28] reviews BV algebras and the divergence operator on supermanifolds, leading to master equation.

Notice that the term in Δ containing only first derivative didn't contribute to the bracket (since it obeys Leibniz rule). In the following we will choose, for simplicity coordinates such that $\mu(x)$ is locally constant [7].

Under the isomorphism i_μ, the de Rham cohomology in the top degree is transferred to the homology in degree 0. This is important for quantum field theory, since in infinite dimensional manifold there is no top form, but we can still look at the degree 0 homology of fields.

2.2.3 Ghosts and master equation

We will not venture into the BRST formalism, as this section serves only as a motivational calculation to introduce the master equation. What suffices for us is the fact that dealing with gauge symmetry requires us to introduce anticommuting variables to the path integral. This can be already seen in the Faddeev Popov ghosts, where they appear because of determinant factors associated with volume of gauge invariant subspace. This determinant can be absorbed into the exponential in path integral by Berezinian integration of anticommuting variables. These anticommuting variables are usually called *ghosts*. The extended space is called superspace, referring to $\mathbb{Z}_2$ grading. To further simplify our discussion, we will be in a flat superspace, following [17]. See [29] for general analysis on manifolds.

On a superspace M, the bundle TM looks like $M \bigoplus \Pi M^\#$, where Π reverses the parity (sends degree i to $(i+1 \mod 2)$) and $^\#$ denotes the dual. This is because there is natural pairing of coordinates on the space, thought of as functions, and vector fields, and that is directional derivative

$$(x_i, \xi_j) \equiv \frac{\partial x_i}{\partial x_j} = \delta_{ij}\,. \tag{2.24}$$

The space $\Pi M^\#$ is assigned the prefix anti-, consisting of antifields and antighosts.

The complex we are now working in is the complex of symmetric powers of $M \bigoplus \Pi M^*$,

[6] Literature ([16]) has Δ without the sign, its origin is probably in different conventions for evaluation of forms on multivectors.

[7] Schwarz [29], theorem 5, proves that this is possible for closed forms, under some assumptions.

Since coordinates can be odd and vector fields ξ even, we need to be more careful with the signs. The divergence operator is still of the form

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i}\,, \tag{2.25}$$

where the two derivatives commute, since x_i and ξ_i have opposing degree. The bracket is now more complicated, as one can readily compute

$$\begin{aligned}
\Delta(fg) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial \xi_i} g + (-1)^{|f||\xi_i|} f \frac{\partial g}{\partial \xi_i}\right) \\
&= (\Delta f)g + \sum_{i=1}^{n} (-1)^{(|f|-|\xi_i|)|x_i|} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} \\
&\quad + (-1)^{|f||\xi_i|}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} + (-1)^{|f||\xi_i|+|f||x_i|} f(\Delta g)\,.
\end{aligned}$$

Using $|\xi_i||x_i| = 0$ and $|\xi_i| + |x_i| = 1 \mod 2$, we can read off the bracket

$$\begin{aligned}
\{f, g\} &= \sum_{i=1}^{n} (-1)^{|f|(|x_i|-1)} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} + (-1)^{|f|(|\xi_i|-1)}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} \\
&= \sum_{i=1}^{n} (-1)^{|f||\xi_i|} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} + (-1)^{|f||x_i|}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}\,.
\end{aligned} \tag{2.26}$$

This is usually written using so-called *left* and *right derivatives*, where the left is the ordinary derivative, commuting from the left,

$$\frac{\partial_l f}{\partial X} := \frac{\partial f}{\partial X} \tag{2.27}$$

and the right derivative has a sign, as if it acted from right

$$\frac{\partial_r f}{\partial X} := (-1)^{|f||X|-|X|}\frac{\partial f}{\partial X}\,. \tag{2.28}$$

In these, X can be either x_i or ξ_i. See also [1], page 22, [17] defines the right derivative differently. Now, we can rewrite the bracket as

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial_r f}{\partial x_i}\frac{\partial_l g}{\partial \xi_i} - \frac{\partial_r f}{\partial \xi_i}\frac{\partial_l g}{\partial x_i} \tag{2.29}$$

With degree 1 coordinates, densities are no longer top forms – the forms associated with degree 1 coordinates commute and the complex of forms is not bounded. If we want to look at the cohomology class of a form, it first needs to be closed.

In the path integral, we are interested in integrals of $\Psi e^{\frac{i}{\hbar}S}$, where Ψ and S are functions on the superspace. The form is closed under the differential d iff the corresponding multivector is closed under Δ. When we take the function $\Psi e^{\frac{i}{\hbar}S}$ as an element of V^0, we get condition

$$\Delta(\Psi e^{\frac{i}{\hbar}S}) = 0\,. \tag{2.30}$$

At first, let's look at the factor $e^{\frac{i}{\hbar}S}$ alone, applying Δ, the condition translates as

$$0 = \Delta(e^{\frac{i}{\hbar}S}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{i}{\hbar}\frac{\partial S}{\partial \xi_j} e^{\frac{i}{\hbar}S}\right) = e^{\frac{i}{\hbar}S}\sum_{j=1}^{n} \frac{i}{\hbar}\frac{\partial^2 S}{\partial \xi_j \partial x_j} - (-1)^{|x_j|(|S|-|\xi_j|)}\frac{1}{\hbar^2}\frac{\partial S}{\partial \xi_j}\frac{\partial S}{\partial x_j}\,. \tag{2.31}$$

For S of even degree[8] , we can write

$$\{S,S\} = 2\sum_{j=1}^{n} \frac{\partial S}{\partial \xi_j}\frac{\partial S}{\partial x_j}\,, \tag{2.32}$$

so our condition reduces to

$$\{S,S\} = 2i\hbar\Delta S\,. \tag{2.33}$$

This is the *quantum master equation*. When quantizing a theory, this is a condition that is imposed on the action extended to ghosts.

For general Ψ, we get

$$\Delta(\Psi e^{\frac{i}{\hbar}S}) = (\Delta\Psi)e^{\frac{i}{\hbar}S} + (-1)^{|\Psi|}\Psi(\Delta e^{\frac{i}{\hbar}S}) + (-1)^{|\Psi|}\{\Psi, e^{\frac{i}{\hbar}S}\} \tag{2.34}$$

Using the definition of the bracket and assumption that S is even, we get

$$\{\Psi, e^{\frac{i}{\hbar}S}\} = \frac{i}{\hbar}\{\Psi, S\}e^{\frac{i}{\hbar}S}\,,$$

so the condition on Ψ reads

$$0 = \Delta\Psi + (-1)^{|\Psi|}\frac{i}{\hbar}\{\Psi, S\} = \Delta\Psi + \frac{i}{\hbar}\{S,\Psi\}\,. \tag{2.35}$$

The operator

$$Q : \Psi \mapsto \Delta\Psi + \frac{i}{\hbar}\{S,\Psi\}\,. \tag{2.36}$$

is nilpotent, as we can calculate

$$Q^2\Psi = \Delta^2\Psi + \frac{i}{\hbar}\Delta\{S,\Psi\} + \frac{i}{\hbar}\{S,\Delta\Psi\} - \frac{1}{\hbar^2}\{S,\{S,\Psi\}\}\,.$$

The last term is, by Jacobi identity

$$\{S,\{S,\Psi\}\} = \{\{S,S\},\Psi\} + (-1)^{(|S|+1)(|S|+1)}\{S,\{S,\Psi\}\}\,,$$

or, since $(-1)^{(|S|+1)(|S|+1)} = -1$,

$$\{S,\{S,\Psi\}\} = \frac{1}{2}\{\{S,S\},\Psi\}\,.$$

Employing 2.7, we get

$$Q^2\Psi = \frac{i}{\hbar}\{\Delta S,\Psi\} - \frac{i}{\hbar}\{S,\Delta\Psi\} + \frac{i}{\hbar}\{S,\Delta\Psi\} - \frac{1}{2\hbar^2}\{\{S,S\},\Psi\} = \frac{1}{2\hbar^2}\{2i\hbar\Delta S - \{S,S\},\Psi\}\,,$$

which vanishes by quantum master equation.

Without going into details of integration over $M \bigoplus \Pi M^*$, we can still operate on facts that Δ-exact functions have zero integrals. This translates to Q-exact functions Ψ, meaning that different quantum observables are the cohomology classes of Q (in the suitable space of functions). Q is called BRST operator.

[8] S is even because the original action of gauge invariant theory is degree 0 (there are no ghosts or antifields) and adding other terms cannot break this property.

3. Modular operads and BV formalism

In the previous chapter, we have seen the significance of actions satisfying the quantum master equation. One surprising way to find a solution to quantum master equation is Barannikov's theory [2]. We will see that each algebra over a *Feynman transform of a modular operad* specifies an algebra and an element S of the algebra satisfying a non-commutative version of quantum master equation.

We will closely follow [1] in this chapter. The original references are [12] and [2]

3.1 Feynman transform and twisted modular operads

Feynman transform was introduced by Getzler and Kapranov in [12]. It takes a modular operad into a *twisted* modular operad, a version of modular operad with signs introduced to some of the axioms.

Definition 3.1. *Twisted modular operad* differs from modular operad from definition 1.19 in the degree of composition maps, ${}_a\circ_b$ and ξ_{ab} have degree $+1$. Furthermore, there are signs introduced to these axioms:

4. two non-colliding ξ operations: $\xi_{ab}\,\xi_{cd} = -\xi_{cd}\,\xi_{ab}$.

5. ξ after $\circ$: $\xi_{ab}\,{}_c\circ_d = -\xi_{cd}\,{}_a\circ_b$.

6. non-colliding ξ and $\circ$: $\xi_{ab}\,{}_c\circ_d = -{}_c\circ_d\,(\xi_{ab}\otimes 1)$.

7. associativity for $\circ$: ${}_a\circ_b(1\otimes {}_c\circ_d) = -{}_c\circ_d({}_a\circ_b\otimes 1)$.

△

The Feynman transform of a modular operad $\mathcal{P}$, denoted $\mathcal{FP}$, is defined as a sum over graphs – this is why [12] calls it Feynman. ${}_a\circ_b$ and ξ_{ab} are defined on graphs, connecting together their half-edges. The differential $\partial_{\mathcal{FP}}$ is a combination of $\partial^{\#}_{\mathcal{P}}$, the dual of differential on $\mathcal{P}$, and ∂_c, which is a sum over graphs with added edge. The elements of $\mathcal{FP}$ carry elements of $\mathcal{P}^{\#}$ on their vertices (also called *decoration*). See section III.B of [1] for a discussion, too.

The endomorphism twisted modular operad on space V is defined as

$$\mathcal{E}nd_V := \mathrm{Hom}(\bigotimes_C V, \Bbbk)\,. \tag{3.1}$$

To define ${}_a\circ_b$ and ξ_{ab} on these morphisms, we need more structure on the vector space – the space V is required to be even-dimensional dg symplectic space with symplectic form ω of degree -1[1] The precise definitions are in definition 6 of [1]. Informally,

${}_a\circ_b$ on $f\otimes g$ contracts the argument a of f with argument b of g using ω and

ξ_{ab} on f contracts the two arguments a and b with ω.

The $\mathbf{Set_f}$-module structure on $\mathcal{E}nd_V$ is defined as in definition 1.16, i.e. the equation 1.13.

[1] This means that $|\omega(u,v)| = |u|+|v|-1$, but symplectic form is $\Bbbk$-valued, $\Bbbk$ has degree 0, so $\omega(u,v)=0$ if $|u|+|v|\neq 1$. Dg symplectic space also implies that $\omega(d\otimes 1+1\otimes d)=0$.

3.2 Algebra over a Feynman transform

In Barannikov's work, an algebra over Feynman transform is a central concept. Similarly to the situation in chapter 1, algebra over the Feynman transform of $\mathcal{P}$ is a twisted modular operad morphism $\mathcal{FP} \to \mathcal{E}nd_V$, i.e. a set of dg vector space morphisms

$$\alpha(C,G) : \mathcal{FP}(C,G) \to \mathcal{E}nd_V(C,G)\,,$$

commuting with operadic operations.

As a warm-up, let's take look at a algebra over twisted modular operad

Definition 3.2. *Algebra over an twisted modular operad $\mathcal{T}$ on a dg symplectic space* (V, d_V) is a collection of dg vector space morphisms

$$\alpha(C,G) : \mathcal{T}(C,G) \to \mathcal{E}nd_V(C,G); \;\; (C,G) \in \mathbf{Cor}$$

satisfying

1. $\alpha(C',G) \circ \mathcal{T}(\rho) = \mathcal{E}nd_V(\rho) \circ \alpha(C,G)$ for any bijection $\rho : C \to C'$.

2. $\alpha(C \sqcup D, G_1 + G_2) \circ ({}_a\circ_b)_{\mathcal{T}} = ({}_a\circ_b)_{\mathcal{E}nd_V} \circ (\alpha(C \sqcup \{a\}, G_1) \otimes \alpha(D \sqcup \{b\}, G_2))$.

3. $\alpha(C, G+1) \circ (\xi_{ab})_{\mathcal{T}} = (\xi_{ab})_{\mathcal{E}nd_V} \circ \alpha(C \sqcup \{a,b\}, G)$.

△

The Feynman transform uses a dual of modular operad $\mathcal{P}$. The algebra over $\mathcal{FP}$ can be therefore described using conditions on morphism $\mathcal{P}^{\#} \to \mathcal{E}nd_V$. This is theorem 10 of [1], but in our context, it is a definition of some of the properties of the Feynman transform[2].

Definition 3.3. *Algebra over the Feynman transform $\mathcal{FP}$ of modular operad $\mathcal{P}$ on dg symplectic space* (V, d_V) is a collection of degree 0 linear maps

$$\alpha(C,G) : \mathcal{P}^{\#}(C,G) \to \mathcal{E}nd_V(C,G); \;\; (C,G) \in \mathbf{Cor}$$

with these compatibility conditions

1. equivariance: $\mathcal{E}nd_V(\rho) \circ \alpha(C,G) = \alpha(C',G) \circ \mathcal{P}(\rho^{-1})^{\#}$ for any bijection $\rho : C \to C'$.

2. Feynman differential:

$$\begin{aligned} d_V \circ \alpha(C,G) &= \alpha(C,G) \circ \partial_{\mathcal{P}}^{\#} + (\xi_{ab})_{\mathcal{E}nd_V} \circ \alpha(C \sqcup \{a,b\}, G-1) \circ (\xi_{ab})_{\mathcal{P}}^{\#} \\ &+ \frac{1}{2} \sum_{\substack{C_1 \sqcup C_2 = C \\ G_1 + G_2 = G}} ({}_a\circ_b)_{\mathcal{E}nd_V} \circ \big(\alpha(C_1 \sqcup \{a\}, G_1) \otimes \alpha(C_2 \sqcup \{b\}, G_2)\big) \circ ({}_a\circ_b)_{\mathcal{P}}^{\#}\,, \end{aligned} \tag{3.2}$$

where the last $({}_a\circ_b)_{\mathcal{P}}^{\#}$ is the dual to

$${}_a\circ_b : \mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \to \mathcal{P}(C,G)\,.$$

The elements a and b can be chosen arbitrarily. △

This is where the intuition about the Feynman differential comes – the second term of RHS of equation 3.2 corresponds to breaking one loop of the graph, third term is a sum over all possible splits of the graph into two.

[2] Defining only the algebra over $\mathcal{FP}$ allows us to circumvent most of the theory of Feynman transform.

3.3 Barannikov's theory

Barannikov's approach for describing solutions of master equation is somewhat indirect – given a modular operad and an algebra over its Feynman transform, he defines S, but also the algebra and operations d, $\{\,,\,\}$ and Δ such that S is a solution to the master equation in these operations. The remarkable property of these definitions is that simple choices of modular operads lead to descriptions of solutions of master equation in physically motivated situations, like string field theory [1].

We will need one technical construction:

Definition 3.4. For a vector space V with action of group G, we denote V^G the subspace of V of *invariants* under the G action, i.e.

$$V^G := \{v \in V\,;\; gv = v,\ \forall g \in G\}\,. \tag{3.3}$$

$\triangle$

The fact that V^G is a subspace of V follows immediately from linearity of the action.

An important example of space of invariants is the isomorphism $(V^{\otimes n})^{\mathbb{S}_n} \cong S^n(V)$, the n-th symmetric power of V. The action is taken to exchange the terms in the tensor product, with the Koszul sign.

For a modular operad $\mathcal{P}$ and endomorphism twisted modular operad $\mathcal{E}$, defined on a dg symplectic space V, we can now define an algebra in which the master equation can be stated. The underlying vector space P is composed of

$$P(n,G) := \big(\mathcal{P}([n],G) \otimes \mathcal{E}([n],G)\big)^{\mathbb{S}_n}\,, \tag{3.4}$$

where the action of $\sigma \in \mathbb{S}_n$ on the tensor product is *diagonal* and is defined as action of $\mathcal{P}(\sigma)$ and $\mathcal{E}(\sigma)$ on the components $P(n,G)$ i.e.

$$\sigma(p \otimes f) := (\sigma p) \otimes (\sigma f) := (\mathcal{P}(\sigma)p) \otimes (\mathcal{E}(\sigma)f)$$

for $p \in \mathcal{P}([n],G)$ and $f \in \mathcal{E}([n],G)$. The whole P is a direct sum with a formal variable $\hbar$

$$P := \bigoplus_{\substack{n,G\geq 0\\ 2(G-1)+n>0}} \hbar^G P(n,G)\,. \tag{3.5}$$

Now we proceed with defining d, Δ and $\{\,,\,\}$ as linear maps on P. The differential d just makes P into a dg vector space:

$$d := 1_{\mathcal{P}([n],G)} \otimes d_{\mathcal{E}([n],G)} - d_{\mathcal{P}([n],G)} \otimes 1_{\mathcal{E}([n],G)}\,. \tag{3.6}$$

For Δ, there is some reordering, which we encode using a function

$$\theta : [n+2] \to [n] \sqcup \{a,b\}\,,$$

defined as

$$\begin{array}{rccccc} i: & 1 & 2 & 3 & \dots & n+2 \\ \theta(i): & a & b & 1 & \dots & n\,. \end{array}$$

With θ, Δ is defined as

$$\Delta := -\big((\xi_{ab})_{\mathcal{P}} \otimes (\xi_{ab})_{\mathcal{E}}\big) \circ \big(\mathcal{P}(\theta) \otimes \mathcal{E}(\theta)\big)\,. \tag{3.7}$$

Since Δ acts on elements invariant under the permutation, we can interpret it as a operator taking any two legs of the input and joining them. Because of this, it is trivially zero on elements of $P(n, G)$ with $n < 2$.

For the bracket, the renumbering is more involved. Let C_1, C_2, G_1, G_2 be such that $C_1 \sqcup C_2 = [n]$. We define two maps, θ_1 and θ_2

$$\begin{aligned} \theta_1 &: [|C_1| + 1] \to C_1 \sqcup \{a\}, \\ \theta_2 &: [|C_2| + 1] \to C_2 \sqcup \{b\}, \end{aligned}$$

defined such that $\theta_1(1) = a$, after that is θ_1 increasing and similarly for θ_2 with $\theta_2(1) = b$. In a more explicit representation,

$$\begin{array}{rccccccccc} r: & a & b & 1 & 2 & 3 & 4 & \ldots & n \\[1em] \theta_1^{-1}(r): & 1 & - & 2 & - & - & 3 & \ldots & \\[1em] \theta_2^{-1}(r): & - & 1 & - & 2 & 3 & - & \ldots & \end{array}$$

where we have chosen (for concreteness) $1, 4 \in C_1$ and $2, 3 \in C_2$. Then, on vectors $g \in P(n_1 + 1, G_1)$ and $h \in P(n_2 + 1, G_2)$

$$\{g, h\} := - \sum_{\substack{C_1 \sqcup C_2 = [n_1 + n_2] \\ |C_1| = n_1, |C_2| = n_2}} \left(({}_a\circ_b)_{\mathcal{P}} \otimes ({}_a\circ_b)_{\mathcal{E}} \right) \circ \overset{\frown}{\tau} \circ \left(\mathcal{P}(\theta_1) \otimes \mathcal{E}(\theta_1) \otimes \mathcal{P}(\theta_2) \otimes \mathcal{E}(\theta_2) \right) (g \otimes h) . \quad (3.8)$$

The $\overset{\frown}{\tau}$ swaps the two middle terms of four-term tensor product. More precisely, at first it's an inclusion

$$\begin{gathered} \left(\mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{E}(C_1 \sqcup \{a\}, G_1) \right)^{\mathbb{S}} \otimes \left(\mathcal{P}(C_2 \sqcup \{b\}, G_2) \otimes \mathcal{E}(C_2 \sqcup \{b\}, G_1) \right)^{\mathbb{S}} \\ \hookrightarrow \\ \mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{E}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \otimes \mathcal{E}(C_2 \sqcup \{b\}, G_1) , \end{gathered}$$

where the $\mathbb{S}$ action is a generalization of symmetric group action to bijections of sets $C_1 \sqcup \{a\}$ and $C_2 \sqcup \{b\}$, respectively. This is then followed by $1 \otimes s_{\mathcal{E}(C_1 \sqcup \{a\}, G_1)\, \mathcal{P}(C_2 \sqcup \{b\}, G_2)} \otimes 1$, bringing the result into

$$\mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \otimes \mathcal{E}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{E}(C_2 \sqcup \{b\}, G_1) ,$$

onto which we can apply the two ${}_a\circ_b$ operations.

The important results we can now state are the axioms of non-commutative BV algebra and the relevance of quantum master equation. These two are written as a theorem 12 in [1].

Theorem 3.5. *The operations d, Δ and $\{\ ,\ \}$ have degree $+1$ and satisfy*

1. symmetry: *(not stated in [1])*

$$\{f, g\} = (-1)^{|f||g|} \{g, f\} .$$

2. Jacobi identity:

$$\{\{f, g\}, h\} + (-1)^{|h|(|f|+|g|)} \{\{h, f\}, g\} + (-1)^{|f|(|g|+|h|)} \{\{g, h\}, f\} = 0$$

for any $f, g, h \in P$.

3. nilpotency:

$$d^2 = \Delta^2 = 0\,.$$

4. derivation of the bracket:

$$d \circ \{\,,\,\} + \{\,,\,\} \circ (d \otimes 1 + 1 \otimes d) = 0$$

and

$$\Delta \circ \{\,,\,\} + \{\,,\,\} \circ (\Delta \otimes 1 + 1 \otimes \Delta) = 0\,.$$

5. d and Δ compatibility:

$$d\Delta + \Delta d = 0\,.$$

These formulas can be proven directly from the definition of the operations. For the property 1., symmetry, the signs come from the Koszul sign of $\overset{\frown}{\tau}$ and symmetry of ${}_a\circ_b$ from the definition of modular operad 1.19. We see that this is a version of BV algebra we developed in a section 2.1.1.

Theorem 3.6. *Elements S of algebra P satisfying the quantum master equation*

$$dS + \hbar\Delta S + \frac{1}{2}\{S, S\} = 0 \tag{3.9}$$

are equivalent to algebras over Feynman transform of modular operad $\mathcal{P}$.

The equivalence is established using the isomorphism

$$\mathrm{Hom}_{\mathbb{S}_C}\big(\mathcal{P}^{\#}(C,G), \mathcal{E}(C,G)\big) \sim \big(\mathcal{P}(C,G) \otimes \mathcal{E}(C,G)\big)^{\mathbb{S}_C}$$

or, on a map α commuting with the $\mathbf{Set_f}$-module induced action,

$$\alpha \mapsto \sum_i p_i \otimes \alpha(p_i^{\#})\,.$$

This way, we relate maps satisfying property 1 from definition 3.3 with elements of P. The master equation is then equivalent to the second property of algebras over a Feynman transform, the equation 3.2.

4. Examples

To make things less abstract, we will apply the theory presented in the previous chapter under some simplifying assumptions. We will see that the structure of BV algebra from chapter 2 is intrinsic to the endomorphism operad. Extending this example will lead us to loop homotopy algebras, governing structure of action of closed string field theory, as presented by Zwiebach [10]. Application of operads to Zwiebach's work is due to Markl, [1] reformulate Markl's result in the terms of Barannikov's theory.

4.1 Endomorphism twisted modular operad

This section relies heavily on the definition of twisted modular operad. We refer the reader to definition 6 of [1].

Let's begin by choosing a simple dg symplectic space V, a span of elements t_i and θ_i:

$$V := \mathrm{Span}_{\Bbbk}\{t_1, \dots, t_m, \theta_1, \dots, \theta_m\}\,, \tag{4.1}$$

with degrees $|t_i| = 0$ and $|\theta_i| = 1$. The symplectic form has a block structure

$$\begin{aligned} \omega(t_i, \theta_j) &= -\omega(\theta_j, t_i) = \delta_{ij} \\ \omega(t_i, t_j) &= \omega(\theta_i, \theta_j) = 0\,. \end{aligned}$$

If we order the basis elements such that $a_i = t_i$ and $a_{i+m} = \theta_i$ for $i = 1 \dots m$, the matrix form of ω is

$$(\omega_{ij}) = \begin{pmatrix} 0 & 1_{m\times m} \\ -1_{m\times m} & 0 \end{pmatrix}.$$

Its inverse, ω^{ij}, is just minus ω_{ij} and the b_i are then $b_1, \dots, b_{2m} = \theta_1, \dots, \theta_m, t_1, \dots, t_m$ (these a_i and b_i are defined in loc.cit.).

Let's take the differential to be $\mathrm{d}_V t_i := \beta\theta_i$, with $\beta \in \Bbbk$ a constant.

Next, we will look at an algebra over a Feynman transform of a modular operad $\mathcal{P}$ on this vector space, with few assumptions on $\mathcal{P}$. Let's assume that automorphisms act trivially on elements of $\mathcal{P}$, i.e.

$$\mathcal{P}(\rho) = \mathcal{P}(1_C)$$

for any $\rho : C \to C$.

The equivariance from definition 3.3, applied to automorphism ρ, reads

$$\mathcal{E}nd_V(\rho) \circ \alpha(C, G) = \alpha(C, G)$$

for operad morphism $\alpha(C, G) : \mathcal{P}^{\#}(C, G) \to \mathcal{E}nd_V(C, G)$.

Let's take a covector $C^{G\#} \in P^{\#}(C, G)$. Applying equation 4.1 on it, we get

$$\mathcal{E}nd_V(\rho)(\alpha(C, G)(C^{G\#})) = \alpha(C, G)(C^{G\#})$$

or, denoting the image of the covector $C^{G\#}$ as $f_C^G := \alpha(C, G)(C^{G\#})$,

$$\mathcal{E}nd_V(\rho) f_C^G = f_C^G\,. \tag{4.2}$$

To get more tractable formulas, we want to work with morphisms $V^{\otimes|C|} \to \Bbbk$, instead of $\bigotimes_C V$. This is possible if we choose set C to be $[k]$, because there is a canonical isomorphism[1]

$$\begin{aligned}\Big(\bigotimes_{[k]} V \to \Bbbk\Big) &\cong \big(V^{\otimes k} \to \Bbbk\big), \\ f &\mapsto f \circ \imath_{1_{[k]}} .\end{aligned} \tag{4.3}$$

Thanks to the equivariance, knowledge of functions $\bigotimes_{[k]} V \to \Bbbk$ suffices to specify the α-image of all elements in the endomorphism operad – see lemma 9 of [1].

Using the definition of $\mathcal{E}nd_V(\rho)$, the equation 1.13, we have

$$f^G_{[k]} \circ \imath_{1_{[k]}} \circ \rho^{-1} = f^G_{[k]} \circ \imath_{1_{[k]}}, \ \forall \rho \in \mathbb{S}_k .$$

Denoting $f^G_{[k]} \circ \imath_{1_{[k]}} =: f^G_k : V^{\otimes k} \to \Bbbk$, we see that f^G_k are graded symmetric functions on V.

Graded symmetric function $V^{\otimes k} \to \Bbbk$ can be represented by a polynomial in elements of dual basis to V, we denote[2] $x_i := t_i^{\#}$ and $\xi_i := \theta_i^{\#}$. The evaluation of symmetric function $X_1 X_2 \dots X_k$ on vectors $T_1, \dots, T_k$, where X represents x_i or ξ_i and T represents t_i or θ_i, is defined as

$$X_1 \dots X_k(T_1, \dots, T_k) := \sum_{\sigma \in \mathbb{S}_k} (-1)^{\epsilon(\sigma; X_1 \dots X_k)} X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(k)}(T_1 \otimes \dots \otimes T_k) , \tag{4.4}$$

where ϵ is the sign of $\mathbb{S}$ action on tensor product of graded vector spaces, defined by equation A.9.

Example 4.1. There is a good amount of signs here, so a simple illustration can be helpful. For pair of degree zero elements, we have

$$\begin{aligned}\xi_1\xi_2(\theta_2, \theta_1) = \xi_2\xi_1(\theta_1, \theta_2) = 1 , \\ \xi_1\xi_2(\theta_1, \theta_2) = \xi_2\xi_1(\theta_2, \theta_1) = -1 .\end{aligned}$$

Let's also evaluate $\xi_1\xi_2 x_1(t_1, \theta_1, \theta_2)$:

$$\begin{aligned}\xi_1\xi_2 x_1(t_1, \theta_1, \theta_2) = \\ &\big(\xi_1 \otimes \xi_2 \otimes x_1 + \xi_1 \otimes x_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 \otimes x_1 \\ &-\xi_2 \otimes x_1 \otimes \xi_1 + x_1 \otimes \xi_1 \otimes \xi_2 - x_1 \otimes \xi_2 \otimes \xi_1\big)(t_1 \otimes \theta_1 \otimes \theta_2) \\ &\quad = (x_1 \otimes \xi_1 \otimes \xi_2)(t_1 \otimes \theta_1 \otimes \theta_2) \\ &\quad = (-1)^{|\xi_2||\theta_1|} x_1(t_1)\xi_1(\theta_1)\xi_2(\theta_2) = -1 .\end{aligned}$$

◇

Turning our attention to operations defined on the endomorphism operad, we want to see how ξ_{ab} acts on a function $f \in \mathcal{E}nd([k] \sqcup \{a, b\}, G)$. It's defined as ([1], equation 6)

$$\xi_{ab} f^G_{[k]} \circ \imath_\psi := (-1)^{|f^G_{[k+2]}|} \sum_{i=1}^{2m} (f^G_{[k]} \circ \imath_{\psi'})(a_i \otimes b_i \otimes 1^{\otimes k}) , \tag{4.5}$$

[1] For general C, the isomorphism is not canonical, since one would need to choose a ordering of the set. We exploit the fact that sets $[k]$ have a canonical order.

[2] There is a minor clash of convention, ξ with two indices is always the loop creating operation and ξ with one index is a dual to degree 1 element of V.

with $\psi'(a) = 1$, $\psi'(b) = 2$ and $\psi'(c) = \psi(c) + 2$, and the equation holds for any $\psi : [k] \to [k]$. Let's take $a, b = k+1, k+2$ and $\psi = 1_{[k]}$:

$$(\xi_{ab} f^G_{[k+2]}) \circ \imath 1_{[k]} = (-1)^{|f^G_{[k+2]}|} \sum_{i=1}^{2m} (f^G_{k+2} \circ \psi'^{-1})(a_i \otimes b_i \otimes 1^{\otimes k}) .$$

Since f^G_k are symmetric, $f^G_k \circ \psi'^{-1} = f^G_{k+2}$ and we have

$$(\xi_{ab} f^G_{[k+2]}) \circ \imath 1_{[k]} = (-1)^{|f^G_{[k+2]}|} \sum_{i=1}^{2m} (f^G_{k+2})(a_i \otimes b_i \otimes 1^{\otimes k}) .$$

If we express f^G_{k+2} as $X_1 \dots X_{k+2}$, composing it with $(t_i \otimes \theta_i \otimes 1^{\otimes k})$ removes one x_i and one ξ_i [3], picking a sign for commuting to the left. In the case of multiple covectors x_i, we get a sum of terms for every covector, which is exactly Leibniz rule. Let's not forget a sign θ_i will cause commuting through all the other covectors X in the term

$$x_i \otimes \xi_i \otimes X_{i_1} \dots X_{i_k} ,$$

which is equal to $\sum_{j=1}^k |X_{i_j}| = |f^G_{k+2}| - 1$. The sign picked commuting ξ_i to the left can be implemented using left derivatives, and we arrive at

$$\begin{aligned}(\xi_{ab} f^G_{[k+2]}) \circ \imath 1_{[k]} &= (-1)^{|f^G_{[k+2]}| + |f^G_{[k+2]}| - 1} \sum_{i=1}^{2m} \frac{\partial_l}{\partial a_i^\#} \frac{\partial_l}{\partial b_i^\#} f^G_{k+2} \\ &= -2 \sum_{i=1}^{m} \frac{\partial_l}{\partial x_i} \frac{\partial_l}{\partial \xi_i} f^G_{k+2} .\end{aligned} \tag{4.6}$$

For the differential, we start by looking at a function with 1 input only:

$$(df^G_{[1]}) \imath 1_{[k]} := (-1)^{|f^G_{[1]}|} f^G_1 \circ d_V .$$

There are four possible combinations of f and the input it gets:

$$\begin{aligned} x_i(d_V t_j) &= \beta x_i(\theta_j) = 0 , \\ x_i(d_V \theta_j) &= \beta x_i(0) = 0 , \\ \xi_i(d_V t_j) &= \beta \xi_i(\theta_j) = \beta \delta_{ij} , \\ \xi_i(d_V \theta_j) &= \beta \xi_i(0) = 0 , \end{aligned}$$

which can be concisely written as a operator d acting on $f^G_{[1]}$ as

$$(df^G_{[k]}) \imath [k] = - \sum_{j=1}^{m} \beta x_j \frac{\partial}{\partial \xi_j} f^G_k . \tag{4.7}$$

The sign $-$ comes from the fact that the expression is non-zero only if $|f^G_{[1]}| = 1$

In the general case, we have

$$(df^G_{[k]}) \circ \imath 1_{[k]} = (-1)^{|f^G_{[1]}|} \sum_{i=1}^{k} f^G_k \circ (1^{\otimes(i-1)} \otimes d_V \otimes 1^{\otimes(k-i)}) .$$

[3] This is abuse of notation, since t_i are vectors, but here we take them as functions $\Bbbk \to V$, assigning $1 \mapsto t_i$.

Looking at one summand at a time, we evaluate it on $T_1 \dots T_k$ with $f_k^G = X_1 \dots X_k$

$$
\begin{aligned}
&X_1 \dots X_k \circ \left(1^{\otimes(i-1)} \otimes d_V \otimes 1^{\otimes(k-i)}\right)\left(T_1 \otimes \dots \otimes T_k\right) \\
&= (-1)^{|T_1|+\dots+|T_{i-1}|}\left(X_1 \dots X_k\right)\left(T_1 \otimes \dots \otimes d_V T_i \otimes \dots \otimes T_k\right) \\
&= (-1)^{|T_1|+\dots+|T_{i-1}|} \sum_{\sigma \in \mathbb{S}_k} (-1)^{\epsilon(\sigma,X)}\left(X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(k)}\right)\left(T_1 \otimes \dots \otimes d_V T_i \otimes \dots \otimes T_k\right) \\
&= (-1)^{|T_1|+\dots+|T_{i-1}|} \sum_{\sigma \in \mathbb{S}_k} (-1)^{\epsilon(\sigma,X)+\epsilon(X \leftarrow T)+|X_{\sigma^{-1}(i+1)}|+\dots+|X_{\sigma^{-1}(k)}|} \\
&\qquad\qquad\qquad \times X_{\sigma^{-1}(1)}(T_1) \dots X_{\sigma^{-1}(i)}(d_V T_i) \dots X_{\sigma^{-1}(k)}(T_k) = \dots
\end{aligned}
\tag{4.8}
$$

where the factor $(-1)^{\epsilon(X \leftarrow T)}$ is the Koszul sign of commuting

$$
X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(k)}(T_1 \otimes \dots \otimes T_k) = (-1)^{\epsilon(X \leftarrow T)} X_{\sigma^{-1}(1)}(T_1) \dots X_{\sigma^{-1}(k)}(T_k)\,,
$$

without the differential; the sign due tu differentiated element is $(-1)^{|X_{\sigma^{-1}(i+1)}|+\dots+|X_{\sigma^{-1}(k)}|}$

We calculated that $X d_V T = \sum_{j=1}^m \beta x_j \frac{\partial X}{\partial \xi_j}(T)$, we can continue in equation 4.8

$$
\begin{aligned}
\dots = (-1)&^{\epsilon(\sigma,X)+|X_{\sigma^{-1}(i+1)}|+\dots+|X_{\sigma^{-1}(k)}|+|X_{\sigma^{-1}(1)}|+\dots+|X_{\sigma^{-1}(i-1)}|} \\
\times &\underbrace{\left(\sum_{j=1}^m \beta x_j \frac{\partial_l}{\partial \xi_j}\right)}_{\text{acts on } i\text{th vector}} X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(k)}(T_1 \otimes \dots \otimes T_k)\,,
\end{aligned}
$$

with these signs: $(-1)^{|T_1|+\dots+|T_{i-1}|}$ cancelled out with the factor from commuting vectors T with the differential, $(-1)^{|X_{\sigma^{-1}(1)}|+\dots+|X_{\sigma^{-1}(i-1)}|}$ comes from commuting the derivation to the ith position. Factors with $|X|$ combine to $(-1)^{|f_k^G|-1}$, because the expression is nonzero only when $|X_{\sigma^{-1}(i)}| = 1$. Summing through all the $i = 1 \dots k$ just removes the condition "acts on ith vector" and we are left with

$$
(df_{[k]}^G) \circ \imath_{1_{[k]}} = -\sum_{i=1}^m \beta x_i \frac{\partial_l f_k^G}{\partial \xi_i}\,. \tag{4.9}
$$

If we tried to find such representation for ${}_a\circ_b$ as well, we would not succeed, because ${}_a\circ_b(f \otimes g) \circ \imath_{1_{[n_1+n_2]}}$ is not a symmetric function any more. Roughly, the result, product

$$
\mu \circ \left(\frac{\partial f}{\partial a_i^{\#}} \otimes \frac{\partial g}{\partial b_i^{\#}}\right)\,,
$$

where $\mu : \mathbb{k} \otimes \mathbb{k} \to \mathbb{k}$ is the multiplication in the field $\mathbb{k}$, is symmetric with respect to arguments of f and g individually, but not with respect to all permutations [4].

4.2 *Quantum Closed* operad

We could continue without explicitly choosing an operad $\mathcal{P}$, but the assumptions would soon become unnatural. This is defined in [1], definition 13.

[4]This sheds some light on the sum we see in the definition of $\{\ ,\ \}$, equation 3.8 – the sum is exactly the symmetrization between the two two functions.

Definition 4.2. Modular operad $\mathfrak{QC}$ (short for *Quantum Closed*) consists of

$$\mathfrak{QC}(C,G) := \mathrm{Span}_{\mathbb{k}}(C^G)\,, \tag{4.10}$$

1-dimensional spans of symbol C^G of degree 0. The maps are defined as follows

$$\mathfrak{QC}(\rho)(C^G) := D^G,\ \ \forall \rho : C \to D\,, \tag{4.11}$$

$${}_a\circ_b(C \sqcup \{a\})^{G_1} \otimes (D \sqcup \{b\})^{G_2} := (C \sqcup D)^{G_1+G_2}\,, \tag{4.12}$$

$$\xi_{ab}(C \sqcup \{a,b\})^G := C^{G+1}\,. \tag{4.13}$$

The differential d_{C^G} on spaces $\mathfrak{QC}(C,G)$ is zero. △

$\mathfrak{QC}$ has a topological interpretation - C^G is a compact 2-dimensional surface with genus G and disks on the surface, labelled by elements of C. ξ_{ab} and ${}_a\circ_b$ then just glue and cut out these disks together (see also images in loc.cit.)

With such operad, we can describe corresponding structures arising from it more explicitly. An algebra over Feynman transform of $\mathfrak{QC}$ is again composed of symmetrical functions ($\mathfrak{QC}$ satisfies all the assumptions of previous section, so the result applies directly). The compatibility with the differential, equation 3.2, would give us identities for maps $\alpha(C,G)(C^{G\#})$, which, after some suspensions, turn out to be defining axioms of *loop homotopy algebras*. This was first shown by Markl in [9] (theorem 5.5). Markl described the loop homotopy algebras as algebras over Feynman transform of a modular envelope of cyclic operad $\mathcal{C}om$. This modular envelope of $\mathcal{C}om$ is in fact $\mathfrak{QC}$. This is discussed in [1], theorem 16, see also ibid., section IV.A and namely theorems 14 and 15, for restating Markl's results in the formalism we follow.

Now, we can continue in describing the structure of operations on $\mathcal{E}nd_V$, only in the algebra P from section 3.3. Due to the simplicity of $\mathfrak{QC}$, the operations d, Δ and $\{\,,\,\}$ will be determined by d_V, ξ_{ab} and ${}_a\circ_b$ of endomorphism operad, respectively.

The elements of P are, by definition

$$\begin{aligned} P(n,G) &= \big(\mathfrak{QC}([n],G) \otimes \mathcal{E}nd_V([n],G)\big)^{\mathbb{S}_n} = \mathfrak{QC}([n],G) \otimes \big(\mathcal{E}nd_V([n],G)\big)^{\mathbb{S}_n} \\ &\cong \big(\mathcal{E}nd_V([n],G)\big)^{\mathbb{S}_n}\,, \end{aligned} \tag{4.14}$$

since automorphisms act trivially on QC. This looks familiar, these are again symmetric functions $\bigotimes_{[n]} V \to \mathbb{k}$. P is then a vector space [5] of polynomials in x_i, ξ_i and $\hbar$, with $|x_i| = |\hbar| = 0$ and $|\xi_i| = 1$.

With $d_{\mathfrak{QC}} = 0$, the differential on P is just

$$d = 1_{\mathfrak{QC}} \otimes d_V\,,$$

which acts on polynomials as in equation 4.9, i.e.

$$d = -\sum_{i=1}^{m} \beta x_i \frac{\partial_l}{\partial \xi_i}\,.$$

Similarly, Δ works through ξ_{ab} as in equation 4.6. The $\mathfrak{QC}$ part increases the G by one, so there is $\hbar$ appearing

$$\Delta = 2\hbar \sum_{i=1}^{m} \frac{\partial_l}{\partial x_i} \frac{\partial_l}{\partial \xi_i}$$

[5] formally, we are working under isomorphism $C^G \otimes f^G_{[n]} \mapsto \hbar^G (f^G_{[n]} \circ \imath_{1_{[n]}})$, with $f^G_{[n]} \circ \imath_{1_{[n]}}$ represented as polynomials in x_i and ξ_i.

Note that here, we take Δ as an operator $P \to P$, hence the $\hbar$ is necessary. [1] takes it as a operator $P(n+2,G) \to P(n,G+1)$, where the $\hbar$ is not even defined.

At last, the bracket. Let's evaluate it on vectors $g = [n_1+1]^{G_1} \otimes g_{[n_1+1]} \in P(n_1+1, G_1)$ and $h = [n_2+1]^{G_2} \otimes h_{[n_2+1]} \in P(n_2+1, G_2)$. Elements of $\mathcal{QC}$ have degree 0, so there are no signs and we can readily apply the swap $\overset{\frown}{\tau}$

$$\begin{aligned}\{g,h\} = -\sum_{C_1,C_2} \big(({}_a\circ_b)_{\mathcal{QC}} \otimes ({}_a\circ_b)_{\mathcal{E}nd_V}\big)\big(\mathcal{P}(\theta_1)([n_1+1]^{G_1}) \otimes \mathcal{P}(\theta_2)([n_2+1]^{G_2}) \otimes \\ \mathcal{E}(\theta_1)(g_{[n_1+1]}) \otimes \mathcal{E}(\theta_2)(h_{[n_2+1]})\big)\,.\end{aligned} \tag{4.15}$$

Since θ_1 is a function $[n_1+1] \to C_1 \sqcup \{a\}$ and $\theta_2 : [n_2+1] \to C_2 \sqcup \{b\}$, can evaluate the first composition, $({}_a\circ_b)_{\mathcal{QC}}$.

$$({}_a\circ_b)_{\mathcal{QC}}\big((C_1 \sqcup \{a\})^{G_1} \otimes (C_2 \sqcup \{b\})^{G_2}\big) = (C_1 \sqcup C_2)^{G_1+G_2} = [n_1+n_2]^{G_1+G_2}\,. \tag{4.16}$$

Now we can move the sum over C_1, C_2 inside the tensor product and write

$$\{g,h\} = -[n_1+n_2]^{G_1+G_2} \otimes \sum_{C_1,C_2} ({}_a\circ_b)_{\mathcal{E}nd_V}\big(\mathcal{E}(\theta_1)(g_{[n_1+1]}) \otimes \mathcal{E}(\theta_2)(h_{[n_2+1]})\big)\,. \tag{4.17}$$

Now we drop the $\mathcal{QC}$ part of the tensor product on both sides and apply isomorphism $\imath 1_{[n_1+n_2]}$. We denote the result $\{g_{n_1+1}, h_{n_2+1}\} := \imath^{\mathcal{E}nd_V} \circ \{g,h\} \circ (1 \otimes \imath 1_{[n_1+n_2]})$, where $\imath^{\mathcal{E}nd_V}$ projects to $\mathcal{E}nd_V$, dropping the factor $[n_1+n_2]^{G_1+G_2}$.

On the right hand side, we cannot apply the definition of $({}_a\circ_b)_{\mathcal{E}nd_V}$ (ibid., equation 5), since it assumes that the isomorphism $\psi : [n_1+n_2] \to [n_1+n_2]$ from $\imath_\psi$ satisfies $\psi(C_1) = [n_1]$, which isn't generally true for identity $1_{[n_1+n_2]}$. If we choose a permutation $\tau : [n_1+n_2] \to [n_1+n_2]$ such that $\tau(C_1) = [n_1]$, we can use equation 1.10 as $\imath 1_{[n_1+n_2]} = \imath_\tau \circ \tau$. We will fix τ by requiring that it's increasing on C_1 and C_2, separately.

We are left with

$$\{g_{n_1+1}, h_{n_2+1}\} = -\sum_{C_1,C_2} ({}_a\circ_b)_{\mathcal{E}nd_V}\big(\mathcal{E}(\theta_1)(g_{[n_1+1]}) \otimes \mathcal{E}(\theta_2)(h_{[n_2+1]})\big) \circ \imath_\tau \circ \tau\,. \tag{4.18}$$

Looking at one summand at a time, we have

$$\begin{aligned}&({}_a\circ_b)_{\mathcal{E}nd_V}\big(\mathcal{E}(\theta_1)(g_{[n_1+1]}) \otimes \mathcal{E}(\theta_2)(h_{[n_2+1]})\big) \circ \imath_\tau \\ &= \sum_{i=1}^{2m} (-1)^{|g|+|h||b_i|} \mu \circ \big((\mathcal{E}(\theta_1)(g_{[n_1+1]}) \circ \imath_{\psi_1})(a_i \otimes 1^{\otimes n_1}) \otimes (\mathcal{E}(\theta_2)(h_{[n_2+1]} \circ \imath_{\psi_2})(b_i \otimes 1^{\otimes n_2})\big)\,.\end{aligned} \tag{4.19}$$

Expressions like $(\mathcal{E}(\theta_1)(g_{[n_1+1]}) \circ \imath_{\psi_1})(a_i \otimes 1^{\otimes n_1})$ were already discussed in previous section; choosing a to be n_1+1, applying the symmetry of g_{n_1+1} and adding a sign for commuting a_i to the beginning, we get

$$(\mathcal{E}(\theta_1)(g_{[n_1+1]}) \circ \imath_{\psi_1})(a_i \otimes 1^{\otimes n_1}) = (-1)^{|a_i|(|g|-|a_i|)} \frac{\partial_l g_{n_1+1}}{\partial a_i^\#} = \frac{\partial_r g_{n_1+1}}{\partial a_i^\#}\,. \tag{4.20}$$

Putting everything back together, we arrive at

$$\{g_{n_1+1}, h_{n_2+1}\} = -\sum_{C_1,C_2} \sum_{i=1}^{2m} (-1)^{|g|+|b_i|} \mu \circ \Big(\frac{\partial_r g_{n_1+1}}{\partial a_i^\#} \otimes \frac{\partial_l h_{n_2+1}}{\partial b_i^\#}\Big) \circ \tau\,. \tag{4.21}$$

The permutation τ depends on C_1 and C_2. To show that the sum over C_1, C_2 symmetrizes the input, we first need to show that we can move τ from acting on input vectors to covectors in g_{n_1+1} and h_{n_2+1}. This is captured in

$$X_1 \otimes \cdots \otimes X_k \tau(T_1 \otimes \cdots \otimes T_k) = \tau^{-1}(X_1 \otimes \cdots \otimes X_k)(T_1 \otimes \cdots \otimes T_k)\,, \tag{4.22}$$

where the τ^{-1} acts on the tensor product of vectors X as explained in appendix A, i.e. the same way as it would act on vectors T. This formula can be proven from the definition of symmetric function evaluation[6]. Now we need to show that

$$\sum_{C_1,C_2} \sum_{\substack{\sigma_1 \in \mathbb{S}_{n_1} \\ \sigma_2 \in \mathbb{S}_{n_2}}} \tau^{-1} \circ (\sigma_1 \otimes \sigma_2) \tag{4.23}$$

is equal to sum over all permutations of $[n_1 + n_2]$. We can view the two permutations $\sigma_1 \otimes \sigma_2$ as an action of element of $\mathbb{S}_{n_1+n_2}$, only the arguments and results of σ_2 need to be increased by n_1

The inner sum has $n_1!n_2!$ terms, the outer sum $\binom{n_1+n_2}{n_1}$ terms, together $(n_1+n_2)!$, exactly the right number. For any τ, we have $\tau^{-1}([n_1]) = C_1$, the permutation $\sigma_1 \otimes \sigma_2$ satisfies $\sigma_1([n_1]) = [n_1]$, so the composite $\tau^{-1} \circ (\sigma_1 \otimes \sigma_2)$ maps $[n_1]$ to C_1. This means that two composite permutations corresponding to different C_1 are different. On the other hand, for fixed τ, all permutations $\tau^{-1} \circ (\sigma_1 \otimes \sigma_2)$ are different, meaning that the permutations in the sum in 4.23 are different. Since the number of these permutations is equal to the order of $\mathbb{S}_{n_1+n_2}$, we have

$$\sum_{C_1,C_2} \sum_{\substack{\sigma_1 \in \mathbb{S}_{n_1} \\ \sigma_2 \in \mathbb{S}_{n_2}}} \tau^{-1} \circ (\sigma_1 \otimes \sigma_2) = \sum_{\sigma \in \mathbb{S}_{n_1+n_2}} \sigma\,. \tag{4.24}$$

This leads us to

$$\begin{aligned} \{g_{n_1+1}, h_{n_2+1}\} &= -\sum_{i=1}^{2m} (-1)^{|g|+|b_i|} \frac{\partial_r g_{n_1+1}}{\partial a_i^{\#}} \frac{\partial_l h_{n_2+1}}{\partial b_i^{\#}} \\ &= (-1)^{|g|} \sum_{i=1}^{m} \frac{\partial_r g_{n_1+1}}{\partial x_i} \frac{\partial_l h_{n_2+1}}{\partial \xi_i} - \frac{\partial_r g_{n_1+1}}{\partial \xi_i} \frac{\partial_l h_{n_2+1}}{\partial x_i} \end{aligned} \tag{4.25}$$

Which is exactly the version of the bracket from equation 2.29, with the sign factor $(-1)^{|g|}$, discussed in section 2.1.1. This bracket corresponds to nilpotent operator $\Delta/2$, as we saw in chapter 2 (we discuss the factor $\frac{1}{2}$ in the next section).

4.2.1 Zwiebach's master equation

We can now now relate this construction to [10]. In section 3.3, Zwiebach states the quantum master equation, equivalent to the master equation in terms of algebra P for $\mathfrak{QC}$.

Then he explores the algebra of string products, physically motivated operations. For these string products, he shows that their properties lead to action satisfying quantum master equation.

In our case, the operations correspond to $\alpha(C,G)(C^{G\#})$, algebra over Feynman transform $\mathfrak{FQC}$. Zwiebach's proof is just a special case of theorem 3.6.

However, our master equation doesn't look the same. There are two issues:

[6] The trickiest part is the sign; the sign factors that appear on both sides look different (these are factors we could denote $\epsilon(X \leftarrow \tau T)$ and $\epsilon(\tau^{-1}X \leftarrow T)$ in spirit of equation 4.8). The key is to express these factors as a functions of $\sum_i |X_i|$ and $\sum_i |X_i|^2$, which are invariant to commuting the vectors X_i to $X_{\tau^{-1}(i)}$.

The term dS of master equation can be absorbed in the action, see [1], IV.B

There is, however, a bigger nuisance, the factor 2 in the operator Δ. In comes from definition of $(\xi_{ab})_{\mathcal{E}nd_V}$, where in the sum $\sum_i a_i \otimes b_i$, there is a a factor $t_i \otimes \theta_i$ as well as $\theta_i \otimes t_i$, which is the same for symmetric functions. The theorem 3.5 is invariant to rescalings of Δ, but the master equation isn't. This means that we fail to reproduce Zwiebach's master equation (and commutative BV algebra) with these $\{\,,\,\}$ and Δ. We think that this might be caused by different conventions for the action[7].

[7] Or, of course, a numerical mistake, most probably in this work.

A. differential graded vector spaces

Barannikov's theory is formulated in the language of differential graded vector spaces (or dg vector spaces for short). Here we review the necessary theory. Bourbaki [30], chapter 2 section 11 gives throughout treatment to a more general case of modules (vector space is a module over a field). Weibel's introduction to homological algebra [31] starts with chain complexes, which are again just a more general case of dg vector spaces.

$\mathbb{Z}$-graded vector space V is a vector space that is written as a direct sum

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \tag{A.1}$$

of its subspaces. The degree of an element $v \in V_i$ is defined as $|v| := i$; elements having a defined degree are called homogeneous. Degree of a linear map $f : V \to W$ is defined only if $\forall i : f(V_i) \subseteq W_{i+k}$ and is defined as the corresponding shift in the degree $|f| := k$. Elements of V_i are called homogeneous, maps having degree are called homogeneous of degree k, or homomorphisms of degree k.

Differential graded vector space (or shortly dg vector space) is a graded vector space with differential d_V of degree $+1$ satisfying $d_V^2 = 0$. That is, the differential is a collection of maps

$$d_{V,i} = d_V\big|_{V_i} : V_i \to V_{i+1}$$

satisfying

$$d_{V,i} \circ d_{V,i-1} = 0;\ \forall i\,.$$

Differential graded vector space is a special case of a chain complex, specifically it's a chain complex of vector spaces.

Vector spaces can be graded with any commutative monoid, not just $\mathbb{Z}$. Other notable examples include $\mathbb{N}$-graded vector spaces and $\mathbb{Z}_2$-graded vector spaces, with $\mathbb{N}$ being natural numbers and Z_2 being the two-element group. The latter case is often called super vector space.

To specify a category of differential graded vector spaces, **dgVec**, one has to specify the morphisms – they are defined as degree 0 linear maps commuting with differential.

Note that morphisms, elements of $\mathrm{Hom}_{\mathbf{dgVec}}(V, W)$, have degree 0 and commute with differential, unlike homomorphisms, homogeneous linear maps of arbitrary (but well defined) degree. Confusingly, all degree p morphisms between V and W are denoted by $\mathrm{Hom}_p(V, W)$ and their direct sum is denoted $\mathrm{Hom}(V, W)$:

$$\mathrm{Hom}(V, W) := \bigoplus_p \mathrm{Hom}_p(V, W)\,. \tag{A.2}$$

$\mathrm{Hom}(V, W)$ can also be given a dg vector space structure with differential

$$Df := d_W \circ f - (-1)^{|f|} f \circ d_V\,. \tag{A.3}$$

A.1 Tensor product

Tensor product of two graded vector spaces is defined (see also [5], section 1.1 of part II) as

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j\,. \tag{A.4}$$

If the underlying spaces are dg, the $V \otimes W$ is made a dg vector space with differential

$$d_{VW}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w) \tag{A.5}$$

for homogeneous elements v and w and extended linearly[1].

For tensor product of vector spaces, two spaces $V \otimes W$ and $W \otimes V$ are isomorphic. For graded spaces, this isomorphism, denoted s_{VW}, is defined with a sign

$$\begin{aligned} s_{VW} : V \otimes W &\to W \otimes V\,, \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v \end{aligned} \tag{A.6}$$

for homogeneous v and w.

When writing out formulas for morphisms in tensor product of dg vector spaces, we will use *Koszul sign convention*: when putting vectors in the formula, we pick a sign $(-1)^{|v|\cdot|f|}$ every time a vector v passes a function f. This allows us to write, for example

$$d_{VW} = d_V \otimes 1_W + 1_V \otimes d_W\,.$$

The convention can be further expanded for commuting graded objects, e.g.

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{|f_2||g_1|} (f_1 \circ f_2) \otimes (g_1 \circ g_2) \tag{A.7}$$

for functions. This follows from a simple calculation

$$\begin{aligned} (f_1 \otimes g_1) \circ (f_2 \otimes g_2)(x \otimes y) &= (-1)^{|x||g_2|} (f_1 \otimes g_1) \circ (f_2(x) \otimes g_2(y)) \\ &= (-1)^{|x||g_2| + |f_2(x)||g_1|} f_1(f_2(x)) \otimes g_1(g_2(y)) \end{aligned}$$

and, using $|f_2(x)| = |f_2| + |x|$ and $|g_1 \circ g_2| = |g_1| + |g_2|$, we can compare this with

$$(f_1 \circ f_2) \otimes (g_1 \circ g_2)(x \otimes y) = (-1)^{|g_1 \circ g_2||x|} f_1(f_2(x)) \otimes g_1(g_2(y))\,.$$

Another interesting exercise in graded signs is the action of $\mathbb{S}_n$ on $V^{\otimes N} \equiv V \otimes \cdots \otimes V$, the nth tensor power of V. We have already seen the action of S_2 on $V \otimes V$, the transposition corresponding to symmetry operator s_{VV}. The general sign of left action

$$\sigma(v_1 \otimes \cdots \otimes v_n) = (-1)^{\epsilon} (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})\,, \tag{A.8}$$

is

$$\epsilon := \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i||v_j| \mod 2\,, \tag{A.9}$$

picking a sign for every two elements that have to commute. Note that ϵ doesn't depend on the σ only, but also on degrees and ordering of vectors v_i, in contrast with the usual sign permutation, corresponding to all vectors being odd.

To see this is a left action, let's evaluate $\tau(\sigma(v_1 \otimes \cdots \otimes v_n))$. Defining $w_i := v_{\sigma^{-1}(i)}$, we get

$$\begin{aligned} \tau(\sigma(v_1 \otimes \cdots \otimes v_n)) &= (-1)^{\epsilon} \tau(w_1 \otimes \cdots \otimes w_n)) \\ &= (-1)^{\epsilon' + \epsilon} \tau(w_1 \otimes \cdots \otimes w_n)) \\ &= (-1)^{\epsilon' + \epsilon} (w_{\tau^{-1}(1)} \otimes \cdots \otimes w_{\tau^{-1}(n)})\,. \end{aligned} \tag{A.10}$$

[1] The minus sign is important to make it a differential, i.e. to have $d_{VW}^2 = 0$. One other construction that comes to mind is $d_{VW}(v \otimes w) = d_V(v) \otimes d_W(w)$. This differential would, amongst other problems, have a degree -2.

Reusing the definition of w_i, we get element $w_{\tau^{-1}(i)} = v_{\sigma^{-1}(\tau^{-1}(i))} = v_{(\tau\sigma)^{-1}(i)}$ on the ith place of the tensor product. So, at least up to a sign, this is a left action.

Using the equation ϵ A.9 to express the sign, we get

$$\begin{aligned}
\epsilon + \epsilon' &= \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i||v_j| + \sum_{\substack{i<j \\ \tau(i)>\tau(j)}} |w_i||w_j| \mod 2 \\
&= \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i||v_j| + \sum_{\substack{i<j \\ \tau(i)>\tau(j)}} |v_{\sigma^{-1}(i)}||v_{\sigma^{-1}(j)}| \mod 2 \\
&= \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i||v_j| + \sum_{\substack{\sigma(i)<\sigma(j) \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| \mod 2
\end{aligned} \tag{A.11}$$

Splitting the first sum as

$$\sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i||v_j| = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j) \\ \tau(\sigma(i))<\tau(\sigma(j))}} |v_i||v_j| + \sum_{\substack{i<j \\ \sigma(i)>\sigma(j) \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| \tag{A.12}$$

and the second sum as

$$\sum_{\substack{\sigma(i)<\sigma(j) \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| = \sum_{\substack{i<j \\ \sigma(i)<\sigma(j) \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| + \sum_{\substack{i>j \\ \sigma(i)<\sigma(j) \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| . \tag{A.13}$$

Now, the second sum in RHS of $A.12$ and the first sum in RHS of $A.13$ add to

$$\sum_{\substack{i<j \\ \tau(\sigma(i))>\tau(\sigma(j))}} |v_i||v_j| , \tag{A.14}$$

the sign factor corresponding to reordering $(\tau\sigma)(v_1 \otimes \cdots \otimes v_n)$. Finally, the first sum in RHS of $A.12$ and the second sum in RHS of $A.13$ are exactly the same, only with exchanged summation indices i and j, so their sum is an even number.

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