

Computability via Recursive Functions

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1 Effective Calculability and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, \mathbb{N} refers to the set $\{0, 1, 2, 3, \dots\}$

Definition. *The following functions from $\mathbb{N} \oplus \dots \oplus \mathbb{N}$ to \mathbb{N} are primitive recursive functions:*

1. *The unary constant function c :*

$$c(x) = 0$$

2. *The unary successor function s :*

$$s(x) = x + 1$$

3. *The n -ary projection function p :*

$$1 \leq i \leq n$$
$$p_i(x_1, \dots, x_n) = x_i$$

4. *Function composition*

Let f be an n -ary primitive recursive function and g_1, g_2, \dots, g_n all be m -ary primitive recursive functions. Then the m -ary composition h of f and g_1, g_2, \dots, g_n given by

$$h(x_1, x_2, \dots, x_m) = f(g_1(x_1, x_2, \dots, x_m), \dots, g_n(x_1, x_2, \dots, x_m))$$

is a primitive recursive function

5. *Primitive recursion* Let g be an n -ary primitive recursive function and f be an $(n + 2)$ -ary primitive recursive function. Then the $(n + 1)$ -ary primitive recursion h of f and g given by

$$\begin{aligned} h(0, x_1, \dots, x_n) &= g(x_1, \dots, x_n) \\ h(s(x), x_1, \dots, x_n) &= f(x, h(x, x_1, \dots, x_n), x_1, \dots, x_n) \end{aligned}$$

is a primitive recursive function

3 The Ackermann Function

Definition (The Ackermann Function). Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:

$$A(n, m) = \begin{cases} m + 1 & n = 0 \\ A(n - 1, 1) & n > 0 \wedge m = 0 \\ A(n - 1, A(n, m - 1)) & n > 0 \wedge m > 0 \end{cases}$$

Theorem 1. For any $n, m \in \mathbb{N}$, $A(n, m) \in \mathbb{N}$

Proof. We proceed by double induction.

Clearly $A(0, k) \in \mathbb{N}$ for every $k \in \mathbb{N}$ because $k + 1 \in \mathbb{N}$. Now assume $A(k, m) \in \mathbb{N}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Then $A(k + 1, 0) = A(k + 1 - 1, 1) = A(k, 1) \in \mathbb{N}$. Now assume $A(k + 1, j) \in \mathbb{N}$ for some $j \in \mathbb{N}$. Then $A(k + 1, j + 1) = A(k + 1 - 1, A(k + 1, j + 1 - 1)) = A(k, A(k + 1, j))$. By our second hypothesis $A(k + 1, j) \in \mathbb{N}$, so by our first hypothesis $A(k, A(k + 1, j)) \in \mathbb{N}$. Therefore $A(k + 1, j + 1) \in \mathbb{N}$. Then $A(k + 1, m) \in \mathbb{N}$ for every $m \in \mathbb{N}$, so $A(n, m) \in \mathbb{N}$ for all choices of $n, m \in \mathbb{N}$ \square

Theorem 2. A is a total function

Proof. By Theorem 1, all n, m in the domain of A have an image $A(n, m) \in \mathbb{N}$. Therefore A is a total function. \square

Theorem 3. For any $m, n, s \in \mathbb{N}$ where $s > n$, $A(m, n) < A(m, s)$

Proof. Use the proof of $A(m, n) < A(m, n + 1)$ \square

Theorem 4. For any $m, n, s \in \mathbb{N}$ where $s > m$, $A(m, n) < A(s, n)$

Proof. □

Theorem 5. For any $m, n, s \in \mathbb{N}$, $A(m, A(s, n)) < A(m + s + 2, n)$

Proof. Proof here □

Theorem 6. For any $x \in \mathbb{N}$, $x + 2 = A(1, x)$

Proof. We proceed by induction.

$A(1, 0) = A(1 - 1, 1) = A(0, 1) = 2 = 0 + 2$. Now assume $A(1, k) = k + 2$ for some $k \in \mathbb{N}$. Then $A(1, k + 1) = A(1 - 1, A(1, k + 1 - 1)) = A(0, A(1, k)) = A(0, k + 2) = k + 3 = (k + 1) + 2$. □

Definition. Let P be a set all primitive recursive functions f so that if $f \in P$, there is a $t \in \mathbb{N}$ such that for any $x_1, \dots, x_n \in \mathbb{N}$, $f(x_1, \dots, x_n) < A(t, \max\{x_1, \dots, x_n\})$.

Theorem 7. $c(x), s(x), p_i(x_1, x_2, \dots, x_n) \in P$

Proof. Let $m = \max\{x_1, \dots, x_n\}$. Then

$$\begin{aligned} c(x) &= 0 < x + 1 = A(0, x) \\ s(x) &= x + 1 < x + 2 = A(1, x) \\ p_i(x_1, x_2, \dots, x_n) &= x_i \leq m < m + 1 = A(0, m) \end{aligned}$$

□

Theorem 8. P is closed under composition

Proof. Let $f, g_1, g_2, \dots, g_k \in P$, where f is k -ary and each g_i is j -ary. Let $x_1, x_2, \dots, x_j \in \mathbb{N}$. Let $m = \max\{x_1, x_2, \dots, x_j\}$. Let h be the j -ary primitive recursive function that results from function composition of f with g_1, g_2, \dots, g_k . Let g_{\max} be the g_i giving the maximum value in $\max\{g_1(x_1, \dots, x_j), \dots, g_k(x_1, \dots, x_j)\}$. Let $m_g = g_{\max}(x_1, \dots, x_j)$. Since $g_{\max} \in P$, there exists some $t_g \in \mathbb{N}$ so that $m_g < A(t_g, m)$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1, \dots, x_j) = f(g_1(x_1, \dots, x_j), \dots, g_k(x_1, \dots, x_j)) < A(t_f, m_g)$. But since $m_g < A(t_g, m)$, by Theorem 3 $A(t_f, m_g) < A(t_f, A(t_g, m))$. By Theorem 5, $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$. Let $t = t_f + t_g + 2 \in \mathbb{N}$. Then $h(x_1, \dots, x_j) < A(t, m)$. So $h \in P$. □

Lemma 1. Let h be the primitive recursion of two functions $f, g \in P$. Let $x_1, \dots, x_n \in \mathbb{N}$. Let $m = \max\{x_1, \dots, x_n\}$. Then there exists a $t \in \mathbb{N}$ such that for every $x \in \mathbb{N}$, $h(x, x_1, \dots, x_n) < A(t, x + m)$.

Proof. Since $f, g \in P$, there exist t_f and t_g so that for every $x \in \mathbb{N}$, $f(x, h(x, x_1, \dots, x_n), x_1, \dots, x_n) < A(t_f, \max\{x, h(x, x_1, \dots, x_n), x_1, \dots, x_n\})$ and $g(x_1, \dots, x_n) < A(t_g, \max\{x_1, \dots, x_n\})$. Choose $t = \max\{t_f, t_g\}$. We proceed by induction. oh man I have no idea where to continue on this one $h(0, x_1, \dots, x_n) = g(x_1, \dots, x_n) < A(t, \max\{0, x_1, \dots, x_n\})$. \square

Theorem 9. *P is closed under primitive recursion*

Proof. Let $f, g \in P$ and h be the primitive recursion of f and g . Let $x, x_1, \dots, x_n \in \mathbb{N}$. Let $m = \max\{x_1, \dots, x_n\}$. Then by Lemma 1, there is a $t \in \mathbb{N}$ so that $h(x, x_1, \dots, x_n) < A(t, x + m)$. Let $m' = \max\{x, x_1, \dots, x_n\} = \max\{x, m\}$. Since $x + m \leq m'$, $A(t, x + m) \leq A(t, 2m')$ by Lemma 3. Since $2m' < 2m' + 3$, $A(t, 2m') < A(t, 2m' + 3)$ also by Lemma 3.

By Lemma 1, $2m' + 3 = A(2, m')$, so $A(t, 2m' + 3) = A(t, A(2, m'))$. By Theorem 5, $A(t, A(2, m')) = A(t+2+2, m') = A(t+4, m')$. So $h(x, x_1, \dots, x_n) < A(t+4, m')$.

Therefore $h \in P$. \square

Theorem 10. *P is precisely the primitive recursive functions*

Proof. All $f \in P$ are primitive recursive by the definition of P . So it remains to show all primitive recursive functions are in P . Let f be a primitive recursive function. Then f can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in P , and P is closed under function composition and primitive recursion. Therefore $f \in P$. \square

Theorem 11. *$A(m, n)$ is not a primitive recursive function*

Proof. Suppose that A is a primitive recursive function. Then $A \in P$. Then there exists a $t \in \mathbb{N}$ so that for any $m, n \in \mathbb{N}$, $A(m, n) < A(t, \max\{m, n\})$. Set $m = n = t$. Then $\max\{m, n\} = m = n = t$. So $A(m, n) = A(t, t) < A(t, t)$ which is a contradiction. Therefore A is not primitive recursive. \square

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions