Computability via Recursive Functions

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1 Effective Calcubility and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$

3 The Ackermann Function

Definition. *Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:*

$$A(m,n) = \begin{cases} n+1 & m = 0 \\ A(m-1, 1) & m > 0 \wedge n = 0 \\ A(m-1, A(m, n-1)) & m > 0 \wedge n > 0 \end{cases}$$

Lemma. *For any $m, n \in mathbbN$, $A(m, n) \in \mathbb{N}$*

Theorem. *A(m, n) is a total function*

Proof. We will proceed inductively to show that $A(m, n)$ is defined for all $m, n \in \mathbb{N}$.

Clearly $A(0, n)$ is defined for all $n \in \mathbb{N}$. Assume $A(k, n)$ is defined for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$. Since $k + 1 > 0$, $A(k + 1, 0) = A(k, 1)$, which is defined.

Now we assume $A(k+1, j)$ is defined for some $j \in \mathbb{N}$. By our lemma, $A(k+1, j) = a$ for some $a \in \mathbb{N}$. Then since $j + 1 > 0$, $A(k+1, j+1) = A(k, A(k+1, j)) = A(k, a)$. Since $A(k, n)$ is defined for every $n \in \mathbb{N}$ by our inductive hypothesis, $A(k, a) = A(k+1, j+1)$ is defined. $\square$

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions