Computability via Recursive Functions

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1 Effective Calculability and Computability

How would one rigorously define what functions can be calculated by means of an algorithm? It is easy to see that the function $f(x) = x^3 + 2$ can be calculated by a combination of the well-known algorithms for evaluating exponents and addition.

But what about $isRing(X, i, +, *)$, a function that accepts a set, a unary operator, and two binary operators? It returns 0 if the set is not a ring under the given operations, and 1 if the set is a ring under the given operations.

Many problems in math employ creativity in solving and may not always have a step-by-step way of solving them, like determining if an arbitrary set under certain operations is a ring or not. Is there a way to separate what can be solved via an algorithm and what cannot?

Enter the concept of effective calculability. An effectively calculable function is one for which an effective method exists to calculate the value of the function. Paraphrased from The Cambridge Dictionary of Philosophy, an effective method:

1. has a finite number of finite instructions

2. always finishes after a finite number of steps

3. always produces a correct answer

4. could be done by a human with a pencil and paper

5. requires no ingenuity to complete

Obviously this definition is not mathematically rigorous enough for us to use in a proof. We can, however, capture the essence in this definition by constructing special functions that are built on steps of an algorithm. These will capture ideas familiar to computer scientists and programmers, such as function "composition" and "loops", all of which are able to be done via pencil and paper by a sufficiently patient individual. These are not new definitions, and were formalized in the first half of the 1900s.

2 Primitive Recursive Functions

For this paper, $\mathbb{N}$ refers to the set $\{0\} \cup \mathbb{Z}^+ = \{0, 1, 2, 3, ...\}$

Definition. *A total function f from a set X to a set Y is a subset of the Cartesian product $X \times Y$ so that the following hold:*

$$\forall x \in X \; \exists y \in Y : (x, y) \in f$$
$$(x, y) \in f \wedge (x, z) \in f \implies y = z$$

All functions hereafter are total unless otherwise indicated.

Definition. *A function $f : X \to Y$ has arity n if $X = X_1 \times X_2 \times ... \times X_n$, the n-ary Cartesian product of sets $X_1, ..., X_n$. If X is not composed of tuples at all, then the f is unary.*

Definition. *The following functions from $\mathbb{N} \times ... \times \mathbb{N}$ to $\mathbb{N}$ are primitive recursive functions:*

1. *The unary constant function c:*

$$c(x) = 0$$

2. *The unary successor function s:*

$$s(x) = x + 1$$

3. *The n-ary projection function p_i:*

$$1 \leq i \leq n$$
$$p_i(x_1, ..., x_n) = x_i$$

4. *Function composition*
   Let f be an n-ary primitive recursive function and $g_1, g_2, ..., g_n$ all be m-ary primitive recursive functions. Then the m-ary composition h of f and $g_1, g_2, ..., g_n$ given by

$$h(x_1, x_2, ..., x_m) = f(g_1(x_1, x_2, ..., x_m), ..., g_n(x_1, x_2, ..., x_m))$$

   is a primitive recursive function

5. *Primitive recursion Let g be an n-ary primitive recursive function and f be an $(n+2)$-ary primitive recursive function. Then the $(n+1)$-ary primitive recursion h of f and g given by*

$$\begin{aligned} h(0, x_1, ..., x_n) &= g(x_1, ..., x_n) \\ h(s(x), x_1, ..., x_n) &= f(x, h(x, x_1, ..., x_n), x_1, ..., x_n) \end{aligned}$$

   is a primitive recursive function

Intuitively the primitive recursive functions are all operations that a human could do with pencil and paper. Besides the requirement that there are a finite number of steps, we have something that seems to adhere to the spirit of effective calculability. Note also that all primitive recursive functions are total. But are there total functions that could conceivably be performed by a human with a pencil and paper, and enumerated by steps that require no "ingenuity", that are not primitive recursive?

3 The Ackermann Function

Definition (The Ackermann Function)**.** *Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:*

$$A(n,m) = \begin{cases} m+1 & n = 0 \\ A(n-1, 1) & n > 0 \wedge m = 0 \\ A(n-1, A(n, m-1)) & n > 0 \wedge m > 0 \end{cases}$$

The Ackermann function looks simple, while lengthy, to compute with pencil and paper. It requires no ingenuity or creativity to compute. As we will see below, it is also a total function. But we will find a startling result with sufficient development of a series of proofs.

Theorem 1. *For any $n, m \in \mathbb{N}$, $A(n, m) \in \mathbb{N}$*

Proof. We proceed by double induction.
Clearly $A(0, k) \in \mathbb{N}$ for every $k \in \mathbb{N}$ because $k + 1 \in \mathbb{N}$.

Now assume $A(k, m) \in \mathbb{N}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Then $A(k+1, 0) = A(k + 1 - 1, 1) = A(k, 1) \in \mathbb{N}$.

Now assume $A(k + 1, j) \in \mathbb{N}$ for some $j \in \mathbb{N}$. Then $A(k + 1, j + 1) = A(k+1-1, A(k+1, j+1-1)) = A(k, A(k+1, j))$. By our second hypothesis $A(k + 1, j) \in \mathbb{N}$, so by our first hypothesis $A(k, A(k + 1, j)) \in \mathbb{N}$. Therefore $A(k + 1, j + 1) \in \mathbb{N}$. Then $A(k + 1, m) \in \mathbb{N}$ for every $m \in \mathbb{N}$, so $A(n, m) \in \mathbb{N}$ for all choices of $n, m \in \mathbb{N}$ $\square$

Theorem 2. *A is a total function*

Proof. By Theorem 1, all n, m in the domain of A have an image $A(n, m) \in \mathbb{N}$. Therefore A is a total function. $\square$

Theorem 3. *For any $m \in \mathbb{N}$, $A(1, m) = m + 2$*

Proof. We proceed by induction.

$A(1, 0) = A(1 - 1, 1) = A(0, 1) = 2 = 0 + 2$.

Now assume $A(1, k) = k + 2$ for some $k \in \mathbb{N}$. Then $A(1, k + 1) = A(1 - 1, A(1, k + 1 - 1)) = A(0, A(1, k)) = A(0, k + 2) = k + 3 = (k + 1) + 2$. $\square$

Theorem 4. *For every $n, m \in \mathbb{N}$, $m < A(n, m)$*

Proof. We proceed by double induction.

$A(0, 0) = 0 + 1 = 1 > 0$. Clearly $A(0, m) = m + 1 > m$ for every $m \in \mathbb{N}$.

Now assume $m < A(k, m)$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. $A(k + 1, 0) = A(k, 1)$. By our inductive hypothesis, $A(k, 1) > 1 > 0$, So $0 < A(k + 1, 0)$.

Now assume $j < A(k+1, j)$ for some $j \in \mathbb{N}$. Then $A(k+1, j+1) = A(k, A(k+1, j))$. By our first inductive hypothesis, $A(k + 1, j) < A(k, A(k + 1, j))$. By our second inductive hypothesis, $j < A(k + 1, j)$. So $j + 1 \leq A(k + 1, j) < A(k, A(k + 1, j)) = A(k + 1, j + 1)$. $\square$

Theorem 5. *For any $n, m, s \in \mathbb{N}$ where $m < s$, $A(n, m) < A(n, s)$*

Proof. Let $m, s, n \in \mathbb{N}$ Since $m < s$ and $m, s \in \mathbb{N}$, we have a $d = s - m$ where $d \in \mathbb{Z}^+$.

We will proceed by induction to first show $A(n, m) < A(n, m + 1)$.

$A(0, m) = m + 1 < m + 1 + 1 = A(0, m + 1)$.

Assume $A(k, m) < A(k, m+1)$ for some $k \in \mathbb{N}$. By Theorem 4, $A(k+1, m) < A(k, A(k + 1, m)) = A(k + 1, m + 1)$. So $A(k + 1, m) < A(k + 1, m + 1)$. Therefore $A(n, m) < A(n, m + 1)$.

Now we use this result to construct a chain of d inequalities $A(n, m) < A(n, m+1) < A(n, m+1+1) < ... < A(n, m+1+1+...+1) = A(n, m+d) = A(n, s)$. $\square$

Theorem 6. *For any $n, m \in \mathbb{N}$, $A(n, m + 1) \leq A(n + 1, m)$*

Proof. We proceed by induction.

$A(n, 0 + 1) = A(n, 1) = A(n + 1, 0)$ by the definition of the Ackermann function.

Now assume $A(n, k + 1) \leq A(n + 1, k)$ for some $k \in \mathbb{N}$. Note that $k + 1 < A(n, k + 1))$ by Theorem 4. So $k + 2 \leq A(n, k + 1)$. Then $A(n, k + 2) \leq A(n, A(n, k + 1))$ by Theorem 5. By our inductive hypothesis, $A(n, k + 1) \leq A(n+1, k)$, so by Theorem 5 again, $A(n, A(n, k+1)) \leq A(n, A(n+1, k))$. By the definition of the Ackermann function, $A(n, A(n + 1, k)) = A(n + 1, k + 1)$. $\square$

Theorem 7. *For any $n, m, s \in \mathbb{N}$ where $n < s$, $A(n, m) < A(s, m)$*

Proof. Let $n, s, m \in \mathbb{N}$. Since $n < s$ and $n, s \in \mathbb{N}$, we have a $d = s - n$ where $d \in \mathbb{Z}^+$. We begin by showing $A(n, m) < A(n + 1, m)$.

$A(n, m) < A(n, m + 1)$ by Theorem 5, and $A(n, m + 1) \leq A(n + 1, m)$ by Theorem 6. So $A(n, m) < A(n + 1, m)$.

Now we use this result to construct a chain of d inequalities $A(n, m) <$

$A(n+1,m) < A(n+1+1,m) < ... < A(n+1+1+...+1,m) = A(n+d,m) = A(s,m)$. $\square$

Theorem 8. *For any $n, m, s \in \mathbb{N}$, $A(n, A(s,m)) < A(n+s+2, m)$*

Proof. $A(n, A(s,m)) < A(n+s, A(s,m))$ by Theorem 7. $A(n+s, A(s,m)) < A(n+s, A(s+n+1,m))$ by Theorem 5 since, by Theorem 7, $A(s,m) < A(s+n+1,m)$.

$A(n+s, A(s+n+1,m)) = A(n+s, A(n+s+1,m)) = A(n+s+1, m+1)$ by the definition of the Ackermann function. Finally, $A(n+s+1, m+1) \leq A(n+s+1+1, m) = A(n+s+2, m)$ by Theorem 6. $\square$

Theorem 9. *For any $m \in \mathbb{N}$, $A(2,m) = 2m+3$*

Proof. We proceed by induction

$A(2,0) = A(1,1) = 1+2 = 3 = 0+3$ by Theorem 3.

Now assume $A(2,k) = 2k+3$ for some $k \in \mathbb{N}$. Then $A(2,k+1) = A(1, A(2,k)) = A(1, 2k+3)$ by Theorem 3. $A(1, 2k+3) = 2k+3+2 = 2k+5 = 2(k+1)+3$ also by Theorem 3. $\square$

Definition. *Let P be a set all primitive recursive functions so that if $f \in P$, there is a $t \in \mathbb{N}$ such that for any $x_1, ..., x_n \in \mathbb{N}$, $f(x_1, ..., x_n) < A(t, max\{x_1, ..., x_n\})$.*

Theorem 10. $c(x)$, $s(x)$, $p_i(x_1, x_2, ..., x_n) \in P$

Proof. Let $m = max\{x_1, ..., x_n\}$. Then

$$c(x) = 0 < x+1 = A(0,x)$$
$$s(x) = x+1 < x+2 = A(1,x)$$
$$p_i(x_1, x_2, ..., x_n) = x_i \leq m < m+1 = A(0,m)$$

$\square$

Theorem 11. *P is closed under composition*

Proof. Let $f, g_1, g_2, ..., g_k \in P$, where f is k-ary and each g_i is j-ary. Let $x_1, x_2, ..., x_j \in \mathbb{N}$. Let $m = max\{x_1, x_2, ..., x_j\}$. Let h be the j-ary primitive recursive function that results from function composition of f with

$g_1, g_2, ..., g_k$. Let g_{max} be the g_i giving the maximum value in $max\{g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)\}$. Let $m_g = g_{max}(x_1, ..., x_j)$.

Since each $g_i \in P$, there is some t_i for each g_i such that $g_i(x_1, ..., x_j) < A(t_i, m)$. Take $t_g = max\{t_1, t_2, ..., t_k\}$. Note this is not dependent on $x_1, ..., x_j$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1, ..., x_j) = f(g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)) < A(t_f, m_g)$. Note this is also not dependent on $x_1, ..., x_j$.

But since $m_g < A(t_g, m)$, by Theorem 5 $A(t_f, m_g) < A(t_f, A(t_g, m))$. By Theorem 8, $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$. Let $t = t_f + t_g + 2 \in \mathbb{N}$. Then $h(x_1, .., x_j) < A(t, m)$. So $h \in P$. □

Lemma 1. *Let h be the primitive recursion of two functions $f, g \in P$. Then there exists a $t \in \mathbb{N}$ such that for every $x, x_1, ..., x_n \in \mathbb{N} h(x, x_1, ..., x_n) < A(t, x + max\{x_1, ..., x_n\})$.*

Proof. Since $f, g \in P$, there exist t_f and t_g so that $f(x, y, x_1, ..., x_n) < A(t_f, max\{x, y, x_1, ..., x_n\})$ and $g(x_1, ..., x_n) < A(t_g, max\{x_1, ..., x_n\})$ for any arguments of f and g. Let $t = 1 + max\{t_f, t_g\}$. Note that t is not dependent on $x, x_1, ..., x_n$. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$.

We proceed by induction.

$h(0, x_1, ..., x_n) = g(x_1, ..., x_n) < A(t_g, m) < A(t, m) = A(t, m + 0)$.

Now assume $h(k, x_1, ..., x_n) < A(t, max\{m, k\})$ for some $k \in \mathbb{N}$. Let $m_{k+1} = max\{k, h(k, x_1, ..., x_n), m\}$. Then $h(k+1, x_1, ..., x_n) = f(k, h(k, x_1, ..., x_n), x_1, ..., x_n) < A(t_f, m_{k+1})$.

Now note that $m_{k+1} < A(t, k + m)$:

1. $m \leq k + m < A(t, k + m)$ by Theorem 4
2. $k \leq k + m < A(t, k + m)$ by Theorem 4
3. $h(k, x_1, ..., x_n) < A(t, max\{m, k\})$ by our induction hypothesis, and $A(t, max\{m, k\}) \leq A(t, k+m)$ by Theorem 5 since $max\{m, k\} \leq k+m$

By Theorem 5, $A(t_f, m_{k+1}) < A(t_f, A(t, k + m))$. Since $t = 1 + max\{t_f, t_g\}$, $t_f \leq t - 1$. Then by Theorem 5 and since $t \neq 0$, $A(t_f, A(t, k + m)) \leq A(t -$

$1, A(t, k+m))$. By the definition of the Ackermann function, $A(t-1, A(t, k+m)) = A(t, k+m+1)$. Therefore $h(k+1, x_1, ..., x_n) < A(t, (k+1)+m)$ □

Theorem 12. *P is closed under primitive recursion*

Proof. Let $f, g \in P$ and h be the primitive recursion of f and g. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$.

Then by Lemma 1, there is a $t \in \mathbb{N}$ so that $h(x, x_1, ..., x_n) < A(t, x+m)$.

Let $m' = max\{x, m\}$. Since $x + m \leq 2m'$, $A(t, x+m) \leq A(t, 2m')$ by Theorem 5. Since $2m' < 2m' + 3$, $A(t, 2m') < A(t, 2m'+3)$ also by Theorem 5. By Theorem 9, $2m'+3 = A(2, m')$. Therefore $A(t, 2m'+3) = A(t, A(2, m'))$. By Theorem 8, $A(t, A(2, m')) < A(t+2+2, m') = A(t+4, m')$. Therefore $h(x, x_1, ..., x_n) < A(t+4, m')$. □

Theorem 13. *P is precisely the primitive recursive functions*

Proof. All $f \in P$ are primitive recursive by the definition of P, so it remains to show all primitive recursive functions are in P.

Let f be a primitive recursive function. Then f can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in P, and P is closed under function composition and primitive recursion. Therefore $f \in P$. □

Theorem 14. *$A(m, n)$ is not a primitive recursive function*

Proof. Suppose that A is a primitive recursive function. Then $A \in P$. Then there exists a $t \in \mathbb{N}$ so that for any $n, m \in \mathbb{N}$, $A(n, m) < A(t, max\{n, m\})$.

Set $m = n = t$. Then $max\{n, m\} = n = m = t$. So $A(n, m) = A(t, t) < A(t, t)$ which is a contradition. Therefore A is not primitive recursive. □

A is not a primitive recursive function! In fact, our rigorous enumeration of calculable functions is incomplete. We need one more operator to capture the Ackermann function. For the curious reader we will enumerate the general recursive functions, but it is beyond the scope of this paper to show the Ackermann function is a general recursive function.

4 General Recursive Functions

4.1 Partial Functions

Definition. *A partial function f from a set X to a set Y is a subset of the Cartesian product $X \times Y$ so that $(x, y) \in f \wedge (x, y) \implies y = z$.*

Note that all total functions are partial functions, but not all partial functions are total functions because a partial function f may be undefined for some $x \in X$.

4.2 Definition of General Recursive Functions

Definition (Minimization)**.** *Let $f : \mathbb{N}^n \to \mathbb{N}$ be an n-ary partial function. Then the minimization operator μ creates a $(n-1)$-ary function $\mu(f)$ given by*

$$\mu(f)(x_1, ..., x_n) = \begin{cases} z & f(z, x_1, ..., x_n) = 0 \wedge f(i, x_1, ..., x_n) > 0 \forall i \in \mathbb{N} \cap [0, z) \\ undefined & f(i, x_1, ..., x_n) \neq 0 \forall i \in \mathbb{N} \end{cases}$$

Definition. *A function f is a general recursive function if it is either:*

1. *A primitive recursive function*
2. *A general recursive function under the minimization operator*

That the Ackermann function is a general recursive function is beyond the scope of this paper.

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