Computability via Recursive Functions

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1 Effective Calcubility and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$

Definition. *The following functions from $\mathbb{N} \oplus ... \oplus \mathbb{N}$ to $\mathbb{N}$ are primitive recursive functions:*

1. *The unary constant function c:*

$$c(x) = 0$$

2. *The unary successor function s:*

$$s(x) = x + 1$$

3. *The n-ary projection function p:*

$$1 \leq i \leq n$$
$$p_i(x_1, ..., x_n) = x_i$$

4. *Function composition*
Let f be an n-ary primitive recursive function and $g_1, g_2, ..., g_n$ all be m-ary primitive recursive functions. Then the m-ary composition h of f and $g_1, g_2, ..., g_n$ given by

$$h(x_1, x_2, ..., x_m) = f(g_1(x_1, x_2, ..., x_m), ..., g_n(x_1, x_2, ..., x_m))$$

is a primitive recursive function

5. *Primitive recursion Let g be an n-ary primitive recursive function and f be an $(n+2)$-ary primitive recursive function. Then the $(n+1)$-ary primitive recursion h of f and g given by*

$$h(0, x_1, ..., x_n) = g(x_1, ..., x_n)$$
$$h(s(x), x_1, ..., x_n) = f(x, h(x, x_1, ..., x_n), x_1, ..., x_n)$$

is a primitive recursive function

3 The Ackermann Function

Definition (The Ackermann Function)**.** *Let $n, m \in \mathbb{N}$. Then define $A(n,m)$ as follows:*

$$A(n,m) = \begin{cases} m+1 & n = 0 \\ A(n-1, 1) & n > 0 \wedge m = 0 \\ A(n-1, A(n, m-1)) & n > 0 \wedge m > 0 \end{cases}$$

Theorem 1. *For any $n, m \in \mathbb{N}, A(n,m) \in \mathbb{N}$*

Proof. We proceed by double induction.
Clearly $A(0,k) \in \mathbb{N}$ for every $k \in \mathbb{N}$ because $k+1 \in \mathbb{N}$. Now assume $A(k,m) \in \mathbb{N}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Then $A(k+1, 0) = A(k+1-1, 1) = A(k,1) \in \mathbb{N}$. Now assume $A(k+1, j) \in \mathbb{N}$ for some $j \in \mathbb{N}$. Then $A(k+1, j+1) = A(k+1-1, A(k+1, j+1-1)) = A(k, A(k+1, j))$. By our second hypothesis $A(k+1, j) \in \mathbb{N}$, so by our first hypothesis $A(k, A(k+1, j)) \in \mathbb{N}$. Therefore $A(k+1, j+1) \in \mathbb{N}$. Then $A(k+1, m) \in \mathbb{N}$ for every $m \in \mathbb{N}$, so $A(n,m) \in \mathbb{N}$ for all choices of $n, m \in \mathbb{N}$ □

Theorem 2. *A(m, n) is a total function*

Proof. We will proceed inductively to show that $A(m,n)$ is defined for all $m, n \in \mathbb{N}$.

Clearly $A(0,n)$ is defined for all $n \in \mathbb{N}$. Assume $A(k,n)$ is defined for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$. Since $k+1 > 0$, $A(k+1, 0) = A(k, 1)$, which is defined.

Now we assume $A(k+1, j)$ is defined for some $j \in \mathbb{N}$. By Lemma 1,

$A(k+1,j) = a$ for some $a \in \mathbb{N}$. Then since $j+1 > 0$, $A(k+1,j+1) = A(k, A(k+1,j)) = A(k,a)$. Since $A(k,n)$ is defined for every $n \in \mathbb{N}$ by our inductive hypothesis, $A(k,a) = A(k+1,j+1)$ is defined. □

Theorem 3. *For any $m,n,s \in \mathbb{N}$ where $s > n$, $A(m,n) < A(m,s)$*

Proof. Use the proof of A(m, n) ¡ A(m, n + 1) □

Theorem 4. *For any $m,n,s \in \mathbb{N}$ where $s > m$, $A(m,n) < A(s,n)$*

Proof. □

Theorem 5. *For any $m,n,s \in \mathbb{N}$, $A(m, A(s,n)) < A(m+s+2, n)$*

Proof. Proof here □

Theorem 6. *For any $x \in \mathbb{N}$, $x+2 = A(1,x)$*

Proof. We proceed by induction.
$A(1,0) = A(1-1,1) = A(0,1) = 2 = 0+2$. Now assume $A(1,k) = k+2$ for some $k \in \mathbb{N}$. Then $A(1,k+1) = A(1-1, A(1,k+1-1)) = A(0, A(1,k)) = A(0,k+2) = k+3 = (k+1)+2$. □

Definition. *Let P be a set all primitive recursive functions f so that if $f \in P$, there is a $t \in \mathbb{N}$ such that for any $x_1, ..., x_n \in \mathbb{N}$, $f(x_1,...,x_n) < A(t, max\{x_1,...,x_n\})$.*

Theorem 7. $c(x)$, $s(x)$, $p_i(x_1,x_2,...,x_n) \in P$

Proof. Let $m = max\{x_1,...,x_n\}$. Then

$$\begin{aligned} c(x) = 0 < x+1 = A(0,x) \\ s(x) = x+1 < x+2 = A(1,x) \\ p_i(x_1,x_2,...,x_n) = x_i \leq m < m+1 = A(0,m) \end{aligned}$$

□

Theorem 8. *P is closed under composition*

Proof. Let $f, g_1, g_2, ..., g_k \in P$, where f is k-ary and each g_i is j-ary. Let $x_1, x_2, ..., x_j \in \mathbb{N}$. Let $m = max\{x_1, x_2, ..., x_j\}$. Let h be the j-ary primitive recursive function that results from function composition of f with $g_1, g_2, ..., g_k$. Let g_{max} be the g_i giving the maximum value in $max\{g_1(x_1,...,x_j), ..., g_k(x_1,...,x_j)\}$.

Let $m_g = g_{max}(x_1, ..., x_j)$ Since $g_{max} \in P$, there exists some $t_g \in \mathbb{N}$ so that $m_g < A(t_g, m)$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1, ..., x_j) = f(g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)) < A(t_f, m_g)$. But since $m_g < A(t_g, m)$, by Theorem 3 $A(t_f, m_g) < A(t_f, A(t_g, m))$. By Theorem 5, $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$. Let $t = t_f + t_g + 2 \in \mathbb{N}$. Then $h(x_1, .., x_j) < A(t, m)$. So $h \in P$. □

Lemma 1. *Let h be the primitive recursion of two functions $f, g \in P$. Let $x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$. Then there exists a $t \in \mathbb{N}$ such that for every $x \in \mathbb{N} h(x, x_1, ..., x_n) < A(t, x + m)$.*

Proof. Since $f, g \in P$, there exist t_f and t_g so that for every $x \in \mathbb{N}$, $f(x, h(x, x_1, ..., x_n), x_1, ..., x_n) < A(t_f, max\{x, h(x, x_1, ..., x_n), x_1, ..., x_n\})$ and $g(x_1, ..., x_n) < A(t_g, max\{x_1, ..., x_n\})$. Choose $t =$ We proceed by induction. oh man I have no idea where to continue on this one $h(0, x_1, ..., x_n) = g(x_1, ..., x_n) <$ □

Theorem 9. *P is closed under primitive recursion*

Proof. Let $f, g \in P$ and h be the primitive recursion of f and g. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$. Then by Lemma 1, there is a $t \in \mathbb{N}$ so that $h(x, x_1, ..., x_n) < A(t, x + m)$. Let $m' = max\{x, x_1, ..., x_n\} = max\{x, m\}$. Since $x + m \leq m'$, $A(t, x + m) \leq A(t, 2m')$ by Lemma 3. Since $2m' < 2m' + 3$, $A(t, 2m') < A(t, 2m' + 3)$ also by Lemma 3.

By Lemma ?, $2m' + 3 = A(2, m')$, so $A(t, 2m' + 3) = A(t, A(2, m'))$. By Theorem 5, $A(t, A(2, m')) = A(t+2+2, m') = A(t+4, m')$. So $h(x, x_1, ..., x_n) < A(t + 4, m')$.

Therefore $h \in P$. □

Theorem 10. *P is precisely the primitive recursive functions*

Proof. All $f \in P$ are primitive recursive by the definition of P. So it remains to show all primitive recursive functions are in P. Let f be a primitive recursive function. Then f can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in P, and P is closed under function composition and primitive recursion. Therefore $f \in P$. □

Theorem 11. *$A(m, n)$ is not a primitive recursive function*

Proof. Suppose that A is a primitive recursive function. Then $A \in P$. Then there exists a $t \in \mathbb{N}$ so that for any $m, n \in \mathbb{N}$, $A(m,n) < A(t, max\{m,n\})$. Set $m = n = t$. Then $max\{m,n\} = m = n = t$. So $A(m,n) = A(t,t) < A(t,t)$ which is a contradition. Therefore A is not primitive recursive. $\square$

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions