Computability via Recursive Functions

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March 2020

1 Effective Calcubility and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$

Definition. *The following functions from $\mathbb{N} \oplus ... \oplus \mathbb{N}$ to $\mathbb{N}$ are primitive recursive functions:*

1. *The unary constant function c:*

$$c(x) = 0$$

2. *The unary successor function s:*

$$s(x) = x + 1$$

3. *The n-ary projection function p:*

$$1 \leq i \leq n$$
$$p(x_1, ..., x_n) = x_i$$

4. *Function composition*
Let f be an n-ary primitive recursive function and $g_1, g_2, ..., g_n$ all be m-ary primitive recursive functions. Then the m-ary composition h of f and $g_1, g_2, ..., g_n$ given by

$$h(x_1, x_2, ..., x_m) = f(g_1(x_1, x_2, ..., x_m), ..., g_n(x_1, x_2, ..., x_m))$$

is a primitive recursive function

5. *Primitive recursion Let g be an n-ary primitive recursive function and f be an $(n+2)$-ary primitive recursive function. Then the $(n+1)$-ary primitive recursion h of f and g given by*

$$h(0, x_1, ..., x_n) = g(x_1, ..., x_n)$$
$$h(s(x), x_1, ..., x_n) = f(x, h(x, x_1, ..., x_n), x_1, ..., x_n)$$

is a primitive recursive function

2.2

3 The Ackermann Function

Definition (The Ackermann Function)**.** *Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:*

$$A(m,n) = \begin{cases} n+1 & m = 0 \\ A(m-1, 1) & m > 0 \land n = 0 \\ A(m-1, A(m, n-1)) & m > 0 \land n > 0 \end{cases}$$

Lemma 1. *For any $m, n \in \mathbb{N}, A(m, n) \in \mathbb{N}$*

Proof. Proof Here □

Theorem 1. *A(m, n) is a total function*

Proof. We will proceed inductively to show that $A(m, n)$ is defined for all $m, n \in \mathbb{N}$.

Clearly $A(0, n)$ is defined for all $n \in \mathbb{N}$. Assume $A(k, n)$ is defined for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$. Since $k + 1 > 0$, $A(k + 1, 0) = A(k, 1)$, which is defined.

Now we assume $A(k + 1, j)$ is defined for some $j \in \mathbb{N}$. By Lemma 1, $A(k + 1, j) = a$ for some $a \in \mathbb{N}$. Then since $j + 1 > 0$, $A(k + 1, j + 1) = A(k, A(k + 1, j)) = A(k, a)$. Since $A(k, n)$ is defined for every $n \in \mathbb{N}$ by our inductive hypothesis, $A(k, a) = A(k + 1, j + 1)$ is defined. □

Theorem 2. *For any $m, n, s \in \mathbb{N}$ where $s > n$, $A(m, n) < A(m, s)$*

Proof. Use the proof of A(m, n) ¡ A(m, n + 1) □

Theorem 3. *For any* $m, n, s \in \mathbb{N}$, $A(m, A(s, n)) < A(m + s + 2, n)$

Proof. Proof here □

Definition. *Let* P *be the set of all primitive recursive functions so that if* $f(x_1, x_2, ..., x_n) \in P$ *and* $m = max\{x_1, x_2, ..., x_n\}$*, then there exists* $t \in \mathbb{N}$ *so that* $f(x_1, x_2, ..., x_n) < A(t, m)$

Theorem 4. $c(x)$, $s(x)$, $p_i(x_1, x_2, ..., x_n) \in P$

Proof.

$$c(x) = 0 < x + 1 = A(0, x)$$
$$s(x) = x + 1 < x + 2 = A(1, x)$$
$$p_i(x_1, x_2, ..., x_n) = x_i \leq m < m + 1 = A(0, m)$$

To verify $x + 2 = A(1, x)$, we proceed by induction.
$A(1, 0) = A(1 - 1, 1) = A(0, 1) = 2 = 0 + 2$. Now assume $A(1, k) = k + 2$ for some $k \in \mathbb{N}$. Then $A(1, k + 1) = A(1 - 1, A(1, k + 1 - 1)) = A(0, A(1, k)) = A(0, k + 2) = k + 3 = (k + 1) + 2$. □

Theorem 5. P *is closed under composition*

Proof. Let $f, g_1, g_2, ..., g_k \in P$, where f is k-ary and each g_i is j-ary. Let $x_1, x_2, ..., x_j \in \mathbb{N}$. Let $m = max\{x_1, x_2, ..., x_j\}$. Let h be the j-ary primitive recursive function that results from function composition of f with $g_1, g_2, ..., g_k$. Let g_{max} be the g_i giving the maximum value in $max\{g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)\}$. Let $m_g = g_{max}(x_1, ..., x_j)$ Since $g_{max} \in P$, there exists some $t_g \in \mathbb{N}$ so that $m_g < A(t_g, m)$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1, ..., x_j) = f(g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)) < A(t_f, m_g)$. But since $m_g < A(t_g, m)$, by Theorem 2 $A(t_f, m_g) < A(t_f, A(t_g, m))$. By Theorem 3, $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$. Let $t = t_f + t_g + 2 \in \mathbb{N}$. Then $h(x_1, .., x_j) < A(t, m)$. So $h \in P$. □

Theorem 6. P *is closed under primitive recursion*

Proof. □

Theorem 7. P *is precisely the primitive recursive functions*

Proof. □

Theorem 8. $A(m, n)$ *is not a primitive recursive function*

Proof. Proof Here □

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions