Computability via Recursive Functions

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1 Effective Calcubility and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$

Definition. *The following functions from $\mathbb{N} \oplus ... \oplus \mathbb{N}$ to $\mathbb{N}$ are primitive recursive functions:*

1. *The unary constant function c:*

$$c(x) = 0$$

2. *The unary successor function s:*

$$s(x) = x + 1$$

3. *The n-ary projection function p:*

$$1 \leq i \leq n$$
$$p_i(x_1, ..., x_n) = x_i$$

4. *Function composition*
Let f be an n-ary primitive recursive function and $g_1, g_2, ..., g_n$ all be m-ary primitive recursive functions. Then the m-ary composition h of f and $g_1, g_2, ..., g_n$ given by

$$h(x_1, x_2, ..., x_m) = f(g_1(x_1, x_2, ..., x_m), ..., g_n(x_1, x_2, ..., x_m))$$

is a primitive recursive function

5. *Primitive recursion Let g be an n-ary primitive recursive function and f be an $(n+2)$-ary primitive recursive function. Then the $(n+1)$-ary primitive recursion h of f and g given by*

$$\begin{aligned} h(0, x_1, ..., x_n) &= g(x_1, ..., x_n) \\ h(s(x), x_1, ..., x_n) &= f(x, h(x, x_1, ..., x_n), x_1, ..., x_n) \end{aligned}$$

is a primitive recursive function

3 The Ackermann Function

Definition (The Ackermann Function)**.** *Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:*

$$A(n,m) = \begin{cases} m+1 & n = 0 \\ A(n-1, 1) & n > 0 \wedge m = 0 \\ A(n-1, A(n, m-1)) & n > 0 \wedge m > 0 \end{cases}$$

Theorem 1. *For any $n, m \in \mathbb{N}, A(n, m) \in \mathbb{N}$*

Proof. We proceed by double induction.
Clearly $A(0, k) \in \mathbb{N}$ for every $k \in \mathbb{N}$ because $k + 1 \in \mathbb{N}$. Now assume $A(k, m) \in \mathbb{N}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Then $A(k+1, 0) = A(k+1-1, 1) = A(k, 1) \in \mathbb{N}$. Now assume $A(k+1, j) \in \mathbb{N}$ for some $j \in \mathbb{N}$. Then $A(k+1, j+1) = A(k+1-1, A(k+1, j+1-1)) = A(k, A(k+1, j))$. By our second hypothesis $A(k+1, j) \in \mathbb{N}$, so by our first hypothesis $A(k, A(k+1, j)) \in \mathbb{N}$. Therefore $A(k+1, j+1) \in \mathbb{N}$. Then $A(k+1, m) \in \mathbb{N}$ for every $m \in \mathbb{N}$, so $A(n, m) \in \mathbb{N}$ for all choices of $n, m \in \mathbb{N}$ □

Theorem 2. *A is a total function*

Proof. By Theorem 1, all n, m in the domain of A have an image $A(n, m) \in \mathbb{N}$. Therefore A is a total function. □

Theorem 3. *For any $x \in \mathbb{N}$, $A(1, x) = x + 2$*

Proof. We proceed by induction.
$A(1, 0) = A(1-1, 1) = A(0, 1) = 2 = 0 + 2$. Now assume $A(1, k) = k + 2$ for some $k \in \mathbb{N}$. Then $A(1, k+1) = A(1-1, A(1, k+1-1)) = A(0, A(1, k)) = A(0, k+2) = k + 3 = (k+1) + 2$. □

Lemma 1. *For every $n, m \in \mathbb{N}$, $m < A(n,m)$*

Proof. We proceed by induction. Let $m \in \mathbb{N}$. $A(0,m) = m+1 > m$, so $m < A(0,m)$. Now assume $m < A(k,m)$ for some $k \in \mathbb{N}$. Then $A(k+1,m) =$ Not done with this one yet □

Theorem 4. *For any $n, m, s \in \mathbb{N}$ where $m < s$, $A(n,m) < A(n,s)$*

Proof. Let $m, s, n \in \mathbb{N}$ Since $m < s$ and $m, s \in \mathbb{N}$, we have a $d = s - m$ where $d \in \mathbb{Z}^+$. We will proceed by induction to first show $A(n,m) < A(n,m+1)$. $A(0,m) = m+1 < m+1+1 = A(0,m+1)$. Assume $A(k,m) < A(k,m+1)$ for some $k \in \mathbb{N}$. By Lemma 1, $A(k+1,m) < A(k, A(k+1,m)) = A(k+1,m+1)$. So $A(k+1,m) < A(k+1,m+1)$. Therefore $A(n,m) < A(n,m+1)$.

Now we use this result to construct a chain of d inequalities $A(n,m) < A(n,m+1) < A(n,m+1+1) < ... < A(n,m+1+1+...+1) = A(n,m+d) = A(n,s)$. □

Theorem 5. *For any $n, m \in \mathbb{N}$, $A(n,m+1) \leq A(n+1,m)$*

Proof. We proceed by induction. $A(n,0+1) = A(n,1) = A(n+1,0)$ by the definition of the Ackermann function. Now assume $A(n,k+1) \leq A(n+1,k)$ for some $k \in \mathbb{N}$. Note that $k+1 < A(n,k+1))$ by Theorem 1. So $k+2 \leq A(n,k+1)$. Then $A(n,k+2) \leq A(n, A(n,k+1))$ by Theorem 4. By our inductive hypothesis, $A(n,k+1) \leq A(n+1,k)$, so by Theorem 4 again, $A(n, A(n,k+1)) \leq A(n, A(n+1,k))$. By the definition of the Ackermann function, $A(n, A(n+1,k)) = A(n+1,k+1)$. □

Theorem 6. *For any $n, m, s \in \mathbb{N}$ where $n < s$, $A(n,m) < A(s,m)$*

Proof. Let $n, s, m \in \mathbb{N}$. Since $n < s$ and $n, s \in \mathbb{N}$, we have a $d = s - n$ where $d \in \mathbb{Z}^+$. We begin by showing $A(n,m) < A(n+1,m)$. $A(n,m) < A(n,m+1)$ by Theorem 4, and $A(n,m+1) \leq A(n+1,m)$ by Theorem 5. So $A(n,m) < A(n+1,m)$. Now we use this result to construct a chain of d inequalities $A(n,m) < A(n+1,m) < A(n+1+1,m) < ... < A(n+1+1+...+1,m) = A(n+d,m) = A(s,m)$. □

Theorem 7. *For any $n, m, s \in \mathbb{N}$, $A(n, A(s,m)) < A(n+s+2,m)$*

Proof. $A(n, A(s,m)) < A(n+s, A(s,m))$ by Theorem 6. $A(n+s, A(s,m)) < A(n+s, A(s+n+1,m))$ by Theorem 4 since, by Theorem 6, $A(s,m) <$

$A(s+n+1,m)$. $A(n+s,A(s+n+1,m)) = A(n+s,A(n+s+1,m)) = A(n+s+1,m+1)$ by the definition of the Ackermann function. Finally, $A(n+s+1,m+1) \leq A(n+s+1+1,m) = A(n+s+2,m)$ by Theorem 5. □

Definition. *Let P be a set all primitive recursive functions f so that if $f \in P$, there is a $t \in \mathbb{N}$ such that for any $x_1,...,x_n \in \mathbb{N}$, $f(x_1,...,x_n) < A(t, max\{x_1,...,x_n\})$.*

Theorem 8. *$c(x)$, $s(x)$, $p_i(x_1,x_2,...,x_n) \in P$*

Proof. Let $m = max\{x_1,...,x_n\}$. Then

$$\begin{aligned} c(x) = 0 < x+1 = A(0,x) \\ s(x) = x+1 < x+2 = A(1,x) \\ p_i(x_1,x_2,...,x_n) = x_i \leq m < m+1 = A(0,m) \end{aligned}$$

□

Theorem 9. *P is closed under composition*

Proof. Let $f, g_1, g_2, ..., g_k \in P$, where f is k-ary and each g_i is j-ary. Let $x_1, x_2, ..., x_j \in \mathbb{N}$. Let $m = max\{x_1,x_2,...,x_j\}$. Let h be the j-ary primitive recursive function that results from function composition of f with $g_1, g_2, ..., g_k$. Let g_{max} be the g_i giving the maximum value in $max\{g_1(x_1,...,x_j),...,g_k(x_1,...,x_j)\}$. Let $m_g = g_{max}(x_1,...,x_j)$ Since $g_{max} \in P$, there exists some $t_g \in \mathbb{N}$ so that $m_g < A(t_g,m)$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1,...,x_j) = f(g_1(x_1,...,x_j),...,g_k(x_1,...,x_j)) < A(t_f,m_g)$. But since $m_g < A(t_g,m)$, by Theorem 4 $A(t_f,m_g) < A(t_f,A(t_g,m))$. By Theorem 7, $A(t_f,A(t_g,m)) < A(t_f+t_g+2,m)$. Let $t = t_f+t_g+2 \in \mathbb{N}$. Then $h(x_1,..,x_j) < A(t,m)$. So $h \in P$. □

Lemma 2. *Let h be the primitive recursion of two functions $f, g \in P$. Let $x_1,...,x_n \in \mathbb{N}$. Let $m = max\{x_1,...,x_n\}$. Then there exists a $t \in \mathbb{N}$ such that for every $x \in \mathbb{N} h(x,x_1,...,x_n) < A(t,x+m)$.*

Proof. Since $f, g \in P$, there exist t_f and t_g so that for every $x \in \mathbb{N}$, $f(x,h(x,x_1,...,x_n),x_1,...,x_n) < A(t_f, max\{x,h(x,x_1,...,x_n),x_1,...,x_n\})$ and $g(x_1,...,x_n) < A(t_g, max\{x_1,...,x_n\})$. Choose $t =$ We proceed by induction. oh man I have no idea where to continue on this one $h(0,x_1,...,x_n) = g(x_1,...,x_n) <$ □

Theorem 10. *P is closed under primitive recursion*

Proof. Let $f, g \in P$ and h be the primitive recursion of f and g. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$. Then by Lemma 2, there is a $t \in \mathbb{N}$ so that $h(x, x_1, ..., x_n) < A(t, x+m)$. Let $m' = max\{x, x_1, ..., x_n\} = max\{x, m\}$. Since $x + m \leq m'$, $A(t, x+m) \leq A(t, 2m')$ by Lemma 4. Since $2m' < 2m' + 3$, $A(t, 2m') < A(t, 2m' + 3)$ also by Lemma 4.

By Lemma ?, $2m' + 3 = A(2, m')$, so $A(t, 2m' + 3) = A(t, A(2, m'))$. By Theorem 7, $A(t, A(2, m')) = A(t+2+2, m') = A(t+4, m')$. So $h(x, x_1, ..., x_n) < A(t+4, m')$.

Therefore $h \in P$. □

Theorem 11. *P is precisely the primitive recursive functions*

Proof. All $f \in P$ are primitive recursive by the definition of P. So it remains to show all primitive recursive functions are in P. Let f be a primitive recursive function. Then f can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in P, and P is closed under function composition and primitive recursion. Therefore $f \in P$. □

Theorem 12. *$A(m, n)$ is not a primitive recursive function*

Proof. Suppose that A is a primitive recursive function. Then $A \in P$. Then there exists a $t \in \mathbb{N}$ so that for any $n, m \in \mathbb{N}$, $A(n, m) < A(t, max\{n, m\})$. Set $m = n = t$. Then $max\{n, m\} = n = m = t$. So $A(n, m) = A(t, t) < A(t, t)$ which is a contradition. Therefore A is not primitive recursive. □

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions

Definition (Minimization)**.** *Let $f : \mathbb{N}^n \to \mathbb{N}$ be an n-ary partial function. Then the minimization operator μ creates a $(n-1)$-ary function $\mu(f)$ given by*

$$\mu(f)(x_1, ..., x_n) = \begin{cases} z & f(z, x_1, ..., x_n) = 0 \wedge f(i, x_1, ..., x_n) > 0 \forall i \in \mathbb{N} \cap [0, z) \\ undefined & f(i, x_1, ..., x_n) \neq 0 \forall i \in \mathbb{N} \end{cases}$$

Definition. *A function f is a general recursive function if it is either:*

1. *A primitive recursive function*
2. *A general recursive function under the minimization operator*