

# Computability via Recursive Functions

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## 1 Effective Calculability and Computability

## 2 Primitive Recursive Functions

### 2.1 Functions

For this paper,  $\mathbb{N}$  refers to the set  $\{0, 1, 2, 3, \dots\}$

**Definition.** *The following functions from  $\mathbb{N} \oplus \dots \oplus \mathbb{N}$  to  $\mathbb{N}$  are primitive recursive functions:*

1. *The unary constant function  $c$ :*

$$c(x) = 0$$

2. *The unary successor function  $s$ :*

$$s(x) = x + 1$$

3. *The  $n$ -ary projection function  $p$ :*

$$1 \leq i \leq n$$
$$p_i(x_1, \dots, x_n) = x_i$$

4. *Function composition*

*Let  $f$  be an  $n$ -ary primitive recursive function and  $g_1, g_2, \dots, g_n$  all be  $m$ -ary primitive recursive functions. Then the  $m$ -ary composition  $h$  of  $f$  and  $g_1, g_2, \dots, g_n$  given by*

$$h(x_1, x_2, \dots, x_m) = f(g_1(x_1, x_2, \dots, x_m), \dots, g_n(x_1, x_2, \dots, x_m))$$

*is a primitive recursive function*

5. *Primitive recursion* Let  $g$  be an  $n$ -ary primitive recursive function and  $f$  be an  $(n + 2)$ -ary primitive recursive function. Then the  $(n + 1)$ -ary primitive recursion  $h$  of  $f$  and  $g$  given by

$$\begin{aligned} h(0, x_1, \dots, x_n) &= g(x_1, \dots, x_n) \\ h(s(x), x_1, \dots, x_n) &= f(x, h(x, x_1, \dots, x_n), x_1, \dots, x_n) \end{aligned}$$

is a primitive recursive function

### 3 The Ackermann Function

**Definition** (The Ackermann Function). Let  $n, m \in \mathbb{N}$ . Then define  $A(n, m)$  as follows:

$$A(n, m) = \begin{cases} m + 1 & n = 0 \\ A(n - 1, 1) & n > 0 \wedge m = 0 \\ A(n - 1, A(n, m - 1)) & n > 0 \wedge m > 0 \end{cases}$$

**Theorem 1.** For any  $n, m \in \mathbb{N}$ ,  $A(n, m) \in \mathbb{N}$

*Proof.* We proceed by double induction.

Clearly  $A(0, k) \in \mathbb{N}$  for every  $k \in \mathbb{N}$  because  $k + 1 \in \mathbb{N}$ . Now assume  $A(k, m) \in \mathbb{N}$  for some  $k \in \mathbb{N}$  for every  $m \in \mathbb{N}$ . Then  $A(k + 1, 0) = A(k + 1 - 1, 1) = A(k, 1) \in \mathbb{N}$ . Now assume  $A(k + 1, j) \in \mathbb{N}$  for some  $j \in \mathbb{N}$ . Then  $A(k + 1, j + 1) = A(k + 1 - 1, A(k + 1, j + 1 - 1)) = A(k, A(k + 1, j))$ . By our second hypothesis  $A(k + 1, j) \in \mathbb{N}$ , so by our first hypothesis  $A(k, A(k + 1, j)) \in \mathbb{N}$ . Therefore  $A(k + 1, j + 1) \in \mathbb{N}$ . Then  $A(k + 1, m) \in \mathbb{N}$  for every  $m \in \mathbb{N}$ , so  $A(n, m) \in \mathbb{N}$  for all choices of  $n, m \in \mathbb{N}$   $\square$

**Theorem 2.**  $A$  is a total function

*Proof.* By Theorem 1, all  $n, m$  in the domain of  $A$  have an image  $A(n, m) \in \mathbb{N}$ . Therefore  $A$  is a total function.  $\square$

**Theorem 3.** For any  $x \in \mathbb{N}$ ,  $A(1, x) = x + 2$

*Proof.* We proceed by induction.

$A(1, 0) = A(1 - 1, 1) = A(0, 1) = 2 = 0 + 2$ . Now assume  $A(1, k) = k + 2$  for some  $k \in \mathbb{N}$ . Then  $A(1, k + 1) = A(1 - 1, A(1, k + 1 - 1)) = A(0, A(1, k)) = A(0, k + 2) = k + 3 = (k + 1) + 2$ .  $\square$

**Lemma 1.** For every  $n, m \in \mathbb{N}$ ,  $m < A(n, m)$

*Proof.* We proceed by induction. Let  $m \in \mathbb{N}$ .  $A(0, m) = m + 1 > m$ , so  $m < A(0, m)$ . Now assume  $m < A(k, m)$  for some  $k \in \mathbb{N}$ . Then  $A(k+1, m) =$   
Not done with this one yet  $\square$

**Theorem 4.** For any  $n, m, s \in \mathbb{N}$  where  $m < s$ ,  $A(n, m) < A(n, s)$

*Proof.* Let  $m, s, n \in \mathbb{N}$  Since  $m < s$  and  $m, s \in \mathbb{N}$ , we have a  $d = s - m$  where  $d \in \mathbb{Z}^+$ . We will proceed by induction to first show  $A(n, m) < A(n, m+1)$ .  $A(0, m) = m+1 < m+1+1 = A(0, m+1)$ . Assume  $A(k, m) < A(k, m+1)$  for some  $k \in \mathbb{N}$ . By Lemma 1,  $A(k+1, m) < A(k, A(k+1, m)) = A(k+1, m+1)$ . So  $A(k+1, m) < A(k+1, m+1)$ . Therefore  $A(n, m) < A(n, m+1)$ .

Now we use this result to construct a chain of  $d$  inequalities  $A(n, m) < A(n, m+1) < A(n, m+1+1) < \dots < A(n, m+1+1+\dots+1) = A(n, m+d) = A(n, s)$ .  $\square$

**Theorem 5.** For any  $n, m, s \in \mathbb{N}$  where  $n < s$ ,  $A(n, m) < A(s, m)$

*Proof.*  $\square$

**Theorem 6.** For any  $n, m \in \mathbb{N}$ ,  $A(1, m) \leq A(n, m+1)$

*Proof.* We proceed by induction.  $A(1, m) = m + 2$  by Theorem 3.  $m + 2 = A(0, m+1)$ . Now assume  $A(1, m) \leq A(k, m+1)$ . Then since  $A(k, m+1) < A(k+1, m+1)$  by Theorem 5,  $A(1, m) \leq A(k, m+1) < A(k+1, m+1)$ , so  $A(1, m) < A(k+1, m+1)$ .  $\square$

**Theorem 7.** For any  $n, m \in \mathbb{N}$ ,  $A(n, m+1) \leq A(n+1, m)$

*Proof.* We proceed by induction.  $A(n, 0+1) = A(n, 1) = A(n+1, 0)$  by the definition of the Ackermann function. Now assume  $A(n, k+1) \leq A(n+1, k)$  for some  $k \in \mathbb{N}$ . Note that  $k+2 = A(1, k)$  by Theorem 3. Then  $A(n, k+1+1) = A(n, k+2) = A(n, A(1, k))$ . By Theorem 6,  $A(1, k) \leq A(n, k+1)$ . By Theorem 4,  $A(n, A(1, k)) \leq A(n, A(n, k+1))$ . By our inductive hypothesis and Theorem 4,  $A(n, A(n, k+1)) \leq A(n, A(n+1, k))$ . And finally by the definition of the Ackermann function,  $A(n, A(n+1, k)) = A(n+1, k+1)$ . So  $A(n, k+1+1) \leq A(n+1, k+1)$ .  $\square$

**Theorem 8.** For any  $n, m, s \in \mathbb{N}$ ,  $A(n, A(s, m)) < A(n+s+2, m)$

*Proof.*  $A(n, A(s, m)) < A(n + s, A(s, m))$  by Theorem 5.  $A(n + s, A(s, m)) < A(n + s, A(s + n + 1, m))$  by Theorem 4 since, by Theorem 5,  $A(s, m) < A(s + n + 1, m)$ .  $A(n + s, A(s + n + 1, m)) = A(n + s, A(n + s + 1, m)) = A(n + s + 1, m + 1)$  by the definition of the Ackermann function. Finally,  $A(n + s + 1, m + 1) \leq A(n + s + 1 + 1, m) = A(n + s + 2, m)$  by Theorem 7.  $\square$

**Definition.** Let  $P$  be a set all primitive recursive functions  $f$  so that if  $f \in P$ , there is a  $t \in \mathbb{N}$  such that for any  $x_1, \dots, x_n \in \mathbb{N}$ ,  $f(x_1, \dots, x_n) < A(t, \max\{x_1, \dots, x_n\})$ .

**Theorem 9.**  $c(x), s(x), p_i(x_1, x_2, \dots, x_n) \in P$

*Proof.* Let  $m = \max\{x_1, \dots, x_n\}$ . Then

$$\begin{aligned} c(x) &= 0 < x + 1 = A(0, x) \\ s(x) &= x + 1 < x + 2 = A(1, x) \\ p_i(x_1, x_2, \dots, x_n) &= x_i \leq m < m + 1 = A(0, m) \end{aligned}$$

$\square$

**Theorem 10.**  $P$  is closed under composition

*Proof.* Let  $f, g_1, g_2, \dots, g_k \in P$ , where  $f$  is  $k$ -ary and each  $g_i$  is  $j$ -ary. Let  $x_1, x_2, \dots, x_j \in \mathbb{N}$ . Let  $m = \max\{x_1, x_2, \dots, x_j\}$ . Let  $h$  be the  $j$ -ary primitive recursive function that results from function composition of  $f$  with  $g_1, g_2, \dots, g_k$ . Let  $g_{\max}$  be the  $g_i$  giving the maximum value in  $\max\{g_1(x_1, \dots, x_j), \dots, g_k(x_1, \dots, x_j)\}$ . Let  $m_g = g_{\max}(x_1, \dots, x_j)$ . Since  $g_{\max} \in P$ , there exists some  $t_g \in \mathbb{N}$  so that  $m_g < A(t_g, m)$ . Similarly since  $f \in P$ , there exists some  $t_f \in \mathbb{N}$  so that  $h(x_1, \dots, x_j) = f(g_1(x_1, \dots, x_j), \dots, g_k(x_1, \dots, x_j)) < A(t_f, m_g)$ . But since  $m_g < A(t_g, m)$ , by Theorem ??  $A(t_f, m_g) < A(t_f, A(t_g, m))$ . By Theorem 8,  $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$ . Let  $t = t_f + t_g + 2 \in \mathbb{N}$ . Then  $h(x_1, \dots, x_j) < A(t, m)$ . So  $h \in P$ .  $\square$

**Lemma 2.** Let  $h$  be the primitive recursion of two functions  $f, g \in P$ . Let  $x_1, \dots, x_n \in \mathbb{N}$ . Let  $m = \max\{x_1, \dots, x_n\}$ . Then there exists a  $t \in \mathbb{N}$  such that for every  $x \in \mathbb{N}$ ,  $h(x, x_1, \dots, x_n) < A(t, x + m)$ .

*Proof.* Since  $f, g \in P$ , there exist  $t_f$  and  $t_g$  so that for every  $x \in \mathbb{N}$ ,  $f(x, h(x, x_1, \dots, x_n), x_1, \dots, x_n) < A(t_f, \max\{x, h(x, x_1, \dots, x_n), x_1, \dots, x_n\})$  and  $g(x_1, \dots, x_n) < A(t_g, \max\{x_1, \dots, x_n\})$ . Choose  $t =$  We proceed by induction. oh man I have no idea where to continue on this one  $h(0, x_1, \dots, x_n) = g(x_1, \dots, x_n) <$   $\square$

**Theorem 11.**  *$P$  is closed under primitive recursion*

*Proof.* Let  $f, g \in P$  and  $h$  be the primitive recursion of  $f$  and  $g$ . Let  $x, x_1, \dots, x_n \in \mathbb{N}$ . Let  $m = \max\{x_1, \dots, x_n\}$ . Then by Lemma 2, there is a  $t \in \mathbb{N}$  so that  $h(x, x_1, \dots, x_n) < A(t, x + m)$ . Let  $m' = \max\{x, x_1, \dots, x_n\} = \max\{x, m\}$ . Since  $x + m \leq m'$ ,  $A(t, x + m) \leq A(t, 2m')$  by Lemma ???. Since  $2m' < 2m' + 3$ ,  $A(t, 2m') < A(t, 2m' + 3)$  also by Lemma ??.

By Lemma ?,  $2m' + 3 = A(2, m')$ , so  $A(t, 2m' + 3) = A(t, A(2, m'))$ . By Theorem 8,  $A(t, A(2, m')) = A(t+2+2, m') = A(t+4, m')$ . So  $h(x, x_1, \dots, x_n) < A(t+4, m')$ .

Therefore  $h \in P$ . □

**Theorem 12.**  *$P$  is precisely the primitive recursive functions*

*Proof.* All  $f \in P$  are primitive recursive by the definition of  $P$ . So it remains to show all primitive recursive functions are in  $P$ . Let  $f$  be a primitive recursive function. Then  $f$  can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in  $P$ , and  $P$  is closed under function composition and primitive recursion. Therefore  $f \in P$ . □

**Theorem 13.**  *$A(m, n)$  is not a primitive recursive function*

*Proof.* Suppose that  $A$  is a primitive recursive function. Then  $A \in P$ . Then there exists a  $t \in \mathbb{N}$  so that for any  $m, n \in \mathbb{N}$ ,  $A(m, n) < A(t, \max\{m, n\})$ . Set  $m = n = t$ . Then  $\max\{m, n\} = m = n = t$ . So  $A(m, n) = A(t, t) < A(t, t)$  which is a contradiction. Therefore  $A$  is not primitive recursive. □

## 4 General Recursive Functions

### 4.1 Partial Functions

### 4.2 Definition of General Recursive Functions

**Definition** (Minimization). *Let  $f : \mathbb{N}^n \rightarrow \mathbb{N}$  be an  $n$ -ary partial function. Then the minimization operator  $\mu$  creates a  $(n - 1)$ -ary function  $\mu(f)$  given by*

$$\mu(f)(x_1, \dots, x_n) = \begin{cases} z & f(z, x_1, \dots, x_n) = 0 \wedge f(i, x_1, \dots, x_n) > 0 \forall i \in \mathbb{N} \cap [0, z) \\ \text{undefined} & f(i, x_1, \dots, x_n) \neq 0 \forall i \in \mathbb{N} \end{cases}$$

**Definition.** *A function  $f$  is a general recursive function if it is either:*

- 1. A primitive recursive function*
- 2. A general recursive function under the minimization operator*