Computability via Recursive Functions

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1 Effective Calcubility and Computability

2 Primitive Recursive Functions

2.1 Functions

For this paper, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$

Definition. *The following functions from $\mathbb{N} \oplus ... \oplus \mathbb{N}$ to $\mathbb{N}$ are primitive recursive functions:*

1. *The unary constant function c:*

$$c(x) = 0$$

2. *The unary successor function s:*

$$s(x) = x + 1$$

3. *The n-ary projection function p:*

$$1 \leq i \leq n$$
$$p_i(x_1, ..., x_n) = x_i$$

4. *Function composition*
Let f be an n-ary primitive recursive function and $g_1, g_2, ..., g_n$ all be m-ary primitive recursive functions. Then the m-ary composition h of f and $g_1, g_2, ..., g_n$ given by

$$h(x_1, x_2, ..., x_m) = f(g_1(x_1, x_2, ..., x_m), ..., g_n(x_1, x_2, ..., x_m))$$

is a primitive recursive function

5. *Primitive recursion Let g be an n-ary primitive recursive function and f be an $(n+2)$-ary primitive recursive function. Then the $(n+1)$-ary primitive recursion h of f and g given by*

$$h(0, x_1, ..., x_n) = g(x_1, ..., x_n)$$
$$h(s(x), x_1, ..., x_n) = f(x, h(x, x_1, ..., x_n), x_1, ..., x_n)$$

is a primitive recursive function

3 The Ackermann Function

Definition (The Ackermann Function)**.** *Let $n, m \in \mathbb{N}$. Then define $A(n, m)$ as follows:*

$$A(n, m) = \begin{cases} m+1 & n = 0 \\ A(n-1, 1) & n > 0 \wedge m = 0 \\ A(n-1, A(n, m-1)) & n > 0 \wedge m > 0 \end{cases}$$

Theorem 1. *For any $n, m \in \mathbb{N}, A(n, m) \in \mathbb{N}$*

Proof. We proceed by double induction.
Clearly $A(0, k) \in \mathbb{N}$ for every $k \in \mathbb{N}$ because $k+1 \in \mathbb{N}$. Now assume $A(k, m) \in \mathbb{N}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Then $A(k+1, 0) = A(k+1-1, 1) = A(k, 1) \in \mathbb{N}$. Now assume $A(k+1, j) \in \mathbb{N}$ for some $j \in \mathbb{N}$. Then $A(k+1, j+1) = A(k+1-1, A(k+1, j+1-1)) = A(k, A(k+1, j))$. By our second hypothesis $A(k+1, j) \in \mathbb{N}$, so by our first hypothesis $A(k, A(k+1, j)) \in \mathbb{N}$. Therefore $A(k+1, j+1) \in \mathbb{N}$. Then $A(k+1, m) \in \mathbb{N}$ for every $m \in \mathbb{N}$, so $A(n, m) \in \mathbb{N}$ for all choices of $n, m \in \mathbb{N}$ □

Theorem 2. *A is a total function*

Proof. By Theorem 1, all n, m in the domain of A have an image $A(n, m) \in \mathbb{N}$. Therefore A is a total function. □

Theorem 3. *For any $m \in \mathbb{N}$, $A(1, m) = m + 2$*

Proof. We proceed by induction.
$A(1, 0) = A(1-1, 1) = A(0, 1) = 2 = 0 + 2$. Now assume $A(1, k) = k + 2$ for some $k \in \mathbb{N}$. Then $A(1, k+1) = A(1-1, A(1, k+1-1)) = A(0, A(1, k)) = A(0, k+2) = k+3 = (k+1)+2$. □

Lemma 1. *For every $n, m \in \mathbb{N}$, $m < A(n,m)$*

Proof. We proceed by double induction. $A(0,0) = 0+1 = 1 > 0$. Clearly $A(0,m) = m+1 > m$ for every $m \in \mathbb{N}$. Now assume $m < A(k,m)$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$. Now we show that $m < A(k+1,m)$ for every $m \in \mathbb{N}$. $A(k+1,0) = A(k,1)$. By our inductive hypothesis, $A(k,1) > 1 > 0$, So $A0 < A(k+1,0)$. Now assume $j < A(k+1,j)$ for some $j \in \mathbb{N}$. Then $A(k+1,j+1) = A(k, A(k+1,j))$. By our first inductive hypothesis, $A(k+1,j) < A(k, A(k+1,j))$. By our second inductive hypothesis, $j < A(k+1,j)$. So $j+1 \leq A(k+1,j) < A(k, A(k+1,j)) = A(k+1,j+1)$. $\square$

Theorem 4. *For any $n, m, s \in \mathbb{N}$ where $m < s$, $A(n,m) < A(n,s)$*

Proof. Let $m, s, n \in \mathbb{N}$ Since $m < s$ and $m, s \in \mathbb{N}$, we have a $d = s - m$ where $d \in \mathbb{Z}^+$. We will proceed by induction to first show $A(n,m) < A(n,m+1)$. $A(0,m) = m+1 < m+1+1 = A(0,m+1)$. Assume $A(k,m) < A(k,m+1)$ for some $k \in \mathbb{N}$. By Lemma 1, $A(k+1,m) < A(k, A(k+1,m)) = A(k+1,m+1)$. So $A(k+1,m) < A(k+1,m+1)$. Therefore $A(n,m) < A(n,m+1)$.

Now we use this result to construct a chain of d inequalities $A(n,m) < A(n,m+1) < A(n,m+1+1) < ... < A(n,m+1+1+...+1) = A(n,m+d) = A(n,s)$. $\square$

Theorem 5. *For any $n, m \in \mathbb{N}$, $A(n,m+1) \leq A(n+1,m)$*

Proof. We proceed by induction. $A(n,0+1) = A(n,1) = A(n+1,0)$ by the definition of the Ackermann function. Now assume $A(n,k+1) \leq A(n+1,k)$ for some $k \in \mathbb{N}$. Note that $k+1 < A(n,k+1))$ by Theorem 1. So $k+2 \leq A(n,k+1)$. Then $A(n,k+2) \leq A(n, A(n,k+1))$ by Theorem 4. By our inductive hypothesis, $A(n,k+1) \leq A(n+1,k)$, so by Theorem 4 again, $A(n, A(n,k+1)) \leq A(n, A(n+1,k))$. By the definition of the Ackermann function, $A(n, A(n+1,k)) = A(n+1,k+1)$. $\square$

Theorem 6. *For any $n, m, s \in \mathbb{N}$ where $n < s$, $A(n,m) < A(s,m)$*

Proof. Let $n, s, m \in \mathbb{N}$. Since $n < s$ and $n, s \in \mathbb{N}$, we have a $d = s - n$ where $d \in \mathbb{Z}^+$. We begin by showing $A(n,m) < A(n+1,m)$. $A(n,m) < A(n,m+1)$ by Theorem 4, and $A(n,m+1) \leq A(n+1,m)$ by Theorem 5. So $A(n,m) < A(n+1,m)$. Now we use this result to construct a chain of d inequalities $A(n,m) < A(n+1,m) < A(n+1+1,m) < ... < A(n+1+1+...+1,m) = A(n+d,m) = A(s,m)$. $\square$

Theorem 7. *For any $n, m, s \in \mathbb{N}$, $A(n, A(s, m)) < A(n + s + 2, m)$*

Proof. $A(n, A(s, m)) < A(n+s, A(s, m))$ by Theorem 6. $A(n+s, A(s, m)) < A(n + s, A(s + n + 1, m))$ by Theorem 4 since, by Theorem 6, $A(s, m) < A(s + n + 1, m)$. $A(n + s, A(s + n + 1, m)) = A(n + s, A(n + s + 1, m)) = A(n + s + 1, m + 1)$ by the definition of the Ackermann function. Finally, $A(n + s + 1, m + 1) \leq A(n + s + 1 + 1, m) = A(n + s + 2, m)$ by Theorem 5. □

Theorem 8. *For any $m \in \mathbb{N}$, $A(2, m) = 2m + 3$*

Proof. □

Definition. *Let P be a set all primitive recursive functions f so that if $f \in P$, there is a $t \in \mathbb{N}$ such that for any $x_1, ..., x_n \in \mathbb{N}$, $f(x_1, ..., x_n) < A(t, max\{x_1, ..., x_n\})$.*

Theorem 9. *$c(x)$, $s(x)$, $p_i(x_1, x_2, ..., x_n) \in P$*

Proof. Let $m = max\{x_1, ..., x_n\}$. Then

$$c(x) = 0 < x + 1 = A(0, x)$$
$$s(x) = x + 1 < x + 2 = A(1, x)$$
$$p_i(x_1, x_2, ..., x_n) = x_i \leq m < m + 1 = A(0, m)$$

□

Theorem 10. *P is closed under composition*

Proof. Let $f, g_1, g_2, ..., g_k \in P$, where f is k-ary and each g_i is j-ary. Let $x_1, x_2, ..., x_j \in \mathbb{N}$. Let $m = max\{x_1, x_2, ..., x_j\}$. Let h be the j-ary primitive recursive function that results from function composition of f with $g_1, g_2, ..., g_k$. Let g_{max} be the g_i giving the maximum value in $max\{g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)\}$. Let $m_g = g_{max}(x_1, ..., x_j)$ Since each $g_i \in P$, there is some t_i for each g_i such that $g_i(x_1, ..., x_j) < A(t_i, m)$. Take $t_g = max\{t_1, t_2, ..., t_k\}$. Note this is not dependent on $x_1, ..., x_j$. Similarly since $f \in P$, there exists some $t_f \in \mathbb{N}$ so that $h(x_1, ..., x_j) = f(g_1(x_1, ..., x_j), ..., g_k(x_1, ..., x_j)) < A(t_f, m_g)$. Note this is also not dependent on $x_1, ..., x_j$. But since $m_g < A(t_g, m)$, by Theorem 4 $A(t_f, m_g) < A(t_f, A(t_g, m))$. By Theorem 7, $A(t_f, A(t_g, m)) < A(t_f + t_g + 2, m)$. Let $t = t_f + t_g + 2 \in \mathbb{N}$. Then $h(x_1, .., x_j) < A(t, m)$. So $h \in P$. □

Lemma 2. *Let h be the primitive recursion of two functions $f, g \in P$. Then there exists a $t \in \mathbb{N}$ such that for every $x, x_1, ..., x_n \in \mathbb{N} h(x, x_1, ..., x_n) < A(t, x + max\{x_1, ..., x_n\})$.*

Proof. Since $f, g \in P$, there exist t_f and t_g so that $f(x, y, x_1, ..., x_n) < A(t_f, max\{x, y, x_1, ..., x_n\})$ and $g(x_1, ..., x_n) < A(t_g, max\{x_1, ..., x_n\})$ for any arguments of f and g. Let $t = 1 + max\{t_f, t_g\}$. Note that t is not dependent on $x, x_1, ..., x_n$. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$. We proceed by induction. $h(0, x_1, ..., x_n) = g(x_1, ..., x_n) < A(t_g, m) < A(t, m) = A(t, m + 0)$. Now assume $h(k, x_1, ..., x_n) < A(t, max\{m, k\})$ for some $k \in \mathbb{N}$. Let $m_{k+1} = max\{k, h(k, x_1, ..., x_n), m\}$. Then $h(k + 1, x_1, ..., x_n) = f(k, h(k, x_1, ..., x_n), x_1, ..., x_n) < A(t_f, m_{k+1})$. Now note that $m_{k+1} < A(t, k + m)$:

1. $m \leq k + m < A(t, k + m)$ by Theorem 1
2. $k \leq k + m < A(t, k + m)$ by Theorem 1
3. $h(k, x_1, ..., x_n) < A(t, max\{m, k\})$ by our induction hypothesis, and $A(t, max\{m, k\}) \leq A(t, k+m)$ by Theorem 4 since $max\{m, k\} \leq k+m$

By Theorem 4, $A(t_f, m_{k+1}) < A(t_f, A(t, k + m))$. Since $t = 1 + max\{t_f, t_g\}$, $t_f \leq t - 1$. Then by Theorem 4 and since $t \neq 0$, $A(t_f, A(t, k + m)) \leq A(t - 1, A(t, k+m))$. By the definition of the Ackermann function, $A(t-1, A(t, k+m)) = A(t, k + m + 1)$. Therefore $h(k + 1, x_1, ..., x_n) < A(t, (k + 1) + m)$ □

Theorem 11. *P is closed under primitive recursion*

Proof. Let $f, g \in P$ and h be the primitive recursion of f and g. Let $x, x_1, ..., x_n \in \mathbb{N}$. Let $m = max\{x_1, ..., x_n\}$. Then by Lemma 2, there is a $t \in \mathbb{N}$ so that $h(x, x_1, ..., x_n) < A(t, x + m)$. Let $m' = max\{x, m\}$. Since $x + m \leq 2m'$, $A(t, x + m) \leq A(t, 2m')$ by Lemma 4. Since $2m' < 2m' + 3$, $A(t, 2m') < A(t, 2m' + 3)$ also by Lemma 4. By Theorem 8, $2m' + 3 = A(2, m')$. Therefore $A(t, 2m' + 3) = A(t, A(2, m'))$. By Theorem 7, $A(t, A(2, m')) < A(t + 2 + 2, m') = A(t + 4, m')$. Therefore $h(x, x_1, ..., x_n) < A(t + 4, m')$. □

Theorem 12. *P is precisely the primitive recursive functions*

Proof. All $f \in P$ are primitive recursive by the definition of P. So it remains to show all primitive recursive functions are in P. Let f be a primitive recursive function. Then f can be created by applying function composition and primitive recursion to the constant, successor, and projection functions. The constant, successor, and projection functions are all in P, and P is closed under function composition and primitive recursion. Therefore $f \in P$. $\square$

Theorem 13. *$A(m, n)$ is not a primitive recursive function*

Proof. Suppose that A is a primitive recursive function. Then $A \in P$. Then there exists a $t \in \mathbb{N}$ so that for any $n, m \in \mathbb{N}$, $A(n, m) < A(t, max\{n, m\})$. Set $m = n = t$. Then $max\{n, m\} = n = m = t$. So $A(n, m) = A(t, t) < A(t, t)$ which is a contradition. Therefore A is not primitive recursive. $\square$

4 General Recursive Functions

4.1 Partial Functions

4.2 Definition of General Recursive Functions

Definition (Minimization)**.** *Let $f : \mathbb{N}^n \to \mathbb{N}$ be an n-ary partial function. Then the minimization operator μ creates a $(n-1)$-ary function $\mu(f)$ given by*

$$\mu(f)(x_1, ..., x_n) = \begin{cases} z & f(z, x_1, ..., x_n) = 0 \wedge f(i, x_1, ..., x_n) > 0 \forall i \in \mathbb{N} \cap [0, z) \\ undefined & f(i, x_1, ..., x_n) \neq 0 \forall i \in \mathbb{N} \end{cases}$$

Definition. *A function f is a general recursive function if it is either:*

1. *A primitive recursive function*

2. *A general recursive function under the minimization operator*