Workshop 8: Markov chains, martingales, and potential theory

August 3, 2024

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ be a filtered space.

An S-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ is an $(\mathcal{F}_n)$-Markov process with transition kernel p if it is $(\mathcal{F}_n)$-adapted and for every $n \in \mathbb{N}_0$,

$$\mathbb{P}(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B) \quad \mathbb{P} - \text{a.s.} \tag{1}$$

We write $\mathbb{P}_x(A)$ instead of $\mathbb{P}(A|X_0 = x)$ and, more generally, $\mathbb{P}_{X_0}(A)$ instead of $\mathbb{P}(A|X_0)$.

Let $F_b(S)$ denote the real vector space of bounded real-valued functions on S. We introduce the *generator* $\mathcal{L} : F_b(S) \to F_b(S)$ given by

$$(\mathcal{L}f)(s) = \sum_{t\in S} p(s,t)(f(t) - f(s)). \tag{2}$$

We proved in class that for every $f \in F_b(S)$ and $n \in \mathbb{N}_0$,

$$\mathbb{E}_x(f(X_{n+1}) - f(X_n)|\mathcal{F}_n) = (\mathcal{L}f)(X_n) \quad \mathbb{P}_x - \text{a.s.} \tag{3}$$

Define $M_n^{[f]} = f(X_n) - \sum_{i=0}^{n-1} (\mathcal{L}f)(X_i)$.

Exercise 1 (cf. Eberle, 0.7)**.** Let $(X_n)_{n\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the filtration $\mathcal{F}_n$. Prove that $X = (X_n)$ is an $(\mathcal{F}_n)$-Markov chain with transition kernel p if and only if $M^{[f]}$ is an $(\mathcal{F}_n)$-martingale for every function $f \in F_b(S)$.

Let D be a subset of S and define the boundary $\partial D = \bigcup_{s\in D}\{\, t \in S \setminus D \,:\, p(s,t) > 0 \,\}$. Introduce the stopping time $T := T_{D^c} := \inf\{\, n \geq 0 \,:\, X_n \in D^c \,\}$. For any nonnegative functions $f, c \in F_b(S)$, set

$$u(x) := \mathbb{E}_x\left(f(X_T) I_{\{T<+\infty\}} + \sum_{n=0}^{T-1} c(X_n) \right), \tag{4}$$

where we understand $\sum_{i=0}^{-1} c(X_i) = 0$. Why is this well-defined?

Exercise 2. Choose D, f and c to recover the following quantities:

1. $u(x) = \mathbb{P}_x(T_A < T_Z)$, the probability of hitting A before hitting Z as a function of a starting point $x \in S$, for A, Z disjoint subsets of S.
2. $u(x) = \mathbb{P}_x(T_A < +\infty)$ for $A \subset S$.
3. The average occupation time of $A \subset D$ before exiting D.
4. The mean exit time $u(x) = \mathbb{E}_x(T_{D^c})$.

Exercise 3. Prove that the function $u : S \to \mathbb{R}$ defined in (4) is a solution of the boundary value problem

$$\mathcal{L}v = -c \quad \text{on } D, \tag{5}$$
$$v = f \quad \text{on } \partial D. \tag{6}$$

Hint: Compute first $\mathbb{E}_x(f(X_T)I_{\{T<+\infty\}} + \sum_{n=0}^{T-1} c(X_n)|X_1)$. Consider the shifted chain $\tilde{X} = (X_{n+1})_{n\geq 0}$ starting at X_1 (random), and the corresponding exit time $\tilde{T}$. What is the relation between $\tilde{T}$ and T? And between X_T and $\tilde{X}_{\tilde{T}}$?

In fact, u is the minimal nonnegative solution of the preceding boundary value problem, in view of the following *maximum principle*:

Exercise 4. Let v be a nonnegative bounded function on S. If $\mathcal{L}v \leq -c$ on D and $v \geq f$ on ∂D then $u \leq v$.
Hint: Prove first that $M_n = v(X_{n\wedge T}) + \sum_{i=0}^{(n\wedge T)-1} c(X_i)$ is a nonnegative supermartingale under $\mathbb{P}_x$. What happens when $n \to \infty$? Bound $\mathbb{E}(\liminf_n M_n)$.

An application of this maximum principle is the following result:

Exercise 5. Let A be a proper subset of S. Suppose that $\psi : S \to [0, \infty)$ satisfies the system of inequalities

$$\mathcal{L}\psi \leq -1 \quad \text{on } A^c. \tag{7}$$

Then $\mathbb{E}_x(T_A) \leq \psi(x)$ for all $x \in S$.

This can be used to characterize *recurrence* in Markov chains, see Eberle, Thm. 1.6.
References:

- S. Roch, "Notes 24: Markov chains: martingale methods", `https://people.math.wisc.edu/~roch/grad-prob/gradprob-notes24.pdf`.
- A. Eberle, "Markov Processes", `https://uni-bonn.sciebo.de/s/kzTUFff5FrWGAay#pdfviewer`.