

Predictive Regressions with Persistent Regressors

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- The regression problem
- Local to unity asymptotics
- Constructing a pretest
- Motivating more efficient tests
- Constructing the test statistic
- Making the test feasible
- Analysis of the power gain
- Empirical results

The Regression Setup

- Campbell and Yogo (2006)¹ (CY) consider the system of equations:

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$$r_t = \alpha + \beta x_{t-1} + u_t \quad (1)$$

$$x_t = \gamma + \rho x_{t-1} + e_t \quad (2)$$

- For the sake of simplicity they further assume normality:

$$w_t = (u_t, e_t)' \sim N(0, \Sigma), \text{ where } \Sigma \text{ is known}$$
$$\Sigma = \begin{bmatrix} \sigma_u^2 & \sigma_{ue} \\ \sigma_{ue} & \sigma_e^2 \end{bmatrix}$$

¹Campbell, J. Y. and M. Yogo (2006). Efficient tests of stock return predictability. Journal of Financial Economics 81, 27-60.

Testing for Significance

- To test β for significance, CY start by considering the maximum likelihood ratio test.

$$\max_{\beta, \rho, \alpha, \gamma} L(\beta, \rho, \alpha, \gamma) - \max_{\rho, \alpha, \gamma} L(\beta_0, \rho, \alpha, \gamma) = t(\beta_0)^2 > C,$$

- where C is some constant and the joint log likelihood (ignoring two constants) is given by

$$\begin{aligned} L(\beta, \rho, \alpha, \gamma) = & -\frac{1}{1-\delta^2} \sum_{t=1}^T \left[\frac{(r_t - \alpha - \beta x_{t-1})^2}{\sigma_u^2} \right. \\ & - 2\delta \frac{(r_t - \alpha - \beta x_{t-1})(x_t - \gamma - \rho x_{t-1})}{\sigma_u \sigma_e} \\ & \left. + \frac{(x_t - \gamma - \rho x_{t-1})^2}{\sigma_e^2} \right] \end{aligned} \quad (3)$$

- The LRT from above turns out to be the same when considering only the marginal likelihood

$$L(\beta, \alpha) = - \sum_{t=1}^T (r_t - \alpha - \beta x_{t-1})^2 \quad (4)$$

- It thus ignores information contained in the system.

- The largest autoregressive root is modeled as $\rho = 1 + c/T$ where c is a fixed constant.
- This does not imply that ρ literally follows this process in practice. It is merely a tool to model autoregressive roots very close to one. It has some nice properties:
- Asymptotic distribution theory is not discontinuous when x_t is $I(1)$.

Pretesting for Size Distortions of the T-Test

- Under local-to-unity asymptotic theory the t-statistic does not converge to a standard normal distribution under the null but to functionals of a diffusion process.
- Under the null it converges to:

$$t(\beta_0) \Rightarrow \delta \frac{\tau_c}{\kappa_c} + (1 - \delta^2)^{1/2} Z, \quad (5)$$

- where

$$\kappa_c = \left(\int J_c^\mu(s)^2 ds \right)^{1/2}, \quad \tau_c = \int J_c^\mu(s) dW_e(s),$$

and $Z \sim N(0, 1)$ independent of $(W_e(s), J_c(s))$

- $(W_u(s), W_e(s))'$ is a two-dimensional Wiener process with correlation δ .
- $J_c(s)$ solving $dJ_c(s) = cJ_c(s)ds + dW_e(s)$ with initial condition $J_c(0) = 0$.
- $J_c^\mu(s) = J_c(s) - \int J_c(r)dr$

Pretesting for Size Distortions of the T-Test (Cont.)

- If $\delta = 0$ the first term of (3) vanishes and the distribution under the null is the usual $N(0, 1)$.
- Likewise if $c \ll 0$, x is not persistent and first-order asymptotics are valid.

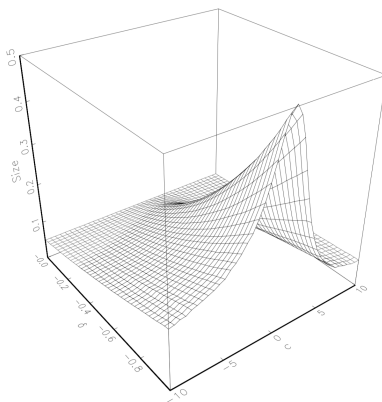


Figure 1: $p(c, \delta; 0.05) = \Pr \left(\delta \frac{\tau_c}{\kappa_c} + (1 - \delta^2)^{1/2} Z > z_{0.05} \right)$ Asymptotic Size of the One-Sided t-test at 5% Significance

Pretesting for Size Distortions of the T-Test (Cont.)

- CY propose that we may accept an actual size below 0.075.
- Test against the null of actual size greater than 0.075.
- Estimate δ from residuals of (1) and (2).
- Construct confidence interval for c by computing Dickey-Fuller generalized least squares (DF-GLS) test statistic² and using its known distribution under the alternative to construct the confidence interval for c .
- Reject the null if the confidence interval for c lies strictly below (or above) the region of the parameter space (c_{min}, c_{max}) .
- Implemented in R package{pr}.

²For a detailed description of the DF-GLS statistic see the appendix

- To improve confidence of inference, we need to increase the signal-to-noise ratio. Here, we focus on reducing the noise.
- Since the innovations of (1) and (2) are correlated, CY propose to subtract off this part of the innovations.
- In contrast to the t-test, this procedure takes advantage of all the information contained in the system.
- Assume for the following $\alpha = \gamma = 0$ and ρ is known.

The Q-test

- Recall the joint log likelihood:

$$L(\beta, \rho, \alpha, \gamma) = -\frac{1}{1-\delta^2} \sum_{t=1}^T \left[\frac{(r_t - \alpha - \beta x_{t-1})^2}{\sigma_u^2} - 2\delta \frac{(r_t - \alpha - \beta x_{t-1})(x_t - \gamma - \rho x_{t-1})}{\sigma_u \sigma_e} + \frac{(x_t - \gamma - \rho x_{t-1})^2}{\sigma_e^2} \right]$$

- By the Neyman–Pearson Lemma the most powerful test against the simple alternative (i.e., a alternative that uniquely specifies the distribution)

$\beta = \beta_1$ rejects the null if the LR is greater than some constant:

$$\sigma_u^2 (1 - \delta^2) (L(\beta_1) - L(\beta_0)) = 2(\beta_1 - \beta_0) \sum_{t=1}^T x_{t-1} [r_t - \beta_{ue} (x_t - \rho x_{t-1})] - (\beta_1^2 - \beta_0^2) \sum_{t=1}^T x_{t-1}^2 > C \quad (6)$$

- Where $\beta_{ue} = \sigma_{ue} / \sigma_e^2$

The Q-test

- Observe that $\sum_{i=1}^T x_{t-1}^2$ does not depend on β .
- CY condition the test on that statistic in order to reduce it to

$$\sum_{t=1}^T x_{t-1} [r_t - \beta_{ue} (x_t - \rho x_{t-1})] > C \quad (7)$$

- To get a standard normal distribution under the null they recenter and rescale:

$$\frac{\sum_{t=1}^T x_{t-1} [r_t - \beta_0 x_{t-1} - \beta_{ue} (x_t - \rho x_{t-1})]}{\sigma_u (1 - \delta^2)^{1/2} \left(\sum_{t=1}^T x_{t-1}^2 \right)^{1/2}} > C \quad (8)$$

- Finally, after CY de-mean x_{t-1} and denote it by x_{t-1}^μ they can eliminate the assumption $\alpha = \gamma = 0$ to get:

$$Q(\beta_0, \rho) = \frac{\sum_{t=1}^T x_{t-1}^\mu [r_t - \beta_0 x_{t-1} - \beta_{ue} (x_t - \rho x_{t-1})]}{\sigma_u (1 - \delta^2)^{1/2} \left(\sum_{t=1}^T x_{t-1}^{\mu 2} \right)^{1/2}} \quad (9)$$

- Consider the case $\beta_0 = 0$.
- Then

$$Q(\beta_0, \rho) = \frac{\sum_{t=1}^T x_{t-1}^{\mu} [r_t - \beta_0 x_{t-1} - \beta_{ue} (x_t - \rho x_{t-1})]}{\sigma_u (1 - \delta^2)^{1/2} \left(\sum_{t=1}^T x_{t-1}^{\mu^2} \right)^{1/2}}$$

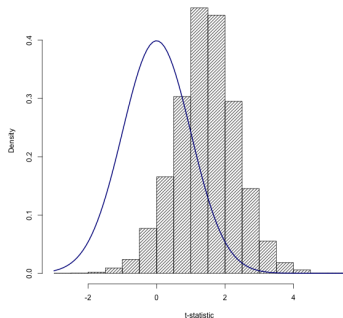
is the t-statistic of the coefficient b regressing

$$r_t - \beta_{ue} (x_t - \rho x_{t-1}) = a + b x_{t-1} + v_t$$

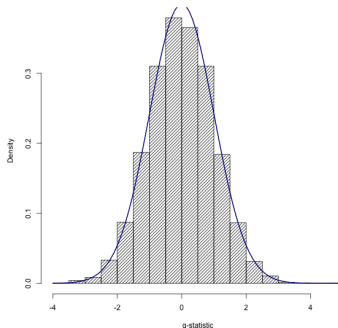
- $x_t - \rho x_{t-1} = e_t + \gamma$
- $\beta_{ue} = \sigma_{ue} / \sigma_e^2$ tells us something about the relation between shocks of (1) and (2).
- The equation above can be interpreted as regressing the de-noised returns onto the regressor x where we exploit the information contained in ρ and the correlation of the shocks.

Making the Test Feasible

standard normal curve over histogram, $c = 0$, $\rho = 1$, $\delta = -0.95$, Obs = 50



standard normal curve over histogram, $c = 0$, $\rho = 1$, $\delta = -0.95$, Obs = 50



- If ρ and δ are known, this Q-test is the best we can do. It's UMP.
- In practice, however, ρ and δ are not known.
- ρ cannot be estimated consistently.
- CY use Bonferroni's inequality to get confidence intervals.

Bonferroni Confidence Intervals

- Construct a $(1 - \alpha_1)$ confidence interval for ρ : $C_\rho(\alpha_1)$
- For each value of ρ in the confidence interval, construct a $(1 - \alpha_2)$ confidence interval for β given ρ : $C_{\beta|\rho}(\alpha_2)$
- Taking the union over all $\rho \in C_\rho(\alpha_1)$ we can marginalize ρ :

$$C_\beta(\alpha) = \bigcup_{\rho \in C_\rho(\alpha_1)} C_{\beta|\rho}(\alpha_2) \quad (10)$$

- with $\alpha = \alpha_1 + \alpha_2$, $C_\beta(\alpha)$ has coverage of at least $(1 - \alpha)$
- This follows from Bonferroni's inequality:

$$P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i)$$

where $P(E_i)$ is the probability that E_i is true and $P\left(\bigcup_{i=1}^n E_i\right)$ is the probability that at least one of E_1, E_2, \dots, E_n is true³.

³Weisstein, Eric W. "Bonferroni Inequalities." From MathWorld—A Wolfram Web Resource.
<http://mathworld.wolfram.com/BonferroniInequalities.html>

Bonferroni Confidence Intervals (Cont.)

- To get the confidence interval for ρ , CY need a unit root test statistic.
- Since they suspect ρ in the neighborhood of 1, the DF-GLS test statistic⁴ is a good choice.
- To get the confidence interval for β they use the Q-test since they know it to be more powerful than the t-test given true ρ by the Neyman–Pearson Lemma. They hope it remains more powerful for other ρ as well. Whether this hope is met will be seen by numerical analysis.

⁴For a detailed description of the DF-GLS statistic see the appendix

$$C_{\beta|\rho}(\alpha_2) = [\underline{\beta}(\rho, \alpha_2), \bar{\beta}(\rho, \alpha_2)]$$

- where

$$\beta(\rho) = \frac{\sum_{t=1}^T x_{t-1}^{\mu} [r_t - \beta_{ue}(x_t - \rho x_{t-1})]}{\sum_{t=1}^T x_{t-1}^{\mu 2}}$$

$$\underline{\beta}(\rho, \alpha_2) = \beta(\rho) - z_{\alpha_2/2} \sigma_u \left(\frac{1 - \delta^2}{\sum_{t=1}^T x_{t-1}^{\mu 2}} \right)^{1/2}$$

$$\bar{\beta}(\rho, \alpha_2) = \beta(\rho) + z_{\alpha_2/2} \sigma_u \left(\frac{1 - \delta^2}{\sum_{t=1}^T x_{t-1}^{\mu 2}} \right)^{1/2}$$

$z_{\alpha_2/2}$ denotes the $1 - \alpha_2/2$ quantile of the standard normal distribution.

- Let $C_\rho(\alpha_1) = [\underline{\rho}(\underline{\alpha}_1), \bar{\rho}(\bar{\alpha}_1)]$ denote the confidence interval for ρ ,
- where $\underline{\alpha}_1 = \Pr(\rho < \underline{\rho}(\underline{\alpha}_1))$, $\bar{\alpha}_1 = \Pr(\rho > \bar{\rho}(\bar{\alpha}_1))$, and $\alpha_1 = \underline{\alpha}_1 + \bar{\alpha}_1$.
- Then the Bonferroni confidence interval is:

$$C_\beta(\alpha) = [\underline{\beta}(\bar{\rho}(\bar{\alpha}_1), \alpha_2), \bar{\beta}(\underline{\rho}(\underline{\alpha}_1), \alpha_2)] \quad (11)$$

- The Bonferroni confidence interval is likely to be conservative, i.e., $\Pr(\beta \notin C_\beta(\alpha)) \leq \alpha_2(1 - \alpha_1) + \alpha_1 \leq \alpha$ is likely a strict inequality.
- To see this consider:

$$\begin{aligned}\Pr(\beta \notin C_\beta(\alpha)) &= \Pr(\beta \notin C_\beta(\alpha) | \rho \in C_\rho(\alpha_1)) \Pr(\rho \in C_\rho(\alpha_1)) \\ &\quad + \Pr(\beta \notin C_\beta(\alpha) | \rho \notin C_\rho(\alpha_1)) \Pr(\rho \notin C_\rho(\alpha_1))\end{aligned}\tag{12}$$

- $\Pr(\beta \notin C_\beta(\alpha) | \rho \notin C_\rho(\alpha_1))$ is unknown. Thus we have to assume the worst case and bound it by one.
- $\Pr(\beta \notin C_\beta(\alpha) | \rho \in C_\rho(\alpha_1)) \leq \alpha_2$ is strict if $C_{\beta|\rho}(\alpha_2)$ depends on ρ .
- Bonferroni confidence interval is conservative since it is built on worst case scenario and reality is likely less harsh (more conservative the smaller is δ in absolute value).
- Refine confidence interval for ρ until the confidence interval for β is exactly at the desired significance level $\tilde{\alpha}$.

- Do this by numerically searching over a grid.
- ④ fix α_2 .
- ② for each ρ , numerically search to find the $\bar{\alpha}_1$ s.t. :
 - $\Pr(\underline{\beta}(\bar{\rho}(\bar{\alpha}_1), \alpha_2) > \beta) \leq \tilde{\alpha}/2$ holds for all values of c on the grid,
 - and $\Pr(\underline{\beta}(\bar{\rho}(\bar{\alpha}_1), \alpha_2) > \beta) = \tilde{\alpha}/2$ at some point on the grid.
- ③ repeat 2. for $\underline{\alpha}_1$ s.t. $\Pr(\bar{\beta}(\underline{\rho}(\underline{\alpha}_1), \alpha_2) < \beta) \leq \tilde{\alpha}/2$
 - $[\underline{\alpha}_1, \bar{\alpha}_1]$ defines a tighter confidence interval for ρ .
 - one-sided Bonferroni test⁵ has exact size $\tilde{\alpha}/2$ for some c .
 - two-sided Bonferroni test has at most size $\tilde{\alpha}$ for all c .

⁵For a detailed description of how to implement this test using OLS refer to the appendix. Alternatively, read the source code of R package{pr}

Comparing Power

- All tests considered should reject the null in favor of alternatives of the form $\beta = \beta_0 + b$, where b is some constant, almost surely as $T \rightarrow \infty$.
- More interesting are alternatives of the form $\beta = \beta_0 + b/T$, where b is again some constant.

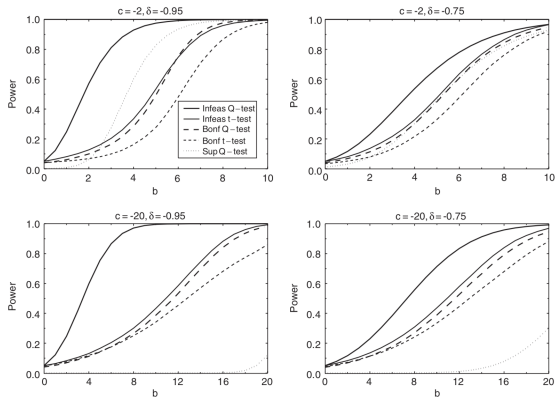


Figure 2: Null: $\beta = \beta_0$ against local alternatives: $b = T(\beta - \beta_0) > 0$ from Campbell and Yogo (2006)

- CY Compare:
 - ① A Bonferroni test based on the ADF test and the t-test.
 - ② A Bonferroni test based on the DF-GLS test and the t-test.
 - ③ A Bonferroni test based on the DF-GLS test and the Q-test.

$$c = -2 \text{ and } \delta = -0.95$$

- Pitman efficiency (relative number of observations needed to achieve 50% power) of test 1 relative to test 2 is 1.03.
- Pitman efficiency of test 2 relative to test 3 is 1.20.
- Close to no difference in power between the unrefined and refined Bonferroni t-test.
- Pitman efficiency of the unrefined relative to the refined Bonferroni Q-test is 1.62.

$$c = -20 \text{ and } \delta = -0.95$$

- Pitman efficiency of test 1 relative to test 2 is 1.07.
- Pitman efficiency of test 2 relative to test 3 is 1.03.
- Pitman efficiency of the unrefined relative to the refined Bonferroni t-test is 1.23.
- Pitman efficiency of the unrefined relative to the refined Bonferroni Q-test is 1.55 .

- When the regressor is highly persistent, the use of the Q-test rather than the t-test is an important source of power gain for the Bonferroni Q-test.
- Bonferroni refinement is an especially important source of power gain for the Bonferroni Q-test since it tries to exploit information about ρ . This makes its confidence interval for β given ρ more sensitive to ρ . Hence, without the refinement the Bonferroni test is too conservative.

Finite Sample Rejection Rates (10,000 Monte Carlo runs)

Table 1: Finite-sample rejection rates for right-tailed tests of predictability at $\alpha=0.05$

	Obs	c	ρ	δ	T-test	Bonf.Q-test	Q-test
1	50	0	1.000	-0.95	0.4160	0.0826	0.0483
2	50	0	1.000	-0.75	0.2916	0.0837	0.0515
3	50	-2	0.961	-0.95	0.2714	0.0868	0.0482
4	50	-2	0.961	-0.75	0.2079	0.0881	0.0532
5	50	-20	0.608	-0.95	0.0977	0.1206	0.0515
6	50	-20	0.608	-0.75	0.0840	0.1078	0.0484
7	100	0	1.000	-0.95	0.4217	0.0616	0.0480
8	100	0	1.000	-0.75	0.2930	0.0616	0.0497
9	100	-2	0.980	-0.95	0.2698	0.0587	0.0505
10	100	-2	0.980	-0.75	0.2104	0.0588	0.0489
11	100	-20	0.802	-0.95	0.1063	0.0622	0.0471
12	100	-20	0.802	-0.75	0.0874	0.0514	0.0500
13	250	0	1.000	-0.95	0.4259	0.0476	0.0483
14	250	0	1.000	-0.75	0.2970	0.0506	0.0536
15	250	-2	0.992	-0.95	0.2866	0.0507	0.0481
16	250	-2	0.992	-0.75	0.2092	0.0466	0.0492
17	250	-20	0.920	-0.95	0.1080	0.0406	0.0517
18	250	-20	0.920	-0.75	0.0944	0.0369	0.0501

Replicated and Updated Results

Table 2: Empirical results

Data was taken from Amit Goyal's Website. Stock returns are the SP 500 index log-returns from 1926 to 2017 from the Center for Research in Security Press (CRSP) minus the rolled over 3-month T-bill rate. ep is the log 10 year moving average earnings/price ratio (1926 to 2017). dp is the log dividend/price ratio (1926 to 2017). tbl is the 3-month T-bill rate (1952 to 2017). tms is the term-spread between long-term government bonds and tbl (1952 to 2017).

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $(\sigma_e/\sigma_u) \hat{\beta}$
Ann	ep	-0.97	[0.827,0.979]	2.12	0	0.114	[-0.01,0.18]
Ann	dp	-0.86	[0.875,0.986]	0.97	0	0.042	[-0.069,0.107]
Ann	tbl	0.08	[0.853,0.908]	-0.41	1	-0.279	[-0.125,0.075]
Ann	tms	-0.06	[0.454,0.575]	0.96	1	1.297	[-0.071,0.275]
Qua	ep	-0.98	[0.973,1.002]	3.16	0	0.048	[0.001,0.039]
Qua	dp	-0.95	[0.976,1.002]	1.82	0	0.023	[-0.012,0.026]
Qua	tbl	-0.09	[0.956,0.975]	-0.64	1	-0.099	[-0.045,0.019]
Qua	tms	0.06	[0.835,0.864]	1.44	1	0.489	[-0.005,0.104]
Mon	ep	-0.99	[0.993,1.002]	2.40	0	0.010	[-0.002,0.009]
Mon	dp	-0.98	[0.993,1.002]	1.20	0	0.004	[-0.005,0.006]
Mon	tbl	-0.13	[0.992,0.997]	-0.91	1	-0.043	[-0.013,0.003]
Mon	tms	0.04	[0.958,0.967]	1.49	1	0.156	[-0.001,0.032]

Results Reported by Campbell and Yogo (2006)

Tests of predictability						
Series	Variable	t -stat	$\hat{\beta}$	90% CI: β		Low CI β ($\rho = 1$)
				t -test	Q -test	
<i>Panel A: S&P 1880–2002, CRSP 1926–2002</i>						
S&P 500	d - p	1.967	0.093	[−0.040, 0.136]	[−0.033, 0.114]	−0.017
	e - p	2.762	0.131	[−0.003, 0.189]	[0.042, 0.224]	−0.023
Annual	d - p	2.534	0.125	[−0.007, 0.178]	[0.014, 0.188]	0.020
	e - p	2.770	0.169	[−0.009, 0.240]	[0.042, 0.277]	0.002
Quarterly	d - p	2.060	0.034	[−0.014, 0.052]	[−0.009, 0.044]	−0.010
	e - p	2.908	0.049	[−0.001, 0.068]	[0.010, 0.066]	0.002
Monthly	d - p	1.706	0.009	[−0.006, 0.014]	[−0.005, 0.010]	−0.005
	e - p	2.662	0.014	[−0.001, 0.019]	[0.002, 0.018]	0.001
<i>Panel C: CRSP 1952–2002</i>						
Annual	d - p	2.289	0.124	[−0.023, 0.178]	[−0.007, 0.183]	0.020
	e - p	1.733	0.114	[−0.078, 0.178]	[−0.031, 0.229]	−0.025
	r_3	−1.143	−0.095	[−0.229, 0.045]	[−0.231, 0.042]	—
Quarterly	y - r_1	1.124	0.136	[−0.087, 0.324]	[−0.075, 0.359]	−0.156
	d - p	2.236	0.036	[−0.011, 0.051]	[−0.010, 0.030]	0.005
	e - p	1.777	0.029	[−0.019, 0.044]	[−0.012, 0.042]	−0.003
Monthly	r_3	−1.766	−0.042	[−0.084, −0.004]	[−0.084, −0.004]	−0.086
	y - r_1	1.991	0.090	[0.009, 0.162]	[0.006, 0.158]	−0.002
	d - p	2.259	0.012	[−0.004, 0.017]	[−0.004, 0.010]	0.001
	e - p	1.754	0.009	[−0.006, 0.014]	[−0.004, 0.012]	−0.001
	r_3	−2.431	−0.017	[−0.030, −0.006]	[−0.030, −0.006]	−0.030
	y - r_1	2.963	0.047	[0.020, 0.072]	[0.020, 0.072]	0.016

This table reports statistics used to infer the predictability of returns. Returns are for the annual S&P 500 index and the annual, quarterly, and monthly CRSP value-weighted index. The predictor variables are the log dividend–price ratio ($d-p$), the log earnings–price ratio ($e-p$), the three-month T-bill rate (r_3), and the long-short yield spread ($y-r_1$). The third and fourth columns report the t -statistic and the point estimate $\hat{\beta}$ from an OLS regression of returns onto the predictor variable. The next two columns report the 90% Bonferroni confidence intervals for β using the t -test and Q -test, respectively. Confidence intervals that reject the null are in bold. The final column reports the lower bound of the confidence interval for β based on the Q -test at $\rho = 1$.

Table 3: Empirical results OOS from 2003 to 2017.

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $(\sigma_e/\sigma_u) \hat{\beta}$
Qua	ep	-0.99	[0.79,1.015]	1.51	0	0.102	[-0.051,0.203]
Qua	dp	-0.97	[0.719,0.966]	0.56	0	0.037	[-0.097,0.198]
Qua	tbl	0.33	[0.95,0.987]	-0.70	1	-0.430	[-0.069,0.021]
Qua	tms	0.18	[0.9,0.949]	0.20	1	0.164	[-0.07,0.097]

- All source code is available at <https://github.com/jpwoeltjen/PersistentRegressors>
- Includes a R package that implements the methods discussed called pr (build it from source).
- Furthermore refer to the appendix for a mathematical implementation of the tests.

Appendix

- Seeks power gain by assuming ρ is in the neighborhood of 1.
- Two possible alternative hypotheses: y_t is stationary around a linear trend or y_t is stationary with no linear time trend.
- Here we assume no linear time trend.
- Generate the following variables:

$$\tilde{y}_1 = y_1$$

$$\tilde{y}_t = y_t - \rho_{GLS} y_{t-1}, \quad t = 2, \dots, T$$

- $x_1 = 1$

$$x_t = 1 - \rho_{GLS}, \quad t = 2, \dots, T$$

$$\rho_{GLS} = 1 - T^{-1}$$

- estimate by OLS $\tilde{y}_t = \delta_0 x_t + \epsilon_t$
- The OLS estimator $\hat{\delta}_0$ is then used to remove the mean from y_t that is, we generate

$$y^* = y_t - \hat{\delta}_0$$

- perform augmented Dickey–Fuller test on the transformed variable by fitting the OLS regression:

$$\Delta y_t^* = \beta y_{t-1}^* + \sum_{j=1}^{k-1} \zeta_j \Delta y_{t-j}^* + \epsilon_t$$

- For AR(1) this reduces to:

$$\Delta y_t^* = \beta y_{t-1}^* + \epsilon_t$$

- The t-statistic of β is the DF-GLS test statistic.
- Test the null hypothesis $H_0 : \beta = 0$ by using tabulated critical values.

Bonferroni Implementation $[\underline{\rho}, \overline{\rho}] = [1 + \underline{c}/T, 1 + \overline{c}/T]$

- Run the regression $r_t = \alpha + \beta x_{t-1} + u_t$ to get $\text{SE}(\hat{\beta})$.
- Run the regression $x_t = \gamma + \rho x_{t-1} + e_t$ to get $\text{SE}(\hat{\rho})$.
- Use the residuals \hat{u}_t and \hat{e}_t to compute:
-

$$\hat{\sigma}_u^2 = \frac{1}{T-2} \sum_{t=1}^T \hat{u}_t^2$$

$$\hat{\sigma}_e^2 = \frac{1}{T-2} \sum_{t=1}^T \hat{e}_t^2$$

$$\hat{\sigma}_{ue} = \frac{1}{T-2} \sum_{t=1}^T \hat{u}_t \hat{e}_t$$

$$\hat{\delta} = \frac{\hat{\sigma}_{ue}}{\hat{\sigma}_u \hat{\sigma}_e}$$

Bonferroni Implementation $[\underline{\rho}, \bar{\rho}] = [1 + \underline{c}/T, 1 + \bar{c}/T]$ (Cont.)

- Compute the DF-GLS statistic.
- Given DF-GLS statistic and $\hat{\delta}$ use lookup tables to get $[\underline{c}, \bar{c}]$.⁶
- Now we can compute the confidence interval for ρ which is given by $[\underline{\rho}, \bar{\rho}] = [1 + \underline{c}/T, 1 + \bar{c}/T]$

⁶Lookup tables are provided by Campbell, J.Y., Yogo, M., 2005. Implementing the econometric methods in “Efficient tests of stock return predictability”. Unpublished working paper. University of Pennsylvania.

Bonferroni Implementation $[\underline{\beta}(\rho), \overline{\beta}(\rho)]$

- Run the regression $r_t^* = \alpha + \beta x_{t-1} + u_t$ for each $\rho = \{\underline{\rho}, \overline{\rho}\}$ to get $\hat{\beta}(\rho)$.
- Where $r_t^* = r_t - \hat{\sigma}_{ue} \hat{\sigma}_e^{-2} (x_t - \rho x_{t-1})$.
- The confidence interval for β given ρ is $[\underline{\beta}(\rho), \overline{\beta}(\rho)]$

- Where

$$\begin{aligned}\underline{\beta}(\rho) &= \hat{\beta}(\rho) - 1.645 \left(1 - \hat{\delta}^2\right)^{1/2} \text{SE}(\hat{\beta}) \\ \overline{\beta}(\rho) &= \hat{\beta}(\rho) + 1.645 \left(1 - \hat{\delta}^2\right)^{1/2} \text{SE}(\hat{\beta})\end{aligned}$$

- The 90% Bonferroni confidence interval $[\underline{\beta}(\overline{\rho}), \overline{\beta}(\rho)]$ corresponds to a 10% two-sided test or a 5% one-sided test of the null hypothesis $\beta = 0$.

- Under the local alternative

$$Q(\beta_0, \tilde{\rho}) = \frac{b \left(T^{-2} \sum_{t=1}^T x_{t-1}^{\mu 2} \right)^{1/2}}{\sigma_u (1 - \delta^2)^{1/2}} + \frac{\delta(\tilde{c} - c) \left(T^{-2} \sum_{t=1}^T x_{t-1}^{\mu 2} \right)^{1/2}}{\omega (1 - \delta^2)^{1/2}} + \frac{T^{-1} \sum_{t=1}^T x_{t-1}^{\mu} (u_t - \sigma_{ue} / (\sigma_e \omega) v_t) + \frac{1}{2} \sigma_{ue} / (\sigma_e \omega) (\omega^2 - \sigma_v^2)}{\sigma_u (1 - \delta^2)^{1/2} \left(T^{-2} \sum_{t=1}^T x_{t-1}^{\mu 2} \right)^{1/2}} \quad (13)$$

- where $\tilde{c} = T(\tilde{\rho} - 1)$

$$Q(\beta_0, \tilde{\rho}) \Rightarrow \frac{b\omega\kappa_c}{\sigma_u (1 - \delta^2)^{1/2}} + \frac{\delta(\tilde{c} - c)\kappa_c}{(1 - \delta^2)^{1/2}} + Z \quad (14)$$

- The power function for a right-tailed test is

$$\pi_Q(b) = \mathbb{E} \left[\Phi \left(z_\alpha - \frac{b\omega\kappa_c}{\sigma_u(1-\delta^2)^{1/2}} - \frac{\delta(\tilde{c} - c)\kappa_c}{(1-\delta^2)^{1/2}} \right) \right] \quad (15)$$

- Where $\Phi(z)$ is one minus the standard normal CDF, z_α is the $1 - \alpha$ quantile.
- The expectation is taken over the distribution of $(W_e(s), J_c(s))$.