Predictive Regressions with Persistent Regressors

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Outline

- The regression problem
- Local to unity asymptotics
- Constructing a pretest
- Motivating more efficient tests
- Constructing the test statistic
- Making the test feasible
- Analysis of the power gain
- Empirical results

The Regression Setup

- Campbell and Yogo (2006)[1] (CY) consider the system of equations:
-

$$r_t = \alpha + \beta x_{t-1} + u_t \tag{1}$$
$$x_t = \gamma + \rho x_{t-1} + e_t \tag{2}$$

- For the sake of simplicity they further assume normality:

$$w_t = (u_t, e_t)' \sim \mathrm{N}(0, \Sigma), \text{ where } \Sigma \text{ is known}$$
$$\Sigma = \begin{bmatrix} \sigma_u^2 & \sigma_{ue} \\ \sigma_{\text{ue}} & \sigma_e^2 \end{bmatrix}$$

[1] Campbell, J. Y. and M. Yogo (2006). Efficient tests of stock return predictability. Journal of Financial Economics 81, 27-60.

Testing for Significance

- To test β for significance, CY start by considering the maximum likelihood ratio test.

$$\max_{\beta,\rho,\alpha,\gamma} L(\beta,\rho,\alpha,\gamma) - \max_{\rho,\alpha,\gamma} L\left(\beta_0,\rho,\alpha,\gamma\right) = t\left(\beta_0\right)^2 > C,$$

- where C is some constant and the joint log likelihood (ignoring two constants) is given by

$$\begin{aligned} L(\beta,\rho,\alpha,\gamma) = & -\frac{1}{1-\delta^2}\sum_{t=1}^{T}\Bigg[\frac{\left(r_t-\alpha-\beta x_{t-1}\right)^2}{\sigma_u^2} \\ & -2\delta\frac{\left(r_t-\alpha-\beta x_{t-1}\right)\left(x_t-\gamma-\rho x_{t-1}\right)}{\sigma_u\sigma_e} \\ & +\frac{\left(x_t-\gamma-\rho x_{t-1}\right)^2}{\sigma_e^2}\Bigg] \end{aligned} \tag{3}$$

The T-Test Ignores Information

- The LRT from above turns out to be the same when considering only the marginal likelihood

$$L(\beta, \alpha) = -\sum_{t=1}^{T} (r_t - \alpha - \beta x_{t-1})^2 \tag{4}$$

.

- It thus ignores information contained in the system.

Local to Unity Asymptotics

- The largest autoregressive root is modeled as $\rho = 1 + c/T$ where c is a fixed constant.
- This does not imply that ρ literally follows this process in practice. It is merely a tool to model autoregressive roots very close to one. It has some nice properties:
- Asymptotic distribution theory is not discontinuous when x_t is $I(1)$.

Pretesting for Size Distortions of the T-Test

- Under local-to-unity asymptotic theory the t-statistic does not converge to a standard normal distribution under the null but to functionals of a diffusion process.
- Under the null it convergences to:

$$t\left(\beta_0\right) \Rightarrow \delta \frac{\tau_c}{\kappa_c} + \left(1-\delta^2\right)^{1/2} Z, \tag{5}$$

- where

$$\kappa_c = \left(\int J_c^{\mu}(s)^2 ds\right)^{1/2}, \tau_c = \int J_c^{\mu}(s) dW_e(s),$$

and $Z \sim \mathrm{N}(0,1)$ independent of $\left(W_e(s), J_c(s)\right)$

- $(W_u(s), W_e(s))'$ is a two-dimensional Wiener process with correlation δ.
- $J_c(s)$ solving $\mathrm{d}J_c(s) = cJ_c(s)\mathrm{d}s + \mathrm{d}W_e(s)$ with initial condition $J_c(0) = 0$.
- $J_c^{\mu}(s) = J_c(s) - \int J_c(r)\mathrm{d}r$

Pretesting for Size Distortions of the T-Test (Cont.)

- If $\delta = 0$ the first term of (3) vanishes and the distribution under the null is the usual $\mathrm{N}(0, 1)$.
- Likewise if $c << 0$, x is not persistent and first-order asymptotics are valid.

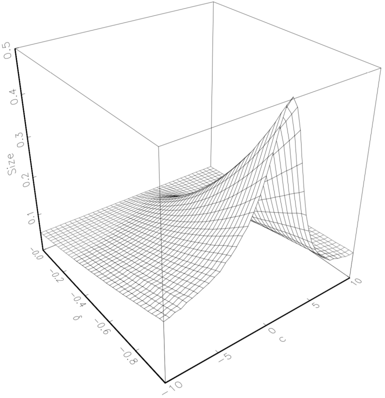


Figure 1: $p(c, \delta; 0.05) = \Pr\left(\delta \frac{\tau_c}{\kappa_c} + \left(1 - \delta^2\right)^{1/2} Z > z_{0.05}\right)$ Asymptotic Size of the One-Sided t-test at 5% Significance

Pretesting for Size Distortions of the T-Test (Cont.)

- CY propose that we may accept an actual size below 0.075.
- Test against the null of actual size greater than 0.075.
- Estimate δ from residuals of (1) and (2).
- Construct confidence interval for c by computing Dickey-Fuller generalized least squares (DF-GLS) test statistic[2] and using its known distribution under the alternative to construct the confidence interval for c.
- Reject the null if the confidence interval for c lies strictly below (or above) the region of the parameter space (c_{min}, c_{max}).
- Implemented in R package{pr}.

[2] For a detailed description of the DF-GLS statistic see the appendix

Motivating the Importance of Efficient Tests

- Once inferred that the t-test is size distorted, we could adjust the rejection region to account for the size distortion. This, however, is very inefficient, i.e., we need a lot of data to reject the null at a decent size.
- Testing efficiency has obviously importance when data is sparse as is the case when predicting index returns.
- When developing trading strategies, however, one could increase the data amount by looking at the cross-section of individual stocks. This could get us above a threshold where first order asymptotics become reliable.
- One could imagine a dynamic trading strategy that takes into account the changing nature of the market. Predictive regressions are performed on an ongoing basis and updated as soon as new information becomes available (Online Learning). Higher Pitman efficiency of our tests could give us the advantage of earlier detection of market anomalies.

Developing a More Efficient Test

- To improve confidence of inference, we need to increase the signal-to-noise ratio. Here, we focus on reducing the noise.
- Since the innovations of (1) and (2) are correlated, CY propose to subtract off this part of the innovations.
- In contrast to the t-test, this procedure takes advantage of all the information contained in the system.
- Assume for the following $\alpha = \gamma = 0$ and ρ is known.

The Q-test

- Recall the joint log likelihood:

$$L(\beta,\rho,\alpha,\gamma) = -\frac{1}{1-\delta^2}\sum_{t=1}^{T}\Bigg[\frac{\left(r_t-\alpha-\beta x_{t-1}\right)^2}{\sigma_u^2} - 2\delta\frac{\left(r_t-\alpha-\beta x_{t-1}\right)\left(x_t-\gamma-\rho x_{t-1}\right)}{\sigma_u\sigma_e} + \frac{\left(x_t-\gamma-\rho x_{t-1}\right)^2}{\sigma_e^2}\Bigg]$$

- By the Neyman–Pearson Lemma the most powerful test against the simple alternative (i.e., a alternative that uniquely specifies the distribution) $\beta = \beta_1$ rejects the null if the LR is greater than some constant:

$$\sigma_u^2\left(1-\delta^2\right)\left(L\left(\beta_1\right)-L\left(\beta_0\right)\right) = 2\left(\beta_1-\beta_0\right)\sum_{t=1}^{T}x_{t-1}\left[r_t-\beta_{ue}\left(x_t-\rho x_{t-1}\right)\right] - \left(\beta_1^2-\beta_0^2\right)\sum_{t=1}^{T}x_{t-1}^2 > C \tag{6}$$

- Where $\beta_{ue} = \sigma_{ue}/\sigma_e^2$

The Q-test

- Observe that $\sum_{i=1}^{T} x_{t-1}^2$ does not depend on β.
- CY condition the test on that statistic in order to reduce it to

$$\sum_{t=1}^{T} x_{t-1} \left[r_t - \beta_{ue} \left(x_t - \rho x_{t-1} \right) \right] > C \tag{7}$$

- To get a standard normal distribution under the null they recenter and rescale:

$$\frac{\sum_{t=1}^{T} x_{t-1} \left[r_t - \beta_0 x_{t-1} - \beta_{ue} \left(x_t - \rho x_{t-1} \right) \right]}{\sigma_u \left(1 - \delta^2\right)^{1/2} \left(\sum_{t=1}^{T} x_{t-1}^2 \right)^{1/2}} > C \tag{8}$$

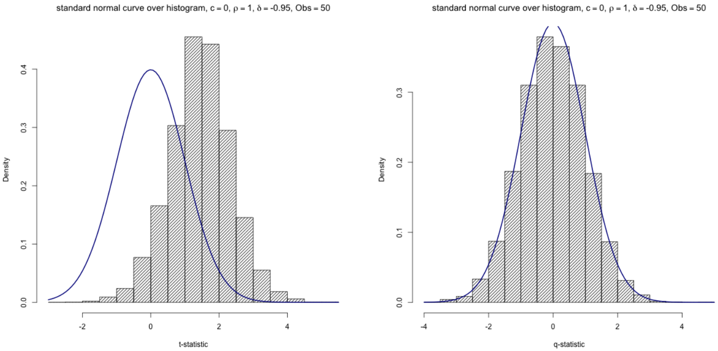
- Finally, after CY de-mean x_{t-1} and denote it by x_{t-1}^{μ} they can eliminate the assumption $\alpha = \gamma = 0$ to get:

$$Q\left(\beta_0, \rho\right) = \frac{\sum_{t=1}^{T} x_{t-1}^{\mu} \left[r_t - \beta_0 x_{t-1} - \beta_{ue} \left(x_t - \rho x_{t-1} \right) \right]}{\sigma_u \left(1 - \delta^2\right)^{1/2} \left(\sum_{t=1}^{T} x_{t-1}^{\mu 2} \right)^{1/2}} \tag{9}$$

Intuition for Q-test

- Consider the case $\beta_0 = 0$.
- Then
$$Q(\beta_0, \rho) = \frac{\sum_{t=1}^{T} x_{t-1}^{\mu} \left[r_t - \beta_0 x_{t-1} - \beta_{ue} \left(x_t - \rho x_{t-1} \right) \right]}{\sigma_u \left(1 - \delta^2\right)^{1/2} \left(\sum_{t=1}^{T} x_{t-1}^{\mu 2} \right)^{1/2}}$$
is the t-statistic of the coefficient b regressing
$$r_t - \beta_{ue} \left(x_t - \rho x_{t-1} \right) = a + b x_{t-1} + v_t$$
- $x_t - \rho x_{t-1} = e_t + \gamma$
- $\beta_{ue} = \sigma_{ue} / \sigma_e^2$ is the correlation between shocks.
- The equation above can be interpreted as regressing the de-noised returns onto the regressor x where we exploit the information contained in ρ and the correlation of the shocks.

Making the Test Feasible



- If ρ and δ are known, this Q-test is the best we can do. It's UMP.
- In practice, however, ρ and δ are not known.
- ρ cannot be estimated consistently.
- CY use Bonferroni's inequality to get confidence intervals.

Bonferroni Confidence Intervals

- Construct a $(1-\alpha_1)$ confidence interval for ρ: $C_\rho(\alpha_1)$
- For each value of ρ in the confidence interval, construct a $(1-\alpha_2)$ confidence interval for β given ρ: $C_{\beta|\rho}(\alpha_2)$
- Taking the union over all $\rho \in C_\rho(\alpha_1)$ we can marginalize ρ:

$$C_\beta(\alpha) = \bigcup_{\rho \in C_\rho(\alpha_1)} C_{\beta|\rho}(\alpha_2) \tag{10}$$

- with $\alpha = \alpha_1 + \alpha_2$, $C_\beta(\alpha)$ has coverage of at least $(1-\alpha)$
- This follows from Bonferroni's inequality:

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$$

where $P(E_i)$ is the probability that E_i is true and $P\left(\bigcup_{i=1}^{n} E_i\right)$ is the probability that at least one of $E_1, E_2, \ldots, E_n$ is true [3].

[3] Weisstein, Eric W. "Bonferroni Inequalities." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BonferroniInequalities.html

Bonferroni Confidence Intervals (Cont.)

- To get the confidence interval for ρ CY need a unit root test statistic.
- Since they suspect ρ in the neighborhood of 1, the DF-GLS test statistic[4] is a good choice.
- To get the confidence interval for β they use the Q-test since they know it to be more powerful than the t-test given true ρ and hope it remains more powerful for other ρ as well. Whether this hope is met will only be seen by Monte Carlo evidence.

[4] For a detailed description of the DF-GLS statistic see the appendix

Bonferroni Confidence Intervals (Cont.)

$$C_{\beta|\rho}(\alpha_2) = \left[\underline{\beta}(\rho, \alpha_2), \overline{\beta}(\rho, \alpha_2)\right]$$

- where

$$\beta(\rho) = \frac{\sum_{t=1}^{T} x_{t-1}^{\mu}\left[r_t - \beta_{ue}\left(x_t - \rho x_{t-1}\right)\right]}{\sum_{t=1}^{T} x_{t-1}^{\mu 2}})$$

$$\underline{\beta}(\rho, \alpha_2) = \beta(\rho) - z_{\alpha_2/2}\sigma_u \left(\frac{1-\delta^2}{\sum_{t=1}^{T} x_{t-1}^{\mu 2}}\right)^{1/2}$$

$$\overline{\beta}(\rho, \alpha_2) = \beta(\rho) + z_{\alpha_2/2}\sigma_u \left(\frac{1-\delta^2}{\sum_{t=1}^{T} x_{t-1}^{\mu 2}}\right)^{1/2}$$

$z_{\alpha_2/2}$ denotes the $1 - \alpha_2/2$ quantile of the standard normal distribution.

- $C_\rho(\alpha_1) = \left[\underline{\rho}(\underline{\alpha}_1), \overline{\rho}(\overline{\alpha}_1)\right]$ denotes the confidence interval for ρ,
- where $\underline{\alpha}_1 = \Pr\left(\rho < \underline{\rho}(\underline{\alpha}_1)\right)$, $\overline{\alpha}_1 = \Pr\left(\rho > \overline{\rho}(\overline{\alpha}_1)\right)$, and $\alpha_1 = \underline{\alpha}_1 + \overline{\alpha}_1$.
- Then the Bonferroni confidence interval is:

$$C_\beta(\alpha) = \left[\underline{\beta}\left(\overline{\rho}(\overline{\alpha}_1), \alpha_2\right), \overline{\beta}\left(\underline{\rho}(\underline{\alpha}_1), \alpha_2\right)\right] \tag{11}$$

We Can Do Better

- The Bonferroni confidence interval is likely to be conservative, i.e., $\Pr\left(\beta \notin C_\beta(\alpha)\right) \leqslant \alpha_2\left(1-\alpha_1\right)+\alpha_1 \leqslant \alpha$ is likely a strict inequality.
- To see this consider:

$$\begin{aligned} \Pr\left(\beta \notin C_\beta(\alpha)\right) = & \Pr\left(\beta \notin C_\beta(\alpha) | \rho \in C_\rho\left(\alpha_1\right)\right) \Pr\left(\rho \in C_\rho\left(\alpha_1\right)\right) \\ & + \Pr\left(\beta \notin C_\beta(\alpha) | \rho \notin C_\rho\left(\alpha_1\right)\right) \Pr\left(\rho \notin C_\rho\left(\alpha_1\right)\right) \end{aligned} \tag{12}$$

- $\Pr\left(\beta \notin C_\beta(\alpha) | \rho \notin C_\rho\left(\alpha_1\right)\right)$ is unknown. Thus we have to assume the worst case and bound it by one.
- $\Pr\left(\beta \notin C_\beta(\alpha) | \rho \in C_\rho\left(\alpha_1\right)\right) \leq \alpha_2$ is strict if $C_{\beta|\rho}\left(\alpha_2\right)$ depends on ρ.
- Bonferroni confidence interval is conservative since it is built on worst case scenario and reality is likely less harsh (more convervative the smaller is δ in absolute value).
- Refine confidence interval for ρ until the confidence interval for β is exactly at the desired significance level $\widetilde{\alpha}$.

Refined Bonferroni Method

- Do this by numerically searching over a grid.

1. fix α_2.
2. for each ρ, numerically search to find the $\overline{\alpha}_1$ s.t. :

- $\Pr\left(\underline{\beta}\left(\overline{\rho}\left(\overline{\alpha}_1\right), \alpha_2\right) > \beta\right) \leq \widetilde{\alpha}/2$ holds for all values of c on the grid,
- and $\Pr\left(\underline{\beta}\left(\overline{\rho}\left(\overline{\alpha}_1\right), \alpha_2\right) > \beta\right) = \widetilde{\alpha}/2$ at some point on the grid.

3. repeat 2. for $\underline{\alpha}_1$ s.t. $\Pr\left(\overline{\beta}\left(\underline{\rho}\left(\underline{\alpha}_1\right), \alpha_2\right) < \beta\right) \leq \widetilde{\alpha}/2$

- $[\underline{\alpha}_1, \overline{\alpha}_1]$ is a tighter confidence interval for ρ.
- one-sided Bonferroni test[5] has exact size $\widetilde{\alpha}/2$ for some permissible value of c.
- two-sided Bonferroni test has at most size $\widetilde{\alpha}$ for all c.

[5] For a detailed description of how to implement this test using OLS refer to the appendix. Alternatively, read the source code of R package{pr}

Comparing Power

- All tests considered should reject alternatives of the form $\beta = \beta_0 + b$, where b is some constant, almost surely as $T \to \infty$.
- More interesting are alternatives of the form $\beta = \beta_0 + b/T$, where b is again some constant.

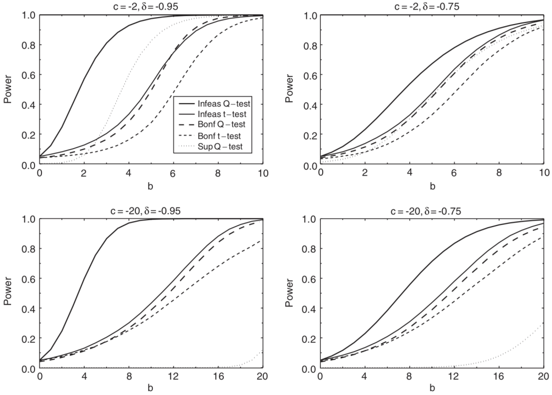


Figure 2: Null: $\beta = \beta_0$ against local alternatives: $b = T(\beta - \beta_0) > 0$ from Campbell and Yogo (2006)

Analysis of the Power Gain

- CY Compare:

1. A Bonferroni test based on the ADF test and the t-test.
2. A Bonferroni test based on the DF-GLS test and the t-test.
3. A Bonferroni test based on the DF-GLS test and the Q-test.

Analysis of the Power Gain (Cont.)

$c = -2$ and $\rho = -0.95$

- Pitman efficiency (relative number of observations needed to achieve 50% power) of test 1 relative to test 2 is 1.03.
- Pitman efficiency of test 2 relative to test 3 is 1.20.
- Close to no difference in power between the unrefined and refined Bonferroni t-test.
- Pitman efficiency of the unrefined relative to the refined Bonferroni Q-test is 1.62.

Analysis of the Power Gain (Cont.)

$c = -20$ and $\rho = -0.95$

- Pitman efficiency of test 1 relative to test 2 is 1.07.
- Pitman efficiency of test 2 relative to test 3 is 1.03.
- Pitman efficiency of the unrefined relative to the refined Bonferroni t-test is 1.23.
- Pitman efficiency of Bonferroni Q-test is 1.55 .

Analysis of the Power Gain (Cont.)

- When the regressor highly persistent, the use of the Q-test rather than the t-test is a relatively important source of power gain for the Bonferroni Q-test.
- When the predictor variable is less persistent, the use of the DF-GLS test rather than the ADF test is a relatively important source of power gain for the Bonferroni Q-test.
- Bonferroni refinement is an especially important source of power gain for the Bonferroni Q-test since it tries to exploit information about ρ. This makes its confidence interval for β given ρ more sensitive to ρ resulting in a too conservative Bonferroni test without the refinement.

Finite Sample Rejection Rates (10,000 Monte Carlo runs)

Table 1: Finite-sample rejection rates for tests of predictability

	Obs	c	ρ	δ	T-test	Bonf.Q-test	Q-test
1	50	0	1.000	-0.95	0.4160	0.0826	0.0483
2	50	0	1.000	-0.75	0.2916	0.0837	0.0515
3	50	-2	0.961	-0.95	0.2714	0.0868	0.0482
4	50	-2	0.961	-0.75	0.2079	0.0881	0.0532
5	50	-20	0.608	-0.95	0.0977	0.1206	0.0515
6	50	-20	0.608	-0.75	0.0840	0.1078	0.0484
7	100	0	1.000	-0.95	0.4217	0.0616	0.0480
8	100	0	1.000	-0.75	0.2930	0.0616	0.0497
9	100	-2	0.980	-0.95	0.2698	0.0587	0.0505
10	100	-2	0.980	-0.75	0.2104	0.0588	0.0489
11	100	-20	0.802	-0.95	0.1063	0.0622	0.0471
12	100	-20	0.802	-0.75	0.0874	0.0514	0.0500
13	250	0	1.000	-0.95	0.4259	0.0476	0.0483
14	250	0	1.000	-0.75	0.2970	0.0506	0.0536
15	250	-2	0.992	-0.95	0.2866	0.0507	0.0481
16	250	-2	0.992	-0.75	0.2092	0.0466	0.0492
17	250	-20	0.920	-0.95	0.1080	0.0406	0.0517
18	250	-20	0.920	-0.75	0.0944	0.0369	0.0501

Results

Table 2: Empirical results
Data was taken from Amit Goyal's Website. Stock returns are the SP 500 index log-returns from 1926 to 2017 from the Center for Research in Security Press (CRSP) minus the rolled over 3-month T-bill rate. ep is the log 10 year moving average earnings/price ratio (1926 to 2017). dp is the log dividend/price ratio (1926 to 2017). tbl is the 3-month T-bill rate (1952 to 2017). tms is the term-spread between long-term government bonds and tbl (1952 to 2017).

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $\hat{\beta}$
Ann	ep	-0.97	[0.827,0.979]	2.12	0	0.114	[-0.01,0.18]
Ann	dp	-0.86	[0.875,0.986]	0.97	0	0.042	[-0.069,0.107]
Ann	tbl	0.08	[0.853,0.908]	-0.41	1	-0.279	[-0.125,0.075]
Ann	tms	-0.06	[0.454,0.575]	0.96	1	1.297	[-0.071,0.275]
Qua	ep	-0.98	[0.973,1.002]	3.16	0	0.048	[0.001,0.039]
Qua	dp	-0.95	[0.976,1.002]	1.82	0	0.023	[-0.012,0.026]
Qua	tbl	-0.09	[0.956,0.975]	-0.64	1	-0.099	[-0.045,0.019]
Qua	tms	0.06	[0.835,0.864]	1.44	1	0.489	[-0.005,0.104]
Mon	ep	-0.99	[0.993,1.002]	2.40	0	0.010	[-0.002,0.009]
Mon	dp	-0.98	[0.993,1.002]	1.20	0	0.004	[-0.005,0.006]
Mon	tbl	-0.13	[0.992,0.997]	-0.91	1	-0.043	[-0.013,0.003]
Mon	tms	0.04	[0.958,0.967]	1.49	1	0.156	[-0.001,0.032]

Results Reported by Campbell and Yogo (2006)

Tests of predictability

Series	Variable	t-stat	$\widehat{\beta}$	90% CI: β t-test	90% CI: β Q-test	Low CI β ($\rho = 1$)
Panel A: S&P 1880–2002, CRSP 1926–2002						
S&P 500	d–p	1.967	0.093	[−0.040, 0.136]	[−0.033, 0.114]	−0.017
	e–p	2.762	0.131	[−0.003, 0.189]	**[0.042, 0.224]**	−0.023
Annual	d–p	2.534	0.125	[−0.007, 0.178]	**[0.014, 0.188]**	0.020
	e–p	2.770	0.169	[−0.009, 0.240]	**[0.042, 0.277]**	0.002
Quarterly	d–p	2.060	0.034	[−0.014, 0.052]	[−0.009, 0.044]	−0.010
	e–p	2.908	0.049	[−0.001, 0.068]	**[0.010, 0.066]**	0.002
Monthly	d–p	1.706	0.009	[−0.006, 0.014]	[−0.005, 0.010]	−0.005
	e–p	2.662	0.014	[−0.001, 0.019]	**[0.002, 0.018]**	0.001
Panel C: CRSP 1952–2002						
Annual	d–p	2.289	0.124	[−0.023, 0.178]	[−0.007, 0.183]	0.020
	e–p	1.733	0.114	[−0.078, 0.178]	[−0.031, 0.229]	−0.025
	r_3	−1.143	−0.095	[−0.229, 0.045]	[−0.231, 0.042]	—
	y–r_1	1.124	0.136	[−0.087, 0.324]	[−0.075, 0.359]	−0.156
Quarterly	d–p	2.236	0.036	[−0.011, 0.051]	[−0.010, 0.030]	0.005
	e–p	1.777	0.029	[−0.019, 0.044]	[−0.012, 0.042]	−0.003
	r_3	−1.766	−0.042	**[−0.084, −0.004]**	**[−0.084, −0.004]**	−0.086
	y–r_1	1.991	0.090	**[0.009, 0.162]**	**[0.006, 0.158]**	−0.002
Monthly	d–p	2.259	0.012	[−0.004, 0.017]	[−0.004, 0.010]	0.001
	e–p	1.754	0.009	[−0.006, 0.014]	[−0.004, 0.012]	−0.001
	r_3	−2.431	−0.017	**[−0.030, −0.006]**	**[−0.030, −0.006]**	−0.030
	y–r_1	2.963	0.047	**[0.020, 0.072]**	**[0.020, 0.072]**	0.016

This table reports statistics used to infer the predictability of returns. Returns are for the annual S&P 500 index and the annual, quarterly, and monthly CRSP value-weighted index. The predictor variables are the log dividend–price ratio (d–p), the log earnings–price ratio (e–p), the three-month T-bill rate (r_3), and the long-short yield spread (y–r_1). The third and fourth columns report the t-statistic and the point estimate $\widehat{\beta}$ from an OLS regression of returns onto the predictor variable. The next two columns report the 90% Bonferroni confidence intervals for β using the t-test and Q-test, respectively. Confidence intervals that reject the null are in bold. The final column reports the lower bound of the confidence interval for β based on the Q-test at $\rho = 1$.

OOS results

Table 3: Empirical results OOS from 2003 to 2017.

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $\hat{\beta}$
Qua	ep	-0.99	[0.79,1.015]	1.51	0	0.102	[-0.051,0.203]
Qua	dp	-0.97	[0.719,0.966]	0.56	0	0.037	[-0.097,0.198]
Qua	tbl	0.33	[0.95,0.987]	-0.70	1	-0.430	[-0.069,0.021]
Qua	tms	0.18	[0.9,0.949]	0.20	1	0.164	[-0.07,0.097]

R Implementation

- All source code is available at http://
- Includes a R package that implements the methods discussed called pr (build it from source).
- Furthermore refer to the appendix for a mathematical implementation of the tests.

Appendix

DF-GLS

- Seeks power gain by assuming ρ is in the neighborhood of 1.
- Two possible alternative hypotheses: y_t is stationary around a linear trend or y_t is stationary with no linear time trend.
- Here we assume no linear time trend.
- Generate the following variables:
- $$\begin{aligned} &\widetilde{y}_1 = y_1 \\ &\widetilde{y}_t = y_t - \rho_{GLS} y_{t-1}, \quad t = 2, \ldots, T \\ &x_1 = 1 \\ &x_t = 1 - \rho_{GLS}, \qquad\quad t = 2, \ldots, T \\ &\rho_{GLS} = 1 - 7/T \end{aligned}$$
- estimate by OLS $\widetilde{y}_t = \delta_0 x_t + \epsilon_t$
- The OLS estimator $\widehat{\delta}_0$ is then used to remove the mean from y_t that is, we generate

$$y^* = y_t - \widehat{\delta}_0$$

DF-GLS (Cont.)

- perform augmented Dickey–Fuller test on the transformed variable by fitting the OLS regression:

$$\Delta y_t^* = \beta y_{t-1}^* + \sum_{j=1}^{k-1} \zeta_j \Delta y_{t-j}^* + \epsilon_t$$

- For AR(1) this reduces to:

$$\Delta y_t^* = \beta y_{t-1}^* + \epsilon_t$$

- The t-statistic of β is the DF-GLS test statistic.
- Test the null hypothesis $H_0 : \beta = 0$ by using tabulated critical values.

Bonferroni Implementation $[\underline{\rho}, \overline{\rho}] = [1 + \underline{c}/T, 1 + \overline{c}/T]$

- Run the regression $r_t = \alpha + \beta x_{t-1} + u_t$ to get SE($\widehat{\beta}$).
- Run the regression $x_t = \gamma + \rho x_{t-1} + e_t$ to get SE($\widehat{\rho}$).
- Use the residuals $\widehat{u}_t$ and $\widehat{e}_t$ to compute:
- $$\widehat{\sigma}_u^2 = \frac{1}{T-2} \sum_{t=1}^{T} \widehat{u}_t^2$$
  $$\widehat{\sigma}_e^2 = \frac{1}{T-2} \sum_{t=1}^{T} \hat{e}_t^2$$
  $$\widehat{\sigma}_{ue} = \frac{1}{T-2} \sum_{t=1}^{T} \widehat{u}_t \widehat{e}_t$$
  $$\widehat{\delta} = \frac{\widehat{\sigma}_{ue}}{\widehat{\sigma}_u \widehat{\sigma}_e}$$

Bonferroni Implementation $[\underline{\rho}, \overline{\rho}] = [1 + \underline{c}/T, 1 + \overline{c}/T]$ (Cont.)

- Compute the DF-GLS statistic.
- Given DF-GLS statistic and $\widehat{\delta}$ use lookup tables to get $[\underline{c}, \overline{c}]$.[6]
- Now we can compute the confidence interval for ρ which is given by $[\underline{\rho}, \overline{\rho}] = [1 + \underline{c}/T, 1 + \overline{c}/T]$

[6] Lookup tables are provided by Campbell, J.Y., Yogo, M., 2005. Implementing the econometric methods in "Efficient tests of stock return predictability". Unpublished working paper. University of Pennsylvania.

Bonferroni Implementation $[\underline{\beta}(\rho), \overline{\beta}(\rho)]$

- Run the regression $r_t^* = \alpha + \beta x_{t-1} + u_t$ for each $\rho = \{\underline{\rho}, \overline{\rho}\}$ to get $\widehat{\beta}(\rho)$.
- Where $r_t^* = r_t - \widehat{\sigma}_{ue}\widehat{\sigma}_e^{-2}(x_t - \rho x_{t-1})$.
- The confidence interval for β given ρ is $[\underline{\beta}(\rho), \overline{\beta}(\rho)]$
- Where

$$\underline{\beta}(\rho) = \widehat{\beta}(\rho) - 1.645\left(1 - \widehat{\delta}^2\right)^{1/2} \mathrm{SE}(\widehat{\beta})$$
$$\overline{\beta}(\rho) = \widehat{\beta}(\rho) + 1.645\left(1 - \widehat{\delta}^2\right)^{1/2} \mathrm{SE}(\widehat{\beta})$$

- The 90% Bonferroni confidence interval $[\underline{\beta}(\overline{\rho}), \overline{\beta}(\underline{\rho})]$ corresponds to a 10% two-sided test or a 5% one-sided test of the null hypothesis $\beta = 0$.

Finite Sample Rejection Rates (10,000 Monte Carlo runs)

Table 4: Finite-sample rejection rates for tests of predictability

	Obs	c	ρ	δ	T-test	Bonf.Q-test	Q-test
1	50	0	1.000	-0.95	0.4160	0.0826	0.0483
2	50	0	1.000	-0.75	0.2916	0.0837	0.0515
3	50	-2	0.961	-0.95	0.2714	0.0868	0.0482
4	50	-2	0.961	-0.75	0.2079	0.0881	0.0532
5	50	-20	0.608	-0.95	0.0977	0.1206	0.0515
6	50	-20	0.608	-0.75	0.0840	0.1078	0.0484
7	100	0	1.000	-0.95	0.4217	0.0616	0.0480
8	100	0	1.000	-0.75	0.2930	0.0616	0.0497
9	100	-2	0.980	-0.95	0.2698	0.0587	0.0505
10	100	-2	0.980	-0.75	0.2104	0.0588	0.0489
11	100	-20	0.802	-0.95	0.1063	0.0622	0.0471
12	100	-20	0.802	-0.75	0.0874	0.0514	0.0500
13	250	0	1.000	-0.95	0.4259	0.0476	0.0483
14	250	0	1.000	-0.75	0.2970	0.0506	0.0536
15	250	-2	0.992	-0.95	0.2866	0.0507	0.0481
16	250	-2	0.992	-0.75	0.2092	0.0466	0.0492
17	250	-20	0.920	-0.95	0.1080	0.0406	0.0517
18	250	-20	0.920	-0.75	0.0944	0.0369	0.0501

Results

Table 5: Empirical results
Data was taken from Amit Goyal's Website. Stock returns are the SP 500 index log-returns from 1926 to 2017 from the Center for Research in Security Press (CRSP) minus the rolled over 3-month T-bill rate. ep is the log 10 year moving average earnings/price ratio (1926 to 2017). dp is the log dividend/price ratio (1926 to 2017). tbl is the 3-month T-bill rate (1952 to 2017). tms is the term-spread between long-term government bonds and tbl (1952 to 2017).

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $\hat{\beta}$
Ann	ep	-0.97	[0.827,0.979]	2.12	0	0.114	[-0.01,0.18]
Ann	dp	-0.86	[0.875,0.986]	0.97	0	0.042	[-0.069,0.107]
Ann	tbl	0.08	[0.853,0.908]	-0.41	1	-0.279	[-0.125,0.075]
Ann	tms	-0.06	[0.454,0.575]	0.96	1	1.297	[-0.071,0.275]
Qua	ep	-0.98	[0.973,1.002]	3.16	0	0.048	[0.001,0.039]
Qua	dp	-0.95	[0.976,1.002]	1.82	0	0.023	[-0.012,0.026]
Qua	tbl	-0.09	[0.956,0.975]	-0.64	1	-0.099	[-0.045,0.019]
Qua	tms	0.06	[0.835,0.864]	1.44	1	0.489	[-0.005,0.104]
Mon	ep	-0.99	[0.993,1.002]	2.40	0	0.010	[-0.002,0.009]
Mon	dp	-0.98	[0.993,1.002]	1.20	0	0.004	[-0.005,0.006]
Mon	tbl	-0.13	[0.992,0.997]	-0.91	1	-0.043	[-0.013,0.003]
Mon	tms	0.04	[0.958,0.967]	1.49	1	0.156	[-0.001,0.032]

Results Reported by Campbell and Yogo (2006)

Tests of predictability

Series	Variable	t-stat	$\widehat{\beta}$	90% CI: β t-test	90% CI: β Q-test	Low CI β ($\rho = 1$)
Panel A: S&P 1880–2002, CRSP 1926–2002						
S&P 500	d–p	1.967	0.093	[−0.040, 0.136]	[−0.033, 0.114]	−0.017
	e–p	2.762	0.131	[−0.003, 0.189]	**[0.042, 0.224]**	−0.023
Annual	d–p	2.534	0.125	[−0.007, 0.178]	**[0.014, 0.188]**	0.020
	e–p	2.770	0.169	[−0.009, 0.240]	**[0.042, 0.277]**	0.002
Quarterly	d–p	2.060	0.034	[−0.014, 0.052]	[−0.009, 0.044]	−0.010
	e–p	2.908	0.049	[−0.001, 0.068]	**[0.010, 0.066]**	0.002
Monthly	d–p	1.706	0.009	[−0.006, 0.014]	[−0.005, 0.010]	−0.005
	e–p	2.662	0.014	[−0.001, 0.019]	**[0.002, 0.018]**	0.001
Panel C: CRSP 1952–2002						
Annual	d–p	2.289	0.124	[−0.023, 0.178]	[−0.007, 0.183]	0.020
	e–p	1.733	0.114	[−0.078, 0.178]	[−0.031, 0.229]	−0.025
	r_3	−1.143	−0.095	[−0.229, 0.045]	[−0.231, 0.042]	—
	y–r_1	1.124	0.136	[−0.087, 0.324]	[−0.075, 0.359]	−0.156
Quarterly	d–p	2.236	0.036	[−0.011, 0.051]	[−0.010, 0.030]	0.005
	e–p	1.777	0.029	[−0.019, 0.044]	[−0.012, 0.042]	−0.003
	r_3	−1.766	−0.042	**[−0.084, −0.004]**	**[−0.084, −0.004]**	−0.086
	y–r_1	1.991	0.090	**[0.009, 0.162]**	**[0.006, 0.158]**	−0.002
Monthly	d–p	2.259	0.012	[−0.004, 0.017]	[−0.004, 0.010]	0.001
	e–p	1.754	0.009	[−0.006, 0.014]	[−0.004, 0.012]	−0.001
	r_3	−2.431	−0.017	**[−0.030, −0.006]**	**[−0.030, −0.006]**	−0.030
	y–r_1	2.963	0.047	**[0.020, 0.072]**	**[0.020, 0.072]**	0.016

This table reports statistics used to infer the predictability of returns. Returns are for the annual S&P 500 index and the annual, quarterly, and monthly CRSP value-weighted index. The predictor variables are the log dividend–price ratio (d–p), the log earnings–price ratio (e–p), the three-month T-bill rate (r_3), and the long-short yield spread (y–r_1). The third and fourth columns report the t-statistic and the point estimate $\widehat{\beta}$ from an OLS regression of returns onto the predictor variable. The next two columns report the 90% Bonferroni confidence intervals for β using the t-test and Q-test, respectively. Confidence intervals that reject the null are in bold. The final column reports the lower bound of the confidence interval for β based on the Q-test at $\rho = 1$.

Table 6: Empirical results OOS from 2003 to 2017.

Prd	Regr	$\hat{\delta}$	CI $\hat{\rho}$	T-stat	Pt	$\hat{\beta}$	CI $\hat{\beta}$
Qua	ep	-0.99	[0.79,1.015]	1.51	0	0.102	[-0.051,0.203]
Qua	dp	-0.97	[0.719,0.966]	0.56	0	0.037	[-0.097,0.198]
Qua	tbl	0.33	[0.95,0.987]	-0.70	1	-0.430	[-0.069,0.021]
Qua	tms	0.18	[0.9,0.949]	0.20	1	0.164	[-0.07,0.097]