A Classical Mechanics Approach to Measuring the Depth of a Vertical Borehole*

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1. INTRODUCTION

The R&D department of Borehole Industries™ is interested in researching alternative procedures that will result in more efficient mining operations.

Well logging, otherwise known as borehole logging, is the most common technique our company utilizes to catalog geologic conditions across the depth of boreholes. Almost all of our operations make use of wireline logging, where a measurement device is lowered via a 'wire line'. However, this process is time intensive, with our older logging equipment only providing accurate and precise measurements while traveling under 600 ft/hr (Ellis & Singer 2007).

Our director has proposed a simple, economical approach to re-verifying the depth our newest 4 km borehole: Dropping a 1 kg test mass and measuring the time it takes to hit the bottom.

In this report, we explore three critical areas of investigation that emphasize the continued importance of wireline logging: the calculation of fall time with and without drag, the affect of Coriolis forces, and the affect of density concentration on operation. Calculations of all three fields made use of graphical software composed of array, numerical integration, and IVP (initial value problem) programming packages discussed in later sections.

2. CALCULATION OF FALL TIME

2.1. *The Ideal Case*

Assuming no drag, the fall time of the test mass can be calculated via the following equation:

$$d = v_0 t + \frac{1}{2} g t^2 \tag{1}$$

Where d is distance (m), v_0 is initial velocity (m/s), g is acceleration (m/s^2), and t is time (s).

To isolate time, we do some algebraic manipulation:

$$t = \frac{-v_0 \pm \sqrt{v_0^2 + 2gd}}{g} \tag{2}$$

Since time is positive, and the test mass will start with 0 initial velocity, the equation reduces to:

$$t = \sqrt{\frac{2d}{g}} \tag{3}$$

Since 4km is so close to the surface of the Earth, we can assume that $g \approx g_0$ (9.81 m/s/s) for the entire free fall duration

The motion of an object experiencing a constant gravitational force, including a drag force, can be described by an appended version of acceleration:

$$\frac{dy^2}{dt^2} = -g + \alpha \left(\frac{dy}{dt}\right)^{\gamma} \tag{4}$$

Where t is time (s), y is depth (m), g is the gravitational acceleration, α is the drag coefficient, and γ is the speed dependence of the drag. From now one, we assume that $\gamma = 2$.

Since $v = \frac{dy}{dt}$, we can do substitution to make:

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -g - \alpha (v)^{\gamma} \end{aligned} \tag{5}$$

If we assume $\alpha = 0$, 5 reduces to 3.

If we assume $g = g_0$, the surface gravity of the Earth at 9.81 m/s/s, both equations will result in a free fall time of approximately 28.6 ± 1s.

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2.2. *Assuming Drag and Variable g*

Since gravitational acceleration can be approximated linearly by the following equation:

$$g(r) = g_0 \frac{r}{R_E} \tag{6}$$

Where g_0 is surface gravity and R_E is the radius of the Earth.

Equation 5 was implemented in an ODE (ordinary differential equation) python package to numerically solve for the velocity and position of the test mass. α was found to be approximately 0.04 to satisfy the general consensus of a 50 m/s terminal velocity.

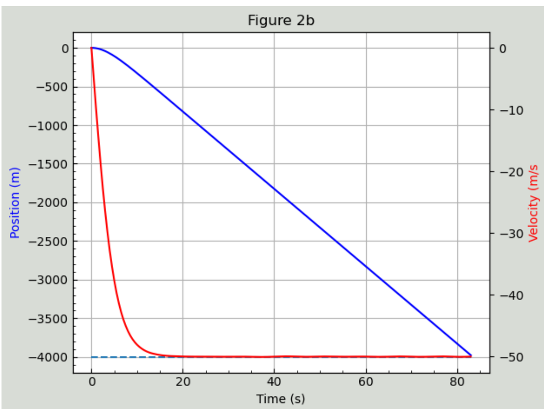

Figure 1. This graph plot represents the depth of the test mass over time. The blue line represents the position (m), while the red line represents the velocity (m/s). The dashed blue line is the borehole floor.

Including drag and variable gravity results in a fall time of approximately 83.5 ± 0.1s.

3. FEASIBILITY OF DEPTH MEASUREMENT APPROACH

Since our new borehole is located at the equator, the Coriolis is a significant variable to account for. It may introduce enough horizontal motion to hit the side of the borehold (which has a radius of 5m) before it hits the bottom. The Coriolis force F_c is:

$$-2m\,(\Omega \times v) \tag{7}$$

Where Ω is the rotation rate of the Earth (rad/s) and v is the velocity of the test mass (m/s).

Updating 5, we get the two equations:

$$\begin{aligned} \frac{dv}{dt_y} &= -g(r) - 2\Omega \times v + \alpha\,(v)^{\gamma} \\ \frac{dv}{dt_x} &= -2\Omega \times v \end{aligned} \tag{8}$$

Where $\frac{dv}{dt_y}$ is the acceleration in the vertical direction, and $\frac{dv}{dt_x}$ is the acceleration in the horizontal direction.

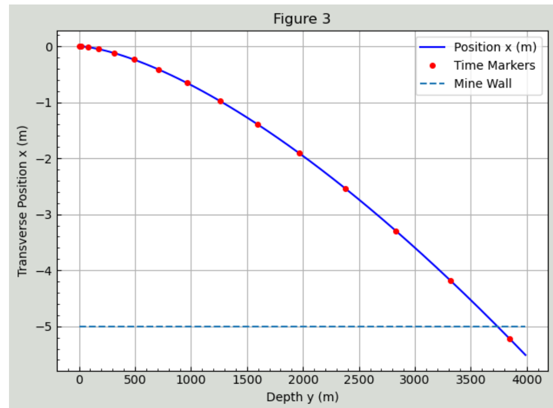

Figure 2. This graph plot represents the depth of the test mass over time, assuming Coriolis force and no drag. The y axis is the transverse position x (m), while the x axis is the depth y (m). The dashed blue line is the borehole wall at 5m.

Plugging 8 back into our IVP software, we find that the test mass hits the wall at $t = 27.6 \pm 0.1$s, at $y = -3736$m with no drag. With $\alpha = 0.04$, the test mass hits the wall at a shorter depth, with $t = 40.5 \pm 0.1$s, at $y = -1848$.

From the plot and calculations, it's clear that simply dropping a test mass down a borehole located on the equator is not feasible without a wider hole, which would be impractical to expand at this time.

4. HOMOGENEOUS VS. NON-HOMOGENEOUS SCENARIOS

The Earth does not have a constant density. We know that as we get closer to the center of the Earth, density increases. How that density increases is a more complicated matter, but can be simply modeled by:

$$\rho(r) = \rho_n \left(1 - \frac{r^2}{R_E^2}\right)^n \tag{9}$$

Where R_E is the radius of the Earth and ρ_n is the normalization factor derived by the total mass of the object.

Gravitational force at a certain distance of r can then be derived:

$$\begin{aligned} M(r) &= 4\pi \int_0^{R_E} \rho_r(r) r^2 dr \\ g(r) &= \frac{GM(r)m}{r^2} \end{aligned} \tag{10}$$

Where G is the gravitational constant.

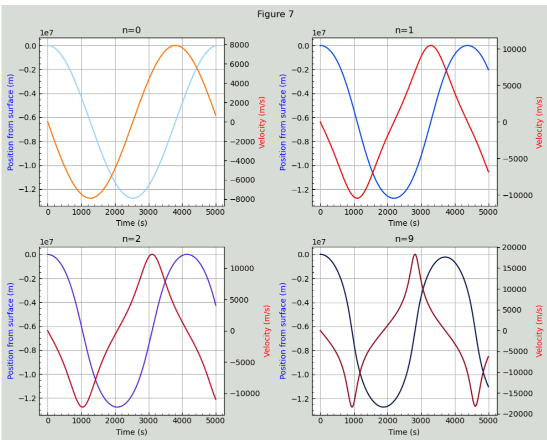

Figure 3. This graph plot represents the depth of the test mass over time, variable density force and no drag. The first y axis is position from the surface (m) while the second y axis is test mass velocity (m/s). Each consecutive plot (clockwise, top left) illustrates an increasing n value.

This would heavily affect the theoretical crossing time for a homogeneous and non-homogeneous Earth. If we compare two extreme cases (n=0, n=9), we find that the center-crossing time of a test mass to be 1267.2s and 943.9s, respectively. This makes sense, as the n=9 case allows for more 'peak' gravitational acceleration for most of the test mass' journey to the core.

For a less dense object, such as the moon, this travel time is greatly increased. Assuming n=0, the travel time to the core would take 1624s. Assuming n=9, it would take 1210s.

5. DISCUSSION AND FUTURE WORK

Many approximations were used to simplify the calculations that were undertaken in this investigation. We assumed that the radius of the Earth was constant, that the Earth was a perfect sphere, and had an ideal density distribution.

We also assumed a constant drag coefficient and speed dependence on drag. In reality, α and γ would increase/decrease based on the atmosphere present in the borehole.

Nevertheless, these calculations indicate the infeasibility of replacing line-logging with dropping a test mass. In order to make the calculations more realistic, we must assume a non-spherical Earth, as well as a non-ideal density profile. This is another reason to reject the use of a falling test mass, as mapping the density profile of the crust requires line logging. Line logging takes care of both depth measurement and geological measurements, making a classical mechanics approach completely redundant. Future work should focus on material science and coming up of better ways to resist the immense pressure probes endure at lower depths.

Software: numpy, scipy, matplotlib

REFERENCES

Ellis, D. V., & Singer, J. M. 2007, Well logging for earth scientists, Vol. 692 (Springer)