Chapter 2

Flows on the Line

2.0 Introduction

Consider general nth-order differential equation (i.e. n sets of first order differential equations).

Begin with the simplest case: **First-order equation**

$$\dot{x} = f(x).$$

What do we mean by **system**? - A dynamical system, not in the sense of a collection of two or more equations

Can f have time-dependence? - No, as we saw in the first chapter, we should treat time as any another variable, which would make this a second-order system.

2.1 A Geometric Way of Thinking

Example of interpreting differential equations as vector fields

$$\dot{x} = \sin x$$

Solve this by separation of variables:

$$t = \ln \left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|.$$

This is exact but untransparent.

But if we plot this in phase space, it's extremely transparent.

We can identify **fixed points** when $\dot{x} = 0$, alternatingly **stable** or **unstable**.

We can plot the trajectories, *qualitatively*.

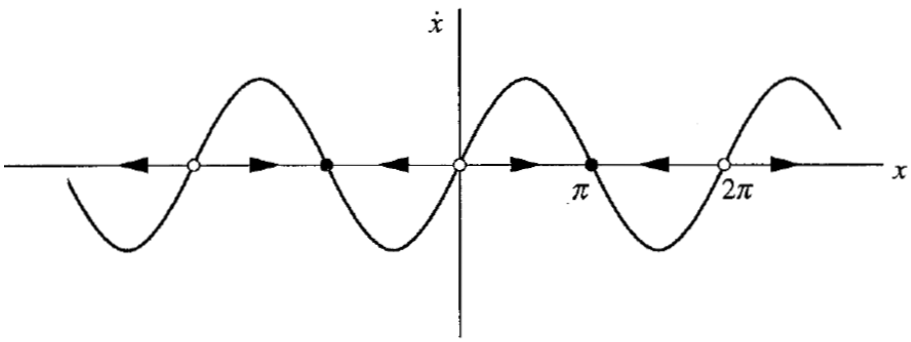


Figure 2.1.1

Figure 1: 2.1.1

2.2 Fixed Points and Stability

Fixed points as **equilibrium solutions**. Fixed points tell us appearance of phase portrait.

Contrast *local* and *global* stability: We usually mean the former (i.e. resilience to small disturbances)

A few good examples on analyzing fixed points: - Parabola - RC circuit

$$\dot{x} = x - \cos x$$

2.3 Population Growth

Simplest model is exponential.

$$\dot{N} = rN$$

with solutions

$$N(t) = N_0 e^{rt}$$

How do we introduce a **carrying capacity**? - Assume that the per capita growth rate $\dot{N}/N$ decreases linearly with N.

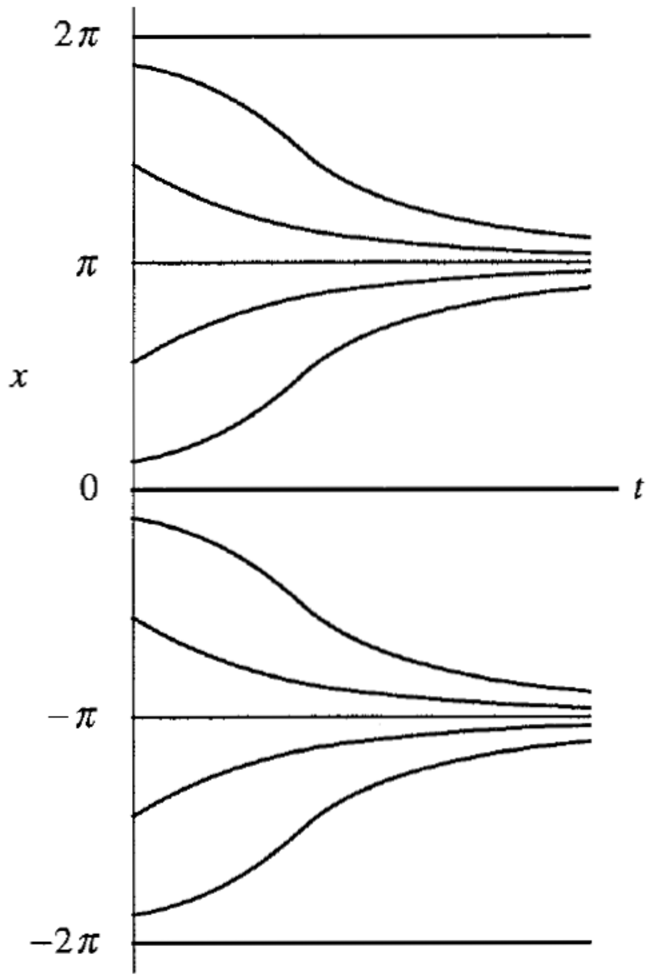
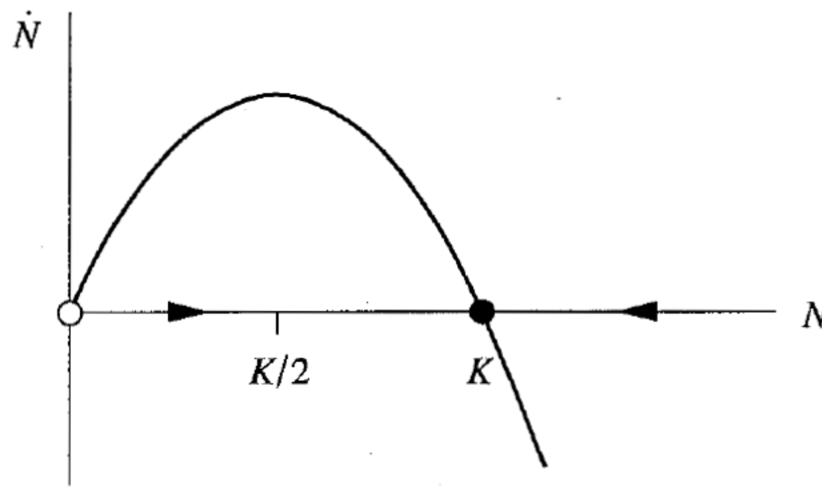


Figure 2.1.3

Figure 2: 2.1.3

The **Logistic equation**

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$



What does it look like in phase space?

Who was the logistic equation proposed by? Verhulst. When? 1838

What is the intellectual history of the logistic equation? Originally it was argued to be universal law of growth (Pearl 1927)

How well does the logistic equation match up to real population growth?
For simple organisms (yiest, bacteria, etc.) - the comparisons are pretty good.
For more complex organisms (fruit flies, flour beetles, etc.) - the populations continue to fluctuate persistently after initial logistic growth period.

2.4 Linear Stability Analysis

How to get a more exact, quantitative measure of stability (e.g. rate of decay to stable fixed point)? - *Linearize* around the fixed point

Consider: Fixed point, x^*. Small perturbation away

$$\eta(t) = x(t) - x^* << 1 \text{ as } t \to 0$$

We want to determine a differential equation for η: $\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x}$ Expand about fixed point: $\dot{\eta} = f(x^*) + \eta f'(x^*) + O(\eta^2)$

The **linearization**: $\dot{\eta} \approx \eta f'(x*)$

How does this determine stability? $f'(x^*) > 0$: perturbation grows with time $f'(x^*) < 0$: perturbation decays with time $f'(x^*) = 0$: nonlinear analysis needed

What is the **characeristic time scale** of a fixed point? Qualitatively? The time required for $x(t)$ to vary significantly in the neighboorhood of the fixed point? Mathematically? $1/|f'(x^*)|$

Examples of calculating fixed point stability quantitatively $\dot{x} = \sin x$ Logistic equation Example where you need to take higher order terms in your Taylor expansion - **half-stable**

2.5 Existence and Uniqueness

So far we've ignored these issues (reflecting applied spirit of book). There are, however, pathological cases

Example of non-unique solution

$$\dot{x} = x^{1/3},$$

where $x(0) = 0$\$.

Has two solutions: 1. $x(t) = 0$ 2. $x(t) = \left(\frac{2}{3}t\right)^{3/2}$

In fact, **infinitely**-many solutions? - Hint: fixed point is very unstable with infinite slope of $f'(0)$

What is the Existence and Uniqueness Theorem? Given some IVP ($\dot{x} = f(x)$, $x(0) = x_0$)?
: If $f(x)$ and $f'(x)$ are continuous on an open interval R containing x_0, then . . .
Then IVP has a unique solution $x(t)$ on somoe time interval $(-\tau, \tau)$ containing $t = 0$. What about outside of these intervals?
This does not guarantee that solutions exist forever.

Example of applying Existence and Uniqueness Thoerem

$$\dot{x} = 1 + x^2$$

$$x(0) = x_0$$

\$ f(x)=1+x^2 \$ is continuous and has continuous derivatives for all of space.

Imposing $x(0) = 0$ leads to (by separation of variables), the solution:

$$\tan^{-1} x = t$$

This only exists for the range of times $(-\pi/2, \pi/2)$.

Why is this incredible? - **Blow-up**: The solution reaches infinity in finite time. Indeed relevant in models of combustion.

2.6 Impossibility of Oscillations

What things can happen for a vector field on the real line? Trajectories either approach fixed points or diverge to $\pm\infty$. They cannot oscillate.

Why can't vector fields on the real line oscillate? They are forced to change monotonically. The phase points never reverse directions. Topologically: monotonic behavior on a line can never bring you back to the starting place.

How does this match up with classical mechanical expectations from damped oscillators? The over-damped limit tin which the second order derivative vanishes does not allow for any overshooting.

2.7 Potentials

What is the potential $V(x) for a given first-order differential equation? It is defined so that: $f(x) = -\frac{dV}{dx}$. Indeed, when $f(x)$ is negative it measn $V has positive curvature and is stable (and positive => negative curvatrue => unstable).

Examples: Single-well and double-well potentials

2.8 Solving Equations on the Computer

What is Euler's Method used for? Approximating solutions $x(t)$ to first order differential equations subject to initial conditions.

How do we use Euler's Method to approximate solutions $x(t)$ to first order differential equation We iteratively update our value of position according to the rule: $x_{n+1} = x_n + f(x_n)\Delta t \approx x(t_n)$

What does the improved Euler method change? TODO: Link card

1. Take a trial step $\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$
2. Average $f(x_n)$ and $f(\tilde{x}_{n+1})$, and use that to take the real step.

How is the improved Euler method an improvement? TODO: Link card

It makes a smaller **error** $E = |x(t_n) - x_n|$ for a given stepsize, i.e. this decreases to 0 quadratically as Δt goese to 0, whereas the original error decreases linearly.

Why not just take much higher order approximations? - Tradeoff between computing higher orders and the accuracy

What is the fourth-order Runge-Kutta method? TODO: links cards

Define

$$k_1 = f(x_n)\Delta t$$

$$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$$

$$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$$

$$k_4 = f(x_n + k_3)\Delta t$$

Calculate $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.