

Chapter 1

Dynamics as the overbearing subject dealing with change. Answers questions relating to: - Does a system settle to **equilibrium**? - Repeat in **cycles**? - Do something **more complicated**?

1.1 Capsule History of Dynamics

1. 1666s: Newton
 - Newton invented calculus and discovered differential equations, laws of motion
 - Solved **2-body problem**
2. 1700s-1800s: Subsequent generations tried to solve **3-body problem**
 - Much harder (not tractable!) but why?
3. 1890s: Poincare
 - Qualitative questions:
 - Is the solar system forever **stable**?
 - Developed *geometric* approach
 - First to glimpse **chaos**
4. 1900s-1950s: Focus on nonlinear oscillators and their applications not on chaos
 - Relevant for technological advances, e.g. radio, lasers, radar
 - Mathematical innovation
5. 1950s as turning point: distribution of computers
 - Allowed for new kinds of experiments
6. 1963: Lorenz discovers Lorenz attractor
 - Trying to make weather predictions (convection rolls in atmosphere model)
 - Process is inherently unpredictable but structured (i.e. points fall on butterfly curves)
7. 1970s: Boom years for chaos
 - 1971 Ruelle and Takens - Turbulence
 - May and population biology
 - Feigenbaum and universal laws governing transitions between regular and chaotic systems
 - Link between chaos and phase transitions
 - Mandelbrot and fractals
 - Winfree and geometric methods applied to biological oscillations
8. 1980s: Widespread attention

Define **chaos** : Property of **deterministic systems** exhibiting **aperiodicity** depending sensitively on initial conditions. Impossible to predict long-term behavior.

1.2 The Importance of Being Nonlinear

What are the two main types of dynamical systems? (1) Differential equations & (2) iterated maps (aka difference equations). How are these contrasted?

Continuous vs discrete time, respectively

Nonlinear differential equations are much harder!

Example) pendulum is described by:

$$\ddot{x} + \frac{g}{L} \sin x = 0$$

Writing in terms of single derivatives:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{L} \sin x_1\end{aligned}$$

How do we get around non-linearity in the pendulum equation? - Small-angle approximation

Is this even necessary? - No, we can actually solve the pendulum analytically with elliptic curves.

How would we go about addressing this problem using geometric techniques? - Suppose we know a solution for a given initial condition (pair of functions $x_1(t)$ and $x_2(t)$). - Construct a space with coordinates (x_1, x_2) , “**phase space**”. - Our solution is a **trajectory** through phase space. - **Goal:** Run process in reverse. Draw trajectories then extract information about solutions.

Nonautonomous Systems

How do we generalize our approach when the equations we are trying to solve have **explicit time dependence**? (called a **nonautonomous** system) - Introduce new variable $x_3 = t$, then $\dot{x}_3 = 1$

Example) Forced HO

$$m\ddot{x} + b\dot{x} + kx = F \cos t$$

becomes . . .

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m} (-kx_1 - bx_2 + F \cos x_3)\end{aligned}$$

$$\dot{x}_3 = 1$$

Why is this change of variables useful? 1. Geometric picture: We can visualize a phase space with *frozen-in* trajectories. 2. Physically: the state of our equation is uniquely determined by the 3 numbers.

What is the cost? - Nontraditional terminology: e.g. forced HO becomes third-order nonlinear system instead of the typical second-order linear equation.

What is the relation between time-dependence and the dimensionality of phase space?

Autonomous systems: n -th order equation yields n -dimensional system

Nonautonomous systems: n -th order equation yields $(n + 1)$ -dimensional system

Why are nonlinear systems so much harder to solve analytically? - Linear systems we can break into parts that we can solve separately.

1.3 A Dynamical View of the World

Logical structure of entire book captured by two axes: 1. Number of variables needed to characterize state of system (dimension of phase space) 2. Linearity / nonlinearity

Figure 1.3.1

		Number of variables →				
		$n = 1$	$n = 2$	$n \geq 3$	$n \gg 1$	Continuum
Linearity ↓ Nonlinearity	Linear	Growth, decay, or equilibrium Exponential growth RC circuit Radioactive decay	Oscillations Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	Collective phenomena Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	Waves and patterns Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
	Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	Chaos Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos?	The frontier Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	Spatio-temporal complexity Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

Figure 1: dynamicsTable