Chapter 3

Bifurcations

3.0 Introduction

What is interesting about 1-D systems if their dynamics are so trivial? - The dependence on parameters. Qualitative structure of flow changes as parameters are varied.

What do we call qualitative changes in the dynamics of 1-D systems as we vary parameters?
Bifurcations. These occur at bifurcation points.

How do we distinguish varying parameters from variables like x and t?
Parameters are uncoupled from x and t. We could introduce another dimension into phase space for a parameter, but each hyperplane spanned by a particular parameter value is self-contained and non-interacting with the other hyperplanes.

3.1 Saddle-Node Bifurcation

What are saddle-node bifurcations? When two fixed points collide and annihilate.

Prototypical example:

$$\dot{x} = r + x^2$$

Bifurcation point at $r = 0$: - $r < 0$: two fixed points (one stable and one unstable) - $r = 0$: one half-fixed point - $r > 0$: zero fixed points.

How do we represent bifurcations graphically? - With **Bifurcation diagrams** with confusing axes: Y-axis for the x value, X-axis for the parameter value. Draw lines (solid/striped) for the fixed points (unstable/stable).

Example with $\dot{x} = r - x - e^{-x}$.

In what sense are the examples $\dot{x} = r \pm x^2$ prototypical? Close to saddle-node bifurcations, the dynamics will be well-approximated by this form.

With preceding example:

$$\dot{x} = r - x - \left[1 - x + \frac{x^2}{2!} + \dots\right]$$

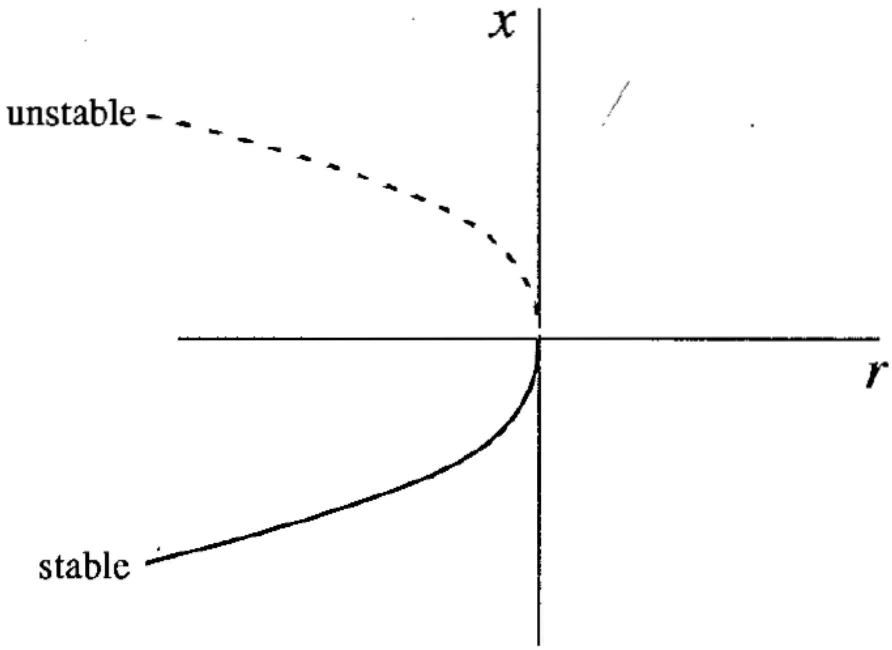


Figure 3.1.4

Figure 1: 3.1.4.png

$$\dot{x} = (r-1) - \frac{x^2}{2} + \dots$$

What do we call "prototypical" examples of bifurcations $\dot{x} = r \pm x^2$?
Normal forms

3.2 Transccritical Bifurcation

What are transcritical bifurcations? The (standard) mechanism by which a fixed point (that exists for all values of a parameter) **changes its stability**.

What is the normal form of a transcritical bifurcation? $\dot{x} = rx - x^2$.
What should this remind you of?
The logistic equation.

What occurs at a transcritical bifurcation? An **exchange of stabilities** between the two fixed points.

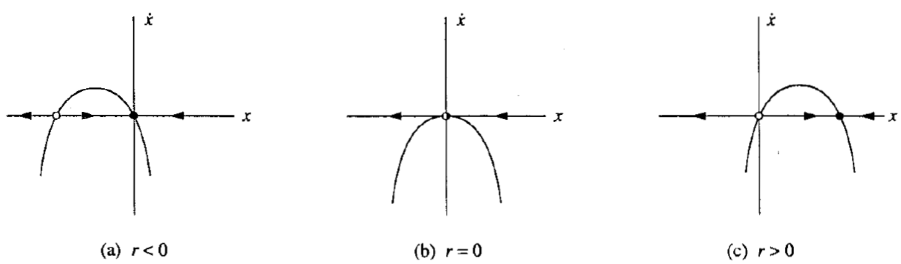


Figure 3.2.1

Figure 2: 3.2.1.png

What is the bifurcation diagram for the transcritical bifurcation?

Examples: - $\dot{x} = x(1-x^2) - a(1-e^{-bx})$ near $x=0$ - $\dot{x} = r \ln x + x - 1$ near $x = 1$

3.3 Laser Threshold

Following Haken (1983)

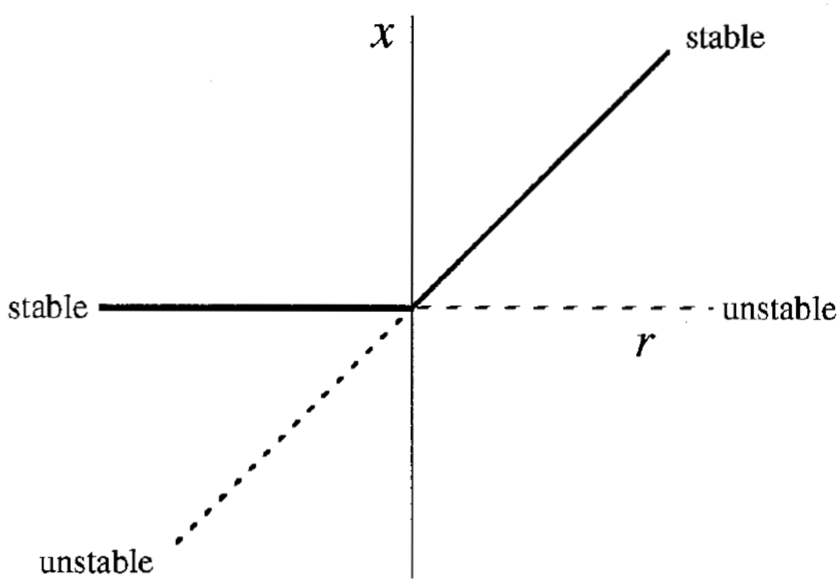


Figure 3.2.2

Figure 3: 3.2.2.png

Physical Background

Subject: Solid-state laser - Collection of "laser-active" atoms embedded in solid state matrix - Matrix is bounded by reflecting mirrors at either end - External energy source pumps atoms out of ground state

Observation: - Phase transition from low pump strength to high pump strength: lamp becomes laser

Model: - $n(t)$ is dynamical variable, the number of photons in field. - $\dot{n} = \text{gain} - \text{loss} = GnN - kn$. - N is number of excited photons. - G is the gain coefficient. This term is due to stimulated emission. - k as rate constant (with reciprocal equal to average photon lifetime.

Insight: - Relate N to n, after emission, an atom drops to lower level and is no longer excited - Assume: $N(t) = N_0 - \alpha n$ - $\alpha > 0$ is the rate of atoms dropping to ground state.

Combining: - $\dot{n} = (gN_0 - k)n - (\alpha G)n^2$

What is the **laser threshold**? - $N_0 = k/G$, the transcritical bifurcation point when $n^* = 0$ becomes unstable

What are the limitations of this model? - Ignores excited atom dynamics, spontaneous emission, etc.

3.4 Pitchfork Bifurcation

Common for problems with **symmetries**.

What are the types of pitchfork bifurcations? Supercritical and Subcritical

How do supercritical and subcritical bifurcations differ? {{Look at the normal forms}}
Supercritical: $\dot{x} = rx - x^3$. Cubic term is *stabilizing*.
Subcritical: $\dot{x} = rx + x^3$. Cubic term is *destabilizing*.

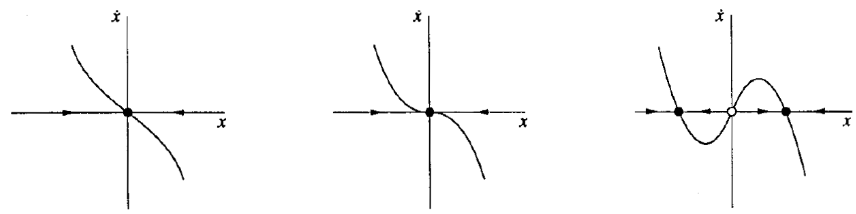
What are the two non-zero fixed points that appear for pitchfork bifurcations?
$x^* = \pm\sqrt{r}$. Are they stable?
Stable for supercritical. Unstable for subcritical.

Type 1) Supercritical Pitchfork Bifurcation

Normal form: $\dot{x} = rx - x^3$. Cubic term is *stabilizing*.

What symmetry do we see in the normal form of the supercritical pitchfork bifurcation ($\dot{x} = rx$
Parity flips $x \to -x$. How do we describe such a property of the vector field?
As an **equivariance**.

How does the vector field of a supercritical pitchfork bifurcation change with r?



(a) $r<0$ (b) $r=0$ (c) $r>0$

What is the bifurcation diagram of a supercritical pitchfork bifurcation change with r? {{Hint:

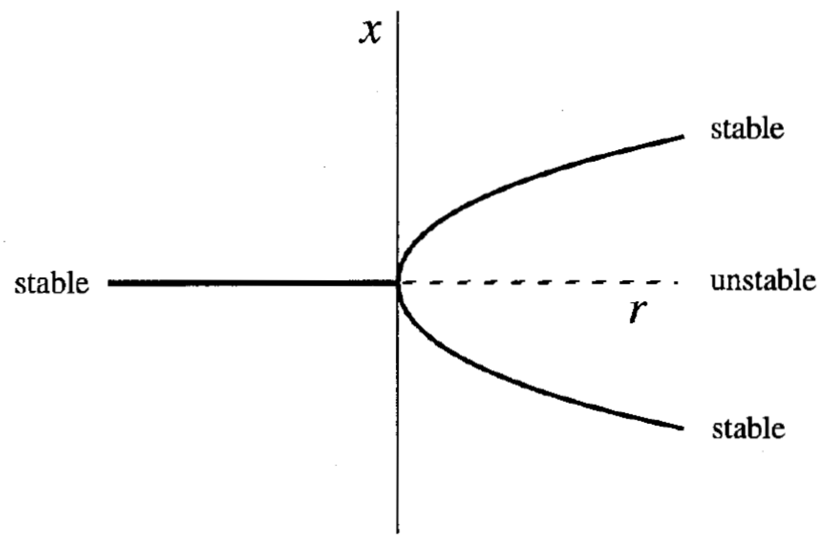


Figure 3.4.2

How to get the bifurcation diagram for $\dot{x} = -x + \beta \tanh x$? Treat x^* as the independent variable, and plot $\beta = x^* / \tanh x^*$. Why does this work? Our function depends jmore simply on β then on x.

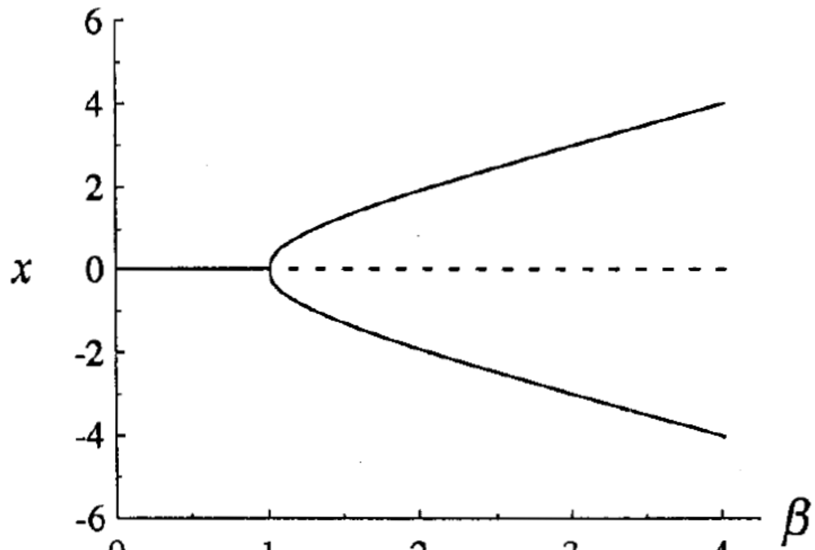


Figure 3.4.4

Type 2) Subcritical Pitchfork Bifurcation

Normal form: $\dot{x} = rx + x^3$. Cubic term is *destabilizing.*

What is the bifurcation diagram for a subcritical pitchform bifurcation?

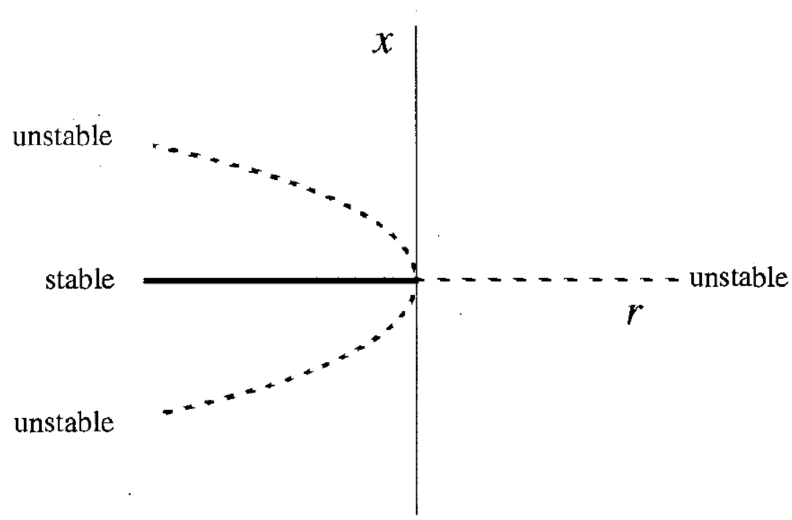


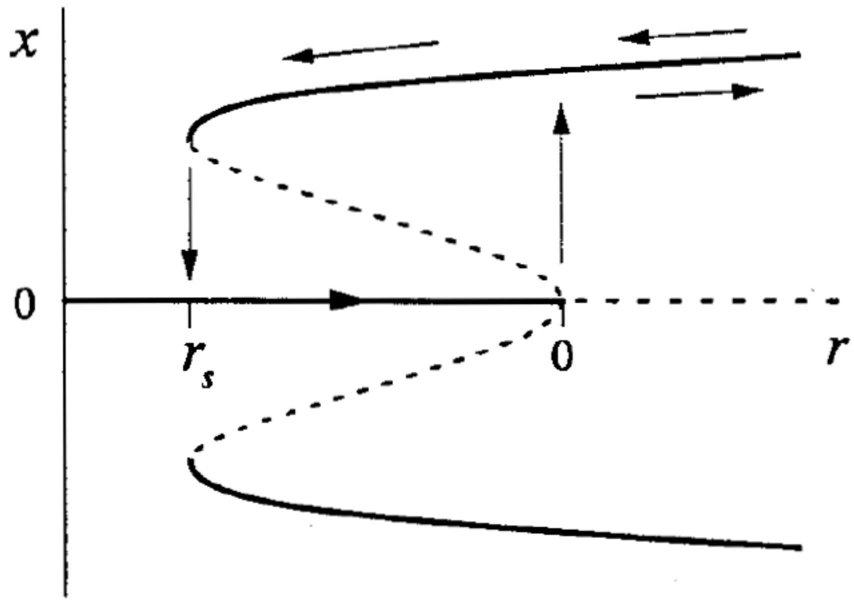
Figure 3.4.6

What happens when $r > 0$ to the fixed point for the subcritical pitchfork bifurcation?

Blow-up, the x term goes to ∞ in finite time. How do we deal with this infinity in physical systems?

Stabilizing higher-order terms

What is the bifurcation diagram when we add the next-highest order (parity - preserving) stabilizing term to the subcritical pitchfork bifurcation? $\dot{x} = rx + x^3 -$



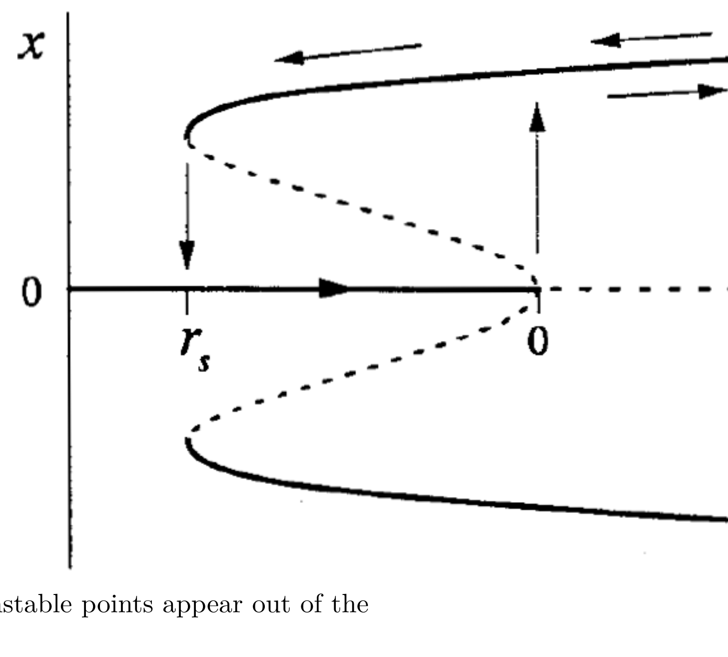
x^5 :

What happens to the origin when we add the next-highest order (parity-preserving) stabilizing

It becomes locally but not globally stable.

How do you identify which stable state a system occupies when we add the next-highest order

Hysteresis: which stable state the system occupies depends on its history.



What kind of bifurcation occurs at r_s?

A saddle-node bifurcation, stable and unstable points appear out of the clear sky.

What else can we call a supercritical pitchfork bifurcation in the physics literature?
Forward bifurcation or continuous/2nd-order phase transition. And in the engineering literature?
Soft or safe in engineering literature.
What else can we call a subcritical pitchfork bifurcation in the physics literature?
Inverted/backward bifurcation or discontinuous/1st-order phase transition.
And in the engineering literature?
Hard or dangerous

3.5 Overdamped Bead on a Rotating Hoop

New twist: Add a frictional force on bead opposing its motion (due to viscous damping)

What is the governing equation for the overdamped bead on a rotating hoop with tangential da
$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$

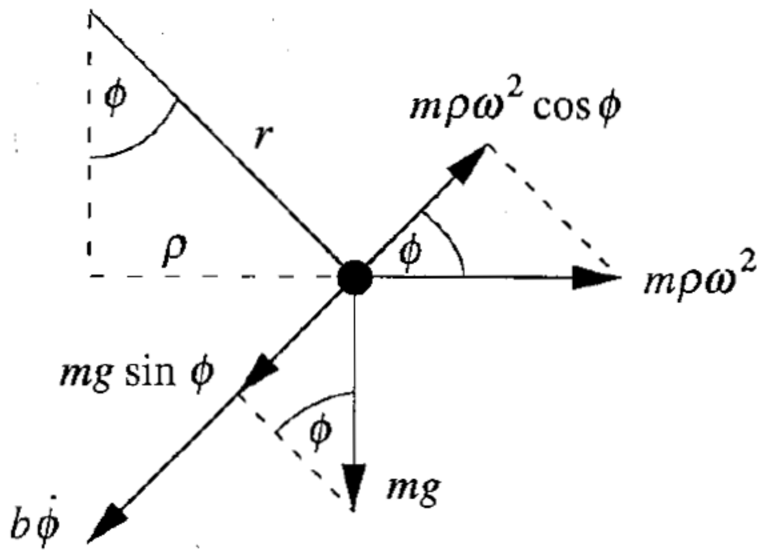


Figure 3.5.3

Where are the fixed points for the first-order system (ignoring the second-order derivative)? - $b\dot{\phi} = mg \sin\phi \left(\frac{r\omega^2}{g} \cos\phi - 1 \right)$. - $\sin\phi = 0$ so the top and bottom ($\phi^* = 0, \phi$) - When $\frac{r\omega^2}{g} > 1$, there are two additional points $\phi^* = \pm \cos^{-1}\left(g/r\omega^2\right)$

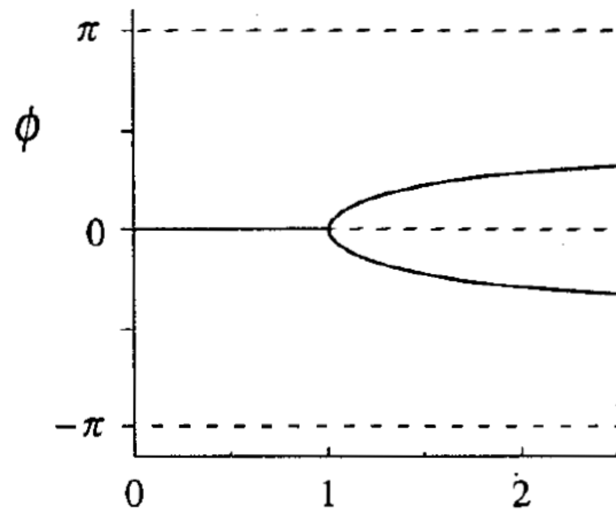


Figure 3.5.6

What is the bifurcation diagram for this system? $\gamma = \frac{r\omega^2}{g}$

Dimensional Analysis

When is it valid to neglect the inertia term $mr\ddot{\phi}$? 1. Express equation in dimensionless form. May not be immediately clear what the best waya to nondimensionalize this is. - We want the dimensionless coefficient on the first-order term to be $O(1)$ and the dimensionless coefficient on the second-order term to be vanishing. 2. Example choice: Introduce **characteristic time scale** to create dimensionless time τ: $\tau = t/T$.0 - Derivatives $d\phi/d\tau$ and $d^2\phi/d\tau^2$ will be $O(1)$.

Go through the exercise, we want $b^2 >> m^2grr$. Strong damping / small mass.

Reduces 5 parameters to 2 dimensionless groups - γ (which we saw already) and $\epsilon = \frac{m^2gr}{b^2}$.

Paradox

What is the fundamental sin we commit in replacing a second-order equation with a first-order equation? - We ignore one of the initial values which we have to choose for a second-order equation.

How do we resolve this for our example? - There is another characteristic timescale much faster than T, a **transient**, during which our system relaxes to the one dimensional case, at which point it can follow the slower drift.

Singular perturbation theory - See Jordan and Smith (1987) or Lin and Seegel (1988)

3.6 Imperfect Bifurcation and Catastrophes

We looked at symmetric cases, What happens when introduce imperfections? - Example) $h : \dot{x} = h + rx - x^3$?

h is an **imperfection parameter**.

To simplify presence of two parameters, we'll treat r as fixed and look at varying h

Use graphical approach to find fixed points (i.e. plot $rx - x^3$ and $-h$ on same axes and look for intersections).

How do the number of fixed points depend on r and h? - $r \leq 0$: one fixed point - $r > 0$: three cases - one fixed point if $|h| > h_c(r)$ - two fixed points if $|h| = h_c(r)$ - three fixed points if $|h| < h_c(r)$

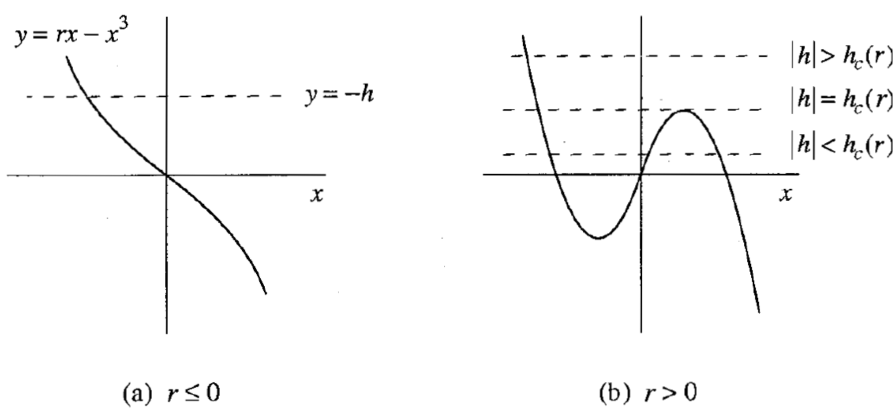


Figure 4: 3.6.1.png

Critical case when horizontal line tangent to min or max of cubic: saddle-nodoe bifurcation. - $x_{\max} = \sqrt{\frac{r}{3}}$ - $h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}$

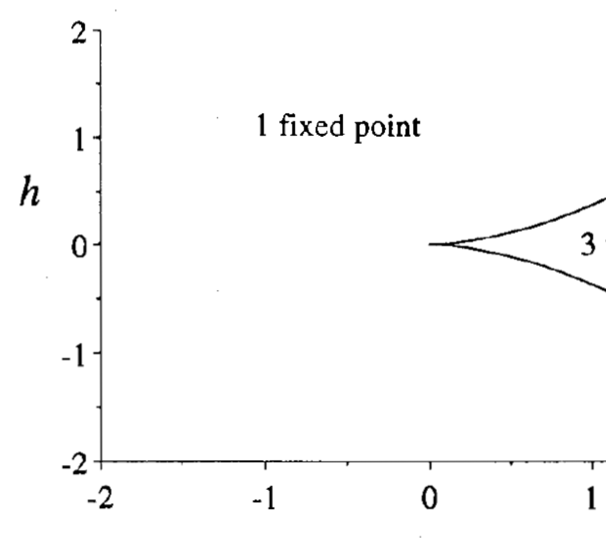


Figure 3.6.2

What do the bifurcation curves look like for this problem?

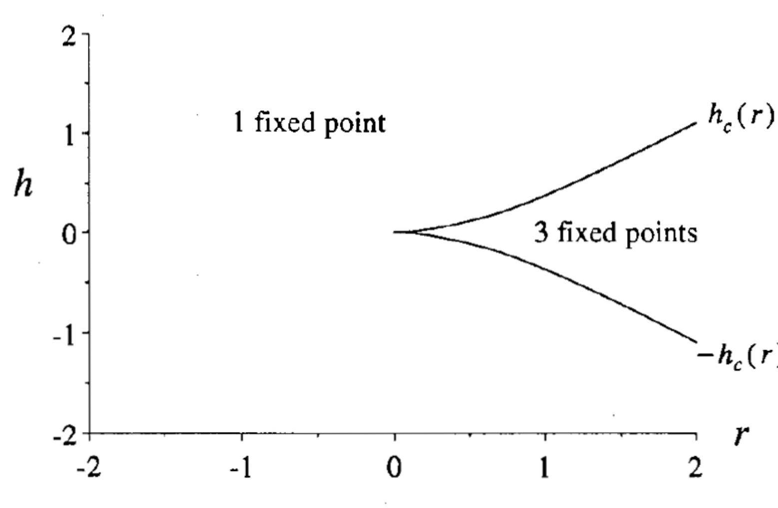


Figure 3.6.2

What do we call such a diagram

A stability diagram

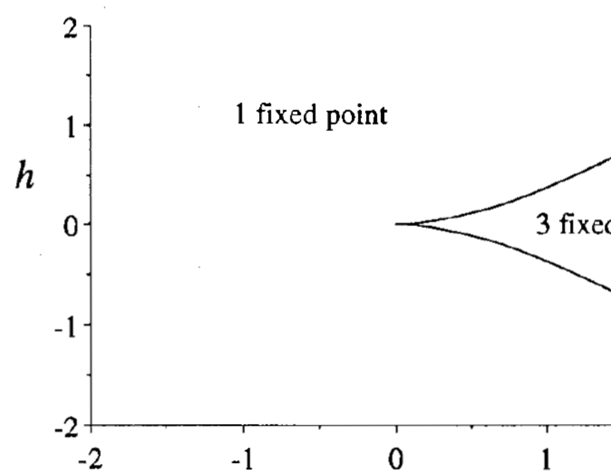


Figure 3.6.2

What is the point $(r,h)=(0,0)$ **in this diagram?**

A **cusp point**, or codimension-2 bifurcation. Why is it called this latter term?

Because it took two parameters to achieve such a bifurcation (contrast with preceding examples, codimension-1 bifurcations).

What does the bifurcation diagram look like? 3.6.5.png What do we call this kind of surface?

A **cusp catastrophe** (the discontinuous jump in the second diagram could be a catastrophe, e.g. for a building's equilibrium).

Example) Bead attached to spring on tilted wire.

3.7 Insect Outbreak

Example of spruce budworm outbreak (pest in eastern Canada of the balsam fir tree)

Analysis by Ludwig et al. (1978)

Model - Budworm population has fast characteristic time scale - Tree population has slow characteristic time scale (essentially constant from the perspective of the former)

Model mathematically - $\dot{N} = RN\left(1-\frac{N}{K}\right) - p(N)$ - N is budworm population - K is the tree-dependent parameter we treat as slowly varying - $p(N)$ is death rate due to predation. - Assume $p(N)=\frac{BN^2}{A^2+N^2}$, with positive constants

What do we mean by outbreak? - Jump in population as parameters drift

How to find this jump? - Dimensionless recasting - Get dimensionless groups out of predation part and into logistic part… - $\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$

Fixed point analysis Fixed point at $x^* = 0$ - Always exists, always unstable (predation extremely weak for small x)

Analyze second fixed points graphically! - Solutions to $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$ - Right-side never depends on parameters so this will always be the same graphically. - Small k, larger r: one intersection - Larger k, smaller r: up to three intersections - When curve is line intersects tangentially a second time, we have a saddle-nodee bifurcations

Because 0 is unstable, and roots are alternatingly stable/unstable, a and c are stable - the **refuge** and **outbreak** levels respectively.

Bifurcation curves With the previous example, we could calculate $r(k)$ for the saddle-node bifurcations exactly. This is no longer explicitly possible. We will use **parametric form** $(k(x), r(x))$

We require: 1. $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$ for the fixed point. 2. $\frac{d}{dx}\left[r\left(1 - \frac{x}{k}\right)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right]$ for the saddle-point bifurcation (tangential intersection).

The second condition will give us an exppression for r/k which we can use in the former to get $r(x)$. Then plug back in for $k(x)$.

$$r = \frac{2x^3}{(1+x^2)^2}$$

$$k = \frac{2x^3}{x^2 - 1}$$

3.7.5 3.7.6

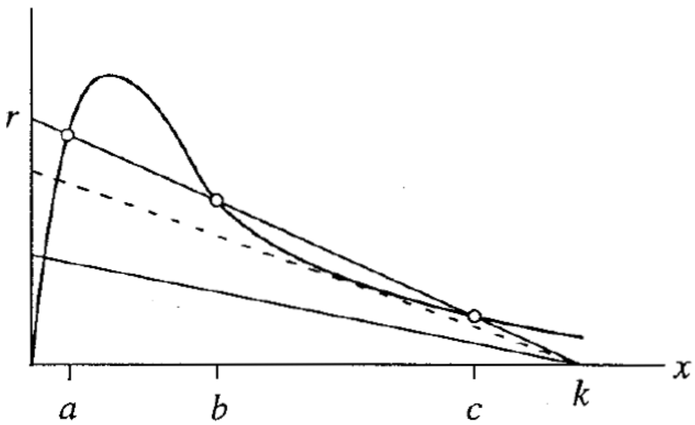


Figure 3.7.3

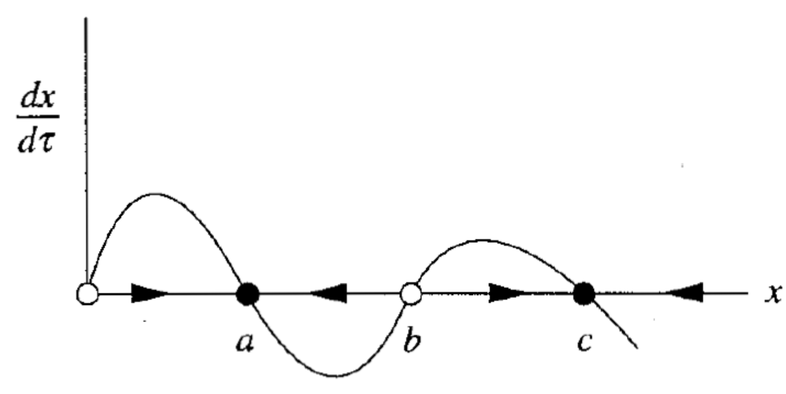


Figure 3.7.4

Figure 5: 3.7.2