ENGG2470A: Probability for Engineers
Spring 2020

Solution for Assignment 7 Date: N.A.

Prob. 8.4.10 Solution. (9 pts) According to $Y = \sqrt{\omega}X_1 + \sqrt{1-\omega}X_2$

$$\begin{aligned} E[Y^2] &= E\left[\left(\sqrt{\omega}X_1 + \sqrt{1-\omega}X_2\right)^2\right] \\ &= E\left[\omega X_1^2 + (1-\omega)X_2^2 + 2\sqrt{\omega}\sqrt{1-\omega}X_1X_2\right] \end{aligned}$$

According to the covariance matrix and $E[\mathbf{X}] = [0,0]'$, we have $E[X_1^2] = 1, E[X_2^2] = 1, E[X_1X_2] = \rho$. So

$$E[Y^2] = \omega + 1 - \omega + 2\rho\sqrt{\omega(1-\omega)} = 1 + 2\rho\sqrt{\omega(1-\omega)}$$

$\rho > 0$, $E[Y^2]$ is maximized when $\omega(1-\omega)$ is maximized, that is $\omega = 1/2$.

$\rho = 0$, $E[Y^2] = 1$ for any ω

$\rho < 0$, $E[Y^2]$ is maximized when $\omega(1-\omega)$ is minimizes, that is $\omega = 0$ or $\omega = 1$.

Prob. 9.1.1 Solution. (9 pts)

(a) $E[X_1 - X_2] = E[X_1] - E[X_2] = 0$

(b) $Var[X_1 - X_2] = Var[X_1] + Var[-X_2] = Var[X_1] + Var[-X_2] = 2Var[X]$

Prob. 9.1.2 Solution. (9 pts) From the context,

$$P_{X_{33}}(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$

X_1 and X_2 are independent. Y is the number of heads that occur o 100 flips.

$$P_Y(y) = \binom{100}{y} p^y (1-p)^{100-y}, y = 0, 1, \cdots, 100$$

$$E[Y] = E[X_1 + X_2 + \cdots + X_{100}] = \sum_{i=1}^{100} E[X_i] = \sum_{i=1}^{100} p = 100p$$

Because $X_i, i = 1, 2, \cdots, 100$ are independent.

$$Var[Y] = Var[X_1 + X_2 + \cdots + X_{100}] = \sum_{i=1}^{100} Var[X_i] = \sum_{i=1}^{100} E[X_i^2] - (E[X_i])^2 = \sum_{i=1}^{100} (p - p^2) = 100p(1-p)$$

Prob. 9.1.4 Solution. (9 pts) As X_1, X_2, X_3 are iid continuous uniform random variables. We have

$$E[Y] = E[X_1 + X_2 + X_3] = 3E[X_1],$$

$$Var[Y] = Var[X_1 + X_2 + X_3] = 3Var[X_1]$$

So, $E[X_1] = \frac{1}{3}E[Y] = 0$, $Var[X_1] = \frac{1}{3}Var[Y] = \frac{4}{3}$.

Suppose X_1 is uniformly distributed among $[a,b]$, $E[X_1] = \frac{a+b}{2}$, $Var[X_1] = \frac{(b-a)^2}{12}$. So,

$$\frac{a+b}{2} = 0$$
$$\frac{(b-a)^2}{12} = \frac{4}{3}$$

After solve the above equations, $a = -2, b = 2$. So X_1 follows a uniform distribution among $[-2,2]$, that is

$$f_{X_1}(x) = \begin{cases} \frac{1}{4}, & x \in [-2,2] \\ 0, & otherwise \end{cases}$$

Prob. 9.1.5 Solution. (10 pts) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & otherwise \end{cases}$$

1. To find $Var(W)$, we can apply

$$Var(W) = Var(X) + Var(Y) + 2Cov(X,Y)$$

We can first find the marginal PDF of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_0^{1-x} 2\mathrm{d}y, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

So,

$$f_X(x) = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

Similarly, we have

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1 \\ 0, & otherwise \end{cases}$$

Then,

$$\begin{aligned} Var[X] &= E[X^2] - E^2[X] \\ &= \int_0^1 x^2 2(1-x)\mathrm{d}x - \left(\int_0^1 x2(1-x)\mathrm{d}x\right)^2 \\ &= \frac{1}{6} - \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{18} \end{aligned}$$

Similarly, $Var(Y) = \frac{1}{18}$. To find Cov(X,Y)

$$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= \int_0^1 \int_0^{1-x} xy \times 2\mathrm{d}y\mathrm{d}x - \left(\frac{1}{3}\right)^2 \\ &= -\frac{1}{36} \end{aligned}$$

So $Var(W) = Var(X) + Var(Y) + 2Cov(X,Y) = \frac{1}{18} + \frac{1}{18} + 2 \times -\frac{1}{36} = \frac{1}{18}$

2. We can also begin with finding the PDF of W.

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\mathrm{d}x \\ &= \begin{cases} \int_0^w 2\mathrm{d}x, & 0 \le w \le 1 \\ 0, & otherwise \end{cases} \\ &= \begin{cases} 2w, & 0 \le w \le 1 \\ 0, & otherwise \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} Var[W] &= E[W^2] - E^2[W] \\ &= \int_0^1 w^2 2w\mathrm{d}w - \left(\int_0^1 w \times 2w\mathrm{d}x\right)^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{18} \end{aligned}$$

Prob. 9.3.1 Solution. (9 pts) As $N \sim Binomial(100, 0.4)$, the moment generation function of N is

$$\phi_N(s) = (1 - 0.4 + 0.4e^s)^{100}.$$

The moment generation function of M is

$$\phi_M(s) = (1 - 0.4 + 0.4e^s)^{50}.$$

As $L = M + N$ and M, N are independent, we have

$$\phi_L(s) = \phi_{M+N}(s) = \phi_M(s)\phi_N(s) = (1 - 0.4 + 0.4e^s)^{150}.$$

Correspondingly, L follows $Binomial(150, 0.4)$.

Prob. 9.3.2 Solution. (9 pts)

(a) As the moment generation function of Y is $\phi_Y(s) = \frac{1}{1-s}$. According to

$$E[Y^n] = \phi_Y^n(s)|_{s=0}.$$

We have

$$\begin{aligned} E[Y] &= \phi_Y'(s)|_{s=0} = \frac{1}{(1-s)^2}|_{s=0} = 1 \\ E[Y^2] &= \phi_Y''(s)|_{s=0} = \frac{2}{(1-s)^3}|_{s=0} = 2 \\ E[Y^3] &= \phi_Y^{(3)}(s)|_{s=0} = \frac{6}{(1-s)^4}|_{s=0} = 6 \end{aligned}$$

(b) As $W = Y + V$ and Y, V are independent. We have

$$\phi_W(s) = \phi_Y(s)\phi_V(s) = \frac{1}{1-s}\frac{1}{(1-s)^4} = \frac{1}{(1-s)^5}$$

$$E[W^2] = \phi_W''(s)|_{s=0} = \frac{30}{(1-s)^7}|_{s=0} = 30$$

Prob. 1 Solution. (9 pts) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. So $Y_1 = a_{11}X_1 + a_{12}X_2$. $Y_2 = a_{21}X_1 + a_{22}X_2$.

We have $X = A^{-1}Y$ and

$$\left| J \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \right| = \left| J \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix} \right|^{-1} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{-1} = [det(A)]^{-1}$$

So

$$f_Y = \left| J \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \right| f_X = [det(A)]^{-1} f_X(A^{-1}Y)$$

Prob. 2 Solution. (9 pts)

$$\begin{aligned} var\left(\sum_{i=1}^n X_i\right) &= E\left(\sum_{i=1}^n X_i\right)^2 - E^2\left(\sum_{i=1}^n X_i\right) \\ &= E\left(\sum_{i=1}^n X_i\right)^2 - \left(\sum_{i=1}^n E[X_i]\right)^2 \\ &= E\left(\sum_{i=1}^n \sum_{j=1}^n X_i X_j\right) - \sum_{i=1}^n \sum_{j=1}^n E[X_i]E[X_j] \\ &= \sum_{i=1}^n \sum_{j=1}^n E[X_i X_j] - \sum_{i=1}^n \sum_{j=1}^n E[X_i]E[X_j] \\ &= \sum_{i=1}^n \sum_{j=1}^n (E[X_i X_j] - E[X_i]E[X_j]) \\ &= \sum_{i=1}^n \sum_{j=1}^n cov(X_i, X_j) \end{aligned}$$

It's a generation of $Var(X+Y) = VarX + VarY + 2cov(X,Y)$ from $n = 2$ to general n.

Prob. 3 Solution. (9 pts)

$$E[(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T]_{ij} = E[(\mathbf{X} - E\mathbf{X})(\mathbf{X} - E\mathbf{X})^T_{ij}] = E[(X_i - EX_i)(X_j - EX_j)] = cov(X_i, X_j)$$

We then easily complete the proof.

Prob. 4 Solution. (9 pts)

(a) As N and $X_1, X_2 \cdots$ are independent, we have

$$E[X_1 + X_2 + \cdots X_N | N = n] = E[X_1 + X_2 + \cdots X_n] = nE[X]$$

By law of total expectation,

$$\begin{aligned} E[X_1 + X_2 + \cdots + X_N] &= E[E[X_1 + X_2 + \cdots + X_N]|N] \\ &= E[NEX] \\ &= E[N]E[X] \end{aligned}$$

(b) No. For example X_i is a Bernoulli distribution with parameter ρ. $N = 1$ if $X_1 = 0$, $N = 2$ if $X_1 = 1$. Then we can find $P_N(1) = 1 - \rho, P_N(2) = \rho$. So $E[N] = 1 + \rho$, $E[X] = \rho$ Then $E[X_1 + X_2 + \cdots + X_N] = P_N(1)E[X_1 + X_2 + \cdots + X_N | N = 1] + P_N(2)E[X_1 + X_2 + \cdots + X_N]|N = 2] = (1-\rho)0 + \rho E[0 + X_2] = \rho^2 \neq E[N]E[X]$