ENGG2470A Assignment 5 Solution

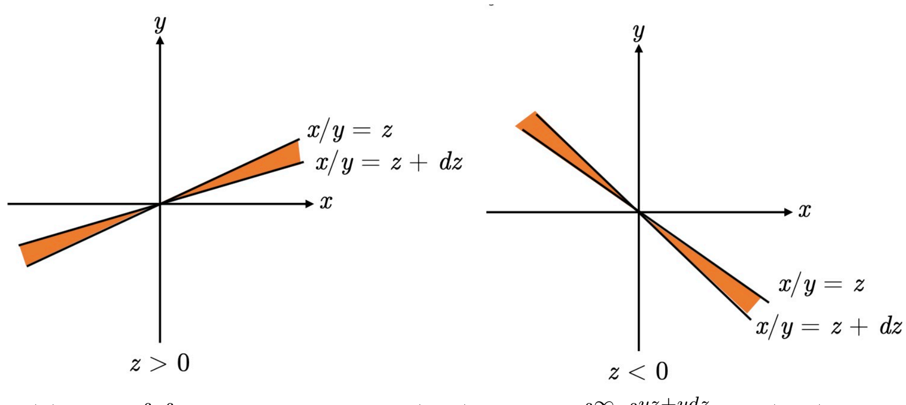
1 Problem 1 Solution (10 pts)

(a) The corresponding pdf is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}$. Let $\mu = \frac{x-m}{\sigma}$, then $du = \frac{1}{\sigma}dx$. Thus, $1 = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \int_{-\infty}^{\infty}f(x)dx$.

(b) The mean $\int_{-\infty}^{\infty}xf(x)dx = \int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \int_{-\infty}^{\infty}\frac{x-m+m}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \int_{-\infty}^{\infty}\frac{x-m}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx + m\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \int_{-\infty}^{\infty}\frac{u\sigma}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du + m\int_{-\infty}^{\infty}f(x)dx = 0(because\ \frac{u\sigma}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\ is\ an\ odd\ function\ in\ terms\ of\ u) + m = m$.

The variance $\int_{-\infty}^{\infty}(x-m)^2 f(x)dx = \int_{-\infty}^{\infty}\frac{(x-m)^2}{\sqrt{2\pi}\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}dx = \int_{-\infty}^{\infty}\frac{\sigma^2u^2}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du = \frac{-\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ud(e^{-\frac{u^2}{2}}) = \frac{-\sigma^2}{\sqrt{2\pi}}([ue^{-\frac{u^2}{2}}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}e^{-\frac{u^2}{2}}du) = \frac{-\sigma^2}{\sqrt{2\pi}}(0 - \sqrt{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du) = \frac{-\sigma^2}{\sqrt{2\pi}}(-\sqrt{2\pi}) = \sigma^2$.

2 Problem 2 Solution (10 pts)



$f_Z(z)dz = \int\int_{\{(x,y):\frac{x}{y}\in[z,z+dz)\}} f_{XY}(x,y)dxdy = \int_0^{\infty}\int_{yz}^{yz+ydz} f_{XY}(x,y)dxdy + \int_{-\infty}^{0}\int_{yz+ydz}^{yz} f_{XY}(x,y)dxdy = \int_0^{\infty} f_{XY}(zy,y)ydzdy - \int_{-\infty}^{0} f_{XY}(zy,y)ydzdy = \int_0^{\infty}|y|f_{XY}(zy,y)dzdy + \int_{-\infty}^{0}|y|f_{XY}(zy,y)dzdy = \int_{-\infty}^{\infty}|y|f_{XY}(zy,y)dzdy$. Thus, $f_Z(z) = \int_{-\infty}^{\infty}|y|f_{XY}(zy,y)dy$.

3 Problem 3 Solution (10 pts)

$Z = \sqrt{X^2+Y^2}$. X and Y are i.i.d. $\sim \mathcal{N}(0,\sigma^2)$. Thus $f_Y(y)$ is an even function.
We have $F_Z(z) = \int_{-z}^{z}\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f_{XY}(x,y)dydx = \int_{-z}^{z}\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f_X(x)f_Y(y)dydx =$
$2\int_{-z}^{z} f_X(x)\int_0^{\sqrt{z-x^2}} f_Y(y)dydx$.
For $z<0$, $f_Z(z)=0$; Otherwise, $f_Z(z) = \frac{d}{dz}F_Z(z) = 2\int_{-z}^{z} f_X(x)\frac{d}{dz}\int_0^{\sqrt{z^2-x^2}} f_Y(y)dydx =$
$2\int_{-z}^{z} f_X(x)f_Y(\sqrt{z^2-x^2})\frac{d\sqrt{z^2-x^2}}{dz}dx = \frac{1}{\pi\sigma^2}e^{-\frac{z^2}{2\sigma^2}}\int_{-z}^{z}\frac{z}{\sqrt{z^2-x^2}}dx = \frac{1}{\pi\sigma^2}e^{-\frac{z^2}{2\sigma^2}}(z\pi) =$
$\frac{z}{\sigma^2}e^{-\frac{z^2}{2\sigma^2}}$.

4 Prob. 5.1.3 Solution (10 pts)

$\mathbf{P}[x_1 \le X < x_2 \cup y_1 \le Y < y_2] = \mathbf{P}[x_1 \le X < x_2] + \mathbf{P}[y_1 \le Y < y_2] - \mathbf{P}[x_1 \le X < x_2 \cap y_1 \le Y < y_2] = F_X(x_2) - F_X(x_1) + F_Y(y_2) - F_Y(y_1) - (F_{XY}(x_2,y_2) + F_{XY}(x_1,y_1) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1))$.

5 Prob. 5.1.4 Solution (10 pts)

$F(x,y) = F_X(x)F_Y(y) \ge 0$. $F(\infty,\infty) = F_X(\infty)F_Y(\infty) = 1\times 1 = 1$. $F(-\infty,-\infty) = F_X(-\infty)F_Y(-\infty) = 0\times 0 = 0$.
$F(x_2,y_2)+F(x_1,y_1)-F(x_1,y_2)-F(x_2,y_1) = F_X(x_2)F_Y(y_2)+F_X(x_1)F_Y(y_1)-F_X(x_1)F_Y(y_2)-F_X(x_2)F_Y(y_1) = (F_X(x_2)-F_X(x_1))(F_Y(y_2)-F_Y(y_1)) \ge 0$.
Thus, $F(x,y)$ is a valid CDF.

6 Prob. 5.2.4 Solution (10 pts)

$$P_{XY}(x,y) = \begin{cases} 0.25, & x=1, y=0 \\ 0.25, & x=2, y=0 \\ 0.25, & x=0, y=1 \\ 0.25, & x=1, y=1 \\ 0, & otherwise. \end{cases}$$

7 Prob. 5.2.6 Solution (10 pts)

Obviously, $y \le x$. When $x=y=n$, $P_{XY}(x,y) = p^n$; When $0 \le y \le x < n$, following the hint, we know that $P(X=x, Y=y) = P(A\cap B\cap C) = P(A)P(B)P(C)$. $P(A) = p^y$. $P(B) = (1-p)$. $P(C) = (\begin{matrix} n-y-1 \\ x-y \end{matrix})p^{x-y}(1-p)^{n-x-1}$. Thus, $P(X=x,Y=y) = (\begin{matrix} n-y-1 \\ x-y \end{matrix})p^x(1-p)^{n-x}$.

8 Prob. 5.3.2 Solution (10 pts)

(a) Let $\Sigma P_{XY}(x, y) = 1$ we have $c = 1/14$. $P_X(-2) = \Sigma_y P_{XY}(-2, y) = 1/14(|-2-1|+|-2+0|+|-2+1|) = 3/7$. The same applies to $P_X(0), P_X(2), P_Y(-1), P_Y(0), P_Y(1)$ we have:

$$P_X(x) = \begin{cases} 3/7, & x = -2 \\ 1/7, & x = 0 \\ 3/7, & x = 2 \\ 0, & otherwise. \end{cases} \quad and\ P_Y(y) = \begin{cases} 5/14, & y = -1 \\ 2/7, & y = 0 \\ 5/14, & y = 1 \\ 0, & otherwise. \end{cases}$$

(b) $E(X) = \Sigma x P_X(x) = 0$. $E(Y) = \Sigma y P_Y(y) = 0$.
(c) $\sigma_X = \sqrt{E(X^2)} = \sqrt{\Sigma x^2 P_X(x)} = \sqrt{24/7}$. $\sigma_Y = \sqrt{E(Y^2)} = \sqrt{\Sigma y^2 P_Y(y)} = \sqrt{5/7}$.

9 Prob. 5.4.2 Solution (10 pts)

(a) Let $\int_0^1 \int_0^1 f(x, y)dxdy = 1$ we have $c = 6$.
(b) $P[X > Y] = \int_0^1 \int_0^x 6xy^2 dydx = 2/5$. $P[Y < X^2] = \int_0^1 \int_0^{x^2} 6xy^2 dydx = 1/4$.
(c) $P[min(X, Y) \le 1/2] = 1 - P[X > 1/2, Y > 1/2] = 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 dydx = 11/32$.
(d) $P[max(X, Y) \le 3/4] = P[X \le 3/4, Y \le 3/4] = \int_0^{3/4} \int_0^{3/4} 6xy^2 dydx = 243/1024$.

10 Prob. 5.6.7 Solution (10 pts)

(a) Let $\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} f(x, y)dxdy = 1$ we have $k = 3/4$.
(b) $\int_{-1/2}^{1/2} (3/4+3x^2)dy = 3/4+3x^2$. Thus, $f_X(x) = \begin{cases} 3/4 + 3x^2, & -1/2 \le x \le 1/2 \\ 0, & otherwise. \end{cases}$.
(c) $\int_{-1/2}^{1/2} (3/4 + 3x^2)dx = 1$. Thus, $f_Y(y) = \begin{cases} 1, & -1/2 \le y \le 1/2 \\ 0, & otherwise. \end{cases}$.
(d) $f_X(x)f_Y(y) = \begin{cases} 3/4 + 3x^2, & -1/2 \le x \le 1/2, -1/2 \le y \le 1/2 \\ 0, & otherwise. \end{cases} = f_{XY}(x, y)$.
Thus, X and Y are independent.