ENGG2470A: Probability for Engineers
Spring 2020

Solution for Assignment 2 — Date: N.A.

Prob. 1. Solution. (10 pts)

(a) From $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we can get

$$
\begin{aligned}
Left &= \binom{n}{r} + \binom{n}{r-1} \\
&= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\
&= \frac{(n+1)!}{r!(n-r+1)!}\frac{n-r+1}{n+1} + \frac{(n+1)!}{r!(n-r+1)!}\frac{r}{n+1} \\
&= \frac{(n+1)!}{r!(n-r+1)!}\left(\frac{n-r+1}{n+1} + \frac{r}{n+1}\right) \\
&= \frac{(n+1)!}{r!(n+1-r)!} \\
&= \binom{n+1}{r} = Right
\end{aligned}
$$

(b) Omitted.

(c) The binomial formula,

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^r b^{n-r}$$

- Base case, $n = 1$, $Right = \sum_{r=0}^{1} C(1,r)a^r b^{1-r} = a + b = Left$
- Induction step: suppose when $n = k$, $(a+b)^k = \sum_{r=0}^{k} C(k,r)a^r b^{k-r}$
- when $n = k+1$, we have

$$
\begin{aligned}
(a+b)^{k+1} &= (a+b)^k(a+b) \\
&= (a+b)\sum_{r=0}^{k} C(k,r)a^r b^{k-r} \\
&= \sum_{r=0}^{k} C(k,r)a^{r+1} b^{k-r} + \sum_{r=0}^{k} C(k,r)a^r b^{k+1-r} \\
&= \sum_{r=1}^{k+1} C(k,r-1)a^r b^{k+1-r} + \sum_{r=0}^{k} C(k,r)a^r b^{k+1-r}
\end{aligned}
$$

$$\begin{aligned}
&= C(k,0)a^0b^{k+1} + \sum_{r=1}^{k} \left(C(k,r-1)+C(k,r)\right) a^r b^{k+1-r} + C(k,k)a^{k+1}b^0 \\
&= C(k+1,0)a^0b^{k+1-0} + \sum_{r=1}^{k} C(k+1,r)a^r b^{k+1-r} + C(k+1,k+1)a^{k+1}b^0 \\
&= \sum_{r=0}^{k+1} C(k+1,r)a^r b^{k+1-r}
\end{aligned}$$

- In summary, it holds for all $n \in Z^+$.

Prob. 2. Solution. (10 pts)

No. The induction step is not true from $n = 1$ to $n = 2$. In particular, horse 1 to 1 has the same color, and horse 2 to 2 has the same color. But there is no common horse in both groups. So we can not conclude that these two horses have the same color.

Prob. 1.2.1 Solution. (10 pts)

(a) $\Omega = \{hm, ht, mm, mt, lm, lt\}$

(b) $A_1 = \{mm, mt\}$

(c) $A_2 = \{hm, mm, lm\}$

(d) $A_3 = \{hm, ht, lm, lt\}$

(e) No. For example $A_1 \cap A_2 = \{mm\} \neq \emptyset$

(f) Yes, $A_1 \cup A_2 \cup A_3 = \Omega$

Prob. 1.2.2 Solution. (10 pts)

(a) $\Omega = \{aaa, aaf, afa, aff, faa, faf, ffa, fff\}$

(b) $Z_F = \{aaf, aff, faf, fff\}, X_A = \{aaa, aaf, afa, aff\}$

(c) No. Clearly, $Z_F \cap X_A = \{aaf, aff\} \neq \emptyset$

(d) No. Clearly, $Z_F \cup X_A \neq \Omega$

(e) $C = \{aaf, afa, faa, aaa\}$, $D = \{aff, faf, ffa, fff\}$

(f) Yes. Clearly, $C \cap D = \emptyset$

(g) Yes. Clearly, $C \cup D = \Omega$

Prob. 1.3.2 Solution (10 pts)

(a) $P[R_3W_2] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(b) $P[S_5] = P[R_1W_4] + P[R_2W_3] + P[R_3W_2] + P[R_4W_1] = 4 \times \frac{1}{36} = \frac{1}{9}$

Prob. 1.3.4 Solution (10 pts)

(a) False, $P[A] = 2P[A^c] \implies P[A] = 2(1 - P[A]) \implies P[A] = \frac{2}{3}$

(b) False. For example, $A \subset B, P[A] = 0.3, P[B] = 0.5$. Then, $P[AB] = P[A] = 0.3 > P[A]P[B] = 0.15$.

(c) True. $P[AB] \leq P[A] < P[B]$.

(d) False. For example, $A \subset B, P[A] = 0.3, P[B] = 0.5$. Then, $P[A \cap B] = P[A] = 0.3 < P[B] = 0.5$.

Prob. 1.3.5 Solution (10 pts)

The sample space is $\Omega = \{BF, BW, LF, LW\}$. We have,

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4$$

$$P[W] = 1 - P[F] = 1 - (P[BF] + P[LF]) = 1 - (0.2 + 0.5) = 0.3$$
$$P[W \cup B] = P[W] + P[B] - P[WB] = 0.3 + 0.4 - 0.2 = 0.5$$

Prob. 1.4.1 Solution (10 pts)

(a) $P[H_0|B] = \frac{P[H_0 \cap B]}{P[B]} = \frac{0.4}{0.4+0.1+0.1} = \frac{2}{3}$

(b) $P[L|H_1] = \frac{P[L \cap H_1]}{P[H_1]} = \frac{0.1}{0.1+0.1} = 0.5$

(c) $P[H_1 \cup H_2|L] = \frac{P[(H_1 \cup H_2) \cap L]}{P[L]} = \frac{0.1+0.2}{0.1+0.1+0.2} = 0.75$

Prob. 1.4.2 Solution (10 pts)

Accordingly, we have $P[R_i] = \frac{1}{6}, \forall i$, $P[G_j] = \frac{6-j}{6}$, $P[E] = 0.5$.

(a) $P[R_3|G_1] = \frac{P[R_3 \cap G_1]}{P[G_1]} \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$

(b) $P[R_6|G_3] = \frac{P[R_6 \cap G_3]}{P[G_3]} = \frac{1/6}{3/6} = \frac{1}{3}$

(c) $P[G_3|E] = \frac{P[G_3 \cap E]}{P[E]} = \frac{2/6}{0.5} = \frac{2}{3}$

(d) $P[E|G_3] = \frac{P[G_3 \cap E]}{P[G_3]} = \frac{2/6}{0.5} = \frac{2}{3}$

Prob. 1.4.5 Solution (5 pts)

According to the illustration, there are four genetic type in the second generation, $\{yy, yg, gy, gg\}$, each with probability $\frac{1}{4}$.

$$P[Y] = P[yy, yg, gy] = \frac{3}{4}.$$

Prob. 1.4.6 Solution (5 pts)

$$P[yy|Y] = \frac{P[yy, Y]}{P[Y]} = \frac{P[yy]}{P[Y]} = \frac{1}{3}$$