1. Let $(X, Y) \sim f_{XY}(x, y)$ and $Z = X + Y$. Show that

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy.$$

Proof:

We first prove $f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy.$

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(X + Y \le z) = \int\int_{\{(x,y):x+y\le z\}} f_{XY}(x, y)\, dy\, dx.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_{-\infty}^{\infty} \frac{d(z-y)}{dz} \frac{d}{d(z-y)} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy.$$

Now we prove that $f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(X + Y \le z) = \int\int_{\{(x,y):x+y\le z\}} f_{XY}(x, y)\, dy\, dx.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x, y) dxdy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_{-\infty}^{\infty} \frac{d(z-x)}{dz} \frac{d}{d(z-x)} \int_{-\infty}^{z-x} f_{XY}(x, y) dydx$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, z-x) dydx$$

2. According to the statistics, among the students coming back from UK and USA, 1% are infected by COVID-19. They all take the saliva test with accuracy 99%. Suppose a particular student's saliva test is negative. A layman makes the following interpretation. The probability that the student is infected in the first place is 0.01 (i.e., 1%). Since the test independently indicates that he is not infected and the error probability of the test is 0.01 (i.e., 1%), the actual probability of the student being infected is $0.01 \times 0.01 = 0.01\%$. Comment on his interpretation.

Solution:

His interpretation is wrong and the probability is also wrong. The probability of the student being infected is the probability of "someone being infected" given "the one's test being negative".

The correct probability is shown as followed.

$$P(infected|-) = \frac{P(infected \cap -)}{P(-)}$$

$$= \frac{P(infected \cap incorrect)}{P(infected \cap incorrect) + P(NOT\ infected \cap correct)}$$

$$= \frac{P(infected) * P(incorrect)}{P(infected) * P(incorrect) + P(NOT\ infected) * P(correct)}$$

$$= \frac{0.01 * 0.01}{0.01 * 0.01 + 0.99 * 0.99} \approx 0.000102 \neq 0.001$$

5.5.5 X and Y are random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & \begin{matrix} -1\le x\le 1; \\ 0\le y\le x^2 \end{matrix} \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?

(b) What is the marginal PDF $f_Y(y)$?

Solution:

(a) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$\int_0^{x^2} \frac{5x^2}{2}\,dy = \frac{5x^4}{2} = \begin{cases} \frac{5x^4}{2}, & -1 \le x \le 1 \\ 0, & otherwise \end{cases}$$

(b) $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad x > \sqrt{y} \text{ or } x < -\sqrt{y}.$$

$$= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\,dx$$

$$= \begin{cases} \frac{5-5\sqrt{y^3}}{3}, & 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

6.5.4 Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

Notes Work out Problem 6.5.4 by two methods:

1. Take advantage of the observation that X and Y are independent random variables and apply the result in Example 4.17.

2. Use the result in the supplementary problem above.

Solution:

- **Method 1:**

$$\begin{aligned} f_W(w) = f_X * f_Y(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx \\ &= \int_0^1 f_Y(w-x)\, dx \end{aligned}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

For $0 < w \le 1$, if $0 \le x \le w$, then $f_Y(w-x) = 1$. Otherwise, $f_Y(w-x) = 0$.

$$f_W(w) = \int_0^w f_Y(w-x)dx = \int_0^w 1\, dx = w$$

For $1 < w \le 2$, if $w - 1 \le x \le 1$, then $f_Y(w-x) = 1$. Otherwise, $f_Y(w-x) = 0$.

$$f_W(w) = \int_{w-1}^1 f_Y(w-x)dx = \int_{w-1}^1 1\, dx = 2 - w$$

For $w \le 0$ or $w \ge 2$, $f_Y(w-x) = 0$.

$$f_W(w) = \begin{cases} w, & 0 < w \le 1 \\ 2 - w, & 1 < w \le 2 \\ 0, & \text{otherwise}. \end{cases}$$

- **Method 2:**

$$f_W(w) = \int_{-\infty}^{\infty} f_{XY}(x, w-x)dx = \int_0^1 f_{XY}(x, w-x)dx$$

For $0 < \mathrm{w} \le 1$, if $0 \le x \le w$, then $f_{XY}(x, w-x) = 1$. Otherwise, $f_{XY}(x, w-x) = 0$.

$$f_W(w) = \int_0^w f_{XY}(x, w-x)dx = \int_0^w 1\, dx = w$$

For $1 < \mathrm{w} \le 2$, if $w-1 \le x \le 1$, then $f_{XY}(x, w-x) = 1$. Otherwise, $f_{XY}(x, w-x) = 0$.

$$f_W(w) = \int_{w-1}^1 f_{XY}(x, w-x)dx = \int_{w-1}^1 1\, dx = 2-w$$

For $\mathrm{w} \le 0$ or $\mathrm{w} \ge 2$, $f_{XY}(x, w-x) = 0$.

$$f_W(w) = \begin{cases} w, & 0 < w \le 1 \\ 2-w, & 1 < w \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

7.1.1 Random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < -3, \\ 0.4 & -3 \le x < 5, \\ 0.8 & 5 \le x < 7, \\ 1 & x \ge 7. \end{cases}$$

Find the conditional CDF $F_{X|X>0}(x)$ and PMF $P_{X|X>0}(x)$.

Solution:

According to the CDF of X, we have

$$P_X(x) = \begin{cases} 0.4, & x = -3 \\ 0.4, & x = 5 \\ 0.2, & x = 7 \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$P_{X|X>0}(x) = \frac{P_X(x)}{\mathbf{P}(\{X > 0\})} = \begin{cases} 2/3, & x = 5 \\ 1/3, & x = 7 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_{X|X>0}(x) = \begin{cases} 0, & x < 5 \\ 2/3, & 5 \le x < 7 \\ 1, & x \ge 7. \end{cases}$$

7.1.5 Every day you consider going jogging. Before each mile, including the first, you will quit with probability q, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let M equal the number of miles that you run on an arbitrary day. Find the PMF $P_M(m)$.

(b) Let r be the probability that you run a marathon on an arbitrary day. Find r.

(c) Let J be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of r found in part (b).

(d) Define $K = M - 26$. Let A be the event that you have run a marathon. Find $P_{K|A}(k)$.

Solution:

(a) Before you run the first mile, you quit with probability q. Thus $P_M(0) = q$.

When you run the first mile, you quit with probability q. Thus

$$P_M(1) = (1-q)q.$$

Therefore,

$$P_M(m) = \begin{cases} (1-q)^m q, & m \in \mathbb{Z}^{\geq} \\ 0, & \text{otherwise.} \end{cases}$$

(b) When you run at least 26 miles, you haven't quit for 26 times. Thus,

$$r = (1-q)^{26}$$

(c) You run marathon for j days and you don't run for $365 - j$ days.

$$P_J(j) = \begin{cases} \binom{365}{j} r^j (1-r)^{365-j}, & j = 0,1,2,\dots,365 \\ 0, & otherwise \end{cases}.$$

(d) $P_{K|A}(k) = \frac{P_M(k+26)}{r}$

Thus,

$$P_{K|A}(k) = \begin{cases} (1-q)^k q, & k = 0,1,2,\dots \\ 0, & otherwise \end{cases}$$

7.2.1 X is the binomial (5, 1/2) random variable. Find $P_{X|B}(x)$, where the condition $B = \{X \geq \mu_X\}$. What are E[$X|B$] and Var[$X|B$]?

Solution:

$$\mathbf{P}(B) = \left(\binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right) 2^{-5} = 0.5.$$

$$P_{X|B}(x) = \frac{P_X(x)}{P(B)} = \begin{cases} \frac{5}{8}, & x = 3 \\ \frac{5}{16}, & x = 4 \\ \frac{1}{16}, & x = 5 \\ 0, & otherwise \end{cases}$$

$$E[X|B] = 3 * \frac{5}{8} + 4 * \frac{5}{16} + 5 * \frac{1}{16} = \frac{55}{16}$$

$$\text{Var}[X|B] = \sum_{x=3}^{5} (x - E[X|B])^2 P_{X|B}(x) = \frac{95}{256}$$

7.2.3 X is the continuous uniform $(-5, 5)$ random variable. Given the event $B = \{|X| \leq 3\}$, find the

(a) conditional PDF, $f_{X|B}(x)$,

(b) conditional expected value, $\mathrm{E}[X|B]$,

(c) conditional variance, $\mathrm{Var}[X|B]$.

Solution:

(a) $P(B) = \int_{-3}^{3} \frac{1}{10} dx = \frac{3}{5}$

$$f_{X|B}(x) = \frac{f_X(x)}{P(B)} = \begin{cases} \frac{1}{6}, & -3 \leq x \leq 3 \\ 0, & otherwise \end{cases}$$

(b) $E[X|B] = \int_{-\infty}^{+\infty} x f_{X|B}(x) dx = \int_{-3}^{3} x f_{X|B}(x) dx = 0$

(c) $Var[X|B] = \int_{-\infty}^{+\infty} (x - E[X|B])^2 f_{X|B}(x) dx = 3$

7.3.3 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the event that $X + Y \leq 1$. Find the conditional PDF $f_{X,Y|A}(x,y)$.

Solution:

$$P(A) = \iint_{(X,Y)\in A} f_{X,Y}(x,y)dxdy = \int_0^1 \int_0^{1-y} 6e^{-(2x+3y)}dxdy$$
$$= 1 - 3e^{-2} + 2e^{-3}$$

$$f_{X,Y|A}(x) = \frac{f_{X,Y}(x)}{P(A)} = \begin{cases} \dfrac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}}, & x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & otherwise \end{cases}$$

7.4.3 X is the continuous uniform (0, 1) random variable. Given $X = x$, Y is conditionally a continuous uniform (0, 1+ x) random variable. What is the joint PDF $f_{X,Y}(x, y)$ of X and Y?

Firstly, we have

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|X) = \begin{cases} 1/(1+x), & y \in [0, 1+x] \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$f_{XY}(x,y) = \begin{cases} 1/(1+x), & x \in [0,1], y \in [0, 1+x] \\ 0, & \text{otherwise.} \end{cases}$$

7.4.8 $Y = ZX$ where X is the Gaussian (0, 1) random variable and Z, independent of X, has PMF

$$P_Z(z) = \begin{cases} 1-p & z=-1, \\ p & z=1. \end{cases}$$

True or False:

(a) Y and Z are independent.

(b) Y and X are independent.

(a) We only need to determine whether $P_{Y|Z}(y|z=1)$ equals $P_{Y|Z}(y|z=-1)$.

$$P_{Y|Z}(y|z=1) = P_X(X=y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$
$$P_{Y|Z}(y|z=-1) = P_X(X=-y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

Thus, $P_{Y|Z}(y|z=1) = P_{Y|Z}(y|z=-1)$. The statement is true.

(b) This is false. We only need find a counter example.

When $x=0$,

$$P_{Y|X}(y=0|x=0) = 1$$

However, when $x=1$,

$$P_{Y|X}(y=0|x=1) = 0$$

Thus, $P_{Y|X}$ depends on X.

7.5.7 Over the circle $X^2 + Y^2 \leq r^2$, random variables X and Y have the uniform PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $f_{Y|X}(y|x)$?

(b) What is $E[Y|X = x]$?

(a) $f_{Y|X}(y|X=x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad = \begin{cases} \displaystyle\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2} dy, & -r \leq x \leq r \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-x^2}}{\pi r^2}, & -r \leq x \leq r \\ 0, & otherwise \end{cases}$$

Thus,

$$f_{Y|X}(y|X=x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \dfrac{1}{2\sqrt{r^2-x^2}}, & 0 \leq x^2 + y^2 \leq r^2 \\ 0, & otherwise \end{cases}$$

(b) $$\mathbf{E}[Y|X=x] = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{y}{2\sqrt{r^2-x^2}} dy = \frac{1}{4\sqrt{r^2-x^2}} [y^2]_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} = 0.$$

7.6.1 You wish to measure random variable X with expected value E[X] = 1 and variance Var[X] = 1, but your measurement procedure yields the noisy observation $Y = X + Z$, where Z is the Gaussian (0, 2) noise that is independent of X.

(a) Find the conditional PDF $f_{Z|X}(z|x)$ of Z given $X = x$.

(b) Find the conditional PDF $f_{Y|X}(y|2)$ of Y given $X = 2$. Hint: Given $X = x$, $Y = x + Z$.

(a) X and Z are independent. Thus,

$$f_{Z|X}(z|X = x) = f_Z(z) \quad = \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}$$

(b) Y=2+Z.

$$E[Y|X = 2] = E[2 + Z|X = 2] = 2 + E[Z|X = 2] = 2 + E[Z] = 0$$
$$Var[Y|X = 2] = Var[2 + Z|X = 2] = Var[Z|X = 2] = Var[Z] = 2$$

Thus,

$$f_{Y|X} = \frac{1}{\sqrt{4\pi}} e^{-\frac{(y-2)^2}{4}}$$