ENGG 2470A: Probability for Engineers

Assignment 4 —- Solution

1. Solution:

Let

$$I = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx$$

Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} f(y)dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(-\frac{x^2+y^2}{2})dxdy \end{aligned}$$

Transform to the polar coordinates, let $x = r\cos\theta, y = r\sin\theta$, we have $dxdy = rdrd\theta$, then

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} \frac{1}{2\pi} r \exp(-\frac{r^2}{2})d\theta dr \\ &= \int_0^{2\pi} \frac{1}{2\pi} d\theta \int_0^{\infty} \frac{1}{2} \exp(-\frac{r^2}{2})dr^2 \\ &= 1 \end{aligned}$$

Note that $f(x) > 0$, therefore $I \geq 0$, and we conclude that $I = \int_{-\infty}^{\infty} f(x)dx = 1$.

2. Solution:

Recall that the probability distribution function of exponential distribution is given by

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

According to the definition, if a random variable X follows the exponential distribution with parameter λ, then its expectation can be calculated by

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)dx \\ &= \int_0^{\infty} \lambda x e^{-\lambda x} dx \\ &= -\int_0^{\infty} x de^{-\lambda x} \\ &= -x e^{-\lambda x}|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\ &= \frac{1}{\lambda} \end{aligned}$$

3. Solution:

Let X be a random variable that follows the given geometric distribution, then the expectation can be given by

$$
\begin{aligned}
E[X] =& \sum_{n=0}^{\infty} np(1-p)^n && (1)\\
=& p\sum_{n=0}^{\infty} n(1-p)^n && (2)\\
=& p(1-p)\sum_{n=1}^{\infty} n(1-p)^{n-1} && (3)\\
=& p(1-p)\frac{1}{p^2} && (4)\\
=& \frac{1-p}{p} && (5)
\end{aligned}
$$

where (3) to (4) follows by derivative of geometric progression.

Textbook Problems

3.6.1.

(a) Solution:

From the solution to problem 3.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4, & y = 1, \\ 1/4, & y = 2, \\ 1/2, & y = 3, \\ 0, & \text{otherwise} \end{cases}$$

Since $Y \in \{1, 2, 3\}$, we know $U = Y^2 \in \{1, 4, 9\}$, and the PMF of U is

$$P_U(u) = \begin{cases} 1/4, & u = 1, \\ 1/4, & u = 4, \\ 1/2, & u = 9, \\ 0, & \text{otherwise} \end{cases}$$

(b) Solution:

From the PMF, it is straightforward to write down the CDF

$$F_U(u) = \begin{cases} 0, & u < 1, \\ 1/4, & 1 \le u < 4, \\ 1/2, & 4 \le u < 9, \\ 1, & u \ge 9. \end{cases}$$

(c) Solution:

Based on the PMF, it is easy to get the expectation according to the definition:

$$\begin{aligned} E[U] =& \sum_u u P_U(u) \\ =& 1 \times \frac{1}{4} + 4 \times \frac{1}{4} + 9 \times \frac{1}{2} \\ =& 5.75 \end{aligned}$$

3.6.7

(a) Solution:

A student is properly counted with probability p, independent of any other student being counted. Therefore, we have 70 Bernoulli trials and N is a binomial $(70, p)$ random variable with PMF

$$P_N(n) = \binom{70}{n} p^n (1-p)^{70-n}$$

(b) Solution:

A student is uncounted with probability $1-p$, hence the number of uncounted students U is a binomial $(70, 1-p)$ random variable with PMF

$$P_U(u) = \binom{70}{n} (1-p)^u p^{70-u}$$

(c) Solution:

The probability of $U \geq 2$ is

$$\begin{aligned} P[U \geq 2] =& 1 - P[U < 2] \\ =& 1 - \Big(P_U(0) + P_U(1)\Big) \\ =& 1 - (p^{70} + 70(1-p)p^{69}) \end{aligned}$$

(d) Solution:

The expectation of U

$$E[U] = 70(1-p) = 2$$

hence we get $p = 34/35$.

3.7.6

It is easy to verify that this statement is false by a counterexample, e.g., let X be a random with following PMF

$$P_X(x) = \begin{cases} 1/4, & x = 1, \\ 3/4, & x = 2, \end{cases}$$

In this case, $E[X] = 7/4$, while $E[1/X] = 5/8$, obviously $1/E[X] \neq E[1/X]$.

3.8.5

(a) Solution:

- Method 1: Recall that the PMF of binomial distribution is given by

$$f(k,n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

 and

$$P_X(x) = \binom{4}{x} (\frac{1}{2})^4 = \binom{4}{x} (\frac{1}{2})^x (\frac{1}{2})^{4-x}$$

 Therefore X is a binomial $(4, 1/2)$ random variable, whose variance is

$$var[X] = np(1-p) = 4 \times \frac{1}{2} \times \frac{1}{2} = 1$$

 And X has standard deviation $\sigma_X = \sqrt{var[X]} = 1$

- Method 2: According to the definition:

$$E[X] = \sum_{x=0}^{4} x P_X(x) = 2$$

 The expected value of X^2 is

$$E[X^2] = \sum_{x=0}^{4} x^2 P_X(x) = 5$$

 Then the variance of X is

$$var[X] = E[X^2] - (E[X])^2 = 1$$

 Hence the standard deviation is 1.

(b) Solution:

The probability that X is within one standard deviation of its expected values is

$$\begin{aligned} P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] =& P[1 \le X \le 3] \\ =& P_X(1) + P_X(2) + P_X(3) \\ =& 7/8 \end{aligned}$$

4.2.2

(a) Solution:

Since X is continuous random variable, its CDF is continuous, we have $c(7+5)^2 = 1$ and get $c = 1/144$.

(b) Solution:

$$P[V > 4] = 1 - P[V \le 4] = 1 - \frac{1}{144}(4+5)^2 = \frac{7}{16}$$

(c) Solution:

$$\begin{aligned} P[-3 < V \leq 0] =& P[V \leq 0] - P[V \leq -3] \\ =& F_V(0) - F_V(-3) = 7/48 \end{aligned}$$

(d) Solution:

$$\begin{aligned} & P[V > a] = 1 - P[V \leq a] = 2/3 \\ \Rightarrow & F_V(a) = (a+5)^2/144 = 1/3 \\ \Rightarrow & a = 4\sqrt{3} - 5 \end{aligned}$$

4.3.2

When CDF is differentiable,

$$f(x) = \frac{dF(x)}{dx}$$

Therefore, for $x < -5$:

$$f(x) = \frac{dF(x)}{dx} = 0$$

For $-5 \leq x < 3$:

$$f(x) = \frac{dF(x)}{dx} = \frac{1}{8}$$

For $-3 \leq x < 3$:

$$f(x) = \frac{dF(x)}{dx} = 0$$

For $3 \leq x < 5$:

$$f(x) = \frac{dF(x)}{dx} = \frac{3}{8}$$

For $x \geq 5$:

$$f(x) = \frac{dF(x)}{dx} = 0$$

In summary, $f(x)$ can be written as a compact form:

$$f(x) = \begin{cases} 0, & x < -5, \\ 1/8, & -5 \leq x < -3, \\ 0, & -3 \leq x < 3, \\ 3/8, & 3 \leq x < 5, \\ 0, & x \geq 5. \end{cases}$$

4.3.4

According to the definition

$$F_X(x) = \int_{-\infty}^{x} f(u)du$$

For $x \leq 0$,

$$F_X(x) = \int_{-\infty}^{x} 0du = 0$$

For $x > 0$,

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} a^2 u e^{-\frac{a^2u^2}{2}} du \\ &= \int_{0}^{x} a^2 u e^{-\frac{a^2u^2}{2}} du \\ &= 1 - e^{-\frac{a^2x^2}{2}} \end{aligned}$$

In summary, the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ 1 - e^{-\frac{a^2x^2}{2}}, & x > 0. \end{cases}$$

6.2.3

Note that T has the continuous uniform PDF

$$f_T(t) = \begin{cases} 1/15, & 60 \leq t < 75, \\ 0, & \text{otherwise.} \end{cases}$$

The rider's maximum possible speed is $V = 3000/60 = 50$ km/hr, while the minimum speed is $V = 3000/75 = 40$ km/hr. For $40 \leq v \leq 50$,

$$\begin{aligned} F_V(v) = P[\frac{3000}{T} \leq v] &= P[T \geq \frac{3000}{v}] \\ &= \int_{3000/v}^{75} \frac{1}{15} dt \\ &= 5 - \frac{200}{v} \end{aligned}$$

Thus the CDF, and via a derivative, the PDF are

$$F_V(v) = \begin{cases} 0, & v < 40, \\ 5 - \frac{200}{v}, & 40 \leq v \leq 50, \\ 1, & v > 50. \end{cases}$$

$$f_V(v) = \begin{cases} 0, & v < 40, \\ 200/v^2, & 40 \leq v \leq 50, \\ 0, & v > 50. \end{cases}$$

6.2.5

Since X is nonnegative, $W = X^2$ is also nonnegative. Hence for $w < 0$, $f_W(w) = 0$. For $w \geq 0$,

$$\begin{aligned} F_W(w) = P[W \leq w] =& P[X^2 \leq w] \\ =& P[X \leq w] \\ =& 1 - e^{-\lambda\sqrt{w}} \end{aligned}$$

Taking the derivative with respect to w yields $f_W(w) = \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w})$. In summary, the complete expression of the PDF is

$$f_W(w) = \begin{cases} \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w}), & w \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$