Strichartz estimate for the Schrödinger equation

Jingwei Qiu

Supervisor: Dr. Zihua Guo

Honours Thesis submitted as part of the B.Sc. (Honours) degree
in the School of Mathematics, Monash University.

May 29, 2019

Abstract

The Strichartz estimate was first discovered by Strichartz [30] in 1977 in the form of a Fourier restriction theorem. It was then proved and generalized by Ginibre and Velo [13], Yajima [36], Keel and Tao [20]. The Strichartz estimate has applications in many partial differential equations, including the nonlinear Schrödinger equation, nonlinear wave equation, kinetic transport equation and many other dispersive equations. The study of the Strichartz estimate uses techniques from harmonic analysis, functional analysis and mathematical physics. In this thesis, we shall review the related techniques and report on the classical results of the Strichartz estimate. We shall discuss the proof of the Strichartz estimate, with some generalizations to the Sobolev space and Besov space. We shall also discuss the mathematical structures of the nonlinear Schrödinger equation, including its Hamiltonian formulation, symmetries, conservation laws and the Viriel's identity. In particular, we attempted to derive the nonlinear Schrödinger equation from its total energy function in a symplectic space. Finally, we shall apply the Strichartz estimate to the nonlinear Schrödinger equation and discuss some classical wellposedness results.

Contents

Acknowledgement

I am grateful to my supervisor A/Prof. Zihua Guo for his guidance and patience during the honours year. Over the past two years, his guidance has helped me gain more maturity and appreciation in mathematics.

Chapter 1

Introduction

It is unsurprising that our everyday intuition fails in the microscopic world of particles. Due to the inderministic nature of the particles at atomic or subatomic scale, we can only use a statistical average to describe their positions. In particular, if we associate each particle with a quantum wave function $u(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}$, then the quantity

$$\int_V |u(t,x)|^2 dx$$

is the probability of finding the particle in the region V. The time evolution of the quantum wave function is governed by the Schrödinger equation. In this thesis, we are interested in the Cauchy problem of the Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = F(u), \\ u(0,x) = \varphi(x), \end{cases}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. This equation has many applications in physics, especially in quantum mechanics, quantum field theory, and nonlinear optics. Although the Schrödinger equation was originally used to describe certain phenomena that occurs in nature, it turns out that some of its mathematical structures are also attractive to more abstract mathematical subjects like analysis. In fact, it has received much attention from analysts, such as Bourgain [4], Colliander [9], Tsutsumi [34], Kato [17], Staffilani[8] and Tao [32].

From the prospective of analysis, one could ask the following question: given a profile of wave at $t = 0$, what predictions can we make regarding the time evolution of the wave profile? Is it possible to prove that there exists a unique wave that lives in the time interval $[0, T]$, satisfies the equation and coincides with the initial profile at $t = 0$? What would happen to this wave at later time? This is an example of the *wellposedness problem.* In order to answer these problems, we shall make use of the Strichartz estimate for the Schrödinger operator. The celebrated Strichartz estimate reads

$$||e^{\frac{it\Delta}{2}}\varphi||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||\varphi||_{L_x^2(\mathbb{R}^n)}, \tag{1.1}$$

for the homogeneous case, and

$$\left\| \int_{s<t} e^{\frac{i(t-s)\Delta}{2}} F(s,x)\, ds \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)}, \tag{1.2}$$

for the inhomogeneous case. Here the pairs (q, r) and $(\tilde{q}, \tilde{r})$ are called *Schrödinger-admissible* pairs, and $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$, $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$. This estimate is the main study object of our thesis, we shall discuss the literature, proof and applications of this estimate.

The objective of this thesis is to report on the classical results and recent progress of the Strichartz estimate in the context of the Schrödinger equation. We shall also discuss the applications of the Strichartz estimate on proving some wellposedness results of the nonlinear Schrödinger equation. Although we did not aim at solving open problems due to the difficulty of these topics, we made an attempt to derive the Hamiltonian formulation of the nonlinear Schrödinger equation in Chapter 4. In particular, we obtained:

Theorem 1. *Let $L^2(\mathbb{R}^n \to \mathbb{C})$ be a symplectic space equipped with the symplectic form*

$$\omega(g, v) = -\operatorname{Im} \int_{\mathbb{R}^n} g(x)\overline{v(x)}\, dx,$$

then the nonlinear Schrödinger equation $iu_t + \Delta u = \lambda |u|^{p-1} u$ is the Hamiltonian flow of the Hamiltonian

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\lambda}{p+1} |u|^{p+1} dx.$$

The thesis is organized as follows. In Chapter 2, We shall first recall some well-known results in functional analysis and Fourier analysis. We shall review some function spaces such as the Schwartz space, Sobolev space and Besov space. We will also introduce some tools from the harmonic analysis that we use throughout the text. In Chapter 3, we shall establish some fundamental properties and estimates of the Schrödinger equation. We will introduce the Strichartz estimate, along with some generalizations to the Sobolev and Besov spaces. Chapter 4 is devoted to the study of the mathematical structures of the nonlinear Schrödinger equation, including its Hamiltonian formulation, symmetries and conservation laws, and their connections with the Noether's theorem. The last chapter is devoted to the wellposedness theory of the nonlinear Schrödinger equation. We shall realize the power of the Strichartz estimate to obtain some local and global wellposedness results in certain function spaces.

Chapter 2

Preliminaries

Let us start by introducing the basic tools and ideas we shall use in this report. The nonlinear Schrödinger equation can be profitably analyzed by viewing it as describing interaction of waves with different frequencies. A rigorous tool for studying these is the Fourier analysis, in particular the Fourier transform and Fourier multiplier. In this section, we shall introduce these tools and briefly mention their utility in PDEs. Besides, in order to obtain wellposedness of the nonlinear Schrödinger equation, we need to make a delicate balance between the solution and its survival space. We shall introduce some commonly used function spaces in PDE, such as Schwartz space, the space of Schwartz distributions, Sobolev space and Besov space.

2.1 Basic functional analysis

In this subsection, we shall review some basic functional analysis. Some good references for this section are Folland[11], Rudin[24] and Yosida[37]. Let us begin with the definition of a complete metric space.

Definition 2.1.1. *Let $\mathcal{X}$ be a non-empty set. the set $\mathcal{X}$ is said to be a metric space if there is a function $d(x,y)$ satisfying*

(1). $d(x,y) \geqslant 0$ and $d(x,y) = 0$ if and only if $x = y$;

(2). $d(x,y) = d(y,x)$;

(3). $d(x,z) \leqslant d(x,y) + d(y,z)$ for any $x, y, z \in \mathcal{X}$;

In addition, if every Cauchy sequence converges with respect to the metric d, we say $(\mathcal{X}, d)$ is complete.

One of the main topic in this thesis is to obtain the existence and uniqueness of solutions to the nonlinear Schrödinger equation. A common technique is to convert the solution of a partial differential equation to the fixed point of a solution map. Thus, to obtain the local existence of the solution for the nonlinear Schrödinger equation, we shall make use of the contraction mapping principle.

Theorem 2.1.1 (Contraction mapping principle)**.** *Let $(\mathcal{X}, d)$ be a complete metric space and $\phi : \mathcal{X} \to \mathcal{X}$. If there exists a $\gamma \in (0,1)$ such that $d(\phi(u), \phi(v)) \leqslant \gamma d(u,v)$ for any $u, v \in \mathcal{X}$, then ϕ has a unique fixed point $\overline{u} \in \mathcal{X}$. In addition, ϕ is said to be a contraction.*

Although the contraction mapping principle has provided us with a direct way to construct solutions to a PDE, it remains to design a supportive solution space. The L^p

spaces are fundamental in functional analysis, we shall first review their definitions and properties.

Definition 2.1.2 (L^p space). *Let $\Omega \subset \mathbb{R}^n$ and let $u : \Omega \to \mathbb{C}$ be a measurable function, then the $L^p(\Omega)$ space consists of functions such that*

$$\int_\Omega |u(x)|^p dx < \infty,$$

for $1 \leqslant p < \infty$, and

$$\operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty.$$

*for $p = \infty$. In particular, we use a **mixed norm notation**: let $q, r \in \mathbb{R}$, if a function $u(t, x)$ is measurable on $\Omega \times \mathbb{R}^n$, then the mixed norm $L^q_t L^r_x$ of u is defined as*

$$||u||_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} = (\int_{\mathbb{R}} (\int_{\mathbb{R}^n} |u(t, x)|^r dx)^{q/r} dt)^{1/q}.$$

Remark 2.1.1. *The L^p space enjoys a number of properties, we shall summarize these classical properties here.*

(i). The L^p spaces are Banach spaces for $1 \leqslant p \leqslant \infty$. In particular, L^2 is a real Hilbert space when equipped with the inner product

$$\langle u, v\rangle := \operatorname{Re} \int_\Omega u\overline{v}\, dx.$$

(ii). The space $C_c^\infty(\Omega)$ is a dense subspace of $L^p(\Omega)$, for $1 \leqslant p \leqslant \infty$.

(iii). If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leqslant 1$, then $uv \in L^r(\Omega)$ and

$$\|uv\|_{L^r(\Omega)} \leqslant \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

(iv). If $u, v \in L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, then $u + v \in L^p(\Omega)$ and

$$\|u + v\|_{L^p(\Omega)} \leqslant \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

(v). If $u \in L^p(\Omega \times \mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, then

$$\left\|\int_\Omega u(x, y)\, dy\right\|_{L^p_x(\Omega\times\mathbb{R}^n)} \leqslant \int_\Omega ||u(x, y)||_{L^p_x(\Omega\times\mathbb{R}^n)}\, dy.$$

(vi). If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then $f * g \in L^r(\Omega)$ and

$$||f * g||_{L^r(\Omega)} \leqslant ||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}.$$

Although the L^p space is very useful for classifying the integrability of a soluion, it does not have enough structures for controling the regularity and decay rate of the solution. In practice, we often require the solution or its derivatives to have enough decay rate so that they vanish at the boundary of the domain. This is one of the reason why L^p space is not good enough to be our solution space. Instead, the Schwartz space offers a better environment for our analysis.

Before giving a definition of the Schwartz space, let us first introduce some notations. We use the *Japanese bracket* $\langle\cdot\rangle$ to denote $\langle x\rangle = (1+|x|^2)^{1/2}$ for any $x \in \mathbb{R}^n$, then a function $u : \mathbb{R}^n \to \mathbb{C}$ is said to be *rapidly decreasing* if $||\langle x\rangle^N u(x)||_{L^\infty(\mathbb{R}^n)} < \infty$ for all $N \geqslant 0$. In particular, this implies that u shall decay at the rate of $\mathcal{O}(\langle x\rangle^{-N})$ for any $N \geqslant 0$, which is faster than any polynomial. What we have seen above is a common way to give a restriction to the growth of functions and similar definition can be given to the derivatives of u. For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in Z^n_+$, we can define the *partial differential operator* as $\partial^\alpha_x u := (\frac{\partial}{\partial x_1})^{\alpha_1} ... (\frac{\partial}{\partial x_1})^{\alpha_n} u$. Now let us define the Schwartz space.

Definition 2.1.3 (Schwartz space)**.** *For any $N \geqslant 0$, multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in Z^n_+$ and $\Omega \subset \mathbb{R}^n$, the Schwartz space is defined as*

$$\mathcal{S}(\Omega) = \{u \in \mathbb{C}^\infty(\Omega) : ||\langle x\rangle^N \partial^\alpha_x u(x)||_{L^\infty(\Omega)} < \infty\},$$

Remark 2.1.2. *In other words, u is Schwartz if and only if $\partial^\alpha_x u(x) = \mathcal{O}(\langle x\rangle^{-N})$, for all $N \geqslant 0$. This implies that the partial derivatives $\partial^\alpha_x u(x)$ will decay faster than any polynomial. In particular, the dual space of $\mathcal{S}(\Omega)$ is the space of Schwartz distributions, denoted by $S'(\Omega)$. This is a generalized function space consisting of continuous linear functionals mapping from $\mathcal{S}(\Omega)$ to $\mathbb{C}$. We shall review the theory of distributions in the next section.*

Remark 2.1.3. *Let us summarize some useful properties of the Schwartz space.*

(i). The Schwartz space $\mathcal{S}(\Omega)$ is a dense subspace of $L^p(\Omega)$, for $1 \leqslant p \leqslant \infty$.

(ii). The Schwartz space $\mathcal{S}(\Omega)$is closed under the operation of multiplication, differentiation, convolution and Fourier transform. More precisely, if $u, v \in \mathcal{S}(\Omega)$, then uv, $\partial^\alpha u$, $u * v$ and $\hat{u}$ are Schwartz functions.

(iii). The Schwartz function u has enough decay rate such that

$$\lim_{|x|\to\infty} u(x) = 0.$$

What this last point suggests to us is that we can eliminate the boundary term when doing integration by parts. For any function $u \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in C^1(\mathbb{R}^n)$. Integration by parts of the following integral gives

$$\int_{\mathbb{R}^n} \Delta u\phi \, dx = \oiint \nabla u\phi \cdot \hat{n} \, dS - \int_{\mathbb{R}^n} \nabla u \cdot \nabla\phi \, dx$$

where the surface integral is evaluated at the infinite radius in $\mathbb{R}^n$. We may now approximate this surface integral by

$$\lim_{R\to\infty} \oiint_{\partial B_R} \nabla u\phi \cdot \hat{n} \, dS.$$

As $\nabla u \in \mathcal{S}(\mathbb{R}^n)$, we have $|\nabla u(R)| \leqslant \langle R\rangle^{-N}$ for any $N \geqslant 0$. In addition, the surface area of the sphere ∂B_R is cR^{n-1} for some constant $c > 0$. Thus we have

$$|\oiint_{\partial B_R} \nabla u\phi \cdot \hat{n} \, dS \,| \lesssim \langle R\rangle^{-(N-n+1)}.$$

Hence, if we select a sufficiently large N such that $N - n + 1 > 0$, the boundary term should converges to 0 as $R \to \infty$.

2.2 Fourier transform and distributions

The Fourier transform is a useful tool for the analysis of PDEs. In this section, we shall first present some basic definitions and properties of the Fourier transform. After that, we shall introduce the space of distributions and generalize the Fourier transform to the space of Schwartz distribution. The reference for this section is Stein & Sakarchi [28].

Definition 2.2.1 (Fourier transform). *The Fourier transform of a function $u \in \mathcal{S}(\mathbb{R}^n)$ is defined as*

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}dx,$$

for all $\xi \in \mathbb{R}^n$. In particular, $\mathcal{F}$ is the Fourier operator.

Remark 2.2.1. *The Fourier operator $\mathcal{F} : u \to \hat{u}$ takes the function $u(x)$ on the physical domain, and produces a function $\hat{u}(\xi)$ on the frequency domain. In particular, x is the variable representing the position, while ξ denotes the frequency. As we have done for Fourier series, we can decompose some "signals" $f(x)$ into a superposition of plane waves $e^{ix\cdot\xi}$. The Fourier transform $\hat{u}(\xi)$ represents the coefficients for plane waves $e^{ix\cdot\xi}$ with a certain frequency ξ.*

Proposition 2.2.1 (Elementary properties of Fourier transform). *Given functions u, $v \in \mathcal{S}(\mathbb{R}^n)$, $x, h, \lambda, \eta \in \mathbb{R}^n$, the Fourier operator $\mathcal{F}$ satisfies the following properties,*

(i). $\mathcal{F}$ *is linear,*
(ii). $\mathcal{F}(u(x-h)) = e^{-ix\cdot\eta}\hat{u}(\xi)$,
(iii). $\mathcal{F}(e^{ix\cdot\eta}u) = \hat{u}(\xi - h)$,
(iv). $\mathcal{F}(u(\lambda x)) = \hat{u}(\frac{\xi}{\lambda})$,
(v). $\mathcal{F}(\partial_x^\alpha u) = (i\xi)^\alpha \hat{u}(\xi)$ *and* $\mathcal{F}((-ix)^\alpha u(x)) = \partial_x^\alpha \hat{u}(\xi)$,
(vi). $\mathcal{F}(u * v) = \hat{u}\hat{v}$,
(vii). $\mathcal{F}(\overline{u(x)}) = \overline{\hat{u}(-\xi)}$.

Remark 2.2.2. *From property (v), we see that Fourier transform can turn a differential operator into a multiplication operator. This converts a differential equation in the physical space to an algebraic equation in the frequency space, which would simplify the calculations. Furthermore, one can use the Fourier inversion formula to transform the results back to the physical domain after performing the desired operations in the frequency domain.*

Now let us give a definition of the Fourier inverse transform,

Definition 2.2.2 (Fourier inversion). *If $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$, then its Fourier inverse is given by*

$$\mathcal{F}^{-1}(\hat{u}) = u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\xi)e^{ix\cdot\xi}d\xi$$

Remark 2.2.3. *We remark here that the Fourier operator $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism, thus the Schwartz space is closed under Fourier transform.*

The following theorem is very important in Fourier analysis as it states that the Fourier operator is an isometry with respect to the L^2 norm. We shall also use it to obtain an L^2 conservation law for the solution of the Schrödinger equation later.

Theorem 2.2.1 (Plancherel's theorem). *If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{u} \in L^2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |u(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 d\xi.$$

Let us finally introduce the concepts of distributions.

Definition 2.2.3 (Distribution). *Let $\Omega \subset \mathbb{R}^n$, a distribution F is a complex-valued continuous linear functional on $C_c^\infty(\Omega)$ that takes $\varphi \to F(\varphi)$, for any $\varphi \in C_c^\infty(\Omega)$. It is common to use the symbol $\mathcal{D}(\Omega)$ for $C_c^\infty(\Omega)$ and the symbol $\mathcal{D}'(\Omega)$ for the space of distribution on $\mathcal{D}(\Omega)$.*

Remark 2.2.4. *For any sequence $(\varphi_n)_{n=1}^\infty \subset C_c^\infty(\Omega)$, we say $\varphi_n \to \varphi$ in $C_c^\infty(\Omega)$ if the supports of φ_n are contained in a common compact set and for each multi-index α, $\partial_x^\alpha \varphi_n \to \partial_x^\alpha \varphi$ uniformly in x as $n \to \infty$. Based on this, the continuity of F means $F(\varphi_n) \to F(\varphi)$ whenever $\varphi_n \to \varphi$ in $C_c^\infty(\Omega)$.*

Example 2.2.1. *An ordinary funcion $f \in L^1_{loc}(\Omega)$ defines a distribution by*

$$F(\varphi) := \mathrm{Re}\left(\int_\Omega f\overline{\varphi}\, dx\right)$$

for all $\varphi \in \mathcal{D}(\Omega)$.

A distribution can be considered to be a generalized function whose value is the average taken with respect to a test function $\varphi \in \mathcal{D}(\Omega)$. Similarly, we can define the Schwartz distribution.

Definition 2.2.4 (Schwartz distribution). *Let $\Omega \subset \mathbb{R}^n$, a Schwartz distribution F is a complex-valued continuous linear functional on $\mathcal{S}(\Omega)$ that takes $\varphi \to F(\varphi)$, for any $\varphi \in \mathcal{S}(\Omega)$.*

Remark 2.2.5. *It is common to use the notation $\mathcal{S}'(\Omega)$ for the space of Schwartz distribution. Similar to $\mathcal{D}'(\Omega)$, the space of Schwartz distribution is considered to be a generalized function space. In particular, any $L^p(\Omega)$ is a subspace of $\mathcal{S}'(\Omega)$.*

Definition 2.2.5 (Derivative of Schwartz distribution). *Given a Schwartz distribution F, its derivative is defined by*

$$\partial_x^\alpha F(\varphi) := (-1)^{|\alpha|} F(\partial_x^\alpha \varphi)$$

for any Schwartz function $\varphi \in \mathcal{S}(\Omega)$.

Definition 2.2.6 (Fourier transform of Schwartz distributions). *Given a Schwartz distribution F, its Fourier transform is defined by*

$$\hat{F}(\varphi) := F(\hat{\varphi})$$

for any Schwartz function $\varphi \in \mathcal{S}(\Omega)$.

2.3 Sobolev space and Besov space

Since we already have the tools of Fourier transform, we are ready to define the fractional Sobolev space and Besov space using Fourier multipliers. Before doing that, let us first review the concepts of the classical Sobolev space. One may consult [5] for more information.

Definition 2.3.1. *Given $m \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial_x^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqslant m\}$$

with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|=0}^{m} \|\partial_x^\alpha u\|_{L^p(\Omega)} .$$

In particular, $H^m(\Omega) = W^{m,2}(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle u, v\rangle := \sum_{|\alpha|=0}^{m} \mathrm{Re} \int_\Omega \partial^\alpha u \overline{\partial^\alpha v}\, dx.$$

In the classical Sobolev space, the functions are assumed to be L^p functions with the regularity $m \in \mathbb{N}$. However, we may relax these restrictions and define the *fractional Sobolev space* for Schwartz distributions with any regularity $s \in \mathbb{R}$. The classical Sobolev space is just a special case of the fractional Sobolev space.

Definition 2.3.2 (Fractional Sobolev space)**.** *Given $s \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, the inhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined by*

$$W^{s,p}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L^p(\mathbb{R}^n)\}$$

with the norm $||u||_{W^{s,p}(\mathbb{R}^n)} = ||\mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u})||_{L^p_x(\mathbb{R}^n)}$. Similarly, the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$ is defined by

$$\dot{W}^{s,p}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \mathcal{F}^{-1}(|\xi|^s \hat{u}) \in L^p(\mathbb{R}^n)\}$$

with the norm $||u||_{\dot{W}^{s,p}(\mathbb{R}^n)} = ||\mathcal{F}^{-1}(|\xi|^s \hat{u})||_{L^p_x(\mathbb{R}^n)}$.

Remark 2.3.1. *It is common to use the **fractional integration operator** $\langle\nabla\rangle^s := \mathcal{F}^{-1}(\langle\xi\rangle^s)\mathcal{F}$, and the **fractional differentiation operator** $|\nabla|^s := \mathcal{F}^{-1}(|\xi|^s)\mathcal{F}$ to simplify the notation. More precisely, we have the relations $||u||_{W^{s,p}} = ||\langle\nabla\rangle^s u||_{L^p}$ and $||u||_{\dot{W}^{s,p}} = |||\nabla|^s u||_{L^p}$.*

Remark 2.3.2. *Here we summarize some fundamental properties of the Sobolev space.*

(i). For $1 < p < \infty$, the spaces $W^{m,p}$, $W^{s,p}$ and $\dot{W}^{s,p}$ are reflexive Banach spaces.

(ii). The index s denotes the regularity and the index p denotes the integrability of the functions. In particular, $W^{s,2} = H^s$ and $W^{0,p} = L^p$.

(iii). Sobolev embeddings: For $1 < p < q < \infty$ and $s > 0$ such that $\frac{1}{p} \leqslant \frac{1}{q} + \frac{s}{n}$ we have the inhomogeneous Sobolev embedding

$$\|u\|_{L^q} \lesssim \|u\|_{W^{s,p}} .$$

At he endpoint $q = \infty$, we also have

$$\|u\|_{L^\infty} \lesssim \|u\|_{W^{s,p}},$$

whenever $1 < p < \infty$, $s > 0$ and $\frac{1}{p} < \frac{s}{n}$.

Before finishing this section, we would like to introduce a more delicate function space, the Besov space, as we will use the properties of the Besov space to obtain a Besov space version of the Strichartz estimate. The definition of the Besov space is based on the Littlewood Paley decomposition as we shall introduce here.

Consider a function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\eta(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}$$

and define a sequence $(\psi_k)_{k\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$ by

$$\psi_k(\xi) = \eta(\frac{\xi}{2^{k+1}}) - \eta(\frac{\xi}{2^k})$$

then the support of ψ_k is near 2^k and we have the partition of unity

$$\sum_{k=-\infty}^{\infty} \psi_k(\xi) = \begin{cases} 1, & \xi \neq 0 \\ 0, & \xi = 0 \end{cases}$$

The main idea of Littlewood Paley theory is to decompose a function u into a sum of functions $\mathcal{F}^{-1}(\psi_k \hat{u})$ which has localized frequency $|\xi| \sim 2^k$. More precisely, for any $u \in \mathcal{S}'(\mathbb{R}^n)$, there exists a littlewood Paley decomposition of u such that

$$u = \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}(\psi_k \hat{u}). \tag{2.1}$$

Based on these concepts, we now define the Besov space.

Definition 2.3.3. *Given $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and η, ψ_k as defined above, the Besov space $B^s_{p,q}$ is a function space such that*

$$B^s_{p,q}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\}$$

with the norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{k=-\infty}^{\infty} \left(2^{sk} \left\| \mathcal{F}^{-1}(\psi_k \hat{u}) \right\|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q}, & 1 \leqslant p < \infty; \\ \sup_{k\ge 1} 2^{sk} \left\| \mathcal{F}^{-1}(\psi_k \hat{u}) \right\|_{L^p(\mathbb{R}^n)}, & p = \infty. \end{cases}$$

Remark 2.3.3. *By setting $P_k = 2^{sk}\mathcal{F}^{-1}(\psi_k \hat{u})$, the Besov norm is equivalent to the mixed norm*

$$\|P_k\|_{\ell^q L^p} = \Big(\sum_{k=-\infty}^{\infty} \|P_k\|^q_{L^p} \Big)^{1/q}.$$

Remark 2.3.4. *Finally, let us summarize some results about the Besov space.*

(i). The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space.

(ii). For $s_1 \geqslant s_2$, we have the embedding $B^{s_1}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^n)$.

(iii). For $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, we have the embedding $B^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^n)$.

(iv). For $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, if $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, we have the embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

(v). The Sobolev space $H^s(\mathbb{R}^n)$ is a special case of the Besov space ny definition. More precisely, $B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

2.4 Some harmonic analysis tools

We shall review some techniques of harmonic analysis in this section. Most of the results in this section can be found in Stein[27]. Let us begin with the Riez-Thorin interpolation theorem for L^p spaces.

Theorem 2.4.1 (Riez-Thorin interpolation theorem)**.** *Given $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, $\theta \in (0,1)$ such that*

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p_\theta} \quad \text{and} \quad \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q_\theta}.$$

If $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ is a linear operator with the boundedness conditions

$$||Tf||_{L^{q_0}} \leqslant M_0 ||f||_{L^{p_0}},$$

and

$$||Tf||_{L^{q_1}} \leqslant M_1 ||f||_{L^{p_1}},$$

for some constants $M_0, M_1 \in \mathbb{R}$. Then T is bounded from L^{q_θ} to L^{p_θ} with

$$||Tf||_{L^{q_\theta}} \leqslant {M_0}^{1-\theta} {M_1}^{\theta} ||f||_{L^{q_\theta}}.$$

Proof. See [11] for a proof.

□

Remark 2.4.1. *It is well known that some L^p spaces have simpler structures than the others. For instance, the L^2 space is a Hilbert space while it is not true for other L^p spaces. In fact, we shall see that it is easier to prove the boundedness of the Schrödinger operator in L^2, and L^∞. Then we can use the Riez-Thorin theorem to interpolate the boundedness to other L^p spaces. We shall make frequent use of this theorem in the thesis.*

Finally, let us introduce the Hardy-Littlewood-Sobolev theorem. It is an important theorem in harmonic analysis as it gives a criteria for the boundedness of a convolution of the form $|x|^{-\alpha} * u$. We shall make use of the following theorem in the proof of the Strichartz estimate.

Theorem 2.4.2 (Hardy-Littlewood-Sobolev theorem of fractional integration)**.** *Given* $1 < p < q < \infty$ *and* $0 < \alpha < n$ *such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{n-\alpha}{n},$$

then for any $u \in L^p(\mathbb{R}^n)$*, we have*

$$\left\| |x|^{-\alpha} * u \right\|_{L_x^q(\mathbb{R}^n)} \lesssim \|u\|_{L_x^p(\mathbb{R}^n)} .$$

Proof. Interested readers may consult [27] for a proof. □

Chapter 3

Strichartz Estimates for Schödinger Equation

3.1 Fundamental solution of Schrödinger equation

We consider Schrödinger equation of the following form:

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = F(u), \\ u(0,x) = \varphi(x), \end{cases} \tag{3.1}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Here φ is the intial data and F is the nonlinearity of the equation. Some examples of the nonlinear term F include the power-type nonlinearity $\lambda|u|^{p-1}u$, exponential-type nonlinearity $\lambda(e^{\rho|u|^2} - 1)u$ or other nonlinear functions like $sin(u)$. Intuitively, Schrödinger equation describes the time evolution of the wavefunction $u(x,t)$ which is the probability amplitude of the momentum of a particle. The term $|u(x,t)|^2$ is interpreted as the probability density of the particle locating at point x in time t, thus the quantity

$$M(t) = \int_V |u(t,x)|^2 dx$$

will be the probability of finding the particle in the region V. The Schrödinger equation is classified as a type of dispersive PDE, as it has the characteritics that the *sup* norm of the solution decays as time increases, while some forms of mass or energy associated to the solution conserve. The conservation of mass and energy is an interesting topic and we shall discuss the connections between conservation laws and symmetries via the Noether's theorem in the next chapter. In this chapter, we will derive the fundamental solutions of the Schrödinger equation using the Fourier transform and deduce some useful estimates of the solution.

Let us first consider the homogeneous case (when $F = 0$) of (3.1). It is easy to check that the plane wave $e^{i(x\cdot\xi - t\frac{\xi^2}{2})}$ is a solution of the homogeneous Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$. By the superposition principle of linear equations, we expect that the general solution can be expressed as the superposition of these plane waves. This provides us with a hint to consider using the Fourier transform for constructions of such a solution. Furthermore, we shall seek solutions in the Schwartz space as Schwartz functions possess a number of nice properties that could help us avoid technicalities when obtaining the wellposedness of the equation. Now suppose $u \in C^1_{t,loc}\mathcal{S}_x(\mathbb{R}\times\mathbb{R}^n)$[1] is a classical solution[2] of the homogeneous equation $iu_t + \frac{1}{2}\Delta u = 0$. After performing a Fourier transform on the physical domain, we obtain

$$i\partial_t\widehat{u(t)}(\xi) - \frac{1}{2}\xi^2\widehat{u(t)}(\xi) = 0.$$

For a fixed frequency ξ, we can consider this as an ordinary differential equation with a single variable t. Hence the techniques of ODEs produce the following unique solution in the frequency space

$$\widehat{u(t)}(\xi) = e^{-i\frac{1}{2}\xi^2 t}\widehat{\varphi}(\xi).$$

After taking an inverse Fourier transform, we get the solution

$$u(t,x) = \mathcal{F}^{-1}_{\xi}(e^{-i\frac{1}{2}\xi^2 t}\widehat{\varphi}(\xi)). \tag{3.2}$$

In particular, we may use the notation of Fourier multiplier $e^{\frac{it\Delta}{2}} := \mathcal{F}^{-1}(e^{-i\frac{1}{2}\xi^2 t})\mathcal{F}$ to simplify the representation of the solution. The Fourier multiplier $e^{\frac{it\Delta}{2}}$ is often called the **Schrödinger operator**. By (3.2), we can represent the solution concisely as $e^{\frac{it\Delta}{2}}\varphi(x)$ and some properties of the solutions are summarized below.

Theorem 3.1.1 ([29]). *Let $e^{\frac{it\Delta}{2}}$ be the Schrödinger operator, then*

(1). $e^{\frac{it\Delta}{2}}$ maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

(2). $e^{\frac{it\Delta}{2}}$ is a unitrary operator on $L^2(\mathbb{R}^n)$.

(3). for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$, $e^{\frac{it\Delta}{2}}\varphi = \varphi * K_t$, where $K_t(x) = (4\pi it)^{-\frac{n}{2}}e^{i\frac{|x|^2}{4t}}$.

Proof. As the derivative of the multiplier $e^{-i\frac{1}{2}\xi^2 t}$ is of at most polynomial growth, the definition of Schwartz space implies that $e^{-i\frac{1}{2}\xi^2 t}$ is a Schwartz function. Since $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ is an isomorphism and the Schwartz space is closed under multiplication, we see that $e^{\frac{it\Delta}{2}}\varphi = \mathcal{F}^{-1}(e^{-i\frac{1}{2}\xi^2 t})\mathcal{F}\varphi$ is a Schwartz function.

For (2), we apply the Plancherel's theorem to get

$$\begin{aligned}
||e^{\frac{it\Delta}{2}}\varphi||_{L^2_x(\mathbb{R}^n)} &= ||\mathcal{F}^{-1}(e^{-i\frac{1}{2}\xi^2 t}\hat{\varphi})||_{L^2_x(\mathbb{R}^n)} \\
&= (2\pi)^{-\frac{n}{2}}||e^{-i\frac{1}{2}\xi^2 t}\hat{\varphi}||_{L^2_\xi(\mathbb{R}^n)} \\
&= (2\pi)^{\frac{n}{2}}(2\pi)^{-\frac{n}{2}}||\varphi||_{L^2_x(\mathbb{R}^n)} \\
&= ||\varphi||_{L^2_x(\mathbb{R}^n)}.
\end{aligned}$$

Thus $e^{\frac{it\Delta}{2}}$ is indeed a bounded linear operator between $L^2(\mathbb{R}^n)$ that preserves the norm.

For (3), we can apply the identity of Fourier transform of the general Gaussian function

$$\mathcal{F}(u^{\frac{n}{2}}e^{-\frac{\pi|x|^2}{u}}) = e^{-u\pi|\xi|^2}$$

and set $u = 4\pi it$ to get $\mathcal{F}(K_t) = (e^{-i\frac{1}{2}\xi^2 t})$. Now by the definition of Fourier multiplier, we have $e^{\frac{it\Delta}{2}}\varphi = (\mathcal{F}^{-1}(e^{-i\frac{1}{2}\xi^2 t})) * \varphi = K_t * \varphi$.

□

[1]In particular, this means $u(t,x)$ is a Schwartz function on the physical domain $\mathbb{R}^n$ and is locally continuously differentiable on the time domain $\mathbb{R}$.

[2]The classical solution of a PDE means that it satisfies the equation pointwise everywhere.

So far we have shown that the linear part of the Schrödinger equation has the solution $e^{\frac{it\Delta}{2}}\varphi$. Our next goal is to obtain the solution for the inhomogeneous Schrödinger equation, that is, when the nonlinear term $F \neq 0$ in (3.1). This uses the Duhamel's principle below.

Theorem 3.1.2 (Duhamel's principle [32])**.** *Given a finite-dimensional vector space $\mathcal{D}$, a time interval $I \subset \mathbb{R}$, and a linear operator $L : \mathcal{D} \to \mathcal{D}$, we have*

$$\partial_t u(t) - Lu(t) = f(t),$$

for all $t \in I$, if and only if

$$u(t) = e^{(t-t_0)L}u(t_0) + \int_{t_0}^{t} e^{(t-s)L} f(s)ds,$$

for all $t \in I$. Here we assume $t_0 \in I$, $u \in C^1(I \to \mathcal{D})$ and $f \in C^0(I \to \mathcal{D})$.

Since the Laplacian operator is linear, a direct application of the Duhamel's principle yields the solution of the inhomogeneous Schrödinger equation (3.1). The solution of this inhomogeneous equation is

$$u(t,x) = e^{\frac{it\Delta}{2}}\varphi(x) - i\int_0^t e^{\frac{i(t-s)\Delta}{2}} F(s,x)\, ds. \tag{3.3}$$

3.2 An introduction to Strichartz estimate

Now we have some nice properties of the Schrödinger operator, our next goal is to obtain some estimates of the solution based on these properties. The most direct one is the *energy estimate*

$$||e^{\frac{it\Delta}{2}}\varphi||_{L^2_x(\mathbb{R}^n)} \lesssim ||\varphi||_{L^2_x(\mathbb{R}^n)}. \tag{3.4}$$

This follows easily from the unitrary property of the Schrödinger operator in Theorem 3.1.1. The energy estimate is also called the L^2 conservation law. The physical interpretation is that the L^2 mass of the solution is conserved over time, but will disperse over an increasingly larger region as time evolves. From the Young's convolution inequality, we can also show the *dispersive estimate*

$$\begin{aligned} ||e^{\frac{it\Delta}{2}}\varphi||_{L^\infty_x(\mathbb{R}^n)} &= ||\varphi * K_t||_{L^\infty_x(\mathbb{R}^n)} \\ &\lesssim ||\varphi||_{L^1_x(\mathbb{R}^n)}||K_t||_{L^\infty_x(\mathbb{R}^n)} \\ &= (4\pi|t|)^{-\frac{n}{2}}||\varphi||_{L^1_x(\mathbb{R}^n)}. \end{aligned} \tag{3.5}$$

This estimate implies that if the initial data φ is integrable in space, then the sup norm of the solution $e^{\frac{it\Delta}{2}}\varphi$ would have decay rate similar to that of the function $t^{-\frac{n}{2}}$. With the energy estimate and the dispersive estimate, we may apply the Riez-Thorin interpolation theorem to obtain the L^p *estimate*

$$||e^{\frac{it\Delta}{2}}\varphi||_{L^p_x(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{L^{p'}_x(\mathbb{R}^n)}, \tag{3.6}$$

for $2 \leqslant p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 3.2.1. *In particular, we see that the energy estimate and dispersive estimate are special cases of the L^p estimate. In addition, we remark that the L^p estimate is untrue* *if we replace the domain $\mathbb{R}^n$ by a general domain $\Omega \subset \mathbb{R}^n$. The reason is that we have the embedding $L^2(\Omega) \hookrightarrow L^{p'}(\Omega)$ for every $p > 2$ in this situation. Thus if we define $\theta(t) := e^{\frac{it\Delta}{2}} \circ e^{\frac{-it\Delta}{2}}$ to be the composition of the Schrödinger operator and its adjoint, then $\theta(t) : L^2(\Omega) \to L^2(\Omega) \hookrightarrow L^{p'}(\Omega) \to L^p(\Omega)$. This is contradictory as it implies $L^2(\Omega) \hookrightarrow L^p(\Omega)$.*

The L^p estimate can be generalized to various spaces involving derivatives. Indeed, using the fact that Schrödinger operator $e^{\frac{it\Delta}{2}}$ commutes with other Fourier multipliers, we can generalize the results to Sobolev space and Besov space.

Theorem 3.2.1 ([5])**.** *Let $e^{\frac{it\Delta}{2}}$ be the Schrödinger operator defined above, then for $t \neq 0$ we have*

(1).
$$||e^{\frac{it\Delta}{2}}\varphi||_{W^{m,r}(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{W^{m,r'}(\mathbb{R}^n)},$$
for all $\varphi \in S'(\mathbb{R}^n)$, $m \in \mathbb{Z}^+ \cup \{0\}$ and $2 \leqslant r \leqslant \infty$

(2).
$$||e^{\frac{it\Delta}{2}}\varphi||_{W^{s,r}(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{W^{s,r'}(\mathbb{R}^n)},$$
for all $\varphi \in S'(\mathbb{R}^n)$, $s \in \mathbb{R}$ and $2 \leqslant r \leqslant \infty$

(3).
$$||e^{\frac{it\Delta}{2}}\varphi||_{B^s_{p,q}(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{B^s_{p',q}(\mathbb{R}^n)},$$
for all $\varphi \in S'(\mathbb{R}^n)$, $s \in \mathbb{R}$, $2 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$.

Remark 3.2.2. *For* (2)*, the same estimate holds if we replace the inhomogeneous Sobolev norm $W^{s,r}$ by the homogeneous Sobolev norm $\dot{W}^{s,r}$. Similarly, the estimate in* (3) *still holds if we replace the inhomogeneous Besov norm $B^s_{p,q}$ and $B^s_{p',q}$ by the homogeneous Besov norm $\dot{B}^s_{p,q}$ and $\dot{B}^s_{p',q}$.*

Proof. For (1), we use the fact $\partial^\alpha e^{\frac{it\Delta}{2}} = e^{\frac{it\Delta}{2}}\partial^\alpha$ for any differential operator ∂^α with order $|\alpha| = m$. Thus, if we replace φ by $\partial^\alpha\varphi$ in the L^p estimate (3.6), we obtain

$$\begin{aligned} ||e^{\frac{it\Delta}{2}}\varphi||_{W^{m,r}} &\leqslant |t|^{-n(\frac{1}{p}-\frac{1}{2})}\sum_{|\alpha|=0}^{m}||\partial^\alpha\varphi||_{L^{r'}} \\ &\leqslant |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{W^{m,r'}} \end{aligned}$$

For (2), use the fact that $e^{\frac{it\Delta}{2}}\langle\nabla\rangle^s = \langle\nabla\rangle^s e^{\frac{it\Delta}{2}}$ for any $s \in \mathbb{R}$, thus

$$\begin{aligned} ||e^{\frac{it\Delta}{2}}\varphi||_{W^{s,r}} &= ||\langle\nabla\rangle^s e^{\frac{it\Delta}{2}}\varphi||_{L^r} \\ &= ||e^{\frac{it\Delta}{2}}\langle\nabla\rangle^s\varphi||_{L^r} \\ &\lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\langle\nabla\rangle^s\varphi||_{L^{r'}} \\ &\lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\varphi||_{W^{s,r'}(\mathbb{R}^n)}. \end{aligned}$$

To obtain the Besov space version of estimate in (3), let $u = e^{\frac{it\Delta}{2}}\varphi$, it follows from the definition of Schrödinger operator that $\widehat{u(t)}(\xi) = e^{-i\frac{1}{2}\xi^2 t}\widehat{\varphi}(\xi)$. Thus given any $\omega \in \mathcal{S}_x(\mathbb{R}^n)$,

$$
\begin{aligned}
\mathcal{F}^{-1}(\omega\widehat{u}) &= \mathcal{F}^{-1}(\omega e^{-i\frac{1}{2}\xi^2 t}\widehat{\varphi}) \\
&= \mathcal{F}^{-1}(e^{-i\frac{1}{2}\xi^2 t}\mathcal{F}\mathcal{F}^{-1}(\omega\widehat{\varphi})) \\
&= e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}(\omega\widehat{\varphi})
\end{aligned}
$$

Therefore,

$$
||\mathcal{F}^{-1}(\omega\widehat{u})||_{L_x^p} \lesssim |t|^{-n(\frac{1}{p}-\frac{1}{2})}||\mathcal{F}^{-1}(\omega\widehat{\varphi})||_{L_x^{p'}}, \tag{3.7}
$$

for any $2 \leqslant p \leqslant \infty$ and the result follows from the definition of Besov space. □

Now we have obtained some fixed-time estimates for the Schrödinger equation. In particular, the dispersive estimate implies that if the initial data φ is assumed to be in L^1, then the solution has a power-type decay for large time. A natural question to ask is whether one can still obtain decay of the solution for large time if the initial data is merely assumed to be in L^2 or H^s. Fortunately, we have the *Strichartz estimate*, for handling this type of data. Before introducing the Strichartz estimate, we need the concept of *Schrödinger-admissible pair.*

Definition 3.2.1 (Schrödinger-admissible pairs [35])**.** *For spatial dimension $n \geq 1$, the exponent pair (q, r) is Schrödinger-admissible if $2 \leqslant q, r \leqslant \infty$, $(q, r, n) \neq (2, \infty, 2)$ and*

$$
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{3.8}
$$

*In particular, the point $P = (2, \frac{2n}{n-2})$ is called the **endpoint**.*

Remark 3.2.3. *As suggested by its name, if (q, r) is Schrödinger-admissible then the Strichartz estimate for the Schrödinger operator is applicable, as we shall see in the next theorem. Although the endpoint is still Schrödinger-admissible, the Strichartz estimate at the endpoint is much harder to obtain. In fact, it was 20 years after the first discovery of the Strichartz estimate that Keel and Tao [20] proved the endpoint case in 1997.*

Theorem 3.2.2 (Strichartz estimate [30] [36])**.** *Let (q, r) and $(\tilde{q}, \tilde{r})$ be any Schrödinger-admissible pairs, then for the Schrödinger equation of the following form*

$$
\begin{cases} iu_t + \frac{1}{2}\Delta u = F(u), \\ u(0, x) = \varphi(x), \end{cases} \tag{3.9}
$$

we have the homogeneous Strichartz estimate

$$
||e^{\frac{it\Delta}{2}}\varphi||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||\varphi||_{L_x^2(\mathbb{R}^n)}, \tag{3.10}
$$

the dual homogeneous Strichartz estimate

$$
\left\| \int_{\mathbb{R}} e^{-\frac{is\Delta}{2}} F(s, x)\, ds \right\|_{L_x^2(\mathbb{R}^n)} \lesssim ||F||_{L_t^{q'} L_x^{r'}(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.11}
$$

and the inhomogenous Strichartz estimate

$$
\left\| \int_{s<t} e^{\frac{i(t-s)\Delta}{2}} F(s, x)\, ds \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.12}
$$

where $s, t \in \mathbb{R}$, $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

Remark 3.2.4. *The Strichartz estimate was first established by Strichartz [30] in 1977 as a Fourier restriction theorem. It was then generalized by Ginibre and Velo [13] where they also gave an elementary proof of the homogeneous Strichartz estimate. The inhomogeneous estimate was proved by Yajima [36] and Cazenave and Weissler [6] in 1988. In contrast, the endpoint case of the Strichartz estimate is much harder to prove and it was not until 1997 that Keel and Tao [20] resolved the endpoint case using the idea of atomic decomposition of the L^p functions. We shall present the proof of the nonendpoint Strichartz estimate in the next section.*

3.3 Nonendpoint Strichartz estimate

In this section, we shall prove the nonendpoint Strichartz estimate. In particular, this means the Theorem 3.2.2 for Schrödinger-admissible pairs (q, r) except for the endpoint $P = (2, \frac{2n}{n-2})$. Before presenting the proof, let us first formulate the Strichartz estimate in a more abstract setting. More precisely, we shall generalize the Schrödinger-admissible pair and Schrödinger operator to the *sharp σ-admissible pair* and the semi-group U.

Definition 3.3.1 (sharp σ-admissible pair [12])**.** *The exponent pair (q, r) is said to be sharp σ-admissible if $2 \leqslant q, r \leqslant \infty$, $(q, r, n) \neq (2, \infty, 2)$ and*

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}. \tag{3.13}$$

*In particular, the point $P = (2, \frac{2\sigma}{\sigma-1})$, $\sigma > 1$, is called the **endpoint**. The endpoint is sharp σ-admissible.*

Remark 3.3.1. *Compared to the definition of Schrödinger-admissible pair, this definition has a wider range of applications. It is worth noting that the Strichartz estimate is not restricted for Schrödinger equation. In fact, there are a great number of Strichartz-type estimates which can be applied to other dispersive equations such as the wave equation. Thus it is necessary to adopt a more general admission criteria here. In particular, (q, r) is Schrödinger-admissible when $\sigma = \frac{n}{2}$ and wave-admissible when $\sigma = \frac{n-1}{2}$.*

Now let (X, dx) be a measure space and H be a Hilbert space, we define the semi-group $U(t) : H \to L^2(X)$ for each parameter $t \in \mathbb{R}$ and require that $U(t)$ satisfies the energy estimate

$$||U(t)f||_{L^2(X)} \lesssim ||f||_H, \tag{3.14}$$

for all $t \in \mathbb{R}$ and $f \in H$; as well as the following dispersive estimate

$$||U(t)U^*(s)f||_{L^\infty(X)} \lesssim |t-s|^{-\sigma}||f||_{L^1(X)}, \tag{3.15}$$

for all $t \neq s$, $g \in L^1(X)$ and some $\sigma > 0$. Here $U(t)$ can be considered as a generalization of the Schrödinger operator $e^{\frac{it\Delta}{2}}$ as they both satisfy the energy and dispersive estimate. Now our goal is to prove the Strichartz estimate for $U(t)$. It turns out that its proof relies on the TT^* argument.

Theorem 3.3.1 (TT^* argument [14])**.** *Let B be a Banach space and H be a Hilbert space, then for any linear operator $T \in \mathcal{L}(B, H)$ with the adjoint $T^* \in \mathcal{L}(H, B^*)$, the following three statements are equivalent.*

(1). There exists a constant $C > 0$ such that for all $F \in B$

$$||TF||_H \leqslant C||F||_B;$$

(2). For all $f \in H$, $T^*f \in B^*$ and we have the same constant C as above such that

$$||T^*f||_{B^*} \leqslant C||f||_H;$$

(3). For all $F \in B$, $T^*TF \in B^*$ and we have the same constant C as above such that

$$||T^*TF||_{B^*} \leqslant C^2||F||_B$$

Proof. See [33]. □

Remark 3.3.2. *We remark here that the operator T^*T is the composition $T^* \circ T$ mapping from B to the dual space of B. In short, the above theorem implies that if a linear operator T is bounded, then its adjoint operator T^* and the operator T^*T are also bounded. In fact, they have the same operator norm.*

Now let us formally prove the Strichartz estimate for $U(t)$.

Theorem 3.3.2 (Strichartz estimate for semi-group $U(t)$ [20])**.** *Given any $t \in \mathbb{R}$ and let $U(t)$ be the semi-group defined above, then for any sharp σ-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$, the semi-group $U(t)$ obeys the following estimates*

$$||U(t)f||_{L_t^q L_x^r} \lesssim ||f||_H \tag{3.16}$$

$$\left\|\int_{\mathbb{R}} U^*(s)F(s,x)\,ds\right\|_H \lesssim ||F||_{L_t^{q'} L_x^{r'}} \tag{3.17}$$

$$\left\|\int_{s<t} U(t)U^*(s)F(s,x)\,ds\right\|_{L_t^q L_x^r} \lesssim ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{3.18}$$

Proof. We shall only show the nonendpoint case of (3.16) and (3.17) here. The endpoint case of this theorem involves much deeper techniques and we shall leave it to the next section. Interested readers may see [20] for a detailed discussion of this theorem. Firstly, these three estimates are connected by the TT^* argument. Indeed, if we let $T : L_t^{q'} L_x^{r'} \to H$ be the operator such that $TF(x) := \int_{\mathbb{R}} U^*(s)F(s,x)\,ds$ then its adjoint is the operator $T^* : H \to L_t^q L_x^r$ such that $T^*f(x) := U(t)f$. By the semi-group properties of $U(t)$, the operator $T^*T : L_t^{q'} L_x^{r'} \to L_t^q L_x^r$ can be defined as $T^*TF(x) := \int_{\mathbb{R}} U(t)U^*(s)F(s,x)\,ds$. By the TT^* argument, these three operators have the same operator norm $||T|| = ||T^*|| = ||T^*T||$. Thus, (3.16) and (3.17) are equivalent and it suffices to prove (3.17). By the TT^* argument again, (3.17) is in turn equivalent to the bilinear estimate

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H\,ds\,dt\right| \lesssim ||F||_{L_t^{q'} L_x^{r'}} ||G||_{L_t^{q'} L_x^{r'}}. \tag{3.19}$$

Here $\langle\cdot,\cdot\rangle_H : H \times H \to \mathbb{C}$ is the inner product in Hilbert space H and $U^*(s) : L^2(X) \to H$ is the adjoint of $U(s)$, for each parameter $s \in \mathbb{R}$. Let us show the equivalency stated above. Firstly, if we assume (3.17) holds, then the TT^* argument implies the boundedness of the operator T^*T, thus

$$\left\| \int_{\mathbb{R}} U(t)U^*(s)F(s,x)\,ds \right\|_{L_t^q L_x^r} \lesssim ||F||_{L_t^{q'} L_x^{r'}}. \tag{3.20}$$

By duality and (3.20), we get the bilinear estimate

$$\begin{aligned}
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H \,ds\,dt \right| &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U(t)U^*(s)F(s,x), G(t,x)\rangle \,ds\,dt \right| \\
&= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_X U(t)U^*(s)F(s,x)\overline{G(t,x)}\,dx\,ds\,dt \right| \\
&= \left| \int_{\mathbb{R}} \int_X (\int_{\mathbb{R}} U(t)U^*(s)F(s,x)\,ds)\overline{G(t,x)}\,dx\,dt \right| \\
&\leqslant \left\| \int_{\mathbb{R}} U(t)U^*(s)F(s,x)\,ds \right\|_{L_t^q L_x^r} ||G(t,x)||_{L_t^{q'} L_x^{r'}} \\
&\lesssim ||F||_{L_t^{q'} L_x^{r'}} ||G||_{L_t^{q'} L_x^{r'}}.
\end{aligned}$$

Now we prove the bilinear estimate (3.19). By symmetry it suffices to restrict our attention to the retarded version of (3.19) below,

$$|T(F,G)| \lesssim ||F||_{L_t^{q'} L_x^{r'}} ||G||_{L_t^{q'} L_x^{r'}}, \tag{3.21}$$

where $T(F,G)$ is the bilinear form

$$T(F,G) = \left| \int \int_{s<t} \langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H \,ds\,dt \right|. \tag{3.22}$$

Applying a real interpolation between the bilinear form of the energy estimate (3.14)

$$|\langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H| \lesssim ||F||_{L_x^2} ||G||_{L_x^2}, \tag{3.23}$$

and the bilinear form of the dispersive estimate (3.15)

$$|\langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H| \lesssim |t-s|^{-\sigma} ||F||_{L_x^1} ||G||_{L_x^1}, \tag{3.24}$$

we obtain

$$|\langle U^*(s)F(s,x), U^*(t)G(t,x)\rangle_H| \lesssim |t-s|^{1-\beta(r,r)} ||F||_{L_x^{r'}} ||G||_{L_x^{r'}}, \tag{3.25}$$

where $\beta(r,\tilde{r}) = \sigma - 1 - \frac{\sigma}{r} - \frac{\sigma}{\tilde{r}}$. For sharp σ-admissible pairs, we have $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$. Now (3.21) follows from (3.25) and the Hardy-littlewood-Sobolev inequality when $q > q'$; that is, when (q,r) is not the endpoint,

$$\begin{aligned}
|T(F,G)| &\lesssim \int \int_{s<t} |t-s|^{1-\beta(r,r)} ||F||_{L_x^{r'}} ||G||_{L_x^{r'}} \,ds\,dt \\
&= \int ||G||_{L_x^{r'}} (\int_{s<t} |t-s|^{1-\beta(r,r)} ||F||_{L_x^{r'}} \,ds)\,dt \\
&\leqslant ||G||_{L_t^{q'} L_x^{r'}} \left\| |t-s|^{1-\beta(r,r)} * ||F(s)||_{L_x^{r'}} \right\|_{L_t^q} \\
&\lesssim ||F||_{L_t^{q'} L_x^{r'}} ||G||_{L_t^{q'} L_x^{r'}}
\end{aligned} \tag{3.26}$$

This concludes the proof of (3.16) and (3.17) when (q,r) is not the endpoint.

□

There are several advantages to formulating the Strichartz estimate at this level of generality. First, it allows us to prove the Strichariz estimate of the Schrödinger equation and the wave equation in a unified manner. Second, we have the following scalings to the Strichartz estimate, which is only apparent in this abstract setting. More presicely, the Strichartz estimates (3.16), (3.17) and (3.18) are invariant under the scaling $U(t) \leftarrow U(t/\lambda)$, $U^*(s) \leftarrow U^*(s/\lambda)$, $dx \leftarrow \lambda^\sigma dx$, $\langle f, g\rangle_H \leftarrow \lambda^\sigma \langle f, g\rangle_H$, for $\lambda \in \mathbb{R}$. This is an important scaling to obtain the endpoint Strichartz estimate.

3.4 Endpoint Strichartz estimate

In this section, we address the endpoint case of the Strichartz estimate. In the previous section, we have proved the nonendpoint case of (3.16) and (3.17) using a one-parameter family of estimate (3.25). However, this estimate is not sufficient to prove the endpoint result, that is, when $(q, r) = P = (2, \frac{2\sigma}{\sigma-1})$, $\sigma > 1$. The proof of the endpoint result is based on the following atomic decomposition of L^p functions.

Lemma 3.4.1. *For $0 < p < \infty$, any $f \in L^p$ can be decomposed as*

$$f = \sum_{k=-\infty}^{\infty} C_k \chi_k,$$

where each χ_k is a characteristic function supported on a set of measure of $\mathcal{O}(2^k)$ and $|\chi_k| \lesssim 2^{-k/p}$. The coefficients C_k are non-negative and $||C_k||_{\ell^p} \lesssim ||f||_{L^p}$.

Proof. See [20]. □

By applying lemma 3.4.1 with $p = r'$ to $F(t)$ and $G(s)$ used in the bilinear estimate (3.19), we obtain the following decomposition

$$F(t) = \sum_{k=-\infty}^{\infty} f_k(t)\chi_k(t) \quad , \quad G(s) = \sum_{\tilde{k}=-\infty}^{\infty} g_{\tilde{k}}(s)\overline{\chi}_{\tilde{k}}(s) \tag{3.27}$$

such that for each k, t, the function $\chi_k(t)$ is supported on a set of measure of $\mathcal{O}(2^k)$ and $|\chi_k| \lesssim 2^{-k/r'}$, and similarly for $\overline{\chi}_{\tilde{k}}(s)$. In addition, the coefficients $f_k(t)$ and $g_{\tilde{k}}(s)$ obey the inequalities

$$\left\| \Big(||f_k||_{\ell^{r'}} \Big) \right\|_{L^2_t} \lesssim ||F||_{L^2_t L^{r'}_x} \quad , \quad \left\| \Big(||g_{\tilde{k}}||_{\ell^{r'}} \Big) \right\|_{L^2_t} \lesssim ||G||_{L^2_t L^{r'}_x}. \tag{3.28}$$

Now we are ready to prove the endpoint result following the basic idea of the split-and-conquer technique. In order to show the endpoint case of (3.16) and (3.17), it suffices to show the endpoint case of (3.21). We first perform a dyadic decomposition of the bilinear form $T(F, G)$,

$$T(F, G) = \sum_{j\in\mathbb{Z}} T_j(F, G),$$

where

$$T_j(F, G) = \int \int_{t-2^{j+1} < s \leqslant t-2^j} \langle U^*(s)F(s), U^*(t)G(t)\rangle \, ds \, dt.$$

for each $j \in \mathbb{Z}$. Then it suffices to prove the estimate

$$\sum_j |T_j(F,G)| \lesssim ||F||_{L_t^2 L_x^{r'}} ||G||_{L_t^2 L_x^{r'}}, \tag{3.29}$$

where $r' = \frac{2\sigma}{\sigma+1}$ such that $(2,r) = (2, \frac{2\sigma}{\sigma-1})$ forms a sharp σ-admissible pair. The proof of (3.29) is derived from the following lemma.

Lemma 3.4.2. *Let $r = \frac{2\sigma}{\sigma-1}$, $\sigma > 1$, then the estimate*

$$|T_j(F,G)| \lesssim 2^{-j\beta(a,b)} ||F||_{L_t^2 L_x^{a'}} ||G||_{L_t^2 L_x^{b'}} \tag{3.30}$$

holds for all $j \in \mathbb{Z}$ and all $(\frac{1}{a}, \frac{1}{b})$ in a neighbourhood of $(\frac{1}{r}, \frac{1}{r})$ in $\mathbb{R}^2$.

Proof. (See [20]) The idea of the proof is based on the observation that the estimate (3.30) is invariant under the scaling $U(t) \leftarrow U(t/\lambda)$, $U^*(s) \leftarrow U^*(s/\lambda)$, $dx \leftarrow \lambda^\sigma dx$, $\langle f, g\rangle_H \leftarrow \lambda^\sigma \langle f, g\rangle_H$, for $\lambda \in \mathbb{R}$. Thus it suffices to prove (3.30) for T_0, that is, when s is localized in the dyadic circle $\{s : t-2 < s \leqslant t-1\}$. For other T_j whose support is localized in another dyadic circle, we can always use the above scaling to transform it back to the dyadic circle of T_0. Then the next goal is to show that

$$|T_0(F,G)| \lesssim ||F||_{L_t^2 L_x^{a'}} ||G||_{L_t^2 L_x^{b'}} \tag{3.31}$$

holds for the following three sets of exponents,

(i). $(\frac{1}{a}, \frac{1}{b}) = (0,0)$
(ii). $\frac{1}{r} < \frac{1}{a} \leqslant \frac{1}{2}$, and $\frac{1}{b} = \frac{1}{2}$
(iii). $\frac{1}{r} < \frac{1}{b} \leqslant \frac{1}{2}$, and $\frac{1}{a} = \frac{1}{2}$;

These three estimates imply that $(\frac{1}{r}, \frac{1}{r})$ is in the interior of the convex hull of estimates $(i) - (iii)$. Thus the lemma will follow by interpolations among $(i) - (iii)$.

Here we only prove the estimate (i). By integrating (3.24) in t and s, we obtain

$$\begin{aligned}
|T_0(F,G)| &= \int \int_{t-2<s\leqslant t-1} \langle U^*(s)F(s), U^*(t)G(t)\rangle \, ds \, dt \\
&\lesssim \int ||G||_{L_x^1} \Big(\int_{t-2<s\leqslant t-1} |t-s|^{-\sigma} ||F||_{L_x^1} \, ds \Big) \, dt \\
&\leq ||G||_{L_t^1 L_x^1} \Big\| |t-s|^{-\sigma} * ||F(s)||_{L_x^1} \Big\|_{L_t^1} \\
&\lesssim ||F||_{L_t^1 L_x^1} ||G||_{L_t^1 L_x^1},
\end{aligned}$$

where the last line follows from the Young's convolution inequality.

□

Combining lemma 3.4.2 and lemma 3.4.1, we see that

$$\sum_j |T_j(F,G)| \leqslant \sum_j \sum_k \sum_{\tilde{k}} |T_j(f_k \chi_k, g_{\tilde{k}} \overline{\chi}_{\tilde{k}})|. \tag{3.32}$$

Then by an optimization of the estimate (3.30), we obtain

$$|T_j(f_k \chi_k, g_{\tilde{k}} \overline{\chi}_{\tilde{k}})| \lesssim 2^{-\epsilon(|k-j\sigma|+|\tilde{k}-j\sigma|)} ||f_k||_{L^2} ||g_{\tilde{k}}||_{L^2} \tag{3.33}$$

for some $\epsilon > 0$. Combining (3.32) and (3.33) and summing in j, we obtain

$$\sum_j |T_j(F,G)| \leqslant \sum_k \sum_{\tilde{k}} (1+|k-\tilde{k}|)2^{-\epsilon(|k-\tilde{k}|)} ||f_k||_{L^2} ||g_{\tilde{k}}||_{L^2}. \tag{3.34}$$

Since the quantity $(1+|k-\tilde{k}|)2^{-\epsilon(|k-\tilde{k}|)}$ is absolutely summable, we may apply the Young's inequality and get

$$\begin{aligned} \sum_j |T_j(F,G)| &\lesssim (\sum_k ||f_k||_{L^2}^2)^{1/2} (\sum_{\tilde{k}} ||g_{\tilde{k}}||_{L^2}^2)^{1/2} \\ &= \left\| \Big(||f_k||_{L^2} \Big) \right\|_{\ell^2} \left\| \Big(||g_{\tilde{k}}||_{L^2} \Big) \right\|_{\ell^2}. \end{aligned} \tag{3.35}$$

Interchanging L^2 and ℓ^2 norms and using the inclusion $\ell^{r'} \subset \ell^2$, for $r' < 2$, we obtain

$$\sum_j |T_j(F,G)| \lesssim \left\| \Big(||f_k||_{\ell^{r'}} \Big) \right\|_{L_t^2} \left\| \Big(||g_{\tilde{k}}||_{\ell^{r'}} \Big) \right\|_{L_t^2}.$$

Thus by (3.28),

$$\sum_j |T_j(F,G)| \lesssim ||F||_{L_t^2 L_x^{r'}} ||G||_{L_t^2 L_x^{r'}}.$$

This concludes the proof of the endpoint case of (3.16) and (3.17).

3.5 Some generalizations of the Strichartz estimate

In this section, we shall discuss some generalizations of the Strichartz estimate. The Strichartz estimate can be generalized to Sobolev space by commuting $e^{\frac{it\Delta}{2}}$ with the fractional integration operator $\langle \nabla \rangle^s$ or the fractional differentiation operator $|\nabla|^s$. Indeed, we have the following Sobolev space version of Strichartz estimate

Theorem 3.5.1 ([5]). *For any Schrödinger-admissible pairs (q,r), $(\tilde{q},\tilde{r})$ and $s \in \mathbb{R}$, we have*

$$||e^{\frac{it\Delta}{2}} \varphi||_{L_t^q W_x^{s,r}(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||\varphi||_{H_x^s(\mathbb{R}^n)}, \tag{3.36}$$

and

$$|| \int_{s<t} e^{\frac{i(t-s)\Delta}{2}} F(s,x)\, ds||_{L_t^q W_x^{s,r}(\mathbb{R}\times\mathbb{R}^n)} \lesssim ||F||_{L_t^{\tilde{q}'} W_x^{s,\tilde{r}'}(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.37}$$

for the Schrödinger equation $iu_t + \frac{1}{2}\Delta u = F(u)$ with initial data φ.

Proof. The idea is straightforward: since $\langle \nabla \rangle^s e^{\frac{it\Delta}{2}} = e^{\frac{it\Delta}{2}} \langle \nabla \rangle^s$, we have

$$\begin{aligned} ||\langle \nabla \rangle^s e^{\frac{it\Delta}{2}} \varphi||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} &= ||e^{\frac{it\Delta}{2}} \langle \nabla \rangle^s \varphi||_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \\ &\lesssim ||\langle \nabla \rangle^s \varphi||_{L_x^2(\mathbb{R}^n)} \\ &= ||\varphi||_{H_x^s(\mathbb{R}^n)} \end{aligned}$$

thus the estimate (3.36). The inhomogeneous estimate () follows by commuting $e^{\frac{i(t-s)\Delta}{2}}$ with $\langle \nabla \rangle^s$. $\square$

Compared to the Sobolev space, the generalization to the Besov space is trickier as it involves the Littlewood-Paley decomposition. We shall present the Besov space version of the Strichartz estimate with a description of its proof.

Theorem 3.5.2 ([5])**.** *For Schrödinger-admissible pairs* (q,r) *and* $(\tilde{q}',\tilde{r}')$ *and* $s\in\mathbb{R}$*, the following properties hold*

$$||e^{\frac{it\Delta}{2}}\varphi||_{L_t^qB^s_{r,2}(\mathbb{R}\times\mathbb{R}^n)}\lesssim||\varphi||_{B^s_{2,2}(\mathbb{R}^n)}, \tag{3.38}$$

and

$$||\int_{s<t}e^{\frac{i(t-s)\Delta}{2}}F(s,x)\,ds||_{L_t^qB^s_{r,2}(\mathbb{R}\times\mathbb{R}^n)}\lesssim||F||_{L_t^{\tilde{q}'}B^s_{\tilde{r}',2}(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.39}$$

for the Schrödinger equation $iu_t+\frac{1}{2}\Delta u=F(u)$ *with initial data* φ*.*

Proof. we shall only illustrate the idea of the proof for the estimate (3.38) here. Let $u=e^{\frac{it\Delta}{2}}\varphi$ and recall that the Besov norm for a function u takes the form

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)}=\Big(\sum_{k=-\infty}^{\infty}\big(2^{sk}\left\|\mathcal{F}^{-1}[\psi_k\hat{u}]\right\|_{L^p(\mathbb{R}^n)}\big)^q\Big)^{1/q}$$

when $1\leqslant p<\infty$. By commuting $e^{\frac{it\Delta}{2}}$ with $\mathcal{F}^{-1}[\psi_k\hat{u}]$, we have

$$\mathcal{F}^{-1}[\psi_k\hat{u}]=e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}[\psi_k\hat{\varphi}], \tag{3.40}$$

thus

$$\begin{aligned}||u||_{L_t^qB^s_{r,2}}&=\big(\int||u||^q_{B^s_{r,2}}\,dt\big)^{1/q}\\&=\Big(\int\big(\sum_{k=-\infty}^{\infty}2^{2sk}\left\|\mathcal{F}^{-1}[\psi_k\hat{u}]\right\|^2_{L^r}\big)^{q/2}\,dt\Big)^{1/q}\\&=\Big(\int\big(\sum_{k=-\infty}^{\infty}2^{2sk}\left\|e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}[\psi_k\hat{\varphi}]\right\|^2_{L^r}\big)^{q/2}\,dt\Big)^{1/q}\end{aligned}$$

Letting $A_k(t)=2^{2sk}\left\|e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}[\psi_k\hat{\varphi}]\right\|^2_{L^r}$ and $p=q/2$, this last expression can be written as

$$\begin{aligned}||u||^2_{L_t^qB^s_{r,2}}&=\Big(\int\big(\sum_{k=-\infty}^{\infty}A_k(t)\big)^p\,dt\Big)^{1/p}\\&\lesssim\sum_{k=-\infty}^{\infty}||A_k(t)||_{L^p}\\&=\sum_{k=-\infty}^{\infty}2^{2sk}\left\|e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}[\psi_k\hat{\varphi}]\right\|^2_{L_t^qL_x^r}.\end{aligned}$$

The last expression can be handled by the homogeneous Strichartz estimate (3.10), thus

$$\left\|e^{\frac{it\Delta}{2}}\mathcal{F}^{-1}[\psi_k\hat{\varphi}]\right\|^2_{L_t^qL_x^r}\lesssim\left\|\mathcal{F}^{-1}[\psi_k\hat{\varphi}]\right\|^2_{L_x^2}.$$

Therefore,

$$||u||_{L_t^q B_{r,2}^s} \lesssim \Big(\sum_{k=-\infty}^{\infty} 2^{2sk} \left\|\mathcal{F}^{-1}[\psi_k \hat{\varphi}]\right\|_{L_x^2}^2 \Big)^{1/2}$$
$$\lesssim ||\varphi||_{B_{2,2}^s}$$

□

Remark 1. *The same result still holds for the homogeneous Besov space. More precisely, we may replace $B_{r,2}^s$ by $\dot{B}_{r,2}^s$, $B_{2,2}^s$ by $\dot{B}_{2,2}^s$ and $B_{\tilde{r}',2}^s$ by $\dot{B}_{\tilde{r}',2}^s$. It is worth noting that the Besov version of the Strichartz estimate is useful for studying the nonlinear Schrödinger equation in the Sobolev space with fractional regularity H^s. In particular, we shall present a wellposedness result of the subcritical nonlinear Schrödinger equation in $H^s(\mathbb{R}^n)$ using this type of Strichartz estimate later. Interested readers may refer to Cazenave and Weissler [5], Kato [18] or Pecher [23] for some applications of this theorem.*

To finish this section, we would like to present a more advanced generalization of the Strichartz estimate, the *refined bilinear Strichartz estimate.*

Theorem 3.5.3 ([4])**.** *Let $n \geqslant 2$, $\delta > 0$, $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$, and let $I \subset \mathbb{R}$, then for any $t_0 \in I$ and $u, v \in \mathcal{S}_x(\mathbb{R}^n)$,*

$$||uv||_{L_t^2 L_x^2(I\times\mathbb{R}^n)} \lesssim (||u(t_0)||_{\dot{H}_x^{\alpha_1}} + ||(i\partial_t + \frac{1}{2}\Delta)u||_{L_t^1 \dot{H}_x^{\alpha_1}}) \times (||v(t_0)||_{\dot{H}_x^{\alpha_2}} + ||(i\partial_t + \frac{1}{2}\Delta)v||_{L_t^1 \dot{H}_x^{\alpha_2}}).$$

Proof. (Sketch) Fix the δ and let $u(t) := e^{\frac{it\Delta}{2}}\zeta$, $v(t) := e^{\frac{it\Delta}{2}}\psi$, the idea is to use a more general homogeneous estimate below due to Bourgain [4],

$$||uv||_{L_t^2 L_x^2} \lesssim ||\zeta||_{\dot{H}_x^{\alpha_1}} ||\psi||_{\dot{H}_x^{\alpha_2}} \tag{3.41}$$

for $\alpha_1 + \alpha_2 = \frac{n}{2} - 1$. In particular, this holds for $\alpha_1 = -\frac{1}{2} + \delta$ and $\alpha_2 = \frac{n-1}{2} - \delta$. Let us skip the proof of the homogeneous estimate (3.41) and only apply it to prove the main result of this theorem. For simplicity, let us set $F := (i\partial_t + \frac{1}{2}\Delta)u$ and $G := (i\partial_t + \frac{1}{2}\Delta)v$ and recall that the solution of the nonlinear Schrödinger equation satisfies

$$u = e^{\frac{i(t-t_0)\Delta}{2}} u(t_0) - i \int_{t_0}^t e^{\frac{i(t-s)\Delta}{2}} F(s)\, ds,$$

and

$$v = e^{\frac{i(t-t_0)\Delta}{2}} v(t_0) - i \int_{t_0}^t e^{\frac{i(t-s')\Delta}{2}} G(s')\, ds'.$$

Using the Mincowski's inequality, we obtain

$$
\begin{aligned}
\|uv\|_{L_t^2L_x^2} \lesssim & \left\| e^{\frac{i(t-t_0)\Delta}{2}} u(t_0) e^{\frac{i(t-t_0)\Delta}{2}} v(t_0) \right\|_{L_t^2L_x^2} \\
& + \left\| e^{\frac{i(t-t_0)\Delta}{2}} u(t_0) \int_{t_0}^t e^{\frac{i(t-s')\Delta}{2}} G(s')\, ds' \right\|_{L_t^2L_x^2} \\
& + \left\| e^{\frac{i(t-t_0)\Delta}{2}} v(t_0) \int_{t_0}^t e^{\frac{i(t-s)\Delta}{2}} F(s)\, ds \right\|_{L_t^2L_x^2} \\
& + \left\| \int_{t_0}^t e^{\frac{i(t-s)\Delta}{2}} F(s)\, ds \int_{t_0}^t e^{\frac{i(t-s')\Delta}{2}} G(s')\, ds' \right\|_{L_t^2L_x^2} \\
& = I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

It follows from the homogeneous estimate (3.41) that

$$
I_1 \lesssim \|u(t_0)\|_{\dot{H}_x^{\alpha_1}} \|v(t_0)\|_{\dot{H}_x^{\alpha_2}} \tag{3.42}
$$

It remains to obtain the bounds for I_2, I_3 and I_4. Since I_2 and I_3 are similar, we only consider I_2. By Mincowski's inequality and the homogeneous estimate (3.41), we have

$$
\begin{aligned}
I_2 &\lesssim \int_{\mathbb{R}} \left\| e^{\frac{i(t-t_0)\Delta}{2}} u(t_0) e^{\frac{i(t-s')\Delta}{2}} G(s') \right\|_{L_t^2L_x^2} ds' \\
&\lesssim \|u(t_0)\|_{\dot{H}_x^{\alpha_1}} \|G\|_{L_t^1\dot{H}_x^{\alpha_2}}.
\end{aligned} \tag{3.43}
$$

Finally, the estimate for I_4 can be obtained by Mincowski's inequality and the homogeneous estimate (3.41),

$$
\begin{aligned}
I_4 &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \left\| e^{\frac{i(t-s)\Delta}{2}} F(s) e^{\frac{i(t-s')\Delta}{2}} G(s') \right\|_{L_t^2L_x^2} ds\, ds' \\
&\lesssim \|F\|_{L_t^1\dot{H}_x^{\alpha_1}} \|G\|_{L_t^1\dot{H}_x^{\alpha_2}}.
\end{aligned} \tag{3.44}
$$

Combining (3.42), (3.43) and (3.44), we get the desired result. □

Remark 1. *This estimate was originally proved by Bourgain [4] when dealing with the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. The basic idea of Bourgain's proof involves decomposing the initial data into the low and high frequency parts. The refined bilinear Strichartz estimate was introduced in Bourgain's paper to obtain smoothing of the high frequency data. This estimate is very good at handling the types of data when u is the high frequency and v is the low frequency, as it moves plenty of derivatives from u to v.*

In particular, one can recover the $L_t^4L_x^4$ Strichartz estimate from the bilinear Strichartz estimate when $n = 2$. To do this, suppose that we have the initial data $u(t_0) = v(t_0) = \varphi$, and let $u(t) = v(t) = e^{\frac{it\Delta}{2}}\varphi$ be the solution of the linear Schrödinger equation $iu_t + \frac{1}{2}\Delta u = 0$, then by the refined bilinear Strichartz estimate above,

$$
\|(e^{\frac{it\Delta}{2}}\varphi)^2\|_{L_t^2L_x^2} \lesssim \|\varphi\|_{\dot{H}_x^{\alpha_1}} \times \|\varphi\|_{\dot{H}_x^{\alpha_2}}.
$$

After setting $n = 2$ and $\delta = \frac{1}{2}$, we have $\alpha_1 = \alpha_2 = 0$, thus

$$
\|(e^{\frac{it\Delta}{2}}\varphi)^2\|_{L_t^2L_x^2} \lesssim \|\varphi\|_{L_x^2}^2,
$$

which is equivalent to the homogeneous Strichartz estimate

$$||(e^{\frac{it\Delta}{2}}\varphi)||_{L_t^4 L_x^4} \lesssim ||\varphi||_{L_x^2}.$$

Chapter 4

Mathematical structure of nonlinear Schrödinger equation

In this chapter, we will study the mathematical structure of the nonlinear Schrödinger equation. The nonlinear Schrödinger equation has received much attention from mathematicians and physicists. The equation appears in many areas such as quantum mechanics, quantum field theory (the Hartree-Fock theory), and nonlinear optics. See, for example, C. Sulem and P.L. Sulem [31], Simon [3, 26], Kato [19], Gogny and Lions [16] for some applications of the nonlinear Schrödinger equation in the Hartree-Fock theory. The nonlinear Schrödinger equation is a good model of the nonlinear dispersive PDEs as it possesses simpler structures than other dispersive equations like the wave equation or the Korteweg–de Vries equation. We shall study in this chapter the global structure of the nonlinear Schrödinger equation. More precisely, we ask the questions about how Schrödinger equation can be derived from the Hamiltonian's perspectives, what symmetries and conservation laws can be deduced from the structures of the equation and what connections exist between these properties. In particular, we shall consider the nonlinear Schrödinger equation in the following form

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, \\ \quad u(0,x) = \varphi(x), \end{cases} \tag{4.1}$$

with $p > 1, t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Here $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is the comlplex-valued quantum wavefunction. By rescaling the value of u, it is possible to reduce the problem to the cases $\lambda = -1$ or $\lambda = 1$. These are called the focusing case and defocusing case respectively. The solution behaviours for the focusing and defocusing equations are different. For instance, the defocusing Schrödinger equation tends to have global wellposedness in H^1, while the focusing Schrödinger equation tends to have blow-up phenomenon. The term $|u|^{p-1}u$ represents the nonlinear interactions between the particles, where the exponent $1 \leqslant p < \infty$ is the power of the nonlinearity. The term $iu_t + \Delta u$ describes the dispersion of the particles. The dispersion and nonlinear interactions are competing in the sense that the dispersion tends to describe the spreading of the solution, while the nonlinear interaction tends to describe the amplification of the solution. As a result of the competing relations between the dispersion and nonlinear interactions, we get various types of behaviours for the solutions.

4.1 Hamiltonian formulation

The nonlinear Schrödinger equation can be considered as an infinite dimensional Hamiltonian system. In ODE settings, the theory of Hamiltonian system has been proven to be extremely useful. In PDE settings, they become guiding principles and each problem requires special treatment. In this section, we shall introduce the concepts of Hamiltonian mechanics on a symplectic vector space and derive the Schrödinger equation from its Hamiltonian. In particular, we shall attempt to show that the nonlinear Schrödinger equation $iu_t+\Delta u = \lambda|u|^{p-1}u$ is the Hamiltonian flow of the Hamiltonian $H(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\lambda}{p+1}|u|^{p+1}dx$.

Definition 4.1.1 (symplectic vector space [2])**.** *A symplectic vector space $(\mathcal{D},\omega)$ is a finite-dimensional real vector space equipped with a symplectic form $\omega : \mathcal{D}\times\mathcal{D}\to\mathbb{R}$ which is biliear, antisymmetric and non-degenerate. For each scalar function $H \in C^1_{loc}(\mathcal{D}\to\mathbb{R})$, the symplectic gradient $\nabla_\omega H \in C^0_{loc}(\mathcal{D}\to\mathcal{D})$ satisfies*

$$\omega(\nabla_\omega H, v) = \frac{d}{d\epsilon}H(u+\epsilon v)\bigg|_{\epsilon=0},$$

for any functions $u, v \in \mathcal{D}$.

Remark 4.1.1. *Here it is worth noting that*

$$\frac{d}{d\epsilon}H(u+\epsilon v)\bigg|_{\epsilon=0} = \langle H'(u), v\rangle$$

is the ***Gateaux derivative*** *of H, and $H'(u)$ is called the* ***variational derivative*** *of H.*

Definition 4.1.2 (Poisson bracket [32])**.** *For any two functions $H,\ E \in C^1_{loc}(\mathcal{D})$, the Poisson bracket is defined as*

$$\{H, E\}(u) := \omega(\nabla_\omega H(u), \nabla_\omega E(u)).$$

Moverover, H and E Poisson commutes if $\{H, E\} = 0$.

Definition 4.1.3 (Hamiltonian function [32])**.** *A Hamiltonian function on a symplectic vector space $(\mathcal{D},\omega)$ is any function $H \in C^2_{loc}(\mathcal{D}\to\mathbb{R})$. For each Hamiltonian function H, its associated Hamiltonian flow is the differential equation of the form*

$$\partial_t u(t) = \nabla_\omega H(u(t)). \tag{4.2}$$

Example 4.1.1. *It can be shown that the Hamiltonian equations of motion is the Hamiltonian flow of any Hamiltonian $H \in C^2_{loc}(\mathbb{R}^{2n}\to\mathbb{R})$. Here we are working on the symplectic vector space*

$$\mathbb{R}^{2n} = \{(q_1, ..., q_n, p_1, ..., p_n) : q_1, ..., q_n, p_1, ..., p_n \in \mathbb{R}\}$$

and the symplectic form is

$$\omega((q_1, ..., q_n, p_1, ..., p_n), (\tilde{q}_1, ..., \tilde{q}_n, \tilde{p}_1, ..., \tilde{p}_n) := \sum_{i=1}^{n} \tilde{p}_i q_i - p_i\tilde{q}_i.$$

Notice that the phase space $(\mathcal{D}, \omega)$ is symplectic under this settings, and the Gateaux derivative of H can be computed using the chain rule

$$\begin{aligned}
&\frac{d}{d\epsilon} H(q_1 + \epsilon\tilde{q}_1, ..., q_n + \epsilon\tilde{q}_n, p_1 + \epsilon\tilde{p}_1, ..., p_n + \epsilon\tilde{p}_n)\Big|_{\epsilon=0} \\
&= \frac{\partial H}{\partial q_1}\tilde{q}_1 + ... + \frac{\partial H}{\partial q_n}\tilde{q}_n + \frac{\partial H}{\partial p_1}\tilde{p}_1 + ... + \frac{\partial H}{\partial p_n}\tilde{p}_n \\
&= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\tilde{p}_i + \frac{\partial H}{\partial q_i}\tilde{q}_i.
\end{aligned}$$

Thus one may check that $\nabla_\omega H(q(t), p(t)) = (\frac{\partial H}{\partial p_1}, ..., \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, ..., -\frac{\partial H}{\partial q_n})$, and the Hamiltonian flow of H is given by the Hamilton's equations of motion,

$$\partial_t q_j(t) = \frac{\partial H}{\partial p_j}(q(t), p(t)); \qquad \partial_t p_j(t) = -\frac{\partial H}{\partial q_j}(q(t), p(t)).$$

With these definitions and examples in mind, we now have a basic understanding of the Hamiltonian functions on a symplectic vector space. Our next goal is to derive the nonlinear Schrödinger equation from its total energy function. Before we do this, we need a lemma about the variational derivatives.

Lemma 4.1.1 ([10])**.** *Suppose that $F :\to \mathbb{R}$ is a functional of the form*

$$F(u(x)) = \int_{\mathbb{R}^n} f(x, u(x), \nabla u(x), \nabla^2 u(x), ..., \nabla^n u(x))\, dx, \tag{4.3}$$

then the variational derivative of F is given by

$$F'(u) = \frac{\partial f}{\partial u} + \sum_{i=1}^{n} (-1)^i \nabla^i \cdot \frac{\partial f}{\partial(\nabla^i u)}. \tag{4.4}$$

Remark 4.1.2. *Here the integrand f is a functional of variables $u(x)$,$\nabla u(x)$,$\nabla^2 u(x)$,..., $\nabla^n u(x)$. Each of them are complex-valued functions and we may treat them as independent variables.*

Now we attempt to show that the nonlinear Schrödinger equation $\partial_t u = i\Delta u - i\lambda|u|^{p-1}u$ is the Hamiltonian flow of the total energy function $H(u) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p+1}|u|^{p+1}\, dx$ on the symplectic space $L_x^2(\mathbb{R}^n \to \mathbb{C})$ equipped with the symplectic form $\omega(g, v) = -\operatorname{Im} \int_{\mathbb{R}^n} g(x)\overline{v(x)}\, dx$. To do this, let us first derive the variational derivative of the total energy function H. Firstly, one may notice $|\nabla u|^2 = \nabla u \overline{\nabla u}$, and $|u|^{p+1}$ is just the product of some combinations of u and $\overline{u}$. Since u, $\overline{u}$, ∇u and $\overline{\nabla u}$ are complex-valued functions, we may treat them as independent variables and write the integrand of H as

$$f(u, \overline{u}, \nabla u, \overline{\nabla u}) = \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{p+1}|u|^{p+1}.$$

Applying the lemma 4.1.1, we obtain

$$\begin{aligned}
H'(u) &= \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \overline{u}} - \nabla \cdot \frac{\partial f}{\partial(\nabla u)} - \nabla \cdot \frac{\partial f}{\partial(\overline{\nabla u})} \\
&= \frac{\lambda}{2}|u|^{p-1}\overline{u} + \frac{\lambda}{2}|u|^{p-1}u - \nabla \cdot (\frac{1}{2}\overline{\nabla u}) - \nabla \cdot (\frac{1}{2}\nabla u) \\
&= \operatorname{Re}(\lambda|u|^{p-1}u - \Delta u).
\end{aligned}$$

Thus the Gateaux derivative of H is

$$\begin{aligned}\frac{d}{d\epsilon}H(u+\epsilon v)\Big|_{\epsilon=0} &= \langle H'(u), v\rangle \\ &= \langle \mathrm{Re}(\lambda|u|^{p-1}u - \Delta u), v\rangle \\ &= \mathrm{Re}\int (\lambda|u|^{p-1}u - \Delta u)\overline{v}\, dx.\end{aligned}$$

Now recall that our symplectic form is $\omega(g,v) = -\mathrm{Im}\int_{\mathbb{R}^n} g\overline{v}\,dx$. Setting $g = i(\Delta u - \lambda|u|^{p-1}u)$, we see that

$$\begin{aligned}\omega(i(\Delta u - \lambda|u|^{p-1}u), v) &= -\mathrm{Im}\int_{\mathbb{R}^n} i(\Delta u - \lambda|u|^{p-1}u)\overline{v}\,dx \\ &= \mathrm{Re}\int_{\mathbb{R}^n} (\lambda|u|^{p-1}u - \Delta u)\overline{v}\,dx.\end{aligned}$$

Hence,

$$\frac{d}{d\epsilon}H(u+\epsilon v)\Big|_{\epsilon=0} = \omega(i(\Delta u - \lambda|u|^{p-1}u), v).$$

By definition 4.1.1, the symplectic gradient $\nabla_\omega H = i(\Delta u - \lambda|u|^{p-1}u)$. Therefore, the Hamiltonian flow $\partial_t u(t) = \nabla_\omega H(u(t))$ is exactly the nonlinear Schrödinger equation

$$\partial_t u = i\Delta u - i\lambda|u|^{p-1}u.$$

This concludes the Hamiltonian formulation of the Schrödinger equation.

4.2 Some symmetries of the Schrödinger equation

One of the remarkable feature of the Schrödinger equation is that its dynamics, while extremely complicated, still possess a great number of stationary scalar quantities throughout its evolution. These stationary scalar quantities are often linked to the *symmetries* of the Schrödinger equation. Usually, symmetries of an equation are groups of transformation that take one solution of the equation to another. They are often very useful structures of the equation. They could give us guidance as to what techniques to use in order to establish the wellposedness of the equation. Besides, they could also give us information about what types of conservation laws are available. For a more detailed discussion of the symmetries of the Schrödinger equation, see [21].

For the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda|u|^{p-1}u, \\ u(0,x) = \varphi(x), \end{cases} \tag{4.5}$$

it is quite straightforward to see that the equation has the *phase rotation symmetry*:

$$u(x,t) \to e^{i\theta}u(x,t) \tag{4.6}$$

for any $\theta \in \mathbb{R}$. Besides, by the chain rule, we also have the *spatial translation symmetry*:

$$u(x,t) \to u(x - x_0, t), \tag{4.7}$$

and the *time translation symmetry*:

$$u(x,t) \to u(x, t - t_0). \tag{4.8}$$

These three symmetries have important physical implications, as they are often related to the conservation of mass, total momentum and total energy. The next symmetry we have is the scaling symmetry:

$$u(x,t) \to \mu^{\frac{2}{p-1}} u(\frac{t}{\mu^2}, \frac{x}{\mu}); \quad \varphi(x) \to \mu^{\frac{2}{p-1}} \varphi(\frac{x}{\mu}), \tag{4.9}$$

for any dilation factor $\mu \in \mathbb{R}$. In particular, this suggests that if u solves the Cauchy problem (4.5), then $\mu^{\frac{2}{p-1}} u(\frac{t}{\mu^2}, \frac{x}{\mu})$ also solves the equation with the rescaled initial data $\mu^{\frac{2}{p-1}} \varphi(\frac{x}{\mu})$. This scaling symmetry also helps us define the criticality of the Schrödinger equation, as we will see in the next chapter. Last but not least, we have the Galilean invariance: if u solves the equation in (4.5), then

$$e^{ix\cdot v} e^{it|v|^2/2} u(t, x - vt) \tag{4.10}$$

solves the equation with the new initial data

$$e^{ix\cdot v} \varphi(x).$$

for any velocity vector $v \in \mathbb{R}^n$.

4.3 Noether's theorem

In this section, we shall introduce the Noether's theorem. This theorem was first proved by the German mathematician and physicist Emmy Noether [22] in 1918 before establishing herself as a leading mathematician through her work in abstract algebra. It plays an important role in connecting the symmetries and conservation laws of a Hamiltonian system. More precisely, conservative quantities of a system are often scalar quantities that remain constant throughout the evolution of the system. On the other hand, symmetries of a system often refer to groups of transformation that take one solution of the system to another [32]. In particular, one part of the Noether's theorem states that for each symmetry transformation of a Hamiltonian system, there corresponds a conservation law. This provides us with an advanced viewpoint when dealing with the conservation laws of the nonlinear Schrödinger equation. However, the Noether's theorem for the Hamiltonian system is often formulated using the language of symplectic geometry which is far beyond the scope of this thesis. Thus, we shall only give an elementary version of the Noether's theorem in this section. A discussion of the Noether's theorem in full generality can be found in [1]. Moverover, We shall also informally state the correspondence between various symmetries and conservation laws of the nonlinear Schrödinger equation. A rigorous derivation of some conservation laws is left to the next section.

Firstly, let us recall that the Poisson bracket between two Hamiltonian functions H and E is given by

$$\{H, E\}(u(t)) := \omega(\nabla_\omega H(u(t)), \nabla_\omega E(u(t))), \tag{4.11}$$

where $\nabla_\omega H(u)$ is the symplectic gradient of H with respect to the symplectic form ω. Now let us assume u satisfies the Hamiltonian flow,

$$\partial_t u(t) = \nabla_\omega H(u(t)). \tag{4.12}$$

Substituting the equation (4.12) into (4.11), and use the definition and anti-symmetry properties of the symplectic form, we shall obtain

$$\begin{aligned}
\{H, E\}(u(t)) &= \omega(\nabla_\omega H(u(t)), \nabla_\omega E(u(t))) \\
&= -\omega(\nabla_\omega E(u(t)), \nabla_\omega H(u(t))) \\
&= -\frac{d}{d\epsilon} E(u + \epsilon u_t)\Big|_{\epsilon=0}
\end{aligned}$$

Then by the chain rule, the last term is just

$$-\frac{d}{d\epsilon} E(u + \epsilon u_t)\Big|_{\epsilon=0} = -\frac{d}{dt} E(u(t)).$$

This gives us the relationship

$$\frac{d}{dt} E(u(t)) = -\{H, E\}(u(t)). \tag{4.13}$$

From this, we see that if $\{H, E\}(u(t)) = 0$, then $E(u(t))$ is a constant of motion of the Hamiltonian flow (4.12). Together with the anti-symmetry property of the Poisson bracket, we conclude the Noether's theorem.

Theorem 4.3.1 (Noether's theorem [32])**.** *Let H and E be two Hamiltonian functions on a symplectic phase space $(\mathcal{D}, \omega)$. The following statements are equivalent,*

- *(i). $\{H, E\}(u(t)) = 0$*
- *(ii). The quantity E is conserved by the Hamiltonian flow of H*
- *(iii). The quantity H is conserved by the Hamiltonian flow of E.*

Now the Noether's theorem implies that if we want to check whether a scalar quantity E is conserved by the Hamiltonian ODE of H below

$$\partial_t u(t) = \nabla_\omega H(u(t)), \tag{4.14}$$

then we should check whether this ODE is invariant under the Hamiltonian flow associated with E. For instance, we can easily conclude that any Hamiltonian H is a conserved quantity of its Hamiltonian flow as H always Poisson commutes with itself. As another example, let us consider the phase space $(\mathbb{C}^n, \omega)$ with the symplectic form

$$\omega := \sum_{j=1}^{n} \frac{1}{2} \operatorname{Im}(dz_j \wedge d\overline{z_j}),$$

and assume that the Hamiltonian H is invariant under the phase rotations such that

$$H(e^{i\theta} z_1, ..., e^{i\theta} z_n) = H(z_1, ..., z_n),$$

for all $z_i \in \mathbb{C}$ and $\theta \in \mathbb{R}$. As the phase rotations is actually just the Hamiltonian flow associated with the total charge $E = \sum_{j=1}^{n} |z_j|^2$, we conclude that the total charge is conserved by the Hamiltonian flow of H via the Noether's theorem.

Table 4.1: Some symmetry groups with their corresponding conserved quantities. Through the Noether's theorem, we have one-to-one correspondence between symmetries and conservation laws (Table adapted from [32]).

Symmetry	**Conserved quantity**
Time translation	Total energy/Hamiltonian
Phase rotation	Total mass
Spatial translation	Total momentum
Scaling	Viriel's identity
Galilean transformation	centre of mass

The Noether's theorem provides us with a way to connect the exact Hamiltonian symmetries with the exact conservation laws. However, different types of symmetries are often linked to different conservation laws and not all of these follow from the Noether's theorem. Generally speaking, we often have the correspondence between symmetries and conservation laws summarized in the table above.

Now let us apply the above ideas in the context of the nonlinear Schrödinger equation below,

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, \\ u(0,x) = \varphi(x), \end{cases} \tag{4.15}$$

In Section 4.1, we have derived that the nonlinear Schrödinger equation is nothing more than the Hamiltonian flow of the total energy function

$$H(u) = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 \, dx + \frac{\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx,$$

thus H is a conserved quantity of the nonlinear Schrödinger equation. In addition, this Hamiltonian function H is invariant under the spatial rotation $u(t,x) \to e^{i\theta} u(t,x)$. This shall lead to the mass conservation

$$\partial_t \int_{\mathbb{R}^n} |u|^2 \, dx = 0. \tag{4.16}$$

Finally, the symmetry of the spatial translation $u(t,x) \to u(t, x - x_0)$ shall lead to the momentum conservation

$$\partial_t \operatorname{Im} \int_{R^n} (u \nabla \overline{u}) dx = 0. \tag{4.17}$$

4.4 Conservation laws

So far we have established the connection between symmetry and conservation laws via the Noether's theorem. In particular, we have shown that spatial rotation symmetry is associated with mass conservation while time-translation symmetry is associated with energy conservation. However, it remains unclear how these conservation laws can be derived rigorously. In this subsection, we shall derive these conservation laws rigorously in the context of nonlinear Schrödinger equation.

Recall that the wavefunction $u(x,t)$ is the probability amplitude for the momentum of a particle. The positive quantity $|u(x,t)|^2$ is interpreted as the probability density that the particle is located at x in time t. Then the quantity

$$M(t) = \int_{\mathbb{R}^n} |u(t,x)|^2 \, dx$$

is said to be the total probability, mass or charge in the literature. In addition, The total energy of a system at time t is defined as the sum of the corresponding kinetic energy $K(t)$ and the potential energy $P(t)$. Here the kinetic energy and the potential energy are defined as

$$K(t) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \tag{4.18}$$

and

$$P(t) = \frac{\lambda}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx. \tag{4.19}$$

Thus, the total energy has the form

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 \, dx + \frac{\lambda}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx. \tag{4.20}$$

We observe that the parameter $\lambda = \pm 1$ could give us completely different situations for the energy of the system. If $\lambda = 1$, the conservation of energy could provide us with a uniform bound for the L^2 norm of ∇u and the L^{p+1} norm of u. This *a priori* control in time is essential for proving that a solution exists for all time. However, if $\lambda = -1$, the kinetic energy part and the potential energy part will be competing and if the latter is more powerful than the former we could have negative energy. This is associated with the blow-up phenomena of the equation. Finally, the total momentum is defined as

$$p(t) = \operatorname{Im} \int_{R^n} (u \nabla \overline{u}) dx. \tag{4.21}$$

Now let us derive the conservation laws for these three quantities.

Mass conservation. Starting from the model equation (4.5), one may multiply the equation by $\overline{u}$, integrate over $\mathbb{R}^n$ and take the imaginary part to obtain

$$\operatorname{Re} \int_{\mathbb{R}^n} u_t \overline{u} \, dx = -\operatorname{Im} \int_{\mathbb{R}^n} \Delta u \overline{u} \, dx.$$

As the solution $u \in \mathcal{S}(\mathbb{R}^n)$, the right-hand-side of the equation shall vanish by the Stokes' theorem. In addition, we may apply the identity $\partial_t |u|^2 = 2 \operatorname{Re}(u_t \overline{u})$ to the remaining integral and obtain the *conservation of mass*

$$\partial_t \int_{\mathbb{R}^n} |u|^2 \, dx = 0. \tag{4.22}$$

Therefore, this identity implies the L^2 mass of the solution remains constant over time.

Energy conservation. To obtain the energy function, we multiply the model equation (4.5) by $\overline{u_t}$, integrate over $\mathbb{R}^n$ and take the real part. This gives

$$\operatorname{Re} \int_{\mathbb{R}^n} \Delta u \overline{u_t} \, dx = \operatorname{Re} \int_{\mathbb{R}^n} \lambda |u|^{p-1} u \overline{u_t} \, dx.$$

Applying the Stokes' theorem to the left-hand-side of the equation and noticing that the boundary term vanishes since $u \in \mathcal{S}(\mathbb{R}^n)$, we find that

$$-\operatorname{Re}\int_{\mathbb{R}^n} \nabla u \partial_t(\overline{\nabla u})\, dx = \operatorname{Re}\int_{\mathbb{R}^n} \lambda |u|^{p-1} u \overline{u_t}\, dx.$$

Now, using the identities $\partial_t |\nabla u|^2 = 2\operatorname{Re}(\nabla u \partial_t(\overline{\nabla u}))$ and $\partial_t |u|^2 = 2\operatorname{Re}(u\overline{u_t})$ we obtain

$$-\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx = \int_{\mathbb{R}^n} \lambda |u|^{p-1} |u| |u|_t\, dx.$$

Finally, by the chain rule, we obtain the *conservation of energy*

$$\partial_t \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2\, dx + \frac{\lambda}{p+1}|u|^{p+1}\, dx = 0. \tag{4.23}$$

Momentum conservation. Starting from the model equation (4.5), multiplying both sides by $\nabla\overline{u}$, and taking the imaginary part, we immediately recover the *conservation of momentum*,

$$\partial_t \operatorname{Im}\int_{R^n} (u\nabla\overline{u}) dx = 0. \tag{4.24}$$

In practice, the integral form of the conservation laws is not very easy to manipulate. Fortunately, the differential form of these conservation laws can be derived easily from the equation itself and they are often more accessible than the integral form. Let us introduce the pseudo-stress-energy tensor $T_{\alpha,\beta}$ for $\alpha, \beta = 0, 1, ..., n$, and let $T_{00} = |u|^2$ be the mass density, $T_{0j} = T_{j0} = Im(\bar{u}\partial_{x_j} u)$ be the momentum density and $T_{jk} = \operatorname{Re}(\partial_{x_j} u \overline{\partial_{x_k} u}) - \frac{1}{4}\delta_{jk}\Delta(|u|^2) + \lambda\frac{p-1}{p+1}\delta_{jk}|u|^{p+1}$ be the momentum current. Here we are using the Einstein convention for summing indices, with the Latin letters summing from 1 to n. Thus, for instance, $Im(\bar{u}\partial_{x_j} u)$ means $\sum_{j=1}^{n} Im(\bar{u}\partial_{x_j} u)$. A direct computation using the equation (4.5) gives us the differential form of the conservation laws,

$$\partial_t T_{00} + \partial_{xj} T_{0j} = 0 \tag{4.25}$$

and

$$\partial_{xj} T_{0j} + \partial_{xk} T_{jk} = 0. \tag{4.26}$$

Proof. To obtain (4.25), we may multiply the equation (4.5) by $\overline{u}$ and take the imaginary part to get

$$2\operatorname{Re}(u_t\overline{u}) + \operatorname{Im}(\overline{u}\partial_{x_j}^2 u) = 0.$$

Now we use the chain rule and the property of complex conjugate to obtain the identities $\partial_t |u|^2 = 2\operatorname{Re}(u_t\overline{u})$ and $\partial_{xj}\operatorname{Im}(\overline{u}\partial_{x_j} u) = \operatorname{Im}(\overline{u}\partial_{x_j}^2 u)$. Therefore, the result follows by direct substitution. The proof of (4.26) is similar and we shall omit it here. □

The conservation of mass, energy and momentum are important in the global well-posedness theory of the Schrödinger equation, as they provide us with *a priori* control that allows us to extend the life span of the solution from a local time interval $[-T, T]$ to the entire real line. However, this is insufficient for some cases, especially when the problem is *critical*. We will need other *a priori* controls on the norms of the solutions. We shall introduce one of them in the next section.

4.5 Viriel identity

The discovery of the *Viriel identity* goes back to the 1970's when Glassey [15] used it to show blow-up phenomena for some focusing ($\lambda = -1$) nonlinear Schrödinger equation. Since then, it has become a useful tool for showing that a positive quantity has monotonic behaviours in time. Within, we shall present the Viriel identity for the quantity $\int_{\mathbb{R}^n} |x|^2|u|^2\, dx$.

Theorem 4.5.1 (Viriel identity [32]). *For any solution u of the nonlinear Schrödinger equation $iu_t + \Delta u = \lambda |u|^{p-1}u$, the following identity holds,*

$$\partial_t^2 \int_{\mathbb{R}^n} |x|^2|u|^2\, dx = 4E(u) + \frac{\lambda[n(p-1)-4]}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}\, dx. \tag{4.27}$$

Here $E(u)$ represents the total energy.

Proof. Multiply both sides of (4.25) by $a(x) = |x|^2$ and do an integration by part to obtain

$$\partial_t \int_{\mathbb{R}^n} a(x)|u|^2\, dx = \int_{\mathbb{R}^n} \partial_{x_j} a(x) T_{0j}\, dx.$$

Here the boundary term shall vanish since u is assumed to be Schwartz. If we differentiate this with respect to t, we obtain that

$$\partial_t^2 \int_{\mathbb{R}^n} a(x)|u|^2\, dx = \partial_t \int_{\mathbb{R}^n} \partial_{x_j} a(x) T_{0j}\, dx.$$

Now, by conservation law (4.26), this expression can be modulated as

$$\begin{aligned}
\partial_t^2 \int_{\mathbb{R}^n} a(x)|u|^2\, dx &= \partial_t \int_{\mathbb{R}^n} \partial_{x_j} a(x) T_{0j}\, dx \\
&= -\int_{\mathbb{R}^n} \partial_{x_j} a(x) \partial_{x_k} T_{jk}\, dx \\
&= \int_{\mathbb{R}^n} \partial_{x_j}\partial_{x_k} a(x) T_{jk}\, dx \\
&= \int_{\mathbb{R}^n} \partial_{x_j}\partial_{x_k} a(x)\, \mathrm{Re}(\partial_{x_j} u \overline{\partial_{x_k} u})\, dx - \frac{1}{4}\int_{\mathbb{R}^n} \Delta^2 a(x)|u|^2\, dx \\
&+ \frac{\lambda(p-1)}{p+1} \int_{\mathbb{R}^n} \Delta a(x)|u|^{p+1}\, dx.
\end{aligned}$$

Thus, by substituting $a(x) = |x|^2$ into the last expression and using the fact that $\partial_{x_j}\partial_{x_k}|x|^2 = 2\delta_{jk}$, we obtain the required result. □

Remark 4.5.1. *We see that the convexity of the positive quantity $\int_{\mathbb{R}^n} |x|^2|u|^2\, dx$ is controlled by the energy $E(u)$ and parameter λ. If $\lambda = -1$, and if one starts with $E < 0$, the function $g(t) = \partial_t^2 \int_{\mathbb{R}^n} |x|^2|u|^2\, dx$ shall be concave down in time, thus there exists a point in time such that the function $g(t)$ ceases to exist. This can be used to show the existence of a blow-up time for the solution of certain focusing equation.*

Chapter 5

Wellposedness theory

This chapter is devoted to the local and global wellposedness theory of the nonlinear schrödinger equation. The local theory regards the existence of the solution on some small neighborhood of $t = 0$. The global theory asks whether the solution exists for long time and if it does, what is the asymptotic behavior of the solution as $t \to \pm\infty$. Roughly speaking, a Cauchy problem is globally wellposed if the solution exists for all time, is unique, and depends continuously on the initial data. Furthermore, whether the global wellposedness is available often depends on the criticality of the nonlinear power p. We shall first discuss the criticality of the equation and give a precise definition for the wellposedness. Then we shall apply the Strichartz estimate and some properties of the Schrödinger equation to obtain some local and global wellposedness results.

5.1 Criticality

As we discussed in the previous chapter, the Cauchy problem of the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u \\ u(0,x) = \varphi(x) \end{cases} \tag{5.1}$$

enjoys the scaling symmetry $u(x,t) \to \mu^{-\frac{2}{p-1}} u(\frac{t}{\mu^2}, \frac{x}{\mu})$. More precisely, if u solves the Cauchy problem (5.1), then $u_\mu(x,t) = \mu^{-\frac{2}{p-1}} u(\frac{t}{\mu^2}, \frac{x}{\mu})$ solves the same equation with the new initial data $\varphi_\mu = \mu^{-\frac{2}{p-1}} \varphi(\frac{x}{\mu})$, for any $\mu \in \mathbb{R}$. If we measure the size of the rescaled initial datum using the homogeneous Sobolev norm, we will get the following relationships

$$||\varphi_\mu||_{\dot{H}^s_x(\mathbb{R}^n)} = \mu^{-s+s_c} ||\varphi||_{\dot{H}^s_x(\mathbb{R}^n)}, \tag{5.2}$$

this helps us define the criticality of the related Cauchy problem. Here $s_c = \frac{n}{2} - \frac{2}{p-1}$ is called the *critical regularity*, and its size relative to the size of the regularity s could give us information about whether global wellposedness of the Cauchy problem (5.1) is available. In particular, we refer to the regularities $s > s_c$ as *subcritical*, the regularities $s = s_c$ as *critical* and the regularities $s < s_c$ as *supercritical.* As higher-regularity solutions normally have better behaviours than the lower-regularity solutions, we expect the subcritical solutions to have less pathological behaviours than the critical and supercritical solutions. This is elaborated in the following three cases,

(1). Subcritical case ($s > s_c$): μ has negative power and the $\dot{H}^s$ norm of the rescaled initial data φ_μ can be made arbitrarily small when $\mu \to \infty$. This is a supportive setting for global wellposedness.

(2). Critical case ($s = s_c$): the $\dot{H}^s$ norm of the rescaled initial data φ_μ is conserved for all μ. This is a problematic situation for proving global wellposedness.

(3). Supercritical case ($s < s_c$): the $\dot{H}^s$ norm of the rescaled initial data φ_μ will become large as μ increases. This will prevent us from getting global wellposedness for the equation.

Certain special regularities are particularly important in mathematical physics. These are the $\dot{H}^1$ norm which is associated with the total energy and the L^2 norm which is associated with the total mass in nonlinear Schrödinger equation. The $\dot{H}^1$ critical (or energy critical) case corresponds to $s_c = 1$, thus $p = 1 + \frac{4}{n-2}$ for $n \geq 3$ and the L^2 critical (or mass critical) case corresponds to $s_c = 0$, thus $p = 1 + \frac{4}{n}$ for $n \geq 1$. The corresponding subcritical and supercritical ranges are defined similarly. For more discussions about the criticality of the nonlinear Schrödinger equation, the readers may consult a reference like [7].

5.2 Definition of wellposedness and Strichartz space

Definition 5.2.1 (Wellposedness [35]). *The Cauchy problem (5.1) is locally wellposed in $H^s_x(\mathbb{R}^n)$ if the following conditions hold*

(1). Existence: for any initial data $\varphi \in H^s_x(\mathbb{R}^n)$, there exists an open ball B_R with radius $R > 0$, a time bound $T > 0$ and a function space $X \subset C^0_t H^s_x([-T,T], \mathbb{R}^n)$[1] *such that, for each $\varphi \in B_R$, there exists a solution $u \in X$ to the Cauchy problem ().*

(2) Uniqueness: the solution u is unique in X. Furthermore, the uniqueness is unconditional if $X = C^0_t H^s_x([-T,T], \mathbb{R}^n)$.

(3) Continuous dependence on initial data: The mapping $\varphi \to u$ is continuous from $H^s_x(\mathbb{R}^n)$ to $C^0_t H^s_x([-T,T], \mathbb{R}^n)$.

(4) Persistence of Regularity: if the initial data $\varphi \in H^{s_1}_x(\mathbb{R}^n)$ for some $s_1 > s$, then the solution $u \in H^{s_1}_x(\mathbb{R}^n)$.

If the above conditions $(1) - (4)$ hold for any large time $T > 0$, we say the Cauchy problem (5.1) is globally wellposed.

Remark 5.2.1. *Here we are primarily working with the fractional Sobolev space H^s for any $s \in \mathbb{R}$. In particular, some regularities have physical meanings. For instance, if we are looking for a solution with finite mass or energy, then we may focus on the cases $s = 0, 1$. Besides, one can also seek wellposedness in other variants of H^s, like the homogeneous Sobolev space $\dot{H}^s$. However, if one wish to choose a space other than the L^2-based Sobolev space, it is better to ensure the solutions of the linear part of the equation is conserved by the function norm, since otherwise there could be little chance to obtain any sort of wellposedness for the equation.*

The basic tool for obtaining wellposedness is the *contraction mapping principle*. As discussed in chapter 1, the equation $iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1} u$ with initial data φ is equivalent to the integral equation

[1] The space of functions $u : I \times \mathbb{R}^n \to \mathbb{C}$ such that u is continuous on $[-T,T]$ and $u(x) \in H^s_x(\mathbb{R}^n)$.

$$u = e^{\frac{it\Delta}{2}}\varphi(x) - i\lambda \int_0^t e^{\frac{i(t-s)\Delta}{2}}(|u|^{p-1}u)(s)\,ds.$$

Thus the basic idea is to construct a contraction mapping $u \to \phi(u)$ with

$$\phi(u) = e^{\frac{it\Delta}{2}}\varphi(x) - i\lambda \int_0^t e^{\frac{i(t-s)\Delta}{2}}(|u|^{p-1}u)(s)\,ds,$$

and show that the mapping has a fixed point. In general, the first obstacle we might encounter is to design a function space which has enough structures for constructing such a contraction. Here we shall introduce the notation of the Strichart space and the dual Strichartz space. We shall see that they are supportive settings for applying the Strichartz estimates and obtaining the wellposedness.

Definition 5.2.2 (Strichartz space [7]). *The Strichartz space $S^0(I \times \mathbb{R}^n)$ is the closure of the Schwartz space under the norm*

$$||u||_{S^0(I\times\mathbb{R}^n)} = \sup_{(q,r)\ \textit{Schrödinger-admissible}} ||u||_{L_t^q L_x^r(I\times\mathbb{R}^n)} \tag{5.3}$$

In particular, $S^0(I \times \mathbb{R}^n)$ is a Banach space and has a dual space $N^0(I \times \mathbb{R}^n)$ with the norm

$$||u||_{N^0(I\times\mathbb{R}^n)} = \inf_{(q,r)\ \textit{Schrödinger-admissible}} ||u||_{L_t^q L_x^r(I\times\mathbb{R}^n)} \tag{5.4}$$

Using these notations, the Strichartz estimate can be expressed in a more concise way.

Theorem 5.2.1 (Strichartz estimate [32]). *For any Schrödinger-admissible pair (q, r) and a compact time domain $I \subset \mathbb{R}$, the solution $u : I \times \mathbb{R}^n \to \mathbb{C}$ of the equation $iu_t + \frac{1}{2}\Delta u = F(u)$ with initial data φ satisfies*

$$\||\nabla|^s u\|_{S^0(I\times\mathbb{R}^n)} \lesssim \|\varphi\|_{\dot{H}_x^s(\mathbb{R}^n)} + \||\nabla|^s F\|_{N^0(I\times\mathbb{R}^n)}, \tag{5.5}$$

where $|\nabla|^s$ is the fractional differentiation operator.

5.3 The local theory

In this section, we shall focus on the local existence theory of the nonlinear Schrödinger equation. We shall start from a review of the abstract iteration argument and develop the local theory for wellposedness in L^2 and H^1.

The main tool for proving wellposedness of the nonlinear Schrödinger equation is the contraction mapping theorem. Let us first work abstractly on the following model equation and introduce the abstract iteration argument which would allow us to construct the desired contraction mapping. The model equation

$$\begin{cases} \partial_t u - Lu = N(u) \\ u(0,x) = \varphi(x) \end{cases} \tag{5.6}$$

with the linear operator L and nonlinear mapping N, is equivalent to the integral equation $u = u_{lin} + DN(u)$. Here $u_{lin}(t) := e^{tL}\varphi(x)$ is the solution for the linear equation $\partial_t u - Lu =$

0 with the initial data φ and D is the Duhamel operator $D(\cdot) := \int_0^t e^{(t-s)L}(\cdot)\, ds$. It is clear that $u = u_{lin} + DN(u)$ has the form

$$u = e^{\frac{it\Delta}{2}}\varphi(t) - i\lambda \int_0^t e^{\frac{i(t-s)\Delta}{2}} |u|^{p-1} u \, ds \tag{5.7}$$

in nonlinear Schrödinger equation. Now let us introduce the abstract iteration argument.

Theorem 5.3.1 (abstract iteration argument [25]). *Let $\mathcal{N}$, $\mathcal{S}$ be two Banach spaces. Consider the integral equation $u = u_{lin} + DN(u)$, where u_{lin} is the solution for the linear part of the equation and $D : \mathcal{N} \to \mathcal{S}$ is a linear operator such that*

$$||DF||_{\mathcal{S}} \leqslant C_0 ||F||_{\mathcal{N}} \tag{5.8}$$

for any $F \in \mathcal{N}$ and some constant $C_0 > 0$. Let $N : \mathcal{S} \to \mathcal{N}$ be a nonlinear mapping such that $N(0) = 0$ and obeys

$$||N(u) - N(v)||_{\mathcal{N}} \leqslant \frac{1}{2C_0} ||u - v||_{\mathcal{S}}, \tag{5.9}$$

for all u, v in the ball $B_r := \{u \in \mathcal{S} : ||u||_{\mathcal{S}} \leq r\}$. Then for all $u_{lin} \in B_{r/2}$, there exists a unique solution $u \in B_r$ to the integral equation $u = u_{lin} + DN(u)$. Moreover, the mapping $u_{lin} \to u$ from $B_{r/2}$ to B_r is Lipschitz continuous.

Remark 5.3.1. *The abstract iteration argument can be considered as an improved version of the contraction mapping principle. In particular, if we have the conditions (5.8) and (5.9), this theorem gives us the existence of solution of the corresponding equation. Later, we shall make use of this theorem to prove a local existence result for the nonlinear Schrödinger equation with initial data in L^2. In particular, we shall see that condition (5.8) can be realized by the inhomogeneous Strichartz estimate, while the condition (5.9) relies on the following inequality.*

Lemma 5.3.1 (Elementary estimate). *Let u, v be complex-valued functions, then for $p > 1$,*

$$\big| |u|^{p-1}u - |v|^{p-1}v \big| \leq C(|u| + |v|)^{p-1} |u - v|, \tag{5.10}$$

for some constants $C > 0$ depending on p.

Proof. Set $F(u) = |u|^{p-1}u$. It is clear that $F : \mathbb{C} \to \mathbb{C}$ is continuously differentiable and the derivative is computed by the chain rule

$$\begin{aligned} F'(u) &= |u|^{p-1} + (p-1)|u|^{p-2} u \frac{d|u|}{du} \\ &= |u|^{p-1} + (p-1)|u|^{p-2} u \frac{\overline{u}}{|u|} \\ &= p|u|^{p-1}. \end{aligned}$$

For any $t \in [0, 1]$, the fundamental theorem of calculus and the triangle inequality yield

$$
\begin{aligned}
|F(u)-F(v)| &= \left|\int_0^1 \frac{d}{dt}F(tu+(1-t)v)\,dt\right| \\
&\le \int_0^1 |(u-v)||F'(tu+(1-t)v)|\,dt \\
&\le p|u-v|\int_0^1 |tu+(1-t)v|^{p-1}\,dt \\
&\le C|u-v|\int_0^1 (|u|+|v|)^{p-1}\,dt \\
&= C(|u|+|v|)^{p-1}|u-v|.
\end{aligned}
$$

□

Having these tools in hand, we are ready to present a local wellposedness theorem for the subcritical nonlinear Schrödinger equation in $L^2(\mathbb{R}^n)$.

Theorem 5.3.2 (Local wellposedness for $L^2(\mathbb{R}^n)$ subcritical case [34])**.** *Let $1 < p < 1+\frac{4}{n}$, then for any $R > 0$, there exists a time bound $T > 0$ such that given any initial data $\varphi(x)$ in the ball $B_R = \{\varphi \in L^2_x(\mathbb{R}^n) : \|\varphi\|_{L^2_x(\mathbb{R}^n)} < R\}$, there exists a unique L^2-solution u to the Cauchy problem (5.1). In addition, the solution $u \in S^0([-T,T]\times\mathbb{R}^n) \subset C^0_t L^2_x([-T,T]\times\mathbb{R}^n)$ and the mapping $\varphi \to u$ from B_R to $S^0([-T,T]\times\mathbb{R}^n)$ is Lipschitz continuous.*

Proof. The proof uses the abstract iteration argument. Firstly, we set $\mathcal{S} = S^0([-T,T]\times\mathbb{R}^n)$ and $\mathcal{N} = N^0([-T,T]\times\mathbb{R}^n)$. They are both Banach spaces as required in the Theorem 5.3.1. Then we use F to denote the nonlinear term $\lambda|u|^{p-1}u$ and let $D(\cdot) = \int_0^t e^{(t-s)L}(\cdot)\,ds$ be the Duhamel operator. By the inhomogeneous Strichartz estimate,

$$
\|DF\|_{S^0([-T,T]\times\mathbb{R}^n)} \lesssim \|F\|_{N^0([-T,T]\times\mathbb{R}^n)}\,.
$$

Thus we see that $D : N \to S$ is a bounded linear operator, which is another required condition. Now we just need to check that the nonlinear operator $N : u \to |u|^{p-1}u$ satisfies the inequality

$$
||N(u)-N(v)||_{N^0([-T,T]\times\mathbb{R}^n)} \le \frac{1}{2C_0}||u-v||_{S^0([-T,T]\times\mathbb{R}^n)}, \tag{5.11}
$$

for all u, v in the ball $B_r \in S^0([-T,T]\times\mathbb{R}^n)$, where r, C_0 are positive. Since we need to perform analysis in the Strichartz space S^0, we need to choose exponent pairs (q,r) such that they are Schödinger-admissible. Here we choose (q,r) that satisfies $\frac{2}{q}+\frac{n}{r}=\frac{n}{2}$ and $\frac{p}{r}=\frac{1}{r'}$. Using the fact that p is in the L^2-subcritical range $(1, 1+\frac{4}{n})$, it follows that $2 < r < q < \infty$. Thus we see that (q,r) is Schrödinger-admissible and $\frac{p}{q} > \frac{1}{q'}$ after some algebraic calculations. Now we can bound the N^0 norm by the $L^{q'}_t L^{r'}_x$ norm, and since the power $\alpha := \frac{1}{q'} - \frac{p}{q}$ is positive we can use the Hölder's inequality to replace the $L^{q'}_t$ norm by the $L^{q'}_t$ norm by paying a factor of $(2T)^\alpha$. Although these procedures are quite mechanical and tedious, we shall present them here in order to fully illustrate the ideas of the proof. We begin with the elementary estimate

$$
\left||u|^{p-1}u - |v|^{p-1}v\right| \le C(|u|+|v|)^{p-1}|u-v|, \tag{5.12}
$$

then since the N^0 norm is just the infimum of the $L^{q'}_t L^{r'}_x$ norm, we have

$$\left\||u|^{p-1}u - |v|^{p-1}v\right\|_{N^0} \le \left\||u|^{p-1}u - |v|^{p-1}v\right\|_{L_t^{q'}L_x^{r'}}.$$

Now we need to repeatedly use the Hölder's inequality and the elementary estimate to obtain the following inequalities,

$$\begin{aligned}\left\||u|^{p-1}u - |v|^{p-1}v\right\|_{L_t^{q'}L_x^{r'}} &\le \left\||u|^{p-1}u - |v|^{p-1}v\right\|_{L_t^{q/p}L_x^{r'}} \|1\|_{L_x^{1/\alpha}} \\ &\lesssim (2T)^\alpha \left\||u-v|(|u|+|v|)^{p-1}\right\|_{L_t^{q/p}L_x^{r'}} \\ &\lesssim (2T)^\alpha \|u-v\|_{L_t^qL_x^r} \left\|(|u|+|v|)^{p-1}\right\|_{L_t^{q/(p-1)}L_x^{r/(p-1)}} \\ &\lesssim (2T)^\alpha ||u-v||_{L_t^qL_x^r}(||u||_{L_t^qL_x^r} + ||v||_{L_t^qL_x^r})^{p-1}.\end{aligned}$$

Summarizing what we have derived above, we find that

$$||N(u) - N(v)||_{N^0} \lesssim (2T)^\alpha ||u-v||_{L_t^qL_x^r}(||u||_{L_t^qL_x^r} + ||v||_{L_t^qL_x^r})^{p-1}.$$

Thus we may now choose $r = C_1R$, for some $C_1, R > 0$, then for all $u,\ v \in B_r$ we obtain

$$||N(u) - N(v)||_{N^0} \lesssim (2T)^\alpha ||u-v||_{S^0}. \tag{5.13}$$

Now choose a sufficiently small T such that $(2T)^\alpha < \frac{1}{2}$ and we get the desired inequality (5.9). By directly applying Theorem 5.3.1, we see that for any $\varphi \in B_{r/2}$, there exists a solution $u \in B_r$, in addition, the mapping $\varphi \to u$ is Lipschitz.

□

The proof of the local existence of solutions in the L^2 subcritical case relies heavily on the non-zero power α. It is due to this non-zero power that we can construct a contraction mapping by shrinking the time T. However, this technique becomes problematic in the situation of the critical equation. One of the reason is that the power $\alpha = \frac{1}{q'} - \frac{p}{q}$ vanishes when p is the critical exponent, i.e. $p = 1 + \frac{4}{n}$, so there will be no time factor appearing in the right-hand-side of estimate (5.13) and we cannot obtain local existence of the solution using the above technique. It turns out that we need a different technique to handle this critical case, as shown in the next theorem.

Theorem 5.3.3 (Local wellposedness for $L^2(\mathbb{R}^n)$ critical case)**.** *Let $p = 1 + \frac{4}{n}$, then for any initial data $\varphi \in L_x^2(\mathbb{R}^n)$, there exists a time bound $T > 0$ and a unique solution $u \in C_t^0L_x^2([-T,T] \times \mathbb{R}^n) \cap L_{loc,t}^{\frac{4+n}{n}}L_x^{\frac{4+n}{n}}([-T,T] \times \mathbb{R}^n)$ to the Cauchy problem (5.1). In addition, the solution u satisfies the following properties,*

(i). For every Schrödinger-admissible pair (q,r), the solution $u \in L_{loc,t}^qL_x^r([-T,T] \times \mathbb{R}^n)$

(ii). Continuous dependence: suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is sequence of initial datum in $L_x^2(\mathbb{R}^n)$, and for each $n \in \mathbb{N}$ let u_n be the corresponding solution to the Cauchy problem (5.1) with the initial data φ_n. If $\varphi_n \to u_n$ in $L_x^2(\mathbb{R}^n)$, then $u_n \to u$ in $L_t^qL_x^r(I \times \mathbb{R}^n)$, where $I \subset [-T,T]$ and (q,r) is arbitrary Schrödinger-admissible pair.

(iii). Blowup alternative: If T is finite, then $\|u\|_{L_t^qL_x^r((0,T)\times\mathbb{R}^n)} = \infty$ for any Schrödinger -admissible pair (q,r) with $r \ge \frac{4+n}{n}$

Proof. Interested readers may see Cazenave and Weissler [5] and Tsutsumi [34] for a proof of this theorem. □

So far we have obtained some local existence results in $L^2_x(\mathbb{R}^n)$. Formally, the local wellposedness requires more conditions like the uniqueness, continuous dependence of initial data and persistence of regularities. Even in this relatively simple case, we will not be able to mention all the findings and results concerning the Schrödinger equation. A classical text about the wellposedness theory for the semilinear Schrödinger equation is Cazenave and Weissler [5]. However, we are not in a position to discuss all these aspects of analysis here. Instead we shall focus on obtaining a similar local wellposedness result in the energy space $H^1_x(\mathbb{R}^n)$. The techniques for the H^1 situation is very similar to those presented above. They are both based on the contraction mapping principle and can be realized by the abstract iteration argument. In order to prove the results for H^1 it is convenient to introduce the following norms,

$$\|u\|_{S^1(I\times\mathbb{R}^n)} = \|u\|_{S^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{S^0(I\times\mathbb{R}^n)}$$

and

$$\|u\|_{N^1(I\times\mathbb{R}^n)} = \|u\|_{N^0(I\times\mathbb{R}^n)} + \|\nabla u\|_{N^0(I\times\mathbb{R}^n)}$$

Using the language of S^1 norm, we can combine the Strichartz estimates into a more compact form,

$$\|u\|_{S^1(I\times\mathbb{R}^n)} \lesssim \|\varphi\|_{H^1_x(\mathbb{R}^n)} + \|F\|_{N^1(I\times\mathbb{R}^n)}, \tag{5.14}$$

for any time domain $I \in \mathbb{R}$ and $iu_t + \frac{1}{2}\Delta u = F(u)$.

Theorem 5.3.4 (Local wellposedness for $H^1(\mathbb{R}^n)$ subcritical case). *Let $1 < p < 1 + \frac{4}{n-2}$, then for any $R > 0$, there exists a time bound $T > 0$ such that given any initial data $\varphi(x)$ in the ball $B_R = \{\varphi \in H^1_x(\mathbb{R}^n) : \|\varphi\|_{H^1_x(\mathbb{R}^n)} < R\}$, there exists a unique strong H^1-solution u to the Cauchy problem (5.1). In addition, the solution $u \in S^1([-T,T]\times\mathbb{R}^n) \subset C^0_t H^1_x([-T,T]\times\mathbb{R}^n)$ and the mapping $\varphi \to u$ from B_R to $S^1([-T,T]\times\mathbb{R}^n)$ is Lipschitz continuous.*

Proof. (Sketch from [7]). Here we will use the contraction mapping principle directy. We first construct a solution map $u \to \phi(u)$ such that

$$\phi(u) = e^{\frac{it\Delta}{2}}\varphi(x) - i\lambda\int_0^t e^{\frac{i(t-s)\Delta}{2}}(|u|^{p-1}u)(s)\,ds.$$

Here we need to choose the exponent pair (q,r) such that

$$\frac{2}{q} + \frac{r}{n} = \frac{n}{2} \quad , \quad (p-1)(\frac{1}{r} - \frac{1}{n}) = \frac{1}{r'} - \frac{1}{r} \tag{5.15}$$

with $\frac{1}{\tilde{r}} := \frac{1}{r} - \frac{1}{n}$. These scalars were carefully chosen so that (q,r) is Schrödinger-admissible and supportive for establishing a contracion mapping. Indeed, since the power p belongs to the subcritical range $(1, 1+\frac{4}{n+2})$, one can deduce that $\frac{1}{q'} > \frac{p}{q}$ and this would help us define a positive power $\alpha = \frac{1}{q'} - \frac{p}{q}$. We will see that this is very helpful when we get to the step $\|\phi(u)\|_{S^1} \lesssim \|\varphi\|_{H^1(\mathbb{R}^n)} + (2T)^\alpha \|u\|^p_{S^1}$ as we will be able to construct a contraction by shrinking the time T. In contrast, the scalar α will be equal to zero in the situation of the critical nonlinear Schrödinger equation, so there will be no time factor appearing in the right-hand-side of this estimate. This shall make the critical case trickier than the subcritical case and we will see this in the next theorem. By the Strichartz estimate stated in (5.14), we obtain

$$
\begin{aligned}
\|\phi(u)\|_{S^1([-T,T]\times\mathbb{R}^n)} &\lesssim \|\varphi\|_{H^1_x(\mathbb{R}^n)} + \big\||u|^{p-1}u\big\|_{N^1([-T,T]\times\mathbb{R}^n)} \\
&\lesssim \|\varphi\|_{H^1_x(\mathbb{R}^n)} + \big\||u|^{p-1}u\big\|_{N^0([-T,T]\times\mathbb{R}^n)} + \big\|\nabla(|u|^{p-1}u)\big\|_{N^0([-T,T]\times\mathbb{R}^n)} \\
&= I_1 + I_2 + I_3.
\end{aligned}
\tag{5.16}
$$

Now our purpose is to get an upper bound for I_2 and I_3. Notice that I_2 should be easier to handle than I_3 as the latter involves a derivative. Applying the elementary estimate (5.10) and Hölder's inequality, we get the following estimate for I_2,

$$
\big\||u|^{p-1}u\big\|_{N^0([-T,T]\times\mathbb{R}^n)} \lesssim (2T)^\alpha \|u\|^p_{S^1([-T,T]\times\mathbb{R}^n)}. \tag{5.17}
$$

For I_3, we observe that $\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u$ by the chain rule. Thus, by repeated Hölder's inequality,

$$
\begin{aligned}
\big\||u|^{p-1}\nabla u\big\|_{N^0([-T,T]\times\mathbb{R}^n)} &\lesssim \big\||u|^{p-1}\nabla u\big\|_{L^{q'}_t L^{r'}_x([-T,T]\times\mathbb{R}^n)} \\
&\lesssim (2T)^\alpha \big\||u|^{p-1}\nabla u\big\|_{L^{q/p}_t L^{r'}_x([-T,T]\times\mathbb{R}^n)} \\
&\lesssim (2T)^\alpha \|u\|^{p-1}_{L^q_t L^{\tilde r}_x([-T,T]\times\mathbb{R}^n)} \|u\|_{S^1([-T,T]\times\mathbb{R}^n)}.
\end{aligned}
\tag{5.18}
$$

The second term in this last expression can be estimated again by the Sobolev embedding. More precisely, since $\frac{1}{\tilde r} = \frac{1}{r} - \frac{1}{n}$, the Sobolev embedding yields

$$
\begin{aligned}
\|u\|^{p-1}_{L^q_t L^{\tilde r}_x} &\lesssim \|u\|^{p-1}_{L^q_t W^{1,r}_x} \\
&\le \big(\|u\|_{L^q_t L^r_x} + \|\nabla u\|_{L^q_t L^r_x} \big)^{p-1} \\
&= \|u\|^{p-1}_{S^1}.
\end{aligned}
\tag{5.19}
$$

Combining estimates (5.16), (5.17), (5.18) and (5.19), we obtain

$$
\|\phi(u)\|_{S^1} \lesssim \|\varphi\|_{H^1(\mathbb{R}^n)} + (2T)^\alpha \|u\|^p_{S^1} \tag{5.20}
$$

which is exactly the estimate we want, in order to construct a contraction mapping. Now it remains to select an appropriate $R > 0$ such that the solution map $\theta : u \to \phi(u)$ is a contraction on the closed ball B_R centred at the origin with a radius R. One of these valid choices is $R = 2\|\varphi\|_{H^1(\mathbb{R}^n)}$, and after selecting a sufficiently small T such that $(2T)^\alpha R^{p-1} < \frac{1}{2}$, we see that

$$
\|\phi(u)\|_{S^1} < \frac{R}{2} + \frac{R}{2} = R. \tag{5.21}
$$

This suggests that $\phi(u) \in B_R$ for any u in this ball, and by a similar argument as above,

$$
\begin{aligned}
\|\phi(u) - \phi(v)\|_{S^1} &\lesssim (2T)^\alpha \big(\|u\|^{p-1}_{S^1} + \|v\|^{p-1}_{S^1} \big) \|u - v\|_{S^1} \\
&\lesssim (2T)^\alpha \|u - v\|_{S^1}
\end{aligned}
\tag{5.22}
$$

for all u, $v \in B_R$. Combining the estimates (5.21) and (5.22), we see that the solution map $\theta : B_R \to B_R$ is indeed a contraction mapping. Furthermore, the estimate $(2T)^\alpha R^{p-1} <$

1/2 suggests that the size of the time bound T is approximately $\|\varphi\|_{H^1}^{\frac{1-p}{\alpha}}$. Therefore, the contraction mapping principle suggests that there exists a solution for the Cauchy problem (5.1) on a local time domain $[-T,T]$, with the half life span $T \sim \|\varphi\|_{H^1}^{\frac{1-p}{\alpha}}$. $\square$

Theorem 5.3.5 (local wellposedness for $H^1(\mathbb{R}^n)$ critical case [7])**.** *Let $p = 1 + \frac{4}{n-2}$, and let the initial data $\varphi \in H^1(\mathbb{R}^n)$, if there exists a time bound $T > 0$ such that*

$$\left\| e^{\frac{it\Delta}{2}} \varphi \right\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} W_x^{1,\frac{2n(n+2)}{n^2+4}}} \leq \epsilon \tag{5.23}$$

for sufficiently small ϵ, then the Cauchy problem (5.1) is locally wellposed on the domain $[-T,T]$.

Proof. Let $p = 1 + \frac{4}{n-2}$, $\beta_1 = \frac{2(n+2)}{n-2}$ and $\beta_2 = \frac{2n(n+2)}{n^2+4}$. We see that the exponent pair (β_1, β_2) is Schrödinger-admissible. Now the first step is to construct a function space like what we did in Theorem 5.3.4 and 5.3.2. Here we shall work on the space $\tilde{S}^1$ equipped with the norm

$$\|u\|_{\tilde{S}^1} := T\|u\|_{S^1} + \|u\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}}. \tag{5.24}$$

Let $\theta : u \to \phi(u)$ be our solution map with

$$\phi(u) = e^{\frac{it\Delta}{2}} \varphi(x) - i\lambda \int_0^t e^{\frac{i(t-s)\Delta}{2}} (|u|^{p-1} u)(s)\, ds.$$

and let us estimate the $\tilde{S}^1$ norm of $\phi(u)$. In particular, we get

$$\begin{aligned}
\|\phi(u)\|_{\tilde{S}^1} &= T\|\phi(u)\|_{S^1} + \|\phi(u)\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}} \\
&\lesssim T\|\varphi\|_{H_x^1} + \left\| |u|^{p-1} \nabla u \right\|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}} + \|\phi(u)\|_{L_{[-T,T]}^{\beta_1} L_x^{\beta_2}} + \|\nabla \phi(u)\|_{L_{[-T,T]}^{\beta_1} L_x^{\beta_2}} \\
&\lesssim T\|\varphi\|_{H_x^1} + \left\| e^{\frac{it\Delta}{2}} \varphi \right\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}} + \left\| |u|^{p-1} \nabla u \right\|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}}
\end{aligned} \tag{5.25}$$

where $(\tilde{q}, \tilde{r})$ is arbitrary Schrödinger-admissible exponent pair. For the purpose of construction, we need to choose $(\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})$. This will allow us to use the Hölder's inequality repeatedly to derive the following results,

$$\left\| |u|^{p-1} \nabla u \right\|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|\nabla u\|_{L_{[-T,T]}^{\beta_1} L_x^{\beta_1}}^{p-1} \left\| |u|^{p-1} \nabla u \right\|_{L_{[-T,T]}^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|\nabla u\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}}^{p-1}. \tag{5.26}$$

As we have the scaling condition $\frac{1}{\beta_2} = \frac{1}{\beta_1} + \frac{1}{n}$ (this is the scaling condition for the Sobolev embedding theorem, see Chapter 2.3), hence the Sobolev embedding implies

$$\|\nabla u\|_{L_{[-T,T]}^{\beta_1} L_x^{\beta_1}} \lesssim \|\nabla u\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}}. \tag{5.27}$$

Therefore, combining (5.25), (5.26) and (5.27), we have

$$\|\phi(u)\|_{\tilde{S}^1} \leq C_0 T\|\varphi\|_{H_x^1} + C_0 \left\| e^{\frac{it\Delta}{2}} \varphi \right\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}} + C_1 \|u\|_{L_{[-T,T]}^{\beta_1} W_x^{1,\beta_2}}^p. \tag{5.28}$$

In order to construct a contraction, we may take $T := \frac{\epsilon}{\|\varphi\|_{H^1_x}}$ and $R := 4C_0\epsilon$ then by (5.28), we have

$$\|\phi(u)\|_{\tilde{S}^1} \leq 2C_0\epsilon + C_1 \|u\|^p_{L^{\beta_1}_{[-T,T]} W^{1,\beta_2}_x}. \tag{5.29}$$

Thus if $u \in B_R := \{u \in \tilde{S}^1 : \|u\|_{\tilde{S}^1} \leq R\}$, then

$$\|u\|^p_{L^{\beta_1}_{[-T,T]} W^{1,\beta_2}_x} \leq R^p = (4C_0\epsilon)^p$$

which implies

$$\|\phi(u)\|_{\tilde{S}^1} \leq \frac{R}{2} + C_1(4C_0\epsilon)^p.$$

Now the subsequent steps are routine: if we take a sufficiently small $\epsilon > 0$ such that $C_1(4C_0\epsilon)^p < \frac{R}{2}$, then

$$\|\phi(u)\|_{\tilde{S}^1} \leq \frac{R}{2} + \frac{R}{2} = R.$$

This establishes a contraction $u \to \phi(u)$ from B_R to B_R. The rest of the proof is similar to that of Theorem 5.3.4.

□

Remark 5.3.2. *From the above theorem, we know that for any initial data $\varphi \in H^1(\mathbb{R}^n)$, the nonlinear Schrödinger equation with the critical exponent $p = 1 + \frac{4}{n-2}$ is locally wellposded on the domain $[-T, T]$. Here the time bound T is finite and is approximately equal to $\epsilon \|\varphi\|^{-1}_{H^1_x}$. The next question is how to extend the existence domain to the entire real line (this is exactly the meaning of global wellposedness). This requires an a priori estimate of the Strichartz norm of the solution u. We shall give a global wellposedness theory for the nonlinear Schrödinger equation in the next section.*

5.4 Discussion: from local to global theory

In the previous sections, we discussed the local wellposedness theory of the nonlinear Schrödinger equation. We first defined the four elements of wellposedness, including existence, uniqueness, continuous dependence on initial data and persistence of regularity. After that, we presented some local wellposedness results of the nonlinear Schrödinger equation in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. The first obstacle is to make a delicate balance between the solution and its survival space. On one hand, if we require too many degrees of regularity of the solution, its survival space could be too small or even its existence is a problem. On the other hand, a weak solution of the equation could exist in a very large space such that its uniqueness is no longer available. In order to realize the balance between the solution and its survival space, we need the Strichartz estimate. For instance, in the Theorem 5.3.4, we have chosen a Schrödinger-admissible pair (q, r) and required that it satisfies the equation

$$(p-1)(\frac{1}{r}-\frac{1}{n}) = \frac{1}{r'} - \frac{1}{r}.$$

This equation might seem mysterious at the first glance. However, they are carefully designed by the mathematicians such that the Strichartz estimate, Hölder's inequality and Sobolev embeddings can be applied to the Duhamel's formulae

$$\phi(u) = e^{\frac{it\Delta}{2}}\varphi(x) - i\lambda \int_0^t e^{\frac{i(t-s)\Delta}{2}}(|u|^{p-1}u)(s)\,ds$$

to yield the inequality

$$\|\phi(u)\|_{L_t^q L_x^r} \lesssim \|\varphi\|_{H_x^1(\mathbb{R}^n)} + (2T)^\alpha \|u\|_{L_t^q L_x^r}^p. \tag{5.30}$$

This helps us construct a contraction mapping by choosing a sufficiently small time bound $T > 0$, and show that the nonlinear Schrödinger equation is wellposed in $[-T, T]$. Here it is also worth mentioning the difference between the subcritical case and the critical case. In general, the critical number for the H^s problem is $s_c = \frac{n}{2} - \frac{2}{p-1}$. In the case of H^1, we need to set $s_c = 1$ and obtain the corresponding critical power $p = 1 + \frac{4}{n-2}$. If p is in the subcritical range $[1, 1 + \frac{4}{n-2})$, we may use the estimate (5.20) to obtain a contraction directly. If p is at the critical range $1 + \frac{4}{n-2}$, the power α in the estimate (5.20) will be zero and we no longer have the time factor T^α appearing on the right-hand-side of the estimate (5.20). We see that this is problematic for obtaining a contraction. In this situation, we need a restriction on the solution of the linear part of the nonlinear Schrödinger equation in order to obtain a local wellposedness result, as shown in Theorem 5.3.5.

The next question we can ask is how to obtain a global solution u on the time domain. Since we already have a local solution u on $[-T, T]$ for some finite $T > 0$, the idea is to iterate the local in time solution to a uniformly global one. As we shall discuss in the next section, the global wellposedness of the subcritical nonlinear Schrödinger equation in $H^1(\mathbb{R}^n)$ uses the conservation of mass and energy. More precisely, for the Cauchy problem

$$\begin{cases} iu_t + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u \\ u(0, x) = \varphi(x) \end{cases} \tag{5.31}$$

with the subcritical power $p \in [1, 1 + \frac{4}{n-2})$. We have the conservation of mass,

$$\|u(t)\|_{L^2}^2 = \|\varphi\|_{L^2}^2$$

and the conservation of energy,

$$\frac{1}{2}\left\|\nabla u(t)\right\|_{L^2}^2 + \frac{2\lambda}{p+1}\left\|u(t)\right\|_{L^{p+1}}^{p+1} = \frac{1}{2}\left\|\nabla\varphi\right\|_{L^2}^2 + \frac{2\lambda}{p+1}\left\|\varphi\right\|_{L^{p+1}}^{p+1}.$$

Now let us assume $\lambda = 1$ and the initial data $\varphi \in H^1(\mathbb{R}^n)$, the conservation of mass and energy would give us the *a priori* uniform bound

$$\left\|u(t)\right\|_{H^1} \leq C^* \left\|\varphi\right\|_{H^1}. \tag{5.32}$$

We can then use this estimate to extend the existence domain from $[-T, T]$ to the entire real line.

5.5 The global theory

As promised before, we shall give some criteria for the global wellposedness of the nonlinear Schrödinger equation in this final section. Let us begin with the $H^1(\mathbb{R}^n)$ subcritical case. In Theorem 5.3.4, we already established the local wellposedness. Thus, given any initial data $\varphi \in H^1(\mathbb{R}^n)$, there exists a solution u for the Cauchy problem (5.1) on the interval $[-T, T]$, where the size of the time bound $T \sim \left\|\varphi\right\|_{H^1}^{\frac{1-p}{\alpha}}$. Now the idea is to use the estimate (5.32) above to extend the lifespan of the solution to the entire real line. Indeed, if we set $T^* \sim (C^*)^{\frac{1-p}{\alpha}}$, we can then repeat the Theorem 5.3.4 to extend the existence domain. In particular, the Theorem 5.3.4 tells us that there exists a solution on $[-T^*, T^*]$. Then we will use $u(T^*)$ as our new initial data and the local existence result shall imply that there exists a solution on $[T^*, 2T^*]$ and $[-2T^*, -T^*]$. By repeating this argument we can cover the entire real line and our wellpoedness becomes global.

Similar ideas apply to the H^1 critical case. The next theorem shows that if a certain Strichartz norm of the solution has an *a priori bound*, then global wellposedness follows.

Theorem 5.5.1 (Global wellposedness for the $H^1(\mathbb{R}^n)$ critical case). *Let $p = 1 + \frac{4}{n+2}$, and assume that the solution u has an a priori bound*

$$\left\|u\right\|_{L_{[-T,T]}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \leq C, \tag{5.33}$$

then for any $\varphi \in H^1(\mathbb{R}^n)$, the Cauchy problem (5.1) is globally wellposed.

Proof. (sketch from [7]). From (5.33) we can find finitely many intervals of time I_j with $j = 1, ..., N$ such that

$$\left\|u\right\|_{L_{I_j}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}} \leq \epsilon \tag{5.34}$$

for all j, and some ϵ to be determined later. Then using the definition of the S^1 norm and the Duhamel's representation of the solution u, we obtain

$$\begin{aligned}\left\|u\right\|_{S^1_{I_j}} &\leq C_1\left\|\varphi\right\|_{H^1} + C_2\left\|u\right\|_{L_{I_j}^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}}^{p-1}\left\|u\right\|_{S^1_{I_j}}\\ &\leq C_1\left\|\varphi\right\|_{H^1} + C_2\epsilon^{p-1}\left\|u\right\|_{S^1_{I_j}}.\end{aligned}$$

Now if we select a sufficiently small ϵ such that $C_2\epsilon^{p-1} < \frac{1}{2}$, we would obtain

$$\|u\|_{S^1_{I_j}} \leq C.$$

Putting all these small interval I_j together, we then obtain a stronger bound

$$\|u\|_{S^1} \leq C. \tag{5.35}$$

Now we may use (5.35) to extend the existence domain of the local solution. In particular, Theorem 5.3.5 tells us that for any initial data φ, there exists a solution $u \in S^1_{[-T,T]}$. Now we may assume $\overline{T} < \infty$ to be the maximum lifespan of the solution, that is, no solution shall exist for all $T > \overline{T}$. However, the estimate (5.35) tells us that

$$\left\|u(\overline{T})\right\| \leq C.$$

Therefore we may now continue our solution until $\overline{T}$ reaches ∞. This is basically the idea for obtaining the global wellposedness of the Schrödinger equation in this setting. □

Finally, let us end this section by claiming that the same result of the Theorem 5.5.1 still holds for the L^2 subcritical case. This result can be found in [32].

Chapter 6

Conclusion

In this report, we have outlined the work that we have completed during the Honours year. Now let us emphasize our themes again. The objective of this project is to present the classical results and recent progress of the Strichartz estimate, along with some of its applications in the nonlinear Schrödinger equation. In particular, Chapter 3 outlined the literature review, proof and generalizations of the Strichartz estimate. We saw that the endpoint Strichartz estimate requires completely different techniques compared to the nonendpoint case. In Chapter 4, we discussed the Hamiltonian formulation of the nonlinear Schrödinger equation, symmetries and conservation laws and the famous Viriel's identity. We have employed the concepts of symplectic geometry to derive the nonlinear Schrödinger equation from the total energy function $H(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\lambda}{p+1}|u|^{p+1}dx$ on the symplectic space $L^2(\mathbb{R}^n \to \mathbb{C})$ equipped with a symplectic form $\omega(g, v) = -\operatorname{Im}\int_{\mathbb{R}^n} g\overline{v}\, dx$. The last chapter focused on the applications of the Strichartz estimate in obtaining the local and global wellposedness of the Schrödinger equation in L^2 and H^1.

We also saw some limitations of our work and findings. In our Hamiltonian formulation of the Schrödinger equation in Chapter 4, we did not find a direct way to compute the symplectic gradient of the total energy function. We expect that there should be a more efficient way to derive the Schrödinger equation from its Hamiltonian instead of using Lemma 4.1.1. We also saw that there is much more that can be done for the part of the wellposedness theory. For the local theory part, further work could consider generalize the local wellposedness results to the Sobolev space H^s with higher regularity s. For the global theory part, one could conduct more systematic researches about the global existence of solutions with small initial data, oscillating initial data and asymptotically homogeneous initial data. It is also possible to investigate other aspects of the nonlinear Schrödinger equation, including the scattering theory, finite-time blowup, and the stability theory of bound states.

Bibliography

[1] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.

[2] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*, volume 3. Springer Science & Business Media, 2007.

[3] J. Avron, I. Herbst, and B. Simon. Separation of center of mass in homogeneous magnetic fields. *Annals of Physics*, 114(1-2):431–451, 1978.

[4] J. Bourgain. Refinements of strichartz'inequality and applications to 2d-nls with critical nonlinearity. *International Mathematics Research Notices*, 1998(5):253–283, 1998.

[5] T. Cazenave. *Semilinear schrödinger equations*, volume 10. American Mathematical Soc., 2003.

[6] T. Cazenave and F. B. Weissler. The cauchy problem for the nonlinear schrödinger equation in h. *manuscripta mathematica*, 61(4):477–494, 1988.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. The theory of nonlinear schrodinger equations: Part i.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for schrödinger equations with derivative. *SIAM Journal on Mathematical Analysis*, 33(3):649–669, 2001.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global existence and scattering for rough solutions of a nonlinear schrödinger equation on 3. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 57(8):987–1014, 2004.

[10] R. Courant and D. Hilbert. Chapter iv. the calculus of variations. *Methods of Mathematical Physics. New York: Interscience Publishers*, pages 164–274, 1953.

[11] G. B. Folland. *Real analysis: Modern techniques and their applications.* John Wiley & Sons, 2013.

[12] D. Foschi. Inhomogeneous strichartz estimates. *Journal of Hyperbolic Differential Equations*, 2(01):1–24, 2005.

[13] J. Ginibre and G. Velo. The global cauchy problem for the non linear schrödinger equation revisited. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 2, pages 309–327. Elsevier, 1985.

[14] J. Ginibre and G. Velo. Smoothing properties and retarded estimates for some dispersive evolution equations. *Communications in mathematical physics*, 144(1):163–188, 1992.

[15] R. T. Glassey. On the blowing up of solutions to the cauchy problem for nonlinear schrödinger equations. *Journal of Mathematical Physics*, 18(9):1794–1797, 1977.

[16] D. Gogny and P.-L. Lions. Hartree-fock theory in nuclear physics. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 20(4):571–637, 1986.

[17] T. Kato. On nonlinear schrödinger equations. In *Annales de l'IHP Physique théorique*, volume 46, pages 113–129, 1987.

[18] T. Kato. On nonlinear schrödinger equations, ii. l^2 -solutions and unconditional well-posedness. *Journal d'Analyse Mathématique*, 67(1):281–306, 1995.

[19] T. Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[20] M. Keel and T. Tao. Endpoint strichartz estimates. *American Journal of Mathematics*, 120(5):955–980, 1998.

[21] R. Killip and M. Visan. Nonlinear schrödinger equations at critical regularity. *Evolution equations*, 17:325–437, 2013.

[22] E. Noether. Invariante variationsprobleme, math-phys. *Klasse, pp235-257*, 1918.

[23] H. Pecher. Solutions of semilinear schrödinger equations in $\hat{h}s$. In *Annales de l'IHP Physique théorique*, volume 67, pages 259–296, 1997.

[24] W. Rudin. *Real and complex analysis*. Tata McGraw-Hill Education, 2006.

[25] I. Segal. Non-linear semi-groups. *Annals of Mathematics*, pages 339–364, 1963.

[26] B. Simon. Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7(3):447–526, 1982.

[27] E. M. Stein. *Harmonic Analysis (PMS-43), Volume 43: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.(PMS-43)*, volume 43. Princeton University Press, 2016.

[28] E. M. Stein and R. Shakarchi. Fourier analysis, volume 1 of princeton lectures in analysis, 2003.

[29] E. M. Stein and R. Shakarchi. *Functional analysis: introduction to further topics in analysis*, volume 4. Princeton University Press, 2011.

[30] R. S. Strichartz et al. Restrictions of fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Mathematical Journal*, 44(3):705–714, 1977.

[31] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation: self-focusing and wave collapse*, volume 139. Springer Science & Business Media, 2007.

[32] T. Tao. *Nonlinear dispersive equations: local and global analysis.* Number 106. American Mathematical Soc., 2006.

[33] P. A. Tomas. A restriction theorem for the fourier transform. *Bulletin of the American Mathematical Society*, 81(2):477–478, 1975.

[34] Y. Tsutsumi. L^2-solutions for nonlinear schrodinger equations and nonlinear groups. 1985.

[35] B. Wang, Z. Huo, Z. Guo, and C. Hao. *Harmonic analysis method for nonlinear evolution equations, I.* World Scientific, 2011.

[36] K. Yajima. Existence of solutions for schrödinger evolution equations. *Communications in Mathematical Physics*, 110(3):415–426, 1987.

[37] K. Yosida. Functional analysis.— springer-verlag. *New York*, 1965.