

A Summations

A.1 Summation formulas and properties

Exercise A.1-1

Find a simple formula for $\sum_{k=1}^n 2k - 1$.

Answer:

$$\begin{aligned}\sum_{k=1}^n 2k - 1 &= 2 \sum_{k=1}^n k - n \\ &= 2 \cdot \frac{n(n+1)}{2} - n \\ &= n\end{aligned}$$

Exercise A.1-2 ★

Show that $\sum_{k=1}^n 1/(2k-1) = \ln \sqrt{n} + O(1)$ by manipulating the harmonic series.

Answer:

$$\begin{aligned}\sum_{k=1}^n \frac{1}{2k-1} &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\ &= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \\ &= \ln 2n + O(1) - \frac{1}{2} \ln n - \frac{1}{2} O(1) \\ &= \ln \sqrt{n} + O(1)\end{aligned}$$

Exercise A.1-3

Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

Answer:

$$\begin{aligned}\sum_{k=0}^{\infty} k x^k &= \frac{x}{(1-x)^2} \\ \frac{d}{dx} \sum_{k=0}^{\infty} k x^k &= \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) \\ \sum_{k=0}^{\infty} k^2 x^{k-1} &= \frac{1+x}{(1-x)^3} \\ \sum_{k=0}^{\infty} k^2 x^k &= \frac{x(1+x)}{(1-x)^3}\end{aligned}$$

Exercise A.1-4 ★

Show that $\sum_{k=0}^{\infty} (k-1)/2^k = 0$.

Answer:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{k-1}{2^k} &= \sum_{k=0}^{\infty} k \left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \\
&= \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} - \frac{1}{1 - \frac{1}{2}} \\
&= 0
\end{aligned}$$

Exercise A.1-5 ★

Evaluate the sum $\sum_{k=1}^{\infty} (2k+1)x^{2k}$.

Answer:

$$\begin{aligned}
\sum_{k=1}^{\infty} (2k+1)x^{2k} &= \sum_{k=0}^{\infty} (2k+1)x^{2k} - 1 \\
&= 2 \sum_{k=0}^{\infty} k(x^2)^k + \sum_{k=0}^{\infty} (x^2)^k - 1 \\
&= 2 \cdot \frac{x^2}{(1-x^2)^2} + \frac{1}{1-x^2} - 1 \\
&= \frac{3x^2 - x^4}{(1-x^2)^2}
\end{aligned}$$

Exercise A.1-6

Prove that $\sum_{k=1}^n O(f_k(i)) = O(\sum_{k=1}^n f_k(i))$ by using the linearity property of summations.

Answer:

$$\begin{aligned}
\sum_{k=1}^n O(f_k(i)) &\leq c_1 f_1(i) + c_2 f_2(i) + c_3 f_3(i) + \dots + c_n f_n(i) \\
&\leq \max(c_1, c_2, \dots, c_n) (f_1(i) + f_2(i) + f_3(i) + \dots + f_n(i)) \\
&= O\left(\sum_{k=1}^n f_k(i)\right)
\end{aligned}$$

Exercise A.1-7

Evaluate the product $\prod_{k=1}^n 2 \cdot 4^k$.

Answer:

$$\begin{aligned}
\prod_{k=1}^n 2 \cdot 4^k &= \prod_{k=1}^n 2^{2k+1} \\
&= 2^{(n^2+2n)}
\end{aligned}$$

Exercise A.1-8 ★

Evaluate the product $\prod_{k=2}^n (1 - 1/k^2)$.

Answer:

$$\begin{aligned}
\prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) &= \prod_{k=2}^n \frac{k-1}{k} \cdot \frac{k+1}{k} \\
&= \left(\frac{1}{2} \cdot \frac{3}{2}\right) \left(\frac{2}{3} \cdot \frac{4}{3}\right) \dots \left(\frac{n-1}{n} \cdot \frac{n+1}{n}\right) \\
&= \frac{1}{2} \left(\frac{3}{2} \cdot \frac{2}{3}\right) \left(\frac{4}{3} \cdot \frac{3}{4}\right) \dots \left(\frac{n}{n-1} \cdot \frac{n-1}{n}\right) \frac{n+1}{n} \\
&= \frac{n+1}{2n}
\end{aligned}$$

A.2 Bounding summations

Exercise A.2-1

Show that $\sum_{k=1}^n 1/k^2$ is bounded above by a constant.

Answer:

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{k^2} &= 1 + \sum_{k=2}^n \frac{1}{k^2} \\
&\leq 1 + \int_1^n \frac{1}{x^2} dx \\
&= 2 - \frac{1}{n} \\
&\leq 2
\end{aligned}$$

Exercise A.2-2

Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n \rfloor} \lceil n/2^k \rceil.$$

Answer:

$$\begin{aligned}
\sum_{k=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^k} \right\rceil &\leq \sum_{k=0}^{\lfloor \lg n \rfloor} \left(\frac{n}{2^k} + 1 \right) \\
&= n \sum_{k=0}^{\lfloor \lg n \rfloor} \left(\frac{1}{2^k} \right) + 1 + \lfloor \lg n \rfloor \\
&\leq n \sum_{k=0}^{\infty} \frac{1}{2^k} + 1 + \lg n \\
&= 2n + 1 + \lg n \\
&= O(n)
\end{aligned}$$

Exercise A.2-3

Show that the n th harmonic number is $\Omega(\lg n)$ by splitting the summation.

Answer:

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{\lfloor \lg n \rfloor - 1} + 1} + \dots + \frac{1}{2^{\lfloor \lg n \rfloor}}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{\lfloor \lg n \rfloor}} + \dots + \frac{1}{2^{\lfloor \lg n \rfloor}}\right) \\
&= 1 + \frac{1}{2} \lfloor \lg n \rfloor \\
&\geq 1 + \frac{1}{2} (\lg n - 1) \\
&= \Omega(\lg n)
\end{aligned}$$

Exercise A.2-4

Approximate $\sum_{k=1}^n k^3$ with an integral.

Answer:

$$\begin{aligned}
\int_0^n k^3 dx &\leq \sum_{k=1}^n k^3 \leq \int_1^{n+1} k^3 dx \\
\frac{n^4}{4} &\leq \sum_{k=1}^n k^3 \leq \frac{(n+1)^4 - 1}{4} \\
\sum_{k=1}^n k^3 &= \Theta(n^4)
\end{aligned}$$

Exercise A.2-5

Why didn't we use the integral approximation (A.12) ($\int_m^{n+1} f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x) dx$) directly on $\sum_{k=1}^n 1/k$ to obtain an upper bound on the n th harmonic number?

Answer:

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{k} &\leq \int_0^n \frac{1}{x} dx \\
&= \infty
\end{aligned}$$

Problems

Problem A-1 Bounding summations

Give asymptotically tight bounds on the following summations. Assume that $r \geq 0$ and $s \geq 0$ are constants.

a.

$$\sum_{k=1}^n k^r.$$

b.

$$\sum_{k=1}^n \lg^2 k.$$

c.

$$\sum_{k=1}^n k^r \lg^s k.$$

Answer:

a.

$$\begin{aligned} \sum_{k=\lfloor n/2 \rfloor + 1}^n \left(\frac{n}{2}\right)^r &\leq \sum_{k=1}^n k^r \leq \sum_{k=1}^n n^r \\ \left(\frac{n}{2}\right)^{r+1} &\leq \sum_{k=1}^n k^r \leq n^{r+1} \\ \sum_{k=1}^n k^r &= \Theta(n^{r+1}) \end{aligned}$$

b.

$$\begin{aligned} \sum_{k=\lfloor n/2 \rfloor + 1}^n \lg^s \frac{n}{2} &\leq \sum_{k=1}^n \lg^s k \leq \sum_{k=1}^n \lg^s n \\ \frac{n}{2} \lg^s \frac{n}{2} &\leq \sum_{k=1}^n \lg^s k \leq n \lg^s n \\ \frac{n}{2} \lg^s \frac{n}{\sqrt{n}} &\leq \sum_{k=1}^n \lg^s k \leq n \lg^s n \quad (n \geq 4) \\ \frac{1}{2^{s+1}} n \lg^s n &\leq \sum_{k=1}^n \lg^s k \leq n \lg^s n \\ \sum_{k=1}^n \lg^s k &= \Theta(n \lg^s n) \end{aligned}$$

c.

$$\sum_{k=1}^n k^r \lg^s k = \Theta(n^{r+1} \lg^s n)$$