A Summations

A.1 Summation formulas and properties

Exercise A.1-1

Find a simple formula for $\sum_{k=1}^{n} 2k - 1$.

Answer:

$$\begin{aligned}
\sum_{k=1}^{n} 2k - 1 &= 2\sum_{k=1}^{n} k - n \\
&= 2 \cdot \frac{n(n+1)}{2} - n \\
&= n
\end{aligned}$$

Exercise A.1-2 ★

Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln\sqrt{n} + O(1)$ by manipulating the harmonic series.

Answer:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{2k-1} &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \\
&= \ln 2n + O(1) - \frac{1}{2}\ln n - \frac{1}{2}O(1) \\
&= \ln\sqrt{n} + O(1)
\end{aligned}$$

Exercise A.1-3

Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

Answer:

$$\begin{aligned}
\sum_{k=0}^{\infty} kx^k &= \frac{x}{(1-x)^2} \\
\frac{d}{dx}\sum_{k=0}^{\infty} kx^k &= \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\
\sum_{k=0}^{\infty} k^2 x^{k-1} &= \frac{1+x}{(1-x)^3} \\
\sum_{k=0}^{\infty} k^2 x^k &= \frac{x(1+x)}{(1-x)^3}
\end{aligned}$$

Exercise A.1-4 ★

Show that $\sum_{k=0}^{\infty} (k-1)/2^k = 0$.

Answer:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k-1}{2^k} &= \sum_{k=0}^{\infty}k\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\ &= \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} - \frac{1}{1-\frac{1}{2}} \\ &= 0\end{aligned}$$

Exercise A.1-5 ★

Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

Answer:

$$\begin{aligned}\sum_{k=1}^{\infty}(2k+1)x^{2k} &= \sum_{k=0}^{\infty}(2k+1)x^{2k} - 1 \\ &= 2\sum_{k=0}^{\infty}k(x^2)^k + \sum_{k=0}^{\infty}(x^2)^k - 1 \\ &= 2\cdot\frac{x^2}{(1-x^2)^2} + \frac{1}{1-x^2} - 1 \\ &= \frac{3x^2-x^4}{(1-x^2)^2}\end{aligned}$$

Exercise A.1-6

Prove that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$ by using the linearity property of summations.

Answer:

$$\begin{aligned}\sum_{k=1}^{n} O(f_k(i)) &\leq c_1f_1(i) + c_2f_2(i) + c_3f_3(i) + \ldots + c_nf_n(i) \\ &\leq \max(c_1, c_2, \ldots, c_n)(f_1(i) + f_2(i) + f_3(i) + \ldots + f_n(i)) \\ &= O\left(\sum_{k=1}^{n} f_k(i)\right)\end{aligned}$$

Exercise A.1-7

Evaluate the product $\prod_{k=1}^{n} 2\cdot 4^k$.

Answer:

$$\begin{aligned}\prod_{k=1}^{n} 2\cdot 4^k &= \prod_{k=1}^{n} 2^{2k+1} \\ &= 2^{(n^2+2n)}\end{aligned}$$

Exercise A.1-8 ★

Evaluate the product $\prod_{k=2}^{n}(1-1/k^2)$.

Answer:

$$
\begin{aligned}
\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) &= \prod_{k=2}^{n}\frac{k-1}{k}\cdot\frac{k+1}{k} \\
&= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{n-1}{n}\cdot\frac{n+1}{n}\right) \\
&= \frac{1}{2}\left(\frac{3}{2}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{3}{4}\right)\cdots\left(\frac{n}{n-1}\cdot\frac{n-1}{n}\right)\frac{n+1}{n} \\
&= \frac{n+1}{2n}
\end{aligned}
$$

A.2 Bounding summations

Exercise A.2-1

Show that $\sum_{k=1}^{n} 1/k^2$ is bounded above by a constant.

Answer:

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k^2} &= 1+\sum_{k=2}^{n}\frac{1}{k^2} \\
&\le 1+\int_{1}^{n}\frac{1}{x^2}\,dx \\
&= 2-\frac{1}{n} \\
&\le 2
\end{aligned}
$$

Exercise A.2-2

Find an asymptotic upper bound on the summation

$$\sum_{k=0}^{\lfloor \lg n\rfloor}\lceil n/2^k\rceil .$$

Answer:

$$
\begin{aligned}
\sum_{k=0}^{\lfloor \lg n\rfloor}\left\lceil\frac{n}{2^k}\right\rceil &\le \sum_{k=0}^{\lfloor \lg n\rfloor}\left(\frac{n}{2^k}+1\right) \\
&= n\sum_{k=0}^{\lfloor \lg n\rfloor}\left(\frac{1}{2^k}\right)+1+\lfloor \lg n\rfloor \\
&\le n\sum_{k=0}^{\infty}\frac{1}{2^k}+1+\lg n \\
&= 2n+1+\lg n \\
&= O(n)
\end{aligned}
$$

Exercise A.2-3

Show that the nth harmonic number is $\Omega(\lg n)$ by splitting the summation.

Answer:

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{\lfloor \lg n \rfloor - 1} + 1} + \cdots + \frac{1}{2^{\lfloor \lg n \rfloor}}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{\lfloor \lg n \rfloor}} + \cdots + \frac{1}{2^{\lfloor \lg n \rfloor}}\right) \\
&= 1 + \frac{1}{2} \lfloor \lg n \rfloor \\
&\geq 1 + \frac{1}{2} (\lg n - 1) \\
&= \Omega(\lg n)
\end{aligned}
$$

Exercise A.2-4

Approximate $\sum_{k=1}^{n} k^3$ with an integral.

Answer:

$$
\begin{aligned}
\int_0^n k^3 dx \leq & \sum_{k=1}^{n} k^3 \leq \int_1^{n+1} k^3 dx \\
\frac{n^4}{4} \leq & \sum_{k=1}^{n} k^3 \leq \frac{(n+1)^4 - 1}{4} \\
& \sum_{k=1}^{n} k^3 = \Theta(n^4)
\end{aligned}
$$

Exercise A.2-5

Why didn't we use the integral approximation (A.12) ($\int_m^{n+1} f(x) dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x) dx$) directly on $\sum_{k=1}^{n} 1/k$ to obtain an upper bound on the nth harmonic number?

Answer:

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &\leq \int_0^n \frac{1}{x} dx \\
&= \infty
\end{aligned}
$$

Problems

Problem A-1 Bounding summations

Give asymptotically tight bounds on the following summations. Assume that $r \geq 0$ and $s \geq 0$ are constants.

a.

$$\sum_{k=1}^{n} k^r .$$

b.

$$\sum_{k=1}^{n} \lg^2 k .$$

c.

$$\sum_{k=1}^{n} k^r \lg^s k .$$

Answer:

a.

$$
\begin{aligned}
\sum_{k=\lfloor n/2 \rfloor +1}^{n} \left(\frac{n}{2}\right)^{r} \;&\leq\; \sum_{k=1}^{n} k^{r} \;\leq\; \sum_{k=1}^{n} n^{r} \\
\left(\frac{n}{2}\right)^{r+1} \;&\leq\; \sum_{k=1}^{n} k^{r} \;\leq\; n^{r+1} \\
\sum_{k=1}^{n} k^{r} \;&=\; \Theta(n^{r+1})
\end{aligned}
$$

b.

$$
\begin{aligned}
\sum_{k=\lfloor n/2 \rfloor +1}^{n} \lg^{s} \frac{n}{2} \;&\leq\; \sum_{k=1}^{n} \lg^{s} k \;\leq\; \sum_{k=1}^{n} \lg^{s} n \\
\frac{n}{2} \lg^{s} \frac{n}{2} \;&\leq\; \sum_{k=1}^{n} \lg^{s} k \;\leq\; n \lg^{s} n \\
\frac{n}{2} \lg^{s} \frac{n}{\sqrt{n}} \;&\leq\; \sum_{k=1}^{n} \lg^{s} k \;\leq\; n \lg^{s} n \qquad (n \geq 4) \\
\frac{1}{2^{s+1}} n \lg^{s} n \;&\leq\; \sum_{k=1}^{n} \lg^{s} k \;\leq\; n \lg^{s} n \\
\sum_{k=1}^{n} \lg^{s} k \;&=\; \Theta(n \lg^{s} n)
\end{aligned}
$$

c.

$$
\sum_{k=1}^{n} k^{r} \lg^{s} k = \Theta(n^{r+1} \lg^{s} n)
$$