Chapter 3 Growth of Functions

3.1 Asymptotic notation

Exercise 3.1-1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Answer:*

First, let's clarify what the function $\max(f(n), g(n))$ is. Let's define the function $h(n) = \max(f(n), g(n))$. Then

$$h(n) = \begin{cases} f(n) & \text{if } f(n) \geq g(n), \\ g(n) & \text{if } f(n) < g(n). \end{cases}$$

Since $f(n)$ and $g(n)$ are asymptotically nonnegative, there exists n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. Thus for $n \geq n_0$, $f(n) + g(n) \geq f(n) \geq 0$ and $f(n) + g(n) \geq g(n) \geq 0$. Since for any particular n, $h(n)$ is either $f(n)$ or $g(n)$, we have $f(n) + g(n) \geq h(n) \geq 0$, which shows that $h(n) = \max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq n_0$ (with $c_2 = 1$ in the definition of Θ).

Similarly, since for any particular n, $h(n)$ is the larger of $f(n)$ and $g(n)$, we have for all $n \geq n_0$, $0 \leq f(n) \leq h(n)$ and $0 \leq g(n) \leq h(n)$. Adding these two inequalities yields $0 \leq f(n) + g(n) \leq 2h(n)$, or equivalently $0 \leq (f(n) + g(n))/2 \leq h(n)$, which shows that $h(n) = \max(f(n), g(n)) \geq c_1(f(n) + g(n))$ for all $n \geq n_0$ (with $c_1 = 1/2$ in the definition of Θ).

Exercise 3.1-2

Show that for any real constants a and b, where $b > 0$, $(n + a)^b = \Theta(n^b)$.

Answer:*

To show that $(n + a)^b = \Theta(n^b)$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1 n^b \leq (n + a)^b \leq c_2 n^b$ for all $n \geq n_0$.

Note that

$$\begin{aligned} n + a &\leq n + |a| \\ &\leq 2n \qquad \text{when } |a| \leq n, \end{aligned}$$

and

$$\begin{aligned} n + a &\geq n - |a| \\ &\geq \frac{1}{2}n \qquad \text{when } |a| \leq \frac{1}{2}n. \end{aligned}$$

Thus, when $n \geq 2|a|$,

$$0 \leq \frac{1}{2}n \leq n + a \leq 2n.$$

Since $b > 0$, the inequality still holds when all parts are raised to the power b:

$$0 \leq \left(\frac{1}{2}n\right)^b \leq (n + a)^b \leq (2n)^b,$$

$$0 \leq \left(\frac{1}{2}\right)^b n^b \leq (n + a)^b \leq 2^b n^b.$$

Thus, $c_1 = (1/2)^b$, $c_2 = 2^b$, and $n_0 = 2|a|$ satisfy the definition.

Exercise 3.1-3

Explain why the statement, "The running time of algorithm A is at least $O(n^2)$," is meaningless.

Answer:*

Let the running time be $T(n)$. $T(n) \geq O(n^2)$ means that $T(n) \geq f(n)$ for some function $f(n)$ in the set $O(n^2)$. This statement holds for any running time $T(n)$, since the function $g(n) = 0$ for all n is in $O(n^2)$, and running times are always nonnegative. Thus, the statement tells us nothing about the running time.

Exercise 3.1-4

Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

Answer:*

$2^{n+1} = O(2^n)$, but $2^{2n} \neq O(2^n)$.

To show that $2^{n+1} = O(2^n)$, we must find constants $c, n_0 > 0$ such that

$0 \leq 2^{n+1} \leq c \cdot 2^n$ for all $n \geq n_0$.

Since $2^{n+1} = 2 \cdot 2^n$ for all n, we can satisfy the definition with $c = 2$ and $n_0 = 1$.

To show that $2^{2n} \neq O(2^n)$, assume there exist constants $c, n_0 > 0$ such that

$0 \leq 2^{2n} \leq c \cdot 2^n$ for all $n \geq n_0$.

Then $2^{2n} = 2^n \cdot 2^n \leq c \cdot 2^n \Rightarrow 2^n \leq c$. But no constant is greater than all 2^n, and so the assumption leads to a contradiction.

Exercise 3.1-5

Prove Theorem 3.1.

(For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(n)$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.)

Exercise 3.1-6

Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

Exercise 3.1-7

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Exercise 3.1-8

We can extend our notation to the case of two parameters n and m that can go to infinity independently at different rates. For a given function $g(n, m)$, we denote by $O(g(n, m))$ the set of functions

$$
\begin{aligned}
O(g(n,m)) = \{f(n,m) : \ & \text{there exist positive constants } c, n_0, \text{ and } m_0 \\
& \text{such that } 0 \leq f(n,m) \leq cg(n,m) \\
& \text{for all } n \geq n_0 \text{ or } m \geq m_0\}.
\end{aligned}
$$

Give corresponding definitions for $\Omega(g(n, m))$ and $\Theta(g(n, m))$.

Answer:*

$$
\begin{aligned}
\Omega(g(n,m)) = \{f(n,m) :\ & \text{there exist positive constants } c, n_0, \text{ and } m_0 \\
& \text{such that } 0 \le cg(n,m) \le f(n,m) \\
& \text{for all } n \ge n_0 \text{ or } m \ge m_0\}. \\
\Theta(g(n,m)) = \{f(n,m) :\ & \text{there exist positive constants } c_1, c_2, n_0, \text{ and } m_0 \\
& \text{such that } 0 \le c_1 g(n,m) \le f(n,m) \le c_2 g(n,m) \\
& \text{for all } n \ge n_0 \text{ or } m \ge m_0\}.
\end{aligned}
$$

3.2 Standard notations and common functions

Exercise 3.2-1

Show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so are the functions $f(n) + g(n)$ and $f(g(n))$, and if $f(n)$ and $g(n)$ are in addition nonnegative, then $f(n) \cdot g(n)$ is monotonically increasing.

Exercise 3.2-2

Prove equation (3.16) ($a^{\log_b c} = c^{\log_b a}$).

Answer:

$$
\begin{aligned}
a^{\log_b c} &= c^{\log_c a \cdot \log_b c} \\
&= c^{\log_b \left(c^{\log_c a}\right)} \\
&= c^{\log_b a}
\end{aligned}
$$

Exercise 3.2-3

Prove equation (3.19) ($\lg(n!) = \Theta(n \lg n)$). Also prove that $n! = \omega(2^n)$ and $n! = o(n^n)$.

Answer:

$$
\begin{aligned}
\lg(n!) &= \lg \left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) \right) \\
&= \lg \sqrt{2\pi n} + \lg \left(\frac{n}{e}\right)^n + \lg \left(1 + \Theta\left(\frac{1}{n}\right)\right) \\
&= \Theta(\lg n) + \Theta(n \lg n) + O(1) \\
&= \Theta(n \lg n)
\end{aligned}
$$

$$
\begin{aligned}
n! &= 4! \cdot \prod_{k=5}^{n-1} k \cdot n \\
&> 2^4 \cdot \prod_{k=5}^{n-1} 2 \cdot 2c \qquad \left(n > \frac{c}{2}, n \ge 5\right) \\
&= 2^n c \\
&= \omega(2^n)
\end{aligned}
$$

$$
\begin{aligned}
n! &= 1 \cdot \prod_{k=2}^{n} k \\
&< cn \cdot \prod_{k=2}^{n} n \qquad \left(n \ge 3, n > \frac{1}{c}\right) \\
&= cn^n \\
&= o(n^n)
\end{aligned}
$$

Exercise 3.2-4 ★

Is the function $\lceil \lg n \rceil !$ polynomially bounded? Is the function $\lceil \lg \lg n \rceil !$ polynomially bounded?

Answer:*

$\lceil \lg n \rceil !$ is not polynomially bounded, but $\lceil \lg \lg n \rceil !$ is.

Proving that a function $f(n)$ is polynomially bounded is equivalent to proving that $\lg(f(n)) = O(\lg n)$ for the following reasons.

- If f is polynomially bounded, then there exist constants c, k, n_0 such that for all $n \geq n_0$, $f(n) \leq cn^k$. Hence, $\lg(f(n)) \leq kc \lg n$, which, since c and k are constants, means that $\lg(f(n)) = O(\lg n)$.
- Similarly, if $\lg(f(n)) = O(\lg n)$, then f is polynomially bounded.

In the following proofs, we will make use of the following two facts:

1. $\lg(n!) = \Theta(n \lg n)$ (by equation (3.19)).
2. $\lceil \lg n \rceil = \Theta(\lg n)$, because

- $\lceil \lg n \rceil \geq \lg n$
- $\lceil \lg n \rceil < \lg n + 1 \leq 2 \lg n$ for all $n \geq 2$

$$
\begin{aligned}
\lg(\lceil \lg n \rceil !) &= \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) \\
&= \Theta(\lg n \lg \lg n) \\
&= \omega(\lg n)
\end{aligned}
$$

Therefore, $\lg(\lceil \lg n \rceil !) \neq O(\lg n)$, and so $\lceil \lg n \rceil !$ is not polynomially bounded.

$$
\begin{aligned}
\lg(\lceil \lg \lg n \rceil !) &= \Theta(\lceil \lg \lg n \rceil \lg \lceil \lg \lg n \rceil) \\
&= \Theta(\lg \lg n \lg \lg \lg n) \\
&= o((\lg \lg n)^2) \\
&= o(\lg^2(\lg n)) \\
&= o(\lg n).
\end{aligned}
$$

The last step above follows from the property that any polylogarithmic function grows more slowly than any positive polynomial function, i.e., that for constants $a, b > 0$, we have $\lg^b n = o(n^a)$. Substitute $\lg n$ for n, 2 for b, and 1 for a, giving $\lg^2(\lg n) = o(\lg n)$.

Therefore, $\lg(\lceil \lg \lg n \rceil !) = O(\lg n)$, and so $\lceil \lg \lg n \rceil !$ is polynomially bounded.

Exercise 3.2-5 ★

Which is asymptotically larger: $\lg(\lg^* n)$ or $\lg^*(\lg n)$?

Answer:*

$\lg^*(\lg n)$ is asymptotically larger because $\lg^*(\lg n) = \lg^* n - 1$.

Exercise 3.2-6

Show that the golden ratio ϕ and its conjugate $\hat{\phi}$ both satisfy the equation $x^2 = x + 1$.

Answer:*

Both ϕ^2 and $\phi + 1$ equal $(3 + \sqrt{5})/2$, and both $\hat{\phi}^2$ and $\hat{\phi} + 1$ equal $(3 - \sqrt{5})/2$.

Exercise 3.2-7

Prove by induction that the ith Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}},$$

where ϕ is the golden ratio and $\widehat{\phi}$ is its conjugate.

Answer:*

We have two base cases: $i = 0$ and $i = 1$. For $i = 0$, we have

$$\begin{aligned}
\frac{\phi^0 - \widehat{\phi}^0}{\sqrt{5}} &= \frac{1-1}{\sqrt{5}} \\
&= 0 \\
&= F_0
\end{aligned}$$

and for $i = 1$, we have

$$\begin{aligned}
\frac{\phi^1 - \widehat{\phi}^1}{\sqrt{5}} &= \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} \\
&= \frac{2\sqrt{5}}{2\sqrt{5}} \\
&= 1 \\
&= F_1
\end{aligned}$$

For the inductive case, the inductive hypothesis is that $F_{i-1} = (\phi^{i-1} - \widehat{\phi}^{i-1})/\sqrt{5}$ and $F_{i-2} = (\phi^{i-2} - \widehat{\phi}^{i-2})/\sqrt{5}$. We have

$$\begin{aligned}
F_i &= F_{i-1} + F_{i-2} && \text{(equation (3.22))} \\
&= \frac{\phi^{i-1} - \widehat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \widehat{\phi}^{i-2}}{\sqrt{5}} && \text{(inductive hypothesis)} \\
&= \frac{\phi^{i-2}(\phi + 1) - \widehat{\phi}^{i-2}(\widehat{\phi} + 1)}{\sqrt{5}} \\
&= \frac{\phi^{i-2}\phi^2 - \widehat{\phi}^{i-2}\widehat{\phi}^2}{\sqrt{5}} && \text{(Exercise 3.2-6)} \\
&= \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}}.
\end{aligned}$$

Exercise 3.2-8

Show that $k \ln k = \Theta(n)$ implies $k = \Theta(n/\ln n)$.

Answer:

$$\begin{aligned}
n &= \Theta(k \ln k) \\
\ln n &= \Theta(\ln k + \ln \ln k) \\
&= \Theta(\ln k) \\
\frac{n}{\ln n} &= \Theta\left(\frac{k \ln k}{\ln k}\right) \\
&= \Theta(k) \\
k &= \Theta\left(\frac{n}{\ln n}\right)
\end{aligned}$$

Problems

Problem 3-1 Asymptotic behavior of polynomials

Let

$$p(n) = \sum_{i=0}^{d} a_i n^i,$$

where $a_d > 0$, be a degree-d polynomial in n, and let k be a constant. Use the definitions of the asymptotic notations to prove the following properties.

a. If $k \geq d$, then $p(n) = O(n^k)$.

b. If $k \leq d$, then $p(n) = \Omega(n^k)$.

c. If $k = d$, then $p(n) = \Theta(n^k)$.

d. If $k > d$, then $p(n) = o(n^k)$.

e. If $k < d$, then $p(n) = \omega(n^k)$.

Problem 3-2 Relative asymptotic growths

Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon \leq 1$ and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
$a.$	$\lg^k n$	n^ϵ	Yes	Yes	No	No	No
$b.$	n^k	c^n	Yes	Yes	No	No	No
$c.$	$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
$d.$	2^n	$2^{n/2}$	No	No	Yes	Yes	No
$e.$	$n^{\lg c}$	$c^{\lg n}$	Yes	No	Yes	No	Yes
$f.$	$\lg n!$	$\lg(n^n)$	Yes	No	Yes	No	Yes

Problem 3-3 Ordering by asymptotic growth rates

a. Rank the following functions by order of growth; that is, find an arrangement $g_1, g_2, \ldots, g_{30}$ of the functions satisfying $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{29} = \Omega(g_{30})$. Partition your list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$.

b. Give an example of a single nonnegative function $f(n)$ such that for all functions $g_i(n)$ in part (a), $f(n)$ is neither $O(g_i(n))$ nor $\Omega(g_i(n))$.

Answer:*

a.

Here is the ordering, where functions on the same line are in the same equivalence class, and those higher on the page are Ω of those below them:

$2^{2^{n+1}}$	
2^{2^n}	
$(n+1)!$	
$n!$	see justification 7
e^n	see justification 1
$n \cdot 2^n$	
2^n	
$(3/2)^n$	
$(\lg n)^{\lg n} = n^{\lg \lg n}$	see identity 1
$(\lg n)!$	see justification 2,8
n^3	
$n^2 = 4^{\lg n}$	see identity 2
$n \lg n$ and $\lg(n!)$	see justification 6
$n = 2^{\lg n}$	see identity 3
$(\sqrt{2})^{\lg n} (= \sqrt{n})$	see identity 6,justification 3
$2^{\sqrt{2 \lg n}}$	see identity 5,justification 4
$\lg^2 n$	
$\ln n$	
$\sqrt{\lg n}$	
$\ln \ln n$	see justification 5
$2^{\lg^* n}$	
$\lg^* n$ and $\lg^*(\lg n)$	see identity 7
$\lg(\lg^*)n$	
$n^{1/\lg n} (= 2)$ and 1	see identity 4

Much of the ranking is based on the following properties:

- Exponential functions grow faster than polynomial functions, which grow faster than polylogarithmic functions.
- The base of a logarithm doesn't matter asymptotically, but the base of an exponential and the degree of a polynomial do matter.

We have the following *identities*:

1. $(\lg n)^{\lg n} = n^{\lg \lg n}$ because $a^{\log_b c} = c^{\log_b a}$.
2. $4^{\lg n} = n^2$ because $a^{\log_b c} = c^{\log_b a}$.
3. $2^{\lg n} = n$.
4. $2 = n^{1/\lg n}$ by raising identity 3 to the power $1/\lg n$.
5. $2^{\sqrt{2 \lg n}} = n^{\sqrt{2/\lg n}}$ by raising identity 4 to the power $\sqrt{2 \lg n}$.
6. $(\sqrt{2})^{\lg n} = \sqrt{n}$ because $(\sqrt{2})^{\lg n} = 2^{(1/2)\lg n} = 2^{\lg \sqrt{n}} = \sqrt{n}$.
7. $\lg^*(\lg n) = \lg^* n - 1$.

The following *justifications* explain some of the rankings:

1. $e^n = 2^n (e/2)^n = \omega(n 2^n)$, since $(e/2)^n = \omega(n)$.
2. $(\lg n)! = \omega(n^3)$ by taking logs: $\lg(\lg n)! = \Theta(\lg n \lg \lg n)$ by Stirling's approximation, $\lg(n^3) = 3 \lg n$. $\lg \lg n = \omega(3)$.
3. $(\sqrt{2})^{\lg n} = \omega\left(2^{\sqrt{2 \lg n}}\right)$ by taking logs: $\lg(\sqrt{2})^{\lg n} = (1/2) \lg n$, $\lg 2^{\sqrt{2 \lg n}} = \sqrt{2 \lg n}$. $(1/2) \lg n = \omega(\sqrt{2 \lg n})$.

4. $2^{\sqrt{2\lg n}} = \omega(\lg^2 n)$ by taking logs: $\lg 2^{\sqrt{2\lg n}} = \sqrt{2\lg n}$, $\lg\lg^2 n = 2\lg\lg n$. $\sqrt{2\lg n} = \omega(2\lg\lg n)$.
5. $\ln\ln n = \omega(2^{\lg^* n})$ by taking logs: $\lg 2^{\lg^* n} = \lg^* n$. $\lg\ln\ln n = \omega(\lg^* n)$.
6. $\lg(n!) = \Theta(n\lg n)$ (equation (3.19)).
7. $n! = \Theta(n^{n+1/2}e^{-n})$ by dropping constants and low-order terms in equation (3.18) ($n! = \sqrt{2\pi n}\,(n/e)^n(1+\Theta(1/n))$).
8. $(\lg n)! = \Theta((\lg n)^{\lg n+1/2}e^{-\lg n})$ by substituting $\lg n$ for n in the previous justification. $(\lg n)! = \Theta((\lg n)^{\lg n+1/2}n^{-\lg e})$ because $a^{\log_b c} = c^{\log_b a}$.

b.

The following $f(n)$ is nonnegative, and for all functions $g_i(n)$ in part (a), $f(n)$ is neither $O(g_i(n))$ nor $\Omega(g_i(n))$.

$$f(n) = \begin{cases} 2^{2^{n+2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Problem 3-4 Asymptotic notation properties

Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

a. $f(n) = O(g(n))$ implies $g(n) = O(f(n))$. (False)

b. $f(n) + g(n) = \Theta(\min(f(n), g(n)))$. (False)

c. $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n. (True)

d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$. (False)

e. $f(n) = O(f(n)^2)$. (False)

f. $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$. (True)

g. $f(n) = \Theta(f(n/2))$. (False)

h. $f(n) + o(f(n)) = \Theta(f(n))$. (True)

Problem 3-5 Variations on O and Ω

Some authors define Ω in a slightly different way than we do; let's use $\overset{\infty}{\Omega}$ (read "omega infinity") for this alternative definition. We say that $f(n) = \overset{\infty}{\Omega}(g(n))$ if there exists a positive constant c such that $f(n) \geq cg(n) \geq 0$ for infinitely many integers n.

a. Show that for any two functions $f(n)$ and $g(n)$ that are asymptotically nonnegative, either $f(n) = O(g(n))$ or $f(n) = \overset{\infty}{\Omega}(g(n))$ or both, whereas this is not true if we use Ω in place of $\overset{\infty}{\Omega}$.

b. Describe the potential advantages and disadvantages of using $\overset{\infty}{\Omega}$ instead of Ω to characterize the running times of programs.

Some authors also define O in a slightly different manner; let's use O' for the alternative definition. We say that $f(n) = O'(g(n))$ if and only if $|f(n)| = O(g(n))$.

c. What happens to each direction of the "if and only if" in Theorem 3.1 if we substitute O' for O but still use Ω?

Some authors define $\widetilde{O}$ (read "soft-oh") to mean O with logarithmic factors ignored:

$$\widetilde{O}(g(n)) = \{f(n) : \text{ there exist positive constants } c, k, \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n)\lg^k(n) \text{ for all } n \ge n_0\}.$$

d. Define $\widetilde{\Omega}$ and $\widetilde{\Theta}$, in a similar manner. Prove the corresponding analog to Theorem 3.1.

Answer:

a.

$$f(n) \ne \overset{\infty}{\Omega}(g(n)) \Rightarrow f(n) = O(g(n))$$

$$f(n) \ne O(g(n)) \Rightarrow f(n) = \overset{\infty}{\Omega}(g(n))$$

c.

Does not change.

d.

$$\widetilde{\Omega}(g(n)) = \{f(n) : \text{ there exist positive constants } c, k, \text{ and } n_0 \text{ such that } 0 \le cg(n)\lg^k(n) \le f(n) \text{ for all } n \ge n_0\}$$

$$\widetilde{\Theta}(g(n)) = \{f(n) : \text{ there exist positive constants } c_1, c_2, k_1, k_2 \text{ and } n_0 \text{ such that } 0 \le c_1 g(n)\lg^{k_1}(n) \le f(n) \le c_2 g(n)\lg^{k_2}(n) \text{ for all } n \ge n_0\}$$

Problem 3-6 Iterated functions

We can apply the iteration operator $*$ used in the $\lg^*$ function to any monotonically increasing function $f(n)$ over the reals. For a given constant $c \in \mathbb{R}$, we define the iterated function f_c^* by

$$f_c^*(n) = \min\{i \ge 0 : f^{(i)}(n) \le c\},$$

which need not be well defined in all cases. In other words, the quantity $f_c^*(n)$ is the number of iterated applications of the function f required to reduce its argument down to c or less.

For each of the following functions $f(n)$ and constants c, give as tight a bound as possible on f_c^*.

Answer:

	$f(n)$	c	$f_c^*(n)$
$a.$	$n-1$	0	$\lceil n \rceil$
$b.$	$\lg n$	1	$\lg^* n$
$c.$	$n/2$	1	$\lceil \lg n \rceil$
$d.$	$n/2$	2	$\lceil \lg n \rceil - 1$
$e.$	$\sqrt{n}$	2	$\lg\lg n$
$f.$	$\sqrt{n}$	1	∞
$g.$	$n^{1/3}$	2	$\log_3 \lg n$
$h.$	$n/\lg n$	2	$\Omega(\lg n/\lg\lg n)$