DATA605

Week1 and Week2 Lecture

Vectors

What are vectors?

- Vectors of n-dimensions are an ordered collection of n numbers.
- These are sometimes called components
- Dimension is n
- They represent a direction and magnitude.
- They are very useful for conveniently representing many quantities.

```
> v <- c(1,4,-3);
> v
[1]  1  4 -3
```

Operations on Vectors

- Addition
  - Addition of vectors happens component by component
- Multiplication by scalar
  - Multiply each component by scalar

```
> v <- c(1,4,-3);
> v
[1]  1  4 -3

> x <- c(3, 4, 5)

> v + x
[1] 4 8 2

> 5 * x
[1] 15 20 25
```

Linear Combination of Vectors

- General form of mult. by scalars and addition of vectors.
- The following in a linear combination of n vectors (which need to be all the same dimension)
    - $\mathbf{V} = a_1 * \mathbf{v_1} + a_2 * \mathbf{v_2} + \ldots a_n * \mathbf{v_n}$

```
> 5 * x + 2 * v
[1] 17 28 19
```

Length, Dot Product

The dot or inner product of two vectors is a scalar that results from summation of multiplication component by component

The norm or length of a vector is the distance from the zero point the vector is. This can be found by using the sqrt of the dot product of the vector to itself

```
> t(v) %*% x
     [,1]
[1,]  4

> t(x) %*% v
     [,1]
[1,]  4

> unlist(t(x) %*% v)
     [,1]
[1,]  4

> (t(x) %*% v)[1]
[1] 4

> y <- c(1,0,0)
> sqrt(t(y) %*% y)
     [,1]
[1,]  1
> y <- c(2,3,0)
> sqrt(t(y) %*% y)
     [,1]
[1,] 3.605551
> y <- c(3,4,0)
> sqrt(t(y) %*% y)
     [,1]
[1,]  5
```

Matrices

In some sense a vector is a matrix with only one dimension. A matrix is a m x n collection of numbers, with m being the number of rows and n being the number of columns.

We can think of matrices as either n column-vectors (each vector being 1 column of the matrix) concatenated together or as m row-vectors concatenated together to form the matrix.

```
> a = matrix(c(1,2,3,4,5,6), ncol=3)
> a
     [,1] [,2] [,3]
[1,] 1    3    5
[2,] 2    4    6

> a = matrix(c(1,2,3,4,5,6), ncol=3,
byrow=T)

> a
     [,1] [,2] [,3]
[1,] 1    2    3
[2,] 4    5    6
```

Matrix Multiplication by Vector

You can multiply an m x n matrix A with an m-dimensional vector x as follows:

- In the resulting m dimensional vector, the ith component is just the result of the dot product between ith row of the matrix and the n-dimensional vector
- Pages 21 - 24 have good explanation
- Use the %*% operator in R

```
> a
     [,1] [,2] [,3]
[1,] 1    2    3
[2,] 4    5    6

> a %*% c(3,2,1)
     [,1]
[1,]   10
[2,]   28
```

Matrix by Matrix Mult

For any multiplication with matrices, you need to remember two things:

- Not communitive, so order matters
- Dimensions have to match:
  (m x **n**) * (**n** x r) = (m x r)
    - Ie. the n's need to be the same

The result of multiplying **A** and **B**, **AB,** is generally a matrix where the component at row i and column j is the dot product of row i in **A** and column j in **B**.

Page 26 has a good picture

```
> a = matrix(c(1,2,3,4,5,6), ncol=3,
byrow=T)
> a
     [,1] [,2] [,3]
[1,] 1    2    3
[2,] 4    5    6

> b = matrix(c(1,1,1,2,2,2), nrow=3)
> b
     [,1] [,2]
[1,] 1    2
[2,] 1    2
[3,] 1    2

> a %*% b
     [,1] [,2]
[1,] 6   12
[2,]   15   30
```

Matrix Mult Diagram

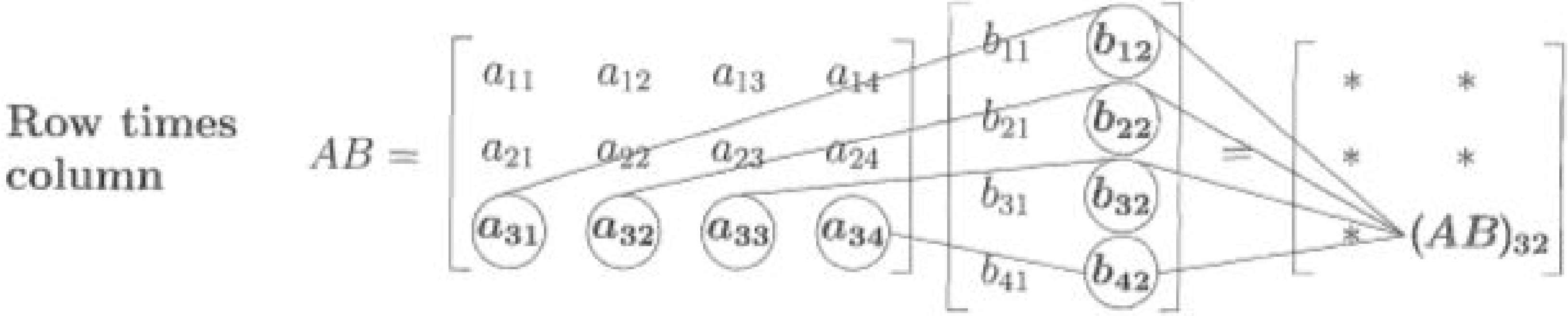

Figure 1.7: A 3 by 4 matrix A times a 4 by 2 matrix B is a 3 by 2 matrix AB.

Laws of Matrix Mult

Matrices follow these basic laws below:

- A + B = B + A
- c(A + B) = cA + cB
- A + (B + C) = (A + B) + C
- C(A + B) = CA + CB
- (A + B)C = AC + BC
- **AB != BA**
- (AB)C = A(BC)

Special Matrices

- **Square** - matrix thats is n x n
- **Diagonal** - Matrix with values in diagonal positions
- **Identity** - A Diagonal where the elements along the diagonal are 1. Denoted as **I**
- **Transpose** - To take a transpose of a matrix, you flip the coordinates of each component (i.e flip over diagonal)
- **Symmetric** - A matrix is symmetric when it's equal to its own transpose
- **Inverse** - The inverse of matrix **A** is the matrix **B** such that **AB = I**
    - Note: $(\mathbf{AB})^{-1} = (\mathbf{B}^{-1})(\mathbf{A}^{-1})$

```
> diag(3)
     [,1] [,2] [,3]
[1,] 1    0    0
[2,] 0    1    0
[3,] 0    0    1
```

Linear Equations

One of the reasons we care about all of this is that we can use matrices to represent systems of linear equation. These equations can be represented in the classical form: **Ax = b** where **A** is the coefficient matrix and b is the set of constraints that are imposed on it.

```
> A = matrix(c(1,1,2,-1), nrow=2,
byrow=T)
> A
     [,1] [,2]
[1,] 1    1
[2,] 2   -1

> b = c(1,2)

> solve(A,b)
[1] 1 0
```

Elimination

We are trying to transform the coeff matrix in such a way that it preserves the linear equation it represents. We are allowed to do only a few things, and each operation can be represented by a matrix:

1. You may swap the positions of any two rows
2. You may multiply any row by a non-zero scalar value (we tend to not use this unless we are getting the reduced row echelon form)
3. You may add a scalar multiple of any one row to another row

Elimination Overview

(1) Start with row 1 of the co-efficient matrix

(2) Pivot: The first non-zero element in the row being evaluated

(3) Multiplier: The element being eliminated divided by the Pivot

(4) Subtract Multiplier times row n from row n+1

(5) Advance to the next row and repeat

Elimination Matrices

Pg 24: The identity matrix I, with 1s on the diagonal and 0s everywhere else, leaves every vector unchanged. The elementary matrix $\mathbf{E}_{ij}$ subtracts ℓ times row j from row i. This $\mathbf{E}_{ij}$ includes $-\ell$ in row i, column j.

```
> A <- matrix(c(1,2,3,1,1,1,2,0,1),nrow=3)
> A
     [,1] [,2] [,3]
[1,] 1    1    2
[2,] 2    1    0
[3,] 3    1    1

> E21 = matrix(c(1,-2,0,0,1,0,0,0,1),nrow=3)
> E21
     [,1] [,2] [,3]
[1,] 1    0    0
[2,] -2   1    0
[3,] 0    0    1

> E21 <- diag(3)
> E21[2,1] <-  -2
> E21
     [,1] [,2] [,3]
[1,] 1    0    0
[2,] -2   1    0
[3,] 0    0    1
```

Elimination Matrices

You can use this matrix to emulate the third rule:
“You may add a scalar multiple of any one row to another row”

```
> E21 %*% A
     [,1] [,2] [,3]
[1,] 1    1    2
[2,] 0   -1   -4
[3,] 3    1    1
```

Permutation Matrix

Sometimes we need to swap rows. This is really only needed when your pivot value is 0. You need to swap a row such that the pivot will be non-zero. This may not be possible, in which case your matrix is **singular**.

We can form a permutation matrix (p42): A permutation matrix P has the same rows as the identity (*in some order*). There is a single “1” in every row and column.

```
> P <- matrix(c(0, 1, 0, 1, 0, 0, 0, 0,
1), ncol=3)
> P
     [,1] [,2] [,3]
[1,] 0    1    0
[2,] 1    0    0
[3,] 0    0    1
> A
     [,1] [,2] [,3]
[1,] 1    1    2
[2,] 2    1    0
[3,] 3    1    1

> P %*% A
     [,1] [,2] [,3]
[1,] 2    1    0
[2,] 1    1    2
[3,] 3    1    1
```

Elimination Example

Check notes for Week2 on a worked example in R

Elimination Links

Some useful links (also at the end)

- https://www.sophia.org/tutorials/gaussian-elimination-and-matrix-equations
- http://mathworld.wolfram.com/GaussianElimination.html
- http://www.gregthatcher.com/Mathematics/GaussJordan.aspx ← calculator

LU Factorization

Some more useful definitions:

- **Upper Triangular:** a matrix that has non-zero values along diagonal plus positions ‘above’ it
- **Lower Triangular:** a matrix that has non-zero values along diagonal plus positions ‘above’ it

Check notes for Week2 on a worked example in R

Inverses

The inverse of matrix **A** is the matrix **B** such that **AB = I**

Useful to remember: $(\mathbf{AB})^{-1} = (\mathbf{B}^{-1})(\mathbf{A}^{-1})$

The Calculation of A^{-1} : The Gauss-Jordan Method

Page 52 shows how to do this; essentially elimination but we augment the coeff matrix with the appropriately sized Identity matrix

Vector Spaces

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers. Addition and multiplication must produce vectors in the space, and follow these properties:

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. There is a unique "zero vector" such that $x + 0 = x$ for all x.
4. For each x there is a unique vector $-x$ such that $x + (-x) = 0$.
5. $1x = x$.
6. $(c_1 c_2)x = c_1(c_2 x)$.
7. $c(x + y) = cx + cy$.
8. $(c_1 + c_2)x = c_1 x + c_2 x$.

Vector Spaces and Subspaces

A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space:

- Linear combinations stay in the subspace.
- Geometrically, think of the usual three-dimensional R 3 and choose any plane through the origin. That plane is a vector space in its own right.
- Notice in particular that the zero vector will belong to every subspace

Column Space

Pg79: We now come to the key examples, the column space and the nullspace of a matrix **A**. The **column space** contains all linear combinations of the columns of **A**. It is a subspace of $\mathbf{R}^m$.

- In some sense, the column space is the 'range' of vectors than can come from matrix **A** being multiplied by vector **x**. Or possible values of **b** in the **Ax = b** equation

Null Space

The second approach to **Ax = b** is “dual” to the first. We are concerned not only with attainable right-hand sides b, but also with the solutions x that attain them. The right-hand side **b = 0** always allows the solution **x = 0**, but there may be infinitely many other solutions. (There always are, if there are more unknowns than equations, $n > m$.)

The solutions to **Ax = 0** form a vector space—the nullspace of **A**. The nullspace of a matrix consists of all vectors x such that **Ax = 0**. It is denoted by **N(A)**. It is a subspace of $\mathbf{R}^n$, just as the column space was a subspace of $\mathbf{R}^m$.

We want to be able, for any system Ax = b, to find C (A) and N (A): all attainable right-hand sides b and all solutions to Ax = 0.

Nullspace, Rank

(pg93) For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains more than the zero vector and/or the column space contains less than all vectors:

Every solution to $Ax = b$ is the sum of one particular solution and a solution to $Ax = 0$: $x_{complete} = x_{particular} + x_{nullspace}$. The particular solution comes from solving the equation with all free variables set to zero. Geometrically, the solutions again fill a two-dimensional surface—but it is not a sub-space. It does not contain $x = 0$. It is parallel to the nullspace we had before, shifted by the particular solution x_p

Elimination reveals the pivot variables and free variables. If there are **r pivots**, there are **r pivot variables** and **n − r free variables**. That important number r will be given a name—it is the **rank** of the matrix.

Example

Page 94 has an example

Linear Independence

Suppose $c_1 v_1 + \cdots + c_k v_k = 0$ only happens when $c_1 = \cdots = c_k = 0$. Then the vectors $v_1, \ldots, v_k$ are linearly independent. If any c's are nonzero, the v's are linearly dependent. One vector is a combination of the others.

A simpler way of saying this is that a set of vectors are linearly independent as long as each vector cannot be seen as a linear combination of the others.

A set of n vectors in R^m must be linearly dependent if $n > m$.

Why do we care? How does it connect to spaces?

The columns of A are independent exactly when $N(A)$ = {zero vector}. To check any set of vectors $v_1, \ldots, v_n$ for independence, put them in the columns of A. Then solve the system $Ac = 0$; the vectors are dependent if there is a solution other than $c = 0$. With no free variables (rank n), there is no nullspace except $c = 0$;

Spanning a Space

What does it mean to span a space or subspace? Basically it means having a set of vectors that can describe all vectors in the space as a linear combination. Formally: If a vector space V consists of all linear combinations of w 1 , . . . , w , then these vectors span the space. Every vector v in V is some combination of the w's.

Think of span as like a coordinate system: The coordinate vectors e 1 , . . . , e n coming from the identity matrix span R n

Basis of Subspace

Combination of spanning and linear independence: A basis for **V** is a sequence of vectors having two properties at once:

1. The vectors are linearly independent (not too many vectors).
2. They span the space V (not too few vectors).

There is one and only one way to write v as a combination of the basis vectors. A vector space has infinitely many different bases.

Connection: The columns that contain pivots (in this case the first and third, which correspond to the basic variables) are a basis for the column space.

Dimension

Any two bases for a vector space V contain the same number of vectors. This number, which is shared by all bases and expresses the number of “degrees of freedom” of the space, is the **dimension of V.**

The point is that a basis is a maximal independent set. It cannot be made larger without losing independence. A basis is also a minimal spanning set. It cannot be made smaller and still span the space.

Four Fundamental Spaces

The author points out that there are four fundamental spaces associated with a matrix A. We went over two of them. We also have spaces associated with the transpose of A.

The following picture shows how they connect.

For more information from the author of the book, checkout:

- https://www.youtube.com/watch?v=nHlE7EgJFds
- http://www.engineering.iastate.edu/~julied/classes/CE570/Notes/strangpaper.pdf
- http://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf

Four Fundamental Spaces

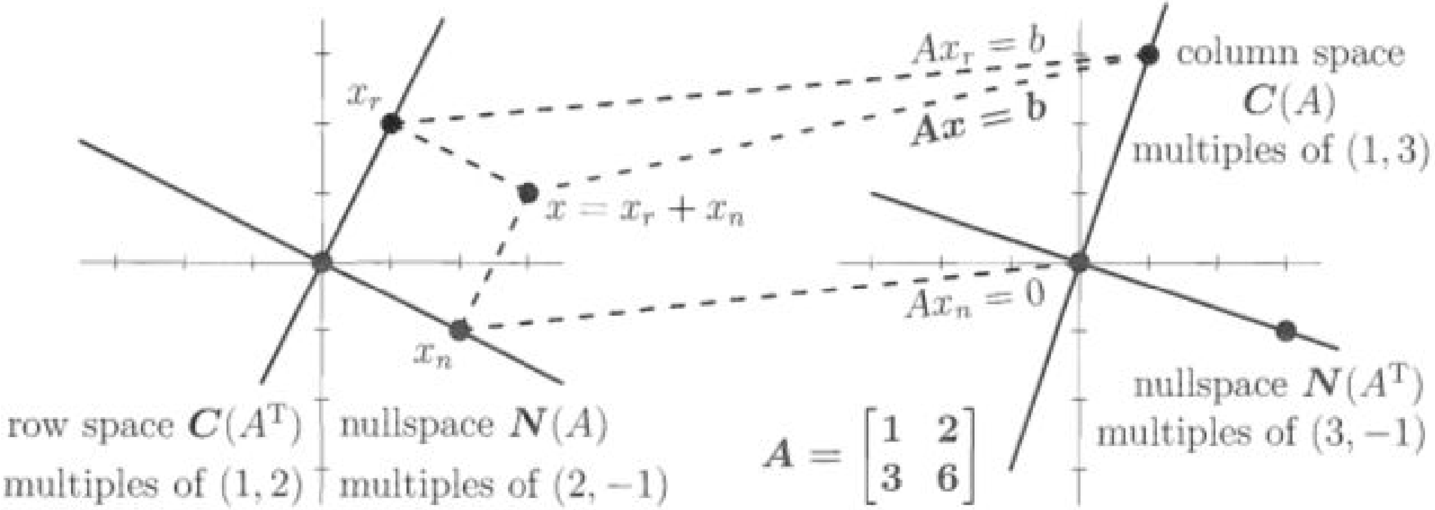

Figure 2.5: The four fundamental subspaces (lines) for the singular matrix *A*.

Useful Links

Links:

- http://www.endmemo.com/program/R/solve.php
- http://www.engineering.iastate.edu/~julied/classes/CE570/Notes/strangpaper.pdf (Fund theory of LA)
- http://web.mit.edu/18.06/www/Essays/newpaper_ver3.pdf (Fund theory of LA)
- https://www.math.uh.edu/~jmorgan/Math6397/day13/LinearAlgebraR-Handout.pdf (linear algebra R)
- http://www.statmethods.net/advstats/matrix.html (linear algebra R)

- http://www.gregthatcher.com/Mathematics/GaussJordan.aspx (Gauss Elimination calc)
- https://www.wolframalpha.com/examples/Matrices.html (Wolfram Alpha on matrices)

Videos:

- https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/
- 4 fundamnetal spaces: https://www.youtube.com/watch?v=nHlE7EgJFds
- https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces