Week4 Discussion

James Quacinella

09/29/2015

Question (Discussion) Exercise 3.5.1

On the space of nonnegative integers, which of the following functions are distance measures? If so, prove it; if not, prove that it fails to satisfy one or more of the axioms.

(a) max(x, y) = the larger of x and y.
(b) diff(x, y) = |x - y| (the absolute magnitude of the difference between x and y).
(c) sum(x, y) = x + y.

Answer

For a distance metric, we need to meet these conditions:

1. $d(x, y) >= 0$ (no negative distances)
2. $d(x, y) = 0$ if and only if $x = y$ (distances are positive, except for the distance from a point to itself).
3. $d(x, y) = d(y, x)$ (distance is symmetric).
4. $d(x, y) <= d(x, z) + d(z, y)$ (the triangle inequality)

Lets look at $diff(x, y) = |x - y|$. The first condition is obviously true from the definition of the absolute value operator. All elements in the range of $|z|$ are greater than 0, so condition one is satisfied.

The second condition is also true since the only time the $abs(z)$ function is 0 is when $z = 0$, in this case $z = x - y$. This infers that $x = y$ for this condition, and property two is satisfied.

The third condition is based on symmetry. To prove this, we need to prove that $d(x, y) = d(y, x)$, or $|x - y| = |y - x|$. We do know that the asolute value has the property that $|z| = | - z|$. Therefore, the left side of the proof, |x - y|, can be rewritten as:

$$\begin{aligned} &|x - y| \\ &= | - (x - y)| \\ &= | - x + y| \\ &= |y - x| \end{aligned}$$

The third condition is therefore satsified. The last condition states that:

$$d(x, y) <= d(x, z) + d(z, y)$$

Therefore, the following must hold:

$$|x - y| <= |x - z| + |z - y|$$

We are working with non-negative integers. Let us assume for now that x < y. The following arguments can be re-made for the y < x case, which will complete the proof. We have three conditions:

1) z < x
2) x < z < y
3) y < z

Lets look at each case. If z is less than x, the condition evaluates to:

$$|x-y| <= |x-z| + |z-y|$$
$$x-y <= (x-z) + (y-z)$$
$$x-y <= x+y-2z$$
$$x <= x+2y-2z$$
$$0 <= 2(y-z)$$
$$0 <= y-z$$
$$z <= y$$

So for case one, the condition holds, since we know by stipulation that $z <= y$. The same reasoning will hold for condition 3, $y < z$:

$$|x-y| <= |x-z| + |z-y|$$
$$x-y <= (z-x) + (z-y)$$
$$x-y <= 2z-x-y$$
$$x <= 2z-x$$
$$2x < 2z$$
$$x <= z$$

Which we know to be true by stipulation. The second condition, when $x < z < y$, is as follows:

$$|x-y| <= |x-z| + |z-y|$$
$$x-y <= (z-x) + (y-z)$$
$$x-y <= y-x$$
$$2x <= 2y$$
$$x <= y$$

Which is also true by stipulation. Without losing rigor, the same arguments can be made for y < x, due to symmetry. Therefore, $d(x,y)$, is a distance metric.